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On Volumes of Arithmetic Quotients of $SO(1,n)$

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Abstract. We apply G. Prasad's volume formula for the arithmetic quotients of semi-simple groups and Bruhat-Tits theory to study the covolumes of arithmetic subgroups of $SO(1,n)$. As a result we prove that for any even dimension n there exists a unique compact arithmetic hyperbolic n-orbifold of the smallest volume. We give a formula for the Euler-Poincaré characteristic of the orbifolds and present an explicit description of their fundamental groups as the stabilizers of certain lattices in quadratic spaces. We also study hyperbolic 4-manifolds defined arithmetically and obtain a number theoretical characterization of the smallest compact arithmetic 4-manifold.

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1. – Introduction

In this article we consider the problem of determining the smallest hyperbolic manifolds and orbifolds defined arithmetically. This problem has a long history which goes back to Klein and Hurwitz. Its solution for the hyperbolic dimension 2 was known to Hurwitz which allowed him to write down his famous bound for the order of the automorphisms group of a Riemann surface. The first extremal example for the bound is the Klein quartic. Many interesting facts about this classical subject and far reaching generalizations can be found in the book [Le]. For the dimension 3 the problem is also completely solved but the results are quite recent [CF], [CFJR]. For the higher dimensions very little is known.

Probably the most interesting case among the dimensions higher than 3 is in dimension 4. Recently it has attracted particular attention due to a possible application in cosmology: closed orientable hyperbolic 4-manifolds arise as the doubles of the real tunnelling geometries if the cosmological constant is assumed to be negative. In this context there are physical arguments in favor of using the smallest volume orientable hyperbolic 4-manifold as a model of the

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Lorentzian spacetime [Gi]. In view of the known facts and conjectures for the small dimensions it is quite natural to look for the smallest manifold among the arithmetic ones.

In this article we apply G. Prasad's volume formula for the arithmetic quotients of semi-simple groups and Bruhat-Tits theory to investigate the particular case of the group $SO(1,n)$ whose symmetric space is the hyperbolic n-space. Since our primary interest lies in dimension 4, at some point we restrict our attention to even dimensions and in the final section even more restrictively, we consider only the $SO(1,4)$-case. The main results are given in Theorems 4.1 and 5.5.

The first theorem says that the smallest compact arithmetic orbifold in any even dimension greater or equal than 4 is unique and defined over the field $\mathbb{Q}[\sqrt{5}]$, it also provides an explicit description for the orbifold and a formula for its volume. Let us remark that the quadratic number field $\mathbb{Q}[\sqrt{5}]$ has the interesting property that its fundamental unit $\epsilon = (1+\sqrt{5})/2$ is the "golden section" unit which was already known to Greek mathematicians. Since for the dimension 2 the situation is different (the field of definition of the smallest hyperbolic 2-orbifold is $\mathbb{Q}[\cos(2\pi/7)]$) this result was a little unexpected for us.

The problem of determining the smallest arithmetic manifold is much more delicate than that for the orbifolds. Here we are currently able to present only partial results and only in dimension 4. Still our Theorem 5.5 gives an explicit classification of all the possible candidates and essentially reduces the problem of finding the smallest arithmetic 4-manifold to an extensive computation, which we hope is practically possible.

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2. – Arithmetic subgroups

2.1. – Hyperbolic n-space can be obtained as a symmetric space associated to the orthogonal group G of type $(1,n)$:

$$\mathcal{H}^n = G/K_G = SO(1,n)^o/SO(n)$$

(K_G denotes a maximal compact subgroup of G). This way the connected component of identity of $SO(1,n)$ acts as a group of isometries of $\mathcal{H}^n$. A discrete subgroup Γ of G defines a locally symmetric space $X = \Gamma\backslash G/K_G$ which in our case will be an orientable hyperbolic orbifold or manifold if Γ is torsion-free. We will be interested in hyperbolic orbifolds which arise from arithmetic subgroups of G.

Let G be a connected semi-simple Lie group, H/k is a semi-simple algebraic group defined over a number field k and $\phi : H(k \otimes \mathbb{R}) \to G$ is a surjective homomorphism with a compact kernel. We consider H as a k-subgroup of $GL(n)$ for n big enough and define a subgroup Λ of $H(k)$ to be *arithmetic* if it is commensurable with the subgroup of k-integral points $H(k) \cap GL(n, \mathcal{O}_k)$, that is, the intersection $\Lambda \cap GL(n, \mathcal{O}_k)$ is of finite index in both Λ and $H(k) \cap GL(n, \mathcal{O}_k)$. The subgroups of G which are commensurable with $\phi(\Lambda)$ are called *arithmetic subgroups* of G defined over the field k. It can be shown that the notion of arithmeticity does not depend on a particular choice of the k-embedding of H into $GL(n)$.

We call an orbifold or manifold $X = \Gamma\backslash G/K_G$ *arithmetic* if Γ is an arithmetic subgroup of G, and we say that X is *defined over* k if k is the field of definition of Γ.

Arithmetic subgroups of the orthogonal groups can be constructed as follows. Let now k be a totally real algebraic number field with the ring of integers $\mathcal{O}$ and let f be a quadratic form of type $(1,n)$ with the coefficients in k such that for any non-identity embedding $\sigma : k \to \mathbb{R}$ the conjugate form f^σ is positive definite (such an f is called *admissible form*). Then given an $\mathcal{O}$-integral lattice L in k^{n+1} the group $\Gamma = G_L = \{\gamma \in G \cong SO(f)^o \mid L\gamma = L\}$ is an arithmetic subgroup of G defined over k.

It can be shown that for even n this construction gives all arithmetic subgroups of $G = SO(1,n)^o$ up to commensurability and conjugation in G. For odd n there is also another construction related to quaternion algebras, and for $n = 7$ there is a special type of arithmetic subgroups related to the Cayley algebra.

2.2. – Looking for the hyperbolic orbifolds and manifolds of the smallest volume we will be interested in the *maximal arithmetic subgroups* of G. The maximal arithmetic subgroups can be effectively classified in terms of the arithmetic data and the local structure of G. In order to discuss the classification picture we will give some more definitions.

Let G/k be a connected semi-simple algebraic group defined over a number field k, and let V_f (resp. V_∞) denote the set of finite (resp. infinite) places of k. By [BT] for a local place $v \in V_f$ the group $G(k_v)$ is endowed with the structure of Tits system of affine type $(G(k_v), B_v, N_v, \Delta_v)$. A subgroup $I_v \subset G(k_v)$ is called *Iwahori subgroup* if it is conjugate to B_v. A subgroup $P_v \subset G(k_v)$ which contains an Iwahori subgroup is called *parahoric*. A collection $P = (P_v)_{v\in V_f}$ of parahoric subgroups P_v is said to be *coherent* if $\prod_{v\in V_\infty} G(k_v) \cdot \prod_{v\in V_f} P_v$ is an open subgroup of the adèle group $G(\mathbb{A})$. A coherent collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ defines an arithmetic subgroup $\Lambda = G(k) \bigcap \prod_{v \in V_f} P_v$ of $G(k)$ which will be called the *principal arithmetic subgroup* determined by P. We will also call the corresponding subgroups of the Lie group G principal arithmetic subgroups.

If the group G/k is simply connected and adjoint then the maximal arithmetic subgroups of G are exactly the principal arithmetic subgroups defined by coherent collections of maximal parahoric subgroups. For the other forms the situation becomes more complicated, but still it is true that any maximal arithmetic subgroup is a normalizer in G of some principal arithmetic subgroup [Pl]. The problem of classification of the principal arithmetic subgroups which give rise to the maximal subgroups was studied in [CR] where, in particular, a criterion for the groups of type B_r (this is the type of $SO(1, 2r)$) is given. However, the criterion of Ryzhkov and Chernousov is a little subtle: it provides explicit conditions on the collections of parahoric subgroups but it does not always guarantee the existence of the global subgroup with the prescribed local properties. Let us consider this more carefully.

2.3. – Let Λ be a principal arithmetic subgroup of G/k defined by $\prod_{v \in V_f} P_v \subset G(\mathbb{A}_f) = \prod'_{v \in V_f} G(k_v)$ (where Π' denotes the restricted product with respect to $G(\mathcal{O}_v)$). For each place v the type of P_v depends on the splitting type of $G(k_v)$. We claim that there is a natural restriction on the possible splitting types of $G(k_v)$ which, in turn, implies a restriction on types of P_v.

In [K] to any reductive group G/k Kottwitz assigned an invariant $\epsilon(G) \in \{\pm 1\}$, which can be computed explicitly for G over the completions of k and for which the product formula holds. Thus, for G over a nonarchimedean local field

$$\epsilon(G(k_v)) = (-1)^{r(G_{qs}) - r(G)}$$

(G_{qs} denotes the quasi-split inner form of G and $r(G)$ is the k_v-rank of the derived group of G), and for the archimedean places

$$\epsilon(G(k_v)) = (-1)^{q(G_{qs}) - q(G)}$$

($q(G)$ is a half of the dimension of the symmetric space attached to $G(k_v)$). From the product formula for $\epsilon(G)$ we immediately obtain that the total number of places for which $\epsilon = -1$ is even. This is what we call the *parity condition* on the number of nonsplit places. Let us see what does it mean for our semi-simple groups of type B_r.

There are two forms B_r and ${}^2B_r$ of type B_r over a nonarchimedean local field, the first form is split and the second is a non quasi-split form [T]. In the first case the k_v-rank $r(G) = r$ and in the second case $r(G) = r - 1$, and also always $r(G_{qs}) = r$. So for $v \in V_f$ we have:

$$\epsilon(G(k_v)) = \begin{cases} 1 & \text{if } G \text{ is split over } k_v\,, \\ -1 & \text{otherwise}\,. \end{cases}$$

Over the archimedean places of k by the admissibility condition $G(k_v)$ is compact for all $v \in V_\infty$ except at one place, say $v = Id$, which implies:

$$q(G(k_v)) = 0 \text{ for } v \neq Id\,,$$
$$q(G(k_{Id})) = r\,.$$

For the quasi-split form we have:

$$\begin{aligned} q(G_{qs}) &= \frac{1}{2}(\dim(G_{qs}) - \dim(K_{G_{qs}})) \\ &= \frac{1}{2}\big(\dim(SO(r+1,r)) - \dim(S(O(r+1)\times O(r)))\big) = (r^2+r)/2\,. \end{aligned}$$

So for the place over which G is non-compact ($v = Id$):

$$\epsilon(G(k_v)) = \begin{cases} 1 & \text{if} \quad r \equiv 0,1\ (\mathrm{mod}\ 4)\,, \\ -1 & \text{if} \quad r \equiv 2,3\ (\mathrm{mod}\ 4)\,; \end{cases}$$

and for all the other infinite places:

$$\epsilon(G(k_v)) = \begin{cases} 1 & \text{if} \quad r \equiv 0,3\ \ (\mathrm{mod}\ 4)\,, \\ -1 & \text{if} \quad r \equiv 1,2\ \ (\mathrm{mod}\ 4)\,. \end{cases}$$

This implies, for example, that over a totally real quadratic field the number of nonarchimedean places over which G does not split is odd for odd r and even for r even. So we have a parity condition on the number of places over which G has type ${}^2B_r$. As we already remarked, this gives a restriction on the possible types of the collections of parahoric subgroups of G. This kind of restriction can not be seen in [CR], but it appears to be important for the applications.

It can be checked using the stabilizers of lattices at least for the orthogonal groups that this condition is also sufficient for the existence of the prescribed collections of parahoric subgroups.

2.4. – From the general theory of arithmetic subgroups of semi-simple Lie groups it follows that any arithmetic subgroup Γ is a discrete subgroup of G and the volume of $\Gamma\backslash G/K_G$ is finite. It is also known that $\Gamma \subset SO(1,2r)^o$ is cocompact if and only if the corresponding quadratic form f does not represent zero non-trivially over k, which for $n = 2r \geq 4$ means that Γ is non-cocompact if and only if it is defined over $\mathbb{Q}$. We are going to investigate the volumes of $\Gamma\backslash\mathcal{H}^n$.

For the future reference we fix some notations. Throughout this paper k will denote a totally real algebraic number field with the discriminant D_k, ring of integers $\mathcal{O}$ and adèle ring $\mathbb{A}$. The set of places V of k is a union of the set V_∞ of archimedean and V_f of finite places. For $v \in V_f$, as usually, k_v denotes the completion of k at v, $\mathcal{O}_v$ is the ring of integers of k_v with the uniformizer π_v and the residue degree $\#\mathcal{O}_v/\pi_v = q_v$.

3. – The volume formula

3.1. – In a fundamental paper [P] G. Prasad obtained a formula for the volume of a principal arithmetic subgroup of an arbitrary quasi-simple, simply connected group. This is an extensive generalization of the results of Siegel, Tamagawa, Harder, Borel and other people who worked in this direction. Gross has extended Prasad's formula to the arithmetic subgroups of reductive groups [Gr]. From these results we can write down a closed formula for the volume of a principal arithmetic subgroup Λ of $G = SO(1, 2r)^o$.

We use the Euler-Poincaré normalization of the Haar measure on $G(\mathbb{A})$ in the sense of Serre [S]. Namely, for a discrete subgroup Λ with finite covolume, we then have:

$$|\chi(\Lambda\backslash G)| = \mu^{EP}(\Lambda\backslash G)\,.$$

For a principal arithmetic subgroup Λ associated to a coherent collection of parahoric subgroups $P = (P_v)_{v\in V_f}$:

$$\text{(1)}\quad \mu^{EP}(\Lambda\backslash G) = \mu(\Lambda\backslash G) = c_\infty D_k^{\frac{1}{2}\dim G}\left(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]}\tau(G)\mathcal{E}\prod_{v\in T}\lambda_v\,,$$

where

- $c_\infty = 2$ is the Euler-Poincaré characteristic of the compact dual of the symmetric space $G/SO(n)$ (see [S], Section 3 and also [BP], Section 4 for the discussion);
- dimension $\dim G$ and exponents $\{m_i\}$ of our group G of type B_r are well known to be $\dim G = 2r^2 + r$ and $m_i = 2i - 1$ $(i = 1, \dots, r)$ [B];
- the Tamagawa number $\tau(G) = 2$ (see [W]);
- $\mathcal{E}$ is an Euler product which in our case is given by $\mathcal{E} = \zeta_k(2)\cdot\ldots\cdot\zeta_k(2r)$ ($\zeta_k(.)$ is the Dedekind zeta function of k);
- finally, the rational factors $\lambda_v \in \mathbb{Q}$ correspond to the (finite) set T of the finite places of k over which $P_v \not\cong G^o_{qs}(\mathcal{O}_v)$, where G^o_{qs} is the identity component of the quasi-split inner form of G.

This formula gives us the (generalized) Euler characteristic of Λ. The hyperbolic volume of $\Lambda\backslash\mathcal{H}^n$ can be obtained from $|\chi(\Lambda\backslash G)|$ by multiplying by the half of the volume of the unit sphere S^n in R^{n+1}:

$$\mathrm{vol}(\Lambda\backslash\mathcal{H}^{2r}) = \frac{(2\pi)^r}{1\cdot 3\cdot\ldots\cdot(2r-1)}\cdot|\chi(\Lambda\backslash G)|\,.$$

Note, that in odd dimensions the Euler characteristic vanishes but we can still obtain a similar formula for the covolume of an arithmetic subgroup without passing through the Euler-Poincaré measure.

3.2. – The λ-factors in (1) are the most subtle matter. Fortunately, we can explicitly compute the factors using the Bruhat-Tits theory. In [GHY] this was done for the parahoric subgroups which arise as the stabilizers of the maximal lattices. We will extend the table from [GHY] for the odd special orthogonal groups to the other maximal parahorics.

Consider orthogonal group $G = SO_{2r+1}$ over a nonarchimedean local field k_v whose residue field $\mathcal{O}/\pi$ has order q. By [T] the group G belongs to one of the two possible types: B_r or ${}^2B_r$. For $r > 2$ the local Dynkin diagrams and relative local index for the nonsplit type are given on Figure 1. For $r = 2$ there is an isogeny between the groups of types B and C, in [T] this case is represented by the diagrams of C_2 and ${}^2C_2$. We leave to the reader to check that (with the suitable notations) all our computations remain to be valid for this case as well.

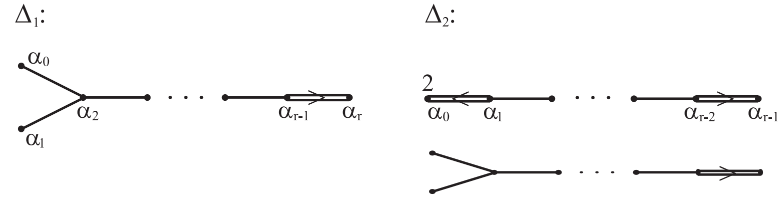

Fig. 1.

Similarly to [CR], having the local diagrams we can enumerate all the types of the maximal parahoric subgroups P of G, and the type defines a parahoric subgroup up to conjugation in G. However, some parahoric subgroups which are not conjugate in the simply connected group can become conjugate in the adjoint group. This happens exactly when the diagrams defining the types of the parahoric subgroups are symmetric with respect to an automorphism of the entire diagram. So, in our case types $\Delta_1 \setminus \{\alpha_0\}$ and $\Delta_1 \setminus \{\alpha_1\}$ define conjugate subgroups.

For each type, using results of Bruhat and Tits ([BT], the account of what we need can be found in [T], Section 3), we can determine the type of the maximal reductive quotient $\overline{G}$ of the special fiber $\underline{G}$ of the Bruhat-Tits group scheme associated with P and also the type of the reductive quotient $\overline{G}^o_{qs}$ of the smooth affine group scheme $\underline{G}^o_{qs}$ which was defined in [Gr], Section 4 for the quasi-split inner form of G (see also [P]). Now, using the tables of orders of finite groups of Lie type (e.g. [Ono], Table 1) for each of the cases the corresponding λ-factor is readily computed by the formula from [GHY], Section 2:

$$\lambda = \lambda(P) = \frac{q^{-N(\overline{G}^o_{qs})} \cdot \#\overline{G}^o_{qs}(\mathcal{O}/\pi)}{q^{-N(\overline{G})} \cdot \#\overline{G}(\mathcal{O}/\pi)}$$

($N(\overline{G})$ denotes the number of positive roots of $\overline{G}$ over the algebraic closure of the residue field $\mathcal{O}/\pi$). We list the results in the following table.

TABLE 1

maximal type θ_v	$\overline{G}$	$\overline{G}^o_{qs}$	λ
$\Delta_1\backslash\{\alpha_0\}$	SO_{2r+1}	SO_{2r+1}	1
$\Delta_1\backslash\{\alpha_0, \alpha_1\}$	$GL_1 \times SO_{2r-1}$	SO_{2r+1}	$\dfrac{q^{2r}-1}{q-1}$
$\Delta_1\backslash\{\alpha_i\},\ i = 2, \dots, r-1$	$O_{2i} \times SO_{2(r-i)+1}$	SO_{2r+1}	$\dfrac{(q^i+1)\prod_{\nu=i+1}^{r}(q^{2\nu}-1)}{2 \cdot \prod_{\nu=1}^{r-i}(q^{2\nu}-1)}$
$\Delta_1\backslash\{\alpha_r\}$	O_{2r}	SO_{2r+1}	$\dfrac{q^r+1}{2}$
$\Delta_2\backslash\{\alpha_0\}$	${}^2O_2 \times SO_{2r-1}$	SO_{2r+1}	$\dfrac{q^{2r}-1}{2(q+1)}$
$\Delta_2\backslash\{\alpha_i\},\ i = 1, \dots, r-2$	${}^2O_{2(i+1)} \times SO_{2(r-i)-1}$	SO_{2r+1}	$\dfrac{(q^{i+1}-1)\prod_{\nu=i+2}^{r}(q^{2\nu}-1)}{2 \cdot \prod_{\nu=1}^{r-i-1}(q^{2\nu}-1)}$
$\Delta_2\backslash\{\alpha_{r-1}\}$	${}^2O_{2r}$	SO_{2r+1}	$\dfrac{q^r-1}{2}$

3.3. – PROPOSITION. *For any rank $r \geq 2$ and $P_v \not\cong \underline{G}^o_{qs}(\mathcal{O}_v)$ we have:*

1) $\lambda_v > 1$;
2) $\lambda_v > 2$ *except for the case* $r = 2$, $q_v = 2$, $\theta_v = \Delta_2\backslash\{\alpha_1\}$.

PROOF. If P_v is a maximal parahoric subgroup then the statement reduces to an easy check of the values of λ in Table 1. For an arbitrary parahoric subgroup $P_v \subset G(k_v)$ there exists a maximal parahoric P which contains P_v. By the formula for λ we get $\lambda(P_v) = [P : P_v]\lambda(P)$, so the λ-factors of non-maximal parahoric subgroups also satisfy the conditions (1) and (2). □

We remark that as in the simply connected case ([BP], Section 3.1 and Appendix A) the minimal values of λ_v correspond to the special parahoric subgroups, i.e. those for which the diagram representing θ_v is the Coxeter diagram of the underlying finite reflection group.

3.4. – EXAMPLE. Let us consider the case when f is the unimodular integral quadratic form $-x_0^2 + x_1^2 + \ldots + x_n^2$ (n is even) and Γ is the stabilizer of a maximal lattice L on which f takes integral values. Then by [GHY] the group Γ is a principal arithmetic subgroup of $SO(f)^o = SO(1,n)^o$ and the types of the corresponding parahoric subgroups can be determined from the local invariants of the quadratic form f. We have $k = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$, v runs through the primes of $\mathbb{Q}$, the determinant $\delta_v(f) = \pm 1 \in \mathbb{Q}^\times$, the Hasse-Witt invariant $w_v(f) = \epsilon(SO(f;k_v)^o)$ is 1 for $v \neq 2$ and $w_2(f) = 1$ if $n \equiv 0, 2 \pmod 8$, $w_2(f) = -1$ if $n \equiv 4, 6 \pmod 8$. So we take the values for λ_v from the Table 4 in [GHY] and immediately obtain:

$$|\chi(\Gamma)| = 2\prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}} \, 2\prod_{i=1}^{r} \zeta(2i)\lambda_2$$

$$= 4\prod_{i=1}^{r} \frac{|B_{2i}|}{4i} \cdot \begin{cases} 1 & \text{if} \quad r \equiv 0, 1 (\text{mod } 4)\,, \\ 6^{-1}(2^{2r}-1) & \text{if} \quad r \equiv 2, 3 (\text{mod } 4) \end{cases}$$

(B_{2i} are Bernoulli numbers: $B_2 = 1/6, \; B_4 = -1/30, \; B_6 = 1/42 \ldots$). This can be compared with [RT] where the authors evaluated the Siegel's limit, but the results will not coincide. The reason is that Ratcliffe and Tschantz consider the arithmetic subgroups $SO(1,n;\mathbb{Z})$ of G which are the stabilizers of not maximal but unimodular lattices in the corresponding quadratic spaces. The relation between these two cases is not straightforward, but it appears that it is still possible to use a similar approach to obtain the covolumes of the stabilizers of the unimodular lattices and, in particular, to deduce the results of [RT]. We will explain this in detail in [BG].

3.5. – We now return to the maximal arithmetic subgroups. As it was already mentioned in Section 2.2 any maximal arithmetic subgroup Γ of G can be obtained as a normalizer in G of a principal arithmetic subgroup. So, in order to have a control over the volumes of $\Gamma = N_G(\Lambda)$ we need an estimate for the index $[\Gamma : \Lambda]$. Following [BP] such an estimate can be obtained from an exact sequence for the Galois cohomology of k due to Rohlfs [R]. We have ([BP], Section 2.10):

$$[\Gamma : \Lambda] \leq \# \prod_{v \in S} C(k_v) \cdot \# H^1(k, \widetilde{C})_\xi \cdot \prod_{v \in V \setminus S} \# \Xi_{\theta_v}\,,$$

where C is the center of G, $\widetilde{C}$ is the center of its simply connected inner form, Ξ_{θ_v} is the subgroup of the group of automorphisms of the local Dynkin diagram of $G(k_v)$ stabilizing the type θ_v and all the other notations can be found in [BP]. In our case the center of G is trivial, $S = V_\infty$, so we get

$$[\Gamma : \Lambda] \leq \# H^1(k, \widetilde{C})_\xi \cdot \prod_{v \in V_f} \# \Xi_{\theta_v}\,. \tag{2}$$

Let T_{ns} denote the (finite) set of places of k for which G does not split over k_v. By [BP], Proposition 5.1 applied to our group

$$\# H^1(k, \widetilde{C})_\xi \leq h_k \cdot 2^{[k:\mathbb{Q}] + \# T_{ns}}\,. \tag{3}$$

By [BP], the proof of Proposition 6.1, the class number h_k can be estimated as

$$h_k \leq 10^2 \left(\frac{\pi}{12}\right)^{[k:\mathbb{Q}]} D_k \tag{4}$$

(this bound follows from the Brauer-Siegel theorem and Zimmert's bound for the regulator of k. We refer to [BP] for the details).

We obtain:

$$\begin{aligned}
&\mu(\Gamma\backslash G)\\
&\geq \frac{1}{h_k 2^{[k:\mathbb{Q}]+\#T_{ns}} \prod_{v\in V_f} \#\Xi_{\theta_v}} c_\infty D_k^{\frac{1}{2}\dim G} \left(\prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]} \tau(G)\mathcal{E} \prod_{v\in T}\lambda_v\\
&\geq \frac{4D_k^{\frac{1}{2}\dim G-1}}{10^2\left(\frac{\pi}{6}\right)^{[k:\mathbb{Q}]}} \left(\prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]} \mathcal{E}\prod_{v\in T}\lambda_v \left(\prod_{v\in V_f}\#\Xi_{\theta_v}\right)^{-1} 2^{-\#T_{ns}}\,.
\end{aligned} \tag{5}$$

The group Ξ_{θ_v} is trivial if $G(k_v)$ is nonsplit or $\theta_v = \Delta_1\backslash\{\alpha_0\}$ and has order at most 2 in all the rest of the cases, so we always have

$$\prod_{v\in T}\lambda_v \left(\prod_{v\in V_f}\#\Xi_{\theta_v}\right)^{-1} 2^{-\#T_{ns}} \geq \prod_{v\in T}\frac{\lambda_v}{2}\,.$$

3.6. – Proposition.

$$\mathcal{E}\prod_{v\in T}\lambda_v \left(\prod_{v\in V_f}\#\Xi_{\theta_v}\right)^{-1} 2^{-\#T_{ns}} > 1\,.$$

Proof. Except for the case $r = 2$ and there exist $v \in V(k)$ with $q_v = 2$ the statement immediately follows from Proposition 3.3 (2). For the remaining case we need to split the factors corresponding to the 2-adic places from the Euler product $\zeta_k(2)$. The meaning of this is that in order to make the estimate we need to consider (almost) the actual volumes of certain parahoric subgroups, not just their quotients by the volume of the standard parahoric which are captured in the λ-factors:

$$\zeta_k(2)\zeta_k(4)\prod_{\substack{v\in T\\ q_v=2}}\frac{\lambda_v}{2}\prod_{\substack{v\in T\\ q_v\neq 2}}\frac{\lambda_v}{2} > \zeta_k(2)\prod_{\substack{v\in T\\ q_v=2}}\frac{\lambda_v}{2} \geq \zeta_k'(2)\prod_{\substack{v\in T\\ q_v=2}}\frac{1}{1-2^{-2}}\frac{2^2-1}{4} > 1. \quad \square$$

4. – Orbifolds

4.1. – Theorem. *For any $n = 2r \geq 4$ there exists a* unique *compact orientable arithmetic hyperbolic n-orbifold $O^n_{\min}$ of the smallest volume. It is defined over the field $\mathbb{Q}[\sqrt{5}]$ and has Euler characteristic*

$$|\chi(O^n_{\min})| = \frac{\lambda(r)}{N(r)4^{r-1}} \prod_{i=1}^{r} |\zeta_{\mathbb{Q}[\sqrt{5}]}(1-2i)|\,,$$

where: $\lambda(r) = 1$ *if r is even and* $\lambda(r) = 2^{-1}(4^r - 1)$ *if r is odd*;
$N(r)$ *is a positive integer*, ≤ 4 *if r is even*, *and* ≤ 8 *if r is odd.*

Proof. **1.** We are looking for a maximal arithmetic subgroup of $G = SO(1,n)^o$ of the smallest volume which is defined over $k \neq \mathbb{Q}$.

Let $k = \mathbb{Q}[\sqrt{5}]$. Then k is a totally real quadratic field of the smallest discriminant $D_k = 5$. By (1) the volume is proportional to ${D_k}^{\dim G/2}$ and depends exponentially on the degree of the field (for big enough r), so the smallest orbifold $O^n_0 = \Gamma^n_0 \backslash \mathcal{H}^n$ defined over k is a good candidate for $O^n_{\min}$. Let $P = (P_v)_{v \in V_f(k)}$ be a coherent collection of parahoric subgroups of $G(k)$ such that:

- if r is even $P_v = G(\mathcal{O}_v)$ for all $v \in V_f$;
- if r is odd $P_v = G(\mathcal{O}_v)$ for all v except one with the residue characteristic 2, for the remaining place v_2 we choose P_{v_2} so that $\lambda_{v_2} = (q^r - 1)/2$.

These collections of parahoric subgroups satisfy the conditions of the maximality criterion [CR] and the parity condition (Section 2.3). Let

$$\Lambda^n_0 = G(k) \cap \prod_v P_v\,.$$

So Λ^n_0 is a principal arithmetic subgroup of G. Let

$$\Gamma^n_0 = N_G(\Lambda^n_0)\,.$$

Then Γ^n_0 is a maximal arithmetic subgroup. We have:

$$\mu(\Lambda^n_0 \backslash G) = 4 \cdot 5^{r^2 + r/2} C(r)^2 \prod_{i=1}^{r} \zeta_k(2i)\, \lambda(r)\,,$$

$$\mu(\Gamma^n_0 \backslash G) = \frac{\mu(\Lambda^n_0 \backslash G)}{[\Gamma^n_0 : \Lambda^n_0]} = \frac{\mu(\Lambda^n_0 \backslash G)}{N(r)}\,.$$

Here $C(r)$ denotes the product $\prod_{i=1}^{r}(2i-1)!/(2\pi)^{2i}$, $\lambda(r) = \lambda_{v_2}$ is as in the statement of the theorem and $N(r)$ is the order of the group of outer automorphisms of Λ^n_0. By (2) and (3), $N(r) \leq 4$ for even r and $N(r) \leq 8$ for odd r. Later on we will state a conjecture about the actual value of $N(r)$.

Consider the other groups defined over quadratic extensions of $\mathbb{Q}$. Note, that for odd r the set T (of "bad places") should contain at least one place due to the parity condition. For Λ_0^n we have chosen T in such a way that the λ-factor in the volume formula has the smallest possible value for the groups defined over $\mathbb{Q}[\sqrt{5}]$. So let Γ^n be a maximal arithmetic subgroup of G defined over a totally real quadratic field k, $k \neq \mathbb{Q}[\sqrt{5}]$. By inequality (5) and Proposition 3.6:

$$\mu(\Gamma^n \backslash G) > \frac{1}{h_k} \cdot D_k^{r^2+r/2} C(r)^2 .$$

Now, except for the case $k = \mathbb{Q}[\sqrt{2}]$, $r = 3$:

$$\begin{aligned} \frac{1}{h_k} D_k^{r^2+r/2} C(r)^2 &\geq 4 \cdot 5^{r^2+r/2} C(r)^2 2\lambda(r) \\ &\geq 4 \cdot 5^{r^2+r/2} C(r)^2 \prod_{i=1}^{r} \zeta_{\mathbb{Q}[\sqrt{5}]}(2i)\lambda(r) \geq \mu(\Gamma_0^n \backslash G) . \end{aligned}$$

In the first inequality for $D_k > 28$ we used the bound (4) for the class number h_k, for the remaining fields of the small discriminants the class numbers are known to be equal to 1. The second inequality is provided by the following property of $\zeta_{\mathbb{Q}[\sqrt{5}]}(s)$:

$$(*) \qquad \prod_{i=1}^{r} \zeta_{\mathbb{Q}[\sqrt{5}]}(2i) < 2 \quad \text{for any } r .$$

The proof of $(*)$ is easy. Let again $k = \mathbb{Q}[\sqrt{5}]$.

$$P := \prod_{i=1}^{r} \zeta_k(2i) \leq \prod_{i=1}^{\infty} \zeta_k(2i) = \zeta_k(2) \prod_{i=2}^{\infty} \zeta_k(2i) \leq \zeta_k(2) \prod_{i=2}^{\infty} \zeta^2(2i) ,$$

$\zeta(n) = 1 + 1/2^n + 1/3^n + \ldots$ is the Riemann zeta function. By induction on n, for $n \geq 4$ we have $\zeta(n) \leq 1 + 2/2^n$. So

$$P \leq \zeta_k(2) \prod_{i=2}^{\infty} (1 + 2/2^{2i})^2 .$$

The right-side product converge and all the factors are > 1 so we can take its logarithm:

$$\log\left(\prod_{i=2}^{\infty}(1 + 2/2^{2i})^2\right) = \sum_{i=2}^{\infty} 2\log(1 + 1/2^{2i-1}) < \sum_{i=2}^{\infty} 1/2^{2i-2} = 1/3 ;$$

$$P < \zeta_k(2) e^{1/3} < 2$$

and $(*)$ is proved.

The remaining case $k = \mathbb{Q}[\sqrt{2}]$, $r = 3$ is checked directly. We use Proposition 3.3 to estimate the λ-factors keeping the Euler product for the next inequality:

$$\mu(\Gamma^n\backslash G) \geq 8^{10,5}C(r)^2\prod_{i=1}^{3}\zeta_k(2i)\prod_{v\in T}\frac{\lambda_v}{2}$$
$$\geq 8^{10,5}C(r)^2\prod_{i=1}^{3}\zeta_k(2i) > 4\cdot 5^{10.5}C(r)^2\prod_{i=1}^{3}\zeta_{\mathbb{Q}[\sqrt{5}]}(2i)\frac{4^3-1}{2}.$$

So we are done with the quadratic fields and can proceed to the higher degrees.

Let $[k:\mathbb{Q}] = 3$. We have:

$$\frac{\mu(\Gamma^n\backslash G)}{\mu(\Gamma_0^n\backslash G)} \geq \left(\frac{D_k}{5}\right)^{r^2+r/2}\frac{C(r)\prod_{i=1}^{r}\zeta_k(2i)\prod_{v\in T}\frac{\lambda_v}{2}}{2^3h_k\prod_{i=1}^{r}\zeta_{\mathbb{Q}[\sqrt{5}]}(2i)\ \lambda(r)}.$$

First consider the totally real cubic field of the smallest discriminant $D_k = 49$. This field has $h_k = 1$, moreover, since its ring of integers does not have prime ideals of norm 2, we can use Proposition 3.3 to estimate the λ-factors in all the cases:

$$\frac{\mu(\Gamma^n\backslash G)}{\mu(\Gamma_0^n\backslash G)} > \left(\frac{49}{5}\right)^{r^2+r/2}\frac{C(r)\prod_{i=1}^{r}\zeta_k(2i)}{8\prod_{i=1}^{r}\zeta_{\mathbb{Q}[\sqrt{5}]}(2i)\ \lambda(r)} > 1.$$

This inequality can be checked directly for $r = 2$; for the higher ranks it is enough to estimate the product of $\zeta_k(2i)$ by 1 from below and the product of $\zeta_{\mathbb{Q}[\sqrt{5}]}(2i)$ by $(*)$ from above, which gives an easy-to-check inequality.

For the other cubic fields by Proposition 3.6 and inequality $(*)$ we have

$$\frac{\mu(\Gamma^n\backslash G)}{\mu(\Gamma_0^n\backslash G)} > \left(\frac{D_k}{5}\right)^{r^2+r/2}\frac{C(r)}{8h_k2\lambda(r)}.$$

Again, using the precise values of h_k for the fields of the small discriminants ($D_k = 81,\ 148,\ 169$) and bound (4) for the other fields, we see that this is always greater then 1.

For $d = [k:\mathbb{Q}] \geq 4$ we will make use of the known lower bounds for the discriminants of the totally real number fields (see [Od]):

if $d = 4$ $\quad D_k > 5^d$;
if $d = 5$ $\quad D_k > 6.5^d$;
if $d \geq 6$ $\quad D_k > 7.9^d$ (and $D_k > 10^d$ if $d \geq 8$).

The cases $d = 4,\ 5$ are considered similar to the previous case $d = 3$ and we allow ourselves to skip the details. Let Γ^n is defined over a field k of degree $d \geq 6$. We have:

$$\frac{\mu(\Gamma^n\backslash G)}{\mu(\Gamma_0^n\backslash G)} > \frac{1}{5}\left(\frac{D_k}{5}\right)^{r^2+r/2-1} \frac{C(r)^{d-2}}{2^d \cdot 100 \cdot \left(\frac{\pi}{12}\right)^d 2\,\lambda(r)}.$$

For $r \geq 3$ we can estimate D_k by 7.9^d and then show that $\mu(\Gamma^n\backslash G)/\mu(\Gamma_0^n\backslash G) > 1$ for any d. This does not work for $r = 2$ since for large d the factor $C(r)^{d-2}$ becomes too small. In order to get rid of it we use the second bound $D_k > 10^d$ for $d \geq 8$. Note, that here we do not need any particular knowledge of the class numbers.

We proved that $O^n_{\min} = O^n_0 = \Gamma^n_0\backslash\mathcal{H}^n$ has the smallest possible volume for each n. Using the functional equation for the Dedekind zeta function we can write down the formula for $\mu(\Gamma_0^n\backslash G) = |\chi(O^n_{\min})|$ in a compact form which is given in the statement. It remains to show the uniqueness of $O^n_{\min}$.

2. Let H_1/k and H_2/k be two algebraic groups defined over $k = \mathbb{Q}[\sqrt{5}]$ such that each $H_i(k \otimes \mathbb{R})$ admits a surjective homomorphism onto G with a compact kernel. Then H_1 and H_2 are k-isogenous and the isogeny takes arithmetic subgroups to arithmetic subgroups. So we can fix an algebraic group H/k and the surjective homomorphism with a compact kernel $\phi : H(k \otimes \mathbb{R}) \to G$, such that H is of type B_r and can be supposed to be the adjoint group since G is centerless. Let Λ_1 and Λ_2 be two arithmetic subgroups of $H(k)$ of the same maximal type $P = (P_v)$. We want to prove that $N_G(\phi(\Lambda_1))$ is conjugate in G to $N_G(\phi(\Lambda_2))$.

For each finite place $v \in V_f$ there exists g_v such that $\Lambda_{1,v} = g_v \Lambda_{2,v} g_v^{-1}$ (see Section 3.2). The set of places

$$S = \{v \in V_f \mid \Lambda_{1,v} \neq H(\mathcal{O}_v) \text{ and } \Lambda_{2,v} \neq H(\mathcal{O}_v)\}$$

is finite. This is a known fact. To prove it one can first show the finiteness of such a set of places for the simply connected inner form of H using the strong approximation property (see e.g. [CR]), and then transfer it to H itself by an inner twist. So the set

$$U = \left\{ g \in H(\mathbb{A}_f) \mid \prod_{v \in V_f} \Lambda_{1,v} = g \left(\prod_{v \in V_f} \Lambda_{2,v} \right) g^{-1} \right\}$$

is not empty. Moreover, for each $g = (g_v) \in U$ we have an open subset

$$\prod_{v \in S} g_v \Lambda_{2,v} \prod_{v \in V_f \setminus S} \Lambda_{2,v} \subset U\,.$$

If $S = \emptyset$ then $\Lambda_1 = \Lambda_2$ and there is nothing to prove. Suppose S is non empty. We consider $H(k)$ diagonally embedded into $H(\mathbb{A}_f)$ and in $\prod_{v\in S} H(k_v)$. By the weak approximation property for H (H is an adjoint group so it has weak approximation [H]), $H(k) \cap \prod_{v\in S} g_v\Lambda_{2,v}$ is dense in $\prod_{v\in S} g_v\Lambda_{2,v}$ with respect to the product topology, so there exists a non empty open subset $X \subset H(k)$ such that for any $x \in X$ and any $v \in S$:

$$\Lambda_{1,v} = x\Lambda_{2,v}x^{-1}\,.$$

Let $p_1, \dots, p_n$ be the set of prime ideals in $\mathcal{O}$ which define the places from S. Consider the ring $R = \mathcal{O}[\frac{1}{p_1}, \dots, \frac{1}{p_n}]$. Since $S \neq \emptyset$ it is a dense subset of k, and so $H(R)$ is dense in $H(k)$. Consequently there exists $r \in X \cap H(R)$. We have:

$$\begin{aligned} r\Lambda_2 r^{-1} &= r\left(H(k)\cap\prod_{v\in V_f}\Lambda_{2,v}\right)r^{-1} = H(k)\cap r\left(\prod_{v\in V_f}\Lambda_{2,v}\right)r^{-1}\\ &= H(k)\cap\prod_{v\in S} r\Lambda_{2,v}r^{-1}\prod_{v\in V_f\setminus S} rH(\mathcal{O}_v)r^{-1} = H(k)\cap\prod_{v\in V_f}\Lambda_{1,v} = \Lambda_1\,;\end{aligned}$$

$$\begin{aligned}\phi(r)\phi(\Lambda_2)\phi(r)^{-1} &= \phi(r\Lambda_2 r^{-1}) = \phi(\Lambda_1)\,;\\ \phi(r)N_G(\phi(\Lambda_2))\phi(r)^{-1} &= N_G(\phi(\Lambda_1))\,.\end{aligned}$$ □

The reader can notice that part 1 of the proof can be simplified if we suppose that $r > 2$ but the case $r = 2$ is important. We will come back to it in the next section. Now let us give some remarks concerning the general case.

4.2. – The value of $N(r)$ is the order of the outer automorphisms group of the principal arithmetic subgroup Λ_0^n. Since we are in a very extremal situation we suppose that, in fact, Λ_0^n has no non-trivial symmetries. This can be checked for small n for which the group Λ_0^n is reflective (see [V]). We do not know how to prove this observation for the higher dimensions, but still we would like to have it as a conjecture.

Conjecture. For all $r \geq 2$, $N(r) = 1$ and so

$$|\chi(O^n_{\min})| = \frac{\lambda(r)}{4^{r-1}}\prod_{i=1}^{r}|\zeta_{\mathbb{Q}[\sqrt{5}]}(1-2i)|\,.$$

4.3. – We can describe groups Γ_0^n of the smallest orbifolds as the stabilizers of lattices in quadratic spaces. Let

$$f = -\frac{1+\sqrt{5}}{2}x_0^2 + x_1^2 + \dots + x_n^2\,,$$

and (V, f) is the corresponding $(n+1)$-dimensional quadratic space. By [GHY] for even $r = n/2$ the coherent collection of parahoric subgroups defining the principal arithmetic subgroups Λ_0^n has the same type as the one that gives the stabilizer of the maximal lattice in (V, f). So by the uniqueness argument from the proof of the theorem, Λ_0^n is the stabilizer of the maximal lattice in (V, f). Similarly, by [BG] for r odd Λ_0^n is the stabilizer of the odd unimodular lattice in (V, f) or, equivalently, it is the stabilizer of the maximal lattice in $(V, 2f)$. Consequently, the groups Γ_0^n are obtained as the normalizers in G of the stabilizers of the lattices.

4.4. – For completeness, let us also consider the non-compact case which is easy because the only possible field of definition is $k = \mathbb{Q}$. Similarly to the previous constructions (see also Example 3.4) we obtain:

For any $n = 2r \geq 4$ there is an unique non-compact orientable arithmetic hyperbolic n-orbifold O'^n_{min} of the smallest volume. It has Euler characteristic

$$|\chi(O'^n_{\mathrm{min}})| = \frac{\lambda'(r)}{N'(r)2^{r-2}} \prod_{i=1}^{r} |\zeta(1-2i)|,$$

where: $\lambda'(r) = 1$ if $r \equiv 0, 1 \pmod 4$,
$\lambda'(r) = 2^{-1}(2^r - 1)$ if $r \equiv 2, 3 \pmod 4$;
$N'(r)$ is a positive integer, ≤ 2 if $r \equiv 0, 1 \pmod 4$,
and ≤ 4 if $r \equiv 2, 3 \pmod 4$.

For $r \equiv 0, 1 \pmod 4$ the group of O'^n_{min} is the normalizer of the stabilizer of the maximal lattice in quadratic space (V, f) defined by

$$f = -x_0^2 + x_1^2 + \ldots + x_n^2;$$

and for $r \equiv 2, 3 \pmod 4$ it is the normalizer of the stabilizer of the odd unimodular lattice in (V, f).

Conjecture 4.2 also applies to O'^n_{min} and says that $N'(r) = 1$ for all r.

4.5. – We will now compute the Euler characteristics of the smallest orbifolds for small n. We will give the values for the principal arithmetic subgroups and then either Conjecture 4.2 is true or one should divide by the actual value of $N(r)$ in order to obtain the Euler characteristic of the smallest orbifolds. In any case, since $N(r)$ is bounded and always smaller then 8, this will not change the qualitative picture. We have:

TABLE 2

$n = 2r \geq 4$	4	6	8	10	12	
$\lvert\chi(\Lambda^n_{\mathrm{min}})\rvert$	$\frac{1}{7200}$	$\frac{67}{576000}$	$\frac{24187}{8709120000}$	$\frac{309479461547}{3483648000000}$	$\frac{7939510008126649607}{3766102179840000000}$	…
$\lvert\chi(\Lambda'^n_{\mathrm{min}})\rvert$	$\frac{1}{960}$	$\frac{1}{207360}$	$\frac{1}{348364800}$	$\frac{1}{91968307200}$	$\frac{691}{191294078976000}$	

	14	16	18
...	$8.1824\ldots \cdot 10^{10}$	$3.3481\ldots \cdot 10^{16}$	$1.7455\ldots \cdot 10^{34}$
	$\frac{87757}{289236647411712000}$	$\frac{2499347}{2360171042879569920000}$	$\frac{109638854849}{678029937198442846617600000}$

The smallest non-compact orbifold for all n (which is also the smallest among all the arithmetic orbifolds) has dimension

$$n = 16 \quad \text{and} \quad \chi = \frac{2499347}{2360171042879569920000} = 1.0589\ldots \cdot 10^{-15}\,.$$

The smallest compact orbifold has dimension $n = 8$ and $\chi = \frac{24187}{8709120000} = 0.00000277\ldots$. For $n = 4$ the volume of the smallest compact orbifold is less then that of the non-compact one and for all bigger n the non-compact orbifolds are smaller. After $n = 10$ in the compact case and $n = 18$ for non-compact (and so for all the arithmetic hyperbolic orbifolds) the minimal volumes start to increase and then grow exponentially with respect to the dimension.

It was first discovered in [Lu] that the minimal covolume can be attained on a non-uniform (that is, not cocompact) lattice. The result was obtained for the groups $SL(2, K)$ over local fields K of a positive characteristic. The natural question which appeared the same time was whether this is a purely local phenomenon or it is also possible for the groups over global fields. Our computation gives the answer to this question for the odd orthogonal groups over the totally real number fields, moreover, the method indicates that the minimality of the covolume of the non-uniform lattices might be always the case for the groups of a high enough rank.

4.6. – The previous remark can be considered in a wider context of [BP] where the discreteness of the set of covolumes of arithmetic subgroups was proved in a very general setting. In particular, it follows from [BP] that there exist "absolutely smallest" among all the S-arithmetic subgroups of G over k when G runs through the algebraic k-groups of absolute rank ≥ 2, the global field k can be either a number or a function field, S is any finite subset of places of k containing all the archimedean places and the Haar measures are chosen in a consistent way. Our results imply that for the adjoint groups G of type B_r and real rank 1 over the totally real number fields the smallest arithmetic subgroup is the unique smallest arithmetic non-cocompact subgroup of the group G of rank $r = 8$. It is not hard to generalize this results (except the uniqueness) to the S-arithmetic subgroups and to the other forms of type B_r over totally real fields. Our previous remark allows to conjecture that in general the smallest group might be non-cocompact which significantly reduces the number of possible candidates. Still the detailed study of this question lies beyond the scope of this paper. We have to point out that the geometric or any other meaning of the absolutely smallest group is completely mysterious, the only thing we know is that such a group or groups exist.

5. – 4-Manifolds

5.1. – Let us now consider the problem of determining the smallest arithmetic manifolds. This is a much more difficult task. From the arithmetic point of view the first difficulty is that we can not just estimate the Euler products but rather we have to deal with their rational structure. So, obtaining the precise values of the Euler characteristic for the groups of interest will be the first step. After this one needs to study the low index subgroups lattice of the distinguished groups and find the torsion-free subgroups. We will restrict our attention to the hyperbolic dimension 4.

5.2. – The smallest known example of a compact orientable hyperbolic 4-manifold was constructed by Davis [D]. It can be shown that the Davis manifold M_D is arithmetic and defined over the field $\mathbb{Q}[\sqrt{5}]$ [EM]. The Euler characteristic $\chi(M_D)$ is equal to 26. We will be looking for smaller examples, so we are interested in the manifolds with $\chi < 26$. It is well-known that the Euler characteristic of a compact orientable 4-manifold is an even positive integer which gives us one more natural restriction on χ.

5.3. – We start with refining some estimates from the proof of Theorem 4.1 for the case $r = 2$ and a larger bound for χ.

LEMMA. *If an orientable compact arithmetic hyperbolic* 4*-manifold defined over a field k has* $\chi \leq 24$ *then one of the following possibilities hold*:

(1) $d = 2$, $D_k \leq 362$;

(2) $d = 3$, $D_k \leq 3104$;

(3) $d = 4$, $D_k \leq 26574$;

(4) $d = 5$, $D_k \leq 227481$;

(5) $d = 6$, $D_k \leq 1947276$

where $d = [k : \mathbb{Q}]$ *is the degree of k and* D_k *is its discriminant.*

PROOF. The group of an arithmetic manifold is a (torsion-free) subgroup of a maximal arithmetic subgroup Γ, so we have $\chi(\Gamma) \leq 24$. From the other side

$$\begin{aligned}\chi(\Gamma) &> \frac{1}{2^d h_k} \cdot 4D_k^5 \left(\frac{6}{2^6\pi^6}\right)^d \\ &\geq \frac{1}{2^d \cdot 10^2 \cdot \left(\frac{\pi}{12}\right)^d D_k} \cdot 4D_k^5 \left(\frac{6}{2^6\pi^6}\right)^d \\ &\geq D_k^4 \cdot \frac{1}{25} \cdot \left(\frac{6^2}{2^6\pi^7}\right)^d .\end{aligned}$$

And thus we get

$$D_k < \left(24 \cdot 25 \cdot \left(\frac{6^2}{2^6\pi^7}\right)^d\right)^{1/4} .$$

For $d \geq 8$ this upper bound becomes smaller then the lower bound 10^d for the discriminant of a totally real field of degree d from [Od]. For $d = 7$ the precise smallest value of D_k is known to be $(11.051\ldots)^7$ [Od], and it again appears to be bigger then our upper bound for D_k. So we are left with the 5 remaining values of d and for each of them we compute the corresponding upper bound for D_k from the above inequality. □

5.4. – Using the tables of number fields of low degree [BFPOD] we can perform a more careful analysis of the groups over the fields which satisfy the conditions of Lemma 5.3. There are many fields which fit the conditions and we used a simple program for *GP/PARI* to perform the calculations. We obtain that among $109 + 98 + 182 + 45 + 32 = 466$ totally real fields which have discriminant in one of the 5 ranges only $21 + 12 + 12 + 2 = 47$ can actually admit the groups with $\chi \leq 24$ if we use the precise values of the class numbers in the volume estimate. For the remaining fields we compute the numerator ν of the Euler characteristic χ of the smallest arithmetic group Γ defined over the field (Γ's correspond to the principal arithmetic subgroups for which all the λ-factors in the volume formula are equal to 1). Since when passing to a finite index subgroup Γ' of Γ the number ν still divides the numerator of the Euler characteristic $\chi(\Gamma')$, we can discard all the groups with $\nu > 24$ or ν is odd and > 12. There are also several maximal groups with non-trivial λ-factors that fit into our range and these need to be checked in an entirely similar way. Finally, we are left with only two groups Γ_1 and Γ_2 which are defined over $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$ and have $\chi = 1/7200,\ 11/5760$, respectively. Group Γ_1 is the group of the smallest arithmetic 4-orbifold (see Section 4), Γ_2 is the smallest group defined over $\mathbb{Q}[\sqrt{2}]$ (the same argument as in the proof of Theorem 4.1 can be used to show that Γ_2 is defined uniquely up to conjugations in G).

Let us summarize the results.

5.5. – THEOREM. *If there exists a compact orientable arithmetic hyperbolic* 4*-manifold M having* $\chi(M) \leq 24$ *then it satisfies one of the following conditions*:

1) *M is defined over* $\mathbb{Q}[\sqrt{5}]$ *and has the form* $\Gamma_M\backslash\mathcal{H}^4$ *with* Γ_M *is a torsion-free subgroup of index* $7200\chi(M)$ *of the group* Γ_1 *of the smallest arithmetic* 4*-orbifold*;
2) *M has Euler characteristic* 22, *is defined over* $\mathbb{Q}[\sqrt{2}]$, *and its group is a torsion-free subgroup of index* 11520 *of* Γ_2 *which is the smallest principal arithmetic subgroup of* $SO(1,4)^o$ *defined over* $\mathbb{Q}[\sqrt{2}]$.

5.6. – This result reduces the problem of finding the smallest compact arithmetic 4-manifold to a computational problem: we need to search for the "low" index torsion-free subgroups of the groups Γ_1 and Γ_2 defined above. The first step to implement this in practice is to find good presentations for the maximal groups. For the group Γ_1 this can be done by identifying it with the orientation-preserving subgroup of a Coxeter group Γ_1' which has the Coxeter diagram given on Figure 2.

Fig. 2.

(It is easy to check that Γ_1' is an arithmetic subgroup of $O(1,4)$ defined over $\mathbb{Q}[\sqrt{5}]$ and $\chi(\Gamma_1') = 1/14400$, so its orientation-preserving subgroup is Γ_1 by the uniqueness of the smallest arithmetic orbifold.)

Now we can search for the torsion-free subgroups of Γ_1' of index $14400\chi(M)$. This, in principle, can be done by using the computer programs like *GAP*. The indexes of the subgroups we are interested in are quite large, but, as it can be checked, the Coxeter group Γ_1' has not many subgroups of low index, and so the computation looks more or less realistic.

We do not know whether or not the group Γ_2 is also reflective in a sense of [V], and we suppose that it is not. Since we are dealing with a stabilizer of not a modular but a maximal lattice the application of Vinberg's algorithm [V] for determining the maximal subgroup generated by reflections is not straightforward here. It is possible to write down an explicit matrix representation for the generators of Γ_2 in $SO(1,4)$, but we will not do it now. In any case this group can provide only an example with $\chi = 22$ which is already quite large.

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