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Uniqueness of entire functions sharing a small function with linear differential polynomials

INDRAJIT LAHIRI AND SHUBHASHISH DAS

Abstract. We consider the situation when an entire function shares a small function with linear differential polynomials. Our result improves a result of H. Zhong.

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1. Introduction, definitions and results

Suppose that f is a meromorphic function in the complex plane $\mathbb{C}$. A meromorphic function $a = a(z)$, defined in $\mathbb{C}$, is called a small function of f if $T(r, a) = S(r, f)$, where $T(r, a)$ is Nevanlinna's characteristic function of a and $S(r, f)$ is any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

We denote by $E(a; f)$ the collection of the zeros of $f - a$, where a zero is counted according to its multiplicity. Also by $\overline{E}(a; f)$ and by $E_{1)}(a; f)$ we denote the collection of distinct zeros of $f - a$ and simple zeros of $f - a$ respectively.

Suppose that f and g are two meromorphic functions in $\mathbb{C}$ and $a = a(z)$ is a small function of f and g. We say that f and g share the small function a CM (counting multiplicities) or IM (ignoring multiplicities) if $E(a; f) = E(a; g)$ or $\overline{E}(a; f) = \overline{E}(a; g)$ respectively.

The investigation of uniqueness of an entire function sharing certain values with its derivatives was initiated by L. A. Rubel and C. C. Yang in 1977, see [6]. They proved the following result.

Theorem A ([6]). *Let f be a nonconstant entire function. If for two values a and b, $E(a; f) = E(a; f^{(1)})$ and $E(b; f) = E(b; f^{(1)})$, then $f \equiv f^{(1)}$.*

Let $f(z) = \exp(e^z) \int_0^z \exp(-e^t)(1 - e^t)dt$. Then $f^{(1)} - 1 = e^z(f - 1)$ and so $E(1; f) = E(1; f^{(1)})$. Clearly $f \not\equiv f^{(1)}$ and we see that the hypothesis of

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two-value sharing in Theorem A is essential. So it appeared to be an interesting problem to investigate the situation of a single value sharing by an entire function with its derivative. To this end, the first result came from G. Jank, E. Mues and L. Volkmann [3], which may be stated as follows.

Theorem B ([3]). *Let f be a nonconstant entire function. If for a nonzero constant a, $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

We easily note that the hypothesis of Theorem B is equivalent to the following: $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$.

It is now a natural query whether the second order derivative can be replaced by a higher order one. H. Zhong [9] answered this query in the negative by means of the following example.

Example 1.1. Let $k(\geq 3)$ be a positive integer and $\omega(\neq 1)$ be a $(k-1)^{th}$ root of unity. If $g(z) = e^{\omega z} + \omega - 1$, then $g, g^{(1)}$ and $g^{(k)}$ share the value ω CM but neither $g \equiv g^{(1)}$ nor $g \equiv g^{(k)}$.

Accommodating the general order derivative, H. Zhong [9] proved the following result.

Theorem C ([9]). *Let f be a nonconstant entire function, $a(\neq 0)$ be a finite value and $n(\geq 1)$ be an integer. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$, then $f \equiv f^{(n)}$.*

Suppose that f is a nonconstant entire function and $a_1, a_2, \dots, a_n(\neq 0)$ are complex numbers.

Then
$$L = L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)} \tag{1.1}$$
is called a linear differential polynomial generated by f.

In 1999, P. Li [4] extended Theorem C to linear differential polynomials and proved the following result.

Theorem D ([4]). *Let f be a nonconstant entire function and L be defined by (1.1). Suppose that a is a nonzero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv f^{(1)} \equiv L$.*

In the present paper we extend Theorem C by considering shared small functions instead of shared values.

For two subsets A and B of $\mathbb{C}$, we denote by $A \Delta B$ the set $(A - B) \cup (B - A)$, which is called the symmetric difference of the sets A and B.

We refer the reader to the monograph [2] for standard definitions and notation of the value distribution theory.

Suppose that f is a meromorphic function and $a = a(z)$ is a small function of f. We denote by $n_{(2}(r, a; f)$ the number of multiple zeros of $f - a$ lying in $|z| \leq r$. The function
$$N_{(2}(r, a; f) = \int_0^r \frac{n_{(2}(t, a; f) - n_{(2}(0, a; f)}{t} dt + n_{(2}(0, a; f) \log r$$
is called the integrated counting function of multiple zeros of $f - a$.

Let $A \subset \mathbb{C}$. Then by $n_A(r, a; f)$ we denote the number of zeros of $f - a$ lying in $A \cap \{z : |z| \leq r\}$. The function

$$N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} dt + n_A(0, a; f) \log r$$

is called the integrated counting function of those zeros of $f - a$ that lie in A.

We now state the results of the present paper.

Theorem 1.2. *Let f be a nonconstant entire function and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f such that $a^{(1)} \not\equiv a$. Suppose that $A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f) \backslash \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$, where L defined by* (1.1) *is nonconstant. Then $f \equiv L = \alpha e^z$, where $\alpha(\neq 0)$ is a constant, provided the following hold:*

(i) $N_{A \cup B}(r, a; f) + N_A(r, a; f^{(1)}) = S(r, f)$;
(ii) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$;
(iii) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity.*

Putting $A = B = \emptyset$ we obtain the following corollary.

Corollary 1.3. *Let f be a nonconstant entire function and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f such that $a^{(1)} \not\equiv a$. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, L being nonconstant, then $f \equiv L = \alpha e^z$, where $\alpha(\neq 0)$ is a constant and L is defined by* (1.1).

The following example shows that the hypothesis $a^{(1)} \not\equiv a$ is essential for Theorem 1.2 and Corollary 1.3.

Example 1.4. Let $f = e^z + \exp(e^z)$ and $a = e^z$. Then $a(\not\equiv 0, \infty)$ is a small function of f. Also $E(a; f) = E(a; f^{(1)}) = \emptyset$ and so $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$. Clearly the conclusion of Theorem 1.2 and Corollary 1.3 does not hold.

We note that the function f of Example 1.4 is of infinite order. In the following theorem we see that the hypothesis "$a^{(1)} \not\equiv a$" can be removed from Corollary 1.3 if we consider an entire function of finite order.

Theorem 1.5. *Let f be a nonconstant entire function of finite order and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv L = \alpha e^z$, where $\alpha(\neq 0)$ is a constant and L is defined by* (1.1).

Let f be a nonconstant meromorphic function in $\mathbb{C}$ and $a_1, a_2, \dots, a_l(\not\equiv 0)$ be small functions of f. A function of the form

$$\psi = \sum_{j=1}^{l} a_j (f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \cdots \left(f^{(k)}\right)^{n_{kj}}$$

is called a differential polynomial generated by f, where $n_{ij} (i = 0, 1, \dots k;\ j = 1, 2, \dots l)$ and k are nonnegative integers.

The numbers $\gamma_\psi = \max_{1\le j\le l}\sum_{i=0}^{k} n_{ij}$ and $\Gamma_\psi = \max_{1\le j\le l}\sum_{i=0}^{k}(i+1)n_{ij}$ are respectively called the degree and weight of ψ.

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2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1 ([1]; see also [7]). *Let f be a meromorphic function and k be a positive integer. Suppose that f is a solution of the following differential equation: $a_0 w^{(k)} + a_1 w^{(k-1)} + \cdots + a_k w = 0$, where $a_0 (\neq 0), a_1, a_2, \ldots, a_k$ are constants. Then $T(r, f) = O(r)$. Furthermore, if f is transcendental, then $r = O(T(r, f))$.*

Lemma 2.2 ([1]). *Let f be a meromorphic function and n be a positive integer. If there exist meromorphic functions $a_0(\not\equiv 0), a_1, \ldots, a_n$ such that*

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n \equiv 0,$$

then

$$m(r, f) \le nT(r, a_0) + \sum_{j=1}^{n} m(r, a_j) + (n-1)\log 2.$$

Lemma 2.3 ([5]; see also [8, page 28]). *Let f be a nonconstant meromorphic function. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q}$$

is an irreducible rational function in f with the coefficients being small functions of f and $a_0 b_0 \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2.4. *Let $f, a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_q$ be meromorphic functions. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q} \quad (a_0 b_0 \not\equiv 0),$$

then

$$T(r, R(f)) = O\left(T(r, f) + \sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} T(r, b_j)\right).$$

Proof. The lemma follows from the first fundamental theorem and the properties of the characteristic function. □

Lemma 2.5 ([2, page 68]). *Let f be a transcendental meromorphic function and $f^n P(z) = Q(z)$, where $P(z)$, $Q(z)$ are differential polynomials generated by f and the degree of Q is at most n. Then $m(r, P) = S(r, f)$.*

Lemma 2.6 ([2, page 69]). *Let f be a nonconstant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(z),$$

where $P_{n-1}(z)$ is a differential polynomial generated by f and of degree at most $n-1$.

If $N(r, \infty; f) + N(r, 0; g) = S(r, f)$, then $g(z) = h^n(z)$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h^{(1)}(z)$ for $f^{(1)}(z)$ etc. *in the terms of degree $n-1$ in $P_{n-1}(z)$.*

Let us note the special case, where $P_{n-1}(z) = a_0(z) f^{n-1} +$ terms of degree $n-2$ at most. Then $h^{n-1}(z)a(z) = a_0(z)h^{n-1}(z)$ and so $a(z) = a_0(z)$. Hence $g(z) = \left(f(z) + \frac{a_0(z)}{n}\right)^n$.

Lemma 2.7 ([2, page 47]). *Let f be a nonconstant meromorphic function and a_1, a_2, a_3 be distinct small functions of f. Then*

$$T(r, f) \le \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).$$

We note that in Lemma 2.7 a_1, a_2, a_3 are allowed to be constants, and one of them may even be ∞.

3. Proofs of the theorems

Proof of Theorem 1.2. Let $\lambda = \frac{f^{(1)} - a}{f - a}$ and $g = f - a$. Then

$$g^{(1)} = \lambda g + a - a^{(1)} = \lambda_1 g + \mu_1, \tag{3.1}$$

where $\lambda_1 = \lambda$ and $\mu_1 = a - a^{(1)} = b$, say.

Differentiating (3.1) and using (3.1) repeatedly we get

$$g^{(k)} = \lambda_k g + \mu_k, \tag{3.2}$$

where $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1 \lambda_k$ and $\mu_{k+1} = \mu_k^{(1)} + \mu_1 \lambda_k$ for $k = 1, 2, \dots$

We now divide the proof into two parts.

Part I

We prove that $T(r,\lambda)=S(r,f)$. If λ is constant, then obviously $T(r,\lambda)=S(r,f)$. So we suppose that λ is nonconstant. By the hypothesis (i), (ii) and (iii) we get

$$N(r,0;\lambda)+N(r,\infty;\lambda)\le N_A(r,0;f-a)+N_A\big(r,0;f^{(1)}-a\big)=S(r,f). \tag{3.3}$$

Putting $k=1$ in $\lambda_{k+1}=\lambda_k^{(1)}+\lambda_1\lambda_k$ we get $\lambda_2=\lambda^2+d_1\lambda$, where $d_1=\frac{\lambda^{(1)}}{\lambda}$. Again putting $k=2$ in $\lambda_{k+1}=\lambda_k^{(1)}+\lambda_1\lambda_k$ we have $\lambda_3=\lambda_2^{(1)}+\lambda_1\lambda_2=\lambda^3+3d_1\lambda^2+d_2\lambda$, where $d_2=d_1^2+d_1^{(1)}$. Similarly $\lambda_4=\lambda_3^{(1)}+\lambda_1\lambda_3=\lambda^4+6d_1\lambda^3+(6d_1^2+3d_1^{(1)}+d_2)\lambda^2+(d_2^{(1)}+d_1d_2)\lambda$. Therefore, in general, we get for $k\ge 2$

$$\lambda_k=\lambda^k+\sum_{j=1}^{k-1}\alpha_j\lambda^j, \tag{3.4}$$

where $T(r,\alpha_j)=O(\overline{N}(r,0;\lambda)+\overline{N}(r,\infty;\lambda))+S(r,\lambda)=S(r,f)$ for $j=1,2,\ldots,k-1$.

Again putting $k=1$ in $\mu_{k+1}=\mu_k^{(1)}+\mu_1\lambda_k$ we get $\mu_2=\mu_1^{(1)}+\mu_1\lambda_1=b\lambda+b^{(1)}$. Also putting $k=2$ in $\mu_{k+1}=\mu_k^{(1)}+\mu_1\lambda_k$ we obtain by (3.4), $\mu_3=b\lambda^2+(b^{(1)}+bd_1+\alpha_1)\lambda+b^{(2)}$. Similarly $\mu_4=b\lambda^3+(2bd_1+b^{(1)}+b\alpha_2)\lambda^2+(b^{(2)}+2b^{(1)}d_1+bd^{(1)}+\alpha_1^{(1)}+bd_1^2+\alpha_1d_1+b\alpha_1)\lambda+b^{(3)}$. Therefore, in general, for $k\ge 2$

$$\mu_k=\sum_{j=1}^{k-1}\beta_j\lambda^j+b^{(k-1)}, \tag{3.5}$$

where $T(r,\beta_j)=O(\overline{N}(r,0;\lambda)+\overline{N}(r,\infty;\lambda))+S(r,\lambda)=S(r,f)$ for $j=1,2,\ldots,k-1$ and $\beta_{k-1}=b$.

Let z_0 be a zero of $f-a$ and $f^{(1)}-a$ with multiplicity $q(\ge 2)$. Then z_0 is a zero of $f^{(1)}-a^{(1)}$ with multiplicity $q-1$. Hence z_0 is a zero of $b=a-a^{(1)}=(f^{(1)}-a^{(1)})-(f^{(1)}-a)$ with multiplicity $q-1$. Since $q\le 2(q-1)$, we have $N_{(2}(r,a;f)\le 2N(r,0;b)+N_A(r,a;f)=S(r,f)$.

We first suppose that either $n\ge 2$ or $n=1$ and $a_1\ne 1$. Let

$$\psi=\frac{(a-L(a))\big(f^{(1)}-a^{(1)}\big)-\big(a-a^{(1)}\big)(L-L(a))}{f-a}. \tag{3.6}$$

From (3.6) we get $N(r,\psi)\le N_{(2}(r,a;f)+N_{A\cup B}(r,a;f)+(n+1)N(r,\infty;a)=S(r,f)$ and so $T(r,\psi)=S(r,f)$ because $m(r,\psi)=S(r,f)$.

Using (3.2), (3.4) and (3.5) we get

$$\begin{aligned}L(g)&=a_1g^{(1)}+\sum_{k=2}^{n}a_kg^{(k)}\\&=a_1(\lambda g+b)+\sum_{k=2}^{n}a_k\left(\lambda^k+\sum_{j=1}^{k-1}\alpha_j\lambda^j\right)g+\sum_{k=2}^{n}a_k\left(\sum_{j=1}^{k-1}\beta_j\lambda^j+b^{(k-1)}\right).\end{aligned}$$

Therefore from (3.6) we get

$$\begin{aligned} 0 \equiv & \left\{\psi + a_1 b\lambda + \sum_{k=2}^{n} a_k b\left(\lambda^k + \sum_{j=1}^{k-1}\alpha_j\lambda^j\right) - \lambda(a - L(a))\right\} g \\ & + b\left\{ba_1 + \sum_{k=2}^{n} a_k\left(\sum_{j=1}^{k-1}\beta_j\lambda^j + b^{(k-1)}\right) - (a - L(a))\right\}. \end{aligned} \tag{3.7}$$

If $\psi + a_1 b\lambda + \sum_{k=2}^{n} a_k b(\lambda^k + \sum_{j=1}^{k-1}\alpha_j\lambda^j) - \lambda(a - L(a)) \equiv 0$, then by Lemma 2.2 we get $m(r, \lambda) = S(r, f)$. Therefore by (3.3) we have $T(r, \lambda) = S(r, f)$.

Suppose that $\psi + a_1 b\lambda + \sum_{k=2}^{n} a_k b(\lambda^k + \sum_{j=1}^{k-1}\alpha_j\lambda^j) - \lambda(a - L(a)) \not\equiv 0$. Then from (3.7) we get

$$g = -\frac{b\left\{ba_1 + \sum\limits_{k=2}^{n} a_k\left(\sum\limits_{j=1}^{k-1}\beta_j\lambda^j + b^{(k-1)}\right) - (a - L(a))\right\}}{\psi + a_1 b\lambda + \sum\limits_{k=2}^{n} a_k b\left(\lambda^k + \sum\limits_{j=1}^{k-1}\alpha_j\lambda^j\right) - \lambda(a - L(a))}. \tag{3.8}$$

From (3.8) we get by Lemma 2.4, $T(r, g) = O(T(r, \lambda)) + S(r, f)$ and so $T(r, f) = O(T(r, \lambda)) + S(r, f)$. This implies that $S(r, f)$ is replaceable by $S(r, \lambda)$.

Also, from (3.8) we see that g is a rational function in λ, which can be made irreducible. We now put

$$g = \frac{P_s(\lambda)}{Q_{s+1}(\lambda)}, \tag{3.9}$$

where $P_s(\lambda)$ and $Q_{s+1}(\lambda)$ are relatively prime polynomials in λ of respective degrees s and $s+1$. The coefficients of both the polynomials are small functions of λ. Without loss of generality we assume that $Q_{s+1}(\lambda)$ is a monic polynomial. We further note that the counting function of the common zeros of $P_s(\lambda)$ and $Q_{s+1}(\lambda)$, if any, is $S(r, \lambda)$, because $P_s(\lambda)$ and $Q_{s+1}(\lambda)$ are relatively prime and the coefficients are small functions of λ.

Since $N(r, \infty; g) = S(r, f) = S(r, \lambda)$, we see from (3.9) that $N(r, 0; Q_{s+1}(\lambda)) = S(r, \lambda)$. Also by (3.3) we know that $N(r, \infty; \lambda) = S(r, f) = S(r, \lambda)$. So by Lemma 2.6 we get

$$Q_{s+1}(\lambda) = \left(\lambda + \frac{c}{s+1}\right)^{s+1}, \tag{3.10}$$

where c is the coefficient of λ^s in $Q_{s+1}(\lambda)$.

If $c \not\equiv 0$, then by Lemma 2.7 we obtain

$$\begin{aligned} T(r, \lambda) &\le \overline{N}(r, 0; \lambda) + \overline{N}(r, \infty; \lambda) + \overline{N}\left(r, -\frac{c}{s+1}; \lambda\right) + S(r, \lambda) \\ &= \overline{N}(r, 0; Q_{s+1}(\lambda)) + S(r, \lambda) \\ &= S(r, \lambda), \end{aligned}$$

a contradiction. Therefore $c \equiv 0$ and we get from (3.9) and (3.10)

$$g = \frac{P_s(\lambda)}{\lambda^{s+1}}. \tag{3.11}$$

Differentiating (3.11) we obtain

$$g^{(1)} = d_1 \frac{\lambda P_s^{(1)}(\lambda) - (s+1)P_s(\lambda)}{\lambda^{s+1}},$$

where $d_1 = \frac{\lambda^{(1)}}{\lambda}$ and $T(r, d_1) = O(\overline{N}(r, 0; \lambda) + \overline{N}(r, \infty; \lambda)) + m(r, d_1) = S(r, f) + S(r, \lambda) = S(r, \lambda)$. So by Lemma 2.3 we have

$$T\left(r, g^{(1)}\right) = (s + 1 - p)T(r, \lambda) + S(r, \lambda), \tag{3.12}$$

for some integer $p, 0 \le p \le s$.

Again since $g^{(1)} = \lambda g + b$, where $b = a - a^{(1)} \not\equiv 0$, we get from (3.11)

$$g^{(1)} = \frac{P_s(\lambda)}{\lambda^s} + b$$

and so by Lemma 2.3 we have

$$T\left(r, g^{(1)}\right) = (s - p)T(r, \lambda) + S(r, \lambda), \tag{3.13}$$

where p is same as in (3.12). Now from (3.12) and (3.13) we get $T(r, \lambda) = S(r, \lambda)$, a contradiction.

Next we suppose that $n = 1$ and $a_1 = 1$. Let

$$\phi = \frac{\left(a - L^{(1)}(a)\right)(L - L(a)) - (a - L(a))\left(L^{(1)} - L^{(1)}(a)\right)}{f - a}.$$

Since in this case $L = f^{(1)}$, we get

$$\begin{aligned} \phi &= \frac{\left(a - a^{(2)}\right)\left(f^{(1)} - a^{(1)}\right) - \left(a - a^{(1)}\right)\left(f^{(2)} - a^{(2)}\right)}{f - a} \\ &= \frac{\left(a - a^{(2)}\right)g^{(1)} - bg^{(2)}}{g}. \end{aligned} \tag{3.14}$$

By the hypothesis we have $T(r, \phi) = S(r, f)$. Using (3.2), (3.4), (3.5) and (3.14) we get

$$\left\{b\lambda^2 + (\alpha_1 b - a + a^{(2)})\lambda + \phi\}g + b\{b^{(1)} + \beta_1\lambda + a^{(2)} - a\right\} \equiv 0. \tag{3.15}$$

Following the similar argument of the preceding case and using (3.15) we can show that $m(r, \lambda) = S(r, f)$. So by (3.3) we have $T(r, \lambda) = S(r, f)$. This completes the proof of Part I.

Part II

First we verify that

$$T(r, f) \leq 3\overline{N}(r, 0; f - a) + S(r, f). \tag{3.16}$$

By the first fundamental theorem we get

$$\begin{aligned} T(r, f) &= T(r, f - a) + S(r, f) \\ &= T\left(r, \frac{1}{f - a}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f - a}\right) + m\left(r, \frac{1}{f - a}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f - a}\right) + m\left(r, \frac{1}{f^{(1)} - a^{(1)}}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f - a}\right) + T\left(r, f^{(1)}\right) - N\left(r, \frac{1}{f^{(1)} - a^{(1)}}\right) + S(r, f). \end{aligned}$$

Now by Lemma 2.7 we get from above

$$\begin{aligned} T(r, f) \leq N(r, 0; f - a) + \overline{N}\left(r, 0; f^{(1)} - a\right) + \overline{N}\left(r, 0; f^{(1)} - a^{(1)}\right) \\ - N\left(r, 0; f^{(1)} - a^{(1)}\right) + S(r, f). \end{aligned} \tag{3.17}$$

Let us denote by $N^{p}_{(k}(r, 0; F)$ the counting function of zeros of F with multiplicities not less than k and a zero of multiplicity $q(\geq k)$ is counted $q - p$ times, where $p \leq k$.

Now

$$\begin{aligned} &N(r, 0; f - a) + \overline{N}\left(r, 0; f^{(1)} - a^{(1)}\right) - N\left(r, 0; f^{(1)} - a^{(1)}\right) \\ &= \overline{N}(r, 0; f - a) + N^{1}_{(2}(r, 0; f - a) - N^{1}_{(2}\left(r, 0; f^{(1)} - a^{(1)}\right) \\ &= \overline{N}(r, 0; f - a) + \overline{N}_{(2}(r, 0; f - a) + N^{2}_{(3}(r, 0; f - a) - N^{1}_{(2}\left(r, 0; f^{(1)} - a^{(1)}\right) \\ &\leq 2\overline{N}(r, 0; f - a) + N^{1}_{(2}\left(r, 0; f^{(1)} - a^{(1)}\right) - N^{1}_{(2}\left(r, 0; f^{(1)} - a^{(1)}\right) + S(r, f) \\ &= 2\overline{N}(r, 0; f - a) + S(r, f), \end{aligned}$$

where $\overline{N}_{(2}(r, 0; f - a)$ is the integrated counting function of distinct multiple zeros of $f - a$.

Therefore from (3.17) we get

$$T(r, f) \leq 2\overline{N}(r, 0; f - a) + \overline{N}\left(r, 0; f^{(1)} - a\right) + S(r, f). \tag{3.18}$$

Since

$$\overline{N}(r, 0; f^{(1)} - a) \le \overline{N}(r, 0; f - a) + N_A(r, 0; f^{(1)} - a) = \overline{N}(r, 0; f - a) + S(r, f),$$

(3.16) is obtained from (3.18).

Since $T(r, \lambda) = S(r, f)$, we see that $T(r, \lambda_k) + T(r, \mu_k) = S(r, f)$ for $k = 1, 2, \dots$, where λ_k and μ_k are defined in (3.2). Now

$$\begin{aligned} L &= \sum_{k=1}^{n} a_k f^{(k)} = \sum_{k=1}^{n} a_k g^{(k)} + L(a) \\ &= \left(\sum_{k=1}^{n} a_k \lambda_k\right) g + \sum_{k=1}^{n} a_k \mu_k + L(a) = \xi g + \eta, \text{ say.} \end{aligned} \tag{3.19}$$

Clearly $T(r, \xi) + T(r, \eta) = S(r, f)$. Differentiating (3.19) we get

$$L^{(1)} = \xi^{(1)} g + \xi g^{(1)} + \eta^{(1)}. \tag{3.20}$$

Let $z_0 \notin A \cup B$, be a zero of $g = f - a$. Then from (3.19) and (3.20) we get $a(z_0) - \eta(z_0) = 0$ and $\xi(z_0)(a(z_0) - a^{(1)}(z_0)) + \eta^{(1)}(z_0) - a(z_0) = 0$.

If $a(z) - \eta(z) \not\equiv 0$, we get

$$\overline{N}(r, 0; f - a) \le N_{A\cup B}(r, 0; f - a) + N(r, 0; a - \eta) + S(r, f) = S(r, f),$$

which contradicts (3.16). Therefore

$$a(z) \equiv \eta(z). \tag{3.21}$$

Again if $\xi(z)(a(z) - a^{(1)}(z)) + \eta^{(1)}(z) - a(z) \not\equiv 0$, we get

$$\begin{aligned} \overline{N}(r, 0; f - a) \le N_{A\cup B}(r, 0; f - a) + N\left(r, 0; \xi\left(a - a^{(1)}\right) + \eta^{(1)} - a\right) \\ + S(r, f) = S(r, f), \end{aligned}$$

which contradicts (3.16). Therefore

$$\xi(z)\left(a(z) - a^{(1)}(z)\right) + \eta^{(1)}(z) - a(z) \equiv 0. \tag{3.22}$$

Since $a(z) \not\equiv a^{(1)}(z)$, from (3.21) and (3.22) we get $\xi(z) \equiv 1$. Hence from (3.19) and (3.21) we get $L \equiv g + a \equiv f$.

By actual calculation we see that $\lambda_2 = \lambda^2 + \lambda^{(1)}$ and $\lambda_3 = \lambda^3 + 3\lambda\lambda^{(1)} + \lambda^{(2)}$. We now verify, in general, that

$$\lambda_k = \lambda^k + P_{k-1}[\lambda], \tag{3.23}$$

where $P_{k-1}[\lambda]$ is a differential polynomial in λ with constant coefficients such that the degree $\gamma_{P_{k-1}} \leq k-1$ and the weight $\Gamma_{P_{k-1}} \leq k$. Also each term of $P_{k-1}[\lambda]$ contains some derivative of λ.

Let (3.23) be true. Then

$$\begin{aligned}\lambda_{k+1} &= \lambda_k^{(1)} + \lambda_1\lambda_k \\ &= \left(\lambda^k + P_{k-1}[\lambda]\right)^{(1)} + \lambda\left(\lambda^k + P_{k-1}[\lambda]\right) \\ &= \lambda^{k+1} + P_k[\lambda],\end{aligned}$$

where we note that differentiation does not increase the degree of a differential polynomial but increases its weight by 1. So (3.23) is verified by mathematical induction.

Since $\xi(z) \equiv 1$, by (3.19) and (3.23) we get

$$\sum_{k=1}^{n} a_k\lambda^k + \sum_{k=1}^{n} a_k P_{k-1}[\lambda] \equiv 1. \tag{3.24}$$

By the hypotheses (ii) and (iii) we see that λ has no simple pole. Let z_0 be a pole of λ with multiplicity $p(\geq 2)$. Then z_0 is a pole of $\sum_{k=1}^{n} a_k\lambda^k$ with multiplicity np and it is a pole of $\sum_{k=1}^{n} a_k P_{k-1}[\lambda]$ with multiplicity at most $(n-1)p+1$. Since $np > (n-1)p+1$, it follows that z_0 is a pole of the left hand side of (3.24) with multiplicity np, which is impossible. So λ is an entire function. If λ is transcendental, then by Lemma 2.5 we get from (3.24) that $T(r,\lambda) = S(r,\lambda)$, a contradiction. If λ is a polynomial of degree $d(\geq 1)$, then the left hand side of (3.24) is a polynomial of degree nd, which is also a contradiction. Therefore λ is a constant and so from (3.23) we get $\lambda_k = \lambda^k$ for $k = 1, 2, \ldots$. We suppose that $\lambda \neq 1$.

Since $L \equiv f$, we see by Lemma 2.1 that $T(r, f) = O(r)$ and so $T(r,a) = o(r)$, because a is a small function of f.

Since λ is a constant, by a simple calculation we get $\mu_k = \sum_{j=0}^{k-1} b^{(k-1-j)}\lambda^j$ for $k = 1, 2, \ldots$. Therefore from (3.19) we have

$$\eta = L(a) + \sum_{k=1}^{n} a_k\mu_k = L(a) + \sum_{k=1}^{n} a_k \left(\sum_{j=0}^{k-1} b^{(k-1-j)}\lambda^j\right). \tag{3.25}$$

From (3.21) and (3.25) we see that $a = a(z)$ is an entire function. Since $T(r,a) = o(r)$, by Lemma 2.1, (3.21) and (3.25) we observe that $a = a(z)$ is a polynomial.

Now from (3.1) we get

$$f^{(1)} = \lambda f + (1-\lambda)a = \lambda f + P_l, \tag{3.26}$$

where P_l is a polynomial of degree l.

Differentiating (3.26) $l+1$ times we get $f^{(l+2)} = \lambda f^{(l+1)}$ and so $f^{(l+1)} = \beta e^{\lambda z}$, where $\beta(\neq 0)$ is a constant. Now integrating $f^{(l+1)} = \beta e^{\lambda z}$, $l+1$ times we get

$$f = \frac{\beta}{\lambda^{l+1}} e^{\lambda z} + Q_t,$$

where Q_t is a polynomial of degree $t(\leq l)$.

Since $\xi(z) \equiv 1$ and $\lambda_k = \lambda^k$, we have $\sum_{k=1}^{n} a_k \lambda^k = 1$. Hence

$$L = \sum_{k=1}^{n} a_k f^{(k)} = \left(\sum_{k=1}^{n} a_k \lambda^k\right) \frac{\beta}{\lambda^{l+1}} e^{\lambda z} + \sum_{k=1}^{n} a_k Q_t^{(k)} = \frac{\beta}{\lambda^{l+1}} e^{\lambda z} + \sum_{k=1}^{n} a_k Q_t^{(k)}.$$

Since $f \equiv L$, we have $Q_t \equiv \sum_{k=1}^{n} a_k Q_t^{(k)}$ and this implies $Q_t \equiv 0$. Therefore $f = \frac{\beta}{\lambda^{l+1}} e^{\lambda z}$ and from (3.26) we get $\frac{\beta}{\lambda^l} e^{\lambda z} = \frac{\beta}{\lambda^l} e^{\lambda z} + (1-\lambda)a$, which is impossible as $\lambda \neq 1$ and $a \not\equiv 0$. Hence $\lambda = 1$ and so from (3.26) we obtain $f \equiv L = \alpha e^z$, where $\alpha(\neq 0)$ is a constant. This proves the theorem. □

Proof of Theorem 1.5. Let $a \equiv a^{(1)}$. Then $a = \beta e^z$, where $\beta(\neq 0)$ is a constant. Since $E(a; f) = E(a; f^{(1)})$ and f is of finite order, there exists a polynomial h such that $\frac{f^{(1)}-a}{f-a} = e^h$ and so $\frac{f^{(1)}-a^{(1)}}{f-a} = e^h$. Integrating we get $f = a + \gamma e^\nu$, where $\gamma(\neq 0)$ is a constant and $\nu^{(1)}(z) = e^{h(z)}$. Since f and so a are of finite order, we see that ν is a polynomial. Again $E(a; f) = E(a; f^{(1)}) = \emptyset$ and $f^{(1)} = a + \gamma \nu^{(1)} e^\nu$ imply that $\nu^{(1)}$ is a constant. So $\nu = cz + d$, where $c(\neq 0)$ and d are constants. Therefore $f = a + \gamma e^{cz+d}$ and this contradicts the fact that $a = \beta e^z$ is a small function of f. Hence $a \not\equiv a^{(1)}$ and the theorem follows from Corollary 1.3. This proves the theorem. □

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Department of Mathematics
University of Kalyani
West Bengal 741235, India
ilahiri@hotmail.com
dshubhashish.90@gmail.com