Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 1521-1549

A moving lemma for cycles with very ample modulus

AMALENDU KRISHNA AND JINHYUN PARK

Abstract. We prove a moving lemma for higher Chow groups with modulus, in the sense of Binda-Kerz-Saito, of projective schemes, when the modulus is given by a very ample divisor. This provides one of the first cases of moving lemmas for cycles with modulus, not covered by the additive higher Chow groups. We apply this to prove a contravariant functoriality of higher Chow groups with modulus. We use our moving techniques to show that the higher Chow groups of a line bundle over a scheme, with the 0-section as the modulus, vanish.

Mathematics Subject Classification (2010): 14C25 (primary); 13F35, 19E15 (secondary).

1. Introduction

The moving lemma is one of the most important technical tools in dealing with algebraic cycles. For usual higher Chow groups, this was established by S. Bloch (see [2,3]). In order to study the relative K-theory of schemes (relative to effective divisors) in terms of algebraic cycles, the theory of additive higher Chow groups (see [5, 9, 10, 14]) and cycles with modulus (see [1, 8]) were recently introduced. But the lack of a moving lemma has been an annoying hindrance in the study of these additive higher Chow groups and the Chow groups with modulus.

A moving lemma for additive higher Chow groups of smooth projective schemes was proven in [10]. A similar moving lemma for the additive higher Chow groups of smooth affine schemes has been very recently established by W. Kai [7], along with some more general results after Nisnevich sheafifications. However, without such modifications, one does not yet know the existence of a moving lemma for the higher Chow groups with modulus which do not arise from additive higher Chow groups.

JP was partially supported by the National Research Foundation of Korea (NRF) grant No. 2015R1A2A2A01004120 funded by the Korean government (MSIP). AK was partially supported by the Swarna Jayanti Fellowship, 2011.

Received September 17, 2015; accepted in revised form September 27, 2016.
Published online December 2017.

1.1. Main results

The goal of this paper is to address the moving lemma problem for the higher Chow groups with modulus of projective schemes when the modulus divisor is very ample. Our main result is the following. The necessary definitions are recalled in Section 2.

Theorem 1.1. *Let X be an equidimensional reduced projective scheme of dimension $d \geq 1$ over a field k. Let $D \subsetneq X$ be a very ample effective Cartier divisor such that $X \setminus D$ is smooth over k. Let $\mathcal{W}$ be a finite collection of locally closed subsets of X. Then, the inclusion $z^q_{\mathcal{W}}(X|D, \bullet) \hookrightarrow z^q(X|D, \bullet)$ is a quasi-isomorphism.*

Our first application of Theorem 1.1 is the following complete solution of the moving lemma for cycles with arbitrary modulus on projective spaces. The analogous question for cycles on affine spaces was solved by W. Kai [7].

Corollary 1.2. *Let k be any field and $r \geq 1$ be any integer. Let $D \subset \mathbb{P}^r_k$ be any effective Cartier divisor. Let $\mathcal{W}$ be a finite collection of locally closed subsets of $\mathbb{P}^r_k$. Then the inclusion $z^q_{\mathcal{W}}(\mathbb{P}^r_k|D, \bullet) \hookrightarrow z^q(\mathbb{P}^r_k|D, \bullet)$ is a quasi-isomorphism.*

In the second application of Theorem 1.1, we prove the following contravariance property of the higher Chow groups with modulus.

Theorem 1.3. *Let $f : Y \to X$ be a morphism of equidimensional reduced quasi-projective schemes over a field k, where X is projective over k. Let $D \subset X$ be a very ample effective Cartier divisor such that $X \setminus D$ is smooth over k. Suppose that $f^*(D)$ is a Cartier divisor on Y (*i.e.*, no minimal or embedded component of Y maps into D). Then there exists a map*

$$f^* : z^q(X|D, \bullet) \to z^q(Y|f^*(D), \bullet)$$

in the derived category of Abelian groups. In particular, for every $p, q \geq 0$, there is a pull-back

$$f^* : \mathrm{CH}^q(X|D, p) \to \mathrm{CH}^q(Y|f^*(D), p).$$

Corollary 1.4. *Let $r \geq 1$ be an integer and let $f : Y \to \mathbb{P}^r_k$ be a morphism of quasi-projective schemes over a field k. Let $D \subset \mathbb{P}^r_k$ be an effective Cartier divisor such that $f^*(D)$ is a Cartier divisor on Y. Then, there exists a pull-back $f^* : \mathrm{CH}^q(\mathbb{P}^r_k|D, p) \to \mathrm{CH}^q(Y|f^*(D), p)$ for every $p, q \geq 0$.*

As a final application of our moving techniques, we prove the following vanishing theorem for the higher Chow groups of a line bundle on a scheme with the modulus given by the 0-section. This provides examples where the higher Chow groups of a variety with a modulus in an effective Cartier divisor are all zero. As one knows, this is not possible for the ordinary higher Chow groups. This also gives an evidence in support of the expectation that the higher Chow groups with modulus are the relative motivic cohomology.

Theorem 1.5. *Let X be a quasi-projective scheme over a field k and let $f : \mathcal{L} \to X$ be a line bundle. Let $\iota : X \hookrightarrow \mathcal{L}$ denote the 0-section embedding. Then, the cycle complex $z_s(\mathcal{L}|X, \bullet)$ is acyclic for all $s \in \mathbb{Z}$.*

1.2. Outline of proofs

We prove Theorem 1.1 by following the classical approach used by Bloch to prove his moving lemma for ordinary higher Chow groups of smooth projective schemes. We first prove the above theorem for projective spaces. The main difficulty here lies in constructing suitable homotopy varieties and to check their modulus condition. We solve this problem by using some blow-up techniques and our homotopy varieties are very different from the one used classically.

To deal with the case of general projective schemes, we use the method of linear projections. However, we need to make more subtle choices of our linear subspaces than in the classical case due to the presence of the modulus.

We show later in this article how this method breaks down if we replace a very ample divisor by just an ample one. We show that the linear projection method cannot be used in general to prove the moving lemma for Chow groups with modulus on either smooth affine or smooth projective schemes, if the modulus divisor is not very ample. This suggests that the general case of the moving lemma for Chow groups with modulus on smooth affine or projective schemes may be a very challenging task.

ACKNOWLEDGEMENTS. The authors are deeply indebted to the referee, who so thoroughly read the paper and suggested many valuable corrections and simplifications.

2. Recalls on cycles with modulus

In this section we recollect some necessary definitions and notation associated with cycles with modulus. Let k be a field and let $\mathbf{Sch}_k$ denote the category of quasi-projective schemes over k. Let $\mathbf{Sm}_k$ denote the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes.

2.1. Notation

Set $\mathbb{A}^1_k := \operatorname{Spec} k[t]$, $\mathbb{P}^1_k := \operatorname{Proj} k[Y_0, Y_1]$ and let $y := Y_0/Y_1$ be the coordinate on $\mathbb{P}^1_k$. We set $\square := \mathbb{A}^1_k$ and $\overline{\square} := \mathbb{P}^1_k$. We use the coordinate system $(y_1, \cdots, y_n)$ on $\overline{\square}^n$ with $y_i := y \circ q_i$, where $q_i : \overline{\square}^n \to \overline{\square}$ is the projection onto the i-th $\overline{\square}$. For $i = 1, \ldots, n$, let $F^\infty_{n,i}$ be the Cartier divisor on $\overline{\square}^n$ defined by $\{y_i = \infty\}$. Let F^∞_n denote the Cartier divisor $\sum_{i=1}^n F^\infty_{n,i}$ on $\overline{\square}^n$. A *face* of $\overline{\square}^n$ is a closed subscheme defined by a set of equations of the form $\{y_{i_1} = \epsilon_1, \ldots, y_{i_s} = \epsilon_s \mid \epsilon_j \in \{0, 1\}\}$. For $\epsilon = 0, 1$, and $i = 1, \cdots, n$, let $\iota_{n,i,\epsilon} : \overline{\square}^{n-1} \hookrightarrow \overline{\square}^n$ be the inclusion

$$\iota_{n,i,\epsilon}(y_1, \ldots, y_{n-1}) = (y_1, \ldots, y_{i-1}, \epsilon, y_i, \ldots, y_{n-1}). \qquad (2.1)$$

A face of $\square^n$ is an intersection of $\square^n$ with a face of $\overline{\square}^n$.

2.2. Cycles with modulus

Let $X \in \mathbf{Sch}_k$. Recall ([11, Section 2]) that for effective Cartier divisors D_1 and D_2 on X, we say $D_1 \leq D_2$ if $D_1 + D = D_2$ for some effective Cartier divisor D on X. A *modulus pair* or a *scheme with an effective divisor* is a pair (X, D), where $X \in \mathbf{Sch}_k$ and D an effective Cartier divisor on X. A morphism $f : (Y, E) \to (X, D)$ of modulus pairs is a morphism $f : Y \to X$ in $\mathbf{Sch}_k$ such that $f^*(D)$ is defined as a Cartier divisor on Y and $f^*(D) \leq E$. In particular, $f^{-1}(D) \subset E$. If $f : Y \to X$ is a morphism of k-schemes, and (X, D) is a modulus pair such that $f^{-1}(D) = \emptyset$, then $f : (Y, \emptyset) \to (X, D)$ is a morphism of modulus pairs.

Definition 2.1 ([1,8]). Let (X, D) and $(\overline{Y}, E)$ be two modulus pairs. Let $Y = \overline{Y} \setminus E$. Let $V \subset X \times Y$ be an integral closed subscheme with closure $\overline{V} \subset X \times \overline{Y}$. We say V *has modulus* D *on* $X \times Y$ (relative to E) if $\nu_V^*(D \times \overline{Y}) \leq \nu_V^*(X \times E)$ on $\overline{V}^N$, where $\nu_V : \overline{V}^N \to \overline{V} \hookrightarrow X \times \overline{Y}$ is the normalization followed by the closed immersion.

Definition 2.2 ([1,8]). Let (X, D) be a modulus pair. For $s \in \mathbb{Z}$ and $n \geq 0$, let $\underline{z}_s(X|D, n)$ be the free Abelian group on integral closed subschemes $V \subset X \times \square^n$ of dimension $s + n$ satisfying the following conditions:

(1) (Face condition) for each face $F \subset \square^n$, V intersects $X \times F$ properly;
(2) (Modulus condition) V has modulus D relative to F_n^∞ on $X \times \square^n$.

We usually drop the phrase "relative to F_n^∞" for simplicity. A cycle in $\underline{z}_s(X|D, n)$ is called an *admissible cycle with modulus* D. The following containment lemma is from [11, Proposition 2.4] (see also [1, Lemma 2.1] and [10, Proposition 2.4]).

Proposition 2.3. *Let* (X, D) *and* $(\overline{Y}, E)$ *be modulus pairs and* $Y = \overline{Y} \setminus E$. *If* $V \subset X \times Y$ *is a closed subscheme with modulus* D *relative to* E*, then any closed subscheme* $W \subset V$ *also has modulus* D *relative to* E.

One checks using Proposition 2.3 that $(n \mapsto \underline{z}_s(X|D, n))$ is a cubical Abelian group. In particular, the groups $\underline{z}_s(X|D, n)$ form a complex with the boundary map $\partial = \sum_{i=1}^n (-1)^i (\partial_i^0 - \partial_i^1)$, where $\partial_i^\epsilon = \iota_{n,i,\epsilon}^*$.

Definition 2.4 ([1,8]). The complex $(z_s(X|D, \bullet), \partial)$ is the nondegenerate complex associated to $(n \mapsto \underline{z}_s(X|D, n))$, *i.e.*, $z_s(X|D, n) := \underline{z}_s(X|D, n)/\underline{z}_s(X|D, n)_{\mathrm{degn}}$. The homology $\mathrm{CH}_s(X|D, n) := \mathrm{H}_n(z_s(X|D, \bullet))$ for $n \geq 0$ is called *higher Chow group of* X *with modulus* D. If X is equidimensional of dimension d, for $q \geq 0$, we write $\mathrm{CH}^q(X|D, n) = \mathrm{CH}_{d-q}(X|D, n)$.

The following is a generalization of [11, Proposition 2.12] (see also [1, Lemma 2.7]). The reader can check that the only requirement in the proof of [11, Proposition 2.12] is that the underlying map be flat over the complement of the modulus divisor. This is because of the fact that an admissible cycle lies completely over this complement.

Lemma 2.5. *Let $f : Y \to X$ be a morphism in* $\mathbf{Sch}_k$. *Let $D \subsetneq X$ be an effective Cartier divisor. Assume that $f^*(D)$ is a Cartier divisor on Y such that the map $f^{-1}(X \setminus D) \to X \setminus D$ is flat of relative dimension d. Then, there is a pull-back map $f^* : z_r(X|D, \bullet) \to z_{d+r}(Y|f^*(D), \bullet)$ such $(f \circ g)^* = g^* \circ f^*$.*

We often use the following result from [11, Lemma 2.2]:

Lemma 2.6. *Let $f : Y \to X$ be a dominant map of normal integral k-schemes. Let D be a Cartier divisor on X such that the generic points of* $\mathrm{Supp}(D)$ *are contained in $f(Y)$. Suppose that $f^*(D) \geq 0$ on Y. Then $D \geq 0$ on X.*

Definition 2.7. Let $\mathcal{W}$ be a finite set of locally closed subsets of X and let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set function. Let $\underline{z}^q_{\mathcal{W},e}(X|D, n)$ be the subgroup generated by integral cycles $Z \in \underline{z}^q(X|D, n)$ such that for each $W \in \mathcal{W}$ and each face $F \subset \square^n$, we have $\mathrm{codim}_{W \times F}(Z \cap (W \times F)) \geq q - e(W)$. They form a subcomplex $\underline{z}^q_{\mathcal{W},e}(X|D, \bullet)$ of $\underline{z}^q(X|D, \bullet)$. Modding out by degenerate cycles, we obtain the subcomplex $z^q_{\mathcal{W},e}(X|D, \bullet) \subset z^q(X|D, \bullet)$. We write $z^q_{\mathcal{W}}(X|D, \bullet) := z^q_{\mathcal{W},0}(X|D, \bullet)$. The number $e(W)$ is called the *excess dimension* of the intersection $Z \cap (W \times F)$. Given a function $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$, define $(e - 1) : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ by $(e - 1)(W) = \max\{e(W) - 1, 0\}$. This gives an inclusion $z^q_{\mathcal{W},e-1}(X|D, \bullet) \subset z^q_{\mathcal{W},e}(X|D, \bullet)$.

We also use the following from [12, Proposition 4.3] in our proof of the moving lemma.

Proposition 2.8 (Spreading lemma). *Let $k \subset K$ be a purely transcendental extension. Let (X, D) be a smooth quasi-projective k-scheme with an effective Cartier divisor, and let $\mathcal{W}$ be a finite collection of locally closed subsets of X. Let (X_K, D_K) and $\mathcal{W}_K$ be the base changes via* $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(k)$. *Let $\mathrm{pr}_{K/k} : X_K \to X_k$ be the base change map. Then for every set function $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$, the pull-back maps*

$$\mathrm{pr}^*_{K/k} : \frac{z^q_{\mathcal{W},e}(X|D, \bullet)}{z^q_{\mathcal{W}}(X|D, \bullet)} \to \frac{z^q_{\mathcal{W}_K,e}(X_K|D_K, \bullet)}{z^q_{\mathcal{W}_K}(X_K|D_K, \bullet)} \tag{2.2}$$

and

$$\mathrm{pr}^*_{K/k} : \frac{z^q_{\mathcal{W},e}(X|D, \bullet)}{z^q_{\mathcal{W},e-1}(X|D, \bullet)} \to \frac{z^q_{\mathcal{W}_K,e}(X_K|D_K, \bullet)}{z^q_{\mathcal{W}_K,e-1}(X_K|D_K, \bullet)} \tag{2.3}$$

are injective on homology.

We remark that Proposition 2.8 is stated in [12, Proposition 4.3] only for (2.2) but the argument given there simultaneously proves (2.3) as well.

3. Moving lemma for projective spaces

In this section we prove our moving lemma for the modulus pair (X, D), where X is a projective space over k and D is a hyperplane in X. We use the following:

Lemma 3.1 ([2, Lemma 1.2]). *Let $X \in \mathbf{Sch}_k$ and let G be a connected algebraic group over k acting on X. Let $A, B \subset X$ be closed subsets. Assume that the fibers of the action map $G \times A \to X$, given by $(g, a) \mapsto g \cdot a$, all have the same dimension and that this map is dominant.*

Assume moreover that there is an overfield $k \hookrightarrow K$ and a K-morphism $\psi : X_K \to G_K$. Let $\emptyset \neq U \subset X$ be open such that for every $x \in U_K$, we have

$$\mathrm{tr.deg}_k\, (\phi \circ \psi(x), \pi(x)) \geq \dim(G),$$

where $\pi : X_K \to X$ and $\phi : G_K \to G$ are the base changes. Define $\theta : X_K \to X_K$ by $\theta(x) = \psi(x) \cdot x$ and assume that θ is an isomorphism. Then, the intersection $\theta(A_K \cap U_K) \cap B_K$ is proper.

Corollary 3.2. *Let $X \in \mathbf{Sch}_k$ and let G be a connected algebraic group over k acting transitively on X. Let $Y \in \mathbf{Sch}_k$ and let $\emptyset \neq A \subset X$ and $B \subset X \times Y$ be closed subsets. Let G act on $X \times Y$ by $g \cdot (x, y) = (g \cdot x, y)$.*

Let $K = k(G)$ and let $\phi : G_K \to G$ be the base change. Suppose $\psi : (X \times Y)_K \to G_K$ is a K-morphism and let $U \hookrightarrow X \times Y$ be an open subset such that:

(1) *the image of every point of U_K under the composite map $(X \times Y)_K \xrightarrow{\psi} G_K \xrightarrow{\phi} G$ is the generic point of G;*
(2) *the map $\theta : (X \times Y)_K \to (X \times Y)_K$ given by $\theta(z) = \psi(z) \cdot z$, is an isomorphism.*

Then the intersection $\theta((A \times Y)_K \cap U_K) \cap (B_K \cap U_K)$ is proper on U_K.

We let $\mathbb{A}^r_k = \mathrm{Spec}\,(k[x_1, \cdots, x_r])$ and let $\mathbb{P}^r_k = \mathrm{Proj}(k[X_1, \cdots, X_r, X_0])$, where we set $x_i = X_i / X_0$ for $1 \leq i \leq r$. This yields an open immersion $j_0 : \mathbb{A}^r_k \hookrightarrow \mathbb{P}^r_k$. Let $H_\infty = \mathbb{P}^r_k \setminus \mathbb{A}^r_k$ be the hyperplane at infinity. We write the homogeneous coordinates of $\mathbb{P}^r_k$ as $(X_1; \cdots; X_r; X_0)$. We fix this choice of coordinates of $\mathbb{A}^r_k$ and $\mathbb{P}^r_k$. Set $u = \prod_{i=1}^r x_i \in k[x_1, \cdots, x_r]$.

Let $K = k(\mathbb{P}^r_k)$ and consider the point $\eta = (u, \cdots, u) \in \mathbb{P}^r_K$ so that its image under the projection $\mathbb{P}^r_K \to \mathbb{P}^r_k$ is the generic point of $\mathbb{P}^r_k$. Let $U_+ \hookrightarrow \mathbb{P}^r_K \times \overline{\square}_K$ be the open subset $(\mathbb{P}^r_K \times \square_K) \cup (\mathbb{A}^r_K \times \overline{\square}_K)$ and set $\mathcal{Y} = H_\infty \times \{\infty\} = (\mathbb{P}^r_K \times \overline{\square}_K) \setminus U_+$. For K-schemes X and X', we write the product $X \times_K X'$ as $X \times X'$.

Lemma 3.3. *Let $\phi_\eta : \mathbb{A}^r_K \times \square_K \to \mathbb{A}^r_K$ denote the map $\phi_\eta(x, t) = x + \eta \cdot t$. Then, ϕ_η uniquely extends to a morphism $\phi_\eta|_{U_+} : U_+ \to \mathbb{P}^r_K$ such that the following hold:*

(1) *U_+ is the largest open subset of $\mathbb{P}^r_K \times \overline{\square}_K$ over which ϕ_η can be extended to a regular morphism;*
(2) *The extension of ϕ_η on $\mathbb{P}^r_K \times \square_K$ is a smooth morphism;*
(3) *$(\phi_\eta|_{U_+})^{-1}(\mathbb{A}^r_K) = \mathbb{A}^r_K \times \square_K$;*
(4) *$(\phi_\eta|_{U_+})^{-1}(H_\infty) = (\mathbb{A}^r_K \times \{\infty\}) + (H_\infty \times \square_K)$.*

Proof. Define the rational map $\phi_\eta : \mathbb{P}^r_K \times \overline{\square}_K \dashrightarrow \mathbb{P}^r_K$ by

$$\begin{aligned}&\phi_\eta((X_1; \cdots ; X_r; X_0), (T_0; T_1)) \\ &= (T_1X_1 + uT_0X_0; \cdots ; T_1X_r + uT_0X_0; T_1X_0).\end{aligned} \tag{3.1}$$

Note that $\phi_\eta((X_1; \cdots ; X_r; 1), (t; 1)) = (X_1 + ut; \cdots ; X_r + ut; 1)$ so that ϕ_η restricts to the given map on $\mathbb{A}^r_K \times \square_K$. One checks that (1), (3) and (4) hold from the shape of ϕ_η in (3.1).

To show (2), note that this map is the composite $\mathbb{P}^r_K \times \square_K \to \mathbb{P}^r_K \times \square_K \to \mathbb{P}^r_K$, where the first one is $((X_1; \cdots ; X_r; X_0), t) \mapsto ((X_1 + utX_0; \cdots ; X_r + utX_0; X_0), t)$ and the second is the projection to $\mathbb{P}^r_K$ (which is smooth). Since the first map is an isomorphism, it follows that ϕ_η is smooth on $\mathbb{P}^r_K \times \square_K$. $\square$

Remark 3.4. The unique extension of ϕ_η to U_+ is not a flat morphism even though it is smooth on $\mathbb{P}^r_K \times \square_K$. If we set $V_i = \{(X_1; \cdots ; X_r; X_0) | X_i \neq 0\} \hookrightarrow \mathbb{P}^r_K$ for $i = 1, \cdots , r$, then the map $\phi_\eta^{-1}(V_i) \to V_i$ is not flat because $\mathbb{A}^r_K \times \{0\}$ lies in one fiber but all other fibers have strictly smaller dimensions.

Our idea is to use the rational map $\phi_\eta : \mathbb{P}^r_K \times \overline{\square}_K \dashrightarrow \mathbb{P}^r_K$ to generate a homotopy between an arbitrary admissible cycle in $z^q(\mathbb{P}^r_k | H_\infty, \bullet)$ and a cycle in $z^q_{\mathcal{W},e}(\mathbb{P}^r_k | H_\infty, \bullet)$. In order to do so, we need to extend ϕ_η to an honest morphism of schemes. We achieve this in the following results via a sequence of blow-ups.

Lemma 3.5. *Let $\pi : \Gamma \to \mathbb{P}^r_K \times \overline{\square}_K$ be the blow-up of $\mathbb{P}^r_K \times \overline{\square}_K$ along the closed subscheme $\mathcal{Y} = H_\infty \times \{\infty\}$. Then, there exists a closed point $P_\infty \in \pi^{-1}(\mathcal{Y})$ and a regular map $\overline{\phi}_\eta : \Gamma_+ := \Gamma \setminus \{P_\infty\} \to \mathbb{P}^r_K$ such that $\pi : \Gamma_+ \to \mathbb{P}^r_K \times \overline{\square}_K$ is surjective, and the diagram*

$$\begin{tikzcd} \pi^{-1}(U_+) \arrow[r, hook, "\overline{j}"] \arrow[d, "\simeq"'] & \Gamma_+ \arrow[d, "\pi"] \arrow[dr, "\overline{\phi}_\eta"] & \\ U_+ \arrow[r, hook, "j"] \arrow[rr, bend right] & \mathbb{P}^r_K \times \overline{\square}_K \arrow[r, dashed, "\phi_\eta"] & \mathbb{P}^r_K \end{tikzcd} \tag{3.2}$$

commutes.

Proof. Let $U_i \subsetneq \mathbb{P}^r_K$ be the open set $\{X_i \neq 0\}$ for $0 \leq i \leq r$. One checks by a direct local calculation the blow-up Γ has the following description. Over U_i, it is defined by

$$\begin{aligned}\pi^{-1}(U_i) = \Big\{\big(&(X_1; \cdots ; X_r; X_0), (T_0; T_1), (Y_{1,i}; Y_{0,i})\big) \\ &\in U_i \times \overline{\square}_K \times \mathbb{P}^1_K | X_0T_0Y_{0,i} = X_iT_1Y_{1,i}\Big\}\end{aligned} \tag{3.3}$$

and these blow-ups glue along their intersections to make up Γ via the change of coordinate $Y_{0,i}/Y_{0,j} = (X_i/X_j)(Y_{1,i}/Y_{1,j})$ over $U_i \cap U_j$. The blow-up map $\pi : \pi^{-1}(U_i) \to U_i \times \overline{\square}_K$ is the composite $\pi^{-1}(U_i) \hookrightarrow U_i \times \overline{\square}_K \times \mathbb{P}^1_K \to U_i \times \overline{\square}_K$.

We now define a rational map $\overline{\phi}^i_\eta : \pi^{-1}(U_i) \dashrightarrow \mathbb{P}^r_K$ by

$$\begin{aligned}&\overline{\phi}_\eta\big((X_1;\cdots;X_r;X_0),(T_0;T_1),(Y_{1,i};Y_{0,i})\big)\\&=\big(Y_{0,i}X_1+uX_iY_{1,i};\cdots;Y_{0,i}X_r+uX_iY_{1,i};Y_{0,i}X_0\big).\end{aligned}\tag{3.4}$$

The blow-up Γ is glued along $U_i\cap U_j$ via the automorphism $\psi_{i,j} : \pi^{-1}(U_i\cap U_j) \xrightarrow{\simeq} \pi^{-1}(U_i\cap U_j)$:

$$\begin{aligned}&\psi_{i,j}\big((X_1;\cdots;X_r;X_0),(T_0;T_1),(Y_{1,i};Y_{0,i})\big)\\&=\Big((X_1;\cdots;X_r;X_0),(T_0;T_1),(X_iX_j^{-1}Y_{1,i};X_jX_i^{-1}Y_{0,i})\Big).\end{aligned}$$

It is clear from this isomorphism that $\psi_{i,j}(Y_{l,i}\neq 0) = (Y_{l,j}\neq 0)$ for $l = 0,1$. Over $(Y_{0,i}\neq 0)$, we can let $Y_{0,i} = Y_{0,j} = 1$, $Y_{1,i} = y_i$ and $Y_{1,j} = y_j$. Over this open subset of $\pi^{-1}(U_i\cap U_j)$, we get

$$\begin{aligned}&\overline{\phi}^j_\eta\circ\psi_{i,j}\,((X_1;\cdots;X_r;X_0),(T_0;T_1),y_i)\\&=\overline{\phi}^j_\eta\Big((X_1;\cdots;X_r;X_0),(T_0;T_1),X_iX_j^{-1}y_i\Big)\\&=\Big(X_1+uX_jX_iX_j^{-1}y_i;\cdots;X_r+uX_jX_iX_j^{-1}y_i;X_0\Big)\\&=(X_1+uX_iy_i;\cdots;X_r+uX_iy_i;X_0)\\&=\overline{\phi}^i_\eta\,((X_1;\cdots;X_r;X_0),(T_0;T_1),y_i).\end{aligned}\tag{3.5}$$

Over the intersection of $\pi^{-1}(U_i\cap U_j)$ with the open subset $(Y_{1,i}\neq 0)$, we have

$$\begin{aligned}&\overline{\phi}^j_\eta\circ\psi_{i,j}\,((X_1;\cdots;X_r;X_0),(T_0;T_1),y_i)\\&=\overline{\phi}^j_\eta\Big((X_1;\cdots;X_r;X_0),(T_0;T_1),X_jX_i^{-1}y_i\Big)\\&=\Big(X_jX_i^{-1}X_1y_i+uX_j;\cdots X_jX_i^{-1}X_ry_i+uX_j;X_i^{-1}X_jX_0y_i\Big)\\&=\big(X_1X_jy_i+uX_iX_j;\cdots;X_rX_jy_i+uX_iX_j;X_jX_0y_i\big)\\&=(X_1y_i+uX_i;\cdots;X_ry_i+uX_i;X_0y_i)\\&=\overline{\phi}^i_\eta\,((X_1;\cdots;X_r;X_0),(T_0;T_1),y_i).\end{aligned}\tag{3.6}$$

It follows from (3.5) and (3.6) that $\overline{\phi}^j_\eta$'s glue together to yield a rational map $\overline{\phi}_\eta : \Gamma \dashrightarrow \mathbb{P}^r_K$ such that $\overline{\phi}_\eta|_{\pi^{-1}(U_i)} = \overline{\phi}^j_\eta$ for $0\le i\le r$.

We next show the commutativity of (3.2). The left square of (3.2) commutes by construction. We thus have to show that $\overline{\phi}_\eta\circ\overline{j} = \phi_\eta\circ\pi$, *i.e.*, the trapezoid in (3.2) commutes. It suffices to show this over each open subset $(U_i\times\overline{\square}_K)\cap U_+$.

If $P = \big((X_1; \cdots; X_r; X_0), (T_0; T_1), (Y_{1,i}; Y_{0,i})\big) \in \pi^{-1}(U_+)$, we have $\pi(P) = ((X_1; \cdots; X_r; X_0), (T_0; T_1))$ such that either $T_1 \neq 0$ or $X_0 \neq 0$.

Suppose first that $T_1 \neq 0$. Then, we can take $T_1 = 1$ and $T_0 = t$. In this case, we must have $Y_{0,i} \neq 0$ so that we can assume $Y_{0,i} = 1$. Thus, the equation $X_0 T_0 Y_{0,i} = X_i T_1 Y_{1,i}$ becomes $Y_{1,i} = t X_0 X_i^{-1}$. This yields

$$\overline{\phi}^i_\eta \circ \overline{j}(P) = (X_1 + utX_0; \cdots; X_r + utX_0; X_0)$$

by (3.4) and

$$\phi_\eta \circ \pi(P) = (X_1 + utX_0; \cdots; X_r + utX_0; X_0)$$

by (3.1).

Suppose next that $X_0 \neq 0$. Since the case $T_1 \neq 0$ was already considered, we may suppose $T_0 \neq 0$. Thus, we may take $T_0 = 1$ and $T_1 = t$. In this case, we must have $Y_{1,i} \neq 0$, so that we may take $Y_{1,i} = 1$. Thus, the equation $X_0 T_0 Y_{0,i} = X_i T_1 Y_{1,i}$ becomes $Y_{0,i} = t X_i X_0^{-1}$. This yields

$$\begin{aligned}\overline{\phi}^i_\eta \circ \overline{j}(P) &= (tX_1X_i + uX_0X_i; \cdots; tX_rX_i + uX_0X_i; tX_iX_0)\\ &= (tX_1 + X_0; \cdots; tX_r + X_0; tX_0)\end{aligned}$$

by (3.4). On the other hand, $\phi_\eta \circ \pi(P) = (tX_1 + uX_0; \cdots; tX_r + X_0; tX_0)$ by (3.1). We have thus shown that $\overline{\phi}_\eta \circ \overline{j}(P) = \phi_\eta \circ \pi(P)$ for $P \in \pi^{-1}(U_+)$.

We now show that $\overline{\phi}_\eta$ is regular on $\Gamma \setminus \{P_\infty\}$, where $P_\infty \in \big(\cap_{i=1}^r \pi^{-1}(U_i)\big)$ is the closed point $((1; \cdots; 1; 0), (1; 0), (1; -u))$ in the coordinates of $\pi^{-1}(U_i)$. Let $Q = \big((X_1; \cdots; X_r; X_0), (T_0; T_1), (Y_{1,i}; Y_{0,i})\big) \in \pi^{-1}(U_i)$ be a point so that $X_0 T_0 Y_{0,i} = X_i T_1 Y_{1,i}$. Then $\overline{\phi}_\eta(Q)$ is not defined if and only if all its coordinates are zero, *i.e.*,

$$Y_{0,i}X_j + uX_iY_{1,i} = 0, \quad \text{for all } 1 \leq j \leq r, \text{ and } Y_{0,i}X_0 = 0. \tag{3.7}$$

If $Y_{0,i} = 0$ then $uX_iY_{1,i} = 0$ for $1 \leq i \leq r$. But $u \in K^\times$ and $Q \in \pi^{-1}(U_i)$ imply that $Y_{1,i} = 0$, which cannot happen since $(Y_{1,i}; Y_{0,i}) \in \mathbb{P}^1_K$. So, $Y_{0,i} \neq 0$ and we must have $X_0 = 0$. Since $X_i \neq 0$, we can assume $X_i = 1$. Since $X_0 = 0$, we also have $T_1 Y_{1,i} = 0$, so that either $Y_{1,i} = 0$ or $T_1 = 0$. If $Y_{1,i} = 0$, then it follows from (3.7) that $Y_{0,i} = -uY_{1,i} = 0$, which again is absurd because $(Y_{1,i}; Y_{0,i}) \in \mathbb{P}^1_K$. So, $Y_{1,i} \neq 0$, and $T_1 = 0$. We may assume $Y_{1,i} = 1$. Combining this with (3.7), we thus have

$$Y_{0,i} = -u, \;\; Y_{0,i}X_j + u = 0 \quad \text{for all } 1 \leq j \neq i \leq r \text{ and } X_0 = T_1 = 0. \tag{3.8}$$

We conclude that $\overline{\phi}_\eta(Q)$ is not defined if and only if $Q = ((1; \cdots; 1; 0), (1; 0), (1; -u))$. This proves the regularity of $\overline{\phi}_\eta$ on $\Gamma \setminus \{P_\infty\}$. Since $P_\infty \in \pi^{-1}(\mathcal{Y})$ and since each fiber of π over $\mathcal{Y}$ is 1-dimensional, we conclude that the map $(\Gamma \setminus \{P_\infty\}) \to \mathbb{P}^r_K \times \overline{\square}_K$ is surjective. This finishes the proof of the lemma. $\square$

Remark 3.6. The reader can check that the map $\phi_\eta : \mathbb{P}^r_K \times \overline{\square}_K \dashrightarrow \mathbb{P}^r_K$ is the one defined by the linear system generated by the global sections $S = \{T_1 X_i + uT_0X_0\}_{1\le i\le r} \cup \{T_1X_0\}$ of the line bundle $\mathcal{O}(1,1)$. The sheaf of ideals $\mathcal{I}_\infty$ on $\mathbb{P}^r_K \times \overline{\square}_K$ defining $\mathcal{Y}$ is generated by $\{X_iT_1, X_0T_0 | 0 \le i \le r\}$. Moreover, $\overline{\phi}_\eta : \Gamma \dashrightarrow \mathbb{P}^r_K$ is the rational map defined by the linear system generated by the global sections $\pi^*(S)$ of the line bundle $\pi^*\mathcal{I}_\infty$.

Let $\pi : \Gamma \to \mathbb{P}^r_K \times \overline{\square}_K$ be the blow-up map as in Lemma 3.5 and let $E = \pi^*(\mathcal{Y})$ denote the exceptional divisor for this blow-up. Note that the map $\pi : E \to \mathcal{Y} \simeq H_\infty$ is the $\mathbb{P}^1_K$-bundle associated to the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}$.

Since $H_\infty \times \overline{\square}_K$ and $\mathbb{P}^r_K \times \{\infty\}$ are smooth schemes, and $\mathcal{Y}$ is a smooth divisor inside these schemes, note that $\mathrm{Bl}_{\mathcal{Y}}(H_\infty \times \overline{\square}_K) \to H_\infty \times \overline{\square}_K$ and $\mathrm{Bl}_{\mathcal{Y}}(\mathbb{P}^r_K \times \{\infty\}) \to \mathbb{P}^r_K \times \{\infty\}$ are isomorphisms.

Lemma 3.7. *Let $\pi : \Gamma \to \mathbb{P}^r_K \times \overline{\square}_K$ be as in Lemma 3.5. Then, we have the following.*

(1) $\mathrm{Bl}_{\mathcal{Y}}(H_\infty \times \overline{\square}_K) \cap \{P_\infty\} = \emptyset = \mathrm{Bl}_{\mathcal{Y}}(\mathbb{P}^r_K \times \{\infty\}) \cap \{P_\infty\}$;
(2) $\mathrm{Bl}_{\mathcal{Y}}(H_\infty \times \overline{\square}_K) \cap \mathrm{Bl}_{\mathcal{Y}}(\mathbb{P}^r_K \times \{\infty\}) = \emptyset$ *inside* Γ;
(3) $\pi^*(H_\infty \times \overline{\square}_K) = (H_\infty \times \overline{\square}_K) + E$ *and* $\pi^*(\mathbb{P}^r_K \times \{\infty\}) = (\mathbb{P}^r_K \times \{\infty\}) + E$ *in the group* $\mathrm{Div}(\Gamma)$ *of Weil divisors.*

Proof. It suffices to verify each statement of the lemma over an open subset $\pi^{-1}(U_i)$ with $0 \le i \le r$. On the other hand, (3.3) shows that over U_i, we have

$$\begin{aligned}
&\mathrm{Bl}_{\mathcal{Y}}(H_\infty \times \overline{\square}_K)\\
&= \{((X_1; \cdots ; X_r; 0), (T_0; T_1), (Y_{1,i}; Y_{0,i})) \in \mathbb{P}^r_K \times \overline{\square}_K \times \mathbb{P}^1_K | Y_{1,i} = 0\}\\
&= H_\infty \times \overline{\square}_K \times \{0\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Bl}_{\mathcal{Y}}(\mathbb{P}^r_K \times \{\infty\})\\
&= \{((X_1; \cdots ; X_r; X_0), (1; 0), (Y_{1,i}; Y_{0,i})) \in \mathbb{P}^r_K \times \overline{\square}_K \times \mathbb{P}^1_K | Y_{0,i} = 0\}\\
&= \mathbb{P}^r_K \times \{\infty\} \times \{\infty\}.
\end{aligned}$$

Since P_∞ does not map to $\{0, \infty\} \subset \mathbb{P}^1_K$ under the projection $\pi^{-1}(U_i) \to \mathbb{P}^1_K$ for any $0 \le i \le r$, we get (1). The parts (2) and (3) of the lemma are immediate. $\square$

Let $\Gamma_1 \hookrightarrow \Gamma_+ \times \mathbb{P}^r_K$ denote the graph of $\overline{\phi}_\eta$ and let $\overline{\Gamma}_1 \hookrightarrow \Gamma \times \mathbb{P}^r_K$ be its closure. Let $\pi^N : \overline{\Gamma}^N_1 \to \overline{\Gamma}_1 \hookrightarrow \Gamma \times \mathbb{P}^r_K$ be the normalization composed with the inclusion, and let $\pi_1 := \mathrm{pr}_1 \circ \pi^N$, $\pi_2 := \mathrm{pr}_2 \circ \pi^N$, where $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections from $\Gamma \times \mathbb{P}^r_K$ to Γ and $\mathbb{P}^r_K$, respectively. Here, π^N is finite and π_1 is projective with $\pi_1^{-1}(\Gamma_+) \xrightarrow{\simeq} \Gamma_+$ such that $\pi_2|_{\Gamma_+} = \overline{\phi}_\eta$.

Since π_1 is a birational projective morphism and Γ is smooth, it follows from [6, Theorem II-7.17, page 166, Exercise II-7.11(c), page 171] that there is a closed subscheme $Z \hookrightarrow \Gamma$ such that $Z_{\mathrm{red}} = \{P_\infty\}$ and $\overline{\Gamma}_1^N = \mathrm{Bl}_Z(\Gamma)$. Let $F \hookrightarrow \overline{\Gamma}_1^N$ denote the exceptional divisor for this blow-up so that $F_{\mathrm{red}} = \pi_1^{-1}(P_\infty)$. Let $E_1 \hookrightarrow \overline{\Gamma}_1^N$ denote the strict transform of E under π_1 so that $\pi_1^*(E) = E_1 + F$.

Letting $\delta := \pi \circ \pi_1 : \overline{\Gamma}_1^N \to \mathbb{P}^r_K \times \overline{\square}_K$ and $E' := \pi_1^*(E) = E_1 + F$, a combination of Lemmas 3.5, 3.7 and the above construction proves the following.

Lemma 3.8. *There exists a commutative diagram*

$$\begin{tikzcd}
\delta^{-1}(U_+) \arrow[r, hook, "j_1"] \arrow[d, "\simeq"'] & \overline{\Gamma}_1^N \arrow[d, "\delta"'] \arrow[dr, "\pi_2"] & \\
U_+ \arrow[r, hook, "j"] \arrow[rr, bend right] & \mathbb{P}^r_K \times \overline{\square}_K \arrow[r, dashed, "\phi_\eta"] & \mathbb{P}^r_K
\end{tikzcd} \tag{3.9}$$

such that δ is a blow-up, and in the group $\mathrm{Div}(\overline{\Gamma}_1^N)$ *of Weil divisors, we have:*

$$\begin{aligned}\delta^*\left(H_\infty \times \overline{\square}_K\right) = \left(H_\infty \times \overline{\square}_K\right) + E' \textit{ and } \delta^*\left(\mathbb{P}^r_K \times \{\infty\}\right) \\ = \left(\mathbb{P}^r_K \times \{\infty\}\right) + E'.\end{aligned} \tag{3.10}$$

For any map $f : X \to X'$ of K-schemes, let f_n denote the map

$$f \times \mathrm{Id}_{\overline{\square}^n_K} : X \times \overline{\square}^n_K \to X' \times \overline{\square}^n_K.$$

We now show how the rational map $\phi_\eta : \mathbb{P}^r_K \times \overline{\square}_K \dashrightarrow \mathbb{P}^r_K$ eventually leads to the desired homotopy.

Proposition 3.9. *Let $n \geq 1$ be an integer. Let $V \hookrightarrow \mathbb{P}^r_K \times \square^n_K$ be an integral closed subscheme. Assume that V has modulus H_∞ relative to F^∞_n. Let $\phi_\eta : \mathbb{A}^r_K \times \square_K \to \mathbb{P}^r_K$ be the map as in Lemma* 3.3. *Then, the closure of $\phi_{\eta,n}^{-1}(V)$ in $\mathbb{P}^r_K \times \square^{n+1}_K$ is an integral closed subscheme of $\mathbb{P}^r_K \times \square^{n+1}_K$ which has modulus H_∞ relative to F^∞_{n+1}.*

Proof. We use notations of the paragraph just before Lemma 3.8 and set $E'_n = E' \times \overline{\square}^n_K \hookrightarrow \overline{\Gamma}_1^N \times \overline{\square}^n_K$. Let $\overline{V} \hookrightarrow \mathbb{P}^r_K \times \overline{\square}^n_K$ denote the closure of V and let $\nu_V : \overline{V}^N \to \mathbb{P}^r_K \times \overline{\square}^n_K$ denote the induced map from the normalization of $\overline{V}$. By the modulus condition, we have

$$\nu_V^*\left(\mathbb{P}^r_K \times F^\infty_n\right) \geq \nu_V^*\left(H_\infty \times \overline{\square}^n_K\right) \text{ in } \mathrm{Div}\left(\overline{V}^N\right). \tag{3.11}$$

Condition (3.11) implies that $V \cap (H_\infty \times \square^n_K) = \emptyset$. Set $V' = \phi^{-1}_{\eta,n}(V)$. Since $\phi_{\eta,n}$ is smooth on $\phi^{-1}_{\eta,n}(\mathbb{A}^r_K \times \square^n_K)$ by Lemma 3.3, it follows that V' is an integral closed subscheme of $U_+ \times \overline{\square}^n_K$ with $\dim_K(V') = \dim_K(V)+1$. Let $\overline{V}' \hookrightarrow \mathbb{P}^r_K \times \overline{\square}^{n+1}_K$ be the Zariski closure of V', and let $\nu_{V'} : \overline{V}'^N \to \overline{V}' \hookrightarrow \mathbb{P}^r_K \times \overline{\square}^{n+1}_K$ be the induced map from the normalization of $\overline{V}'$. Let $W \hookrightarrow \overline{\Gamma}^N_1 \times \overline{\square}^n_K$ be the strict transform of $\overline{V}'$. It follows from Lemma 3.5 that $\pi_{2,n}(W \cap \delta_n^{-1}(U_+ \times \square^n_k)) = V$. Since $\pi_{2,n}$ is projective, we must have $\pi_{2,n}(W) = \overline{V}$. This yields a commutative diagram

$$\begin{tikzcd}
W^N \arrow[rr, "f"] \arrow[dr, "\nu_W"] \arrow[dd, "g"'] & & \overline{V}^N \arrow[d, "\nu_V"] \\
& \overline{\Gamma}^N_1 \times \overline{\square}^n_K \arrow[r, "\pi_{2,n}"] \arrow[d, "\delta_n"] & \mathbb{P}^r_K \times \overline{\square}^n_K \\
\overline{V}'^N \arrow[r, "\nu_{V'}"'] & \mathbb{P}^r_K \times \overline{\square}^{n+1}_K, &
\end{tikzcd} \tag{3.12}$$

where ν_W is the normalization of W composed with its inclusion into $\overline{\Gamma}^N_1 \times \overline{\square}^n_K$, and f and g are the maps induced by the universal property of normalization for dominant maps. Since f is a surjective map of integral schemes, condition (3.11) implies that $(\nu_V \circ f)^*(\mathbb{P}^r_K \times F^\infty_n) \geq (\nu_V \circ f)^*(H_\infty \times \overline{\square}^n_K)$ on W^N. In particular, we get $(\pi_{2,n} \circ \nu_W)^*(\mathbb{P}^r_K \times F^\infty_n) \geq (\pi_{2,n} \circ \nu_W)^*(H_\infty \times \overline{\square}^n_K)$ on W^N. Equivalently,

$$\nu^*_W\left(\overline{\Gamma}^N_1 \times F^\infty_n\right) \geq \nu^*_W\left(\pi^*_2(H_\infty) \times \overline{\square}^n_K\right). \tag{3.13}$$

Since $(\phi_\eta|_{U_+})^*(H_\infty) = (\mathbb{A}^r_K \times \{\infty\}) + (H_\infty \times \square_K)$ by Lemma 3.3, we get

$$j^*_{1,n} \circ \pi^*_{2,n}\left(H_\infty \times \overline{\square}^n_K\right) = j^*_{1,n}\left(\mathbb{P}^r_K \times F^\infty_{n,n+1}\right) + j^*_{1,n}\left(H_\infty \times \overline{\square}^{n+1}_K\right),$$

where $j_1 : U_+ \hookrightarrow \overline{\Gamma}^N_1$ is the inclusion. Since $\mathbb{P}^r_K \times F^\infty_{n,n+1}$ and $H_\infty \times \overline{\square}^{n+1}_K$ are irreducible, we get $\pi^*_2(H_\infty) \times \overline{\square}^n_K \geq (\mathbb{P}^r_K \times F^\infty_{n,n+1}) + (H_\infty \times \overline{\square}^{n+1}_K)$ on $\overline{\Gamma}^N_1 \times \overline{\square}^n_K$. Combining this with (3.13), we get

$$\begin{aligned}
\nu^*_W\left(\overline{\Gamma}^N_1 \times F^\infty_n\right) &\geq \nu^*_W\left(\mathbb{P}^r_K \times F^\infty_{n,n+1}\right) + \nu^*_W\left(H_\infty \times \overline{\square}^{n+1}_K\right) \\
&\geq \nu^*_W\left(H_\infty \times \overline{\square}^{n+1}_K\right).
\end{aligned} \tag{3.14}$$

This in turn implies that

$$\begin{aligned}(\delta_n\circ\nu_W)^*\left(\mathbb{P}^r_K\times F^\infty_{n+1}\right) &= (\delta_n\circ\nu_W)^*\left(\mathbb{P}^r_K\times F^\infty_n\times\overline{\square}_K\right)\\ &\quad+(\delta_n\circ\nu_W)^*\left(\mathbb{P}^r_K\times\overline{\square}^n_K\times\{\infty\}\right)\\ &=\nu_W^*\left(\overline{\Gamma}^N_1\times F^\infty_n\right)+(\delta_n\circ\nu_W)^*\left(\mathbb{P}^r_K\times\overline{\square}^n_K\times\{\infty\}\right)\\ &\geq\nu_W^*\left(H_\infty\times\overline{\square}^{n+1}_K\right)+(\delta_n\circ\nu_W)^*\left(\mathbb{P}^r_K\times\overline{\square}^n_K\times\{\infty\}\right)\\ &=^\dagger\nu_W^*\left(H_\infty\times\overline{\square}^{n+1}_K\right)+\nu_W^*(E'_n)+\nu_W^*\left(\mathbb{P}^r_K\times\overline{\square}^n_K\times\{\infty\}\right)\\ &=^\ddagger(\delta_n\circ\nu_W)^*\left(H_\infty\times\overline{\square}^{n+1}_K\right)+\nu_W^*\left(\mathbb{P}^r_K\times\overline{\square}^n_K\times\{\infty\}\right)\\ W&\geq(\delta_n\circ\nu_W)^*\left(H_\infty\times\overline{\square}^{n+1}_K\right),\end{aligned}$$

where $=^\dagger$ and $=^\ddagger$ follow from Lemma 3.8. Using (3.12), this gives $g^*(\nu_{V'}^*(\mathbb{P}^r_K\times F^\infty_{n+1}))\geq g^*(\nu_{V'}^*(H_\infty\times\overline{\square}^{n+1}_K))$. Since g is a surjective map of integral normal schemes, we conclude by Lemma 2.6 that $\nu_{V'}^*(\mathbb{P}^r_K\times F^\infty_{n+1})\geq\nu_{V'}^*(H_\infty\times\overline{\square}^{n+1}_K)$. $\square$

Theorem 3.10. *Given an integer $r\geq 1$, let $D\hookrightarrow\mathbb{P}^r_k$ be a hyperplane. Let $\mathcal{W}=\{W_1,\cdots,W_s\}$ be a finite collection of locally closed subsets of $\mathbb{P}^r_k$ and let $e:\mathcal{W}\to\mathbb{Z}_{\geq 0}$ be a set function. Then, the inclusion $z^q_{\mathcal{W}}(\mathbb{P}^r_k|D,\bullet)\hookrightarrow z^q_{\mathcal{W},e}(\mathbb{P}^r_k|D,\bullet)$ is a quasi-isomorphism. In particular, the inclusion $z^q_{\mathcal{W}}(\mathbb{P}^r_k|D,\bullet)\hookrightarrow z^q(\mathbb{P}^r_k|D,\bullet)$ is a quasi-isomorphism.*

Proof. The second part follows easily from the first part because $z^q(\mathbb{P}^r_k|D,\bullet)=z^q_q(X|D,\bullet)$. We shall prove the first part of the theorem in several steps. We can find a linear automorphism $\tau:\mathbb{P}^r_k\xrightarrow{\simeq}\mathbb{P}^r_k$ such that $\tau(D)=H_\infty$. Replacing $\mathcal{W}$ by $\tau(\mathcal{W})$, we reduce to the case when $D=H_\infty$, condition that we assume from now on. In view of Proposition 2.8, we only need to show that the map

$$\mathrm{pr}^*_{K/k}:\frac{z^q_{\mathcal{W},e}(\mathbb{P}^r_k|D,\bullet)}{z^q_{\mathcal{W}}(\mathbb{P}^r_k|D,\bullet)}\to\frac{z^q_{\mathcal{W}_K,e}(\mathbb{P}^r_K|D_K,\bullet)}{z^q_{\mathcal{W}_K}(\mathbb{P}^r_K|D_K,\bullet)}$$

is zero on the homology, where we choose $K=k(\mathbb{P}^r_k)$.

Following the notation we used so far in this section, consider the maps

$$\mathbb{A}^r_K\times\square^{n+1}_K\xrightarrow{\phi_{\eta,n}}\mathbb{P}^r_K\times\square^n_K\xrightarrow{\mathrm{pr}_{K/k}}\mathbb{P}^r_k\times\square^n_k.$$

For any irreducible cycle $V\hookrightarrow\mathbb{P}^r_k\times\square^n_k$, let $H^*_n(V)=(\mathrm{pr}_{K/k}\circ\phi_{\eta,n})^{-1}(V)$ and let $\overline{H}^*_n(V)$ be its closure in $\mathbb{P}^r_K\times\square^{n+1}_K$. We can extend this linearly to cycles in $z^q(\mathbb{P}^r_k|D,n)$.

Suppose V is an irreducible cycle in $z^q_{\mathcal{W},e}(\mathbb{P}^r_k|D, n)$. We claim that:

(1) $\overline{H}^*_n(V) \in z^q_{\mathcal{W}_K,e}(\mathbb{P}^r_K|D_K, n+1)$;

(2) $\overline{H}^*_n(V) \in z^q_{\mathcal{W}_K}(\mathbb{P}^r_K|D_K, n+1)$ if $V \in z^q_{\mathcal{W}}(\mathbb{P}^r_k|D, n)$;

(3) $\iota^*_{n+1,n+1,0}(\overline{H}^*_n(V)) = V$ and $\iota^*_{n+1,n+1,1}(\overline{H}^*_n(V)) \in z^q_{\mathcal{W}_K}(\mathbb{P}^r_K|D_K, n)$.

We now prove this claim using the previous results of this section. Since V has modulus D on $\mathbb{P}^r_k \times \square^n_k$, it follows that V is a closed subscheme of $\mathbb{A}^r_k \times \square^n_k$. In particular, $V \in z^q_{\mathcal{W}^0,e}(\mathbb{A}^r_k, n)$, where $\mathcal{W}^0 = \{W_1 \cap \mathbb{A}^r_k, \cdots, W_s \cap \mathbb{A}^r_k\}$. Since $\overline{H}^*_n(V)$ has modulus D on $\mathbb{P}^r_K \times \square^{n+1}_K$ by Proposition 3.9, it follows that $\overline{H}^*_n(V)$ is an integral closed subscheme of $\mathbb{A}^r_K \times \square^{n+1}_K$. In particular, $\overline{H}^*_n(V) = H^*_n(V)$. This shows that we can replace $\mathbb{P}^r_k$, $\overline{H}^*_n(V)$ and $\mathcal{W}$ by $\mathbb{A}^r_k$, $H^*_n(V)$ and $\mathcal{W}^0$ respectively, to prove the claim.

We prove (3) first. By the definition of ϕ_η, we have $\iota^*_{n+1,n+1,0}(H^*_n(V)) = V$. In particular, $H^*_n(V)$ intersects $F_{n+1,n+1,0}$ and all its faces properly. We thus have to show that $\iota^*_{n+1,n+1,1}(H^*_n(V)) \in z^q_{\mathcal{W}^0_K}(\mathbb{A}^r_K|D_K, n)$ to prove (3).

Let $\mathbb{A}^r_k$ act on itself by translation and let it act on $\mathbb{A}^r_k \times \square^n_k$ by acting trivially on $\square^n_k = \square^n_k \times \{1\} \hookrightarrow \square^{n+1}_k$. Consider the map $\psi : \mathbb{A}^r_K \times \square^n_K \to \mathbb{A}^r_K$ defined by $\psi(x, y) = \eta$. The reader can check that the assumptions of Corollary 3.2 are satisfied. Applying this corollary to each $A = \overline{W_i \cap \mathbb{A}^r_k}$ (where the closure is taken in $\mathbb{A}^r_k$) and $B = \mathbb{A}^r_k \times F$ for any face F of $\square^n_k \times \{1\}$, we deduce $\iota^*_{n+1,n+1,1}(H^*_n(V)) \in z^q_{\mathcal{W}^0_K}(\mathbb{A}^r_K|D_K, n)$. We have thus proven (3). Since (2) is a special case of (1) where we take $e = 0$, we are left with proving (1).

To prove (1), it is enough to consider the case when $\mathcal{W} = \{W\}$ is a singleton. Note $V \in z^q_{W,e}(\mathbb{A}^r_k, n)$ and let $F \hookrightarrow \square^{n+1}_K$ be any face. If $F \hookrightarrow \square^n_K \times \{0\}$, then the intersection $H^*_n(V) \cap (W \times F)$ has the desired dimension because $\iota^*_{n+1,n+1,0}(H^*_n(V)) = V$ and $V \in z^q_{W,e}(\mathbb{A}^r_k, n)$. We have already proven in (3) that the intersection $H^*_n(V) \cap (W \times F)$ is proper if $F \hookrightarrow \square^n_K \times \{1\}$. We can thus assume that $F = F'_K \times \square_K$, where F' is a face of $\square^n_k$.

Set $Z = V \cap (\mathbb{A}^r_k \times F')$. Consider the map $\psi : \mathbb{A}^r_K \times \square_K \times F'_K \to \mathbb{A}^r_K$ defined by $\psi(x, t, y) = \eta t$ and let $\theta : \mathbb{A}^r_K \times \square_K \times F'_K \to \mathbb{A}^r_K \times \square_K \times F'_K$ be given by $\theta(x, t, y) = (x + \eta t, t, y)$. Let $\mathbb{A}^r_k$ act by translation on itself and trivially on $\square_k \times F'$. Then $\theta(x, t, y) = \psi(x, t, y) \cdot (x, t, y)$. Applying Lemma 3.1 with $X = \mathbb{A}^r_k \times \square_k \times F'$, $A = \overline{W} \times \square_k \times F'$, $U = \mathbb{A}^r_k \times \mathbb{G}_{m,k} \times F'$, and $B = (V \times \square_k) \cap F_k = Z \times \square_k \hookrightarrow X \times F'$, it follows that the intersection $\theta(A_K) \cap B_K$ is proper away from $\mathbb{A}^r_K \times \{0\} \times F'_K$, *i.e.*, the intersection $(H^*_n(V) \cap F) \cap (W_K \times F)$ is proper away from $\mathbb{A}^r_K \times \{0\} \times F'_K$.

On the other hand, since $V \in z^q_{W,e}(\mathbb{A}^r_k, n)$ and hence V meets $W \times F'$ in excess dimension at most $e(W)$, it follows that $H^*_n(V) \cap F$ must meet $W \times F$ in excess dimension at most $e(W)$ along $\mathbb{A}^r_K \times \{0\} \times F'_K$. Thus $H^*_n(V)$ intersects $W_K \times F_K$

in excess dimension at most $e(W)$ for all faces $F_K \hookrightarrow \square_K^{n+1}$. In other words, $H_n^*(V) \in z^q_{W_K,e}(\mathbb{A}^r_K, n+1)$. This proves (1) and hence the claim.

It follows from the claim that there is a chain homotopy

$$H^*_\eta : \frac{z^q_{\mathcal{W},e}\left(\mathbb{P}^r_k|D,\bullet\right)}{z^q_{\mathcal{W}}\left(\mathbb{P}^r_k|D,\bullet\right)} \to \frac{z^q_{\mathcal{W}_K,e}\left(\mathbb{P}^r_K|D_K,\bullet\right)}{z^q_{\mathcal{W}_K}\left(\mathbb{P}^r_K|D_K,\bullet\right)}[-1],$$

and composed with the restriction map $\{1\} \hookrightarrow \square_k$, there is a chain map

$$H^*_{\eta,1} : \frac{z^q_{\mathcal{W},e}\left(\mathbb{P}^r_k|D,\bullet\right)}{z^q_{\mathcal{W}}\left(\mathbb{P}^r_k|D,\bullet\right)} \to \frac{z^q_{\mathcal{W}_K,e}\left(\mathbb{P}^r_K|D_K,\bullet\right)}{z^q_{\mathcal{W}_K}\left(\mathbb{P}^r_K|D_K,\bullet\right)}$$

such that $H^*_\eta \circ \partial + \partial \circ H^*_\eta = \mathrm{pr}^*_{K/k} - H^*_{\eta,1}$. Since $H^*_{\eta,1} = 0$ by the claim, we see that $\mathrm{pr}^*_{K/k}$ is zero on the homology. The proof of the theorem is complete. $\square$

Corollary 3.11. *Given an integer $r \geq 1$, let $D \hookrightarrow \mathbb{P}^r_k$ be a hyperplane. Let $\mathcal{W} = \{W_1, \cdots, W_s\}$ be a finite collection of locally closed subsets of $\mathbb{P}^r_k$ and let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set function. Then the inclusion $z^q_{\mathcal{W},e-1}(\mathbb{P}^r_k|D,\bullet) \hookrightarrow z^q_{\mathcal{W},e}(\mathbb{P}^r_k|D,\bullet)$ is a quasi-isomorphism.*

Proof. For every $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$, there is a short exact sequence of chain complexes

$$0 \to \frac{z^q_{\mathcal{W},e-1}\left(\mathbb{P}^r_k|D,\bullet\right)}{z^q_{\mathcal{W}}\left(\mathbb{P}^r_k|D,\bullet\right)} \to \frac{z^q_{\mathcal{W},e}\left(\mathbb{P}^r_k|D,\bullet\right)}{z^q_{\mathcal{W}}\left(\mathbb{P}^r|D,\bullet\right)} \to \frac{z^q_{\mathcal{W},e}\left(\mathbb{P}^r_k|D,\bullet\right)}{z^q_{\mathcal{W},e-1}\left(\mathbb{P}^r_k|D,\bullet\right)} \to 0. \quad (3.15)$$

The first two quotient complexes are acyclic by Theorem 3.10. Hence the last one must be acyclic as well. $\square$

4. Moving lemma for projective schemes

In this section we prove the moving lemma for the higher Chow groups of projective schemes with very ample modulus. We assume for a while that the base field k is infinite. This is only a temporary assumption and will be removed in the final statement of the moving lemma (see Theorem 4.7).

We fix a closed embedding $\iota_X : X \hookrightarrow \mathbb{P}^N_k$ of an equidimensional reduced projective scheme X over k of dimension $d \geq 1$, with $d < N$. We fix two distinct hyperplanes $H_m, H_\infty \hookrightarrow \mathbb{P}^N_k$ and let $L_{m,\infty} = H_m \cap H_\infty \in \mathrm{Gr}(N-2, \mathbb{P}^N_k)$. We may assume that $X \not\subset H_m \cup H_\infty$. We set

$$X_0 = X \setminus H_\infty \overset{j_0}{\hookrightarrow} X,\ U = X \setminus H_m,\ U_0 = U \cap X_0,\ D = \iota^*_X(H_m) \text{ and } D_0 = j^*_0(D)$$

so that $X = U \cup D$ and $X_0 = U_0 \cup D_0$. We shall assume that U is smooth over k.

Remark 4.1. The hyperplane H_m could have been just called H, but we insist on using the subscript m to keep in mind that H_m later induces the modulus divisor.

Given a locally closed subset $S \subsetneq \mathbb{P}^N_k$, let $\mathrm{Gr}(S, n, \mathbb{P}^N_k)$ denote the set of n-dimensional linear subspaces of $\mathbb{P}^N_k$ which do not intersect S. Denote the set of n-dimensional linear subspaces of $\mathbb{P}^N_k$ containing a locally closed subscheme $S \subsetneq \mathbb{P}^N_k$ by $\mathrm{Gr}_S(n, \mathbb{P}^N_k)$. We let $\dim(\emptyset) = -1$. Given two locally closed subsets $Z_1, Z_2 \hookrightarrow \mathbb{P}^N_k$, let $\mathrm{Sec}(Z_1, Z_2)$ denote the union of all lines $\ell_{xy} \hookrightarrow \mathbb{P}^N_k$, joining $x \in Z_1$ and $y \in Z_2$ with $x \neq y$. One checks that $\dim(\mathrm{Sec}(Z_1, Z_2)) = \dim(Z_1) + \dim(Z_2) - \dim(Z_1 \cap Z_2)$ if Z_1 and Z_2 are linear subspaces of $\mathbb{P}^N_k$. In general, we have $\dim(\mathrm{Sec}(Z_1, Z_2)) \le \dim(Z_1) + \dim(Z_2) + 1$. Given a closed point $x \in X$, let $T_x(X)$ denote the union of lines in $\mathbb{P}^N_k$ which are tangent to X at x. For any locally closed subset $Y \subseteq X$, let $T_Y(X) = \bigcup_{x \in Y} T_x(X)$. It is clear that $\dim(T_Y(X)) \le \dim(Y) + d$ if $Y \subseteq U$. With this notation, we first prove the following:

Lemma 4.2. *Let $W \hookrightarrow \mathbb{P}^N_k$ be a closed subscheme of dimension at most d such that $W \not\subset H_m$. Then,* $\mathrm{Gr}(W, N-d-1, H_m)$ *is a dense open subset of* $\mathrm{Gr}(N-d-1, H_m)$*. If $L_{m,\infty}$ intersects W properly, then* $\mathrm{Gr}(W, N-d-1, L_{m,\infty})$ *is a dense open subset of* $\mathrm{Gr}(N-d-1, L_{m,\infty})$.

Proof. Consider the incidence variety $S = \{(x, L) \in W \times \mathrm{Gr}(N-d-1, H_m) | x \in L\}$. We have the projection maps of projective schemes

$$W \xleftarrow{\pi_1} S \xrightarrow{\pi_2} \mathrm{Gr}(N-d-1, H_m). \tag{4.1}$$

The fiber of π_1 over $W \setminus H_m$ is empty and it is a smooth fibration over $(W \cap H_m)_{\mathrm{red}}$ with each fiber isomorphic to $\mathrm{Gr}(N-d-2, \mathbb{P}^{N-2}_k)$. It follows that

$$\dim(S) = \dim(W \cap H_m) + d(N-d-1) \le d + d(N-d-1) - 1 = d(N-d) - 1.$$

Thus $\pi_2(S)$ is a closed subscheme of $\mathrm{Gr}(N-d-1, H_m)$ of dimension at most $d(N-d)-1$. On the other hand, $\dim(\mathrm{Gr}(N-d-1, H_m)) = d(N-d)$ so that $\mathrm{Gr}(W, N-d-1, H_m)$ is dense and open in $\mathrm{Gr}(N-d-1, H_m) \setminus \pi_2(S)$.

If $L_{m,\infty}$ intersects W properly, then we can argue as above with H_m replaced by $L_{m,\infty}$. We find in this case that

$$\begin{aligned}\dim(\pi_2(S)) \le \dim(S) &= \dim(W \cap L_{m,\infty}) + (d-1)(N-d-1)\\ &\le d + (d-1)(N-d-1) - 2 = (d-1)(N-d) - 1.\end{aligned}$$

Since $\dim(\mathrm{Gr}(N-d-1, L_{m,\infty})) = (d-1)(N-d)$, we get the desired conclusion. $\square$

Given an inclusion of linear subspaces $L \subsetneq L' \subseteq \mathbb{P}^N_k$ such that $\dim(L) \le N-d-1$ and $X \cap L = \emptyset$, the linear projection away from L defines a Cartesian diagram

$$
\begin{tikzcd}
X \setminus L' \arrow[r] \arrow[d] & X \arrow[d, "\phi_L"] & X \cap L' \arrow[l] \arrow[d] \\
\mathbb{P}^d_k \setminus L' \arrow[r] & \mathbb{P}^d_k & \mathbb{P}^d_k \cap L' \arrow[l]
\end{tikzcd}
\tag{4.2}
$$

of finite maps, where $\mathbb{P}^d_k \hookrightarrow \mathbb{P}^N_k$ is a linear subspace complementary to L. Let $R_L(X) \subset X$ denote the ramification locus of ϕ_L.

For an irreducible locally closed subset $A \subsetneq X$, let $L^+(A)$ denote the closure of $\phi_L^{-1}(\phi_L(A)) \setminus A$ in $\phi_L^{-1}(\phi_L(A))$. We linearly extend this definition to all cycles on X. We shall use similar notation for locally closed subsets of $X \times \square^n$ with ϕ_L replaced by $\phi_L \times \mathrm{Id}_{\square^n}$.

For two locally closed subsets $A, C \subset X$, let

$$e(A, C) = \max\{\dim(Z) - \dim(A) - \dim(C) + d\},$$

where the maximum is taken over all irreducible components Z of $A \cap C$, if these numbers are non-negative. We take $e(A, C)$ to be zero it they are not.

Lemma 4.3. *Let $A \subsetneq X \setminus H_m$ be an irreducible locally closed subset and let $C \subsetneq X \setminus H_m$ be any locally closed subset. Let $\Sigma = \{x_1, \cdots, x_s\}$ be a set of distinct closed points of X contained in A. Then, there is a dense open subset $\mathcal{U}^{A,C}_X \hookrightarrow \mathrm{Gr}(N - d - 1, H_m)$ such that the following hold for every $L \in \mathcal{U}^{A,C}_X$:*

(1) *$X \cap L = \emptyset$;*
(2) *$R_L(X)$ contains no irreducible component of A, C or $A \cap C$;*
(3) *$R_L(X) \cap \Sigma = \emptyset$;*
(4) *$e(L^+(A) \cap C) \le \max\{e(A, C) - 1, 0\}$;*
(5) *The map $k(\phi_L(x)) \to k(x)$ is an isomorphism for $x \in \Sigma$.*

Proof. Item (1) follows from Lemma 4.2, so we prove the remaining ones. We may assume that C is irreducible. Let $L \in \mathrm{Gr}(X, N - d - 1, H_m)$. Set $T^L_r = R_L(X) \cap A \cap C = R_L(U) \cap A \cap C$ and $T^L_{ur} = (L^+(A) \cap C) \setminus T^L_r$. Note that '$r$' stands for ramified and 'ur' for unramified. Then we must have $L^+(A) \cap C \subseteq T^L_{ur} \cup T^L_r$ and hence $\dim(L^+(A) \cap C) \le \max\{\dim(T^L_{ur}), \dim(T^L_r)\}$. Since the left square in (4.2) is Cartesian (where $L' = H_m$) and $A, C \subset U = X \setminus H_m$, it follows that the loci T^L_r and T^L_{ur} are contained in U.

Let $S \hookrightarrow ((A \times C) \setminus \Delta_X) \times \mathrm{Gr}(N - d - 1, H_m)$ be the incidence variety $S = \{(a, c, L) | \ell_{ac} \cap L \neq \emptyset\}$. We have the projections $A \times C \xleftarrow{\mathrm{pr}_1} S \xrightarrow{\mathrm{pr}_2} \mathrm{Gr}(N - d - 1, H_m)$. Since $L \cap X = \emptyset$, we see that for any point $(a, c) \in ((A \times C) \setminus \Delta_X)$, $\mathrm{pr}_1^{-1}((a, c)) = \{L \in \mathrm{Gr}(N - d - 1, H_m) | \dim(L \cap \ell_{ac}) = 0\}$. Consider the map $\pi : \mathrm{pr}_1^{-1}((a, c)) \to \ell_{ac}$ given by $\pi(L) = L \cap \ell_{ac}$.

Our hypothesis says that $(A \cup C) \cap H_m = \emptyset$ and this implies that $\ell_{ac} \not\subset H_m$. In particular, $x_{ac} = \ell_{ac} \cap H_m$ is a single closed point of $\mathbb{P}^N_k$. This implies that $\pi^{-1}(\ell_{ac} \setminus \{x_{ac}\}) = \emptyset$ and $\pi^{-1}(\{x_{ac}\}) = \mathrm{pr}_1^{-1}((a,c)) = \{L \in \mathrm{Gr}(N-d-1, H_m) | x_{ac} \in L\} \simeq \mathrm{Gr}(N-d-2, \mathbb{P}^{N-2}_k)$. It follows that $\dim(\mathrm{pr}_1^{-1}((a,c))) = (N-d-1)(N-2-(N-d-2)) = d(N-d-1)$. We conclude from this that

$$\begin{aligned}\dim(S) &\leq \dim(A) + \dim(C) + d(N-d-1)\\ &= \dim(A) + \dim(C) + d(N-d) - d\\ &= \dim(A) + \dim(C) + \dim(\mathrm{Gr}(N-d-1, H_m)) - d.\end{aligned} \tag{4.3}$$

Let $p_C : S \to A \times C \to C$ be the composite projection. We now observe that $c \in T^L_{ur}$ if and only if there exists $a \in A$ such that $a \neq c$ and $\ell_{ac} \cap L \neq \emptyset$. Since $c \in C$ as well, this means that $(a,c) \in \mathrm{pr}_2^{-1}(L)$. In other words, $T^L_{ur} \subset p_C(\mathrm{pr}_2^{-1}(L))$. On the other hand, it follows from (4.3) that there is a dense open subset $\mathcal{U}^{A,C}_{ur} \subseteq \mathrm{Gr}(N-d-1, H_m)$ such that $\mathrm{pr}_2^{-1}(L)$ is either empty or has dimension $\dim(A)+\dim(C)-d$ for every $L \in \mathcal{U}^{A,C}_{ur}$. We conclude that:

($\star$) There is a dense open subset $\mathcal{U}^{A,C}_{ur} \subseteq \mathrm{Gr}(N-d-1, H_m)$ such that $\dim(T^L_{ur}) \leq \dim(A) + \dim(C) - d$ for each $L \in \mathcal{U}^{A,C}_{ur}$.
Since U is smooth, given any point $x \in A \cap C$, our hypothesis implies that $T_x(X)$ is a locally closed subscheme of $\mathbb{P}^N_k$ of dimension d such that $T_x(X) \not\subset H_m$. We can therefore apply Lemma 4.2 to find a dense open subset of $\mathrm{Gr}(N-d-1, H_m)$ whose elements do not meet $T_x(X)$. But this means that $x \notin R_L(X)$ for every L in this dense open subset. We can repeat this for any chosen point in A and C as well. Since $\Sigma \subset A$, we therefore conclude that:

($\star\star$) There is a dense open subset $\mathcal{U}^{A,C}_r \subseteq \mathrm{Gr}(N-d-1, H_m)$ such that $R_L(X)$ does not contain any component of A, C or $A \cap C$ and it does not intersect Σ, whenever $L \in \mathcal{U}^{A,C}_r$.
For any $L \in \mathcal{U}^{A,C}_r$, we have $\dim(T^L_r) = \dim(R_L(X) \cap A \cap C) \leq \max\{\dim(A \cap C) - 1, 0\}$. Combining ($\star$) and ($\star\star$) with Lemma 4.2 and setting $\mathcal{U}^{A,C}_X = \mathcal{U}^{A,C}_{ur} \cap \mathcal{U}^{A,C}_r$, we conclude that $\mathcal{U}^{A,C}_X$ is a dense open subset of $\mathrm{Gr}(N-d-1, H_m)$ such that $e(L^+(A) \cap C) \leq \max\{e(A,C) - 1, 0\}$ for $L \in \mathcal{U}^{A,C}_X$.

The proof of (5) is clear if k is algebraically closed. In general, let $\overline{k}$ be an algebraic closure of k and let $\pi_Y : Y_{\overline{k}} \to Y$ denote the base change to $\overline{k}$ for any $Y \in \mathbf{Sch}_k$. For any $x \in \Sigma$, let $S_x = \pi_X^{-1}(x)$ and let $S = \bigcup_{x \in \Sigma} S_x$. Then $S \hookrightarrow X_{\overline{k}}$ is a finite set of closed points contained in $A_{\overline{k}}$. Let W' be the union of lines l_{xy} in $\mathbb{P}^N_{\overline{k}}$ such that $x \neq y \in S$. Since $S \subset A_{\overline{k}}$ and $A \cap H_m = \emptyset$, we see that $W' \not\subset H_{m,\overline{k}}$. Since $d \geq 1 = \dim(W')$, we can apply Lemma 4.2 to assume that $W' \cap L = \emptyset$ for all $L \in \mathcal{U}^{A,C}_{X_{\overline{k}}} := \mathcal{U}^{A_{\overline{k}}, C_{\overline{k}}}_{X_{\overline{k}}}$.

Since $\mathrm{Gr}(N-d-1, H_{m,\overline{k}})$ contains an affine space $\mathbb{A}^{d(N-d)}_{\overline{k}}$ as a dense open subset, we can replace $\mathcal{U}^{A,C}_{X_{\overline{k}}}$ by $\mathcal{U}^{A,C}_{X_{\overline{k}}} \cap \mathbb{A}^{d(N-d)}_{\overline{k}}$ and assume that $\mathcal{U}^{A,C}_{X_{\overline{k}}} \subseteq \mathbb{A}^{d(N-d)}_{\overline{k}}$.

Since k is infinite, the set of points in $\mathbb{A}_{\overline{k}}^{d(N-d)}$ with coordinates in k is dense in $\mathbb{A}_{\overline{k}}^{d(N-d)}$. Hence, there is a dense subset of $\mathcal{U}_{X_{\overline{k}}}^{A,C}$ each of whose points L is defined over k, *i.e.*, $L \in \mathrm{Gr}(N-d-1, H_m)$. Let $L \in \mathrm{Gr}(N-d-1, H_m)$ be such that (1) $\sim$ (4) hold and $W' \cap L_{\overline{k}} = \emptyset$. We consider the Cartesian square

$$\begin{array}{ccc} X_{\overline{k}} & \xrightarrow{\phi_{L_{\overline{k}}}} & \mathbb{P}^d_{\overline{k}} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbb{P}^d}} \\ X & \xrightarrow{\phi_L} & \mathbb{P}^d_k. \end{array} \tag{4.4}$$

Claim. For a closed point $x \in U$ and $y := \phi_L(x)$, one has $|\pi_{\mathbb{P}^d}^{-1}(y)| \le |\pi_X^{-1}(x)|$, and the equality holds if and only if $[k(x) : k(y)]^{\mathrm{sep}} = 1$. Furthermore, this equality holds if the map $\phi_{L_{\overline{k}}} : \pi_X^{-1}(x) \to \pi_{\mathbb{P}^d}^{-1}(y)$ is injective.

It is an elementary fact that $|\pi_X^{-1}(x)| = [k(x) : k]^{\mathrm{sep}}$ and $|\pi_{\mathbb{P}^d}^{-1}(y)| = [k(y) : k]^{\mathrm{sep}}$. The inclusions $k \hookrightarrow k(y) \hookrightarrow k(x)$ and therefore the equality $[k(x) : k]^{\mathrm{sep}} = [k(y) : k]^{\mathrm{sep}} \cdot [k(x) : k(y)]^{\mathrm{sep}}$ implies the first assertion. Next, the injectivity of the map $\phi_{L_{\overline{k}}} : \pi_X^{-1}(x) \to \pi_{\mathbb{P}^d}^{-1}(y)$ implies that $|\pi_{\mathbb{P}^d}^{-1}(y)| \ge |\pi_X^{-1}(x)|$. The second part of the Claim follows.

To prove (5) in general, it suffices to show that the finite field extension $k(\phi_L(x)) \hookrightarrow k(x)$ is separable as well as purely inseparable for each $x \in \Sigma$. Now, the separability of this extension is equivalent to the assertion $x \notin R_L(X)$, and this is guaranteed by (3). To prove inseparability, it is enough to show, using the above claim, that $\phi_{L_{\overline{k}}} : \pi_X^{-1}(x) \to \pi_{\mathbb{P}^d}^{-1}(\phi_L(x))$ is injective. But this follows immediately from the fact that $W' \cap L_{\overline{k}} = \emptyset$. The proof of the lemma is complete. $\square$

Lemma 4.4. *Let $\alpha \in z^q(X|H_m, n)$ be an admissible cycle. Let $C \subset X \setminus H_m$ be a locally closed subset as in Lemma 4.3. We can then find a dense open subset $\mathcal{U}_X^{Z,C} \subset \mathrm{Gr}(N-d-1, H_m)$ such that the following hold for every $L \in \mathcal{U}_X^{Z,C}$.*

(1) *$X \cap L = \emptyset$;*
(2) *For every irreducible component Z of α, no irreducible component of the support of the cycle $\phi_L^* \circ \phi_{L*}([Z]) - [Z]$ coincides with Z.*

Proof. It is enough to consider the case when $\alpha = [Z]$ is an irreducible admissible cycle. For any $L \in \mathrm{Gr}(N-d-1, H_m)$ satisfying (1), we need to prove the following to obtain (2):

(i) the ramification locus $R_L^n(X)$ of ϕ_L^n does not contain Z, where $\phi_L^n := \phi_L \times \mathrm{Id}_{\square_k^n}$;
(ii) $\phi_L^n|_Z : Z \to \phi_L^n(Z)$ is birational.

Let $\mathrm{pr}_X : X \times \square^n_k \to X$ and $\mathrm{pr}_{\square^n_k} : X \times \square^n_k \to \square^n_k$ be the projection maps. We fix a closed point $z \in Z$ and set $x = \mathrm{pr}_X(z)$, $y = \mathrm{pr}_{\square^n_k}(z)$, $W = \phi^n_L(Z)$ and $A = \mathrm{pr}_X(Z)$. Then A is a finite disjoint union of locally closed subsets of X. Since Z is an admissible cycle having modulus H_m, we must have $A \cap H_m = \emptyset$. In particular, $x \in U$. It is shown in the proof of Theorem 5.4 that $(\{y\} \times X) \cap Z$ is a finite set of closed points away from $(\{y\} \times H_m)$. In particular, $D := \mathrm{pr}_X((\{y\} \times X) \cap Z)$ is a finite set of closed points of X containing x and contained in A. This implies that $\mathrm{Sec}(x, D)$ is a closed subset of $\mathbb{P}^N_k$ of dimension one which is not contained in H_m. Hence, we conclude from Lemma 4.2 that $\mathrm{Gr}(\mathrm{Sec}(x, D), N-d-1, H_m)$ is dense open in $\mathrm{Gr}(N-d-1, H_m)$.

We have shown in the proof of Lemma 4.3 that there is a dense open subset $\mathcal{U}_{Z,1} \subset \mathrm{Gr}(N-d-1, H_m)$ such that $T_x(X) \cap L = \emptyset$ for each $L \in \mathcal{U}_{Z,1}$. Since the left square in (4.2) is Cartesian and ϕ_L is finite, it follows that its restriction $\phi^U_L : U \to \mathbb{P}^d_k \setminus H_m$ is also finite. Since U is furthermore smooth, it follows that ϕ^U_L is a finite and flat morphism of smooth schemes.

The flatness of ϕ^U_L now implies that there is an open neighborhood $V \subset U$ of x such that $\phi_L : V \to \mathbb{P}^d_k$ is étale. In particular, $\phi^n_L : V \times \square^n_k \to \mathbb{P}^d_k \times \square^n_k$ is étale. This implies that there is an open subset V' of Z containing z such that $\phi^n_L|_{V'} : V' \to W$ is unramified. We set $\mathcal{U}^{Z,C}_X = \mathrm{Gr}(\mathrm{Sec}(x, D), N-d-1, H_m) \cap \mathcal{U}_{Z,1} \cap \mathcal{U}^{A,C}_X$, where $\mathcal{U}^{A,C}_X$ is as in Lemma 4.3.

We fix any $L \in \mathcal{U}^{A,C}_X$. Since $R^n_L(X) = R_L(X) \times \square^n_k$ and no component of A is in $R_L(X)$ by Lemma 4.3, it follows that $Z \not\subset R^n_L(X)$, proving (i). To prove (ii), it suffices to show that $z \notin R^n_L(Z)$, $\{z\} = (\phi^n_L)^{-1}(\phi^n_L(z)) \cap Z$ and $k(\phi^n_L(z)) \xrightarrow{\simeq} k(z)$, because they imply that the map $\mathcal{O}_{W,\phi^n_L(z)} \to \mathcal{O}_{Z,z}$ is an isomorphism, and hence induces isomorphism of the function fields.

We have shown above that $z \notin R^n_L(Z)$. Since the map $k(\phi_L(x)) \to k(x)$ is an isomorphism by Lemma 4.3, it follows that the map $\phi^n_L : \square^n_{k(x)} \to \square^n_{k(\phi_L(x))}$ is also an isomorphism. In particular, the map $k(\phi^n_L(z)) \to k(z)$ is an isomorphism. To show $\{z\} = (\phi^n_L)^{-1}(\phi^n_L(z)) \cap Z$, note that if there is a closed point $z' \in ((\phi^n_L)^{-1}(\phi^n_L(z)) \cap Z) \setminus \{z\}$, then $x' := \mathrm{pr}_X(z') \in D \cap L^+(x)$, where we recall that $L^+(x) = \phi_L^{-1}(\phi_L(x)) \setminus \{x\}$. But this can happen only if $\ell_{xx'} \cap L \neq \emptyset$, which is not the case because $L \in \mathrm{Gr}(\mathrm{Sec}(x, D), N-d-1, H_m)$. This concludes the proof of (ii) and the lemma. □

Remark 4.5. We a make few comments on Lemma 4.3. To some readers, this result may appear similar to [13, Lemma 3.5.4]. But we caution the reader that the context, the underlying hypotheses and the proofs of the two results are different. We explain these differences:

(1) The proof of Lemma 4.3 does not work if we replace X by $X \cap \mathbb{A}^N_k$. The reason is that even if X intersects $L_{m,\infty}$ properly, we may not be able to find points on $A \cap C$ whose tangent spaces will intersect $L_{m,\infty}$ properly, and this breaks the second part of the proof of Lemma 4.3.

Since [13] considers the affine case, Levine cannot therefore use the above argument. Instead, he uses the idea of reimbedding X into a big enough projective space which allows him to take care of the above intersection problem associated to the tangent spaces;

(2) Contrary to [13], we cannot use the reimbedding idea. The reason is that we may not be able to realize our modulus H_m as pull-back of any hypersurface on the bigger projective space under the reimbedding. This in turn may not allow us to realize H_m as pull-back of a hypersurface under a linear projection;

(3) The modulus condition imposes more severe restrictions on the choice of L than in the situation of [13]. Thus we need to make more refined choices and without changing the given embedding of X.

Let $\mathcal{W} = \{W_1, \cdots, W_s\}$ be a finite collection of locally closed subsets of $X \setminus H_m$ and let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set function. Let K denote the function field of $\mathrm{Gr}(N-d-1, H_m)$ and let $L_{\mathrm{gen}} \in \mathrm{Gr}(N-d-1, H_m)(K)$ be the generic point of $\mathrm{Gr}(N-d-1, H_m)$. This can be seen as a K-rational point of $\mathrm{Gr}(N-d-1, H_m)$.

Lemma 4.6. *The linear projection away from L_{gen} defines a finite map $\phi_{L_{\mathrm{gen}}} : X_K \to \mathbb{P}^d_K$ satisfying the following conditions:*

(1) *The restriction $\phi^U_{L_{\mathrm{gen}}} : U_K \to \mathbb{P}^d_K \setminus H_{m,K}$ is finite and flat;*

(2) *$D_K = \phi^*_{L_{\mathrm{gen}}}(H_{\mathrm{gen}})$ for the hyperplane $H_{\mathrm{gen}} = (H_m \cap \mathbb{P}^d)_K$ in $\mathbb{P}^d_K$;*

(3) *The pull-back $\phi^*_{L_{\mathrm{gen}}} : z^q(\mathbb{P}^d_K | H_{\mathrm{gen}}, \bullet) \to z^q(X_K | D_K, \bullet)$ is defined;*

(4) *$(\phi^*_{L_{\mathrm{gen}}} \circ \phi_{L_{\mathrm{gen}}*} \circ \mathrm{pr}^*_{K/k} - \mathrm{pr}^*_{K/k})$ maps $z^q_{\mathcal{W},e}(X|D, \bullet)$ to $z^q_{\mathcal{W}_K, e-1}(X_K|D_K, \bullet)$.*

Proof. Having established Lemmas 4.3 and 4.4, the proof of this lemma is identical to that of [13, Lemma 3.5.6]. The modulus condition plays no role in this deduction. Using Lemmas 4.3 and 4.4 and the argument of [13] *verbatim*, one shows that given a cycle $\alpha \in z^q_{\mathcal{W},e}(X|D, p)$, there exists a dense open subset $\mathcal{U}^\alpha_X \subseteq \mathrm{Gr}(N-d-1, H_m)$ such that for each $L \in \mathcal{U}^\alpha_X$, the linear projection away from L defines a finite map $\phi_L : X \to \mathbb{P}^d_k$ satisfying the required conditions. This map is flat on U as shown in the proof of Lemma 4.4. Taking $L = L_{\mathrm{gen}}$ and using Lemma 2.5, we get (1), (3) and (4). The map $\phi_{L_{\mathrm{gen}}*}$ is defined by [11, Proposition 2.10].

Item (2) follows at once from our choice of L_{gen} and the elementary property of linear projection that a hyperplane section $X \cap H$ in $\mathbb{P}^N_k$ is a pull-back of a hyperplane of $\mathbb{P}^d_k$ via ϕ_L if and only if $L \subset H$. □

We are now ready to prove our main theorem on the moving lemma for the higher Chow groups of projective schemes with very ample modulus.

Theorem 4.7. *Let k be any field and let X be an equidimensional reduced projective scheme of dimension $d \geq 1$ over k. Let $D \subset X$ be a very ample effective Cartier divisor such that $X \setminus D$ is smooth over k. Let $\mathcal{W} = \{W_1, \cdots, W_s\}$ be a finite collection of locally closed subsets of X and let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set function.*

Then, the inclusion $z^q_{\mathcal{W},e-1}(X|D,\bullet) \hookrightarrow z^q_{\mathcal{W},e}(X|D,\bullet)$ *is a quasi-isomorphism. In particular, the inclusion* $z^q_{\mathcal{W}}(X|D,\bullet) \hookrightarrow z^q(X|D,\bullet)$ *is a quasi-isomorphism.*

Proof. The second part easily follows from the first part by induction because $z^q_{\mathcal{W}}(X|D,\bullet) = z^q_{\mathcal{W},0}(X|D,\bullet)$ and $z^q(X|D,\bullet) = z^q_{\mathcal{W},q}(X|D,\bullet)$. We thus need to show that the quotient complex $\frac{z^q_{\mathcal{W},e}(X|D,\bullet)}{z^q_{\mathcal{W},e-1}(X|D,\bullet)}$ is acyclic.

First suppose that the theorem is true for all infinite fields and let k be a finite field. Take a homology class α in this quotient. We choose two distinct primes ℓ_1 and ℓ_2, other than $\mathrm{char}(k)$, and take pro-ℓ_i-extensions $\iota_i : \mathrm{Spec}\,(k_i) \to \mathrm{Spec}\,(k)$ for $i = 1, 2$. Then the case of infinite fields tells us that $\iota_i^*(\alpha) = 0$ for $i = 1, 2$. Hence, a descent argument implies that there are finite extensions $\tau_i : \mathrm{Spec}\,(k'_i) \to \mathrm{Spec}\,(k)$ of relatively prime degrees such that $\tau_i^*(\alpha) = 0$ for $i = 1, 2$. Using the projection formula for finite and flat morphisms (see [11, Theorem 3.12]), this implies that $d_1\alpha = 0 = d_2\alpha$, where $(d_1, d_2) = 1$. We conclude that $\alpha = 0$.

We can now assume that k is infinite. We set $\mathcal{W}^0 = \{W_1 \setminus D, \cdots, W_s \setminus D\}$. Since a cycle in $z^q(X|D, p)$ does not intersect $D \times \square^p$, we see that $z^q_{\mathcal{W}}(X|D,\bullet) = z^q_{\mathcal{W}^0}(X|D,\bullet)$, and we may assume that $W \cap D = \emptyset$ for each $W \in \mathcal{W}$.

Since D is very ample, we can choose a closed embedding $\iota_X : X \hookrightarrow \mathbb{P}^N_k$ and a hyperplane $H_m \subset \mathbb{P}^n_k$ such that $D = \iota^*(H_m)$. If $X = \mathbb{P}^N_k$, we are done by Theorem 3.10. So we can assume that $1 \le d \le N-1$.

It follows from Lemma 4.6 that the map

$$\left(\phi^*_{L_{\mathrm{gen}}} \circ \phi_{L_{\mathrm{gen}}*} \circ \mathrm{pr}^*_{K/k} - \mathrm{pr}^*_{K/k}\right) : \frac{z^q_{\mathcal{W},e}(X|D,\bullet)}{z^q_{\mathcal{W},e-1}(X|D,\bullet)} \to \frac{z^q_{\mathcal{W}_K,e}(X_K|D_K,\bullet)}{z^q_{\mathcal{W}_K,e-1}(X_K|D_K,\bullet)} \tag{4.5}$$

is zero. On the other hand, each $\phi^*_{L_{\mathrm{gen}}} \circ \phi_{L_{\mathrm{gen}}*}$ factors as

$$\frac{z^q_{\mathcal{W}_K,e}(X_K|D_K,\bullet)}{z^q_{\mathcal{W}_K,e-1}(X_K|D_K,\bullet)} \xrightarrow{\phi_{L_{\mathrm{gen}}*}} \frac{z^q_{\phi_{L_{\mathrm{gen}}}(\mathcal{W}_K),e'}\left(\mathbb{P}^d_K|H_{\mathrm{gen}},\bullet\right)}{z^q_{\phi_{L_{\mathrm{gen}}}(\mathcal{W}_K),e'-1}\left(\mathbb{P}^d_K|H_{\mathrm{gen}},\bullet\right)} \xrightarrow{\phi^*_{L_{\mathrm{gen}}}} \frac{z^q_{\mathcal{W}_K,e}(X_K|D_K,\bullet)}{z^q_{\mathcal{W}_K,e-1}(X_K|D_K,\bullet)}$$

for some e' (see [10, Section 6C]). It follows from Corollary 3.11 that the middle complex is acyclic. This in turn implies that $\phi^*_{L_{\mathrm{gen}}} \circ \phi_{L_{\mathrm{gen}}*} = 0$ is zero on the level of homology. Combining this with (4.5), we conclude that $\mathrm{pr}^*_{K/k}$ is zero on the level of homology. By Proposition 2.8, the complex $\frac{z^q_{\mathcal{W},e}(X|D,\bullet)}{z^q_{\mathcal{W},e-1}(X|D,\bullet)}$ is acyclic. This concludes the proof of the theorem. $\square$

5. Applications and remarks

In this section we apply our moving lemma to prove certain contravariant functoriality for higher Chow groups with modulus. We prove a vanishing theorem on higher Chow groups with ample modulus. We end the section by explaining why the very ampleness condition is crucial for proving the moving lemma.

5.1. Contravariance

Let X be a quasi-projective scheme over a field k and let $D \subset X$ be a very ample effective Cartier divisor. Recall from [11, Theorem 3.12] if that X is smooth, there is a cap product $\cap_X : \mathrm{CH}^q(X, p) \otimes_{\mathbb{Z}} \mathrm{CH}^{q'}(X|D, p') \to \mathrm{CH}^{q+q'}(X|D, p+p')$. We prove the following contravariant functoriality for cycles with modulus.

Theorem 5.1. *Let $f : Y \to X$ be a morphism of quasi-projective schemes over a field k, where X is projective over k. Let $D \subset X$ be a very ample effective Cartier divisor such that $X \setminus D$ is smooth over k. Suppose that $f^*(D)$ is a Cartier divisor on Y* (i.e., *no minimal or embedded component of Y maps into D*). *Then there exists a map*

$$f^* : z^q(X|D, \bullet) \to z^q\big(Y|f^*(D), \bullet\big)$$

in the derived category of Abelian groups. In particular, there is a pull-back f^* : $\mathrm{CH}^q(X|D, p) \to \mathrm{CH}^q(Y|f^*(D), p)$ for every $p, q \geq 0$. If X and Y are smooth and projective, then for every $a \in \mathrm{CH}^*(Y, \bullet)$ and $b \in \mathrm{CH}^*(X|D, \bullet)$, there is a projection formula $f_*(a \cap_Y f^*(b)) = f_*(a) \cap_X b$.

Proof. The proof is a standard application of the moving lemma for Chow groups. Set $E = f^*(D)$. For $0 \leq i \leq \dim(Y)$, let X_i be the set of points $x \in X$ such that $\dim(f^{-1}(x)) \geq i$, where we assume $\dim(\emptyset) = -1$. Let $\mathcal{W}$ be the collection of the irreducible components of all X_i. The reader can check that $\mathcal{W}$ is a finite collection and the pull-back $f^* : z^q_{\mathcal{W}}(X|D, \bullet) \to z^q(Y|E, \bullet)$ is defined (see [10, Theorem 7.1]). We thus have maps $z^q(X|D, \bullet) \xleftarrow{q.iso} z^q_{\mathcal{W}}(X|D, \bullet) \xrightarrow{f^*} z^q(Y|E, \bullet)$ and Theorem 4.7 says that the arrow on the left is a quasi-isomorphism. This proves the first part of the theorem.

To prove the projection formula, we can assume using Theorem 4.7 that $b \in \mathrm{CH}^*(X|D, \bullet)$ is represented by a cycle $Z \in z^q_{\mathcal{W}}(X|D, \bullet)$, where $\mathcal{W}$ is as constructed above. By [11, Lemma 3.10], there is a finite collection of locally closed subsets $\mathcal{C}$ of Y such that $Z' \boxtimes f^*(Z) \in z^q_{\Delta_Y}(Y|E, \bullet)$ for all $Z' \in z^q_{\mathcal{C}}(Y, \bullet)$. By the moving lemma for Bloch's higher Chow groups, we can assume that $a \in \mathrm{CH}^*(Y, \bullet)$ is represented by a cycle $Z' \in z^q_{\mathcal{C}}(Y, \bullet)$. In this case, it is straightforward to check that $f_*(Z') \boxtimes Z \in z^q_{\Delta_X}(X|D, \bullet)$ and $f_* \circ \Delta^*_Y(Z' \boxtimes f^*(Z)) = \Delta^*_X(f_*(Z') \boxtimes Z)$. This finishes the proof. □

Remark 5.2. We remark that a pull-back map on higher Chow groups with modulus was constructed in [11, Theorem 4.3]. But Theorem 5.1 cannot be deduced from [11, Theorem 4.3]. The reason is that we make no assumption on the map f while [11] assumes D and E to be the pull-backs of a divisor on a base scheme S over which both X and Y should be smooth.

We also remark that Theorem 5.1 proves a stronger statement than giving a pull-back map on the higher Chow groups with modulus. This stronger version of [11, Theorem 4.3] is not yet known.

Corollary 5.3. *Let $r \geq 1$ be an integer and let $f : Y \to \mathbb{P}^r_k$ be a morphism of quasi-projective schemes over a field k. Let $D \subset \mathbb{P}^r_k$ be an effective Cartier divisor such that $f^*(D)$ is a Cartier divisor on Y. Then, there exists a pull-back $f^* : \mathrm{CH}^q(\mathbb{P}^r_k|D, p) \to \mathrm{CH}^q(Y|f^*(D), p)$ for every $p, q \geq 0$. If Y is also smooth and projective, then for every $a \in \mathrm{CH}^*(Y, \bullet)$ and $b \in \mathrm{CH}^*(\mathbb{P}^r_k|D, \bullet)$, there is a projection formula $f_*(a \cap_Y f^*(b)) = f_*(a) \cap_X b$.*

Proof. If $D = 0$, then it is just an application of the moving lemma for usual higher Chow groups. If $D \neq 0$ then it is very ample, so that Theorem 5.1 applies. □

5.2. A vanishing theorem

The following result shows that the higher Chow groups of projective schemes (not necessarily smooth) with ample modulus are nontrivial only in high codimension. More precisely:

Theorem 5.4. *Let X be a projective scheme of dimension $d \geq 1$ over a field k. Let $D \subset X$ be an ample effective Cartier divisor. Then $z_s(X|D, p) = 0$ for $s > 0$. In particular, $\mathrm{CH}_s(X|D, p) = 0$ for $s > 0$.*

Proof. We can find a closed embedding $\iota_X : X \hookrightarrow \mathbb{P}^N_k$ and a hyperplane $H \hookrightarrow \mathbb{P}^N_k$ such that $nD = \iota_X^*(H)$ for some $n \gg 0$. Suppose $z_s(X|D, p) \neq 0$ for some $s \in \mathbb{Z}$. Let $\alpha \in z_s(X|D, p)$ be a nonzero admissible cycle and let Z be an irreducible component of α. Let $\mathrm{pr}_{\mathbb{P}^N_k} : \mathbb{P}^N_k \times \square^p_k \to \mathbb{P}^N_k$ and $\mathrm{pr}_{\square^p_k} : \mathbb{P}^N_k \times \square^p_k \to \square^p_k$ denote the projection maps. Let $y \in \square^p_k$ be any scheme point. For any map $W \to \square^p_k$, let W_y denote the fiber $\mathrm{Spec}\,(k(y)) \times_{\square^p_k} W$ over y. The modulus condition for Z implies that Z_y is a closed subscheme of $\mathbb{P}^N_y$ disjoint from H_y. In particular, Z_y is a projective $k(y)$-scheme which is a closed subscheme of $(\mathbb{P}^N_y \setminus H_y) \simeq \mathbb{A}^N_{k(y)}$. Hence, it must be finite. We have thus shown that the projection map $Z \to \square^p_k$ is projective and quasi-finite, and hence finite. In other words, we must have $\dim(Z) = s + p \leq p$, *i.e.*, $s \leq 0$. Thus $z_s(X|D, p) = 0$ if $s > 0$, as desired. □

5.3. Sharpness of the very ampleness condition

We now show by an example that we cannot weaken the very ampleness condition to mere ampleness for the modulus divisor $D \subset X$. This also shows that the moving lemma for cycles with modulus on smooth affine schemes cannot be proven using the method of linear projections, in general. This partly explains the need for the Nisnevich sheafification of the cycle complex for the moving lemma of W. Kai [7].

Let X be an elliptic curve over an algebraically closed field k and let $D \subset X$ be a closed point. It is clear that $\mathcal{O}_X(D)$ is ample. We claim that there exists no pair (f, D') consisting of a map $f : X \to \mathbb{P}^1_k$ and an effective Cartier divisor $D' \in \mathrm{Div}(\mathbb{P}^1_k)$ such that $D = f^*(D')$.

Suppose there does exist such a pair (f, D'). Observe that we must have $d := \deg(D') > 0$ and D' is very ample. Let $\iota : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^d_k$ denote the closed embedding such that $\mathcal{O}_{\mathbb{P}^1_k}(D') \simeq \iota^*(\mathcal{O}_{\mathbb{P}^d_k}(1))$. This gives a regular map $\iota \circ f : X \to \mathbb{P}^d_k$ such that $(\iota \circ f)^*(\mathcal{O}_{\mathbb{P}^d_k}(1)) = \mathcal{O}_X(D)$. This implies that $\mathcal{O}_X(D)$ is globally generated. However, by Riemann-Roch, one checks immediately that $h^0(D) = 1$ in our case, *i.e.*, $\dim(|D|) = 0$ and the unique element of $|D|$ vanishes at D, a contradiction.

The only technique yet available in the literature to prove the moving lemma for Bloch's higher Chow groups of smooth affine schemes is the method of *linear projections*. Bloch proved the moving lemma for higher Chow groups of all smooth quasi-projective schemes (see [3] and [4, Proposition 2.5.2]). But his proof depends on the moving lemma for smooth affine schemes proven in [2] using linear projections.

Let us now consider the case of moving lemma for higher Chow groups with modulus on smooth affine schemes. Let U be a smooth affine scheme over an algebraically closed field k of characteristic zero. Let $D \subset U$ be a principal effective divisor (u) such that the induced map $u : U \setminus D \to \mathbb{A}^1_k$ is smooth. We use the above example to show that even in this special case, the method of linear projections cannot be used to prove the moving lemma for the higher Chow groups on U with modulus D. This makes proving the moving lemma for cycles with modulus on smooth affine or projective schemes very subtle and challenging.

Let X be an elliptic curve over k as above and let $D \hookrightarrow X$ be a closed point. There exists an affine neighborhood $V \hookrightarrow X$ of D such that $D = (u)$ is principal on V. Let $u : V \to \mathbb{A}^1_k$ be the induced dominant map. We can find an affine neighborhood $U \hookrightarrow V$ of D such that $u : U \setminus D \to \mathbb{A}^1_k$ is étale.

Proposition 5.5. *There exists no pair (f, D') consisting of a finite map $f : U \to \mathbb{A}^1_k$ and effective Cartier divisor $D' \hookrightarrow \mathbb{A}^1_k$ such that $D = f^*(D')$.*

Proof. If such pair (f, D') exists, then we get a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j'"] \arrow[d, "f"'] & X \arrow[d, "f'"] \\ \mathbb{A}^1_k \arrow[r, "j"] & \mathbb{P}^1_k, \end{tikzcd} \tag{5.1}$$

where the horizontal maps are open inclusions and the vertical maps are finite. This finiteness implies that the above square is Cartesian. This in turn implies that we have a finite map $f' : X' \to \mathbb{P}^1_k$ and effective Cartier divisor $D' \hookrightarrow \mathbb{P}^1_k$ such that $D = f'^*(D')$ on X. But we have previously shown that this is not possible. $\square$

6. Higher Chow groups with modulus of a line bundle

Let X be a quasi-projective scheme of dimension $d \geq 0$ over a field k. Let $f : \mathcal{L} \to X$ be a line bundle and let $\iota : X \hookrightarrow \mathcal{L}$ be the 0-section embedding. In this case, one knows that there is an isomorphism $\iota^* : \mathrm{CH}_*(\mathcal{L}, \bullet) \xrightarrow{\simeq} \mathrm{CH}_*(X, \bullet)$ (up to a shift in dimension) of ordinary higher groups. Since the Chow groups with modulus are supposed to be the 'relative motivic cohomology' of the pair $(\mathcal{L}, \iota(X))$, one expects $\mathrm{CH}_*(\mathcal{L}|X, \bullet)$ to be trivial.

As an application of the moving techniques of Section 3, we show in this section that every cycle in $z_s(\mathcal{L}|X, \bullet)$ can be moved to a trivial cycle so that this complex is acyclic. This gives an evidence supporting the expectation that the Chow groups with modulus are the relative motivic cohomology. It also provides examples where the higher Chow groups of a variety with a modulus in an effective Cartier divisor are all zero. Note that this can never happen for the ordinary higher groups. The proof closely follows the arguments of Lemmas 3.5 and 3.8, and Proposition 3.9.

Let $H : \mathcal{L} \times \mathbb{A}^1_k \to \mathcal{L}$ be the standard fiberwise contraction given explicitly as follows: for an affine open subset $U = \mathrm{Spec}\,(R) \subset X$ such that $f|_U$ is trivial, *i.e.*, of the form $f|_U : U \times \mathbb{A}^1_k \to U$, write $\mathcal{L}|_U = \mathrm{Spec}\,(R[t])$. Then, $H|_U : U \times \mathbb{A}^1_k \times \mathbb{A}^1_k \to U \times \mathbb{A}^1_k$ is induced by the polynomial map $R[x] \to R[t, x]$, given by $x \mapsto tx$.

For $n \geq 0$, let $H_n : \mathcal{L} \times \mathbb{A}^1_k \times \overline{\square}^n_k \to \mathcal{L} \times \overline{\square}^n_k$ be the map $H \times \mathrm{Id}_{\overline{\square}^n_k}$. For any irreducible closed admissible cycle $V \in z_s(\mathcal{L}|X, n)$, let $H^*(V)$ denote the cycle associated to the flat pull-back $H_n^{-1}(V)$. Set $V' = (H^*(V))_{\mathrm{red}}$. We extend H^* linearly to all cycles. Let $\overline{V} \hookrightarrow \mathcal{L} \times \overline{\square}^n_k$ denote the closure of V and let $\nu_V : \overline{V}^N \to \mathcal{L} \times \overline{\square}^n_k$ be the composition of the normalization and the inclusion. Let $\overline{V}'$ denote the closure of V' in $\mathcal{L} \times \overline{\square}^{n+1}_k$ and let $\nu_{V'} : \overline{V}'^N \to \mathcal{L} \times \overline{\square}^{n+1}_k$ denote the map induced by the normalization of $\overline{V}'$.

Lemma 6.1. *$V' \hookrightarrow \mathcal{L} \times \square^{n+1}_k$ has modulus X.*

Proof. Since the modulus condition is local on $\mathcal{L}$, it is enough to show that $V' \cap (f^{-1}(U) \times \square^{n+1}_k)$ has modulus U for every affine open subset $U \subset X$ over which f is trivial. So we may assume $X = \mathrm{Spec}\,(R)$ is affine and $\mathcal{L} = \mathrm{Spec}\,(R[X])$ is trivial. In this case, $H : U \times \mathbb{A}^1_k \times \mathbb{A}^1_k \to U \times \mathbb{A}^1_k$ is given by $H(u, x, y) = (u, xy)$. Since U plays no role in this map, we can drop it and assume $U = \mathrm{Spec}\,(k)$ so that $H : \mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ is the multiplication map. This map uniquely extends to a rational map $H : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^1_k$, given by $H((X_0; X_1), (T_0; T_1)) = (X_0T_0; X_1T_1)$, which is regular on $W = (\mathbb{P}^1_k \times \mathbb{P}^1_k) \setminus \{(0, \infty), (\infty, 0)\}$.

We next observe that since the modulus divisor is $U = \{0\} \hookrightarrow \mathbb{A}^1_k$, to check the modulus condition for $H^{-1}(V)$ is equivalent to check the modulus $(\{0\} \times \mathbb{A}^1_k)$ for $(H|_{W \times \square^n_k})^{-1}(V_1)$, where V_1 is the closure of V in $\mathbb{P}^1_k \times \square^n_k$. We can thus replace $\mathbb{A}^1_k$ by $\mathbb{P}^1_k$ as the target space of H and $\overline{V}'$ by its closure in $\mathbb{P}^1_k \times \overline{\square}^{n+1}_k$ in order to check the modulus condition for V'.

Let $\pi : \Gamma \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ be the blow-up along $\Sigma = \{(0, \infty), (\infty, 0)\}$. It is easily checked (see the proof of Lemma 3.5) that $\Gamma \hookrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k$ is the closed subscheme given by $\Gamma = \{((X_0; X_1), (T_0; T_1), (Y_1; Y_0)) \,|\, X_0 T_0 Y_0 = X_1 T_1 Y_1\}$. Define a map $\overline{H} : \Gamma \to \mathbb{P}^1_k$ by $\overline{H}((X_0; X_1), (T_0; T_1), (Y_1; Y_0)) = (Y_1; Y_0)$.

We claim that $\overline{H}|_W = H$. To check this, let $U_1 = \{((X_0; X_1), (T_0; T_1)) \,|\, X_1 \neq 0 \neq T_0\}$ and $U_2 = \{((X_0; X_1), (T_0; T_1)) \,|\, X_0 \neq 0 \neq T_1\}$ be two open subsets of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. In the affine coordinates $(x_0, t_1) \in U_1 \simeq \mathbb{A}^2_k$, the restriction of H on $U_1 \cap W$ is given by $H(x_0, t_1) = (x_0; t_1)$ and the restriction of $\overline{H}$ on $\pi^{-1}(U_1) \cap W \cap (x_0 \neq 0)$ is given by $\overline{H}\big((x_0, t_1, (1; x_0^{-1} t_1)\big) = (1; x_0^{-1} t_1) = (x_0; t_1) = H(x_0, t_1)$. The restriction of $\overline{H}$ on $\pi^{-1}(U_1) \cap W \cap (t_1 \neq 0)$ is given by $\overline{H}\big((x_0, t_1, (x_0 t_1^{-1}; 1)\big) = (x_0 t_1^{-1}; 1) = (x_0; t_1) = H(x_0, t_1)$.

The restriction of H on $U_2 \cap W$ is given by $H(x_1, t_0) = (t_0; x_1)$ and the restriction of $\overline{H}$ on $\pi^{-1}(U_2) \cap W \cap (x_1 \neq 0)$ is given by $\overline{H}\big((x_1, t_0, (x_1^{-1} t_0; 1)\big) = (x_1^{-1} t_0; 1) = (t_0; x_1) = H(x_1, t_0)$. The restriction of $\overline{H}$ on $\pi^{-1}(U_1) \cap W \cap (t_0 \neq 0)$ is given by $\overline{H}\big((x_1, t_0, (1; x_1 t_0^{-1})\big) = (1; x_1 t_0^{-1}) = (t_0; x_1) = H(x_1, t_0)$. Since π is an isomorphism away from $U_1 \cup U_2$, we have shown that $\overline{H}|_W = H$.

It follows from the claim that there is a commutative diagram

$$\begin{tikzcd}
\pi^{-1}(W) \arrow[r, hook, "j_1"] \arrow[d, "\simeq"'] & \Gamma \arrow[d, "\pi"] \arrow[dr, "\overline{H}"] & \\
W \arrow[r, hook, "j"] \arrow[rr, bend right] & \mathbb{P}^1_k \times \mathbb{P}^1_k \arrow[r, dotted, "H"] & \mathbb{P}^1_k.
\end{tikzcd} \tag{6.1}$$

Let $E = \pi^*((0, \infty))$ denote one of the two components of the exceptional divisor for π and let $D = U = \{0\} \hookrightarrow \mathbb{P}^1_k$. We have $\pi^*(D \times \mathbb{P}^1_k) = (D \times \mathbb{P}^1_k) + E$. Similarly, we have $\pi^*(\mathbb{P}^1_k \times \{\infty\}) = (\mathbb{P}^1_k \times \{\infty\}) + E$ in $\mathrm{Div}(\Gamma)$. Set $E_n = E \times \overline{\square}^n_k$.

Let $Z \hookrightarrow \Gamma \times \overline{\square}^n_k$ denote the strict transform of $\overline{V}'$. Since $\overline{H}_n(Z \cap (\pi^{-1}(W) \times \square^n_k)) = V$ and since $\overline{H}_n$ is projective, we must have $\overline{H}_n(Z) = \overline{V}$. We remark at this stage that ensuring the projectivity of $\overline{H}_n$ was the reason for us to replace $\mathbb{A}^1_k \times \mathbb{A}^1_k$ by $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and $\mathbb{A}^1_k$ by $\mathbb{P}^1_k$ as the source and the target of H.

We now have a commutative diagram

$$\begin{tikzcd}
Z^N \arrow[rr, "f"] \arrow[dr, "\nu_Z"] \arrow[dd, "g"'] & & \overline{V}^N \arrow[d, "\nu_V"] \\
& \Gamma \times \overline{\square}^n_k \arrow[r, "\overline{H}_n"] \arrow[d, "\pi_n"] & \mathbb{P}^1_k \times \overline{\square}^n_k \\
\overline{V}'^N \arrow[r, "\nu_{V'}"'] & \mathbb{P}^1_k \times \overline{\square}^{n+1}_k, &
\end{tikzcd} \tag{6.2}$$

where f and g are the unique maps induced by the universal property of normalization for dominant maps. Since f is a surjective map of integral schemes, the modulus condition for V implies that $(\nu_V \circ f)^*(\mathbb{P}^1_k \times F_n^\infty) \geq (\nu_V \circ f)^*(D \times \overline{\square}^n_k)$ on Z^N. In particular, we get $(\overline{H}_n \circ \nu_Z)^*(\mathbb{P}^1_k \times F_n^\infty) \geq (\overline{H}_n \circ \nu_Z)^*(D \times \overline{\square}^n_k)$ on Z^N. Equivalently, we have

$$\nu_Z^*\big(\Gamma \times F_n^\infty\big) \geq \nu_Z^*\left(\overline{H}^*(D) \times \overline{\square}^n_K\right). \tag{6.3}$$

Since $H^*(D) = (\mathbb{A}^1_k \times \{0\}) + (\{0\} \times \square_k)$, we get $j^*_{1,n} \circ \overline{H}^*_n(D \times \overline{\square}^n_k) = j^*_{1,n}(\mathbb{A}^1_k \times F^0_{n,n+1}) + j^*_{1,n}(D \times \overline{\square}^{n+1}_k)$, where $j_1 : W \hookrightarrow \Gamma$ is the inclusion. Since $\mathbb{A}^1_k \times F^0_{n,n+1}$ and $D \times \overline{\square}^{n+1}_k$ are irreducible, we get $\overline{H}^*(D) \times \overline{\square}^n_k \geq (\mathbb{P}^1_k \times F^0_{n,n+1}) + (D \times \overline{\square}^{n+1}_k)$ on $\Gamma \times \overline{\square}^n_k$. Combining this with (6.3), we get

$$\nu_Z^*\big(\Gamma \times F_n^\infty\big) \geq \nu_Z^*\left(D \times \overline{\square}^{n+1}_k\right). \tag{6.4}$$

This in turn implies that

$$\begin{aligned}
(\pi_n \circ \nu_Z)^*\left(\mathbb{P}^1_k \times F^\infty_{n+1}\right) &= (\pi_n \circ \nu_Z)^*\left(\mathbb{P}^1_k \times F^\infty_n \times \overline{\square}_k\right) \\
&\quad + (\pi_n \circ \nu_Z)^*\left(\mathbb{P}^1_k \times \overline{\square}^n_k \times \{\infty\}\right) \\
&= \nu_Z^*\big(\Gamma \times F_n^\infty\big) + (\pi_n \circ \nu_Z)^*\left(\mathbb{P}^1_k \times \overline{\square}^n_k \times \{\infty\}\right) \\
&\geq \nu_Z^*\left(D \times \overline{\square}^{n+1}_k\right) + (\pi_n \circ \nu_Z)^*\left(\mathbb{P}^1_k \times \overline{\square}^n_k \times \{\infty\}\right) \\
&= \nu_Z^*\left(D \times \overline{\square}^{n+1}_k\right) + \nu_Z^*(E_n) + \nu_Z^*\left(\mathbb{P}^1_k \times \overline{\square}^n_k \times \{\infty\}\right) \\
&= (\pi_n \circ \nu_Z)^*\left(D \times \overline{\square}^{n+1}_k\right) + \nu_Z^*\left(\mathbb{P}^1_k \times \overline{\square}^n_k \times \{\infty\}\right) \\
&\geq (\pi_n \circ \nu_Z)^*\left(D \times \overline{\square}^{n+1}_k\right).
\end{aligned}$$

Using (6.2), this gives $g^*(\nu^*_{V'}(\mathbb{P}^1_k \times F^\infty_{n+1})) \geq g^*(\nu^*_{V'}(D \times \overline{\square}^{n+1}_k))$. We now apply Lemma 2.6 to conclude that $\nu^*_{V'}(\mathbb{P}^1_k \times F^\infty_{n+1}) \geq \nu^*_{V'}(D \times \overline{\square}^{n+1}_k)$ and this is the modulus condition for V'. $\square$

Lemma 6.2. *$V' \hookrightarrow \mathcal{L} \times \square^{n+1}_k$ intersects all the faces properly.*

Proof. Since H is flat, V' intersects properly all the faces of $\square^{n+1}_k$ of the form $F \times \square_k$. Since $\iota^*_{n+1,n+1,1}(V') = V$, which intersects the faces of $\square^n_k$ properly, we see that V' intersects $F^1_{n+1,n+1}$ properly. Since $V \cap (X \times \square^n_k) = \emptyset$, we must have $\iota^*_{n+1,n+1,0}(V') = 0$. We have thus shown that V' satisfies the face condition. $\square$

Theorem 6.3. *Let X be a quasi-projective scheme over a field k and let $f : \mathcal{L} \to X$ be a line bundle. Let $\iota : X \hookrightarrow \mathcal{L}$ denote the 0-section embedding. Then the cycle complex $z_s(\mathcal{L}|X, \bullet)$ is acyclic for all $s \in \mathbb{Z}$.*

Proof. It follows from Lemmas 6.1 and 6.2 that $H : \mathcal{L} \times \mathbb{A}^1_k \to \mathcal{L}$ defines a chain homotopy $H^* : z_s(\mathcal{L}|X, \bullet) \to z_s(\mathcal{L}|X, \bullet)[-1]$ between $H_0^* = (H|_{\mathcal{L}\times 0})^*$ and $H_1^* = (H|_{\mathcal{L}\times 1})^*$. It is clear that $H_1^* = \mathrm{Id}_{z_s(\mathcal{L}|X,\bullet)}$ and the modulus condition implies that $H_0^* = 0$. It follows that $z_s(\mathcal{L}|X, \bullet)$ is acyclic. □

References

[1] F. BINDA and S. SAITO, *Relative cycles with moduli and regulator maps*, J. Inst. Math. Jussieu (2017), to appear.

[2] S. BLOCH, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[3] S. BLOCH, *The moving lemma for higher Chow groups*, J. Algebraic Geom. **3** (1994), 537–568.

[4] S. BLOCH, "Basic Definition and Properties of Higher Chow Groups", web page of Spencer Bloch, University of Chicago, 2005.

[5] S. BLOCH and H. ESNAULT, *The additive dilogarithm*, Doc. Math., Extra Vol. (2003), 131–155.

[6] R. HARTSHORNE, "Algebraic Geometry", Graduate Texts in Math., Vol. 52, Springer-Verlag, New York., 1977.

[7] W. KAI, *A moving lemma for algebraic cycles with modulus and contravariance*, arXiv:1507.07619v1, (2015).

[8] M. KERZ and S. SAITO, *Chow group of* 0*-cycles with modulus and higher dimensional class field theory*, Duke Math. J. **165** (2016), 2811–2897.

[9] A. KRISHNA and M. LEVINE, *Additive higher Chow groups of schemes*, J. Reine Angew. Math. **619** (2008), 75–140.

[10] A. KRISHNA and J. PARK, *Moving lemma for additive higher Chow groups*, Algebra Number Theory **6** (2012), 293–326.

[11] A. KRISHNA and J. PARK, *A module structure and a vanishing theorem for cycles with modulus*, Math. Res. Lett., to appear, arXiv:1412.7396v2, (2015).

[12] A. KRISHNA and J. PARK, *On additive higher Chow groups of affine schemes*, Doc. Math. **21** (2016), 49–89.

[13] M. LEVINE, "Mixed Motives", Mathematical Surveys and Monographs, Vol. 57, American Mathematical Society, Providence, RI, 1998.

[14] J. PARK, *Regulators on additive higher Chow groups*, Amer. J. Math. **131** (2009), 257–276.

School of Mathematics
Tata Institute of Fundamental Research
1 Homi Bhabha Road
Colaba, Mumbai, India
amal@math.tifr.res.in

Department of Mathematical Sciences
KAIST, 291 Daehak-ro Yuseong-gu
Daejeon, 305-701, Republic of Korea (South)
jinhyun@mathsci.kaist.ac.kr