Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 1461-1489

Twisted cohomology of arrangements of lines and Milnor fibers

MARIO SALVETTI AND MATTEO SERVENTI

Abstract. Let $\mathcal{A}$ be an arrangement of affine lines in $\mathbb{C}^2$, with complement $\mathcal{M}(\mathcal{A})$. The (co)homology of $\mathcal{M}(\mathcal{A})$ with twisted coefficients is strictly related to the cohomology of the Milnor fibre associated to the conified arrangement, endowed with the geometric monodromy. Although several partial results are known, even the first Betti number of the Milnor fiber is not understood. We give here a vanishing conjecture for the first homology, which is of a different nature with respect to the known results. Let Γ be the graph of *double points* of $\mathcal{A}$: we conjecture that if Γ is connected then the geometric monodromy acts trivially on the first homology of the Milnor fiber (so that the first Betti number is combinatorially determined in this case). This conjecture depends only on the combinatorics of $\mathcal{A}$. We prove it in some cases with stronger hypothesis.

In the final parts, we introduce a new description in terms of the group given by the quotient of the commutator subgroup of $\pi_1(\mathcal{M}(\mathcal{A}))$ by the commutator of its *length-zero subgroup*. We use that to deduce some new interesting cases of a-monodromicity, including a proof of the conjecture under some extra conditions.

Mathematics Subject Classification (2010): 55N25 (primary); 57M05 (secondary).

1. Introduction

Let $\mathcal{A} := \{\ell_1, \ldots, \ell_n\}$ be an arrangement of affine lines in $\mathbb{C}^2$, with complement $\mathcal{M}(\mathcal{A})$. Let $\mathcal{L}$ be a rank-1 local system on $\mathcal{M}(\mathcal{A})$, which is defined by a unitary commutative ring R and an assignment of an invertible element $t_i \in R^*$ for each line $\ell_i \in \mathcal{A}$. Equivalently, $\mathcal{L}$ is defined by a module structure on R over the fundamental group of $\mathcal{M}(\mathcal{A})$ (such structure factorizes through the first homology of $\mathcal{M}(\mathcal{A})$). By "coning" $\mathcal{A}$ one obtains a three-dimensional central arrangement, with complement fibering over $\mathbb{C}^*$. The Milnor fiber F of such fibration is a surface of degree $n+1$, endowed with a natural monodromy automorphism of order $n+1$. It

Partially supported by INdAM and by Università di Pisa under the "PRA – Progetti di Ricerca di Ateneo" (Institutional Research Grants) – Project no. PRA_2016_67 "Geometria, Algebra e Combinatoria di Spazi di Moduli e Configurazioni".

Received August 7, 2015; accepted in revised form September 27, 2016.
Published online December 2017.

is well known that the trivial (co)homology of F with coefficients in a commutative ring A, as a module over the monodromy action, is obtained by the (co)homology of $\mathcal{M}(\mathcal{A})$ with coefficients in $R := A[t^{\pm 1}]$, where here the structure of R as a $\pi_1(\mathcal{M}(\mathcal{A}))$-module is given by taking all the t_i's equal to t and the monodromy action corresponds to t-multiplication. For reflection arrangements, relative to a Coxeter group $\mathbf{W}$, many computations were done, especially for the *orbit space* $\mathcal{M}_{\mathbf{W}}(\mathcal{A}) := \mathcal{M}(\mathcal{A})/\mathbf{W}$, which has an associated Milnor fiber $F_{\mathbf{W}} := F/\mathbf{W}$: in this case we know a complete answer for $R = \mathbb{Q}[t^{\pm 1}]$, for all groups of finite type (see [11,12,21]), and for some groups of affine type [6–8] (based on the techniques developed in [13,30]). For $R = \mathbb{Z}[t^{\pm 1}]$ a complete answer is known in case A_n (see [5]). Some results are known for (non quotiented) reflection arrangements (see [25,31]). A big amount of work in this case has been done on related questions, when $R = \mathbb{C}$, in that case the t_i's being non-zero complex numbers, trying to understand the jump-loci (in $(\mathbb{C}^*)^n$) of the cohomology (see for example [9,10,16,18,24,34]).

Some algebraic complexes computing the twisted cohomology of $\mathcal{M}(\mathcal{A})$ are known (see for example the above cited papers). In [22], the minimal cell structure of the complement which was constructed in [33] (see [15,28]) was used to find an algebraic complex which computes the twisted cohomology, in the case of real defined arrangements (see also [23]). The form of the boundary maps depends not only on the lattice of the intersections associated to $\mathcal{A}$ but also on its *oriented matroid:* for each singular point P of multiplicity m there are $m-1$ generators in dimension 2 whose boundary has non vanishing components along the lines contained in the "cone" of P and passing above P.

Many of the specific examples of arrangements with non-trivial cohomology (*i.e.*, having non-trivial monodromy) which are known are based on the theory of *nets* and *multinets* (see [19]): there are relatively few arrangements with non trivial monodromy in cohomology and some conjecture claim very strict restrictions for line arrangements (see [37]).

In this paper we state a vanishing conjecture of a very different nature, which is very easily stated and which involves only the lattice associated to the arrangement. Let Γ be the graph with vertex set $\mathcal{A}$ and edge set which is given by taking an edge (ℓ_i, ℓ_j) iff $\ell_i \cap \ell_j$ is a double point. Then our conjecture is as follows:

Conjecture 1.

$$\textit{Assume that } \Gamma \textit{ is connected; then } \mathcal{A} \textit{ has trivial monodromy.} \tag{1.1}$$

This conjecture is supported by several "experiments", since all computations we made confirm it. Also, all non-trivial monodromy examples we know have disconnected graph Γ. We give here a proof holding with further restrictions. Our method uses the algebraic complex given in [22] so our arrangements are real.

An arrangement with trivial monodromy will be called *a-monodromic*. We also introduce a notion of monodromic triviality over $\mathbb{Z}$. By using free differential calculus, we show that $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$ iff the fundamental group of the complement $\mathcal{M}(A)$ of the arrangement is commutative modulo the commutator subgroup of the length-zero subgroup of the free group F_n. As a consequence, we deduce that if $G := \pi_1(\mathcal{M}(\mathcal{A}))$ modulo its second derived group is commutative, then $\mathcal{A}$ has trivial monodromy over $\mathbb{Z}$.

In the final section we give an intrinsic characterization of the a-monodromicity. Let K be the kernel of the *length map* $G \to \mathbb{Z}$. We introduce the group $H := \frac{[G,G]}{[K,K]}$, and we show that such group exactly measures the "non-triviality" of the first homology of the Milnor fiber F, as well as its torsion. Any question about the first homology of F is actually a question about H. To our knowledge, H appears here for the first time (a preliminary partial version is appearing in [32]). We use this description to give some interesting new results about the a-monodromicity of the arrangement. First, we show that if G decomposes as a direct product of two groups, each of them containing an element of length 1, then $\mathcal{A}$ is a-monodromic (Theorem 8.11). This includes the case when G decomposes as a direct product of free groups. As a further interesting consequence, an arrangement which decomposes into two subarrangements which intersect each other transversally, is a-monodromic.

Also, we use this description to prove our conjecture under the hypothesis that we have a connected *admissible* graph of commutators (Theorem 8.13): essentially, this means to have enough double points $\ell_i \cap \ell_j$ which give as relation (mod $[K, K]$) the commutator of the fixed geometric generators β_i, β_j of G.

After having finished our paper, we learned about the paper [2] were the graph of double points is introduced and some partial results are shown, by very different methods.

2. Some recalls

We recall here some general constructions (see [36], also as a reference to most of the recent literature). Let M be a space with the homotopy type of a finite CW-complex with $H_1(M; \mathbb{Z})$ free Abelian of rank n, having basis $e_1, \ldots, e_n$. Let $\underline{t} = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ and denote by $\mathbb{C}_{\underline{t}}$ the Abelian rank one local system over M given by the representation

$$\phi : H_1(M; \mathbb{Z}) \longrightarrow \mathbb{C}^* = \mathrm{Aut}(\mathbb{C})$$

assigning t_i to e_i.

Definition 2.1. With this notation one calls

$$V(M) = \left\{\underline{t} \in (\mathbb{C}^*)^n : \dim_{\mathbb{C}} H_1(M; \mathbb{C}_{\underline{t}}) \geq 1\right\}$$

the *(first) characteristic variety* of M.

There are several other analogous definitions in all (co)homological dimensions, as well as refined definitions keeping into account the dimension actually reached by the local homology groups. For our purposes here we need to consider only the above definition.

The characteristic variety of a CW-complex M turns out to be an algebraic subvariety of the algebraic torus $(\mathbb{C}^*)^{b_1(M)}$ which depends only on the fundamental group $\pi_1(M)$ (see for example [10]).

Let now $\mathcal{A}$ be a complex hyperplane arrangement in $\mathbb{C}^n$. One knows that the complement $\mathcal{M}(A) = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$ has the homotopy type of a finite CW-complex of dimension n. Moreover, in this case one knows by a general result (see [1]) that the characteristic variety of M is a finite union of torsion translated subtori of the algebraic torus $(\mathbb{C}^*)^{b_1(M)}$.

Now we need to briefly recall two standard constructions in arrangement theory (see [26] for details).

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine hyperplane arrangement in $\mathbb{C}^n$ with coordinates $z_1, \dots, z_n$ and, for every $1 \le i \le n$, let α_i be a linear polynomial such that $H_i = \alpha_i^{-1}(0)$. The *cone* $c\mathcal{A}$ of $\mathcal{A}$ is a central arrangement in $\mathbb{C}^{n+1}$ with coordinates $z_0, \dots, z_n$ given by $\{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ where $\widetilde{H}_0$ is the coordinate hyperplane $z_0 = 0$ and, for every $1 \le i \le n$, $\widetilde{H}_i$ is the zero locus of the homogenization of α_i with respect to z_0.

Now let $\widetilde{\mathcal{A}} = \{\widetilde{H}_0, \dots, \widetilde{H}_n\}$ be a central arrangement in $\mathbb{C}^{n+1}$ and choose coordinates $z_0, \dots, z_n$ such that $\widetilde{H}_0 = \{z_0 = 0\}$; moreover, for every $1 \le i \le n$; let $\widetilde{\alpha}_i(z_0, \dots, z_n)$ be such that $\widetilde{H}_i = \widetilde{\alpha}_i^{-1}(0)$. The *deconing* of $\widetilde{\mathcal{A}}$ is the arrangement $d\widetilde{\mathcal{A}}$ in $\mathbb{C}^n$ given by $\{H_1, \dots, H_n\}$ where, if we set for every $1 \le i \le n$, $\alpha_i(z_1, \dots, z_1) = \widetilde{\alpha}_i(1, z_1, \dots, z_n)$, $H_i = \alpha_i^{-1}(0)$. One easily sees that $\mathcal{M}(c\mathcal{A}) = \mathcal{M}(\mathcal{A}) \times \mathbb{C}^*$ (and conversely $\mathcal{M}(\widetilde{\mathcal{A}}) = \mathcal{M}(d\widetilde{\mathcal{A}}) \times \mathbb{C}^*$).

The fundamental group $\pi_1(\mathcal{M}(\widetilde{\mathcal{A}})))$ is generated by elementary loops β_i, $i = 0, \dots, n$, around the hyperplanes and in the decomposition $\pi_1(\mathcal{M}(\mathcal{A})) \simeq \pi_1(\mathcal{M}(d\mathcal{A})) \times \mathbb{Z}$ the generator of $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ corresponds to a loop going around all the hyperplanes. The generators can be ordered so that such a loop is represented by $\beta_0 \dots \beta_n$. Choosing $\widetilde{H}_0$ as the hyperplane at infinity in the deconing $\mathcal{A} = d\widetilde{\mathcal{A}}$, one has (see [10])

$$V(\widetilde{\mathcal{A}}) = \big\{\underline{t} \in (\mathbb{C}^*)^{n+1} : (t_1, \dots, t_n) \in V(d\mathcal{A}) \text{ and } t_0 \cdots t_n = 1\big\}.$$

It is still an open question whether the characteristic variety $V(\widetilde{\mathcal{A}})$ is combinatorially determined, that is, determined by the intersection lattice $L(\widetilde{\mathcal{A}})$. Actually, the question is partially solved: thanks to the above description we can write

$$V(\widetilde{\mathcal{A}}) = \check{V}(\widetilde{\mathcal{A}}) \cup T(\widetilde{\mathcal{A}}),$$

where $\check{V}(\widetilde{\mathcal{A}})$ is the union of all the components of $V(\widetilde{\mathcal{A}})$ passing through the unit element $\underline{1} = (1, 1, \dots, 1)$ and $T(\widetilde{\mathcal{A}})$ is the union of the translated tori of $V(\widetilde{\mathcal{A}})$.

The "homogeneous" part $\check{V}(\widetilde{\mathcal{A}})$ is combinatorially described through the *resonance variety*

$$\mathcal{R}^1(\widetilde{\mathcal{A}}) := \big\{a \in A^1 : H^1(A^\bullet, a \wedge \cdot) \ne 0\big\}$$

introduced in [18]. Here $A^\bullet$ is the Orlik-Solomon algebra over $\mathbb{C}$ of $\widetilde{\mathcal{A}}$. Denote by $\mathcal{V}(\widetilde{\mathcal{A}})$ the tangent cone of $V(\widetilde{\mathcal{A}})$ at $\underline{1}$; it turns out that $\mathcal{V}(\widetilde{\mathcal{A}}) \cong \mathcal{R}^1(\widetilde{\mathcal{A}})$. So, from $\mathcal{R}^1(\widetilde{\mathcal{A}})$ we can obtain the components of $V(\widetilde{\mathcal{A}})$ containing $\underline{1}$ by exponentiation.

It is also known (see [10,24]) that $\mathcal{R}^1(\widetilde{\mathcal{A}})$ is a subspace arrangement: $\mathcal{R}^1(\widetilde{\mathcal{A}}) = C_1 \cup \dots \cup C_r$ with $\dim C_i \geq 2, C_i \cap C_j = 0$ for every $i \neq j$.

One makes a distinction between *local components* C_I of $R^1(\widetilde{\mathcal{A}})$, associated to a codimensional-2 flat I in the intersection lattice, which are contained in some coordinate hyperplanes; and *global components*, which are not contained in any coordinate hyperplane of A^1. Global components of dimension $k-1$ are known to correspond to (k,d)-*multinets* [19]. Let $\overline{\mathcal{A}}$ be the projectivization of $\widetilde{\mathcal{A}}$. A (k,d)-*multinet* on a multi-arrangement $(\overline{\mathcal{A}}, m)$, is a pair $(\mathcal{N}, \mathcal{X})$ where $\mathcal{N}$ is a partition of $\overline{A}$ into $k \geq 3$ classes $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_k$ and $\mathcal{X}$ is a set of multiple points with multiplicity greater than or equal to 3 which satisfies a list of conditions. We just recall that $\mathcal{X}$ determines $\mathcal{N}$: construct a graph $\Gamma' = \Gamma'(\mathcal{X})$ with $\overline{\mathcal{A}}$ as vertex set and an edge from l to l' if and only if $l \cap l' \notin \mathcal{X}$. Then the connected components of Γ' are the blocks of the partition $\mathcal{N}$.

3. The Milnor fibre and a conjecture

Let $Q : \mathbb{C}^3 \to \mathbb{C}$ be a homogeneous polynomial (of degree $n+1$) which defines the arrangement $\widetilde{\mathcal{A}}$. Then Q gives a fibration

$$Q_{|\mathcal{M}(\tilde{\mathcal{A}})} : \mathcal{M}(\tilde{\mathcal{A}}) \to \mathbb{C}^* \tag{3.1}$$

with *Milnor fibre* $\mathbf{F} = Q^{-1}(1)$ and *geometric monodromy* $\pi_1(\mathbb{C}^*, 1) \to \mathrm{Aut}(F)$ induced by $x \to e^{\frac{2\pi i}{n+1}} \cdot x$ (see for example [35,38]).

Let A be any unitary commutative ring and

$$R := A\big[t, t^{-1}\big].$$

Consider the Abelian representation

$$\pi_1(\mathcal{M}(\widetilde{\mathcal{A}})) \to H_1(\mathcal{M}(\widetilde{\mathcal{A}}); \mathbb{Z}) \to \mathrm{Aut}(R) : \ \beta_j \to t\cdot$$

taking a generator β_j into t-multiplication. Let R_t be the ring R endowed with this $\pi_1(\mathcal{M}(\widetilde{\mathcal{A}}))$-module structure. Then the following it is well-known:

Proposition 3.1. *One has an R-module isomorphism*

$$H_*(\mathcal{M}(\widetilde{\mathcal{A}}), R_t) \cong H_*(F, A)$$

where t-multiplication on the left corresponds to the monodromy action on the right.

In particular for $R = \mathbb{Q}[t, t^{-1}]$, which is a PID, one has

$$H_*\left(\mathcal{M}(\widetilde{\mathcal{A}}), \mathbb{Q}[t^{\pm 1}]\right) \cong H_*(F, \mathbb{Q}).$$

Since the monodromy operator has order dividing $n+1$, $H_*(\mathcal{M}(\widetilde{\mathcal{A}}); R_t)$ decomposes into cyclic modules either isomorphic to R or to $\frac{R}{(\varphi_d)}$, where φ_d is a cyclotomic polynomial with $d|n+1$. It is another open problem to find a (possibly combinatorial) formula for the Betti numbers of F.

It derives from the spectral sequence associated to (3.1) that

$$n + 1 = \dim(H_1(\mathcal{M}(\widetilde{\mathcal{A}}); \mathbb{Q})) = 1 + \dim \frac{H_1(F; \mathbb{Q})}{(\mu - 1)}$$

where on the right one has the coinvariants with respect to the monodromy action. Therefore

$$b_1(F) \geq n;$$

actually

$$b_1(F) = n \quad \Leftrightarrow \quad \mu = id.$$

Definition 3.2. An arrangement $\widetilde{\mathcal{A}}$ with trivial monodromy will be called *a-monodromic*.

Remark 3.3. The arrangement $\widetilde{\mathcal{A}}$ is a-monodromic if and only if

$$H_1(F; \mathbb{Q}) \cong \mathbb{Q}^n \left(\text{equivalently: } H_1(\mathcal{M}(\widetilde{\mathcal{A}}); R) \cong \left(\tfrac{R}{(t-1)}\right)^n\right)$$

Let $\mathcal{A} = d\widetilde{\mathcal{A}}$ be the affine part. In analogy with Definition 3.2 we say

Definition 3.4. The affine arrangement $\mathcal{A}$ is *a-monodromic* if

$$H_1(\mathcal{M}(\mathcal{A}); R) \cong \left(\frac{R}{(t-1)}\right)^{n-1}.$$

By the Kunneth formula one easily gets (with $R = \mathbb{Z}[t^{\pm 1}]$ or $R = \mathbb{Q}[t^{\pm 1}]$)

$$H_1(\mathcal{M}(\widetilde{\mathcal{A}}); R) \cong H_1(\mathcal{M}(\mathcal{A}); R) \otimes \frac{R}{(t^{n+1} - 1)} \oplus \frac{R}{(t-1)}. \tag{3.2}$$

It follows that if $\mathcal{A}$ has trivial monodromy then $\widetilde{\mathcal{A}}$ does. The converse is not true in general (see the example in Figure 6.7).

We can now state the conjecture presented in the introduction.

Conjecture 1. let Γ be the graph with vertex set $\mathcal{A}$ and edge-set all pairs (ℓ_i, ℓ_j) such that $\ell_i \cap \ell_j$ is a double point. Then if Γ is connected then $\mathcal{A}$ is a-monodromic.

Conjecture 2. let Γ be as before. Then if Γ is connected then $\widetilde{\mathcal{A}}$ is a-monodromic.

By formula (3.2) Conjecture 1 implies Conjecture 2.

A partial evidence of these conjecture is that the connectivity condition on the graph of double points gives strong restrictions on the characteristic variety, as we now show.

Remark 3.5. Let $\underline{t} = (t, \ldots, t) \in (\mathbb{C}^*)^{n+1}$ give non-trivial monodromy for the arrangement $\widetilde{\mathcal{A}}$. Then $\underline{t} \in V(\widetilde{\mathcal{A}})$. Moreover, $\underline{t}$ can intersect $\check{V}(\widetilde{\mathcal{A}})$ only in some global component.

The next theorem shows how the connectivity of Γ is an obstruction to the existence of multinet structures.

Theorem 3.6. *If the above graph Γ is connected then the projectivized $\overline{\mathcal{A}}$ of $\widetilde{\mathcal{A}}$ does not support any multinet structure.*

Proof. Choose a set $\mathcal{X}$ of points of multiplicity greater than or equal to 3 and build $\Gamma'(\mathcal{X})$ as we said at the end of Section 2. This graph $\Gamma'(\mathcal{X})$ has $\overline{\mathcal{A}}$ as the set of vertices and the set of edges of Γ is contained in the set of edges of $\Gamma'(\mathcal{X})$. Since by hypothesis Γ is connected then $\Gamma'(\mathcal{X})$ has at most two connected components and so $\mathcal{X}$ cannot give a multinet structure an $\overline{\mathcal{A}}$. $\square$

Corollary 3.7. *If the graph Γ is connected, there is no global resonance component in $\mathcal{R}^1(\widetilde{\mathcal{A}})$.*

So, according to Remark 3.5, if Γ is connected then non trivial monodromy could appear only in the presence of some translated subtori in the characteristic variety.

4. Algebraic complexes

We shall prove the conjectures with extra assumptions on the arrangement. Our tool will be an algebraic complex which was obtained in [22], as a 2-dimensional refinement of that in [33], where the authors used the explicit construction of a *minimal cell complex* which models the complement. Since these complexes work for real defined arrangements, this will be our first restriction.

Of course, there are other algebraic complexes computing local system cohomology (see the references listed in the introduction). The one in [22] seemed to us particularly suitable to attack the present problem (even if we were not able to solve it in general).

First, the complex depends on a fixed and generic system of "polar coordinates". In the present situation, this just means to take an oriented affine real line ℓ which is transverse to the arrangement. We also assume (even if it is not strictly necessary) that ℓ is "far away" from $\mathcal{A}$, meaning that it does not intersect the closure of the bounded facets of the arrangement. This is clearly possible because the union of bounded chambers is a compact set (the arrangement is finite). The choice of ℓ induces a labelling on the lines $\{\ell_1, \dots, \ell_n\}$ in $\mathcal{A}$, where the indices of the lines agree with the ordering of the intersection points with ℓ, induced by the orientation of ℓ.

Let us choose a basepoint $O \in \ell$, coming before all the intersection points of ℓ with $\mathcal{A}$ (with respect to the orientation of ℓ). We recall the construction in [22] in the case of the Abelian local system defined before.

Let $\mathrm{Sing}(\mathcal{A})$ be the set of singular points of the arrangement. For any point $P \in \mathrm{Sing}(\mathcal{A})$, let $S(P) := \{\ell \in \mathcal{A} : \ P \in \ell\}$; so $m(P) = |S(P)|$ is the multiplicity of P.

Let $i_P,\ i^P$ be the minimum and maximum index of the lines in $S(P)$ (so $i_P < i^P$). We denote by $C(P)$ the subset of lines in $\mathcal{A}$ whose indices belong to the closed interval $[i_P, i^P]$. We also denote by

$$U(P) := \{\ell \in \mathcal{A} : \ \ell \text{ does not separate } P \text{ from the basepoint } O\}.$$

Let $(\mathcal{C}_*, \partial_*)$ be the 2-dimensional algebraic complex of free R-modules having one 0-dimensional basis element e^0, n 1-dimensional basis elements $e^1_j,\ j = 1, \dots, n$, (e^1_j corresponding to the line ℓ_j) and $\nu_2 = \sum_{P \in \mathrm{Sing}(\mathcal{A})} m(P) - 1$ 2-dimensional basis elements: to the singular point P of multiplicity $m(P)$ we associate generators $e^2_{P,h},\ h = 1, \dots, m(P) - 1$. The lines through P will be indicized as $\ell_{j_{P,1}}, \dots, \ell_{j_{P,m(P)}}$ (with growing indices).

As a dual statement to [22, Theorem 2], we obtain:

Theorem 4.1. *The local system homology $H_*(\mathcal{M}(\mathcal{A}); R)$ is computed by the complex $(\mathcal{C}_*, \partial_*)$ above, where*

$$\partial_1\big(e^1_j\big) = (t_j - 1)\, e^0$$

and

$$\begin{aligned}\partial_2(e^2_{P,h}) = &\sum_{\ell_j \in S(P)} \Bigg(\prod_{\substack{i<j \text{ so that} \\ l_i \in U(P)}} t_i\Bigg)\Bigg(\prod_{i \in [j_{P,h+1} \to j)} t_i - \prod_{\substack{i<j \text{ so that} \\ l_i \in S(P)}} t_i\Bigg) e^1_j \\ &+ \sum_{\ell_j \in C(P) \cap U(P)} \Bigg(\prod_{\substack{i<j \text{ so that} \\ l_i \in U(P)}} t_i\Bigg)\Bigg(1 - \prod_{\substack{i \le j_{P,h}, i<j \\ l_i \in S(P)}} t_i\Bigg)\Bigg(\prod_{\substack{i \ge j_{P,h+1},\ i<j \\ l_i \in S(P)}} t_i - \prod_{\substack{i \ge j_{P,h+1} \\ l_i \in S(P)}} t_i\Bigg) e^1_j,\end{aligned} \tag{4.1}$$

where $[j_{P,h+1} \to j)$ is the set of indices of the lines in $S(P)$ which run from $j_{P,h+1}$ (included) to j (excluded) in the cyclic ordering of $1, \dots, n$.

By convention, a product over an empty set of indices equals 1.

When $R = A[t^{\pm 1}]$ and $t_i = t,\ i = 1, \dots, n$, we obtain the local homology $H_*(\mathcal{M}(\mathcal{A}); R)$ by using an analogue algebraic complex, where all t_i's equal t in the formulas. In particular (4.1) becomes

$$\begin{aligned}\partial_2\left(e^2_{P,h}\right) = &\sum_{\ell_j \in S(P)} t^{\#\{\ell_i \in U(P): i<j\}}\left(t^{\#[j_{P,h+1}\to j)} - t^{\#\{\ell_i\in S(P):\ i<j\}}\right) e^1_j \\ &+ \sum_{\ell_j \in C(P)\cap U(P)} t^{\#\{\ell_i\in U(P): i<j\} + \#\{\ell_i \in S(P): i\geq j_{P,h+1}, i<j\}} \\ &\times \left(1 - t^{\#\{\ell_i\in S(P): i\leq j_{P,h}, i<j\}}\right)\left(1 - \ t^{\#\{\ell_i\in S(P):\ i\geq j_{P,h+1},\ i\geq j\}}\right) e^1_j.\end{aligned} \tag{4.2}$$

By separating in the first sum the case $j \geq j_{P,h+1}$ from the case $j \leq j_{P,h}$ we have:

$$\begin{aligned}\partial_2\left(e^2_{P,h}\right) = &\sum_{\substack{\ell_j \in S(P)\\ j\geq j_{P,h+1}}} t^{\#\{\ell_i \in U(P): i<j\} + \#\{\ell_i\in S(P): j_{P,h+1}\leq i<j\}}\left(1 - t^{\#\{\ell_i\in S(P): i\leq j_{P,h}\}}\right) e^1_j \\ &+ \sum_{\substack{\ell_j \in S(P)\\ j\leq j_{P,h}}} t^{\#\{\ell_i \in U(P): i<j\} + \#\{\ell_i\in S(P): i<j\}}\left(t^{\#\{\ell_i\in S(P): j_{P,h+1}\leq i\}} - 1\right) e^1_j \\ &+ \sum_{\ell_j \in C(P)\cap U(P)} t^{\#\{\ell_i\in U(P): i<j\} + \#\{\ell_i \in S(P): i\geq j_{P,h+1}, i<j\}} \\ &\times \left(1 - t^{\#\{\ell_i\in S(P): i\leq j_{P,h}, i<j\}}\right)\left(1 - \ t^{\#\{\ell_i\in S(P):\ i\geq j_{P,h+1},\ i\geq j\}}\right)\ e^1_j.\end{aligned} \tag{4.3}$$

In particular, let P be a double point. Then h takes only the value 1, and $j_{P,1}$, $j_{P,2}$ are the indices of the two lines passing through P. So formula (4.3) becomes

$$\begin{aligned}\partial_2\left(e^2_{P,1}\right) = &\, t^{\#\{\ell_i\in U(P): i<j_{P,2}\}}(1-t)\, e^1_{j_{P,2}} + t^{\#\{\ell_i\in U(P):\ i<j_{P,1}\}}\,(t-1)\ e^1_{j_{P,1}} \\ &+ \sum_{\ell_j\in C(P)\cap U(P)} t^{\#\{\ell_i\in U(P):\ i<j\}}\,(t-1)^2\ e^1_j.\end{aligned} \tag{4.4}$$

Since ∂_2 is divisible by $t-1$ we can rewrite (4.4) as

$$\partial_2\left(e^2_{P,1}\right) = (t-1)\ \tilde{\partial}_2\left(e^2_{P,1}\right) \tag{4.5}$$

where

$$\begin{aligned}\tilde{\partial}_2\left(e^2_{P,1}\right) = &\, t^{\#\{\ell_i\in U(P):\ i<j_{P,2}\}} e^1_{j_{P,2}} - t^{\#\{\ell_i\in U(P): i<j_{P,1}\}} e^1_{j_{P,1}} \\ &+ \sum_{\ell_j\in C(P)\cap U(P)} t^{\#\{\ell_i\in U(P):\ i<j\}(1-t)\ e^1_j}.\end{aligned} \tag{4.6}$$

5. A proof in particular cases

We give here a proof of Conjecture 1 with further hypothesis on $\mathcal{A}$.

Notice that the rank of ∂_2 is $n-1$ (the sum of all rows vanishes). Then the arrangement has no monodromy if and only if the only elementary divisor of ∂_2 is $\varphi_1 := t-1$, so ∂_2 diagonalizes to $\oplus_{i=1}^{n-1} \varphi_1$. This is equivalent to the reduced boundary $\tilde{\partial}_2$ having an invertible minor of order $n-1$.

Let Γ be the graph of double points. A choice of an admissible coordinate system gives a total ordering on the lines so it induces a labelling, varying between 1 and n, on the set of vertices $V\Gamma$ of Γ. Let T be a spanning tree of Γ (with induced labelling on VT).

Definition 5.1. We say that the induced labelling on $VT = V\Gamma$ is *very good* (with respect to the given coordinate system) if the sequence $n, \dots, 1$ is a *collapsing ordering* on T. In other words, the graph obtained by T by removing all vertices with label $\geq i$ and all edges having both vertices with label $\geq i$, is a tree, for all $i = n, \dots, 1$.

We say that the spanning tree T is *very good* if there exists an admissible coordinate system such that the induced labelling on VT is very good (see Figure 6.1).

Remark 5.2.

(1) A labelling over a spanning tree T gives a collapsing ordering if and only if for each vertex v, the number of adjacent vertices with lower label is ≤ 1. In this case, only the vertex labelled with 1 has no lower labelled adjacent vertices (by the connectness of T).
(2) Given a collapsing ordering over T, for each vertex v with label $i_v > 1$, let $\ell(v)$ be the edge which connects v with the unique adjacent vertex with lower label; by giving to $\ell(v)$ the label $i_v + \frac{1}{2}$, we obtain a discrete Morse function on the graph T (see [20]) with unique critical cell given by the vertex with label 1. The set of all pairs $(v, \ell(v))$ is the acyclic matching which is associated to this Morse function.

Let us indicate by Γ_0 the linear tree with n vertices: we consider Γ_0 as a CW-decomposition of the real segment $[1, n]$, with vertices $\{j\}$, $j = 1, \dots, n$, and edges the segments $[j, j+1]$, $j = 1, \dots, n-1$.

Definition 5.3. We say that a labelling induced by some coordinate system on the tree T is *good* if there exists a permutation $i_1, \dots, i_n$ of $1, \dots, n$ which gives a collapsing sequence both for T and for Γ_0. In other words, at each step we always remove either the maximum labelled vertex or the minimum, and this is a collapsing sequence for T.

We say that T is *good* if there exists an admissible coordinate system such that the induced labelling on VT is good (see Figure 6.2).

Notice that a very good labelling is a good labelling where at each step one removes the maximum vertex.

Consider some arrangement $\mathcal{A}$ with graph Γ and labels on the vertices which are induced by some coordinate system. Notice that changes of coordinates act on the labels by giving all possible *cyclic permutations*, which are generated by the transformation $i \to i + 1 \bmod n$. So, given a labelled tree T, checking if T is very good (respectively good) consists in verifying if some cyclic permutation of the labels is very good (respectively good). This property depends not only on the "shape" of the tree, but also on how the lines are disposed in $\mathbb{R}^2$ (the associated oriented matroid). In fact, one can easily find arrangements where some "linear" tree is very good, and others where some linear tree is not good.

Definition 5.4. We say that an arrangement $\mathcal{A}$ is *very good* (respectively *good*) if Γ is connected and has a very good (respectively good) spanning tree.

It is not clear if this property is combinatorial, *i.e.*, if it depends only on the lattice. Of course, $\mathcal{A}$ very good implies $\mathcal{A}$ good.

Theorem 5.5. *Let $\mathcal{A}$ be a good arrangement. Then $\mathcal{A}$ is a-monodromic.*

Proof. We use induction on the number n of lines, the claim being trivial for $n = 1$. Take a suitable coordinate system as in Definition 5.4, such that the graph Γ has a spanning tree T with good labelling. Assume for example that at the first step we remove the last line, so the graph Γ' of the arrangement $\mathcal{A}' := \mathcal{A} \setminus \{\ell_n\}$ is connected and the spanning tree T' obtained by removing the vertex $\{\ell_n\}$ and the "leaf-edge" (ℓ_n, ℓ_j) (for some $j < n$) has a good labelling.

There are $n - 1$ double points which correspond to the edges of T : only one of these is contained in ℓ_n, namely $\ell_n \cap \ell_j$ (see Remark 5.2). Let $\mathcal{D} := \{d_1, \dots, d_{n-1}\}$ be the set of such double points, with $d_{n-1} = \ell_n \cap \ell_j$. Let also $\mathcal{D}' := \{d_1, \dots, d_{n-2}\}$, which corresponds to the edges of T'. Let $(\mathcal{C}(\mathcal{D})_*, \partial_*)$ (respectively $(\mathcal{C}(\mathcal{D}')_*, \partial'_*)$) be the subcomplex of $\mathcal{C}(\mathcal{A})_*$ generated by the 2-cells which correspond to $\mathcal{D}$ (respectively $\mathcal{D}'$): then $\mathcal{C}(\mathcal{D})_2 = \oplus_{1 \le i \le n-1} R e_j$, and $\mathcal{C}(\mathcal{D}')_2 = \oplus_{1 \le i \le n-2} R e'_j$. Notice that, by the explicit formulas given in Section 4, the component of the boundary $\partial_2(e_j)$ along the 1-dimensional generator corresponding to ℓ_n equals $-\varphi_1$ for $j = n - 1$, and vanishes for $j = 1, \dots, n - 2$. Actually, the natural map taking e'_j into e_j, $j = 1, \dots, n - 2$, identifies $\mathcal{C}(\mathcal{D}')_*$ with the sub complex of $\mathcal{C}(\mathcal{D})_*$ generated by the e_j's, $j = 1, \dots, n - 2$,

$$\partial_2 = \left[\begin{array}{c|c} \partial'_2 & * \\ \hline 0 & -\varphi_1 \end{array}\right]. \tag{5.1}$$

Then by induction ∂'_2 diagonalizes to $\oplus_{j=1}^{n-2} \varphi_1$. Therefore ∂_2 diagonalizes to $\oplus_{j=1}^{n-1} \varphi_1$, which gives the thesis. If at the first step we remove the first line, the argument is similar, because $\partial_2\,(e_j)$ has no non-vanishing components along the generator corresponding to ℓ_1. □

Let us consider a different situation.

Definition 5.6. We say that a subset Σ of the set of singular points $\mathrm{Sing}(\mathcal{A})$ of the arrangement $\mathcal{A}$ is *conjugate-free* (with respect to a given admissible coordinate system) if $\forall P \in \Sigma$ the set $U(P) \cap C(P)$ is empty.

An arrangement $\mathcal{A}$ will be called *conjugate-free* if Γ is connected and contains a spanning tree T such that the set of points in $\mathrm{Sing}(\mathcal{A})$ that correspond to the edges ET of T is conjugate-free (see Figure 6.3).

Let Σ be conjugate-free: it follows from formula (4.3) that the boundary of all generators $e^2_{P,h}$, $P \in \Sigma$, can have non-vanishing components only along the lines which contain P.

Theorem 5.7. *Assume that $\mathcal{A}$ is conjugate-free. Then $\mathcal{A}$ is a-monodromic.*

Proof. The sub matrix of ∂_2 which corresponds to the double points ET is φ_1-times the incidence matrix of the tree T. Such matrix is the boundary matrix of the complex which computes the $\mathbb{Z}$-homology of T : it is a unimodular rank-$(n-1)$ integral matrix (see for example [3]). From this the result follows straightforwardly. □

We can have a mixed situation between Definitions 5.4 and 5.6 (see Figure 6.4).

Theorem 5.8. *Assume that Γ is connected and contains a spanning tree T which reduces, after a sequence of moves where we remove either the maximum or the minimum labelled vertex, to a subtree T' which is conjugate-free. Then $\mathcal{A}$ is a-monodromic.*

Proof. The thesis easily follows by induction on the number n of lines. In fact, either T is conjugate-free, and we use Theorem 5.7, or one of the subtrees $T \setminus \{\ell_n\}$, $T \setminus \{\ell_1\}$ satisfies again the hypothesis of the theorem. Assume that it is $T'' = T \setminus \{\ell_n\}$. Then the boundary map ∂_2 restricted to the 2-cells corresponding to ET'' has a shape similar to (5.1). Therefore by induction we conclude. □

Some examples are given in Section 6.

Remark 5.9. In all the theorems in this section, we have proven a stronger result: namely, the subcomplex spanned by the generators corresponding to the double points is a-monodromic.

6. Examples

In this section we give examples corresponding to the various definitions of Section 5. We include the computations of the local homology of the complements.

In Figure 6.1 we show an arrangement having a very good tree (Definition 5.1) and the associated sequence of contractions.

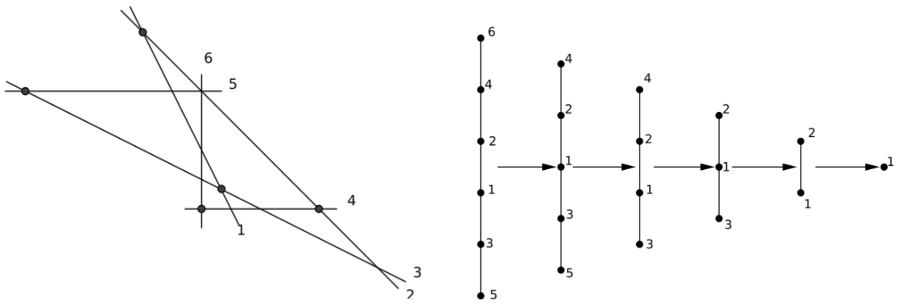

Figure 6.1.

In Figure 6.2 an arrangement with a good tree is given (Definition 5.3) together with its sequence of contractions.

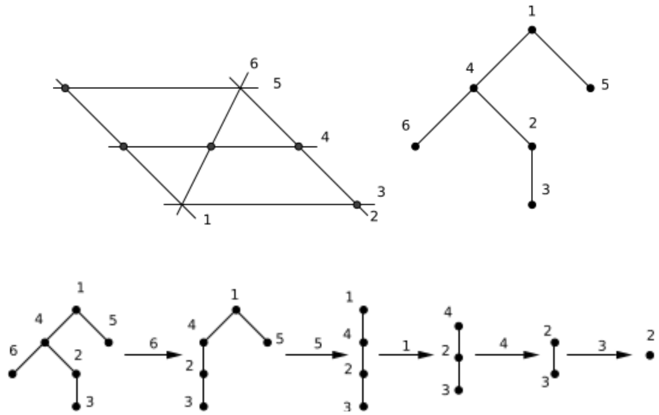

Figure 6.2.

An arrangement having a tree which is both conjugate-free (see Definition 5.6) and good is depicted in Figure 6.3

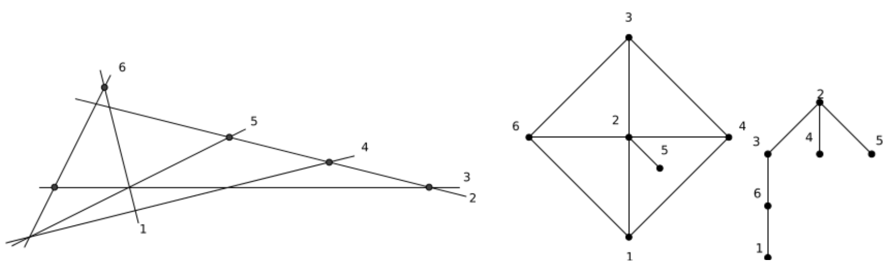

Figure 6.3.

In Figure 6.4 we show an arrangement with a tree which after 2 admissible contractions becomes conjugate free (see Theorem 5.8).

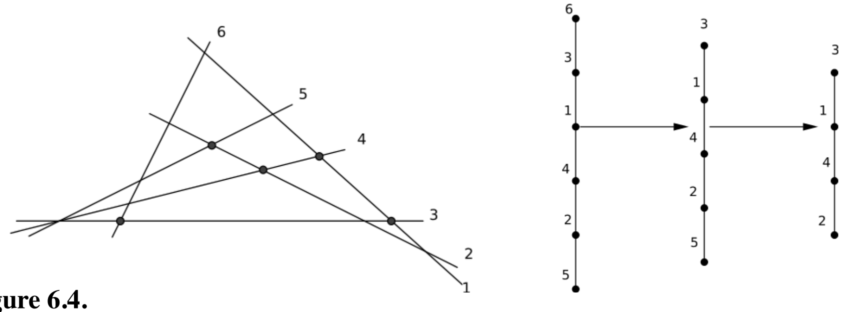

Figure 6.4.

Next we give some example of arrangements with non-trivial monodromy. Notice that the graph of double points is disconnected in these cases.

Notice also that in the first two examples one has non-trivial monodromy both for the given affine arrangement and its conifed arrangement in $\mathbb{C}^3$; in the last example, the given affine arrangement has non trivial monodromy while its conification is a-monodromic.

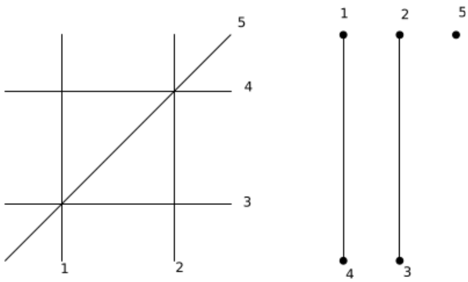

Figure 6.5. Deconed A3 arrangement

$$H_1(\mathcal{M}(\mathcal{A}), \mathbb{Q}[t^{\pm 1}]) \simeq \left(\frac{\mathbb{Q}[t^{\pm 1}]}{(t-1)}\right)^3 \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t^3-1)}.$$

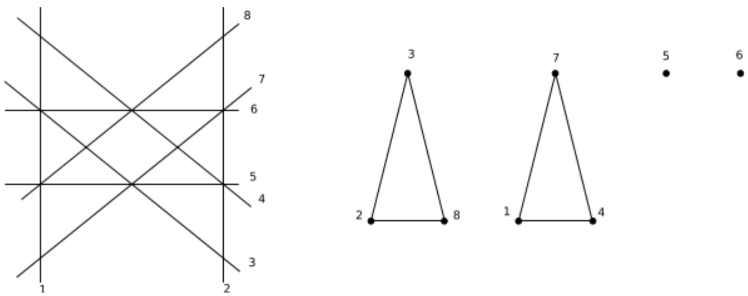

Figure 6.6. Deconed Pappus arrangement

$$H_1(\mathcal{M}(\mathcal{A}), \mathbb{Q}[t^{\pm 1}]) \simeq \left(\frac{\mathbb{Q}[t^{\pm 1}]}{(t-1)}\right)^6 \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t^3-1)}.$$

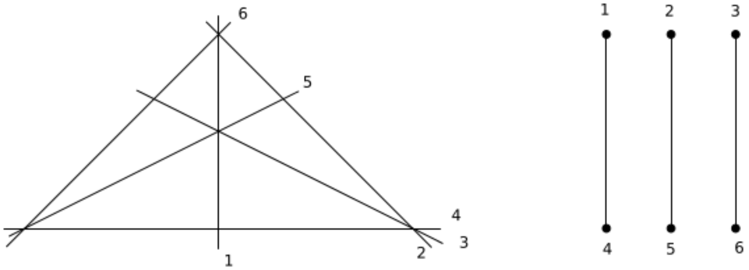

Figure 6.7. The "complete triangle" has non-trivial monodromy but its conification is a-monodromic

$$H_1(\mathcal{M}(\mathcal{A}), \mathbb{Q}[t^{\pm 1}]) \simeq \left(\frac{\mathbb{Q}[t^{\pm 1}]}{(t-1)}\right)^4 \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t^3-1)}.$$

We focus here on the structure of the fundamental groups of the above examples, in particular in case of a-monodromic arrangements.

For arrangement in Figure 6.1: after taking line 5 to infinity we obtain an affine arrangement having only double points with two pairs of parallel lines, namely (the new) lines 2, 6 and 4, ∞. Therefore

$$\pi_1(\mathcal{M}(\mathcal{A})) = \mathbb{Z} \times \mathbb{Z} \times F_2 \times F_2.$$

We consider arrangement in Figure 6.2 and in Figure 6.5 together. The deconed $A3$ arrangement in Figure 6.5 is a well known $K(\pi, 1)$-arrangement: the fundamental group of the complement is the *pure braid group* P_4 in 4 strands. Notice that the projection onto the y coordinate fibers the complement over $\mathbb{C} \setminus \{2 \text{ pts}\}$ with fiber $\mathbb{C} \setminus \{3 \text{ pts}\}$. It is well known that this fibering is not trivial and we obtain a semi-direct product decomposition

$$\pi_1(\mathcal{M}(\mathcal{A})) = F_3 \rtimes F_2.$$

The same projection gives a fibering of the complement of the arrangement in Figure 6.2 over $\mathbb{C} \setminus \{3 \text{ pts}\}$ with fiber $\mathbb{C} \setminus \{3 \text{ pts}\}$. Notice that this is also a non-trivial fibering, so we have a semi-direct decomposition

$$\pi_1(\mathcal{M}(\mathcal{A})) = F_3 \rtimes F_3.$$

In particular, we have an a-monodromic arrangement such that the fundamental group of the complement is not a direct product of free groups.

In the arrangement of Figure 6.3 the line at infinity is transverse to the other lines. If we take line 5 at infinity we get an affine arrangement with only double points, with two pairs of parallel lines 1, 3 and 4, 6. Therefore we obtain a decomposition of $\pi_1(\mathcal{M}(\mathcal{A}))$ as in case of Figure 6.1.

The arrangement of Figure 6.4 has only one triple point. By taking line 5 to infinity we get an affine arrangement with only double points and one pair of parallel lines 3, 4. Therefore

$$\pi_1(\mathcal{M}(\mathcal{A})) = \mathbb{Z}^4 \times F_2.$$

The *complete triangle* in Figure 6.7 becomes, after taking any line at infinity, the affine arrangement $\mathcal{A}'$ which is obtained from the $A3$ deconed arrangement in Figure 6.5 by adding one more line $\bar{l}$ which is transverse to all the others. Therefore

$$\pi_1(\mathcal{M}(\mathcal{A})) = \mathbb{Z} \times (F_3 \rtimes F_2).$$

Remark 6.1. It turns out that the arrangement $\mathcal{A}'$ is a-monodromic. This is not a contradiction: in fact, one is considering two different local systems on $\mathcal{M}(\mathcal{A}') = \mathcal{M}(\mathcal{A})$. The a-monodromic one associates to an elementary loop around $\bar{l}$ the t-multiplication. This is different from the one obtained by exchanging one of the affine lines of the arrangement $\mathcal{A}$ in Figure 6.7 with the infinity line. In this case we should associate to an elementary loop around $\bar{l}$ the t^6-multiplication, and then apply formula (4.1).

7. Free calculus

In this section we reformulate our conjecture in terms of Fox calculus. Let $\mathcal{A} = \{l_1, \ldots, l_n\}$ be as above; if we denote by β_i an elementary loop around l_i we have that the fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ is generated by $\beta_1, \ldots, \beta_n$ and a presentation of this group is given for example in [29]. Let $R = \mathbb{Q}[t^{\pm 1}]$ be as above with the given structure of $\pi_1(\mathcal{M}(\mathcal{A}))-$module.

We denote by F_n the free group generated by $\beta_1, \ldots, \beta_n$. Let $\varphi : F_n \to < t >$ be the group homomorphism defined by $\varphi(\beta_i) = t$ for every $1 \leq i \leq n$ where $< t >$ is the multiplicative subgroup of R generated by t. As in [4], if w is a word in the β_j's, we use the notation w^φ for $\varphi(w)$. Consider the algebraic complex which computes the local homology of $\mathcal{M}(\mathcal{A})$ introduced in Section 4. The following remark is crucial for the rest of this section: if $e^2_{P,j}$ is a two-dimensional generator corresponding to a two-cell which is attached along the word w in the β_j's, then $\left(\frac{\partial w}{\partial \beta_i}\right)^\varphi$ is the coefficient of e^1_i of the border of $e^2_{P,j}$. This easily follows from the combinatorial calculation of local system homology.

Let $l : F_n \longrightarrow \mathbb{Z}$ be the length function, given by

$$l(\beta_{i_1}^{\epsilon_1} \cdots \beta_{i_r}^{\epsilon_r}) = \sum_{k=1}^{r} \epsilon_k.$$

Then kerφ is the normal subgroup of F_n given by the words of lenght 0.

Each relation in the fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ is a commutator (*cf.* [17, 29]), so it lies in kerφ. So, in the sequel, we consider only words in kerφ.

Remark 7.1. The arrangement $\mathcal{A}$ is a-monodromic if and only if (by definition) the $\mathbb{Q}[t^{\pm 1}]$-module generated by $\partial_2(e_j^2),\ j = 1, \dots, \nu_2$, equals $(t-1)\ker\partial_1$. One has: $\ker\partial_1 = \{\sum_{j=1}^n\ x_j\ e_j^1\ :\ \sum_{j=1}^n\ x_j = 0\}$.

Let $R_j,\ j = 1\dots,\nu_2$, be a complete set of relations in $\pi_1(\mathcal{M}(\mathcal{A}))$. We use now e_j^2 to indicate the two-dimensional generator corresponding to a two-cell which is attached along the word R_j. Then the boundary of e_j^2 is given by

$$\partial_2(e_j^2) = \sum_{i=1}^n \left(\frac{\partial R_j}{\partial\beta_i}\right)^{\varphi} e_i^1, \quad j = 1, \dots, \nu_2. \tag{7.1}$$

Then by Remark 7.1 $\mathcal{A}$ is a-monodromic if and only if each element of the shape

$$P(t) := (1-t)\sum_{i=1}^n\ P_i(t)\ e_i^1, \quad \sum_{i=1}^n P_i(t) = 0, \quad (P_i(t) \in \mathbb{Q}[t^{\pm 1}], i = 1\dots,n) \tag{7.2}$$

is a linear combination with coefficients in $\mathbb{Q}[t^{\pm 1}]$ of the elements in (7.1), *i.e.*:

$$P(t) = \sum_{j=1}^{\nu_2} Q_j(t)\partial_2(e_j^2), \quad Q_j(t) \in \mathbb{Q}[t^{\pm 1}]. \tag{7.3}$$

It is natural to wonder about solutions with coefficients in $\mathbb{Z}[t^{\pm 1}]$ instead of $\mathbb{Q}[t^{\pm 1}]$. We say that $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$ if there is a solution to (7.3) over $\mathbb{Z}[t^{\pm 1}]$ (when all the $P_i(t)$'s in (7.2) are in $\mathbb{Z}[t^{\pm 1}]$).

Theorem 7.2. *The arrangement $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$ if and only if $\pi_1(\mathcal{M}(\mathcal{A}))$ is commutative modulo* $[\ker\varphi, \ker\varphi]$. *More precisely, $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$ if and only if*

$$[F_n, F_n] = N[\ker\varphi, \ker\varphi],$$

where N is the normal subgroup generated by the relations R_j's .

Proof. A set of generators for $(t-1)\ker\partial_1$ as $\mathbb{Z}[t^{\pm 1}]$-modulo is given by all elements of the type

$$P_{rs} := (1-t)(e_r^1 - e_s^1)\ , \quad r \neq s.$$

Such an element can be re-written in the form (7.1) as

$$P_{rs} = \sum_{i=1}^n \left(\frac{\partial[\beta_r, \beta_s]}{\partial\beta_i}\right)^{\varphi} e_i^1,$$

where $[\beta_r, \beta_s] = \beta_r\beta_s\beta_r^{-1}\beta_s^{-1}$. Now there exists an expression like (7.3) for P_{rs}, with all $Q_j(t) \in \mathbb{Z}[t^{\pm 1}]$ if and only if

$$\left(\frac{\partial[\beta_r, \beta_s]}{\partial\beta_i}\right)^{\varphi} = \left(\frac{\partial\prod_{j=1}^{\nu_2}\ R_j^{Q_j(\beta_1)}}{\partial\beta_i}\right)^{\varphi}, \quad i = 1, \dots, n. \tag{7.4}$$

Here $Q_j(\beta_1) \in \mathbb{Z}[F_n]$ is obtained by substituting t with β_1 (any word of length one would give the same here). Moreover, for R, w any words in $\ker\varphi$ we set $R^w := wRw^{-1}$, and for $a \in \mathbb{Z}$ we set $R^{aw} := R^w \dots R^w$ (a factors) if $a > 0$ and $(R^{-1})^w \dots (R^{-1})^w$ ($|a|$ factors) for $a < 0$. Also, we set $R^{aw+bu} := R^{aw}R^{bu}$. Then equalities (7.4) come from standard Fox calculus.

Then from Blanchfield theorem (see [4, Chapter 3]) it follows that

$$[F_n, F_n] \subset N[\ker\varphi, \ker\varphi].$$

The opposite inclusion follows because, as we said before Remark 7.1, for any arrangement one has $N \subset [F_n, F_n]$. $\square$

Remark 7.3. The condition in Theorem 7.2 is equivalent to the equality

$$\frac{F_n}{N[\ker\varphi, \ker\varphi]} = \frac{F_n}{[F_n, F_n]} = H_1(\mathcal{M}(\mathcal{A}); \mathbb{Z}).$$

Since $\ker\varphi \supset [F_n, F_n]$, so $[\ker\varphi, \ker\varphi] \supset [[F_n, F_n], [F_n, F_n]]$, the next corollary follows immediately from Theorem 7.2.

Corollary 7.4. *Assume that $\pi_1(\mathcal{M}(\mathcal{A}))/\pi_1(\mathcal{M}(\mathcal{A}))^{(2)}$ is Abelian, which is equivalent to the condition $\pi_1(\mathcal{M}(\mathcal{A}))^{(1)} = \pi_1(\mathcal{M}(\mathcal{A}))^{(2)}$, where $\pi_1(\mathcal{M}(\mathcal{A}))^{(i)}$ is the i-th element of the derived series of $\pi_1(\mathcal{M}(\mathcal{A}))$, for $i \geq 0$. Then $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$.*

The condition of Corollary 7.4 corresponds to the vanishing of the so-called *Alexander invariant* of $\pi_1(\mathcal{M}(\mathcal{A}))$.

As a subgroup of the free group F_n, the group $\ker\varphi$ is a free group We use the Reidemeister-Schreier method to write an explicit list of generators. Notice that for any fixed $1 \leq j \leq n$, the set $\{\beta_j^k : k \in \mathbb{Z}\}$ is a Schreier right coset representative system for $F_n/\ker\varphi$. Denote briefly by $s_{k,i}$ the element $s_{\beta_j^k, \beta_i} = \beta_j^k \beta_i \overline{(\beta_j^k \beta_i)}^{-1} = \beta_j^k \beta_i \beta_j^{-(k+1)}$. Then

$$\ker\varphi = \left\langle \{s_{k,i} : 1 \leq i \leq n,\ k \in \mathbb{Z}\}; s_{k,i} \right\rangle,$$

where $s_{k,i}$ is a relation if and only if $\beta_j^k \beta_i$ is freely equal to β_j^{k+1}; this happens if and only if $i = j$. So $\ker\varphi$ is the free group generated by $\{s_{k,i} : k \in \mathbb{Z}, 1 \leq i \leq n, i \neq j\}$. Its Abelianization

$$\mathrm{ab}\,(\ker\varphi) = \ker\varphi/[\ker\varphi, \ker\varphi]$$

is the free Abelian group on the classes $\overline{s}_{k,i}$ of the generators $s_{k,i}$'s, $i \neq j$. Let

$$\mathrm{ab} : \ker\varphi \longrightarrow \mathrm{ab}\,(\ker\varphi)$$

be the Abelianization homomorphism. Now we define the automorphism σ of $\ker\varphi$ by

$$\sigma(s_{k,i}) = s_{k+1,i},$$

which passes to the quotient, so it defines an automorphism (call it again σ) of $\mathrm{ab}(\ker\varphi)$. Therefore we may view $\mathrm{ab}(\ker\varphi)$ as a finitely genereted free $\mathbb{Z}[\sigma^{\pm 1}]$-module, with basis $\overline{s}_{0,i}$ with $1 \leq i \leq n$ and $i \neq j$. In this language Theorem 7.2 translates as:

Theorem 7.5. *The arrangement $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$ if and only if the submodule* $(1-\sigma)\mathrm{ab}(\ker\varphi)$ *of* $\mathrm{ab}(\ker\varphi)$ *is generated by* $\mathrm{ab}(R_j)$, $j = 1, \ldots, \nu_2$, *as* $\mathbb{Z}[\sigma^{\pm 1}]$*-module.*

Of course, one can give a conjecture holding over $\mathbb{Z}$.

Conjecture 3. Assume that Γ is connected; then $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$.
Conjecture 3 clearly implies Conjectures 1 and 2. Our experiments agree with this stronger conjecture.

We give explicit computations for the arrangements in Figures 6.1 and 6.5. The $\mathbb{Z}[\sigma^{\pm 1}]$-module $(1-\sigma)\mathrm{ab}(\ker\varphi)$ is generated by $\{(1-\sigma)\overline{s}_{0,i}, \ , i \neq j\}$. We choose j as the last index in the natural ordering. All Abelianized relations are divisible by $(1-\sigma)$, so we just divide everything by $1-\sigma$ and verify that $\mathrm{ab}(\ker\varphi)$ is generated by $\mathrm{ab}(R_j)/(1-\sigma)$.

For the arrangement in Figure 6.1 we have to rewrite 13 relations coming from 11 double points and 1 triple point. After Abelianization we obtain:

(a) $\overline{s}_{0,2} - \overline{s}_{0,3}$;

(b) $\overline{s}_{0,2} - \overline{s}_{0,4}$;

(c) $\overline{s}_{0,3} - \overline{s}_{0,4}$;

(d) $\overline{s}_{0,1} - \overline{s}_{0,4}$;

(e) $\overline{s}_{0,1} - \overline{s}_{0,3}$;

(f) $\sigma\overline{s}_{0,2} + \overline{s}_{0,5}$;

(g) $(1+\sigma)\overline{s}_{0,2} - \sigma\overline{s}_{0,5}$;

(h) $\overline{s}_{0,1} + \sigma^{-1}(1-\sigma)\overline{s}_{0,2} - \sigma^{-2}(1+\sigma)\overline{s}_{0,5}$;

(i) $\overline{s}_{0,3} + (\sigma^{-1} - 1)\overline{s}_{0,5}$;

(j) $\overline{s}_{0,4} + (\sigma^{-1} - 1)\overline{s}_{0,5}$;

(k) $\overline{s}_{0,1} + (\sigma^{-1} - 1)\overline{s}_{0,2} - \sigma^{-1}\overline{s}_{0,5}$;

(l) $\overline{s}_{0,1} - \overline{s}_{0,2}$;

(m) $\overline{s}_{0,3} - \overline{s}_{0,5}$.

The generator $\overline{s}_{0,5}$ is obtained as $\sigma(\ (i) - (m)\ )$. From $\overline{s}_{0,5}$ we obtain in sequence all the other generators $\overline{s}_{0,3},\ \overline{s}_{0,1},\ \overline{s}_{0,4},\ \overline{s}_{0,2}$. According to Theorem 7.5 this gives the a-monodromicity of the arrangement in Figure 6.1.

For the arrangement $A3$ *deconed* in Figure 6.5 we have to rewrite two relations for each triple point and one relation for each double point. Their Abelianization is given by:

(a) $\overline{s}_{0,2} - \overline{s}_{0,3}$;
(b) $\sigma\overline{s}_{0,2} + \overline{s}_{0,4}$;
(c) $(\sigma+1)\overline{s}_{0,2} - \sigma\overline{s}_{0,4}$;
(d) $\sigma\overline{s}_{0,1} + (1-\sigma)\overline{s}_{0,2} + (\sigma^{-1})\overline{s}_{0,3} + (\sigma^{-2}-1)\overline{s}_{0,4}$;
(e) $\overline{s}_{0,1} + (\sigma^{-1}-1)\overline{s}_{0,2} - (\sigma^{-1})\overline{s}_{0,4}$;
(f) $(\sigma+1)\overline{s}_{0,1} + (\sigma^{-1}-\sigma)\overline{s}_{0,2} - \overline{s}_{0,3} + (\sigma^{-2}-\sigma^{-1})\overline{s}_{0,4}$.

We perform the following base changes:

$(\mathrm{a}') = (\mathrm{a})$;
$(\mathrm{b}') = (\mathrm{b}) - \sigma(\mathrm{a})$;
$(\mathrm{c}') = (\mathrm{c}) - (\mathrm{b}) - (\mathrm{a})$;
$(\mathrm{d}') = (\mathrm{d}) - \sigma^{-2}(\mathrm{b}) + \sigma^{-1}(\mathrm{a}) - \sigma(\mathrm{e})$;
$(\mathrm{e}') = (\mathrm{e}) + \sigma^{-1}(\mathrm{b}) - \sigma^{-1}(\mathrm{a})$;
$(\mathrm{f}') = (\mathrm{f}) - (\sigma^{-2}+\sigma^{-1}+1)(\mathrm{b}) + \sigma(\mathrm{a}) + \sigma^{-1}(\mathrm{c}) - (\sigma+1)(\mathrm{e})$

and

$$\overline{s}'_{0,1} = \overline{s}_{0,1} + \sigma^{-1}\,\overline{s}_{0,3}; \qquad \overline{s}'_{0,2} = \overline{s}_{0,2} - \overline{s}_{0,3};$$
$$\overline{s}'_{0,3} = \overline{s}_{0,3}; \qquad \overline{s}'_{0,4} = \overline{s}_{0,4} + \sigma\,\overline{s}_{0,3}.$$

It is straighforward to verify, after these changes, that the submodule M generated by $\langle \mathrm{ab}(R_j) : \ j = 1, \dots, 6\rangle$ equals

$$\left\langle \overline{s}'_{0,1}, \overline{s}'_{0,2}, (1+\sigma+\sigma^2)\overline{s}'_{0,3}, \overline{s}'_{0,4} \right\rangle.$$

So $M \subsetneq (1-\sigma)\mathrm{ab}(\ker\varphi)$, in accordance with Theorem 7.5.

8. Further characterizations

In this section we give a more intrinsic picture. Let $\widetilde{\mathcal{A}} = \{H_0, H_1, \dots, H_n\}$ be the conified arrangement in $\mathbb{C}^3$. The fundamental group

$$\mathbf{G} = \pi_1(\mathcal{M}(\widetilde{\mathcal{A}})) \quad (= \pi_1(\mathcal{M}(\mathcal{A})) \times \mathbb{Z})$$

is generated by elementary loops $\beta_0, \dots, \beta_n$ around the hyperplanes. Let

$$\mathbf{F} = \mathbf{F_{n+1}}[\beta_0, \dots, \beta_n]$$

be the free group and $\mathbf{N}$ be the normal subgroup generated by the relations, so we have a presentation

$$1 \longrightarrow \mathbf{N} \longrightarrow \mathbf{F} \xrightarrow{\pi} \mathbf{G} \longrightarrow 1$$

The length map $\varphi : \mathbf{F} \to \langle t \rangle \cong \mathbb{Z}$ factors through π by a map

$$\psi : \mathbf{G} \to \mathbb{Z}.$$

Next, ψ factorizes through the Abelianization

$$\frac{\mathbf{G}}{[\mathbf{G}, \mathbf{G}]} \cong H_1(\mathcal{M}(\widetilde{\mathcal{A}}); \mathbb{Z}) \cong \mathbb{Z}^{n+1} \cong \frac{\mathbf{F}}{[\mathbf{F}, \mathbf{F}]}.$$

Let now $\mathbf{K} = \ker \psi$ so we have

$$1 \longrightarrow \mathbf{K} \longrightarrow \mathbf{G} \xrightarrow{\psi} \mathbb{Z} \longrightarrow 1, \tag{8.1}$$

and ψ factorizes through

$$\mathbf{G} \xrightarrow{\text{ab}} \frac{\mathbf{G}}{[\mathbf{G}, \mathbf{G}]} \cong \mathbb{Z}^{n+1} \xrightarrow{\lambda} \mathbb{Z}.$$

We have the following commutative diagram:

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & \mathbf{N} \arrow[d] & & 1 \\
1 \arrow[r] & \ker(\varphi) \arrow[r] & \mathbf{F} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{Z} \arrow[r] \arrow[ur] & 1 \\
1 \arrow[r] & {[\mathbf{G}, \mathbf{G}]} \arrow[r] \arrow[d] & \mathbf{G} \arrow[r, "\text{ab}"] \arrow[ur, "\psi"] \arrow[d] & \frac{\mathbf{G}}{[\mathbf{G},\mathbf{G}]} \arrow[r] \arrow[u, "\lambda"'] & 1 \\
 & \mathbf{K} \arrow[ur] & 1 & & \\
1 \arrow[ur] & & & &
\end{tikzcd} \tag{8.2}$$

Remark 8.1. One has $\ker(\lambda) = \frac{\mathbf{K}}{[\mathbf{G},\mathbf{G}]}$ so $\frac{\mathbf{K}}{[\mathbf{G},\mathbf{G}]} \cong \mathbb{Z}^n$.

Therefore diagram (8.2) extends to

$$
\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & \mathbf{N} \arrow[d] & 1 & 1 \\
1 \arrow[r] & \ker(\varphi) \arrow[r] & \mathbf{F} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{Z} \arrow[r] \arrow[u] \arrow[ur] & 1 \\
1 \arrow[r] & {[\mathbf{G},\mathbf{G}]} \arrow[r] \arrow[d] & \mathbf{G} \arrow[r, "\mathrm{ab}"] \arrow[ur, "\psi"] \arrow[d] & \frac{\mathbf{G}}{[\mathbf{G},\mathbf{G}]} \arrow[r] \arrow[u, "\lambda"'] & 1 \\
 & \mathbf{K} \arrow[ur] & 1 & \frac{\mathbf{K}}{[\mathbf{G},\mathbf{G}]} \arrow[u] & \\
1 \arrow[ur] & & & 1 \arrow[u] & .
\end{tikzcd}
\tag{8.3}
$$

Recall the $\mathbb{Z}[t^{\pm 1}]$−module isomorphism

$$H_1(\mathbf{G}; \mathbb{Z}[t^{\pm 1}]) \;\cong\; H_1(F; \mathbb{Z}) \tag{8.4}$$

where F is the Milnor fibre, and (by the Shapiro Lemma):

$$H_1(F; \mathbb{Z}) \cong H_1(\mathbf{K}; \mathbb{Z}) = \frac{\mathbf{K}}{[\mathbf{K}, \mathbf{K}]}. \tag{8.5}$$

Remark 8.2. There is an exact sequence

$$1 \longrightarrow \frac{[\mathbf{G}, \mathbf{G}]}{[\mathbf{K}, \mathbf{K}]} \longrightarrow \frac{\mathbf{K}}{[\mathbf{K}, \mathbf{K}]} \longrightarrow \frac{\mathbf{K}}{[\mathbf{G}, \mathbf{G}]} \cong \mathbb{Z}^n \longrightarrow 1. \tag{8.6}$$

From the definition before Theorem 7.2 one has

Lemma 8.3. *The arrangement $\widetilde{\mathcal{A}}$ is a-monodromic over $\mathbb{Z}$ if and only if*

$$H_1(F; \mathbb{Z}) \cong \mathbb{Z}^n.$$

It follows:

Theorem 8.4. *The arrangement $\widetilde{\mathcal{A}}$ is a-monodromic over $\mathbb{Z}$ if and only if*

$$\frac{[\mathbf{G}, \mathbf{G}]}{[\mathbf{K}, \mathbf{K}]} = 0. \tag{8.7}$$

Proof. It immediately follows from sequence (8.6) and from the property that a surjective endomorphism of a finitely generated free Abelian group is an isomorphism. □

Since $\mathbf{K} \supset [\mathbf{G}, \mathbf{G}]$ it follows immediately (see Corollary 7.4) that:

Corollary 8.5. *If*

$$\mathbf{G}^{(1)} = [\mathbf{G}, \mathbf{G}] = \mathbf{G}^{(2)} = [[\mathbf{G}, \mathbf{G}], [\mathbf{G}, \mathbf{G}]],$$

then the arrangement $\widetilde{\mathcal{A}}$ is a-monodromic.

We also have:

Corollary 8.6. *Let* $\mathbf{G}$ *have a central element of length* 1*. Then the arrangement $\widetilde{\mathcal{A}}$ is a-monodromic.*

Proof. Let $\gamma \in \mathbf{G}$ be a central element of length 1. From sequence (8.1) the group splits as a direct product

$$\mathbf{G} \cong \mathbf{K} \times \mathbb{Z}$$

where $\mathbb{Z} =< \gamma >$. Therefore clearly $[\mathbf{G}, \mathbf{G}] = [\mathbf{K}, \mathbf{K}]$. □

An example of the situation of the corollary is when one of the generators β_j commutes with all the others, *i.e.*, one hyperplane is transversal to the others. So, we find again in this way a well-known fact.

Consider again the exact sequence (8.6). Remind that the arrangement $\widetilde{\mathcal{A}}$ is a-monodromic (over $\mathbb{Q}$) if and only if $H_1(F; \mathbb{Q}) \cong \mathbb{Q}^n$. By tensoring sequence (8.6) by $\mathbb{Q}$ we obtain:

Theorem 8.7. *The arrangement $\widetilde{\mathcal{A}}$ is a-monodromic (over $\mathbb{Q}$) if and only if*

$$\frac{[\mathbf{G}, \mathbf{G}]}{[\mathbf{K}, \mathbf{K}]} \otimes \mathbb{Q} = 0.$$

Remark 8.8. All remarkable questions about the H_1 of the Milnor fibre F are actually questions about the group

$$\frac{[\mathbf{G}, \mathbf{G}]}{[\mathbf{K}, \mathbf{K}]}.$$

In particular:

(1) $H_1(F;\mathbb{Z})$ has torsion if and only if $\frac{[\mathbf{G},\mathbf{G}]}{[\mathbf{K},\mathbf{K}]}$ has torsion;

(2) $b_1(F) \;=\; n \;+ rk\left(\frac{[\mathbf{G},\mathbf{G}]}{[\mathbf{K},\mathbf{K}]}\right)$.

(There are only complicated examples with torsion in higher homology of the Milnor fiber, recently found in [14].)

Corollary 8.9. *One has*

$$n \le b_1(F) \le n + rk\left(\frac{[\mathbf{G},\mathbf{G}]}{[[\mathbf{G},\mathbf{G}],[\mathbf{G},\mathbf{G}]]}\right) = n \;+\; rk\left(\frac{\mathbf{G}^{(1)}}{\mathbf{G}^{(2)}}\right).$$

Now we consider again the affine arrangement $\mathcal{A}$. Denoting by $\mathbf{G}' := \pi_1(\mathcal{M}(\mathcal{A}))$, we have

$$\mathbf{G} \cong \mathbf{G}' \times \mathbb{Z},$$

where the factor $\mathbb{Z}$ is generated by a loop around all the hyperplanes in $\widetilde{\mathcal{A}}$. As already said, it follows by the Kunneth formula that if $\mathcal{A}$ has trivial monodromy over $\mathbb{Z}$ (respectively $\mathbb{Q}$), so does $\widetilde{\mathcal{A}}$. Conversely, in Figure 6.7 we have an example where $\widetilde{\mathcal{A}}$ is a-monodromic but $\mathcal{A}$ has non-trivial monodromy.

The a-monodromicity of $\mathcal{A}$ (over $\mathbb{Z}$) is equivalent to

$$H_1(\mathcal{M}(\mathcal{A});R) \cong \left(\frac{R}{(t-1)}\right)^{n-1}, \tag{8.8}$$

($R = \mathbb{Z}[q^{\pm 1}]$). By considering a sequence as in (8.1),

$$1 \longrightarrow \mathbf{K}' \longrightarrow \mathbf{G}' \xrightarrow{\psi} \mathbb{Z} \longrightarrow 1, \tag{8.9}$$

we can repeat the above arguments: in particular condition (8.8) is equivalent to

$$H_1(\mathbf{K}';\mathbb{Z}) = \frac{\mathbf{K}'}{[\mathbf{K}',\mathbf{K}']} = \mathbb{Z}^{n-1},$$

and we get an exact sequence like in (8.6) for $\mathbf{K}'$ and $\mathbf{G}'$. So we obtain:

Theorem 8.10. *The arrangement $\mathcal{A}$ is a-monodromic over $\mathbb{Z}$ (respectively over $\mathbb{Q}$) if and only if*

$$\frac{[\mathbf{G}',\mathbf{G}']}{[\mathbf{K}',\mathbf{K}']} = 0 \left(\textit{respectively } \tfrac{[\mathbf{G}',\mathbf{G}']}{[\mathbf{K}',\mathbf{K}']} \otimes \mathbb{Q} = \; 0\right).$$

By considering a presentation for $\mathbf{G}'$

$$1 \longrightarrow \mathbf{N}' \longrightarrow \mathbf{F}' \xrightarrow{\pi} \mathbf{G}' \longrightarrow 1$$

where $\mathbf{F}'$ is the group freely generated by $\beta_1, \dots, \beta_n$, we have a diagram similar to (8.3) for $\mathbf{G}'$. From $\mathbf{N}' \subset [\mathbf{F}', \mathbf{F}'] \subset \ker\varphi$ we have isomorphisms

$$\frac{[\mathbf{G}', \mathbf{G}']}{[\mathbf{K}', \mathbf{K}']} \cong \frac{\pi^{-1}[\mathbf{G}', \mathbf{G}']}{\pi^{-1}[\mathbf{K}', \mathbf{K}']} \cong \frac{[\mathbf{F}', \mathbf{F}']}{\mathbf{N}'[\ker\varphi, \ker\varphi]},$$

which gives again Theorem 7.2.

Corollary 8.6 extends clearly to the affine case: therefore, if one line of $\mathcal{A}$ is in general position with respect to the others, then $\mathcal{A}$ is a-monodromic. This result has the following useful generalization, which has both a central and an affine versions. We give here the affine one.

Theorem 8.11. *Assume that the fundamental group G' decomposes as a direct product*

$$G' = A \times B$$

of two subgroups, each one having at least one element of length one. Then $\mathcal{A}$ is a-monodromic.

In particular, this applies to the case when G' decomposes as a direct product of free groups,

$$G' = F_{i_1} \times F_{i_2} \times \cdots \times F_{i_k}$$

where (at least) two of them have an element of length one.

Proof. First, observe that any commutator $[ab, a'b'] \in [G', G']$ equals $[a, a'][b, b']$. Therefore it is sufficient to show that $[A, A] \subset [K', K']$, and $[B, B] \subset [K', K']$. Let $a_0 \in A,\ b_0 \in B$ be elements of length one. Let $l = \psi(a),\ l' = \psi(a')$ be the lengths of a and a' respectively. Then

$$[a, a'] = [ab_0^{-l}, a'b_0^{-l'}],$$

and the second commutator lies in $[K', K']$ by construction. This proves that $[A, A] \subset [K', K']$.

In the same way, by using a_0, we show that $[B, B] \subset [K', K']$. □

Remark 8.12. This theorem includes the case when the arrangement is a disjoint union $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$ of two subarrangements which intersect each other transversally. It is known that $\pi_1(\mathcal{M}(\mathcal{A}))$ is the direct product of $\pi_1(\mathcal{M}(\mathcal{A}'))$ with $\pi_1(\mathcal{M}(\mathcal{A}''))$ (see [27]) therefore by Theorem 8.11 the arrangement $\mathcal{A}$ is a-monodromic. This remark also seems new in the literature.

We can use this result (or even Corollary 8.6) to prove the a-monodromicity of those examples in Section 6 for which the fundamental group splits as a direcy product of free groups.

Another example is given by any affine arrangement having only double points: in this case $\mathcal{A} = \cup_{i=1}^{k} \mathcal{A}_i$ where the $\mathcal{A}_i$'s are sets of parallel lines. Then $\pi_1(\mathcal{A}) = \times_{i=1}^{k} F_{n_i}$ where F_{n_i} is the free group in $n_i = |A_i|$ generators. This gives an easy prove of the following known fact: if there exists a line in a projective arrangement $\mathcal{A}$ which contains all the points of multiplicity greater than 3, then $\mathcal{A}$ is a-monodromic.

To take care also of examples as that in Figure 6.2, where the fundamental group is not a direct product of free groups, let us introduce another class of graphs $\tilde{\Gamma}$ as follows. Let the affine arrangement $\mathcal{A}$ have n lines. Then:

(1) the vertex set of $\tilde{\Gamma}$ corresponds to the set of generators $\{\beta_i,\ i = 1, \dots, n\}$ of G';
(2) for each edge (β_i, β_j) of $\tilde{\Gamma}$, the commutator $[\beta_i, \beta_j]$ belongs to $[K', K']$;
(3) $\tilde{\Gamma}$ is connected.

We call a graph $\tilde{\Gamma}$ satisfying the previous conditions an *admissible* graph.

Theorem 8.13. *If $\mathcal{A}$ allows an admissible graph $\tilde{\Gamma}$ then $\mathcal{A}$ is a-monodromic.*

We need the following lemma.

Lemma 8.14. *Let $F_n = F[\beta_1, \dots, \beta_n]$ be the free group in the generators β_i's. Let φ be the length function (see Section 7) on F_n. Then for any sequence of indices $i_0, \dots, i_k$ one has*

$$[\beta_{i_0}, \beta_{i_1}][\beta_{i_1}, \beta_{i_2}] \dots [\beta_{i_{k-1}}, \beta_{i_k}][\beta_{i_k}, \beta_{i_0}] \in [\ker(\varphi), \ker(\varphi)]$$

for each "closed" product of commutators.

Proof of lemma. If $k \leq 2$ the result is trivial. If $k = 3$, a straighforward application of Blanchfield theorem ([4]) gives the result. For $k > 3$, we can write

$$\begin{aligned}&[\beta_{i_0}, \beta_{i_1}][\beta_{i_1}, \beta_{i_2}] \dots [\beta_{i_{k-1}}, \beta_{i_k}][\beta_{i_k}, \beta_{i_0}]\\ &\quad = ([\beta_{i_0}, \beta_{i_1}][\beta_{i_1}, \beta_{i_2}][\beta_{i_2}, \beta_{i_0}])([\beta_{i_0}, \beta_{i_2}] \dots [\beta_{i_{k-1}}, \beta_{i_k}][\beta_{i_k}, \beta_{i_0}]),\end{aligned}$$

and we conclude by induction on k. □

Remark 8.15. Clearly, Lemma 8.14 applied to the generators of G' gives that

$$[\beta_{i_0}, \beta_{i_1}][\beta_{i_1}, \beta_{i_2}] \dots [\beta_{i_{k-1}}, \beta_{i_k}][\beta_{i_k}, \beta_{i_0}] \in [K', K']$$

for each closed product of commutators.

Proof of Theorem 8.13. According to Theorem 8.10 what we have to prove is that any commutator $[\beta_i, \beta_j]$ belongs to $[K', K']$.

If i, j corresponds to an edge (β_i, β_j) of $\tilde{\Gamma}$, the result follows by definition. Otherwise, let $\beta_i = \beta_{i_0}, \beta_{i_1}, \ldots, \beta_{i_k} = \beta_j$ be a path in $\tilde{\Gamma}$ connecting β_i with β_j. By definition, $[\beta_{i_j}, \beta_{i_{j+1}}] \in [K', K']$, $j = 0, \ldots, k-1$, so $\prod_{j=0}^{k-1}[\beta_{i_j}, \beta_{i_{j+1}}] \in [K', K']$. By Lemma 8.14 and Remark 8.15

$$[\beta_{i_0}, \beta_{i_1}][\beta_{i_1}, \beta_{i_2}] \ldots [\beta_{i_{k-1}}, \beta_{i_k}][\beta_{i_k}, \beta_{i_0}] \in [K', K'].$$

It follows that $[\beta_{i_0}, \beta_{i_k}] = [\beta_i, \beta_j] \in [K', K']$, which gives the thesis. □

We can use Theorem 8.13 to prove Conjecture 1 under further hypothesis.

Corollary 8.16. *Let $\mathcal{A}$ be an affine arrangement and let Γ be its associated graph of double points. Assume that Γ contains an admissible spanning tree $\tilde{\Gamma}$. Then $\mathcal{A}$ is a-monodromic.*

Of course, under the hypothesis of Corollary 8.16, the graph Γ is connected. Examples where Γ contains an admissible spanning tree are the conjugate-free arrangements in Definition 5.6. Here all commutators (corresponding to the edges of T) of the geometric generators are simply equal to 1 in the group G'. Therefore Theorem 8.13 is a generalization of Theorem 5.7.

Very little effort is needed to show that the whole graph Γ of double points in the arrangement of Figure 6.2 is admissible: therefore Corollary 8.16 applies to this case.

For the sake of completeness, we also mention that, for all the examples in Section 6 which have non trivial monodromy, all the quotient groups $[G', G']/[K', K']$ are free Abelian of rank 2. This fact is in accordance with the monodromy computations given in Section 6, since in all these cases one has φ_3-torsion. It also follows that, for such examples, the first homology group of the Milnor fiber has no torsion.

Remark 8.17. When the graph Γ of double points is not connected, then we can consider its decomposition into connected components $\Gamma = \sqcup_i \Gamma_i$. We have a corresponding decomposition $\mathcal{A} = \sqcup_i \mathcal{A}_i$ of the arrangement. By definition, every double point of $\mathcal{A}$ is a double point of exactly one of the $\mathcal{A}_i$'s, while each pair of lines in different $\mathcal{A}_i$'s either intersect in some point of multiplicity greater than 2, or are parallel (we are considering the affine case here). If our conjecture is true, then each $\mathcal{A}_i$ is a-monodromic. At the moment we are not able to speculate about how the monodromy of $\mathcal{A}$ is influenced by these data: apparently, the only knowledge of such decomposition gives little control on the multiplicities of the intersection points of different components, which can assume very different values. We are going to address these interesting problems in future work.

References

[1] D. ARAPURA, *Geometry of cohomology support loci for local systems I*, J. Algebraic Geom. **6** (1997), 563–597.

[2] P. BAILET, *On the monodromy of the Milnor fiber of hyperplane arrangements*, Canad. Math. Bull. **57** (2014), 697–707.

[3] N. BIGGS, "Algebraic Graph Theory", Cambridge University Press, 1974.

[4] J. BIRMAN, "Braids, Links and Mapping Class Groups", Princeton Univ. Press, 1975.

[5] F. CALLEGARO, *The homology of the Milnor fibre for classical braid groups*, Algebr. Geom. Topol. **6** (2006), 1903–1923.

[6] F. CALLEGARO, D. MORONI and M. SALVETTI, *Cohomology of Artin groups of type $\tilde{A}_n$, $\tilde{B}_n$ and applications*, In: "Proceedings of the Conference on Groups Homotopy and Configuration Spaces", N. Iwase (ed.), Geometry and Topology Publications, Coventry, 2008, 85–104.

[7] F. CALLEGARO, D. MORONI and M. SALVETTI, *Cohomology of affine Artin groups and applications*, Trans. Amer. Math. Soc. **360** (2008), 4169–4188.

[8] F. CALLEGARO, D. MORONI and M. SALVETTI, *The $K(\pi, 1)$ problem for the affine Artin group of type $\tilde{B}_n$ and its cohomology*, J. Eur. Math. Soc. (JEMS) **12** (2010), 1–22.

[9] D. COHEN and P. ORLIK, *Arrangements and local systems*, Math. Res. Lett. **7** (2000), 299–316.

[10] D. COHEN and A. SUCIU, *Characteristic varieties of arrangements*, Math. Proc. Cambridge Philos. Soc. **127** (1999), 33–53.

[11] C. DE CONCINI, C. PROCESI and M. SALVETTI, *Arithmetic properties of the cohomology of braid groups*, Topology **40** (2001), 739–751.

[12] C. DE CONCINI, C. PROCESI, M. SALVETTI and F. STUMBO, *Arithmetic properties of the cohomology of Artin groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 695–717.

[13] C. DE CONCINI and M. SALVETTI, *Cohomology of Artin groups*, Addendum: "The homotopy type of Artin groups", Math. Res. Lett. **3** (1996), 293–297.

[14] G. DENHAM and A. SUCIU, *Multinets, parallel connections, and Milnor fibrations of Arrangements*, Proc. Lond. Math. Soc. (3) **108** (2014), 1435–1470.

[15] A. DIMCA and S. PAPADIMA, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*, Ann. of Math. (2) **158** (2003), 473–507.

[16] A. DIMCA, S. PAPADIMA and A. SUCIU, *Topology and geometry of cohomology jump loci*, Duke Math. J. **148** (2009), 405–457.

[17] M. ELIYAHU, D. GARBER and M. TEICHER, *A conjugation-free geometric presentation of fundamental groups of arrangements*, Manuscripta Math. **133** (2010), 247–271.

[18] M. J. FALK, *Arrangements and cohomology*, Ann. Comb. **1** (1997), 135–157.

[19] M. J. FALK and S. YUZVINSKY, *Multinets, resonance varieties and pencils of plane curves*, Compos. Math. **143** (2007), 10069–1088.

[20] R. FORMAN, *Morse theory for cell complexes*, Adv. Math. **134** (1998), 90–145.

[21] E. V. FRENKEL, *Cohomology of the commutator subgroup of the braid group*, Funct. Anal. Appl. **22** (1988), 248–250.

[22] G. GAIFFI and M. SALVETTI *The Morse complex of a line arrangement*, J. Algebra **321** (2009), 316–337.

[23] G. GAIFFI, F. MORI and M. SALVETTI, *Minimal CW-complexes for complements to line arrangements via Discrete Morse Theory*, In: "Topology of Alg. Var. and Sing.", Amer. Math. Soc., Contemporary Math., Vol. 538, 2011, 293–308.

[24] A. LIBGOBER and S. YUZVINSKY, *Cohomology of the Orlik-Solomon algebras and local systems*, Compos. Math. **21** (2000), 337–361.

[25] A. MĂCINIC and S. PAPADIMA, *On the monodromy action on Milnor fibers of graphic arrangements*, Topology Appl. **156** (2009), 761–774.

[26] P. ORLIK and M. TERAO, "Arrangements of Hyperplanes", Grundlehren der mathematischen Wissenschaften, Vol. 300, Springer-Verlag, 1992.

[27] M. OKA and K. SAKAMOTO, *Product theorem of the fundamental group of a reducible curve*, J. Math. Soc. Japan **30** (1978), 599–602.
[28] R. RANDELL, *Morse theory, Milnor fibers and minimality of a complex hyperplane arrangement*, Proc. Amer. Math. Soc. **130** (2002), 2737–2743.
[29] M. SALVETTI, *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Invent. Math. **88** (1987), 603–618.
[30] M. SALVETTI, *The homotopy type of Artin groups*, Math. Res. Lett. **1** (1994), 567–577.
[31] S. SETTEPANELLA, *Cohomology of pure braid groups of exceptional cases*, Topol. Appl. **156** (2009), 1008–1012.
[32] M. SALVETTI and M. SERVENTI, *Arrangements of lines and monodromy of associated Milnor fibers*, J. Knot Theory Ramif. **26** (2016).
[33] M. SALVETTI and S. SETTEPANELLA, *Combinatorial Morse theory and minimality of hyperplane arrangements*, Geom. Topol. **11** (2007), 1733–1766.
[34] A. SUCIU, *Translated tori in the characteristic varieties of complex hyperplane arrangements*, Topol. Appl. **118** (2002), 209–223.
[35] A. SUCIU, *Hyperplane arrangements and Milnor fibrations*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), 417–481.
[36] A. SUCIU, *Around the tangent cone theorem*, In "Configuration Spaces: Geometry, Topology and Representation Theory", Springer INdAM Series, F. Callegaro, F. Cohen, C. De Concini, E. Feichtner, G. Gaiffi, M. Salvetti (eds.), 2016, 1–39.
[37] M. YOSHINAGA, *Resonant bands and local system cohomology groups for real line arrangements*, Vietnam J. Math. **42** (2014), 377–392.
[38] M. YOSHINAGA, *Milnor fibers of real line arrangements*, J. Singul. **7** (2013), 220–237.

Departimento di Matematica
Università di Pisa
Largo Pontecorvo, 5
56127 Pisa, Italia
salvetti@dm.unipi.it
serventi@mail.dm.unipi.it