

Weak- and semi-normalization in real algebraic geometry

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Abstract. We define the weak-normalization and the seminormalization of a real algebraic variety relative to its central locus. The study is related to the properties of the rings of continuous rational functions and hereditarily rational functions on real algebraic varieties. We provide in particular several characterizations (algebraic or geometric) of these varieties, and give a full description of centrally seminormal curves.

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The present paper is devoted to the study of the seminormalization of a real algebraic variety. The complex setting is settled for more than fifty years, first in the analytic case in [4] for studying the Chow variety of a complex analytic variety, and more generally for schemes in [3]. Recently, the concept of seminormalization appears in the study of the singularities in the minimal model program by Kollár and Kovács [14], and also in [15]. If the complex setting is well known, the different attempts in real geometry have been unsuccessful so far [3, 24]. The two main difficulties one has to face in the real algebraic case are the complexity of the topology of the real points of algebraic varieties, notably with regards to centrality issues, and the intriguing behavior of rational functions (as illustrated in [17]), whose understanding is somehow the cornerstone of this work. These issues lead to two major features of our construction.

Let us review the classical case. The weak-normalization of a complex analytic variety has been introduced by Andreotti and Norguet [4] in order to study the space of analytic cycles associated with a complex algebraic variety. The operation of weak-normalization consists in enriching the sheaf of holomorphic functions with those continuous functions which are also meromorphic. Later Andreotti and Bombieri [3] defined the notion of weak-normalization in the context of schemes. For algebraic varieties, it consists roughly speaking of an intermediate algebraic variety between an algebraic variety X and its normalization, in such a way that the

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weak-normalization of X is in bijection with X . The construction goes by identifying in the normalization all points in the fibers over X . It gives rise to a variety satisfying a universal property among those varieties in birational bijection (meaning a regular morphism which is birational and a bijection at the level of closed points) via a universal homeomorphism onto X for the Zariski topology. The theory of seminormalization, closely related to that of weak-normalization, have been developed later by Traverso [31] for commutative rings, with subsequent work notably by Swan [30] or Leahy and Vitulli [22] (see also [32]), with a particular focus on the algebraic approach and the study of the singularities. Note that in the geometric context of complex algebraic variety, weak-normalization and seminormalization lead to the same notion. We refer to Vitulli [33] for a survey on weak normality and seminormality for commutative rings and algebraic varieties.

In the context of real geometry, the first occurrence of weak normality or seminormality is the work by Acquistapace, Broglia and Tognoli [1] in the case of real analytic spaces. In [24] the Traverso-style seminormalization of real algebraic varieties is studied by considering the ring of regular functions, showing that such notion does not provide a natural universal property. Seminormalization in the Nash context is introduced in [28]. Our aim in this paper is to provide appropriate definitions for weak normalization and seminormalization in real algebraic geometry, leading to natural universal properties. Two major differences with the complex case appear. First, the normalization is no longer surjective on the real closed points in general, but only on the central loci, namely the Euclidean closure of the set of non-singular points. Moreover, the notions of weak-normalization and seminormalization that we consider in the paper are distinct, the difference being witnessed by the behavior of continuous rational functions on the central locus of real algebraic varieties.

Dealing with continuous functions on real algebraic varieties, we are interested mainly in the real closed points of real varieties. Note that the real closed points of a quasi-projective variety defined over $\mathbb{R}$ are always included in an affine variety, so that we will restrict ourselves to this setting in the paper. In particular, consider now a real algebraic variety as an algebraic subset of $\mathbb{R}^n$ (in the sense of [6]). The first focus on continuous rational function in real geometry is due to Kreisel [18] who proved that a positive answer to Hilbert seventeenth problem of representing a positive polynomial as a sum of squares of rational functions, can always be chosen among continuous functions. Besides, Kucharz [19] used this class of functions to approximate as algebraically as possible continuous maps between spheres, whereas Kollár and Nowak [17] initiated the proper study of these functions, proving notably that the restriction of a continuous function defined on a central real algebraic variety (in the sense of [6]), which is also rational, does not remain rational in general. It is however the case as soon as the ambient variety is nonsingular. As a consequence, on a singular real algebraic variety X one may consider the ring $\mathcal{K}^0(X)$ of continuous rational functions, or its subring $\mathcal{R}^0(X)$ consisting of those continuous rational functions which remain rational under restriction. This class, called hereditarily rational in [17], has been systematically studied in [10] under the name of regulous functions. When the real algebraic variety is no longer central,

a rational function may admit several continuous extensions to the whole variety. This is the reason why we will consider continuous rational functions on the central locus of algebraic varieties. The continuous rational functions are now extensively studied in real geometry, we refer for example to [11,16,20,26] for further readings related to the subject of the paper.

Let us sketch the construction of the weak- and semi-normalizations. The rational functions on a real algebraic variety X that satisfy a monic polynomial equation with coefficients in the ring $\mathcal{P}(X)$ of polynomial functions on X , form the integral closure of $\mathcal{P}(X)$ in the ring $\mathcal{K}(X)$ of rational functions on X , which is a finite module over $\mathcal{P}(X)$. This ring is the polynomial ring of the normalization X' of X , coming with a finite birational morphism onto X . The resulting morphism from the complex points of X' to the complex points of X is well-known to be closed with respect to the Zariski topology, hence surjective. However this is no more the case in restriction to the real points! The *real* behavior of normalization, together with various other integral closures, has been investigated in [12]. In particular, it has been noted there that the lack of surjectivity is related to the notion of central locus of a real algebraic variety.

So in the present paper, we push further the study and restrict ourselves to the central part of the real varieties. When requiring that the rational functions admit a continuous extension to the central locus $\text{Cent } X$ of X , the integral closure of $\mathcal{P}(X)$ in $\mathcal{K}^0(\text{Cent } X)$ is still a finite module over $\mathcal{P}(X)$, and therefore it coincides with the polynomial ring of a real algebraic variety. The study of the properties of this variety is the main subject of the paper. We call this variety the weak-normalization X^{wc} of X relative to the central locus of X . It comes again with a finite birational polynomial morphism $\pi^{\text{wc}} : X^{\text{wc}} \rightarrow X$, which is an homeomorphism for the Euclidean topology in restriction to the central loci. We provide several characterizations of X^{wc} , notably from a geometric point of view that X^{wc} is the biggest intermediate variety between X and X' whose central locus is in bijection with $\text{Cent } X$, or in an algebraic point of view introducing the notion of centrally weakly subintegral extension of rings. Notice that every bijection in the paper is in fact an homeomorphism for the Euclidean topology. It satisfies a universal property as follows.

Theorem. *Let X be an affine real algebraic variety. For each affine real algebraic variety Y together with a finite birational morphism $\pi : Y \rightarrow X$ then $\pi : \text{Cent } Y \rightarrow \text{Cent } X$ is bijective if and only if the morphism $\pi^{\text{wc}} : X^{\text{wc}} \rightarrow X$ factorizes through π .*

The justification for calling X^{wc} the weak-normalization of X comes from Theorem 4.8, which illustrates that X^{wc} satisfies analogue properties in the real setting as the weak-normalization for complex algebraic varieties. Notice that complex algebraic varieties are central i.e equal to their central locus since the semi-algebraic dimension at any point is maximal (in its irreducible component). Note also that this restriction to the central locus is necessary in the real situation, because in general a biggest real algebraic variety in finite birational bijection with a given real algebraic variety does not exist, as illustrated by Example 5.5 at the end of the paper.

Now, replacing the ring of continuous rational functions with the ring of hereditarily rational functions leads similarly to the definition of the seminormalization X^{sc} of X relative to the central locus of X , whose ring of polynomial functions is given by the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}^0(\text{Cent } X)$. The study of the properties of this seminormalization, in comparison with the weak normalization, is the second main topic of the present paper. The seminormalization of X relative to its central locus is an intermediate variety between X and X^{wc} , so that X^{sc} admits a finite birational morphism π^{sc} onto X which is an homeomorphism on the central loci for the Euclidean topology. It is moreover the biggest intermediate variety between X and X' whose polynomial functions are hereditarily rational on $\text{Cent } X$. It is also the biggest intermediate variety between X and X' whose central locus is in bijection with $\text{Cent } X$ with an inverse map which is hereditarily rational. At the level of algebra, we characterize $\mathcal{P}(X^{\text{sc}})$ as the maximal centrally subintegral extension of $\mathcal{P}(X)$. Similarly for the universal property, we have:

Theorem. *Let X be an affine real algebraic variety. For each affine real algebraic variety Y together with a finite birational morphism $\pi : Y \rightarrow X$ then $\pi : \text{Cent } Y \rightarrow \text{Cent } X$ is bijective with an inverse map which is hereditarily rational if and only if the morphism $\pi^{\text{sc}} : X^{\text{sc}} \rightarrow X$ factorizes through π .*

The normalization, weak-normalization and seminormalization relative to the central locus of a real algebraic variety are different in general. The latter two coincide on varieties where all continuous rational functions are hereditarily rational, for instance in the case of curves. We provide in this particular case a full description of the singularities of centrally weakly normal curves, in the spirit of [8] in the complex context.

Note also that, if it is clear from the very construction that isomorphic real algebraic varieties have isomorphic weak normalization and seminormalization, it is still true, similarly to the case of the normalization, that biregular real algebraic varieties have biregular weak normalization and seminormalization (cf. Theorem 4.20).

At this stage, the reader should imagine that the weak normalization and the seminormalization relative to the central loci share very similar properties and somehow behave in a parallel manner. We would like to focus however on a crucial difference. If adding a continuous assumption in the construction of the normalization naturally leads to a variety in (universal) bijection (on the central loci), the additional assumption of hereditary rationality implies moreover that the residual real local fields are isomorphic. This major difference is responsible for the fact that, as in the complex case, the seminormalization process relative to the central locus commutes with localization at a prime ideal, but this is no longer the case for the weak normalization process!

The paper is organized as follows. The first section is devoted to some preliminaries on real algebraic varieties, notably concerning central loci, normalization process or continuous rational functions. The second section deals with topological properties of integral morphisms in the real context, where the lying-over and going-up properties are revisited. We introduce there the algebraic treatment of weak normalization and seminormalization in the Traverso style. The third section

deals geometrically with finite birational morphisms, and details how (hereditary) continuous rational functions behaves under such morphisms. It is in the fourth section that we define the weak and seminormalization relative to the central locus, using (hereditary) continuous rational functions. We make the link with the algebraic notions developed in section three, and tackle the issue of the commutation with the localization. The final section gives a full treatment of the curves case.

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1. Preliminaries on real algebraic varieties

In this section we review the basic definition of a real algebraic variety together with the properties of its normalization, and recall the concept of continuous rational function.

1.1. Real algebraic sets and varieties

For a commutative ring A we denote by $\text{Spec } A$ the Zariski spectrum of A , the set of all prime ideals of A . We denote by $\text{Max } A$ the set of maximal ideals of A . Recall that an ideal I of A is called real if, for every sequence $a_1, \dots, a_k$ of elements of A , then $a_1^2 + \dots + a_k^2 \in I$ implies $a_i \in I$ for $i = 1, \dots, k$. In this work, we also consider the real Zariski spectrum $\text{R-Spec } A$ which consists of all the real prime ideals of A .

An affine real algebraic variety is a Zariski spectrum of the form $\text{Spec}(\mathbb{R}[x_1, \dots, x_n]/I)$ where I is an ideal of $\mathbb{R}[x_1, \dots, x_n]$. To a real algebraic variety given by the ideal I in $\mathbb{R}[x_1, \dots, x_n]$ one associates the real algebraic set $X = \mathcal{Z}(I)$ of all points in $\mathbb{R}^n$ which cancel any polynomial in I . Conversely, to any real algebraic set $X \subset \mathbb{R}^n$ one may associate the real algebraic variety given by the ideal $\mathcal{I}(X) \subset \mathbb{R}[x_1, \dots, x_n]$ of all polynomials which vanish at all points of X . Given an ideal I of $\mathbb{R}[x_1, \dots, x_n]$ then the real Nullstellensatz [6, Theorem 4.1.4] says that $\mathcal{I}(\mathcal{Z}(I)) = I$ if and only if I is a real ideal. In particular, for a real algebraic set $X \subset \mathbb{R}^n$ then $\mathcal{I}(X)$ is a real ideal and X corresponds to $\text{R-Spec}(\mathbb{R}[x_1, \dots, x_n]/\mathcal{I}(X)) \cap \text{Max}(\mathbb{R}[x_1, \dots, x_n]/\mathcal{I}(X))$.

We are interested in this text in the geometry of the real closed points of real algebraic varieties. In this context, it is natural to consider only varieties which are affine since almost all real algebraic varieties are affine [6, Remark 3.2.12]. In the sequel we mostly consider algebraic sets (as in [6]) rather than real algebraic varieties (as in [23], with an emphasis on $\mathbb{R}$ -schemes) and unless specified, these algebraic sets are real.

In complex affine algebraic geometry, polynomial and regular functions coincide and thus we have a unique and natural definition of morphism between complex algebraic sets. In the real setting no such natural definition exists. Usually, real algebraic geometers prefer working with the ring $\mathcal{O}(X)$ of regular functions, *i.e.*, rational functions with no real poles (see [6, Section 3.2] for details), rather than the ring of polynomial functions $\mathcal{P}(X) = \mathbb{R}[x_1, \dots, x_n]/I$ where $I = \mathcal{I}(X)$ on a real algebraic set X . In this paper however, we work rather with the ring of polynomial functions due to its better properties with respect to the normalization process (see [12]).

Let X be a real algebraic set. We denote by $\mathfrak{m}_x$ the maximal ideal of functions in $\mathcal{P}(X)$ that vanish at $x \in X$. With the real Nullstellensatz [6, Theorem 4.1.4], we have a natural correspondence between the points of X and the real maximal ideals of $\mathcal{P}(X)$.

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be real algebraic sets. A polynomial morphism from X to Y is a map whose coordinate functions are polynomial. A polynomial morphism $\varphi : X \rightarrow Y$ induces an $\mathbb{R}$ -algebra homomorphism $\phi : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ defined by $\phi(f) = f \circ \varphi$. The map $\varphi \mapsto \phi$ gives a bijection between the set of polynomial morphisms from X to Y and the $\mathbb{R}$ -algebra homomorphisms from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. We say that a polynomial morphism $\varphi : X \rightarrow Y$ is an isomorphism if φ is bijective with a polynomial inverse, or in another words if $\phi : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ is an isomorphism. We define the analog notions with regular functions in place of polynomial ones. In that situation, an isomorphism will be called a biregular isomorphism. Unless specified, a morphism is polynomial.

Let $X \subset \mathbb{R}^n$ be a real algebraic set. The complexification of X , denoted by $X_{\mathbb{C}}$, is the complex algebraic set $X_{\mathbb{C}} \subset \mathbb{C}^n$, whose ring of polynomial functions is $\mathcal{P}(X_{\mathbb{C}}) = \mathcal{P}(X) \otimes_{\mathbb{R}} \mathbb{C}$. As already mentioned, we have $\mathcal{P}(X_{\mathbb{C}}) = \mathcal{O}(X_{\mathbb{C}})$. We say that X is geometrically smooth if $X_{\mathbb{C}}$ is smooth. Remark that if X is irreducible, then $X_{\mathbb{C}}$ is automatically irreducible because X is an algebraic set. The situation is different when we consider an affine real algebraic variety X , actually X can be irreducible and $X \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$ reducible when the set of real points of X is not Zariski dense in the set of complex points, as illustrated by the example of the affine real algebraic variety corresponding to the ideal $(x^2 + y^2)$ in $\mathbb{R}[x, y]$.

Let $\varphi : X \rightarrow Y$ be a polynomial morphism between real algebraic sets. The tensor product by $\mathbb{C}$ of the morphism of $\mathbb{R}$ -algebras $\phi : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ gives a morphism of $\mathbb{C}$ -algebras $\phi_{\mathbb{C}} : \mathcal{P}(Y_{\mathbb{C}}) \rightarrow \mathcal{P}(X_{\mathbb{C}})$ and by duality we get a polynomial morphism $\varphi_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ called the complexification of φ . Since $\mathcal{I}(X)$ and $\mathcal{I}(Y)$ are real ideals then φ is an isomorphism if and only if $\varphi_{\mathbb{C}}$ is an isomorphism. The situation is again different if we consider affine real algebraic varieties. Indeed, two non-isomorphic affine real algebraic varieties can be isomorphic over the complex numbers, for example the empty conic $\text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1))$ with the circle $\text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 - 1))$.

1.2. Normalization and central locus

Let $A \rightarrow B$ be an extension of rings. An element $b \in B$ is integral over A if b is the root of a monic polynomial with coefficients in A . By [5, Proposition 5.1], b is integral over A if and only if $A[b]$ is a finite A -module. This equivalence allows to prove that $A'_B = \{b \in B \mid b \text{ is integral over } A\}$ is a ring called the integral closure of A in B . The extension $A \rightarrow B$ is said to be integral if $A'_B = B$. In case A is reduced with a finite number of minimal prime ideals and B is the total ring of fractions K of A (see below), the ring A'_K is denoted by A' and is simply called the integral closure of A . The ring A is called integrally closed (in B) if $A = A'$ ($A = A'_B$). If A is the ring of polynomial functions on an algebraic set X over a field k then A' is a finite A -module [25, Section 33, after Lemma 2]. In this situation then A' is a finitely generated k -algebra and so A' is the ring of polynomial functions of an algebraic set, denoted by X' , called the normalization of X . We recall that a morphism $X \rightarrow Y$ between two algebraic sets over a field k is said finite if the ring morphism $\mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ makes $\mathcal{P}(X)$ a finitely generated $\mathcal{P}(Y)$ -module. The inclusion $\mathcal{P}(X) \subset \mathcal{P}(X')$ induces a finite morphism which we denote by $\pi' : X' \rightarrow X$, called the normalization morphism, which is a birational equivalence. We say that an algebraic variety X over a field k is normal if its ring of polynomial functions is integrally closed.

Let us give a property which we will need to deal with the normalization of a non irreducible algebraic variety. Namely, one has to deal with coordinate rings which are not necessarily domains but only reduced rings.

Recall first that if A is a reduced ring with minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, then $(0) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$ and one has the canonical injections $A \rightarrow A_1 \times \dots \times A_r \rightarrow K_1 \times \dots \times K_r = K$ where $A_i = A/\mathfrak{p}_i$ and K_i is the fraction field of A_i for any i . The product of fields K is called the total ring of fractions of A and the A_i 's are called the irreducible components of A . In case A is the ring of polynomial functions $\mathcal{P}(X)$ on a real algebraic set X then we denote by $\mathcal{K}(X)$ the total ring of fractions. If $X_1, \dots, X_t$ denote the irreducible components of X then $\mathcal{K}(X)$ is the product of fields $\mathcal{K}(X_1) \times \dots \times \mathcal{K}(X_t)$. Notice that $\mathcal{K}(X)$ corresponds also to classes of rational functions on X i.e classes of functions without poles on a *dense* Zariski open subset of X . In the following, we prefer to call $\mathcal{K}(X)$ the ring of rational functions on X .

Proposition 1.1 ([12, Proposition 1.2]). *Let A be a reduced ring with minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Then, $A' = A'_1 \times \dots \times A'_r$ where A' is the integral closure of A in its total ring of fractions K , and, for any i , A'_i is the integral closure of $A_i = A/\mathfrak{p}_i$ in its ring of fractions K_i .*

Corollary 1.2. *Let X be an algebraic set. If $X_1, \dots, X_t$ denote the irreducible components of X then $\mathcal{P}(X') = \mathcal{P}(X'_1) \times \dots \times \mathcal{P}(X'_t)$ and X' is the disjoint union of $X'_1, \dots, X'_t$.*

Notice that the previous corollary shows that it is sufficient to understand the normalization for irreducible algebraic sets. It will no longer be the case with the weak and semi-normalization studied in the paper.

A real algebraic set X is said geometrically normal if the associated complex algebraic set $X_{\mathbb{C}}$ is normal. It is well known that X is normal if and only if X is geometrically normal. Note that, if the normality of X implies that the ring of regular functions on X is integrally closed, the converse is not true in general. Note also that biregular real varieties have biregular normalizations [6, page 75]. For more about the integral closure of the ring of regular functions on a real algebraic set, we refer to [12].

Recall that a morphism $\varphi : Y \rightarrow X$ is birational if φ induces an isomorphism between $\mathcal{K}(Y)$ and $\mathcal{K}(X)$. Note that the normalization of an algebraic set X is the biggest algebraic set finitely birational to X . More precisely, for any finite birational morphism $\varphi : Y \rightarrow X$, there exists $\psi : X' \rightarrow Y$ such that $\pi' = \phi \circ \psi$.

The normalization can be thought as a kind of weak desingularization of an algebraic variety, but much closer to the original variety due to the finiteness property. Note however that strange phenomena may appear in the real case. For instance, the normalization of a cubic with an isolated point (e.g., given by the equation $y^2 = t^2(t-1)$ in $\mathbb{R}^2$) is a smooth curve such that no real point lies over the singular point of the cubic. This phenomenon leads us to consider the central loci of real algebraic sets.

Definition 1.3. Let X be an algebraic set. If X is irreducible, we define the central locus of X , denoted by $\text{Cent } X$, to be the Euclidean closure $\overline{X_{\text{reg}}}^{\text{eucl}}$ in X of the set X_{reg} of non-singular points of X . Otherwise, if X is an algebraic set with irreducible components $X_1, \dots, X_t$, we define the central locus of X to be $\text{Cent } X = \bigcup_{i=1}^t \text{Cent } X_i$.

We say that X is central if X is equal to $\text{Cent } X$.

Remark 1.4.

- (1) By [6, Proposition 7.6.2], the central locus of an irreducible algebraic set is the locus of points where the local semi-algebraic dimension is maximal;
- (2) An algebraic set can be central without its irreducible components being so (consider for example $X = \mathcal{Z}((y^2 - x^2(x-1))x)$);
- (3) This definition of the central locus of a real algebraic set, rather than taking the Euclidean closure of the regular locus, is better adapted to normalization issue. Indeed the normalization of the union $P \cup L$ of a plane with a line is the disjoint union of P and L , whereas the Euclidean closure of the regular locus of $P \cup L$ is simply P .

The normalization of any real algebraic curve is central. This is true since the normalization of a curve is even non-singular. However, it may happen that modifying a central curve via a finite birational morphism creates a non-central curve. Even worst, the normalization of a central surface may create isolated points (cf. [12, Example 1.7])! These pathologies are the main reason why in the paper, we define a concept for weak- and semi-normalization of real algebraic varieties *relative* to the central locus.

1.3. Rational and continuous functions

The intriguing behaviour of rational functions on a real algebraic set admitting a continuous extension to the whole algebraic set has been investigated in [17]. Among them, the special class of hereditarily rational functions is of special interest.

Let $X \subset \mathbb{R}^n$ be an algebraic set. A rational function $f \in \mathcal{K}(X)$ is regular on a Zariski-dense open subset $U \subset X$ if there exist polynomial functions p and q on $\mathbb{R}^n$ such that $\mathcal{Z}(q) \cap U = \emptyset$ and $f = p/q$ on U . The pair $(U, f|_U)$ is called a regular presentation of f . The biggest such U is called the domain $\text{dom}(f)$ of f , and its complementary in X is the indeterminacy locus $\text{indet}(f)$ of f .

Definition 1.5. Let $f : X \rightarrow \mathbb{R}$ be a continuous function. We say that f is a continuous rational function on X if there exists a Zariski-dense open subset $U \subset X$ such that $f|_U$ is regular. We denote by $\mathcal{K}^0(X)$ the ring of continuous rational functions on X .

Let $f : \text{Cent } X \rightarrow \mathbb{R}$ be a continuous function. We say that f is a continuous rational function on $\text{Cent } X$ if there exist a Zariski-dense open subset $U \subset X$ and a regular function g on U such that the restriction $f|_{U \cap \text{Cent } X}$ is equal to the restriction $g|_{U \cap \text{Cent } X}$. The ring of continuous rational functions on $\text{Cent } X$ is denoted by $\mathcal{K}^0(\text{Cent } X)$.

A map $Y \rightarrow X$ between real algebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is called rational continuous if its components are rational continuous functions on Y .

A map $\text{Cent } Y \rightarrow \text{Cent } X$ between the central loci of real algebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is called rational continuous if its components are rational continuous functions on $\text{Cent } Y$.

A typical continuous rational function is provided by the function defined by $(x, y) \mapsto x^3/(x^2 + y^2)$ on $\mathbb{R}^2$ minus the origin, and by zero at the origin.

Consider also Cartan umbrella $X = \mathcal{Z}(z(x^2 + y^2) - x^3)$ and the function $f = x^3/(x^2 + y^2)$ extended by 0 along the z -axis. Then f is rational continuous on X and so on $\text{Cent } X$, but f is also polynomial on $\text{Cent } X$ since it coincides there with z . Of course f and z are different on X . Remark that a polynomial function on a real algebraic set X is the restriction to X of a unique polynomial function on $X_{\mathbb{C}}$ and uniqueness comes from the Zariski density of X in $X_{\mathbb{C}}$.

Note also that on a curve with isolated points like the cubic curve $\mathcal{Z}(y^2 - x^2(x - 1))$, a function regular on the one-dimensional branches can be extended continuously by any real value at the isolated points. In particular, the natural ring morphism $\mathcal{K}^0(X) \rightarrow \mathcal{K}(X)$ which sends $f \in \mathcal{K}^0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where $(U, f|_U)$ is a regular presentation of f , is not injective in general. However, restricting our attention to the central locus, note that the canonical map $\mathcal{K}^0(\text{Cent } X) \rightarrow \mathcal{K}(X)$ is now injective.

Considering now a reducible algebraic set X with irreducible components $X_1, \dots, X_t$. By restriction, one gets a well defined natural morphism $\Xi_0 : \mathcal{K}^0(\text{Cent } X) \rightarrow \mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t)$. Indeed, a function $f \in \mathcal{K}^0(\text{Cent } X)$ is in particular rational on each irreducible component X_i of X

as a rational function, and its restriction $f|_{\text{Cent } X_i}$ is continuous. Moreover, this morphism is clearly injective and one also has:

Proposition 1.6. *Let X be an algebraic set with irreducible components $X_1, \dots, X_t$. Then $\mathcal{K}^0(\text{Cent } X)$ is isomorphic to the subring of $\mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t)$ consisting of t -tuples $(f_1, \dots, f_t)$ such that f_i and f_j coincide on the intersection $\text{Cent } X_i \cap \text{Cent } X_j$ for all $i, j \in \{1, \dots, t\}$.*

Proof. Denote by A the subring of $\mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t)$ consisting of t -tuples $(f_1, \dots, f_t)$ such that f_i and f_j coincide on the intersection $\text{Cent } X_i \cap \text{Cent } X_j$ for all $i, j \in \{1, \dots, t\}$.

The image of the natural morphism $\mathcal{K}^0(\text{Cent } X) \rightarrow \mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t)$ is included in A since $\text{Cent } X = \bigcup_{i=1}^t \text{Cent } X_i$.

Let us show the converse inclusion. An element in A gives rise to t rational functions defined on $X_1, \dots, X_t$ respectively, so it coincides with a unique rational function on X . Moreover, this rational function admits a continuous extension to $\text{Cent } X$ by definition of A and since $\text{Cent } X = \bigcup_{i=1}^t \text{Cent } X_i$. $\square$

Another strange phenomenon appearing only on singular sets is illustrated by Kollár's example [17]. Consider the surface $S = \mathcal{Z}(y^3 - (1 + z^2)x^3)$ in $\mathbb{R}^3$. The continuous function defined by $(x, y, z) \mapsto \sqrt[3]{1 + z^2}$ is regular on S minus the z -axis, however its restriction to the z -axis is no longer rational. The continuous rational functions do not having this pathological behavior have been called hereditarily rational functions. Originally defined on the whole (central) algebraic set, one may even define hereditarily rational functions over semi-algebraic subsets as in [16, Def. 1.4]. For our purposes, we will mainly consider hereditarily rational functions over central loci, in the sense of the following definition. Note that one has to be precocious in such a definition because the central locus is just a semialgebraic set in general.

For a subset A of an algebraic set, we denote by $\overline{A}^Z$ the closure of A with respect to the Zariski topology. We introduce a notion of centrality relative to a given algebraic set.

Definition 1.7. Let X be a real algebraic set and V be an irreducible algebraic subset of X . We say that V is central in X if the Zariski closure of $V \cap \text{Cent } X$ is equal to V .

The relative centrality of V in X assures that V intersects sufficiently the central locus of X . This is the case for example for the stick of the Whitney umbrella (i.e., the z -axis in the surface of $\mathbb{R}^3$ defined by $x^2 = y^2z$) while it is not the case for the stick of the Cartan umbrella (i.e., the z -axis in the surface of $\mathbb{R}^3$ defined by $x^3 = (x^2 + y^2)z$). Remark that any real algebraic subset $X \subset \mathbb{R}^n$ is central in $\mathbb{R}^n$ even if X is not central by itself, take for example a plane cubic with an isolated point.

Definition 1.8. Let X be an algebraic set. A rational function f on X is hereditarily rational on $\text{Cent } X$ if $f \in \mathcal{K}^0(\text{Cent } X)$ and, for every irreducible algebraic subset V

central in X , the restriction $f|_{V \cap \text{Cent } X}$ is *rational on V* in the following sense: there exist a Zariski-dense open subset W of V and a rational function g on V which is regular on W , such that the restrictions of $f|_{V \cap \text{Cent } X}$ and g to $\text{Cent } X \cap W$ coincide.

We denote by $\mathcal{R}^0(\text{Cent } X)$ the ring of hereditarily rational functions on $\text{Cent } X$.

A map $\text{Cent } Y \rightarrow \text{Cent } X$ between the central loci of real algebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is called hereditarily rational if its components are hereditarily rational functions on $\text{Cent } Y$.

The relative centrality behaves well with respect to irreducible components.

Lemma 1.9. *Let X be an algebraic set, and V be central in X . There exists an irreducible component Y of X such that $V \subset Y$ and V is central in Y .*

Proof. The relative centrality of V in X implies that the semialgebraic dimension of V and of $V \cap \text{Cent } X$ coincide. In particular there exists an irreducible component Y of X such that $\dim V = \dim(V \cap \text{Cent } Y)$. Moreover $V \subset Y$ by irreducibility of V , and finally $V \cap \text{Cent } Y$ is Zariski dense in V . $\square$

In the case of curves, the rings $\mathcal{K}^0(\text{Cent } X)$ and $\mathcal{R}^0(\text{Cent } X)$ coincide. For non-singular algebraic sets, it is known that any continuous rational function is also hereditarily rational [17], and this is even the case in the presence of isolated singularities [26].

Remark that the canonical map $\mathcal{R}^0(\text{Cent } X) \rightarrow \mathcal{K}(X)$ is again injective. Moreover, if X has irreducible components $X_1, \dots, X_t$, one gets, a natural injective morphism

$$\mathcal{R}^0(\text{Cent } X) \rightarrow \mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t).$$

Actually one can even replace $\mathcal{K}^0$ by $\mathcal{R}^0$ in the image and describe the image as follows.

Proposition 1.10. *Let X be an algebraic set with irreducible components $X_1, \dots, X_t$. Then, $\mathcal{R}^0(\text{Cent } X)$ is canonically isomorphic to the subring of $\mathcal{R}^0(\text{Cent } X_1) \times \dots \times \mathcal{R}^0(\text{Cent } X_t)$ consisting of t -tuples $(f_1, \dots, f_t)$ such that f_i and f_j coincide on the intersection $\text{Cent } X_i \cap \text{Cent } X_j$ for all $i, j \in \{1, \dots, t\}$.*

Proof. Denote by A the subring of $\mathcal{R}^0(\text{Cent } X_1) \times \dots \times \mathcal{R}^0(\text{Cent } X_t)$ consisting of t -tuples $(f_1, \dots, f_t)$ such that f_i and f_j coincide on the intersection $\text{Cent } X_i \cap \text{Cent } X_j$ for all $i, j \in \{1, \dots, t\}$. By Proposition 1.6, we have an injective morphism

$$\mathcal{K}^0(\text{Cent } X) \rightarrow \mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t), \quad f \mapsto (f|_{\text{Cent } X_1}, \dots, f|_{\text{Cent } X_t}).$$

Let $f \in \mathcal{R}^0(\text{Cent } X)$. Let V_i be an algebraic subset central in X_i and denote the restriction $f|_{\text{Cent } X_i}$ by f_i . Note that V_i is central in X since $\text{Cent } X_i \subset \text{Cent } X$ and thus $f|_{V_i \cap \text{Cent } X}$ is rational on V_i because f is hereditarily rational. It implies that $f|_{V_i \cap \text{Cent } X_i}$ is also rational on V_i . Since $(f_i)|_{V_i \cap \text{Cent } X_i} = f|_{V_i \cap \text{Cent } X_i}$ then it follows that $(f_i)|_{V_i \cap \text{Cent } X_i}$ is also rational on V_i . So $f_i \in \mathcal{R}^0(\text{Cent } X_i)$ and we get indeed an injective morphism $\mathcal{R}^0(\text{Cent } X) \rightarrow A$.

To prove the surjectivity, we have to check that, given $(f_1, \dots, f_t)$ in A , the induced continuous rational function f on $\text{Cent } X$ is still hereditarily rational. Let V be an irreducible algebraic subset central in X . There exists an i such that $V \cap \text{Cent } X_i$ is Zariski dense in V by Lemma 1.9, therefore $f|_{V \cap \text{Cent } X_i} = (f_i)|_{V \cap \text{Cent } X_i}$ is rational on V . As a consequence $f|_{V \cap \text{Cent } X}$ is also rational on V , so that f is hereditarily rational on $\text{Cent } X$. $\square$

2. Some topological properties of integral morphisms

In real algebraic geometry, it is common to use various topologies, like the Zariski topology or the Euclidean topology. When dealing with algebra, the same situation appears, and in this section we study topological properties of integral morphisms with respect to the Zariski topology, the topology of the real spectrum, and the real Zariski topology.

The aim of this section is to deal with the general algebraic properties of integral ring homomorphisms between two reduced rings with same total ring of fractions. The results will be applied in the geometric settings in the following sections.

From now on, all our rings will contain $\mathbb{Q}$.

2.1. Several topologies on a ring

Our main interest is the study of the real Zariski topology which can be seen as the real part of the classical Zariski topology. We also introduce the real spectrum topology since it has been intensively studied in the literature and hence it provides some tools to study the real Zariski topology.

Zariski topology

Let A be a commutative ring and denote by $\text{Spec } A$ the Zariski spectrum of A , i.e. the set of all prime ideals of A . The set $\text{Spec } A$ can be endowed with the Zariski topology whose basis of open subsets is given by the sets $D(a) = \{\mathfrak{p} \in \text{Spec } A \mid a \notin \mathfrak{p}\}$ for $a \in A$. The closed subsets are given by the sets $V(I) = \{\mathfrak{p} \in \text{Spec } A \mid I \subset \mathfrak{p}\}$ where I is an ideal of A .

Let us denote $\text{Max } A \subset \text{Spec } A$ the subset of all maximal ideals of A .

Real spectrum topology

To a commutative ring A one may also associate a topological subspace $\text{Spec}_r A$ which takes into account only prime ideals $\mathfrak{p}$ whose residual field admits an ordering. Let us detail this construction a bit.

An order α in A is given by a real prime ideal $\mathfrak{p}$ of A (called the support of α and denoted by $\text{supp}(\alpha)$) and an ordering on the residue field $k(\mathfrak{p})$ at $\mathfrak{p}$. An order can equivalently be given by a morphism ϕ from $A/\mathfrak{p}$ to a real closed field.

One has a natural support mapping $\text{Spec}_r A \rightarrow \text{Spec } A$ which sends α to $\text{supp}(\alpha)$.

The value $a(\alpha)$ of $a \in A$ at the ordering α is just $\phi(a)$. The set of orders of A is called the real spectrum of A and denoted by $\text{Spec}_r A$. It is empty if and only if -1 is a sum of squares in A . One endows $\text{Spec}_r A$ with a natural topology whose open subsets are generated by the sets $\{\alpha \in \text{Spec}_r A \mid a(\alpha) > 0\}$ where $a \in A$. Let α, α' be two points of $\text{Spec}_r A$, then we say that α is a specialization of α' if α is in the closure of the singleton $\{\alpha'\}$. We denote this property by $\alpha' \rightarrow \alpha$.

For more details on the real spectrum, the reader is referred to [6].

Real Zariski topology

We also consider the set $\text{R-Spec } A$ which is just the image of the support mapping, namely it consists of all the real prime ideals of A . We endow it with the induced Zariski topology.

We set $D_R(a) = D(a) \cap (\text{R-Spec } A)$ and $V_R(I) = V(I) \cap (\text{R-Spec } A)$.

Then, the closed subsets of $\text{R-Spec } A$ have the form $V_R(I)$ where I is an ideal of A and a basis of open subsets is given by the subsets $D_R(a)$ for $a \in A$.

Functoriality

Let $\phi : A \rightarrow B$ be a ring morphism. It canonically induces a map $\psi : \text{Spec } B \rightarrow \text{Spec } A$ which is continuous for the Zariski topology.

It also induces a map $\psi_r : \text{Spec}_r B \rightarrow \text{Spec}_r A$ which is continuous for the real spectrum topology.

And also:

Proposition 2.1. *The morphism $\phi : A \rightarrow B$ induces a map $\psi_r : \text{R-Spec } B \rightarrow \text{R-Spec } A$ which is continuous for the real Zariski topology.*

Proof. Let us see first that this is a well-defined map. Indeed, let $\mathfrak{q} \in \text{R-Spec } B$ and $\mathfrak{p} = \psi(\mathfrak{q})$. Then, there exists an ordering on $k(\mathfrak{q})$ that one may define by giving a morphism $B/\mathfrak{q} \rightarrow R$ into a real closed field R . Hence, one gets the following commutative diagram:

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ A/\mathfrak{p} & \rightarrow & B/\mathfrak{q} \rightarrow R \end{array}$$

which defines an ordering on $k(\mathfrak{p})$ and hence $\mathfrak{p}$ is a real prime ideal.

The continuity comes from the following sequence of equalities:

$$\begin{aligned} \psi_r^{-1}(D_R(a)) &= \psi_r^{-1}(D(a) \cap \text{R-Spec } A) = \psi^{-1}(D(a) \cap \text{R-Spec } A) \cap \text{R-Spec } B = \\ &= \psi^{-1}(D(a)) \cap \psi^{-1}(\text{R-Spec } A) \cap \text{R-Spec } B = D(\phi(a)) \cap \text{R-Spec } B = D_R(\phi(a)). \end{aligned}$$

□

From now on, we will deal with ring extensions, namely ϕ will be injective.

2.2. Lying over and going-up

Definition 2.2. We say that a ring extension $\phi : A \rightarrow B$ satisfies the lying-over property if ψ is surjective. Likewise, we say that ϕ satisfies the real lying-over property if ψ_R is surjective.

Recall, for instance from [25, Theorem 9.3], that an integral ring extension $\phi : A \rightarrow B$ satisfies the lying-over property, and ψ induces a map from $\text{Max} B$ to $\text{Max} A$ which is surjective.

One has also, induced by ψ_R , a map from $R - \text{Max} B$ to $R - \text{Max} A$ but the real counterpart of the last property is false in general, namely ψ_R is not necessarily surjective. For instance, the normalization morphism is surjective for complex algebraic sets but this is no longer the case for real algebraic sets, as illustrated by the example of the cubic with an isolated singularity $y^2 - x^2(x - 1) = 0$. Indeed, its normalization has only complex points over the isolated point. The same example says also that the lying-over property does not imply the real lying-over property.

Definition 2.3. We say that a ring extension $\phi : A \rightarrow B$ satisfies the going-up property if, for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ in $\text{Spec} A$ and a prime ideal $\mathfrak{q} \in \text{Spec} B$ lying-over $\mathfrak{p}$, there exists a prime ideal $\mathfrak{q}' \in \text{Spec} B$ over $\mathfrak{p}'$ and such that $\mathfrak{q} \subset \mathfrak{q}'$.

The going-up property is stronger than the lying-over property: it is obvious in the case where A and B are domains and it follows from a theorem by Kaplansky in full generality. An integral ring extension $\phi : A \rightarrow B$ satisfies the going-up property (cf. [25, Theorem 9.4] for instance).

Note moreover that if a ring extension $\phi : A \rightarrow B$ is integral, then ψ is a closed mapping with respect to the Zariski topology. The real counterpart of this fact is false, and this is one motivation to consider a real going-up property for the real spectrum.

Definition 2.4. We say that a ring extension $\phi : A \rightarrow B$ satisfies the going-up property for the real spectrum if, for any pair of points $\alpha, \alpha' \in \text{Spec}_r A$ such that $\alpha' \rightarrow \alpha$ and a point $\beta' \in \text{Spec}_r B$ lying-over α' , there exists a point $\beta \in \text{Spec}_r B$ over α and such that $\beta' \rightarrow \beta$.

We recall from [2, Chapter 2, Propositions 4.2 and 4.3]:

Proposition 2.5. Assume that the ring extension $\phi : A \rightarrow B$ is integral. Then, ϕ satisfies the real going-up property for the real spectrum and ψ_r is a closed mapping.

Likewise, one may define a going-up property for real prime ideals. Looking at the normalization of a non-central irreducible real algebraic curve, we see that integral extensions do not necessarily satisfy the real going-up property since they do not necessarily satisfy the real lying-over property. This leads us to restrict ourselves to central loci in the sequel.

2.3. Central real lying-over

Let X be an algebraic set. Then $\mathcal{P}(X)$ and $\mathcal{O}(X)$ are both reduced rings with a finite number of minimal prime ideals that are all real ideals. To generalize to an abstract setting, we say that a ring A satisfies the condition **(MP)** if A is reduced with a finite number of minimal primes that are all real ideals.

Before giving the definition of the central locus of a ring, let us recall that, for a domain A with fraction field K , one can see $\text{Spec}_r K$ as the subset of orders of $\text{Spec}_r A$ whose support is (0) . Likewise, when A is satisfying **(MP)** with total fraction field K , one can see $\text{Spec}_r K$ as the subset of orders of $\text{Spec}_r A$ whose support is a minimal prime ideal of A .

Definition 2.6. Let A be a ring satisfying **(MP)** with total ring of fractions K . We define the central locus of A , denoted by $\text{Cent } A$, to be the set of all points in $\text{Spec}_r A$ which belong to the closure of $\text{Spec}_r K$. We say that A is a central ring if $\text{Cent } A = \text{Spec}_r A$. We denote by $\text{R-Cent } A$ the subset of $\text{R-Spec } A$ given by all supports of points in $\text{Cent } A$.

An illustrative example is given by the polynomial ring of the Whitney umbrella $X = \mathcal{Z}(x^2 - y^2z)$, for which the ideal (x, y) defining the stick of the umbrella belongs to $\text{R-Cent } \mathcal{P}(X)$, even if the stick is only half-contained in the central locus of X . Actually, consider the semialgebraic half-curve γ given by $t \mapsto (0, 0, t)$ for positive values of t , which is contained in the half-stick inside the central locus of X . This half-curve γ defines an element α of the real spectrum of $\mathcal{P}(X)$ (whose support has zero set the line supporting γ and ordering is the set of all one variable polynomials which are non negative for small $t \geq 0$). Since γ is included in the central locus, we can draw a two-dimensional semialgebraic sheet T on the umbrella containing γ in its closure. This sheet may be associated to a point β of the real spectrum of K whose support is the nil ideal and whose ordering is the set of all polynomials which are non negative for small $(x, y, t) \in T$. As a consequence the point β specializes to α and hence α lies in $\text{Cent } \mathcal{P}(X)$.

Notice for the sequel that if $\mathfrak{p}$ is a minimal prime ideal of a ring A satisfying **(MP)**, then $\mathfrak{p}$ belongs to $\text{R-Cent } A$.

In the geometric setting, one has already defined the notion of central locus of an algebraic set (Definition 1.3). Of course, when X is an algebraic set, then X is a central set if and only if $\mathcal{P}(X)$ is a central ring (as it can be deduced from [6, Propositions 7.6.2, 7.6.4]). For instance, if X is a cubic with an isolated point, then X is not a central set. Since there does not exist any arc included in the one-dimensional component with origin the isolated point, the ring $\mathcal{P}(X)$ is not central neither. In fact, one gets a little bit more: for any algebraic set X , the maximal ideals of $\mathcal{P}(X)$ that lie in the central locus can be associated to points in $\text{Cent } X$.

The real going-up for the real spectrum recalled in Proposition 2.5 implies a real lying-over with respect to the central loci. We first state a version for domains:

Proposition 2.7. *Let $\phi : A \rightarrow B$ be an integral injective morphism of domains. Then:*

- (1) *The morphism ψ_R induces a mapping from $\text{R-Cent } B$ to $\text{R-Cent } A$;*

- (2) *If moreover A and B have same fraction field, then ψ_R induces a surjective mapping from $\text{R-Cent } B$ onto $\text{R-Cent } A$.*

Proof. Property (1) comes from the continuity of ψ_r . Namely, let us start with $\mathfrak{q} \in \text{R-Cent } B$. There is an ordering $\beta \in \text{Spec}_r B$ whose support is $\mathfrak{q}$ and moreover there is a specialization $\beta' \rightarrow \beta$ in $\text{Spec}_r B$ such that the support of β' is the nil ideal. Taking inverse images, one gets a specialization $\alpha' \rightarrow \alpha$ in $\text{Spec}_r A$. By injectivity of ϕ , the support of α' is also the nil ideal and hence, the support of α lies in $\text{R-Cent } A$. It follows that $\psi_R(\mathfrak{q}) = \text{supp}(\alpha) \in \text{R-Cent } A$.

Let us prove (2). Let $\mathfrak{p}' \in \text{R-Cent } A$. So $\mathfrak{p}'$ is the support of $\alpha' \in \text{Spec}_r A$ and moreover there exists $\alpha \in \text{Spec}_r K \cap \text{Spec}_r A$ such that $\alpha \rightarrow \alpha'$.

Since A and B have same fraction field K , there is $\beta \in \text{Spec}_r B \cap \text{Spec}_r K$ over α . By the real going-up for the real spectrum (Proposition 2.5), one deduces the existence of $\beta' \in \text{Spec}_r B$ over α' such that $\beta \rightarrow \beta'$ and hence $\beta' \in \text{Cent } B$. It implies also that the support of β' is a real prime ideal $\mathfrak{q}' \in \text{R-Cent } B$ lying-over $\mathfrak{p}'$. $\square$

Note that property (2) remains valid as soon as any ordering on the total ring of fractions of A does extend to an ordering on the total ring of fractions of B . This is the case, for instance, if the fraction fields extension has odd degree.

Now, we come to the reducible case. Recall first that ([25, Theorem 9.3 (ii)]), given an integral extension $A \rightarrow B$, a prime ideal of B lying-over a minimal prime ideal of A is a minimal prime ideal of B . Note also that a minimal prime ideal of B does not necessarily lies over a minimal prime of A .

Naturally, we will moreover assume that our rings have real minimal primes, namely that they satisfy **(MP)**. One main difference with Proposition 2.7 for domains is that we do not necessarily get an induced morphism $\text{R-Cent } B \rightarrow \text{R-Cent } A$. Take the example for A of the coordinate ring of a cubic curve with isolated point and $B = A \times A/\mathfrak{p}$ where $\mathfrak{p}$ is the maximal ideal corresponding to the isolated point.

Nevertheless, one gets such a morphism when A and B have same total ring of fractions. This context will be sufficient for us in the paper.

Proposition 2.8. *Let $\phi : A \rightarrow B$ be an integral injective morphism of rings satisfying **(MP)**. If A and B have same total ring of fractions, then the morphism ψ_R induces a mapping from $\text{R-Cent } B$ to $\text{R-Cent } A$ and this mapping is surjective.*

Proof. It suffices to argue on each component and use Proposition 2.7.

Namely, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ all the minimal primes of A and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ all the minimal primes of B . Since A and B have same total field of fractions, one gets $r = s$ and, up to re-indexation, $A/\mathfrak{p}_i \rightarrow B/\mathfrak{p}_i$ is an integral extension of domains having the same fraction field.

As a particular consequence, any real minimal prime in B lies over a real minimal prime in A . $\square$

Our standard geometric setting will be when $A = \mathcal{P}(X)$ and $B = \mathcal{P}(Y)$ are the polynomial rings of two given algebraic subsets X and Y together with a birational polynomial morphism $Y \rightarrow X$.

The fact that the maximal ideals of $\mathcal{P}(X)$ that lie in the central locus can be associated to points in $\text{Cent } X$ can be recovered by the following key lemma which relates the geometric and algebraic notions of centrality. Let us emphasize that its proof is really of a semi-algebraic nature, generalizing the example of the Whitney umbrella exposed above.

Lemma 2.9. *Let $V \subset X \subset \mathbb{R}^n$ be two algebraic sets where $V = \mathcal{Z}(\mathfrak{p})$ with $\mathfrak{p}$ a non-zero real prime ideal of $\mathcal{P}(X)$. Then, $\mathfrak{p} \in \text{R-Cent } \mathcal{P}(X)$ if and only if $\mathcal{Z}(\mathfrak{p})$ is central in X .*

Proof. Let us assume first that $V = \mathcal{Z}(\mathfrak{p})$ is central in X . By Lemma 1.9, we may assume X is irreducible. Note that then V is also the Zariski closure of $\text{Cent } V \cap \text{Cent } X$. Set $T = \text{Cent } X$ and $S = \text{Cent } V \cap \text{Cent } X$. Note that S and T are two closed semi-algebraic subsets of X . Our aim is to exhibit two orderings α and β respectively represented as ultrafilters in S and T and such that α is a specialization of β . To do so, we refer to the description of orderings in $\mathbb{R}[x_1, \dots, x_n]$ as ultrafilters of semi-algebraic sets given in [6, Proposition 7.2.4 and Remark 7.5.5].

Let x be an arbitrary point in S . The question being local and semi-algebraic, up to a semi-algebraic triangulation ([6, Theorem 9.2.1]), one may assume that there is a semi-algebraic neighborhood U of x which can be taken to be the origin of $\mathbb{R}^n$, S contains $U \cap ((\mathbb{R}^+)^{\dim V} \times 0)$ and T contains $U \cap (\mathbb{R}^+)^{\dim X}$. It is then classical to construct an ordering α whose support is $\mathfrak{p}$ and an ordering β whose support is (0) such that α specializes β (which itself specializes to x). It shows that $\mathfrak{p} \in \text{R-Cent } \mathcal{P}(X)$.

Let us assume now that $\mathfrak{p} \in \text{R-Cent } \mathcal{P}(X)$. Take α an ordering whose support is $\mathfrak{p}$ and β another ordering whose support is a minimal prime ideal $\mathfrak{q}$ of $\mathcal{P}(X)$ and such that β specializes into α . Replacing X by the irreducible component of X corresponding to $\mathfrak{q}$ then we may assume X is irreducible and $\mathfrak{q} = (0)$ and it is clearly sufficient to consider that situation. Note first that α specializes to a maximal point γ of the real spectrum $\mathcal{P}(X)$ but this γ does not necessarily correspond to a geometric point, for instance γ could corresponds to a branch going to infinity. Nevertheless, one may use the same arguments as previously. Thinking at points of the real spectrum of $\mathcal{P}(X)$ as ultrafilters of semi-algebraic subsets in X (see again [6, Proposition 7.2.4 and Remark 7.5.5]), there are semi-algebraic subsets $A \subset V$ and $B \subset \text{Cent } X$ representing α and β such that $\overline{A}^Z = V$ and $A \subset B$. As a consequence

$$V = \overline{A}^Z \subset \overline{V \cap \text{Cent } X}^Z \subset V$$

therefore V is central in X as required. $\square$

Definition 2.10. Let $\pi : Y \rightarrow X$ be a finite birational morphism and $V \subset X$ be an irreducible algebraic subset. We say that an irreducible algebraic subset W of Y lies over V if $\mathcal{I}(W)$ lies over $\mathcal{I}(V)$ in the integral extension $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$.

The following lemma states that for a finite birational morphism with target X then, generically, the fibers over points in $V \cap \text{Cent } X$, for V central in X , are

contained in the union of the central sets lying-over V . This result will be crucial for the proof of two important results, Proposition 3.13 and Theorem 4.21.

Lemma 2.11. *Let $\pi : Y \rightarrow X$ be a finite birational morphism and $V \subset X$ be an irreducible algebraic set central in X . Let $W_1, \dots, W_r$ be the irreducible algebraic subsets of Y lying-over V and central in Y . There exists an algebraic subset $Z \subset V$ with $\dim Z < \dim V$ such that the following inclusion holds:*

$$\pi^{-1}((V \setminus Z) \cap \text{Cent } X) \cap \text{Cent } Y \subset W_1 \cup \dots \cup W_r.$$

Proof. Consider the decomposition of the algebraic subset $\pi^{-1}(V)$ of Y into irreducible components. Let W be one of this components. Then $\dim W \leq \dim V$ by finiteness of π , and assume that the dimension of the semialgebraic set $W \cap \text{Cent } Y$ is $\dim V$. Then this dimension is equal to $\dim W$ too, so that W is central in Y by Lemma 2.9. In particular W is equal to one of the W_i , with $i \in \{1, \dots, r\}$.

Denote by W' the union of the irreducible components of $\pi^{-1}(V)$ different from $W_1, \dots, W_r$. Then we have just proven that

$$\dim W' \cap \text{Cent } Y < \dim V.$$

Thus the algebraic subset Z of V defined by

$$Z = \overline{\pi(W' \cap \text{Cent } Y)}^Z$$

satisfies the required conditions. □

Example 2.12. Consider the normalization $X' = \mathcal{Z}(\tau^2 - (1 + x^2)) \subset \mathbb{R}^3$ of the surface defined by $X = \mathcal{Z}(\tau^2 - (1 + x^2)(x^2 + y^2)^2)$ in $\mathbb{R}^3$. Note that X together with X' are central, X has a unique singular point at the origin, and that $\pi : X' \rightarrow X$ is defined by $(x, y, \tau) \mapsto (x, y, \tau(x^2 + y^2))$.

There is a unique irreducible curve W lying-over the irreducible curve $V \subset X$ defined by $x = 0$ and $\tau = y^2$, and W is the line given by $x = 0$ and $\tau = 1$. However the preimage of the origin in V consists of the points $(0, 0, 1)$ and $(0, 0, -1)$, with $(0, 0, -1)$ that does not belong to W . Here Z consists of the origin in V .

To end this section let us note that from Proposition 2.8 and Lemma 2.9, we deduce some central lying-over properties for integral extensions of geometric rings.

Proposition 2.13. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. Then the mapping $\text{R-Cent } \mathcal{P}(Y) \rightarrow \text{R-Cent } \mathcal{P}(X)$ is surjective. When X is central, $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the real lying-over property.*

In the forthcoming subsection, we come back to the algebraic setting and study a property stronger than the central lying-over.

2.4. Central subintegral extensions

Recall [33] that an integral extension $A \rightarrow B$ is said subintegral if for any prime ideal $\mathfrak{p} \in \text{Spec } A$, there exists a unique prime ideal $\mathfrak{q} \in \text{Spec } B$ lying-over $\mathfrak{p}$ and, moreover, the induced injective map on the residue fields $k(\mathfrak{p}) \rightarrow k(\mathfrak{q})$ is an isomorphism.

One natural way to define a real subintegral extension $A \rightarrow B$ would be: it is an integral extension and given any real prime ideal $\mathfrak{p}$ of A , there exists a unique real prime ideal $\mathfrak{q}$ of B lying-over $\mathfrak{p}$ and the induced map on the residue fields $k(\mathfrak{p}) \rightarrow k(\mathfrak{q})$ is an isomorphism. Due to centrality issues, we have to take into account central loci also and this leads to the following:

Definition 2.14. Let $A \rightarrow B$ be an integral extension of rings satisfying **(MP)**.

- (1) We say that $A \rightarrow B$ is centrally weakly subintegral (w_c -subintegral for short) if, given any real maximal ideal $\mathfrak{p} \in \text{R-Cent } A \cap \text{Max } A$, there exists a unique real maximal $\mathfrak{q} \in \text{R-Cent } B \cap \text{Max } B$ lying-over $\mathfrak{p}$ and the induced injective map on the residue fields $k(\mathfrak{p}) \rightarrow k(\mathfrak{q})$ is an isomorphism;
- (2) We say that $A \rightarrow B$ is centrally subintegral (s_c -subintegral for short) if, given any real prime ideal $\mathfrak{p} \in \text{R-Cent } A$, there exists a unique real prime $\mathfrak{q} \in \text{R-Cent } B$ lying-over $\mathfrak{p}$ and the induced injective map on the residue fields $k(\mathfrak{p}) \rightarrow k(\mathfrak{q})$ is an isomorphism.

The definitions of w_c and s_c -subintegral extensions will lead to the notion of weak-normalization and seminormalization relative to the central loci introduced in 4. It is worth mentioning that replacing prime ideals with maximal ideals in the definition of subintegral extension gives rise to the same notion, at least for geometric rings. On the contrary, if a s_c -subintegral extension is indeed a w_c -subintegral extension, the converse is false in general. Example 3.9 illustrates for instance that we may keep a bijection at the level of prime ideals, but loosing the equiresiduality condition. Example 3.11 shows that we may loose the bijection between central real prime ideals too.

Centrally (weakly) subintegral extensions are stable under composition:

Proposition 2.15. Let $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ be two integral extensions of rings. Assume either that A, B, C are domains, or A, B, C satisfy **(MP)** and have same total ring of fractions. One has:

- (1) If ϕ and ψ are both w_c -subintegral, then $\psi \circ \phi$ is also w_c -subintegral;
- (2) If ϕ and ψ are both s_c -subintegral, then $\psi \circ \phi$ is also s_c -subintegral.

Proof. The existence and the equiresiduality properties are clear by transitivity. The uniqueness property comes from Propositions 2.7.1) and 2.8. $\square$

In [12] has been introduced the concept of biregular extensions of rings; it happens that these extensions are examples of w_c and s_c -subintegral extensions. Let us explain this fact now.

Recall that the ring $\mathcal{O}(A)$ of regular fractions of elements in A is obtained from A by inverting all elements in $1 + \sum A^2$, and moreover that $\mathcal{O}(\mathcal{P}(X))$ coincides with the usual ring of regular functions when X is an algebraic set.

In case $\mathcal{O}(A) \rightarrow \mathcal{O}(B)$ is an isomorphism we say that the extension $A \rightarrow B$ is biregular.

When $A \rightarrow B$ is an integral extension of rings satisfying (MP), by [12, Proposition 4.11], we know that the extension $A \rightarrow B$ is biregular if and only if one of the following equivalent conditions holds:

- (1) Given any ideal $\mathfrak{m} \in R - \text{Max}A$, there exists a unique maximal $\mathfrak{m}' \in \text{Max}B$ lying-over $\mathfrak{m}$ and moreover $\mathfrak{m}'$ is real and the map $A_{\mathfrak{m}} \rightarrow B_{\mathfrak{m}'}$ is an isomorphism.
- (2) Given any real prime ideal $\mathfrak{p} \in R\text{-Spec } A$, there exists a unique prime $\mathfrak{q} \in \text{Spec } B$ lying-over $\mathfrak{p}$ and moreover $\mathfrak{q}$ is real and the map $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is an isomorphism.

It is clear from this characterization that:

Corollary 2.16. *Let $A \rightarrow B$ be a biregular integral extension of rings satisfying (MP). Then $A \rightarrow B$ is s_c -subintegral and w_c -subintegral.*

2.5. Weak-normalization and seminormalization relative to the central locus of a ring

In close relation with the notion of normalization, Traverso [31] has introduced the seminormalization of a ring A with integral closure denoted by A' , to be the ring

$${}^+A = \{f \in A' \mid \forall \mathfrak{p} \in \text{Spec } A, f_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(A'_{\mathfrak{p}})\}$$

where $f_{\mathfrak{p}}$ is the image of f by $A' \rightarrow A'_{\mathfrak{p}}$ and where $\text{Rad}(B)$ denotes the Jacobson radical of B , i.e., the intersection of all maximal ideals in the ring B . Notice that for a ring C and a multiplicative closed subset S of C , there is no confusion in writing C'_S since $(C_S)' \simeq (C')_S$ ([5, Proposition 5.12]).

Inspiring from [4], where only complex analytic varieties are considered, one may also define the weak-normalization of A as the ring

$$\{f \in A' \mid \forall \mathfrak{m} \in \text{Max}A, f_{\mathfrak{m}} \in A_{\mathfrak{m}} + \text{Rad}(A'_{\mathfrak{m}})\}.$$

Note that for finite type algebras over an algebraically closed field of characteristic zero, these two notions of semi and weak normalization coincide as it is shown in [22, Theorem 2.2] (when the characteristic is not zero see [33]).

We are going to define real counterparts of these two notions, relatively to the central locus of a ring; it is worth to advertise already that in our context, the two notions will be distinct.

We start with defining, for any ring A , the central radical $\text{Rad}^C A$ of A to be the intersection of all real maximal ideals $\mathfrak{m}$ which are also in $R\text{-Cent } A$. In case the

ring A is central, the central radical of A coincides with the real Jacobson radical of A , i.e., the intersection of all real maximal ideals.

Now, we particularly focus on properties of integral extensions of rings contained in the integral closure. Let A be a ring satisfying **(MP)** and with minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$. By Proposition 1.1, the total ring of fractions K of A is given by $K = \prod_{i=1}^t k(\mathfrak{p}_i)$ (where $k(\mathfrak{p}_i)$ denotes the residue field at $\mathfrak{p}_i$) and the integral closure A' of A is given by $\prod_{i=1}^t (A/\mathfrak{p}_i)'$. It follows that A' satisfies also **(MP)**, the minimal prime ideals of A' are in bijection with those of A and K is also the total ring of fractions of A' . The same properties remain for any intermediate ring between A and A' which yield:

Proposition 2.17. *Let $A \rightarrow B \rightarrow A'$ be a sequence of ring extensions where A satisfies **(MP)** and A' is the integral closure of A . Then, one has a surjective mapping from $\text{R-Cent } B$ to $\text{R-Cent } A$.*

Proof. According to [12, Lemma 2.8], B satisfies also **(MP)**, the minimal prime ideals of B and A are in bijection and $A \rightarrow B$ induces an isomorphism between the total rings of fractions. It suffices then to apply Proposition 2.8. $\square$

We prove that sub-extensions of (weakly) centrally subintegral extensions contained in the integral closure are (weakly) centrally subintegral.

Proposition 2.18. *Let $A \rightarrow C \rightarrow B \rightarrow A'$ be a sequence of ring extensions where A satisfies **(MP)** and A' is the integral closure of A . One has:*

- (1) *If $A \rightarrow B$ is w_c -subintegral then $A \rightarrow C$ is also w_c -subintegral;*
- (2) *If $A \rightarrow B$ is s_c -subintegral then $A \rightarrow C$ is also s_c -subintegral.*

Proof. Notice that the rings A , B , C and A' satisfy **(MP)** and have isomorphic total rings of fractions [12, Lemma 2.8]. Assume $A \rightarrow B$ is w_c -subintegral and let $\mathfrak{p} \in \text{R-Cent } A \cap \text{Max } A$. By Proposition 2.17 we know that the maps $\text{R-Cent } C \cap \text{Max } C \rightarrow \text{R-Cent } A \cap \text{Max } A$ and $\text{R-Cent } B \cap \text{Max } B \rightarrow \text{R-Cent } C \cap \text{Max } C$ are surjective and moreover $\text{R-Cent } B \cap \text{Max } B \rightarrow \text{R-Cent } A \cap \text{Max } A$ is bijective ($A \rightarrow B$ is w_c -subintegral). It follows that there exists a unique central maximal ideal of C lying-over $\mathfrak{p}$ and this ideal is $\mathfrak{q} \cap C$ where $\mathfrak{q}$ is the unique central maximal ideal of B lying-over $\mathfrak{p}$. We get a sequence of finite extensions of fields $k(\mathfrak{p}) \rightarrow k(\mathfrak{q} \cap C) \rightarrow k(\mathfrak{q})$ and since $k(\mathfrak{p})$ and $k(\mathfrak{q})$ are isomorphic then the proof of (1) is done. The proof of (2) is similar replacing maximal ideals by prime ideals. $\square$

Here is the central version of weak-normalization:

Definition 2.19. Let A be a ring which has an integral closure denoted by A' . The ring

$$A^{w_c} = \{f \in A' \mid \forall \mathfrak{m} \in \text{R-Cent } A \cap \text{Max } A, f_{\mathfrak{m}} \in A_{\mathfrak{m}} + \text{Rad}^C(A'_{\mathfrak{m}})\}$$

is called the weak-normalization of A relative to its central locus, or w_c -normalization for short. In case $A = A^{w_c}$, we say that A is centrally weakly-normal.

The weak-normalization relative to the central locus satisfies a universal property, that will be an important step for the study of the geometric w_c -normalization in the sequel.

Proposition 2.20. *Let A be a ring satisfying (MP) which has an integral closure denoted by A' . For an extension $A \rightarrow B$ which injects into A' the following properties are equivalent:*

- (i) *The extension $A \rightarrow B$ is w_c -subintegral;*
- (ii) *$B \subset A^{w_c}$.*

Proof. We begin by showing that $A \rightarrow A^{w_c}$ is w_c -subintegral. Let us mention first that, according to Proposition 2.17, one has a canonical surjection from $\text{R-Cent } A^{w_c} \cap \text{Max } A^{w_c}$ onto $\text{R-Cent } A \cap \text{Max } A$. To show the injectivity, let us consider q_1 and q_2 in $\text{R-Cent } A^{w_c} \cap \text{Max } A^{w_c}$ lying-over $\mathfrak{p} \in \text{R-Spec } A \cap \text{Max } A$. Using again Proposition 2.8, one has two central maximal ideals $\mathfrak{r}_1$ and $\mathfrak{r}_2$ in $\text{R-Cent } A' \cap \text{Max } A'$ lying-over q_1 and q_2 respectively. Let $f \in q_1$, then $f_{\mathfrak{p}} = \alpha + \beta$ where $\alpha \in A_{\mathfrak{p}}$ and $\beta \in \text{Rad}^C A'_{\mathfrak{p}}$. Then, $\beta \in \mathfrak{r}_1 \cap \mathfrak{r}_2$ and hence $\beta \in q_2(A^{w_c})_{\mathfrak{p}}$ and $\alpha \in \mathfrak{p}A_{\mathfrak{p}}$. We get then $f \in q_2(A^{w_c})_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ in A (if $\mathfrak{m} \neq \mathfrak{p}$, one has $q_2(A^{w_c})_{\mathfrak{m}} = (A^{w_c})_{\mathfrak{m}}$). This shows (using for instance [25, Theorem 4.6]) that $f \in q_2$ and hence $q_1 \subset q_2$. By symmetry $q_1 = q_2$.

It remains to show that A^{w_c} is equiresidual over A at any central maximal ideal $\mathfrak{m}$ in A . Let us consider the ideal $\mathfrak{m}^{w_c} = (\text{Rad}^C A'_{\mathfrak{m}}) \cap (A^{w_c})_{\mathfrak{m}}$. Using the natural identification between the prime ideals in A' lying-over $\mathfrak{m}$ and the maximal ideals of $A'_{\mathfrak{m}}$ lying-over $\mathfrak{m}A_{\mathfrak{m}}$, it is clear that, over $\mathfrak{m}^{w_c}$ in $A'_{\mathfrak{m}}$ lie all the central maximal ideals of $A'_{\mathfrak{m}}$ and hence $\mathfrak{m}^{w_c}$ is maximal and central. Moreover $\mathfrak{m}^{w_c}$ lies over $\mathfrak{m}A_{\mathfrak{m}}$. Then, $\mathfrak{m}^{w_c}$ is the only central real maximal ideal of $(A^{w_c})_{\mathfrak{m}}$ lying-over $\mathfrak{m}A_{\mathfrak{m}}$. Moreover, one clearly has $A/\mathfrak{m} \simeq (A^{w_c})_{\mathfrak{m}}/\mathfrak{m}^{w_c}$ and thus $A \rightarrow A^{w_c}$ is w_c -subintegral.

Consider an extension $A \rightarrow B$ which injects into A' . If $B \subset A^{w_c}$ then $A \rightarrow B$ is w_c -subintegral by 1) of Proposition 2.18. It follows that (ii) implies (i).

Let us show now that (i) implies (ii). Assume $A \rightarrow B$ is w_c -subintegral and $B \subset A'$. By definition of A^{w_c} , one has to show that $B_{\mathfrak{p}} \subset A_{\mathfrak{p}} + \text{Rad}^C A'_{\mathfrak{p}}$ for any maximal $\mathfrak{p}$ in $\text{R-Cent } A$. Let us denote by q the unique maximal ideal in $\text{R-Cent } B$ lying-over $\mathfrak{p}$. By Proposition 2.17, one has $qA'_{\mathfrak{p}} \subset \text{Rad}^C A'_{\mathfrak{p}}$ which shows that $qB_{\mathfrak{p}} \subset \text{Rad}^C A'_{\mathfrak{p}}$. We use the following commutative diagram

$$\begin{array}{ccc} k(\mathfrak{p}) & \simeq & k(q) \\ \uparrow & & \uparrow \\ A_{\mathfrak{p}} & \rightarrow & B_{\mathfrak{p}}. \end{array}$$

Let $b \in B$. There exist $a \in A$ and $s \in A \setminus \mathfrak{p}$ such that $a/s = b$ in $k(q)$. Hence $a - sb \in q$ and thus $b - a/s \in qB_{\mathfrak{p}} = \ker(B_{\mathfrak{p}} \rightarrow k(q))$. We get $b \in A_{\mathfrak{p}} + \text{Rad}^C A'_{\mathfrak{p}}$. This concludes the proof. $\square$

Using Propositions 2.15 and 2.20, one readily derives an idempotency property:

Proposition 2.21. *For any ring A satisfying (MP), the ring A^{wc} is centrally weakly normal.*

We introduce now the concept of seminormalization relative to the central locus.

Definition 2.22. Let A be a ring which has an integral closure denoted by A' . The ring

$$A^{\text{sc}} = \{f \in A' \mid \forall \mathfrak{p} \in \text{R-Cent } A, f_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}^C A'_{\mathfrak{p}}\}$$

is called the seminormalization of A relative to its central locus, or s_c -normalization for short. In case $A = A^{\text{sc}}$, we say that A is centrally seminormal.

The seminormalization relative to the central locus satisfies a universal property in the same spirit as the weak normalization to the central locus, and the proof is similar as that of Proposition 2.20 (it suffices to replace maximal ideals with prime ideals). It will be an important result for the geometric s_c -normalization in the sequel.

Proposition 2.23. *Let A be a ring satisfying (MP) which has an integral closure denoted by A' . For an extension $A \rightarrow B$ which injects into A' the following properties are equivalent:*

- (i) *The extension $A \rightarrow B$ is s_c -subintegral;*
- (ii) *$B \subset A^{\text{sc}}$.*

Remark 2.24. We have a sequence of inclusions

$$A \subset {}^+A \subset A^{\text{sc}} \subset A^{\text{wc}} \subset A',$$

so that a centrally weakly-normal ring is automatically centrally seminormal, and similarly a centrally seminormal ring is automatically seminormal.

Using Propositions 2.15, 2.21 and 2.23, one readily gets an idempotency property:

Proposition 2.25. *For any ring A satisfying (MP) one has:*

- (1) *A^{sc} is centrally seminormal;*
- (2) *$(A^{\text{wc}})^{\text{sc}} = A^{\text{wc}}$;*
- (3) *$(A^{\text{sc}})^{\text{wc}} = A^{\text{wc}}$.*

We end this section with a comparison with the biregular integral closure that has been defined in [12], in relation with the notion of biregular extension recalled in Section 2.4.

Let A be a ring satisfying (MP). The biregular integral closure A^b of A is the integral closure of A in $\mathcal{O}(A)$, namely $A^b = A'_{\mathcal{O}(A)}$. By [12, Theorem 4.12], A^b is

the biggest ring contained in A' which is a biregular extension of A . Since biregular extensions are s_c -subintegral by Corollary 2.16, we have a sequence of inclusions

$$A \subset A^b \subset A^{s_c} \subset A^{w_c} \subset A'.$$

It follows also that a centrally seminormal ring (which is automatically seminormal) is equal to its biregular integral closure.

3. Finite birational morphisms and continuous functions

This section is devoted to a topological study of finite birational morphisms in real geometry. We begin with considering Euclidean topology, and the restriction of the morphisms to the central loci. We study later the action of finite birational morphisms on continuous rational functions, respectively hereditarily continuous rational functions, on the central loci. The main result is Theorem 3.6 (and Theorem 3.14 for the hereditary case) which gives necessary and sufficient conditions on the restriction to the central loci to be an homeomorphism.

3.1. Properties of finite birational morphisms

Recall that a birational morphism $\pi : Y \rightarrow X$ between algebraic sets is a polynomial morphism which is a birational map i.e that induces an isomorphism from a Zariski-dense open subset of Y to a Zariski-dense open subset of X . Equivalently, π is birational if and only if π induces an isomorphism $\mathcal{K}(X) \simeq \mathcal{K}(Y)$ at the level of the total rings of fractions. Such a map π is defined everywhere, but its inverse is only rational and may have indeterminacy points. Recall that π is finite if the associated ring morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ makes $\mathcal{P}(Y)$ a finite $\mathcal{P}(X)$ -module, a property which implies that the ring extension $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is integral.

Lemma 3.1. *Let $\pi : Y \rightarrow X$ be a birational morphism between algebraic sets. The induced ring morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is injective. If moreover π is finite, then the map π is proper (and hence closed) for the Euclidean topology.*

Proof. The ring morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is injective since π is birational and thus dominant.

Assume moreover that π is a finite birational morphism. We show that the map π is closed and proper with respect to the real spectrum topology, then with respect to the semi-algebraic topology and finally with respect to the Euclidean topology. By [2, Propositions 4.2-4.3], the induced map $\text{Spec}_r \mathcal{P}(Y) \rightarrow \text{Spec}_r \mathcal{P}(X)$ is closed for the real spectrum topology. According to [6, Theorem 7.2.3], there is a bijective correspondence between open (resp. closed) semi-algebraic subsets of X (resp. Y) and open (resp. closed) constructible subsets of the real spectrum $\text{Spec}_r \mathcal{P}(X)$ (resp. the real spectrum $\text{Spec}_r \mathcal{P}(Y)$). It follows that the image by π of every closed semi-algebraic subset of Y is a closed semi-algebraic subset of X . Now it is classical ([9] for instance) to conclude that π is closed for the Euclidean topology.

The morphism π being finite, it has compact fibers and hence is proper. $\square$

We will often consider intermediate algebraic sets between a given algebraic set and its normalization, in the sense of the next statement. Before recall that, given a ring extension $A_1 \rightarrow A_2$ (an injective ring homomorphism), the ring A is said to be an intermediate ring between A_1 and A_2 if one has a factorization $A_1 \rightarrow A \rightarrow A_2$ where the morphisms are extension rings.

Proposition 3.2. *Let X be an algebraic set and $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. Let A be an intermediate ring between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. There exists a unique algebraic set Z such that $A = \mathcal{P}(Z)$. Moreover the induced morphisms $Y \rightarrow Z$ and $Z \rightarrow X$ are finite and birational.*

When the conditions of Proposition 3.2 are satisfied, we say that Z is an intermediate algebraic set between X and Y .

Proof. Note that A is integral over $\mathcal{P}(X)$ since $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y)$ is an integral morphism. Moreover $\mathcal{P}(X) \subset A \subset \mathcal{K}(X)$ since π is birational and A is integral over $\mathcal{P}(X)$ therefore $A \subset \mathcal{P}(X')$. In particular A has $\mathcal{K}(X)$ as total ring of fractions [12, Lemma 2.8]. First note that $\mathcal{P}(X')$ is a finite $\mathcal{P}(X)$ -module (this property is known to be satisfied for the so-called Mori rings, a sup-class of finite-type $\mathbb{R}$ -algebras). Since A is integral over $\mathcal{P}(X)$ then A is a submodule of the Noetherian $\mathcal{P}(X)$ -module $\mathcal{P}(X')$ so that A is a finite $\mathcal{P}(X)$ -module. As a consequence A is a finitely generated $\mathbb{R}$ -algebra, so that there exists an ideal I of some $\mathbb{R}[X_1, \dots, X_m]$ such that A is isomorphic to $\mathbb{R}[X_1, \dots, X_m]/I$. Note that I is a real ideal since the total ring of fractions of A is $\mathcal{K}(X)$, so that A is the ring of polynomial functions of an algebraic set Z by the real Nullstellensatz (see Subsection 1.1). The induced morphisms $Y \rightarrow Z$ and $Z \rightarrow X$ are clearly finite by the above arguments. $\square$

As already mentioned a finite birational morphism is not necessarily surjective in general. This is the motivation to restrict our attention to the central loci.

Proposition 3.3. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. Then*

- (1) $\pi : \text{Cent } Y \rightarrow \text{Cent } X$ is well defined and surjective;
- (2) $\pi : \text{Cent } Y \rightarrow \text{Cent } X$ is a quotient map for the Euclidean topology.
- (3) The composition by π gives an isomorphism between the ring $\mathcal{K}^0(\text{Cent } X)$ and the subring of functions in $\mathcal{K}^0(\text{Cent } Y)$ that are constant on the fibers of $\pi : \text{Cent } Y \rightarrow \text{Cent } X$.

Proof. The fact that π maps surjectively $\text{Cent } Y$ onto $\text{Cent } X$ is given by Proposition 2.8.

By (1) and Lemma 3.1 the map π is continuous, surjective and a closed map for the Euclidean topology; this gives (2). It means that a map is continuous on $\text{Cent } X$ if and only if the composition by π is continuous on $\text{Cent } Y$. Consequently, for a continuous function $f : \text{Cent } Y \rightarrow \mathbb{R}$ constant on the fibers of π then there exists a (unique) continuous function $g : \text{Cent } X \rightarrow \mathbb{R}$ such that $f = g \circ \pi$. Using moreover that π is birational, we get (3). $\square$

Note that we cannot replace continuous rational functions by continuous hereditarily rational functions in the third point as illustrated by Kollár's surface. Note moreover that if X is central, this point says that the composition by π gives a injective ring morphism $\pi_0 : \mathcal{K}^0(X) \hookrightarrow \mathcal{K}^0(Y)$ whose image is the subring of functions in $\mathcal{K}^0(Y)$ that are constant on the fibers of π .

Given an algebraic set X , we investigate now the action of adding to $\mathcal{P}(X)$ a continuous rational function f which is integral over $\mathcal{P}(X)$. Forgetting first about the continuity, this is done as follows. The canonical morphism $\mathcal{P}(X) \rightarrow \mathcal{K}(X)$ factorizes through the morphism $\phi : \mathcal{P}(X)[t] \rightarrow \mathcal{K}(X)$ defined by $t \mapsto f$, inducing a morphism $\mathcal{P}(X)[t]/\text{Ker } \phi \rightarrow \mathcal{K}(X)$. Since f is integral, the ring homomorphism $\mathcal{P}(X) \rightarrow \mathcal{P}(X)[t]/\text{Ker } \phi$ is finite and $\mathcal{P}(X)[t]/\text{Ker } \phi$ is the coordinate ring $\mathcal{P}(Y)$ of an algebraic set Y . In this setting, f corresponds to the new variable t .

Taking into account the continuity, we obtain the following:

Proposition 3.4. *Let X be an algebraic set. Let f be a rational function on X that is integral on $\mathcal{P}(X)$ and assume that $f \in \mathcal{K}^0(\text{Cent } X)$. Denote by Y the algebraic set such that $\mathcal{P}(Y) = \mathcal{P}(X)[f]$, t the polynomial function in $\mathcal{P}(Y)$ that corresponds to f and $\pi : Y \rightarrow X$ the associated finite birational morphism. Then the continuous rational function $f \circ \pi$ coincides with the polynomial function t on $\text{Cent } Y$.*

Proof. The function t is a polynomial extension to Y of $f|_U \circ \pi|_{\pi^{-1}(U)}$ where $(f|_U, U)$ is a regular presentation of f . It follows that $f \circ \pi = t$ on a Zariski-dense open subset of Y and we conclude by density with respect to Euclidean topology that both functions coincide on $\text{Cent } Y$. $\square$

Note that in general $f \circ \pi$ does not coincide with t on the whole of Y even if f is continuous rational on the whole X .

In general, the issue whether a given continuous rational function may be lifted to a polynomial one via a finite birational morphism is crucial in our discussion. Next lemma illustrates two situations that will be useful in the sequel.

Lemma 3.5. *Let X be an algebraic set with normalization $\pi' : X' \rightarrow X$. Let Y be an algebraic set such that there exist finite birational morphisms $\pi : Y \rightarrow X$ and $\varphi : X' \rightarrow Y$ satisfying $\pi' = \pi \circ \varphi$. Then*

- (1) *Let $f \in \mathcal{P}(Y)$ and $\tilde{f} \in \mathcal{K}^0(\text{Cent } X)$. Then $f = \tilde{f} \circ \pi$ on $\text{Cent } Y$ if and only if $f \circ \varphi = \tilde{f} \circ \pi'$ on $\text{Cent } X'$;*
- (2) *Assume Y is central. Let $f \in \mathcal{P}(Y)$ and $\tilde{f} \in \mathcal{K}^0(X)$. Then $f = \tilde{f} \circ \pi$ on Y if and only if $f \circ \varphi = \tilde{f} \circ \pi'$ on X' .*

Proof. The proof follows from the surjectivity of the maps $\text{Cent } X' \rightarrow \text{Cent } X$, $\text{Cent } X' \rightarrow \text{Cent } Y$, $\text{Cent } Y \rightarrow \text{Cent } X$, and also $X' \rightarrow Y$ in the case Y is central, given by Proposition 3.3. $\square$

3.2. Finite birational bijection on the central locus

The normalization process allows to simplify the singularities of an algebraic set X by adding integral rational functions to the polynomial ring of X . Requiring an additional continuity of the integral functions leads to simplify the singularities without losing some bijectivity property with some part of X .

This is the main result of this section.

Theorem 3.6. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets, and denote by $\pi|_{\text{Cent } Y} : \text{Cent } Y \rightarrow \text{Cent } X$ its restriction to the central loci. The following properties are equivalent:*

- (i) $\pi|_{\text{Cent } Y}$ is a bijection;
- (ii) The ring morphism $\pi_0 : \mathcal{K}^0(\text{Cent } X) \rightarrow \mathcal{K}^0(\text{Cent } Y)$ is an isomorphism;
- (iii) For all $g \in \mathcal{P}(Y)$ there exists f in $\mathcal{K}^0(\text{Cent } X)$ such that $g = f \circ \pi$ on $\text{Cent } Y$;
- (iv) $\pi|_{\text{Cent } Y}$ is a homeomorphism for the Euclidean topology;
- (v) The morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally weakly subintegral;
- (vi) The rational map π^{-1} admits a (rational) continuous extension to $\text{Cent } X$.

Note that in (vi), the restriction of π^{-1} to $\text{Cent } X$ is not defined entirely on $\text{Cent } X$, but only on a subset dense in $\text{Cent } X$ with respect to Euclidean topology. So that (vi) means precisely that there exists a continuous map on $\text{Cent } X$ which admits π^{-1} as a rational model.

Proof. Let's prove first the equivalence between (i) and (ii). The fact that (i) implies (ii) is a direct consequence of 3) of Proposition 3.3. To prove the converse implication, assume that $\pi_0 : \mathcal{K}^0(\text{Cent } X) \rightarrow \mathcal{K}^0(\text{Cent } Y)$ is an isomorphism whereas $\pi|_{\text{Cent } Y}$ is not bijective. There exists $x \in \text{Cent } X$ such that we have $\{y_1, y_2\} \subset \pi^{-1}(x) \cap \text{Cent } Y$ and $y_1 \neq y_2$. There exists $p \in \mathcal{P}(Y)$ such that $p(y_1) \neq p(y_2)$. By 3) of Proposition 3.3, we get that $p \in \mathcal{K}^0(\text{Cent } Y) \setminus \pi_0(\mathcal{K}^0(\text{Cent } X))$ since p is not constant on the fibers of $\pi|_{\text{Cent } Y}$, a contradiction. We have proved that (ii) implies (i).

Since (ii) implies clearly (iii) then (i) implies (iii). To show that (iii) implies (i) it suffices to see that for any y_1, y_2 in $\text{Cent } Y$ such that $\pi(y_1) = \pi(y_2) = x$ and for any $g \in \mathcal{P}(Y)$ such that $g(y_1) = 0$, one has also $g(y_2) = 0$. Since there exists f in $\mathcal{K}^0(\text{Cent } X)$ such that $g = f \circ \pi$ on $\text{Cent } Y$, one gets the result.

Note that (iv) implies trivially (i), whereas (i) implies (iv) since π is closed with respect to Euclidean topology by Lemma 3.1.

We clearly have that (v) and (i) are equivalent since $\text{R-Cent } \mathcal{P}(X) \cap \text{Max } \mathcal{P}(X) = \text{Cent } X$ and $\text{R-Cent } \mathcal{P}(Y) \cap \text{Max } \mathcal{P}(Y) = \text{Cent } Y$.

Note that (vi) implies (i) since the rational inverse π^{-1} admits a continuous extension along $\text{Cent } X$, which is indeed an inverse for $\pi|_{\text{Cent } Y}$ by continuity with respect to Euclidean topology. To show the converse, we need to prove that $\pi|_{\text{Cent } Y}^{-1}$ is a rational continuous map. Consider $Y \subset \mathbb{R}^n$ and choose a coordinate function y_i on Y for $i \in \{1, \dots, n\}$. We want to prove that the rational function $z_i = y_i \circ \pi|_{\text{Cent } Y}^{-1}$ is continuous on $\text{Cent } X$. However $z_i \circ \pi|_{\text{Cent } Y}$ is polynomial on $\text{Cent } Y$, so that,

by (ii), z_i belongs to $\mathcal{K}^0(\text{Cent } X)$ and thus $\pi|_{\text{Cent } Y}^{-1} : \text{Cent } X \rightarrow \text{Cent } Y$ is a rational continuous map. $\square$

The centrality of an algebraic set is not preserved under finite birational morphisms in general. However it will be the case if we assume moreover the morphism to be bijective. The following result enumerates the properties of a bijective finite birational morphism onto a central algebraic set.

Proposition 3.7. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets where X is central. Let us assume that π is a bijection. Then, one has the following properties:*

- (1) Y is central;
- (2) The canonical morphism $\mathcal{K}^0(X) \rightarrow \mathcal{K}^0(Y)$ is an isomorphism;
- (3) π is an homeomorphism for the constructible topology;
- (4) The morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally weakly subintegral;
- (5) π^{-1} is rational continuous.

Proof. Since X is assumed to be central, we know by Proposition 3.3 that $\pi|_{\text{Cent } Y}$ is surjective onto X . In particular, if π is assumed to be bijective, then Y is automatically central.

Note that (2), (4) and (5) are direct consequences of Theorem 3.6.

Let us show (3). By Lemma 3.1, π is closed for the Euclidean topology so that using [21, Corollary 4.9], the image by π of a Zariski constructible closed subset of Y is a Zariski constructible closed subset of X . It follows that π is an homeomorphism for the constructible topology. $\square$

Remark 3.8. Note that a bijective finite birational polynomial morphism onto a central algebraic set is not necessarily an isomorphism while it is an isomorphism in the category of rational continuous maps by property 5). For instance, let X be the cuspidal curve given by $y^2 = x^3$ in $\mathbb{R}^2$, and X' be its normalization. The normalization morphism $\pi : X' \rightarrow X$ is birational, finite and bijective. It is even an homeomorphism with respect to the Zariski topology (the curves are irreducible, so the Zariski subsets are just points). However X is singular whereas X' is smooth.

3.3. Finite hereditarily birational bijection on the central locus

In this subsection we restrict our attention to continuous hereditarily rational functions.

Recall that the restriction of a rational continuous functions does not remain rational in general. This phenomenon appears also for birational morphisms. Namely, let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. By Lemma 3.1 the corresponding morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is injective and integral. Let W be an irreducible algebraic subset of Y . There exists $\mathfrak{q} \in \text{R-Spec } \mathcal{P}(Y)$ such that $W = \mathcal{Z}(\mathfrak{q})$. We denote by $\mathfrak{p}$ the real prime ideal $\mathfrak{q} \cap \mathcal{P}(X)$ and by V the real irreducible algebraic subset of X given by $\mathcal{Z}(\mathfrak{p})$. The restriction of π to W gives clearly

a morphism $\pi|_W : W \rightarrow V$ which is finite since the corresponding morphism of polynomial functions

$$\frac{\mathcal{P}(X)}{\mathfrak{p}} \rightarrow \frac{\mathcal{P}(Y)}{\mathfrak{q}}$$

is integral. The residue field $k(\mathfrak{q}) = \mathcal{K}(W)$ is an algebraic extension of $k(\mathfrak{p}) = \mathcal{K}(V)$. Then, $\pi|_W$ remains birational if and only if π induces an isomorphism $k(\mathfrak{p}) \simeq k(\mathfrak{q})$.

Example 3.9. Consider Kollár's surface $X = \mathcal{Z}(x^3 - y^3(1 + z^2))$. Its normalization is given by $\mathcal{P}(X') = \mathcal{P}(X)[x/y] = \mathcal{P}(X)/(t^3 - (1 + z^2), yt - x) \simeq \mathbb{R}[t, y, z]/(t^3 - (1 + z^2))$, setting $t = x/y$. Let $\mathfrak{p} = (x, y) \in \text{R-Spec } \mathcal{P}(X)$ and let $\mathfrak{q} \in \text{R-Spec } \mathcal{P}(X')$ be the unique real prime ideal of $\mathcal{P}(X')$ such that $\mathfrak{q} \cap \mathcal{P}(X) = \mathfrak{p}$. We have $k(\mathfrak{p}) = \mathbb{R}(z)$ and $k(\mathfrak{q}) = \mathbb{R}(z)(\sqrt[3]{1 + z^2}) \not\simeq k(\mathfrak{p})$. Here, the normalization morphism $\pi' : X' \rightarrow X$ ($y, t, z \mapsto (ty, y, z)$) is a bijective finite birational morphism which is not hereditarily birational in the sense that π' is birational but $\pi'|_{\mathcal{Z}(\mathfrak{q})} : \mathcal{Z}(\mathfrak{q}) \rightarrow \mathcal{Z}(\mathfrak{p})$ is not birational. The morphism π' has an inverse bijection π'^{-1} given by $(x, y, z) \mapsto (y, x/y, z)$ if $y \neq 0$ (the inverse of π' as a birational map) and $(0, 0, z) \mapsto (0, \sqrt[3]{1 + z^2}, z)$. We see that π'^{-1} is rational continuous but not hereditarily rational. Moreover, $\mathcal{P}(X) \hookrightarrow \mathcal{P}(X')$ is w_c -subintegral but not s_c -subintegral and one may also notice that $\mathcal{P}(X) \hookrightarrow \mathcal{P}(X)[x^2/y]$ is s_c -subintegral.

This consideration leads us to the definition of hereditarily birational maps on the central locus, inspired by [17].

Definition 3.10. Let $\pi : Y \rightarrow X$ be a birational morphism between algebraic sets. We say that π is hereditarily birational on the central locus of Y if for every irreducible algebraic subset W central in Y , the restriction $\pi|_W$ is birational with the Zariski closure of its image.

Example 3.9 provides a finite birational bijection between central algebraic sets such that the natural map $\text{R-Cent } \mathcal{P}(Y) \rightarrow \text{R-Cent } \mathcal{P}(X)$ is a bijection, but which is not hereditarily birational.

Note moreover that a finite birational morphism $\pi : Y \rightarrow X$ such that $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally weakly subintegral and hereditarily birational on the central locus of Y does not necessarily give rise to a centrally subintegral extension:

Example 3.11. Consider the surface $X = \mathcal{Z}(yz^4 - (y^2 - x^4)^2)$ in $\mathbb{R}^3$. The rational function $f = (\frac{y^2 - x^4}{z})^2$ is integral over $\mathcal{P}(X)$ and admits a continuous extension to $\text{Cent } X$ by $+\sqrt{y}$. The algebraic set obtained by adding f to $\mathcal{P}(X)$ as in Proposition 3.4 is isomorphic to $Y = \mathcal{Z}(yz^2 - (y^4 - x^4)^2) \subset \mathbb{R}^3$. The restriction of the induced finite birational morphism $\pi : Y \rightarrow X$ to the central loci is a bijection, so that the extension $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is w_c -subintegral.

Consider $V \subset X$ the parabola defined by $z = 0$ and $y = x^2$. Then $V \subset \text{Cent } X$, and there exist two irreducible curves central in Y and lying-over V : the lines $W_{\pm}$ given by $z = 0$ and $y = \pm x$. In particular the extension $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is not centrally subintegral. However both of these lines are isomorphic to V via the restriction of π , so that π is hereditarily birational by restriction to W_+ and W_- .

and moreover π is hereditarily birational on the central locus of Y since $W_+ \cup W_-$ is the inverse image by π of the indeterminacy locus of f .

Note moreover that just half of each line is included in $\text{Cent } Y$, and that the restriction of π^{-1} to V is given by the semialgebraic function $x \mapsto (x, |x|, 0)$. It shows that the inverse bijection $(\pi|_{\text{Cent } Y})^{-1} : \text{Cent } X \rightarrow \text{Cent } Y$ is rational continuous but not hereditarily rational.

However, we readily get:

Lemma 3.12. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. The morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally subintegral if and only if π is hereditarily birational on the central locus of Y and the map $\text{R-Cent } \mathcal{P}(Y) \rightarrow \text{R-Cent } \mathcal{P}(X)$ is a bijection.*

We aim to characterize the central subintegrality of a finite birational morphism by hereditary properties of the inverse birational map. More precisely, we are going to consider birational morphisms of the form $\pi : Y \rightarrow X$ such that π^{-1} is hereditarily rational on the center of X . This property is strictly stronger than π^{-1} being only rational continuous as noticed in Examples 3.9 and 3.11. Beware moreover that a birational morphism bijective between the central loci and hereditarily birational does not necessarily satisfy that π^{-1} is hereditarily rational on the center of X as illustrated by Example 3.11.

The subtle differences between hereditarily birational maps and rational morphisms satisfying that π^{-1} is hereditarily rational is crucial in this section.

Proposition 3.13. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. Then π^{-1} is hereditarily rational on the center of X if and only if π is hereditarily birational on the central locus of Y and $\text{R-Cent } \mathcal{P}(Y) \rightarrow \text{R-Cent } \mathcal{P}(X)$ is a bijection.*

Proof. Assume π^{-1} is hereditarily rational on the center of X . Let W be an irreducible subset central in Y , and denote by $V \subset X$ the Zariski closure of $\pi(W)$. Note that $\pi|_W$ is still a finite polynomial map, and that V is central in X .

The restriction of $(\pi|_{\text{Cent } Y})^{-1}$ to $V \cap \text{Cent } X$ coincides on $V \cap \text{Cent } X$ with the continuous extension of a rational map ϕ on V by assumption on π . In particular ϕ is a regular inverse of $\pi|_W$ on a Zariski dense open subset $V^o \subset V$ intersected with $\text{Cent } X$. Since V is central in X , it implies that ϕ is a rational inverse for $\pi|_W$. Therefore π is hereditarily birational on $\text{Cent } Y$.

For V irreducible and central in X , let W_i be the central subsets of Y lying over V for $i = 1, \dots, r$, all equipped with a rational inverse $\phi_i : V \rightarrow W_i$ of $\pi|_{W_i}$ (we have already shown that π is hereditarily birational on $\text{Cent } Y$). However these inverses coincide with $(\pi|_{\text{Cent } Y})^{-1}$ on a Zariski dense open subset of V intersected with $V \cap \text{Cent } X$, so that they coincide with a common rational map ϕ . In particular there exists a semialgebraic subset S of $V \cap \text{Cent } X$ of maximal dimension on which ϕ is injective, and $\phi(S) \subset W_i$ for $i = 1, \dots, r$. As a consequence $W_1 = \dots = W_r$ by centrality of the W_i 's in Y . We have proved that $\text{R-Cent } \mathcal{P}(Y) \rightarrow \text{R-Cent } \mathcal{P}(X)$ is a bijection.

Conversely, let V be central in X and denote by W the unique central set in Y lying-over V . By assumption $\pi|_W : W \rightarrow V$ is birational, and we aim to prove that its inverse is a rational model for the restriction of $(\pi|_{\text{Cent } Y})^{-1}$ to $V \cap \text{Cent } X$.

By Lemma 2.11, there exists $Z \subset V$ with $\dim Z < \dim V$ such that the inverse image by π of $(V \setminus Z) \cap \text{Cent } X$ is included in $W \cap \text{Cent } Y$. In other terms

$$(V \setminus Z) \cap \text{Cent } X \subset \pi(W \cap \text{Cent } Y),$$

so that the restriction to $V \cap \text{Cent } X$ of the continuous map $(\pi|_{\text{Cent } Y})^{-1}$ coincides with the rational map $(\pi|_W)^{-1}$ on the intersection of $V \cap \text{Cent } X$ with a Zariski dense open subset of V . It means that π^{-1} is hereditarily rational on the central locus of X . $\square$

To emphasize a crucial point in the preceding proof, note that the very last argument fails in Example 3.11 since for both W over V , the image $\pi(W \cap \text{Cent } Y)$ does not contains generically $V \cap \text{Cent } X = V$, but only half of it.

Note that a finite birational morphism $\pi : Y \rightarrow X$ induces a morphism $\pi^0 : \mathcal{R}^0(\text{Cent } X) \rightarrow \mathcal{R}^0(\text{Cent } Y)$ given by composition with π . With this in mind, we characterize centrally subintegral extensions using hereditarily rational functions. This provides an analog of Theorem 3.6 in the hereditary context.

Theorem 3.14. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets. The following properties are equivalent:*

- (i) *The ring morphism $\pi^0 : \mathcal{R}^0(\text{Cent } X) \rightarrow \mathcal{R}^0(\text{Cent } Y)$ is an isomorphism;*
- (ii) *For all $g \in \mathcal{P}(Y)$ there exists an hereditarily rational function f in $\mathcal{R}^0(\text{Cent } X)$ such that $g = f \circ \pi$ on $\text{Cent } Y$;*
- (iii) *The rational map π^{-1} is hereditarily rational on the center of X ;*
- (iv) *The extension $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally subintegral.*

Proof. Note first that (iii) and (iv) are equivalent by Lemma 3.12 and Proposition 3.13. It is immediate that (i) implies (ii) since $\mathcal{P}(Y) \subset \mathcal{R}^0(\text{Cent } Y)$.

Assume (ii). Note first that $\pi|_{\text{Cent } Y}$ is bijective by Theorem 3.6. Assume moreover $Y \subset \mathbb{R}^n$ and consider a coordinate function y_i on Y . By assumption there exists f in $\mathcal{R}^0(\text{Cent } X)$ such that $y_i = f \circ \pi$ on $\text{Cent } Y$. This implies that the function $y_i \circ (\pi|_{\text{Cent } Y})^{-1}$ is an hereditarily rational function on $\text{Cent } X$. In particular we have (iii).

Assume (iii). The proof of (i) comes down to prove the surjectivity of π^0 . So for $g \in \mathcal{R}^0(\text{Cent } Y)$, we have to prove that $f \in \mathcal{K}^0(\text{Cent } X)$ given by $f = g \circ (\pi|_{\text{Cent } Y})^{-1}$ is in fact in $\mathcal{R}^0(\text{Cent } X)$. So let V be central in X . Using Proposition 3.13, there exists a unique central set W in Y lying-over V , and $\pi|_W : W \rightarrow V$ is birational. The restriction of g to W is rational by assumption on g , so that the composition is again rational, giving (i). $\square$

In particular in the central case, we deduce from Theorem 3.14 an hereditary version of Proposition 3.7.

Proposition 3.15. *Let $\pi : Y \rightarrow X$ be a finite birational morphism between algebraic sets where X is central. Let us assume that π is bijective and hereditarily birational. Then, one has the following properties:*

- (1) Y is central.
- (2) The morphism $\mathcal{R}^0(X) \rightarrow \mathcal{R}^0(Y)$ $f \mapsto f \circ \pi$ is an isomorphism;
- (3) The extension $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally subintegral;
- (4) The rational map π^{-1} is hereditarily rational.

Proof. By Proposition 3.13 and Theorem 3.14, we only have to prove that $\text{R-Spec } \mathcal{P}(Y) \rightarrow \text{R-Spec } \mathcal{P}(X)$ is injective. Let $\mathfrak{p} \in \text{R-Spec } \mathcal{P}(X)$ and $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{R-Spec } \mathcal{P}(Y)$ lying-over $\mathfrak{p}$. Since $\pi|_{\mathcal{Z}(\mathfrak{q}_i)} : \mathcal{Z}(\mathfrak{q}_i) \rightarrow \mathcal{Z}(\mathfrak{p})$ is birational, for $i \in \{1, 2\}$, and π is bijective, it follows that $\mathcal{Z}(\mathfrak{q}_1) = \mathcal{Z}(\mathfrak{q}_2)$ and the real Nullstellensatz [6, Theorem 4.1.4] implies that $\mathfrak{q}_1 = \mathfrak{q}_2$. $\square$

Remark 3.16. An algebraic set X has a totally real normalization if $(\pi')_{\mathbb{C}}^{-1}(X) = X'$ where $\pi' : X' \rightarrow X$ is the normalization morphism. It is shown in [12] that finite birational morphisms onto an algebraic set with totally real normalization have remarkable properties. Notably, from [12, Theorem 3.9] we know that a finite birational bijection with target an algebraic set with totally real normalization admits a rational inverse which is hereditarily rational on the central locus, and moreover it is an homeomorphism for the Zariski topology.

4. Weak-normalization and seminormalization relative to the central locus

We introduce in this section the weak-normalization and seminormalization of an algebraic set, relative to its central locus. We develop the theory in parallel with the classical notion of normalization (and make some connection too with the biregular normalization defined in [12]), meaning in particular that we define them via integral closures in well-chosen rings of functions. We make latter the connection with the algebraic notions of weak-normalization and seminormalization relative to its central locus considered in Section 2.5.

4.1. Integral closure in rings of continuous functions

Let X be an algebraic set. The normalization X' of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{K}(X)$. The biregular normalization X^b of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{O}(X)$. Taking advantage of the sequence of inclusions

$$\mathcal{P}(X) \hookrightarrow \mathcal{O}(X) \hookrightarrow \mathcal{R}^0(\text{Cent } X) \hookrightarrow \mathcal{K}^0(\text{Cent } X) \hookrightarrow \mathcal{K}(X)$$

we define, using Proposition 3.2, two intermediate algebraic sets between a given algebraic set and its normalization.

Definition 4.1. Let X be an algebraic set.

- (1) The weak-normalization relative to the central locus (or w_c -normalization) X^{w_c} of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{K}^0(\text{Cent } X)$;
- (2) The seminormalization relative to the central locus (or s_c -normalization) X^{s_c} of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}^0(\text{Cent } X)$.

The natural finite birational morphisms $\pi^{w_c} : X^{w_c} \rightarrow X$ and $\pi^{s_c} : X^{s_c} \rightarrow X$ are respectively called the w_c -normalization morphism and the s_c -normalization morphism. The algebraic set X is called centrally weakly-normal if $X = X^{w_c}$ and centrally seminormal if $X = X^{s_c}$.

In particular, the sets X^{w_c} and X^{s_c} are intermediate algebraic sets between X^b and X' , and we derive a sequence of finite birational morphisms

$$X' \rightarrow X^{w_c} \rightarrow X^{s_c} \rightarrow X^b \rightarrow X$$

that will be the object of study in the rest of the paper.

Example 4.2. As a first example, consider Kollár's surface $X = \mathcal{Z}(x^3 - y^3(1 + z^2))$. The polynomial ring of the normalization is given by $\mathcal{P}(X') = \mathcal{P}(X)[x/y]$ and the rational function x/y can be extended to a continuous function on X . As a consequence X' coincides with X^{w_c} , whereas x/y is not hereditarily rational. Note that $X^{s_c} \neq X^{w_c}$, more precisely X^{s_c} has coordinate ring $\mathcal{P}(X)[x^2/y]$.

Remark 4.3. It would make sense algebraically to replace $\mathcal{P}(X)$ by $\mathcal{O}(X)$ in Definition 4.1. However in general, the integral closure of $\mathcal{O}(X)$ in $\mathcal{K}(X)$ or $\mathcal{K}^0(\text{Cent } X)$ or $\mathcal{R}^0(\text{Cent } X)$ is not the ring of regular functions of an intermediate algebraic set between X and X' (see [12]). This justifies why we have decided to work with $\mathcal{P}(X)$ rather than $\mathcal{O}(X)$ (contrarily to [24]).

From the very definitions of w_c -normalization and s_c -normalization, we get that, for an irreducible algebraic set X , if $\mathcal{K}^0(\text{Cent } X) = \mathcal{R}^0(\text{Cent } X)$ then $X^{s_c} = X^{w_c}$. In particular, this is the case for curves or more generally varieties with isolated singularities.

On a complex algebraic variety which is normal, a rational function admitting a continuous extension at its poles is automatically regular by Riemann extension theorem. We characterize, by universal properties closed to the Riemann extension theorem, the algebraic varieties that are “normal” for the different normalizations we have encountered.

Proposition 4.4. Let X be a real algebraic set. Then:

- (1) X is normal if and only if every rational function in $\mathcal{K}(X)$ bounded on $X_{\mathbb{C}}$ is polynomial on X ;
- (2) X is seminormal if and only if every rational function in $\mathcal{K}(X)$ integral over $\mathcal{P}(X)$ and continuous on $X_{\mathbb{C}}$ is polynomial on X ;

- (3) X is biregularly normal if and only if every regular function on X integral over $\mathcal{P}(X)$ is polynomial on X ;
- (4) X is centrally seminormal if and only if every function in $\mathcal{R}^0(\text{Cent } X)$ integral over $\mathcal{P}(X)$ is the restriction to $\text{Cent } X$ of a polynomial function on X ;
- (5) X is centrally weakly-normal if and only if every function in $\mathcal{K}^0(\text{Cent } X)$ integral over $\mathcal{P}(X)$ is the restriction to $\text{Cent } X$ of a polynomial function on X .

Proof. Notice that a regular function on $X_{\mathbb{C}}$ is polynomial on $X_{\mathbb{C}}$. By the Riemann extension theorem then we get (1). Statement (2) follows from [3, 4] and [31]. The last three statements are direct consequences from the definitions of the biregular, centrally weakly and centrally semi-normalizations. $\square$

Remark 4.5. We see by Proposition 4.4 that w_c -normalization does not correspond to w_c -normalization of the irreducible components like classical normalization does (and the same remark holds for the s_c -normalization). More precisely, it may happen that every irreducible component of a non centrally weakly-normal algebraic set is centrally weakly-normal. Consider for instance the union of three distinct two-by-two lines passing through the origin in $\mathbb{R}^2$. In that case the w_c -normalization is the union of the three axes in $\mathbb{R}^3$ and it is also the s_c -normalization (see the next section).

We can identify the functions in $\mathcal{P}(X^{w_c})$ with the rational functions in $\mathcal{K}(X)$ that are integral over $\mathcal{P}(X)$ and that admit a continuous extension to $\text{Cent } X$. Similarly, the functions in $\mathcal{P}(X^{s_c})$ are the rational functions in $\mathcal{K}(X)$ that are integral over $\mathcal{P}(X)$ and that admit an hereditarily rational extension to $\text{Cent } X$. We state this fact as a lemma for further reference.

Lemma 4.6. *Let X be an algebraic set and $\pi' : X' \rightarrow X$ be the normalization morphism.*

- (1) *The polynomial functions on X^{w_c} form the subring of $\mathcal{P}(X')$ given by*

$$\mathcal{P}(X^{w_c}) = \{g \in \mathcal{P}(X') \mid \exists f \in \mathcal{K}^0(\text{Cent } X) \text{ such that } f \circ \pi' = g \text{ on } \text{Cent } X'\};$$

- (2) *The polynomial functions on X^{s_c} form the subring of $\mathcal{P}(X')$ given by*

$$\mathcal{P}(X^{s_c}) = \{g \in \mathcal{P}(X') \mid \exists f \in \mathcal{R}^0(\text{Cent } X) \text{ such that } f \circ \pi' = g \text{ on } \text{Cent } X'\}.$$

We aim to prove that the ring $\mathcal{P}(X^{w_c})$ is the weak normalization relative to the central locus of the ring $\mathcal{P}(X)$, together with the analogous statement for $\mathcal{P}(X^{s_c})$ with respect to the seminormalization. Next result can be viewed as a first step in this direction.

Proposition 4.7. *Let X be an algebraic set.*

- (1) *The morphism $\pi^{w_c} : X^{w_c} \rightarrow X$ is a bijection between the central loci and its inverse map is rational continuous on the center of X ;*

- (2) *The morphism $\pi^{\text{sc}} : X^{\text{sc}} \rightarrow X$ is a bijection between the central loci and its inverse map is hereditarily rational on the center of X .*

Proof. Consider the case of the weak-normalization relative to the central locus. Denote by ψ the morphism $X' \rightarrow X^{\text{wc}}$. Let $g \in \mathcal{P}(X^{\text{wc}})$. By Lemma 4.6, there exists a continuous function $f \in \mathcal{K}^0(\text{Cent } X)$ such that $g \circ \psi = f \circ \pi'$ on $\text{Cent } X'$. By Lemma 3.5, we get that $g = f \circ \pi^{\text{wc}}$ on $\text{Cent } X^{\text{wc}}$. As a consequence the map $\pi^{\text{wc}} : \text{Cent } X^{\text{wc}} \rightarrow \text{Cent } X$ is bijective or equivalently $\pi^{\text{wc}} : X^{\text{wc}} \rightarrow X$ is a bijection on the central loci and its inverse is rational continuous by Theorem 3.6. The proof in the seminormal case is similar using Theorem 3.14. $\square$

Before establishing the universal property satisfied by the weak-normalization (respectively seminormalization) relative to the central locus, and make the connection with its algebraic counterpart, we discuss several examples of curves and surfaces in order to give the reader some familiarity with the new sets introduced.

Let us begin the discussion with the case of curves, where the w_c -normalization and the s_c -normalization coincide. To illustrate the reducible case, note that two crossing lines in the plane give a centrally seminormal curve contrarily to three lines crossing in a common point.

Concerning irreducible plane curves, the s_c -normalization of the cuspidal plane cubic $X = \mathcal{Z}(y^2 - x^3)$ coincides with its normalization, and $X = X^b \subset X^{\text{sc}} = X'$. The nodal plane curve $X = \mathcal{Z}(y^2 - x^2(x + 1))$ is centrally seminormal, and $X = X^b = X^{\text{sc}} \subset X'$. Of course the seminormalization with respect to the central locus can be different from X and X' , consider the curve $X = \mathcal{Z}(y^2 - x^4(x + 1))$ (whose unique singular point is the origin, where two distinct branches intersect with tangency).

Considering surfaces in $\mathbb{R}^3$, remember that for Kollár's surface the weak-normalization relative to the central locus differs from the seminormalization relative to the central locus. Note however that in the case of isolated singularities (which are not necessarily normal if the complex singularities are not isolated), the rational continuous and hereditarily rational functions coincide, so that the w_c -normalization and the s_c -normalization will be the same.

The union of two intersecting planes in $\mathbb{R}^3$ defines a centrally seminormal surface, or more generally the union of the coordinates hyperplanes in $\mathbb{R}^n$ is centrally seminormal. Consider finally two classical umbrellas. The normalization of the (non-central) Whitney umbrella $X = \mathcal{Z}(x^2 - y^2z)$ is the affine plane, and the normalization morphism is not a bijection on the central loci, so that $X = X^b = X^{\text{wc}} = X^{\text{sc}}$. The normalization of the (non-central) Cartan umbrella $X = \mathcal{Z}(x^3 - (x^2 + y^2)z)$ is given by $\mathcal{P}(X') = \mathcal{P}(X)[yz/x]$. The rational function yz/x admits a continuous extension by zero along the y -axis, which is the intersection of its indeterminacy locus with the central part of X , so that $X' = X^{\text{wc}}$. Moreover the restriction of yz/x to the y -axis is a constant function so it is still rational, so that $X^{\text{wc}} = X^{\text{sc}}$. Since yz/x is not regular on the Cartan umbrella, it can be concluded that $X = X^b$.

4.2. Universal properties for the w_c -normalization

We give the universal property satisfied by the w_c -normalization in light of the universal property of the normalization.

Theorem 4.8. *Let X be an algebraic set. Let Y be an algebraic set equipped with a finite birational morphism $\pi : Y \rightarrow X$.*

Then π induces a bijection from $\text{Cent } Y$ to $\text{Cent } X$ if and only if $\pi^{w_c} : X^{w_c} \rightarrow X$ factorizes through π .

Proof. Let X' be the normalization of X . Notice first that from Proposition 4.7 we know that the restriction of $\pi^{w_c} : X^{w_c} \rightarrow X$ to the central loci is a bijection. Let Y be an algebraic set with finite birational morphisms $\pi : Y \rightarrow X$ and $\varphi : X' \rightarrow Y$.

Assume first that π^{w_c} factorizes through π . By Theorem 3.6 the extension $\mathcal{P}(X) \rightarrow \mathcal{P}(X^{w_c})$ is w_c -subintegral and thus $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is also w_c -subintegral (Proposition 2.18). From Theorem 3.6 again then it follows that $\pi : \text{Cent } Y \rightarrow \text{Cent } X$ is a bijection.

Assume now that $\pi : \text{Cent } Y \rightarrow \text{Cent } X$ is a bijection. Let $g \in \mathcal{P}(Y)$. By Theorem 3.6 there exists a continuous function $f \in \mathcal{K}^0(\text{Cent } X)$ such that $g = f \circ \pi$ on $\text{Cent } Y$. By Lemma 3.5, then $g \circ \varphi = f \circ \pi'$ on $\text{Cent } X'$ and thus $g \circ \varphi \in \mathcal{P}(X^{w_c})$ (Lemma 4.6). Since the composition by φ gives the inclusion $\mathcal{P}(Y) \subset \mathcal{P}(X')$ then we get an inclusion $\mathcal{P}(Y) \subset \mathcal{P}(X^{w_c})$ and thus $\pi^{w_c} : X^{w_c} \rightarrow X$ uniquely factors through $\pi : Y \rightarrow X$. $\square$

From Theorems 3.6 and 4.8, we deduce that the w_c -normalization X^{w_c} of X is the biggest set among the intermediate algebraic sets between X and X' whose central loci are birationally continuously isomorphic with the central locus of X .

Note that if the w_c -normalization morphism $\pi^{w_c} : X^{w_c} \rightarrow X$ is bijective by restriction to the central locus, it is not a bijection in general even if we assume X to be central.

Remark 4.9. The w_c -normalization of a (non-normal) toric variety coincides with its normalization. Indeed, the normalization is obtained by saturation of the semi-groups, giving rise to a normal toric variety with the same torus decomposition into orbits. In particular a toric variety and its normalization are in bijection.

We prove that in the geometric setting we recover the algebraic w_c -normalization.

Theorem 4.10. *Let X be an algebraic set. We have*

$$\mathcal{P}(X)^{w_c} = \mathcal{P}(X^{w_c}).$$

Proof. Let X' be the normalization of X . The morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(X)^{w_c}$ being w_c -subintegral (see Proposition 2.20), it induces a bijection in restriction to the central maximal ideals and thus a bijection from $\text{Cent } Y$ onto $\text{Cent } X$ where Y is the algebraic set with $\mathcal{P}(X)^{w_c}$ as ring of polynomial functions (see Proposition 3.2). By the universal property Theorem 4.8, one gets that Y is an intermediate algebraic set

between X and X^{w_c} , and hence $\mathcal{P}(X)^{w_c} = \mathcal{P}(Y) \subset \mathcal{P}(X^{w_c})$ (recall that $\mathcal{P}(X)^{w_c}$ and $\mathcal{P}(X^{w_c})$ are both viewed in $\mathcal{P}(X')$).

To show the converse inclusion, let us note that it follows from Theorem 3.6 and Proposition 4.7 that the extension ring $\mathcal{P}(X) \rightarrow \mathcal{P}(X^{w_c})$ is w_c -subintegral. Then, by the universal property Proposition 2.20, one derives a factorization morphism $\mathcal{P}(X^{w_c}) \rightarrow \mathcal{P}(X)^{w_c}$. $\square$

As direct consequence of Theorem 4.10 and Proposition 2.21, one gets the idempotency of the w_c -normalization:

Corollary 4.11. *Let X be an algebraic set. Then, $(X^{w_c})^{w_c} = X^{w_c}$.*

The inverse of a finite birational morphism that induces a bijection between the central loci is known to be rational continuous (see Theorem 3.6). Next result gives a stronger statement, when we assume that the target variety is centrally weakly normal. It is a direct application of our universal properties (Theorem 4.8 and Proposition 2.20) and it gives a real version of [22, Corollary 2.8].

Proposition 4.12. *Let X be an algebraic set. Suppose that X is centrally weakly-normal and that $\varphi : Y \rightarrow X$ is a finite birational polynomial morphism with Y an algebraic set. Then φ is a bijection on the central loci if and only if φ is an isomorphism.*

Note that it is not possible to remove the “finite” hypothesis in the previous proposition. For instance, let X be the nodal curve given by $y^2 = x^2(x + 1)$ in $\mathbb{R}^2$, and Y be the hyperbola given by $xy = 1$ in $\mathbb{R}^2$. They are both centrally weakly normal and both in polynomial bijection with the punctured line $\mathbb{R} \setminus \{1\}$ however they are not isomorphic curves since X is singular whereas Y is smooth.

Centrally weakly normal sets are stable under the product of varieties.

Proposition 4.13. *Let X and Y be centrally weakly-normal algebraic sets. Then $X \times Y$ is centrally weakly-normal.*

Proof. We use the same strategy as in [22, Corollary 2.13]. Let f be a rational continuous function on $\text{Cent}(X \times Y)$ which is integral over $\mathcal{P}(X \times Y)$. By Proposition 4.4, we have to show that f is polynomial on $\text{Cent}(X \times Y)$. Then, for any $x \in \text{Cent } X$, the restriction f_x of f to $\{x\} \times Y$ satisfies an integral equation over $\mathcal{P}(Y)$. Note however that, if f_x is not necessarily a rational function on Y , there exists a Zariski dense subset U in X such that f_x is rational for any $x \in U$. By intersecting U with the union of the non-singular loci of the irreducible components of X then we may assume $U \subset \text{Cent } X$. By w_c -normality of Y , it follows that f_x is polynomial on $\text{Cent } Y$ for any $x \in U$. Similarly, there exists a Zariski dense subset V in Y such that $V \subset \text{Cent } Y$ and f_y is polynomial on $\text{Cent } X$ for any $y \in V$.

We want to conclude that f coincides with a polynomial function on $\text{Cent}(X \times Y)$. We know by Palais [27] that f is polynomial on $U \times V$, so that there exists a polynomial function $p \in \mathcal{P}(X \times Y)$ such that $f = p$ on $U \times V$. Since f is continuous, it implies that $f = p$ on $\text{Cent}(X \times Y)$ i.e f is polynomial on $\text{Cent}(X \times Y)$. As a consequence $X \times Y$ is centrally weakly-normal. $\square$

4.3. Universal property for the s_c -seminormalization

Here is the universal property for the seminormalization relative to the central locus.

Theorem 4.14. *Let X be an algebraic set. Let Y be any algebraic set equipped with a finite birational morphism $\pi : Y \rightarrow X$.*

Then π^{-1} is hereditarily rational on the center of X if and only if $\pi^{s_c} : X^{s_c} \rightarrow X$ factorizes through π .

Proof. Let X' be the normalization of X . Notice first that from Proposition 4.7 we know that the rational inverse of π^{s_c} is hereditarily rational on the center of X .

Assume $\pi^{s_c} : X^{s_c} \rightarrow X$ factorizes through π . We know by Proposition 4.7 and Theorem 3.14 that $\mathcal{P}(X) \rightarrow \mathcal{P}(X^{s_c})$ is centrally subintegral. Then it follows from Proposition 2.18 that $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is centrally subintegral. From Theorem 3.14 we conclude that the rational inverse of $\pi : Y \rightarrow X$ is hereditarily rational on the center of X .

Assume now $\pi : Y \rightarrow X$ is a finite birational morphism whose inverse is hereditarily rational on the center of X . By the universal property of the normalization, the normalization morphism π' factorizes by π and let $\psi : X' \rightarrow Y$ be the finite birational morphism such that $\pi' = \pi \circ \psi$. Let $g \in \mathcal{P}(Y)$. By Theorem 3.14, there exists $f \in \mathcal{R}^0(\text{Cent } X)$ such that $g = f \circ \pi$ on $\text{Cent } Y$. By Lemma 3.5, we get $g \circ \psi = f \circ \pi'$ on $\text{Cent } X'$ and thus $g \circ \psi \in \mathcal{P}(X^{s_c})$ (Lemma 4.6). It shows that $\mathcal{P}(Y) \subset \mathcal{P}(X^{s_c})$ and the proof is done. $\square$

From Theorems 3.14 and 4.14, we deduce that the s_c -normalization X^{s_c} of X is the biggest set among the intermediate algebraic sets between X and X' equipped with a morphism to X whose rational inverse is hereditarily rational on the center of X .

Let us now show how we recover the algebraic s_c -seminormalization:

Theorem 4.15. *Let X be an algebraic set. We have $\mathcal{P}(X)^{s_c} = \mathcal{P}(X^{s_c})$.*

Proof. Let $\pi' : X' \rightarrow X$ be the normalization morphism. Let us see first that we have $\mathcal{P}(X)^{s_c} \subset \mathcal{P}(X^{s_c})$. Let Y be the algebraic set associated to the ring $\mathcal{P}(X)^{s_c}$ (see Proposition 3.2). The morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(X)^{s_c}$ being centrally subintegral (see Proposition 2.23), from Theorem 3.14 it follows that the finite birational morphism $Y \rightarrow X$ induces a bijection from $\text{Cent } Y$ onto $\text{Cent } X$ and its rational inverse is hereditarily rational on the center of X . By the universal property of Theorem 4.14, one gets that Y is an intermediate algebraic set between X and X^{s_c} , and hence $\mathcal{P}(X)^{s_c} = \mathcal{P}(Y) \subset \mathcal{P}(X^{s_c})$.

To show the converse inclusion, it follows from Proposition 4.7 and Theorem 3.14 that the extension ring $\mathcal{P}(X) \rightarrow \mathcal{P}(X^{s_c})$ is centrally subintegral. Then, by the universal property Proposition 2.23, one derives a factorization morphism $\mathcal{P}(X^{s_c}) \rightarrow \mathcal{P}(X)^{s_c}$. $\square$

Combined with the idempotency properties of the algebraic s_c -seminormalization (Proposition 2.25), we obtain:

Corollary 4.16. *Let X be an algebraic set. Then*

- (1) $(X^{s_c})^{s_c} = X^{s_c}$;
- (2) $(X^{w_c})^{s_c} = X^{w_c}$;
- (3) $(X^{s_c})^{w_c} = X^{w_c}$.

An immediate application of Theorem 4.14 gives a seminormal version of Proposition 4.12.

Proposition 4.17. *Let X be an algebraic set. Suppose that X is centrally seminormal and that $\varphi : Y \rightarrow X$ is a finite birational polynomial morphism with Y an algebraic set. Then φ^{-1} is hereditarily rational on the center of X if and only if φ is an isomorphism.*

Centrally seminormal sets are stable under the product of varieties.

Proposition 4.18. *Let X and Y be centrally seminormal algebraic sets. Then $X \times Y$ is centrally seminormal.*

Proof. We use the same proof as in Proposition 4.13. Note that the proof is even simpler since the restriction of an hereditarily rational function is rational so that we can choose U and V to be respectively the union of the non-singular loci of the irreducible components of X and Y . $\square$

4.4. Up to biregular isomorphisms

From Subsection 1.1, the constructions of the weak- and semi-normalizations we have made for real algebraic sets can be extended to affine real algebraic varieties (such that the set of real closed points is Zariski dense in the set of closed points). It is in particular clear that isomorphic affine real algebraic varieties have isomorphic weak- and semi-normalizations. We prove moreover in this section that biregular affine real algebraic varieties have biregular weak- and semi-normalizations. Combined with the fact that the real closed points of a quasi-projective variety defined over $\mathbb{R}$ are always included in an affine variety via a regular morphism, this proved that the weak- and semi-normalizations are well defined for quasi-projective real algebraic varieties in the sense of [6].

To go toward the proof, we begin with the fact that our constructions commute with the localization along the multiplicative part S defining the regular functions on an algebraic set $X \subset \mathbb{R}^n$, namely $S = 1 + \sum \mathcal{P}(X)^2$. This result has to be compared with the results in Subsection 4.5.

Lemma 4.19. *Let X be an algebraic set and $S = 1 + \sum \mathcal{P}(X)^2$. Then $S^{-1}\mathcal{P}(X^{w_c}) \simeq \mathcal{O}(X)^{w_c}$ and $S^{-1}\mathcal{P}(X^{s_c}) \simeq \mathcal{O}(X)^{s_c}$.*

Proof. Let us deal with w_c -normalization. We aim to prove that the second inclusion in the sequence of natural inclusions

$$\mathcal{O}(X) \rightarrow S^{-1}\mathcal{P}(X^{w_c}) \rightarrow \mathcal{O}(X)^{w_c}$$

is an isomorphism. Let f be an element in $\mathcal{O}(X)^{w_c}$. Note that $\mathcal{O}(X)^{w_c}$ is included in $\mathcal{O}(X)'$, this latter being isomorphic to $S^{-1}\mathcal{P}(X')$ since normalization commutes with localization, so that there exists $s \in S$ such that sf belongs to $\mathcal{P}(X')$. So it is sufficient to prove that sf belongs to $\mathcal{P}(X^{w_c}) = \mathcal{P}(X)^{w_c}$. Note that we have a natural bijection $\text{R-Cent } \mathcal{P}(X) \rightarrow \text{R-Cent } \mathcal{O}(X)$, $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ that gives also a bijection by restriction on maximal ideals on both sides (see [12]). Let $\mathfrak{m}$ be a central maximal ideal in $\mathcal{P}(X)$, and denotes by $\tilde{\mathfrak{m}}$ the corresponding maximal ideal in $\mathcal{O}(X)$. Note that $\mathcal{P}(X)_{\mathfrak{m}} \simeq \mathcal{O}(X)_{\tilde{\mathfrak{m}}}$ and $\mathcal{P}(X')_{\mathfrak{m}} \simeq \mathcal{O}(X')_{\tilde{\mathfrak{m}}}$. Then by assumption

$$(sf)_{\tilde{\mathfrak{m}}} \in \mathcal{O}(X)_{\tilde{\mathfrak{m}}} + \text{Rad}^C \mathcal{O}(X')_{\tilde{\mathfrak{m}}}$$

so that

$$(sf)_{\mathfrak{m}} \in \mathcal{P}(X)_{\mathfrak{m}} + \text{Rad}^C \mathcal{P}(X')_{\mathfrak{m}},$$

proving that $sf \in \mathcal{P}(X^{w_c})$ as expected.

For s_c -normalization, we proceed similarly considering central prime ideals rather than maximal ones. $\square$

With this in hand one can prove the desired result.

Theorem 4.20. *Let X and Y be biregular real algebraic sets. Then X^{w_c} is biregular to Y^{w_c} , and X^{s_c} is biregular to Y^{s_c} .*

Proof. Let us prove the result about w_c -normalization, we proceed likewise for s_c -normalization.

Using Theorem 4.10 and Lemma 4.19, one has a canonical commutative diagram of injective morphisms

$$\begin{array}{ccccccc} \mathcal{P}(X) & \rightarrow & \mathcal{O}(X) & \simeq & \mathcal{O}(Y) & \leftarrow & \mathcal{P}(Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}(X)^{w_c} & \rightarrow & S^{-1}\mathcal{P}(X)^{w_c} \simeq \mathcal{O}(X)^{w_c} \simeq \mathcal{O}(Y)^{w_c} \simeq T^{-1}\mathcal{P}(Y)^{w_c} & \leftarrow & \mathcal{P}(Y)^{w_c} \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & S_{w_c}^{-1}\mathcal{P}(X^{w_c}) = \mathcal{O}(X^{w_c}) & & T_{w_c}^{-1}\mathcal{P}(Y^{w_c}) = \mathcal{O}(Y^{w_c}) & & \end{array}$$

where we have set $S = 1 + \sum \mathcal{P}(X)^2$, $S_{w_c} = 1 + \sum \mathcal{P}(X^{w_c})^2$, $T = 1 + \sum \mathcal{P}(Y)^2$, $T_{w_c} = 1 + \sum \mathcal{P}(Y^{w_c})^2$.

Producing a canonical morphism from one bottom ring to the other one will show, by symmetry, that they both are canonically isomorphic.

Using the horizontal middle morphisms, one has a canonical morphism

$$\mathcal{P}(X)^{w_c} \rightarrow T_{w_c}^{-1}\mathcal{P}(Y^{w_c}).$$

To show that it factorizes through $S_{w_c}^{-1}\mathcal{P}(X^{w_c})$, it is enough to prove that any element of S_{w_c} becomes invertible in $T_{w_c}^{-1}\mathcal{P}(Y^{w_c})$.

Let us then consider $s = 1 + \sum a_i^2$ where $a_i \in \mathcal{P}(X^{w_c})$. There exist $b_i \in \mathcal{P}(Y^{w_c})$ and $t_i \in T$ such that a_i is sent to b_i/t_i in $T^{-1}\mathcal{P}(Y)^{w_c}$. Since $t_i^2 \in T \subset T_{w_c}$ is invertible in $T_{w_c}^{-1}\mathcal{P}(Y^{w_c})$, it is enough to show that $f = t^2 + \sum c_i^2$ is invertible in $T_{w_c}^{-1}\mathcal{P}(Y^{w_c})$ for $t \in T$ and $c_i \in \mathcal{P}(Y)^{w_c}$. Since $t = 1 + \sum u_i^2$ with $u_i \in \mathcal{P}(Y)$, we get that $f \in T_{w_c}$. This gives the result. $\square$

Note that the same proof adapts to the case of the normalization (stated in [6, page 75]) since the normalization process commutes with localization.

4.5. Commutation with localization

It is known, for example from [22, Corollary 1.6], that the standard seminormalization ^+A commutes with localization. The proof of this result follows quite directly from the conductor criterion for seminormal extensions as stated in [22, Proposition 1.4]. Unfortunately, this criterion appears not to have a clear counterpart in our real central setting. Hence, we produce here a new argument based on an induction on the dimension. Namely, we get that the central seminormalization commutes with localization at a prime ideal:

Theorem 4.21. *Let X be an algebraic set and $\mathfrak{p}$ be a prime ideal in $\mathcal{P}(X)$. Then*

$$(\mathcal{P}(X)_{\mathfrak{p}})^{s_c} = \mathcal{P}(X^{s_c})_{\mathfrak{p}}.$$

Remind that the commutation of the integral closure with the standard normalization is formal. It is no more formal for s_c -normalization although it is still an integral closure by Theorem 4.15. One reason is that the localized ring is no more the polynomial ring of a real algebraic variety. Another reason is that the commutation is false for central w_c -normalization although this latter ring is still an integral closure by Theorem 4.10.

Remark 4.22. The result is false for the w_c -normalization. Consider for instance the Whitney umbrella $X = \mathcal{Z}(y^2 - zx^2)$, for which the inclusion $\mathcal{P}(X^{w_c})_{\mathfrak{m}} \subset (\mathcal{P}(X)_{\mathfrak{m}})^{w_c}$ is strict with $\mathfrak{m} = (x, y, z)$ (the rational function $f = y/x \in \mathcal{K}(X)$ belongs to the right-hand side, but not to the left-hand side).

We cut the proof of Theorem 4.21 into several lemmas, starting with the first key Lemma 2.11 and also the following one which translates geometrically the belonging of a rational function to the s_c -normalization of the localization of $\mathcal{P}(X)$ at some prime ideal:

Lemma 4.23. *Let $\pi : X' \rightarrow X$ denote the normalization morphism, and $f \in \mathcal{K}(X)$ be a rational function integral over $\mathcal{P}(X)$. Let V be an irreducible component of the polar locus of f , with V central in X . Assume that the localization $(f \circ \pi)_{\mathcal{I}(V)}$ along $\mathcal{I}(V)$ of the polynomial function $f \circ \pi$ on X' satisfies*

$$(f \circ \pi)_{\mathcal{I}(V)} \in \mathcal{P}(X)_{\mathcal{I}(V)} + \text{Rad}^C \mathcal{P}(X')_{\mathcal{I}(V)}.$$

Then there exists an algebraic subset $Z \subset V$ with $\dim Z < \dim V$ such that

- (1) The rational function f can be extended continuously along $\text{Cent } X \cap (V \setminus Z)$;
- (2) The extension remains rational restricted to any irreducible subset of V central in X and not included in Z .

Proof. Let $W_1, \dots, W_r \subset X'$ be the irreducible algebraic sets central in X' lying over V .

By assumption on f , there exist $q \in \mathcal{P}(X) \setminus \mathcal{I}(V)$, $p \in \mathcal{P}(X)$ and $h \in \mathcal{P}(X')$ (we see h as a rational function on X (integral over $\mathcal{P}(X)$)) such that

$$h \circ \pi \in \bigcap_{i=1}^r \mathcal{I}(W_i)$$

and

$$qf = p + h$$

as rational functions on X .

Let $\tilde{Z}$ be the algebraic subset of V given by Lemma 2.11. Let $x \in (V \setminus \tilde{Z}) \cap \text{Cent } X$. By Lemma 2.11, the set $\pi^{-1}(x) \cap \text{Cent } X'$ is included in $W_1 \cup \dots \cup W_r$, so that $h \circ \pi$ vanishes on $\pi^{-1}(x) \cap \text{Cent } X'$. This means that h can be extended continuously at x by setting $h(x) = 0$. Note moreover that property (2) holds for h with respect to $\tilde{Z}$.

Now let Z be equal to $\tilde{Z} \cup (\mathcal{Z}(q) \cap V)$. We still have $\dim Z < \dim V$ since $q \notin \mathcal{I}(V)$, and p being polynomial on X , properties (1) and (2) hold for $f = \frac{p+h}{q}$. $\square$

The next lemma is crucial for the induction step in the proof of Theorem 4.21.

Lemma 4.24. *Let $\pi : X' \rightarrow X$ denote the normalization morphism, and let $\mathfrak{p}$ be a prime ideal in $\mathcal{P}(X)$. Let $f \in \mathcal{K}(X)$ be a rational function integral over $\mathcal{P}(X)$ such that for any $\mathfrak{q} \in \text{R-Cent } \mathcal{P}(X)$ included in $\mathfrak{p}$, the localization $(f \circ \pi)_{\mathfrak{q}}$ along $\mathfrak{q}$ of the polynomial function $f \circ \pi$ on X' satisfies*

$$(f \circ \pi)_{\mathfrak{q}} \in \mathcal{P}(X)_{\mathfrak{q}} + \text{Rad}^C \mathcal{P}(X')_{\mathfrak{q}}.$$

Then there exist $q \in \mathcal{P}(X) \setminus \mathfrak{p}$ such that $qf \in \mathcal{R}^0(\text{Cent } X)$.

Proof. Consider the central part $\text{indet}^C(f)$ of the polar locus of f , namely

$$\text{indet}^C(f) = \overline{\text{Cent } X \cap \text{indet}(f)}^Z.$$

Denote by $V_1, \dots, V_s$ the irreducible components of $\text{indet}^C(f)$ which are central in X , and set $\mathfrak{q}_i = \mathcal{I}(V_i)$. Assume $\mathfrak{q}_i \subset \mathfrak{p}$ if and only if $i \leq r'$, with $0 \leq r' \leq r$. For any $i > r'$, choose $q_i \in \mathfrak{q}_i \setminus \mathfrak{p}$ and consider $f_1 = f \prod_{i > r'} q_i$.

For $i > r'$, the function f_1 can be extended continuously along V_i by imposing the value zero (the rational function f is locally bounded on X).

For $i \leq r'$, take $Z_i \subset V_i$ given by Lemma 4.23 applied to f_1 , and set $Z^1 = \bigcup_{i \leq r'} Z_i$. Remark that $\dim Z^1 < \dim \text{indet}^C(f)$.

As a consequence:

- (1) The rational function f_1 can be extended continuously along $\text{Cent } X \cap (\text{indet}^C(f) \setminus Z^1)$;
- (2) The extension remains rational restricted to any irreducible subset of $\text{indet}^C(f)$ not included in Z^1 , and central in X .

We apply now the same construction to f_1 in place of f , and to Z^1 in place of $\text{indet}^C(f)$. We obtain likewise a polynomial function $q_2 \notin \mathfrak{p}$ vanishing of the irreducible components of Z^1 central in X and not containing $\mathcal{Z}(\mathfrak{p})$, and an algebraic subset $Z^2 \subset Z^1$ with $\dim Z^2 < \dim Z^1$ such that:

- (1) The rational function $f_2 = f_1 q_2$ can be extended continuously along $\text{Cent } X \cap (Z^1 \setminus Z^2)$;
- (2) The extension remains rational restricted to any irreducible subset of Z^1 not included in Z^2 , and central in X .

It remains to repeat finitely many times the construction to achieve the proof. $\square$

Proof of Theorem 4.21. Let $\pi : X' \rightarrow X$ be the normalization of X . Note that $(\mathcal{P}(X)_{\mathfrak{q}})' = \mathcal{P}(X')_{\mathfrak{q}}$ for any prime ideal $\mathfrak{q}$ in $\mathcal{P}(X)$ since localization and normalization commutes.

We aim to prove that the natural inclusion $\mathcal{P}(X^{\text{sc}})_{\mathfrak{p}} \subset (\mathcal{P}(X)_{\mathfrak{p}})^{\text{sc}}$ is an equality. Let ϕ belongs to the right-hand side, so that by definition

$$\forall \mathfrak{q} \in \text{R-Cent } \mathcal{P}(X)_{\mathfrak{p}}, \quad \phi_{\mathfrak{q}} \in (\mathcal{P}(X)_{\mathfrak{p}})_{\mathfrak{q}} + \text{Rad}^C(\mathcal{P}(X')_{\mathfrak{p}})_{\mathfrak{q}} = \mathcal{P}(X)_{\mathfrak{q}} + \text{Rad}^C \mathcal{P}(X')_{\mathfrak{q}}.$$

In other words, there exists $a \in \mathcal{P}(X) \setminus \mathfrak{p}$ such that $f = a\phi \in \mathcal{K}(X)$ is an integral function verifying:

$$\forall \mathfrak{q} \in \text{R-Cent } \mathcal{P}(X), \quad \mathfrak{q} \subset \mathfrak{p}, \quad (f \circ \pi)_{\mathfrak{q}} \in \mathcal{P}(X)_{\mathfrak{q}} + \text{Rad}^C \mathcal{P}(X')_{\mathfrak{q}}.$$

By Lemma 4.24, there exists $q \in \mathcal{P}(X) \setminus \mathfrak{p}$ such that $qf \in \mathcal{R}^0(\text{Cent } X)$, meaning that qf is in $\mathcal{P}(X^{\text{sc}})$ by Theorem 4.15. As a consequence f belongs to $\mathcal{P}(X^{\text{sc}})_{\mathfrak{p}}$ as required. $\square$

5. Curves

The last section is devoted to centrally seminormal curves. In that situation, several particular phenomena appear. First of all, as already mentioned, continuous rational functions and hereditarily rational functions coincides on real curves, so the w_c -normalization and s_c -normalization coincide. We will show moreover that centrally seminormal curves are central. Nonetheless, the irreducible components of a centrally seminormal curve are also centrally seminormal. We end the section with a characterization of the singularities of centrally seminormal curves.

We are going to prove that a centrally seminormal curve is central. In particular, the seminormalization of any curve, relative to its central locus, becomes a central curve.

Proposition 5.1. *A centrally seminormal curve is central.*

Proof. Let $X \subset \mathbb{R}^n$ be a centrally seminormal curve. Assume X is not central, with $x \in X$ a non-central point. Let $Y \rightarrow X$ be the finite birational morphism obtained by normalizing the point x . The fiber of $Y \rightarrow X$ over x is empty and $\text{Cent } Y$ is in bijection with $\text{Cent } X$. This is in contradiction with the universal property of the w_c -normalization (Theorem 4.8). $\square$

It may happen that an irreducible component of a seminormal complex algebraic set is not seminormal [13, Example 2.11] but never for a curve [13, Corollary 2.9]. We prove a similar result for centrally seminormal real curves.

Proposition 5.2. *Let X be a real algebraic curve. If X is centrally seminormal then every irreducible component of X is centrally seminormal.*

Notice that the converse of the statement of Proposition 5.2 is false, as illustrated by the example of three lines in the plane intersecting in a common point.

Proof of Proposition 5.2. Let $X_1, \dots, X_t$ be the irreducible components of X , with $t \geq 2$. Note that X is central by Proposition 5.1 and that X is the union of the central locus of its irreducible components, which intersect two-by-two in at most finite of points.

Let $f \in \mathcal{P}(X_1^{w_c})$, we see the function f both as a polynomial function on $X_1^{w_c}$ or X'_1 and as a continuous rational function on $\text{Cent } X_1$ which is integral over $\mathcal{P}(X_1)$. We complete f with polynomial functions $p_k \in \mathcal{P}(X_k)$ such that the rational continuous functions of the t -tuple $(f, p_2, \dots, p_t) \in \mathcal{K}^0(\text{Cent } X_1) \times \dots \times \mathcal{K}^0(\text{Cent } X_t)$ coincide on the finite (if not empty) sets $\text{Cent } X_i \cap \text{Cent } X_j$ for all $i, j \in \{1, \dots, t\}$. From Proposition 1.6, we see that $(f, p_2, \dots, p_t)$ induces a continuous rational function $g \in \mathcal{K}^0(\text{Cent } X)$, which is integral over $\mathcal{P}(X)$ because $g \in \mathcal{P}(X'_1) \times \dots \times \mathcal{P}(X'_t) = \mathcal{P}(X')$ (Proposition 1.1). Since X is centrally seminormal then g is polynomial on X as a consequence of Proposition 4.4. By continuity, it follows that f is polynomial on $\text{Cent } X_1$ and we conclude that X_1 is centrally seminormal again by Proposition 4.4. $\square$

To finish to characterize centrally seminormal curves, we are lead to determine their singular points. In the case of seminormal complex curves, the singularities are only ordinary k -fold points. We say that a point x of a complex algebraic curve X is an ordinary k -fold point if x is a point of multiplicity k with k linearly independent tangents. In other words, the singularity at x is analytically isomorphic to the union of the k coordinate axes in $\mathbb{C}^k$ (see [15] for instance). Then by [7], a complex algebraic curve X is seminormal if and only if the singularities of X are ordinary k -fold points.

Coming back to a centrally seminormal real algebraic curve X , we know that $\mathcal{P}(X)$ is a seminormal ring (in Traverso's sense). As a consequence $\mathcal{P}(X_{\mathbb{C}})$ and $X_{\mathbb{C}}$ are seminormal by [13, Corollary 5.7] and therefore the singularities of the complexification $X_{\mathbb{C}}$ are ordinary k -fold points.

We will see below that when these complex singularities are moreover real, and with an additional condition it gives a characterization of centrally seminormal real curves.

Proposition 5.3. *Let X be a real algebraic curve and $\pi' : X' \rightarrow X$ be the normalization morphism. Then X is centrally seminormal if and only if the following properties are satisfied:*

- (1) *The singularities of $X_{\mathbb{C}}$ are ordinary k -fold points;*
- (2) *The singular points of $X_{\mathbb{C}}$ are real;*
- (3) *For any singular point in $X_{\mathbb{C}}$, the fiber $\pi'^{-1}(x)$ is totally real.*

Proof. Assume X is centrally seminormal. Then X is central by Proposition 5.1, and (1) holds by seminormality of $\mathcal{P}(X)$ as explained above.

Assume $X_{\mathbb{C}}$ admits as singularities two complex conjugated points (which are then non-real ordinary k -fold points). We normalize at these two points and we get an algebraic curve Y such that the morphism $Y \rightarrow X$ is birational, finite, and bijective since the real points are not impacted, but it is not an isomorphism (the map $Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is not bijective). This is in contradiction with Proposition 4.12, so we have proved (2).

Assume now there exists $x \in \text{Sing}(X)$ such that the $\pi'^{-1}(x)$, which contains at least a real point by centrality of X , is not totally real. We normalize at the point x and then we glue together the real points over x (as explained in [29]). We get a real algebraic curve Y such that the morphism $Y \rightarrow X$ is birational, finite, bijective but not an isomorphism since $Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is again not bijective. By Proposition 4.12 we get a contradiction and it proves (3).

Assume now the curve X satisfies the three properties of the proposition. Remark that (3) implies that X is central. From (2) and (3), it follows that X is centrally seminormal if and only if X is seminormal. Then (1) implies that $X_{\mathbb{C}}$ is a seminormal algebraic curve by [7]. As a consequence $\mathcal{P}(X_{\mathbb{C}})$ is seminormal, which implies that $\mathcal{P}(X)$ is seminormal by [13, Corollary 5.7] as expected. $\square$

In the situation of Proposition 5.3, we say that the real curve admits only real ordinary k -fold singularities.

We recover from Proposition 5.3 that the nodal curve $\mathcal{Z}(y^2 - x^2(x + 1))$ is centrally seminormal, whereas the cuspidal curve $\mathcal{Z}(y^2 - x^3)$ or the union $\mathcal{Z}(xy(x - y))$ of three lines in a plane are not centrally seminormal.

The notion of central seminormalization enables to initiate the classification of central real algebraic sets up to finite birational homeomorphism (a finite birational polynomial morphism which is an homeomorphism for the Euclidean topology), by establishment of the classification in the case of curves. This classification can be seen as a bridge between the topological study of (smooth) real algebraic sets, personalized for curves by the first part of the famous Hilbert sixteen problem, and birational geometry.

Theorem 5.4. *Let $X \subset \mathbb{R}^n$ be a central algebraic curve. Then X is finitely birationally homeomorphic to a central algebraic curve with only real ordinary k -fold points.*

Proof. The finite birational model homeomorphic to X having only ordinary k -fold real singularities is provided by its central seminormalization. $\square$

A consequence of the classification is that a topologically smooth real algebraic curve is finitely birationally homeomorphic to a nonsingular real algebraic curve. It would be interesting to investigate further the surface case.

We end the paper with the promised example showing that there does not exist in general a biggest algebraic set in finite birational bijection with a given algebraic set, contrary to the situation in complex algebraic geometry.

Example 5.5. We are going to construct a curve X which does not admit a biggest algebraic curve in finite birational bijection with itself. We produce X from a nonsingular real curve Y by contracting pairs of complex conjugated points. The key point is to choose for Y a curve without non-trivial automorphism, like for instance $Y = \mathbb{P}_{\mathbb{R}}^1 \setminus \{4 \text{ real points}\}$.

The construction goes as follows. Let $(P_1, \overline{P_1})$ and $(P_2, \overline{P_2})$ be two distinct pairs of conjugated points of $Y_{\mathbb{C}}$. Let Y_i be the nodal curve obtained from Y by gluing together P_i and $\overline{P_i}$, for $i \in \{1, 2\}$, and X be the curve obtained from Y by gluing together the four points $P_1, P_2, \overline{P_1}, \overline{P_2}$. The curves Y_1, Y_2 and X are real curves with a unique singular point which is an isolated real point, and Y is their common normalization:

$$\begin{array}{ccc} Y & \rightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \rightarrow & X. \end{array}$$

Assume there exists a biggest element Z among the real algebraic curves with a finite birational morphism onto X that is bijective in restriction to the real points. Remark that Z has necessarily a unique singular point which is real and isolated.

By assumption on Z , there exist finite birational morphisms $Z \rightarrow Y_1$ and $Z \rightarrow Y_2$, and they induce bijections at the level of complex points. Moreover Y_1 and Y_2 are seminormal (as complex curves) and thus the maps $Z \rightarrow Y_1$ and $Z \rightarrow Y_2$ are automatically isomorphisms. We derive an isomorphism $\varphi : Y_1 \rightarrow Y_2$ which necessarily maps the unique singular point of Y_1 onto the unique singular point of Y_2 .

We are going to lift φ to a non-trivial automorphism of Y , obtaining therefore a contradiction. Denote by $\pi_1 : Y \rightarrow Y_1$ and $\pi_2 : Y \rightarrow Y_2$ the normalization morphisms. Since $\varphi \circ \pi_1 : Y \rightarrow Y_2$ is finite and birational, there exists a finite and birational morphism $\psi : Y \rightarrow Y$ such that $\pi_2 = \varphi \circ \pi_1 \circ \psi$ by the universal property of the normalization of Y_2 . Note that $\psi(\{P_2, \overline{P_2}\}) = \{P_1, \overline{P_1}\}$, so that ψ is non trivial. Moreover it is an automorphism, whose inverse is constructed similarly by applying the universal property of the normalization of Y_1 to the map $\varphi^{-1} \circ \pi_2 : Y \rightarrow Y_1$.

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