Regular *F*-manifolds: initial conditions and Frobenius metrics

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Abstract. A regular *F*-manifold is an *F*-manifold (with Euler field) $(M, \circ, e, E)$, such that the endomorphism $\mathcal{U}(X) := E \circ X$ of TM is regular at any $p \in M$. We prove that the germ $((M, p), \circ, e, E)$ is uniquely determined (up to isomorphism) by the conjugacy class of $\mathcal{U}_p : T_p M \to T_p M$. We obtain that any regular *F*-manifold admits a preferred system of local coordinates and we find conditions, in these coordinates, for a metric to be Frobenius. We study the Lie algebra of infinitesimal symmetries of regular *F*-manifolds. We show that any regular *F*-manifold is locally isomorphic to the parameter space of a Malgrange universal connection. We prove an initial condition theorem for Frobenius metrics on regular *F*-manifolds.

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1. Introduction

Frobenius manifolds were defined in [2], by Boris Dubrovin, as a geometrization of the so called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV)-equations and appear in many areas of mathematics (quantum cohomology, singularity theory, integrable systems etc). Later on, the weaker notion of *F*-manifold was introduced in the literature by Hertling and Manin [6] and was intensively studied since then (see, *e.g.*, [1,4,9,14,18]). Rather than the usual definition of Frobenius manifolds [2], we prefer the alternative one [4] where Frobenius manifolds are viewed as an enrichment of *F*-manifolds.

Definition 1.1.   i) An *F-manifold* is a manifold M together with a (fiber preserving) bilinear commutative, associative multiplication $\circ$ on TM, with unit field e, and an additional field E (called the Euler field), such that the following

conditions hold:

$$L_{X \circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ) \qquad (1.1)$$

and

$$L_E(\circ)(X, Y) = X \circ Y,$$

for any vector fields $X, Y \in \mathcal{T}_M$;

ii) A *Frobenius manifold* is an F-manifold $(M, \circ, e, E)$ together with a (non-degenerate) flat, multiplication invariant metric g (*i.e.*, $g(X \circ Y, Z) = g(X, Y \circ Z)$, for any $X, Y, Z \in TM$), such that $L_E(g) = Dg$ (with $D \in \mathbb{C}$) and $\nabla^{LC}(e) = 0$ (where ∇^{LC} is the Levi-Civita connection of g);

iii) A *Frobenius metric* on an F-manifold $(M, \circ, e, E)$ is a metric g which makes $(M, \circ, e, E, g)$ a Frobenius manifold.

There are examples of F-manifolds which do not support locally any Frobenius metric (see Remark 8.3). In fact, it is difficult to construct explicitly Frobenius manifolds (one strong obstruction being the flatness of the metric). The semisimple case is understood. More precisely, a semisimple F-manifold admits, by definition, a co-ordinate system (u^i), called canonical, in which the multiplication takes the simple form $\partial_i \circ \partial_j = \delta_{ij}\partial_j$. Any multiplication invariant metric is diagonal in canonical coordinates, its flatness is expressed by the Darboux-Egoroff equations and Frobenius metrics exist locally on the open subset $M^{\text{tame}} := \{(u^i), \ u^i \neq u^j, \ i \neq j\}$ of tamed points (see, *e.g.*, [2]). More general classes of Frobenius manifolds can be obtained from the so called initial condition theorems, developed by Hertling and Manin in [7]. It turns out that the germ $((M, p), \circ, e, E, g)$ of certain Frobenius manifolds is determined (modulo isomorphism) by the linear data induced on the tangent space $T_p M$ and conversely, starting with an abstract linear data (called 'initial condition') one obtains a unique (up to isomorphism) germ of such Frobenius manifolds. In the semisimple case, this was already proved in [2], Lecture 3. Generalizations, where the point was replaced by an entire submanifold, were also developed in [7].

We shall be particularly interested in the relation between F-manifolds and meromorphic connections. The parameter space M of a meromorphic connection ∇ on a vector bundle over $M \times D$ (where $D \subset \mathbb{C}$ is a small disc around the origin), with poles of Poincaré rank one along $M \times \{0\}$, in Birkhoff normal form $(\frac{B_0(x)}{\tau} + B_\infty)\frac{d\tau}{\tau} + \frac{C_i(x)dx^i}{\tau}$, inherits (under additional conditions) an F-manifold structure. On the other hand, if at a point $x_0 \in M$ the matrix $B_0(x_0)$ is regular (see the comments after Definition 1.2), then, for a small neighbourhood U of x_0, $\nabla|_{U \times D}$ is uniquely determined (up to pull-backs $(f \times \text{Id})^*$ and isomorphisms) by its restriction $\nabla^0 := \nabla|_{\{x_0\} \times D}$. (For this reason, ∇^0 can be considered as the 'initial condition' for ∇). This follows from the existence of a universal integrable deformation ∇^{can} of a meromorphic connection (in our case ∇^0), on a vector bundle over D, in Birkhoff normal form, with a pole of Poincaré rank one in $\{0\}$, with regular residue. Such a universal deformation was constructed by Magrange in [10, 11]. The parameter space of more general meromorphic connections (not necessarily

in Birkhoff normal form), the so called (TE)-structures, is also an F-manifold (see [5]).

In this paper we are concerned with a large class of F-manifolds, namely the regular ones, and their relation with Frobenius metrics and meromorphic connections.

Definition 1.2. An F-manifold $(M, \circ, e, E)$ is called *regular* if the endomorphism $\mathcal{U} : TM \to TM, \mathcal{U}(X) := E \circ X$, is regular at any $p \in M$.

(An endomorphism $A : V \to V$ of a complex vector space V is regular if one of the following equivalent conditions holds: 1) any two distinct Jordan blocks from its Jordan normal form have distinct eigenvalues; 2) the characteristic and minimal polynomials of A coincide; 3) the vector space of endomorphisms of V commuting with A has dimension $n = \dim(V)$ and basis $\{\mathrm{Id}, A, \cdots, A^{n-1}\}$; 4) there is a cyclic vector for A, *i.e.*, a vector $v \in V$ such that $\{v, A(v), \cdots, A^{n-1}(v)\}$ is a basis of V.)

Our main result from this paper is as follows. Its second part can be understood as an initial condition theorem for regular F-manifolds.

Theorem 1.3.   i) *Any germ $((M, p), \circ, e, E)$ of regular F-manifolds is isomorphic to a product $\mathcal{P} := \Pi_{\alpha=1}^{n}((\mathbb{C}^{m_\alpha}, 0), \circ_\alpha, e_\alpha, E_\alpha)$ of germs of (regular) F-manifolds. Here m_α are the dimensions of the Jordan blocks of the endomorphism $\mathcal{U}_p(X) = X \circ E_p$ of T_pM. For each such block, let a_α be the corresponding eigenvalue of $\mathcal{U}_p$. In the canonical frame field $\{\partial_i := \frac{\partial}{\partial t^i}, \ 0 \leq i \leq m_\alpha - 1\}$, determined by coordinates $(t^0, \cdots, t^{m_\alpha-1}) \in \mathbb{C}^{m_\alpha}$, the multiplication $\circ_\alpha$ is given by*

$$\partial_i \circ_\alpha \partial_j = \begin{cases} \partial_{i+j}, & i + j \leq m_\alpha - 1 \\ 0, & i + j \geq m_\alpha, \end{cases} \tag{1.2}$$

and the unit field and Euler field by

$$e_\alpha = \partial_0, \quad E_\alpha = (t^0 + a_\alpha)\partial_0 + (t^1 + 1)\partial_1 + t^2\partial_2 + \cdots + t^{m_\alpha-1}\partial_{m_\alpha-1}. \tag{1.3}$$

The product $\mathcal{P}$ is canonically associated to $((M, p), \circ, e, E)$ (up to ordering of its factors) and the isomorphism between $((M, p), \circ, e, E)$ and $\mathcal{P}$ is unique (when such an ordering is fixed);

 ii) *In particular, there is a unique (up to unique isomorphism) germ of regular F-manifolds $((M, p), \circ, e, E)$, with given conjugacy class for the endomorphism $\mathcal{U}_p(X) := X \circ E_p$ of T_pM.*

(The conjugacy class of an endomorphism is determined by its Jordan normal form; two endomorphisms, defined on not necessarily the same vector space, belong to the same conjugacy class if they can be reduced to the same Jordan normal form.)

Structure of the paper.   The paper is structured as follows. In Section 2 we recall, following [2–4, 10, 11, 15], the basic definitions and results we need on Frobenius and F-manifolds, Saito bundles and meromorphic connections.

Section 3 represents a first step in the proof of Theorem 1.3. Here we prove that a regular F-manifold $(M, \circ, e, E)$ for which the endomorphism $\mathcal{U}_{p_0} = \mathcal{C}_{E_{p_0}} \in \mathrm{End}(T_{p_0}M)$ has exactly one eigenvalue (for $p_0 \in M$ fixed), is globally nilpotent around p_0 (see Definition 2.2 and Proposition 3.1).

In Section 4 we prove Theorem 1.3. One can check "by hand" that each factor $(\mathbb{C}^{m_\alpha}, \circ_\alpha, e_\alpha, E_\alpha)$ in Theorem 1.3 i) is an F-manifold, for which the multiplication at $0 \in \mathbb{C}^{m_\alpha}$ by the Euler field E_α is a Jordan block in the canonical frame $\{\partial_i\}$ of $\mathbb{C}^{m_\alpha}$, with eigenvalue a_α. This shows the existence of the germ in Theorem 1.3 ii). In Propositions 4.1 and 4.2 from this section we prove the uniqueness of the germ in Theorem 1.3 ii). The argument is based on Hertling's decomposition of F-manifolds [4], the local classification of $\{e\}$-structures [17] and the material from Section 3. The uniqueness of the germ in Theorem 1.3 ii) implies that any germ $((M, p), \circ, e, E)$ of regular F-manifolds is isomorphic to a product $\mathcal{P}$, as required in Theorem 1.3 i). The uniqueness of the isomorphisms in Theorem 1.3 i) and ii) is a consequence of the fact that any automorphism of a germ of regular F-manifolds is the identity map (see Lemma 4.3).

The next sections are devoted to applications of Theorem 1.3. The local coordinate system on any regular F-manifold $(M, \circ, e, E)$, provided by Theorem 1.3 i), is similar to the canonical coordinate system on semisimple F-manifolds. In Section 5 we study Frobenius metrics on $(M, \circ, e, E)$, in these coordinates. We find conditions for the coidentity $e^\flat$ to be closed, the unit e to be flat and, respectively, the Euler field E to preserve the metric (see Proposition 5.5). The picture is similar to the semisimple case. To express the flatness, we use Dubrovin's description of Frobenius manifolds (without Euler fields) with a maximal Abelian group of algebraic symmetries [2]. We find an alternative formulation for this description (see Proposition 5.3) and we apply it in order to obtain the conditions for a multiplication invariant metric on a regular globally nilpotent F-manifold to be Frobenius (see Theorem 5.5). The conditions are more involved than in the semisimple case, owing to the generalized Darboux-Egoroff equations (see Example 5.6).

In Section 6 we define the Lie algebra of infinitesimal symmetries of a regular F-manifold and we compute it using the coordinate system provided by Theorem 1.3 i) (see Definition 6.1 and Proposition 6.3).

In Section 7 we study the relation between regular F-manifolds and meromorphic connections. As stated above, the parameter space of certain meromorphic connections are F-manifolds, but the converse is not true (not every F-manifold can be locally obtained in this way, see Remark 8.3 b)). We prove that the converse is, however, true, under the regularity assumption (see Corollary 7.2). Namely, we determine the F-manifold structure of the parameter spaces M^{can} of the Malgrange universal deformations ∇^{can}, mentioned above, and we show that any regular F-manifold is locally isomorphic to such a parameter space (see Proposition 7.1).

In Section 8 we prove an initial condition theorem for Frobenius metrics on regular F-manifolds (see Theorem 8.2). This follows from our Theorem 1.3, combined with [7, Theorem 4.5]. While the arguments from [7] work in high generality, they are also quite technical. For completeness of our exposition, we provide in the appendix (Section A) an alternative, simple and self-contained proof for the existence of the extension of the metric in Theorem 8.2, based on our treatment of regular F-manifolds.

2. Preliminary material

This section is intended to fix notation. We work in the holomorphic category: the manifolds are complex and the vector bundles, sections, connections etc are holomorphic. We denote by $\mathcal{T}_M$ the sheaf of holomorphic vector fields on a complex manifold M, by $\mathcal{O}_M$ the sheaf of holomorphic functions on M and by $\Omega^1(M, V)$ the sheaf of holomorphic 1-forms with values in a vector bundle V. In our conventions, the connection form of a connection ∇ on a vector bundle $V \to M$, in a local basis of sections $\{s_1, \cdots, s_n\}$ of V, is the matrix valued 1-form $\Omega = (\Omega_{ij})$, defined by $\nabla_X(s_i) = \sum_{j=1}^{n}(\Omega_{ji})_X s_j$. The representation of an endomorphism $A \in \mathrm{End}(V)$ (where V is a vector space), in a basis $\{v_1, \cdots, v_n\}$ of V, is the matrix $A = (A_{ij})$, where $A(v_i) = \sum_{j=1}^{n} A_{ji} v_j$. To simplify notation, we often use the Einstein summation convention.

2.1. F-manifolds

2.1.1. Hertling's decomposition of F-manifolds

The following theorem due to Hertling (see [4, page 16]) plays an essential role in the proof of our main result. We shall use it for regular F-manifolds, but we remark that the regularity condition is not required for its statement in full generality.

Theorem 2.1. Let $((M, p), \circ, e, E)$ be a germ of F-manifolds and $a_1, \cdots, a_n$ the distinct eigenvalues of the endomorphism $\mathcal{U}(X) := X \circ E$ of TM at p. Then $((M, p), \circ, e, E)$ is isomorphic to a product $\Pi_{\alpha=1}^{n}((M_\alpha, p_\alpha), \circ_\alpha, e_\alpha, E_\alpha)$ of germs of F-manifolds. For any $1 \le \alpha \le n$, the endomorphism $\mathcal{U}_\alpha(X) := X \circ_\alpha E_\alpha$ of TM_α has precisely one eigenvalue at p_α, namely a_α.

2.1.2. Basic facts on globally nilpotent F-manifolds

Definition 2.2. An F-manifold $(M, \circ, e, E)$ is called *globally nilpotent* if, for any $p \in M$ and $X_p \in T_pM$, the endomorphism $\mathcal{C}_{X_p} : T_pM \to T_pM$, $\mathcal{C}_{X_p}(Y_p) := X_p \circ Y_p$, has exactly one eigenvalue. Equivalently, if

$$\mathcal{C}_{X_p} = \mu(X_p)\mathrm{Id} + N_{X_p},$$

with $\mu(X_p) \in \mathbb{C}$ and $N_{X_p} \in \mathrm{End}(T_pM)$ nilpotent.

For any globally nilpotent F-manifold $(M, \circ, e, E)$ and $X \in \mathcal{T}_M$, the function $\mu(X) : M \to \mathbb{C}$, $\mu(X)(p) := \mu(X_p)$ is holomorphic. Indeed, the characteristic polynomial P of $\mathcal{C}_X$ is given by $P(z, p) = (z - \mu(X_p))^n$, for any $z \in \mathbb{C}$ and $p \in M$ (where $n = \dim(M)$). Therefore, $\mu(X) = -\frac{1}{n!} P^{(n-1)}(0, \cdot) \in \mathcal{O}_M$, as P is holomorphic (the superscript $(n-1)$ denotes the $(n-1)$ derivatives with respect to z). In particular, the eigenfunction $a := \mu(E)$ of $\mathcal{U} = \mathcal{C}_E$ is holomorphic (not necessarily constant).

Recall Definition 1.2 of regular F-manifolds, from the introduction.

Lemma 2.3. *Let $(M, \circ, e, E)$ be a regular F-manifold of dimension n.*

  i) *For any $k \geq 0$, let $X_k := E \circ \cdots \circ E$ (k-times), with $X_0 := e$. The vector fields $\{X_0, X_1, \cdots, X_{n-1}\}$ are linearly independent (at any point) and*

$$\left[X_i, X_j \right] = (j - i) X_{i+j-1}, \quad i, j \geq 0; \tag{2.1}$$

  ii) *Suppose that the endomorphism $\mathcal{U}_p(X_p) = X_p \circ E_p$ of $T_p M$ has exactly one eigenvalue, for any $p \in M$. Then $(M, \circ, e, E)$ is globally nilpotent.*

Proof. The linear independence of $\{X_0, \cdots, X_{n-1}\}$ follows from regularity. Formula (2.1) was proved in [6] (and holds on any F-manifold, not necessarily regular). Claim i) follows. We now prove claim ii). By hypothesis, $\mathcal{U} = a \mathrm{Id} + N$ on TM, where $a \in \mathcal{O}_M$ and $N : TM \to TM$ is nilpotent (at any point). Let $X := f_0 X_0 + \cdots + f_{n-1} X_{n-1} \in \mathcal{T}_M$, where $f_i \in \mathcal{O}_M$. We obtain

$$\mathcal{C}_X = \left(\sum_{k=0}^{n-1} f_k a^k \right) \mathrm{Id} + \sum_{k=1}^{n-1} f_k \sum_{p=1}^{k} C_k^p a^{k-p} N^p.$$

The second term in the right hand side of the above relation is a nilpotent endomorphism. Claim ii) follows. □

Definition 2.4. The vector fields $\{X_0, \cdots, X_{n-1}\}$ from Lemma 2.3 i) form the *canonical frame* of the regular F-manifold $(M, \circ, e, E)$.

2.1.3. *Frobenius metrics on constant F-manifolds*

The F-manifolds and Frobenius manifolds we are interested in come, by definition, with an Euler field. However, often in the literature the quasi-homogeneity condition imposed by the Euler field is considered as an additional obstruction and is not required in the definition of these structures. This is true for example in Dubrovin's description of Frobenius manifolds which underlie constant F-manifolds [2]. Since we need this description in Subsection 5.2, we recall it here. We begin with the definition of constant F-manifolds.

Definition 2.5. An F-manifold $(N, \circ, e)$ (without Euler field) is called *constant* if it admits a coordinate system (called *canonical*) in which the multiplication is constant.

Such an F-manifold has an n-dimensional Abelian group of algebraic symmetries, where $n = \dim(N)$ (see [2, page 69]). In a canonical coordinate system $(t^0, \cdots, t^{n-1})$, $\partial_i \circ \partial_j = c_{ij}^k \partial_k$, where $\partial_i := \frac{\partial}{\partial t^i}$ and $c_{ij}^k \in \mathbb{C}$. (Our notation is different from that used in [2]: in this reference, (t^i) denote the flat coordinates, rather than the canonical ones). We assume that there is a constant (in canonical coordinates), multiplication invariant, (non-degenerate) metric on TN and we fix such a metric ϵ. Using ϵ, we identify TN with T^*N. The multiplication $\circ$ on TN induces a multiplication, also denoted by $\circ$, on T^*N. It is given by: $dt^i \circ dt^j = c_k^{ij} dt^k$, where $c_k^{ij} = \epsilon^{is} c_{sk}^j$. A 1-form $\psi \in \Omega^1(N)$ is called invertible if, for any $p \in N$, the covector $\psi_p \in T_p^*N$ is invertible with respect to $\circ$ (acting on T_p^*N).

Let $\gamma \in \mathrm{End}(TN)$ be an ϵ symmetric endomorphism which satisfies the generalized Darboux-Egoroff equations:

$$\left[\mathcal{C}_i, L_{\partial_j}(\gamma)\right] - \left[\mathcal{C}_j, L_{\partial_i}(\gamma)\right] + \left[[\mathcal{C}_i, \gamma], [\mathcal{C}_j, \gamma]\right] = 0, \ 0 \le i, j \le n-1, \quad (2.2)$$

where $\mathcal{C}_i := \mathcal{C}_{\partial_i}$ and $[.,.]$ denotes the commutator of endomorphisms. Such an endomorphism is called, by analogy with the semisimple case, a rotation coefficient operator. The following description of Frobenius metrics on constant F-manifolds is due to Dubrovin (see [2, Theorem 3.1]). Below $\psi[\mathcal{C}_i, \gamma] \in \Omega^1(N)$ is the composition (as maps) of $[\mathcal{C}_i, \gamma] \in \mathrm{End}(TN)$ with the 1-form $\psi : TN \to \mathbb{C}$.

Theorem 2.6 ([2])**.** *Let γ be a rotation coefficient operator on $(N, \circ, e)$ and $\psi \in \Omega^1(N)$ invertible, satisfying*

$$L_{\partial_i}(\psi) = \psi[\mathcal{C}_i, \gamma], \ 0 \le i \le n-1. \quad (2.3)$$

Then the metric

$$g(X, Y) := (\psi \circ \psi)(X \circ Y), \ X, Y \in TN \quad (2.4)$$

is Frobenius on $(N, \circ, e)$. Conversely, any Frobenius metric on $(N, \circ, e)$ is of this form, for a certain rotation coefficient operator γ and 1-form ψ.

2.2. F-manifolds, Frobenius manifolds and Saito bundles

In this paragraph we recall the relation between Frobenius or F-manifolds and Saito bundles (see, *e.g.*, [15, Chapter VII]).

Definition 2.7.   i) A *Saito bundle* is a vector bundle $(\pi : V \to M, \nabla, \Phi, R_0, R_\infty)$ with a connection ∇, a 1-form $\Phi \in \Omega^1(M, \mathrm{End}(V))$ and two endomorphisms $R_0, R_\infty \in \mathrm{End}(V)$, such that the following conditions are satisfied:

$$R^\nabla = 0, \quad \Phi \wedge \Phi = 0, \quad [R_0, \Phi] = 0$$

and

$$d^\nabla \Phi = 0, \quad \nabla R_0 + \Phi = [\Phi, R_\infty], \quad \nabla R_\infty = 0. \quad (2.5)$$

Above R^∇ is the curvature of ∇ and the End(V)-valued forms $[R, \Phi]$ (with $R := R_0$ or R_∞), $d^\nabla \Phi$ and $\Phi \wedge \Phi$ are defined by: for any $X, Y \in \mathcal{T}_M$,

$$[R, \Phi]_X := [R, \Phi_X]$$
$$(d^\nabla \Phi)_{X,Y} := \nabla_X(\Phi_Y) - \nabla_Y(\Phi_X) - \Phi_{[X,Y]}$$
$$(\Phi \wedge \Phi)_{X,Y} := \Phi_X \Phi_Y - \Phi_Y \Phi_X;$$

ii) A *Saito bundle with metric* is a Saito bundle $(V, \nabla, \Phi, R_0, R_\infty)$ with a (non-degenerate) metric $g \in S^2(V^*)$, such that the following conditions are satisfied:

$$\nabla g = 0, \quad R_\infty + R_\infty^* = 0, \quad R_0 = R_0^*, \quad \Phi_X = \Phi_X^*, \ \forall X \in TM,$$

where the superscript "$*$"denotes the g-adjoint.

Let $(\pi : V \to M, \nabla, \Phi, R_0, R_\infty)$ be a Saito bundle. Suppose there is a section s of V, such that

$$I : TM \to V, \quad I(X) := \Phi_X(s), \ X \in TM \qquad (2.6)$$

is a bundle isomorphism. Define an (associative, commutative, with unit field $e_M := I^{-1}(s)$) multiplication $\circ_M$ on TM and a vector field $E_M \in \mathcal{T}_M$ by the conditions $\Phi_{X \circ_M Y}(s) = \Phi_X \Phi_Y(s)$ and $E_M = -I^{-1}R_0(s)$. (We remark that our conventions differ from those used in [3, 15]; in these references, the identification between TM and V is done via $-I$; the induced multiplication on TM is then $-\circ_M$, but the induced fields E_M are the same). The following holds (see, *e.g.*, [3, Lemmas 4.1 and 4.3]):

Proposition 2.8. *Both $\circ_M$ and E_M are independent of the section s and $(M, \circ_M, e_M, E_M)$ is an F-manifold. The endomorphism $\mathcal{U}_M(X) = X \circ E_M$ of TM coincides with $-I^{-1}R_0 I$. In particular, $(\mathcal{U}_M)_p$ and $-(R_0)_p$ belong to the same conjugacy class, for any $p \in M$.*

Suppose now that g is a metric on V, which makes $(V, \nabla, \Phi, R_0, R_\infty, g)$ a Saito bundle with metric. Suppose that the section s from the isomorphism (2.6) is ∇-parallel and $R_\infty(s) = qs$, for $q \in \mathbb{C}$. Such a section is called primitive homogeneous. Then $g_M(X, Y) := g(I(X), I(Y))$ is a Frobenius metric on $(M, \circ_M, e_M, E_M)$. The Levi-Civita ∇^{LC} of g_M is given by $\nabla^{LC} = I^{-1} \circ \nabla \circ I$ and

$$\nabla^{LC} E_M = I^{-1} R_\infty I + (1 - q)\text{Id}. \qquad (2.7)$$

(see [15, page 239]). Conversely, any Frobenius manifold arises in this way (see [15, page 240]).

2.3. *F*-manifolds and flat meromorphic connections

Let ∇ be a flat meromorphic connection on a vector bundle E over $M \times D$ (where $D \subset \mathbb{C}$ is a small disc around the origin), with poles of Poincaré rank one along

$M \times \{0\}$, in Birkhoff normal form. By definition, this means that $E = (M \times D) \times \mathbb{C}^n \to M \times D$ is the trivial bundle and the connection form of ∇ in the standard trivialization of E is given by

$$\Omega = \left(\frac{B_0(x)}{\tau} + B_\infty \right) \frac{d\tau}{\tau} + \frac{\mathcal{C}_i(x)dx^i}{\tau}, \tag{2.8}$$

where (x^i) are coordinates on M, τ is the coordinate on D, $\mathcal{C}_i : M \to M_n(\mathbb{C})$, $B_0 : M \to M_n(\mathbb{C})$ and $B_\infty \in M_n(\mathbb{C})$. We consider B_0, B_∞ as endomorphisms (the latter, constant) of the trivial bundle $V = M \times \mathbb{C}^n \to M$ and $\mathcal{C} = \mathcal{C}_i dx^i$ as an $\mathrm{End}(V)$-valued 1-form on M. In its simplest form, the relation between F-manifolds and meromorphic connections is the following (for the more general relation between F-manifolds and (TE)-structures, see [5, Theorem 3.1]).

Proposition 2.9. *Let ∇ be a flat meromorphic connection on the trivial bundle $E = (M \times D) \times \mathbb{C}^n \to M \times D$, in Birkhoff normal form (2.8).*

i) *The trivial bundle $V = M \times \mathbb{C}^n \to M$, together with $(D^c, \mathcal{C}, B_0, -B_\infty)$, where D^c is the canonical flat connection of V and $\mathcal{C}, B_0, B_\infty$ are as above, is a Saito bundle;*

ii) *In particular, if there is a section s of V, such that (2.6) is an isomorphism, then M inherits an F-manifold structure.*

Proof. Claim i) follows from the flatness condition $d\Omega + \Omega \wedge \Omega = 0$. Claim ii) follows from Proposition 2.8. $\qquad\square$

2.4. Malgrange universal deformation

Let ∇^0 be a connection on the trivial bundle $V^0 = D \times \mathbb{C}^n \to D$, with connection form

$$\Omega^0 = \left(\frac{B_0^o}{\tau} + B_\infty \right) \frac{d\tau}{\tau}, \tag{2.9}$$

where $B_0^o, B_\infty \in M_n(\mathbb{C})$. To keep the text self-contained, we recall the definition of an integrable deformation of ∇^0.

Definition 2.10. An *integrable deformation of ∇^0* is a flat meromorphic connection ∇ on the trivial vector bundle $E = (M \times D) \times \mathbb{C}^n \to M \times D$, in Birkhoff normal form (2.8), which coincides with ∇^0 when restricted to $\{p_0\} \times D$ (where $p_0 \in M$), *i.e.* $B_0(p_0) = B_0^\circ$.

Assume now that B_0^o is regular. Then ∇^0 admits an integrable deformation ∇^{can}, constructed by Malgrange [10, 11], which is universal (see, *e.g.*, [15, page 208], for the definition of universal integrable deformations). Following Sabbah [15, Chapter VI, Section 3.a], we now recall its construction. Let $\mathcal{D} \subset T(M_n(\mathbb{C}^n))$ be defined by

$$\mathcal{D}_\Gamma := \mathrm{Span}_{\mathbb{C}} \left\{ \mathrm{Id}, (B_0)_\Gamma, \cdots, (B_0)_\Gamma^{n-1} \right\} \subset T_\Gamma M_n(\mathbb{C}) = M_n(\mathbb{C}), \tag{2.10}$$

where

$$(B_0)_\Gamma := B_0^o - \Gamma + [B_\infty, \Gamma]. \tag{2.11}$$

Because B_0^o is regular, so is $(B_0)_\Gamma$, for any $\Gamma \in W$, where W is a small open neighbourhood of 0 in $M_n(\mathbb{C})$. For any $\Gamma \in W$, $\mathcal{D}_\Gamma$ is the (n-dimensional) vector space of polynomials in $(B_0)_\Gamma$ and the distribution $\mathcal{D} \to W$ is integrable. The Malgrange universal deformation of ∇^0 is defined as follows [10,11] (also [15]):

Definition 2.11. i) The *parameter space* $M^{\mathrm{can}} = M^{\mathrm{can}}(B_0^o, B_\infty)$ of the universal deformation ∇^{can} of ∇^0 is the maximal integral submanifold of $\mathcal{D}|_W$, passing through 0;

ii) The connection ∇^{can} of ∇^0 is defined on the trivial bundle $E = (M^{\mathrm{can}} \times D) \times \mathbb{C}^n \to M^{\mathrm{can}} \times D$, with connection form in the standard trivialization of E given by

$$\Omega^{\mathrm{can}} = \left(\frac{B_0}{\tau} + B_\infty \right) \frac{d\tau}{\tau} + \frac{\mathcal{C}}{\tau}. \tag{2.12}$$

Here $B_0 : M^{\mathrm{can}} \to M_n(\mathbb{C})$, $(B_0)(\Gamma) := (B_0)_\Gamma$ is given by (2.11) and $\mathcal{C}_X := X$ is the action of the matrix X on $\mathbb{C}^n$, for any $X \in T_\Gamma M^{\mathrm{can}} \subset M_n(\mathbb{C})$.

3. Globally nilpotent regular F-manifolds

Our aim in this section is to prove the following result.

Proposition 3.1. *Let $(M, \circ, e, E)$ be an F-manifold of dimension $n \geq 2$, such that at a point $p_0 \in M$, the endomorphism $\mathcal{U}_{p_0} = (\mathcal{C}_E)_{p_0} : T_{p_0} M \to T_{p_0} M$ is regular, with exactly one eigenvalue. Then there is a neighbourhood U of p_0, such that $(U, \circ, e, E)$ is globally nilpotent (and regular).*

Proof. Let U be a small neighbourhood of p_0, such that, for any $p \in U$, the endomorphism $\mathcal{U}_p = (\mathcal{C}_E)_p : T_p M \to T_p M$ is regular, and let $P(p, z) = z^n + \sum_{k=0}^{n-1} \lambda_k(p) z^k$ be the characteristic (or minimal) polynomial of $\mathcal{U}_p$. As before, let $X_i := E \circ \ldots \circ E$ (i-times, $i \geq 0$) with $X_0 = e$. Since $P(p, \mathcal{U}_p) = 0$,

$$X_n + \sum_{k=0}^{n-1} \lambda_k X_k = 0. \tag{3.1}$$

Define the functions

$$f_k := \lambda_k - \frac{C_n^k}{n^{n-k}} \lambda_{n-1}^{n-k}, \quad 0 \leq k \leq n - 2.$$

(Remark that $f_k(p) = 0$ for $p \in U$ and all $0 \leq k \leq n - 2$, if and only if $P(p, z) = (z + \frac{\lambda_{n-1}(p)}{n})^n$, if and only if $\mathcal{U}_p$ has exactly one eigenvalue.) By hypothesis, $f_k(p_0) = 0$, for any $0 \leq k \leq n - 2$. Our aim is to compute the derivatives

$X_i(f_k)$ (with $0 \leq i \leq n - 1$) and to show, using the Cauchy-Kovalevskaia theorem, that $f_k = 0$ on U, for any $0 \leq k \leq n - 2$. This implies that $\mathcal{U}$ has exactly one eigenvalue at any point of U and we conclude from Lemma 2.3 that $(U, \circ, e, E)$ is globally nilpotent, as required. Details are as follows.

We take the Lie derivative of (3.1) with respect to X_s ($s \geq 0$) and we use (2.1). We obtain

$$(n - s)X_{s+n-1} + \sum_{k=0}^{n-1} (X_s(\lambda_k)X_k + (k - s)\lambda_k X_{s+k-1}) = 0, \qquad (3.2)$$

which is equivalent, by taking $\lambda_n := 1$, to

$$\sum_{k=0}^{n-1} X_s(\lambda_k)X_k + \sum_{k=0}^{n}(k - s)\lambda_k X_{s+k-1} = 0. \qquad (3.3)$$

Relation (3.3), with $s = 0$, gives

$$X_0(\lambda_k) = -(k + 1)\lambda_{k+1}, \ 0 \leq k \leq n - 1. \qquad (3.4)$$

Relation (3.3), with $s = 1$, gives

$$\sum_{k=0}^{n-1} X_1(\lambda_k)X_k + \sum_{k=0}^{n}(k - 1)\lambda_k X_k = 0.$$

From (3.1), $X_n = -\sum_{k=0}^{n-1} \lambda_k X_k$. Replacing this expression of X_n into the above relation we obtain

$$X_1(\lambda_k) = (n - k)\lambda_k, \quad 0 \leq k \leq n - 1. \qquad (3.5)$$

The computation of $X_2(\lambda_k)$ is done in the same way, but is a bit more complicated. Taking in (3.3) $s = 2$ we obtain

$$\sum_{k=0}^{n-1} X_2(\lambda_k)X_k + \sum_{k=0}^{n}(k - 2)\lambda_k X_{k+1} = 0$$

or, equivalently,

$$\sum_{k=0}^{n-1} X_2(\lambda_k)\mathcal{U}^k + \sum_{k=0}^{n}(k - 2)\lambda_k \mathcal{U}^{k+1} = 0.$$

Since $P(p, \cdot)$ is the minimal polynomial of $\mathcal{U}_p$ (for any $p \in U$) there are holomorphic functions $b_0, b_1 \in \mathcal{O}_U$ such that

$$\sum_{k=0}^{n-1} X_2(\lambda_k)z^k + \sum_{k=0}^{n}(k - 2)\lambda_k z^{k+1} = (b_0 + b_1 z) \sum_{k=0}^{n}\lambda_k z^k. \qquad (3.6)$$

Identifying in (3.6) the coefficients of z^{n+1} we obtain $b_1 = n - 2$. Relation (3.6) becomes

$$\sum_{k=0}^{n}(X_2(\lambda_k) - b_0\lambda_k)z^k - \sum_{k=1}^{n}(n - k + 1)\lambda_{k-1}z^k = 0. \tag{3.7}$$

Then (from the coefficient of z^0), $X_2(\lambda_0) = b_0\lambda_0$, and the remaining terms in (3.7) give

$$\sum_{k=1}^{n}(X_2(\lambda_k) - b_0\lambda_k - (n - k + 1)\lambda_{k-1})z^k = 0. \tag{3.8}$$

Using $\lambda_n = 1$, we obtain $b_0 = -\lambda_{n-1}$ (from the coefficient of z^n) and relation (3.8) becomes

$$X_2(\lambda_k) = -\lambda_{n-1}\lambda_k + (n - k + 1)\lambda_{k-1}, \ 0 \leq k \leq n - 1. \tag{3.9}$$

(We use the convention $\lambda_i = 0$ for $i < 0$; similarly, below $f_i = 0$ whenever $i < 0$.)

Next, we compute the derivatives $X_3(\lambda_k)$. Relation (3.3), with $s = 3$, gives, by a similar argument as for $s = 2$,

$$X_3(\lambda_k) = \left(\lambda_{n-1}^2 - 2\lambda_{n-2}\right)\lambda_k + (n-k+2)\lambda_{k-2} - \lambda_{n-1}\lambda_{k-1}, \ 0 \leq k \leq n-1. \tag{3.10}$$

From relations (3.4), (3.5), (3.9) and (3.10), we obtain, from long but straightforward computations, the expressions for the derivatives $X_i(f_k)$ (for any $0 \leq i \leq 3$ and $0 \leq k \leq n - 2$):

$$X_0(f_k) = -(k+1)f_{k+1} \ (k \leq n-3), \ X_0(f_{n-2}) = 0, \quad X_1(f_k) = (n - k)f_k \tag{3.11}$$

and

$$X_2(f_k) = -\lambda_{n-1}f_k + (n - k + 1)f_{k-1} - \frac{2C_n^k(n - k)}{n^{n-k}}\lambda_{n-1}^{n-k-1}f_{n-2},$$

$$X_3(f_k) = \left(\lambda_{n-1}^2 - 2\lambda_{n-2}\right)f_k + \frac{C_n^k(3n - 3k - 2)}{n^{n-k}}\lambda_{n-1}^{n-k}f_{n-2} \tag{3.12}$$

$$+ (n - k + 2)f_{k-2} - \lambda_{n-1}f_{k-1} - \frac{3C_n^k(n - k)}{n^{n-k}}\lambda_{n-1}^{n-k-1}f_{n-3}.$$

In particular, both (3.11) and (3.12) are of the form

$$X_i(f_k) = \sum_{s=0}^{n-2} a_{ks}^{(i)} f_s, \ 0 \leq i \leq 3, \ 0 \leq k \leq n - 2, \tag{3.13}$$

for some $a_{ks}^{(i)} \in \mathcal{O}_U$. Since $[X_i, X_j] = (j - i)X_{i+j-1}$ (see relation (2.1)), we obtain that the derivatives $X_i(f_k)$, for any $0 \leq i \leq n - 1$ (and $0 \leq k \leq n - 2$),

are of the form (3.13). In a coordinate chart $\chi = (y^0, \cdots, y^{n-1}) : U \to \mathbb{C}^n$ with $\chi(p_0) = 0$ we obtain

$$\frac{\partial \left(f_k \circ \chi^{-1} \right)}{\partial y^i} = \sum_{s=0}^{n-2} b_{ks}^{(i)} \left(f_s \circ \chi^{-1} \right), \quad 0 \le i \le n-1, \quad 0 \le k \le n-2,$$

for some $b_{ks}^{(i)} \in \mathcal{O}_{\chi(U)}$. Also, $(f_k \circ \chi^{-1})(0) = 0$ for any $0 \le k \le n-2$. Applying successively the uniqueness statement of the Cauchy-Kovalevskaia theorem (in the form stated, *e.g.*, in [7], relations (2.42) and (2.43), with no (t^i)-parameters in the notation of this reference), we obtain that $f_k = 0$ on U, as required. $\qquad \square$

The computations from the above proof imply the following corollary.

Corollary 3.2. *Let $(M, \circ, e, E)$ be an n-dimensional globally nilpotent regular F-manifold and $\{X_0, \cdots, X_{n-1}\}$ its canonical frame. Let $a \in \mathcal{O}_M$ be the eigenfunction of $\mathcal{U} = \mathcal{C}_E$. Then*

$$[X_i, X_j] = \begin{cases} (j-i)X_{i+j-1}, & i+j \le n \\ (i-j) \sum_{k=0}^{n-1} c_k^{(i+j-1-n)} a^{i+j-1-k} X_k, & i+j > n, \end{cases} \tag{3.14}$$

where $c_k^{(p)}$ ($p \ge 0$ and $0 \le k \le n-1$) are constants, defined inductively by $c_k^{(0)} = (-1)^{n-k} C_n^k$ and for any $s \ge 0$, $c_k^{(s+1)} = c_{k-1}^{(s)} - c_k^{(0)} c_{n-1}^{(s)}$ (when $k \ge 1$) and $c_0^{(s+1)} = -c_0^{(0)} c_{n-1}^{(s)}$. Moreover,

$$X_i(a) = a^i, \quad i \ge 0. \tag{3.15}$$

Proof. Relation (3.14) for $i+j \le n$ is just (2.1). We now prove (3.14) for $i+j > n$. Since $(\mathcal{U} - a\mathrm{Id})^n = 0$,

$$\mathcal{U}^n + \sum_{k=0}^{n-1} c_k^{(0)} a^{n-k} \mathcal{U}^k = 0. \tag{3.16}$$

Multiplying the above relation with $\mathcal{U}, \mathcal{U}^2$, etc, and using an induction argument, we obtain

$$\mathcal{U}^{n+s} + \sum_{k=0}^{n-1} c_k^{(s)} a^{n-k+s} \mathcal{U}^k = 0, \quad s \ge 0,$$

or, equivalently,

$$X_{n+s} = - \sum_{k=0}^{n-1} c_k^{(s)} a^{n-k+s} X_k, \quad s \ge 0. \tag{3.17}$$

Relations (2.1) and (3.17) imply (3.14) for $i+j > n$, as required.

It remains to prove (3.15). With the notation from the proof of Proposition 3.1, $a = -\frac{\lambda_{n-1}}{n}$ and $\lambda_k = C_n^k(-a)^{n-k}$, for any $0 \leq k \leq n-1$ (because $f_k = 0$, $(M, \circ, e, E)$ being globally nilpotent). On the other hand, in the proof of Proposition 3.1 we computed the following derivatives:

$$X_0(\lambda_{n-1}) = -n, \; X_1(\lambda_{n-1}) = \lambda_{n-1}$$
$$X_2(\lambda_{n-1}) = -\lambda_{n-1}^2 + 2\lambda_{n-2}$$
$$X_3(\lambda_{n-1}) = \lambda_{n-1}^3 + 3\lambda_{n-3} - 3\lambda_{n-1}\lambda_{n-2}.$$

These expressions, written in terms of a, give (3.15), for $0 \leq i \leq 3$. Using (2.1) we obtain (3.15), for any $i \geq 0$. $\qquad\square$

Remark 3.3. For the proof of Theorem 1.3 (next section), it is convenient to express the Lie brackets $[X_i, X_j]$, computed in Corollary 3.2, in a unified form (not as in (3.14), where the cases $i + j \leq n$ and $i + j > n$ are separated). This can be done as follows. Consider the constants $c_k^{(p)}$ from Corollary 3.2. They were defined for $p \geq 0$ and $0 \leq k \leq n-1$. For $p < 0$ (and $0 \leq k \leq n-1$), let $c_k^{(p)} := 0$, unless $p = k - n$, in which case $c_k^{(k-n)} := -1$. With this notation, the two relations (3.14) reduce to the single one

$$[X_i, X_j] = (i - j) \sum_{k=0}^{n-1} c_k^{(i+j-1-n)} a^{i+j-1-k} X_k, \quad i, j \geq 0.$$

4. Proof of Theorem 1.3

Using the material from the previous section, we now prove Theorem 1.3. With the explanations from the introduction, the only statements which need to be proved are the uniqueness (up to isomorphism) of the germ in Theorem 1.3 ii) and the uniqueness of the isomorphisms in Theorem 1.3 i) and ii).

We begin by proving the uniqueness of the germ. Consider two germs $((M, p), \circ_M, e_M, E_M)$ and $((N, q), \circ_N, e_N, E_N)$ of n-dimensional F-manifolds. Let $\mathcal{U}_M \in \text{End}(TM)$ and $\mathcal{U}_N \in \text{End}(TN)$ be the endomorphisms given by the multiplication with the Euler fields. We assume that $(\mathcal{U}_M)_p : T_pM \to T_pM$ and $(\mathcal{U}_N)_q : T_qN \to T_qN$ are regular and belong to the same conjugacy class. Our aim is to show that the two germs are isomorphic. Owing to Hertling's decomposition of F-manifolds (see Theorem 2.1), we can (and will) assume that $(\mathcal{U}_M)_p$ and $(\mathcal{U}_N)_q$ have exactly one eigenvalue. From Proposition 3.1, the germs $((M, p), \circ_M, e_M, E_M)$ and $((N, q), \circ_N, e_N, E_N)$ are globally nilpotent. Let $a \in \mathcal{O}_M$ and $b \in \mathcal{O}_N$ be the eigenfunction of $\mathcal{U}_M$ and $\mathcal{U}_N$, respectively. Since $(\mathcal{U}_M)_p$ and $(\mathcal{U}_N)_q$ belong to the same conjugacy class, $a(p) = b(q)$. We denote by $\{X_i := (E_M)^i, 0 \leq i \leq n-1\}$ and $\{Y_i := (E_N)^i, \; 0 \leq i \leq n-1\}$ the canonical frames of the two germs.

Proposition 4.1. *In the above setting, there is a biholomorphic transformation ψ : $(M, p) \to (N, q)$, such that $\psi_*(X_i) = Y_i$ for any $0 \leq i \leq n - 1$ and $b \circ \psi = a$.*

Proof. The statement follows from the classification of $\{e\}$-structures (see [17, Theorem 4.1, page 344]). For completeness of our exposition we present the argument in detail, by adapting the proof of [17, Theorem 4.1] to our setting. Let $\{\omega_i^X\ 0 \leq i \leq n - 1\}$ and $\{\omega_i^Y,\ 0 \leq i \leq n - 1\}$ be the 1-forms dual to $\{X_i,\ 0 \leq i \leq n - 1\}$ and $\{Y_i,\ 0 \leq i \leq n - 1\}$, respectively. From Remark 3.3,

$$d\omega_k^X = \sum_{0 \leq i, j \leq n-1} \left(c^X\right)_{ij}^k \omega_i^X \wedge \omega_j^X, \quad \left(c^X\right)_{ij}^k := -\frac{1}{2}(i-j)c_k^{(i+j-1-n)}a^{i+j-1-k} \quad (4.1)$$

and similarly

$$d\omega_k^Y = \sum_{0 \leq i, j \leq n-1} \left(c^Y\right)_{ij}^k \omega_i^Y \wedge \omega_j^Y, \quad \left(c^Y\right)_{ij}^k := -\frac{1}{2}(i-j)c_k^{(i+j-1-n)}b^{i+j-1-k}. \quad (4.2)$$

From relation (3.15),

$$da = \sum_{i=0}^{n-1} a^i \omega_i^X, \quad db = \sum_{i=0}^{n-1} b^i \omega_i^Y. \quad (4.3)$$

In particular, $d_p a \in T_p^* M$ and $d_q b \in T_q^* N$ are non-trivial. We restrict M and N such that $d_x a \neq 0$ and $d_y b \neq 0$, for any $x \in M$ and $y \in N$. Let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the natural projections, $\theta_i^X := (\pi_1)^*(\omega_i^X)$ and $\theta_i^Y := (\pi_2)^*(\omega_i^Y)$ for any i. Pulling back (4.3) to $M \times N$, we obtain

$$d\left(\pi_1^* a\right) = \sum_{i=0}^{n-1} \left(\pi_1^* a\right)^i \theta_i^X, \quad d\left(\pi_2^* b\right) = \sum_{i=0}^{n-1} \left(\pi_2^* b\right)^i \theta_i^Y. \quad (4.4)$$

Let S be the $(2n - 1)$-dimensional submanifold of $M \times N$, defined by

$$S := \{(x, y) \in M \times N,\ a(x) = b(y)\}.$$

Remark that $(p, q) \in S$. Let $\vec{i} : S \to M \times N$ be the inclusion. We will show that

$$\mathcal{D}_S := \mathrm{Span}\left\{(\vec{i})^* \left(\theta_i^X - \theta_i^Y\right),\ 0 \leq i \leq n - 1\right\} \subset T^* S$$

is a rank $(n - 1)$ subbundle of $T^* S$. For this, we remark from (4.4) that

$$\theta_0^X - \theta_0^Y = d\left(\pi_1^* a - \pi_2^* b\right) - \sum_{i=1}^{n-1} \left(\left(\pi_1^* a\right)^i \theta_i^X - \left(\pi_2^* b\right)^i \theta_i^Y\right). \quad (4.5)$$

Restricting (4.5) to TS and using that $\pi_1^*a = \pi_2^*b$ on S (and $d(\pi_1^*a - \pi_2^*b) = 0$ on TS), we obtain that $(\vec{i})^*(\theta_0^X - \theta_0^Y)$ is a linear combination of the remaining $(\vec{i})^*(\theta_i^X - \theta_i^Y)$, $1 \leq i \leq n-1$. We deduce that $\mathrm{rank}(\mathcal{D}_S) \leq n-1$. On the other hand, since S is of codimension one in $M \times N$, the kernel of the map $(\vec{i})^* : T_{(x,y)}^*(M \times N) \to T_{(x,y)}^*S$ is one dimensional (for any $(x, y) \in S$). We deduce that the kernel of the restriction of this map to $\mathrm{Span}\{(\theta_i^X - \theta_i^Y)_{(x,y)}, \ 0 \leq i \leq n-1\} \subset T_{(x,y)}^*(M \times N)$ is at most one dimensional. Since $\{(\theta_i^X - \theta_i^Y)_{(x,y)}\}$ are linearly independent, we obtain that $\dim(\mathcal{D}_S)_{(x,y)} \geq n - 1$. We conclude that $\dim(\mathcal{D}_S)_{(x,y)} = n - 1$, for any $(x, y) \in S$, i.e., $\mathcal{D}_S$ is of rank $n - 1$, as needed.

We now prove that $\mathrm{Ker}(\mathcal{D}_S)$ is integrable. Pulling back the first relations (4.1) and (4.2) to S, and using $\pi_1^*(c^X)_{ij}^k = \pi_2^*(c^Y)_{ij}^k$ on S (which follows from the definition of $(c^X)_{ij}^k$ and $(c^Y)_{ij}^k$ and from $\pi_1^*a = \pi_2^*b$ on S), we obtain that

$$d\left(\theta_k^X - \theta_k^Y\right) = \sum_{i,j} \pi_1^*\left(c^X\right)_{ij}^k \left(\theta_i^X - \theta_i^Y\right) \wedge \theta_j^X + \sum_{i,j} \pi_2^*\left(c^X\right)_{ij}^k \theta_i^Y \wedge \left(\theta_j^X - \theta_j^Y\right), \ \forall k$$

i.e., $\mathrm{Ker}(\mathcal{D}_S)$ is an integrable distribution on S.

Consider now the integral submanifold S' of $\mathrm{Ker}(\mathcal{D}_S)$ which contains (p, q). It is of dimension $\dim(S) - \mathrm{rank}(\mathcal{D}_S) = (2n - 1) - (n - 1) = n$. We claim that $\{(\theta_i^X)|_{TS'}, \ 0 \leq i \leq n - 1\}$ is a basis of forms on S'. Indeed, since the forms $\{(\theta_i^X)_{(x,y)}, (\theta_i^X - \theta_i^Y)_{(x,y)}, \ 0 \leq i \leq n - 1\}$ are a basis of $T_{(x,y)}^*(M \times N)$, their restriction to $T_{(x,y)}S'$ generate $T_{(x,y)}^*S'$. Therefore, $\{(\theta_i^X)|_{T_{(x,y)}S'}, \ 0 \leq i \leq n - 1\}$ generate $T_{(x,y)}^*S'$, hence form a basis of $T_{(x,y)}^*S'$ (because $(\theta_i^X - \theta_i^Y)_{(x,y)}$ vanishes on $T_{(x,y)}S' = \mathrm{Ker}(\mathcal{D}_S)_{(x,y)}$ and $\dim(S') = n$). We proved that $\{(\theta_i^X)|_{TS'}, \ 0 \leq i \leq n - 1\}$ is a basis of forms on S', as needed. From this fact and $(\pi_1|_{S'})^*(\omega_i^X) = (\theta_i^X)|_{TS'}$, we obtain that $\pi_1|_{S'} : S' \to M$ is locally a biholomorphic transformation. A similar argument shows that $\pi_2|_{S'} : S' \to N$ is also, locally, a biholomorphic transformation. We restrict the representatives M and N of the germs, such that $\pi_1 : S' \to M$ and $\pi_2 : S' \to N$ are biholomorphic transformations and we define $\psi := \pi_2 \circ \pi_1^{-1}$. Since $(p, q) \in S'$, $\psi(p) = q$. Since any $(x, y) \in S'$ satisfies $a(x) = b(y)$, we obtain that $b \circ \psi = a$. Since $\pi_1^*(\omega_i^X) = \pi_2^*(\omega_i^Y)$ on TS', we obtain that $\psi^*(\omega_i^Y) = \omega_i^X$, i.e., $\psi_*(X_i) = Y_i$, for any $0 \leq i \leq n - 1$. $\square$

Proposition 4.2. *The map* $\psi : ((M, p), \circ_M, e_M, E_M) \to ((N, q), \circ_N, e_N, E_N)$ *from Proposition* 4.1 *is an isomorphism of germs of F-manifolds.*

Proof. From Proposition 4.1 the map ψ preserves the unit and Euler fields. It remains to check that it preserves the multiplications, too. From regularity, this is equivalent to $\psi_*(E_M^i) = E_N^i$, for any $i \geq 0$. The statement for $i \leq n - 1$ follows from Proposition 4.1 (since $E_M^i = X_i$ and $E_N^i = Y_i$ for such i). We need to prove that $\psi_*(E_M^i) = E_N^i$ also for $i \geq n$. For this, we notice that the characteristic polynomials of $(\mathcal{U}_M)_x$ and $(\mathcal{U}_N)_{\psi(x)}$ coincide, for any $x \in M$ (both $(\mathcal{U}_M)_x$ and $(\mathcal{U}_N)_{\psi(x)}$

are regular, defined on vector spaces of the same dimension, with the same (unique) eigenvalue $a(x) = (b \circ \psi)(x))$. Therefore, for any $i \geq n$, the coordinates of $(E_M)_x^i$ in the basis $\{(X_j)_x, \, 0 \leq j \leq n-1\}$ coincide with the coordinates of $(E_N)_{\psi(x)}^i$ in the basis $\{(Y_j)_{\psi(x)}, \, 0 \leq j \leq n-1\}$. Using that $\psi_*(X_j) = Y_j$ we deduce that $\psi_*((E_M^i)_x) = (E_N^i)_{\psi(x)}$, i.e., $\psi_*(E_M^i) = E_N^i$, as needed. $\qquad\square$

The uniqueness of the isomorphisms required by Theorem 1.3 i) and ii) is a consequence of the following simple lemma, which concludes the proof of Theorem 1.3.

Lemma 4.3. *Any automorphism of a germ* $((M,p), \circ, e, E)$ *of regular F-manifolds is the identity map.*

Proof. Let ψ be such an automorphism. Then $\psi_*(E^i) = E^i$ for any $i \geq 0$. From regularity, $\psi_*(X) = X$, i.e., $\phi_t^X \circ \psi = \psi \circ \phi_t^X$, where ϕ_t^X is the flow of X and $X \in \mathcal{T}_M$ is arbitrary. Since $\psi(p) = p$, we obtain that ψ is the identity map. $\qquad\square$

In the following sections we develop applications of Theorem 1.3.

5. Frobenius metrics in canonical coordinates

In this section we study Frobenius metrics in the coordinate system provided by Theorem 1.3 i). In Subsection 5.1 we express the conditions which involve the unit and Euler fields. The flatness of the metric will be treated in Subsection 5.2.

5.1. The unit and Euler fields

Let $M := \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_n}$. We denote by $(t^{i(\alpha)})$ $(0 \leq i \leq m_\alpha - 1, 1 \leq \alpha \leq n)$ the canonical coordinates on M and by $\{\partial_{i(\alpha)} := \frac{\partial}{\partial t^{i(\alpha)}}\}$ the associated vector fields. According to Theorem 1.3 i), the multiplication

$$\partial_{i(\alpha)} \circ \partial_{j(\beta)} = \begin{cases} \partial_{(i+j)(\alpha)} & \alpha = \beta, \, i+j \leq m_\alpha - 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the vector field

$$E = \sum_{\alpha=1}^n \left(\left(t^{0(\alpha)} + a_\alpha\right) \partial_{0(\alpha)} + \left(t^{1(\alpha)} + 1\right) \partial_{1(\alpha)} + \sum_{i=2}^{m_\alpha-1} t^{i(\alpha)} \partial_{i(\alpha)} \right),$$

give M the structure of an F-manifold, with unit field

$$e = \sum_{\alpha=1}^n \partial_{0(\alpha)}.$$

Any multiplication invariant metric on M is of the form

$$g = \delta_{\alpha\beta}\eta_{(i+j)(\alpha)}dt^{i(\alpha)} \otimes dt^{j(\beta)}, \tag{5.1}$$

for some functions $\eta_{i(\alpha)}$, where $1 \le \alpha, \beta \le n, 0 \le i \le m_\alpha - 1$ and $\eta_{i(\alpha)} = 0$, for $i \ge m_\alpha$.

Proposition 5.1.   i) *The coidentity $e^\flat := g(e, \cdot)$ is closed if and only if there is a function H (called a metric potential) such that $\eta_{i(\alpha)} = \partial_{i(\alpha)}(H)$ for any $i(\alpha)$;*

  ii) *The unit field e is flat (with respect to the Levi-Civita connection ∇^{LC} of g) if and only if $d(e^\flat) = 0$ and $e(\eta_{i(\alpha)}) = 0$, for any $i(\alpha)$;*

 iii) *The Euler field rescales g (i.e., $L_E(g) = Dg$ for a constant D) if and only if $E(\eta_{i(\alpha)}) = (D - 2)\eta_{i(\alpha)}$ for any $i(\alpha)$.*

Proof. Since $e = \sum_{\alpha=1}^n \partial_{0(\alpha)}$, the coidentity is given by $e^\flat = \eta_{i(\alpha)}dt^{i(\alpha)}$. It is closed if and only if it is exact, *i.e.*, $e^\flat = dH$, for a function H. Claim i) follows. For claim ii), we use that $\nabla^{\text{LC}}(e) = 0$ if and only if $d(e^\flat) = 0$ and $L_e(g) = 0$. But

$$L_e(g)\left(\partial_{i(\alpha)}, \partial_{j(\beta)}\right) = \delta_{\alpha\beta}e\left(\eta_{(i+j)(\alpha)}\right) - g\left([e, \partial_{i(\alpha)}], \partial_{j(\beta)}\right) - g\left(\partial_{i(\alpha)}, [e, \partial_{j(\beta)}]\right)$$
$$= \delta_{\alpha\beta}e\left(\eta_{(i+j)(\alpha)}\right),$$

where in the second line we used $e = \sum_{\alpha=1}^n \partial_{0(\alpha)}$ and the fact that the vector fields $\{\partial_{i(\alpha)}\}$ commute. Claim ii) follows. Claim iii) follows equally easily. $\qquad\square$

5.2. The flatness condition

It remains to study the flatness. For this, let us consider again Dubrovin's description of Frobenius metrics on constant (not necessarily regular) F-manifolds, recalled in Subsection 2.1.3. In Lemma 5.2 we prove that the rotation coefficient operator is determined (modulo a term $\mathcal{C}_X$, for $X \in \mathcal{T}_N$) by the Frobenius metric. Therefore, the generalized Darboux-Egoroff equations (2.2) may be written directly in terms of the metric (rather than the rotation coefficient operator). Proposition 5.3 below is a rewriting of [2, Theorem 3.1]. We will apply it in order to obtain a description of Frobenius metrics on regular, globally nilpotent F-manifolds (see Theorem 5.5).

We use the notation from Subsection 2.1.3. In particular, we identify TN with T^*N using $\epsilon = \epsilon_{ij}dt^i \otimes dt^j$. We denote by $\epsilon^{-1} : T^*N \to TN$ this isomorphism. The induced metric on T^*N will also be denoted by ϵ. It is given by $\epsilon = \epsilon^{ij}\partial_i \otimes \partial_j$ where (ϵ^{ij}) is the inverse of (ϵ_{ij}).

Lemma 5.2. *Let $(N, \circ, e)$ be a constant F-manifold, with constant multiplication invariant metric $\epsilon \in S^2(T^*N)$, $\psi = \psi_j dt^j \in \Omega^1(N)$ an invertible 1-form and $T = \epsilon^{-1}(\psi) \in \mathcal{T}_N$ the ϵ dual vector field. There is an ϵ symmetric endomorphism $\tilde{\gamma} \in \text{End}(TN)$ which satisfies*

$$L_{\partial_i}(\psi) = \psi[\mathcal{C}_i, \tilde{\gamma}], \ \forall i, \tag{5.2}$$

if and only if $\epsilon(\psi, \psi)$ is constant and the endomorphism $\gamma \in \text{End}(TN)$, defined by

$$\gamma = L_{\epsilon^{-1}(dt^i)}(\psi) \otimes \left(\partial_i \circ T^{-1}\right), \tag{5.3}$$

is ϵ symmetric. If $\epsilon(\psi, \psi)$ is constant and γ is ϵ symmetric, then γ satisfies (5.2), and, moreover, any other ϵ symmetric endomorphism $\tilde{\gamma}$, which satisfies (5.2), is of the form $\tilde{\gamma} = \gamma + \mathcal{C}_X$, for $X \in \mathcal{T}_N$.

Proof. We divide the proof into several steps.

Step 1. We claim that the operator γ defined by (5.3) satisfies

$$\gamma(T) = \frac{1}{2}\epsilon^{-1}\left(dt^i\right)(\epsilon(\psi, \psi))\partial_i \circ T^{-1}. \tag{5.4}$$

To prove (5.4), we use the definition of γ and $L_{\partial_i}(\epsilon) = 0$:

$$\gamma(T) = L_{\epsilon^{-1}(dt^i)}(\psi)(T)\partial_i \circ T^{-1} = \epsilon\left(L_{\epsilon^{-1}(dt^i)}(\psi), \psi\right)\partial_i \circ T^{-1}$$
$$= \frac{1}{2}\epsilon^{-1}\left(dt^i\right)(\epsilon(\psi, \psi))\partial_i \circ T^{-1}.$$

Step 2. We claim that any ϵ symmetric endomorphism $\tilde{\gamma}$ of TN satisfies

$$\tilde{\gamma} = \epsilon^{ik}\psi[\mathcal{C}_k, \tilde{\gamma}](\partial_j)dt^j \otimes \left(\partial_i \circ T^{-1}\right) + \mathcal{C}_{\tilde{\gamma}(T)\circ T^{-1}}. \tag{5.5}$$

To prove (5.5), let $\tilde{\gamma}$ be such an endomorphism. Using that $\mathcal{C}_i$ and $\tilde{\gamma}$ are ϵ symmetric, we obtain

$$\psi[\mathcal{C}_i, \tilde{\gamma}]\left(\partial_j\right) = \epsilon\left(T, [\mathcal{C}_i, \tilde{\gamma}](\partial_j)\right) = \epsilon\left(T \circ \tilde{\gamma}(\partial_j) - \tilde{\gamma}(T) \circ \partial_j, \partial_i\right). \tag{5.6}$$

From $X = \epsilon^{ik}\epsilon(X, \partial_k)\partial_i$, for any $X \in \mathcal{T}_N$, and relation (5.6), we obtain

$$T \circ \tilde{\gamma}\left(\partial_j\right) - \tilde{\gamma}(T) \circ \partial_j = \epsilon^{ik}\epsilon\left(T \circ \tilde{\gamma}(\partial_j) - \tilde{\gamma}(T) \circ \partial_j, \partial_k\right)\partial_i$$
$$= \epsilon^{ik}\psi[\mathcal{C}_k, \tilde{\gamma}]\left(\partial_j\right)\partial_i,$$

which implies (5.5).

Step 3. We claim that if there is an ϵ symmetric endomorphism $\tilde{\gamma}$ of TN, which satisfies (5.2), then $\epsilon(\psi, \psi)$ is constant and the operator γ, defined by (5.3), is ϵ symmetric. Let $\tilde{\gamma}$ be such an endomorphism. From (5.2) and $L_{\partial_i}(\epsilon) = 0$,

$$\frac{1}{2}\partial_i(\epsilon(\psi, \psi)) = \epsilon(L_{\partial_i}(\psi), \psi) = \epsilon(\psi[\mathcal{C}_i, \tilde{\gamma}], \psi) = -\epsilon([\mathcal{C}_i, \tilde{\gamma}](T), T) = 0. \tag{5.7}$$

(In the third equality we used that $[\mathcal{C}_i, \tilde{\gamma}] \in \text{End}(TN)$ is ϵ skew-symmetric; owing to this, the 1-form $\psi[\mathcal{C}_i, \tilde{\gamma}] \in \Omega^1(N)$ is ϵ dual to $-[\mathcal{C}_i, \tilde{\gamma}](T) \in \mathcal{T}_N$. In the fourth equality we used again that $[\mathcal{C}_i, \tilde{\gamma}]$ is ϵ skew-symmetric). Relation (5.7) shows that

$\epsilon(\psi, \psi)$ is constant. Using (5.5) ($\tilde{\gamma}$ is ϵ symmetric), (5.2) and $\epsilon^{-1}(dt^i) = \epsilon^{ik}\partial_k$, we obtain

$$\tilde{\gamma} = \epsilon^{ik}\partial_k(\psi_j)dt^j \otimes \left(\partial_i \circ T^{-1}\right) + \mathcal{C}_{\tilde{\gamma}(T)\circ T^{-1}} = \gamma + \mathcal{C}_{\tilde{\gamma}(T)\circ T^{-1}}. \qquad (5.8)$$

Since $\tilde{\gamma}$ is ϵ symmetric, so is γ. Our claim follows.

Step 4. We assume that $\epsilon(\psi, \psi)$ is constant and γ is ϵ symmetric. We claim that γ satisfies (5.2). Since $\epsilon(\psi, \psi)$ is constant, $\gamma(T) = 0$ (see relation (5.4)). Since γ is symmetric and $\gamma(T) = 0$, relation (5.5) implies that

$$\gamma = \epsilon^{ik}\psi[\mathcal{C}_k, \gamma](\partial_j)dt^j \otimes \left(\partial_i \circ T^{-1}\right). \qquad (5.9)$$

On the other hand, from its definition (5.3),

$$\gamma = \epsilon^{ik}\partial_k(\psi_j)dt^j \otimes \left(\partial_i \circ T^{-1}\right). \qquad (5.10)$$

Combining (5.9) with (5.10) we obtain that $\partial_k(\psi_j) = \psi[\mathcal{C}_k, \gamma](\partial_j)$, *i.e.*, γ satisfies (5.2), as claimed.

Step 5. In the hypothesis from Step 4, we claim that any other ϵ symmetric endomorphism $\tilde{\gamma}$, which satisfies (5.2), is equal to $\gamma + \mathcal{C}_{\tilde{\gamma}(T)\circ T^{-1}}$. Let $\tilde{\gamma}$ be such an endomorphism. Since it is ϵ symmetric, it satisfies (5.5). Using (5.2), relation (5.5) becomes $\tilde{\gamma} = \gamma + \mathcal{C}_{\tilde{\gamma}(T)\circ T^{-1}}$, as needed. $\qquad \square$

Proposition 5.3. *Let $(N, \circ, e)$ be a constant F-manifold and $\epsilon \in S^2(T^*N)$ a constant, multiplication invariant metric. Let $\psi = \psi_j dt^j \in \Omega^1(N)$ be an invertible 1-form. Then the metric*

$$g(X, Y) := (\psi \circ \psi)(X \circ Y)$$

is Frobenius on $(N, \circ, e)$ if and only if $\epsilon(\psi, \psi)$ is constant and the endomorphism

$$\gamma = L_{\epsilon^{-1}(dt^i)}(\psi) \otimes \left(\partial_i \circ T^{-1}\right) \qquad (5.11)$$

is ϵ symmetric and satisfies the generalized Darboux-Egoroff equations (2.2). Above $T = \epsilon^{-1}(\psi) \in \mathcal{T}_N$ is ϵ dual to ψ.

Proof. From Theorem 2.6, g is Frobenius if and only if there is an ϵ symmetric endomorphism $\tilde{\gamma} \in \text{End}(TN)$ (a rotation coefficient operator), which satisfies (5.2) and the generalized Darboux-Egoroff equations (2.2). From Lemma 5.2, the existence of an ϵ symmetric endomorphism $\tilde{\gamma}$, which satisfies (5.2), is equivalent to the ϵ symmetry of γ and to $\epsilon(\psi, \psi)$ being constant. Suppose that these equivalent conditions hold. From Lemma 5.2 again, $\gamma = \tilde{\gamma} - \mathcal{C}_{\tilde{\gamma}(T)\circ T^{-1}}$. Therefore, $\tilde{\gamma}$ satisfies the generalized Darboux-Egoroff equations if and only if γ does. Our claim follows. $\qquad \square$

Remark 5.4. There is an alternative formula for the endomorphism γ from Lemma 5.2, which is more suitable for computations. Let c_{ij}^k and c_k^{ij} be the structure constants, in canonical coordinates (t^i), of the multiplications on TN and T^*N. We claim that

$$\gamma = \partial_k(\psi_j)\beta_s\epsilon^{ik}c_i^{st}dt^j \otimes \partial_t, \tag{5.12}$$

where $\beta = \sum_i \beta_i dt^i \in \Omega^1(N)$ is the inverse of ψ. Relation (5.12) is obtained from the following computation: from (5.3),

$$\gamma = \epsilon^{ik}\partial_k(\psi_j)dt^j \otimes (\partial_i \circ \epsilon^{-1}(\beta)) = \epsilon^{ik}\partial_k(\psi_j)\beta_s dt^j \otimes (\partial_i \circ \epsilon^{-1}(dt^s))$$

$$= \partial_k(\psi_j)\beta_s\epsilon^{ik}\epsilon^{sf}dt^j \otimes (\partial_i \circ \partial_f) = \partial_k(\psi_j)\beta_s\epsilon^{ik}\epsilon^{sf}c_{if}^t dt^j \otimes \partial_t$$

$$= \partial_k(\psi_j)\beta_s\epsilon^{ik}c_i^{st}dt^j \otimes \partial_t,$$

where we used $\epsilon^{-1}(dt^s) = \epsilon^{sf}\partial_f$, $T^{-1} = \epsilon^{-1}(\beta)$ and $c_i^{st} = \epsilon^{sf}c_{if}^t$.

We now return to the setting of regular F-manifolds. For simplicity, we assume that $(M, \circ, e, E)$ is globally nilpotent (and regular, of dimension m). Let $(t^0, \cdots, t^{m-1})$ be the coordinate system of M provided by Theorem 1.3 i) and $\epsilon \in S^2(T^*M)$ the (multiplication invariant) metric given by

$$\epsilon = \epsilon_{ij}dt^i \otimes dt^j, \quad \epsilon_{ij} = \epsilon(\partial_i, \partial_j) := \delta_{i+j, m-1}. \tag{5.13}$$

We identify TM with T^*M using ϵ. The induced multiplication on T^*M is given by $dt^i \circ dt^j = dt^{i+j-(m-1)}$ (with the convention $dt^s = 0$ when $s \geq m$ or $s < 0$) and dt^{m-1} is the unit. A 1-form $\psi = \psi_j dt^j \in \Omega^1(M)$ is invertible if and only if ψ_{m-1} is non-vanishing. If ψ is invertible and $\beta = \beta_j dt^j$ is its inverse, then

$$\beta_{m-1}\psi_{m-1} = 1, \quad \sum_{r+s=k} \beta_s\psi_r = 0, \ m-1 \leq k < 2(m-1). \tag{5.14}$$

The following theorem is our main result from this section.

Theorem 5.5. *Let $(M, \circ, e, E)$ be a regular, globally nilpotent, m-dimensional F-manifold, with fixed constant metric $\epsilon \in S^2(T^*M)$ given by (5.13). Let g be a multiplication invariant metric, given by*

$$g = \eta_{i+j}dt^i \otimes dt^j. \tag{5.15}$$

We fix a branch of $(\eta_{m-1})^{1/2}$.

i) *There is a unique invertible 1-form $\psi = \psi_j dt^j \in \Omega^1(M)$, related to g by*

$$g(X, Y) = (\psi \circ \psi)(X \circ Y), \ X, Y \in TM. \tag{5.16}$$

Its $(m-1)$-component is given by $\psi_{m-1} = (\eta_{m-1})^{1/2}$ and its remaining components are determined inductively by the conditions:

$$\sum_{s+t=(m-1)+k} \psi_s\psi_t = \eta_k, \ 0 \leq k \leq m-2; \tag{5.17}$$

ii) *The metric g is Frobenius on $(M, \circ, e)$ if and only if*

$$\epsilon(\psi, \psi) = \sum_{i+j=m-1} \psi_i \psi_j$$

is constant and

$$\gamma := \sum_j \sum_{i \leq s} \beta_s \partial_{m-1-i}(\psi_j) \partial_{m-1+i-s} \otimes dt^j$$

is ϵ symmetric and satisfies the generalized Darboux-Egoroff equations (2.2), *where $\beta := \beta_j dt^j \in \Omega^1(M)$ is the inverse of ψ. In particular, if g is Frobenius then there is (locally) a function H such that $\eta_i = \partial_i(H)$, for any i, and $\partial_i(H)$ is independent of t^0;*

iii) *The metric g is Frobenius on $(M, \circ, e, E)$ if and only if the conditions from ii) hold and, moreover, $E(\eta_i) = (D - 2)\eta_i$ for any i.*

Proof. The proof follows from Propositions 5.1 and 5.3 and relation (5.12). □

Example 5.6. We consider the setting of Theorem 5.5.

i) The metric ϵ itself is Frobenius with $\psi = dt^{m-1}$ and $\gamma = 0$;

ii) Assume that $m = 2$. The 1-form ψ, its inverse β, the operator γ and $\epsilon(\psi, \psi)$ are given by

$$\psi = \frac{1}{2}\eta_0(\eta_1)^{-1/2}dt^0 + (\eta_1)^{1/2}dt^1, \quad \beta = -\frac{1}{2}(\eta_1)^{-3/2}\eta_0 dt^0 + (\eta_1)^{-1/2}dt^1$$

$$\gamma(\partial_i) = \beta_1 \partial_1(\psi_i)\partial_0 + (\beta_0 \partial_1(\psi_i) + \beta_1 \partial_0(\psi_i))\partial_1, \quad 0 \leq i \leq 1$$

$$\epsilon(\psi, \psi) = 2\psi_0\psi_1 = \eta_0.$$

The generalized Darboux-Egoroff equations reduce to $[\mathcal{C}_1, L_{\partial_0}(\gamma)] = 0$. The unit field e is flat if and only if $\partial_0(\eta_1) = \partial_1(\eta_0)$ and η_i are independent of t^0 ($i = 1, 2$). Suppose that e is flat. Then η_0 is constant, the generalized Darboux-Egoroff equations are automatically satisfied and γ is ϵ symmetric. A metric is Frobenius on $(M, \circ, e)$ if and only if it is of the form $g = \dot{f}(dt^1 \otimes dt^1 + dt^1 \otimes dt^0)$, where $f = f(t^1)$ and its derivative $\dot{f}$ (with respect to t^1) is non-vanishing. The metric g is Frobenius on $(M, \circ, e, E)$ if, moreover, $t^1 \ddot{f} = (D - 2)\dot{f}$, for a constant $D \in \mathbb{C}$;

iii) Assume that $m = 3$. The 1-form $\psi = \psi_j dt^j$, its inverse $\beta = \beta_j dt^j$ and $\epsilon(\psi, \psi)$ are given by

$$\psi = \left(\frac{1}{2}\eta_0(\eta_2)^{-1/2} - \frac{1}{8}(\eta_1)^2(\eta_2)^{-3/2}\right)dt^0 + \frac{1}{2}\eta_1(\eta_2)^{-1/2}dt^1 + (\eta_2)^{1/2}dt^2$$

$$\beta = \left(-\frac{1}{2}\eta_0(\eta_2)^{-3/2} + \frac{3}{8}(\eta_1)^2(\eta_2)^{-5/2}\right)dt^0 - \frac{1}{2}\eta_1(\eta_2)^{-3/2}dt^1$$
$$+ (\eta_2)^{-1/2}dt^2$$

$$\epsilon(\psi, \psi) = 2\psi_0\psi_2 + (\psi_1)^2 = \eta_0.$$

Suppose that e is flat. Like in the case $m = 2$, η_i are independent of t^0 and $\partial_i(\eta_j) = \partial_j(\eta_i)$, for any i, j. In particular, η_0 is constant. The operator γ is given by: for any $0 \leq i \leq 2$,

$$\gamma(\partial_i) = \partial_2(\psi_i)\beta_2\partial_0 + (\partial_2(\psi_i)\beta_1 + \partial_1(\psi_i)\beta_2)\partial_1 + (\beta_0\partial_2(\psi_i) + \beta_1\partial_1(\psi_i))\partial_2.$$

It is ϵ symmetric if and only if $\gamma_{10} = \gamma_{21}$, $\gamma_{00} = \gamma_{22}$ and $\gamma_{01} = \gamma_{12}$, where $\gamma(\partial_i) = \gamma_{ji}\partial_j$. Suppose that these relations are satisfied. The generalized Darboux-Egoroff equations become the highly non-trivial condition

$$\big[\mathcal{C}_1, L_{\partial_2}(\gamma)\big] - \big[\mathcal{C}_2, L_{\partial_1}(\gamma)\big] + \big[[\mathcal{C}_1, \gamma], [\mathcal{C}_2, \gamma]\big] = 0,$$

which, in terms of γ_{ij}, gives

$$\partial_2(\gamma_{11} - \gamma_{00}) - \partial_1(\gamma_{01}) + (\gamma_{01})^2 - (\gamma_{11} - \gamma_{00})\gamma_{02} = 0$$
$$\partial_2(\gamma_{01}) - \partial_1(\gamma_{02}) - \gamma_{02}\gamma_{01} = 0$$
$$\partial_2(\gamma_{02}) + (\gamma_{02})^2 = 0.$$

6. Infinitesimal symmetries in canonical coordinates

Definition 6.1. An *infinitesimal symmetry* of an F-manifold $(M, \circ, e, E)$ is a vector field X which preserves the multiplication and the Euler field:

$$L_X(\circ) = 0, \quad [X, E] = 0.$$

Using the Jacobi identity and the general formula $L_{[X,Y]} = [L_X, L_Y]$ for the Lie derivative, we obtain that the set $\mathcal{L}$ of infinitesimal symmetries of any F-manifold is a subalgebra of the Lie algebra of vector fields. In this section we compute the Lie algebra $\mathcal{L}$ of germs of regular F-manifolds. According to [4, Theorem 2.11], an infinitesimal symmetry of a product F-manifold decomposes into a product of infinitesimal symmetries of the factors. The Lie algebra $\mathcal{L}$ decomposes accordingly and, from Theorem 1.3, there is no loss of generality to assume that the germ is the standard model $((\mathbb{C}^m, 0), \circ, e, E)$, with coordinates $(t^0, \cdots, t^{m-1})$ and F-manifold structure given by (1.2) and (1.3) (with no index α). We begin with the following lemma.

Lemma 6.2. *A vector field X on $((\mathbb{C}^m, 0), \circ, e, E)$ satisfies $L_X(\circ) = 0$ if and only if*

$$[\partial_0, X] = 0, \quad [\partial_1, X] \circ \partial_{m-1} = 0,$$
$$[\partial_i, X] = i\partial_{i-1} \circ [\partial_1, X], \quad 2 \leq i \leq m-1. \tag{6.1}$$

Proof. For any vector field X,

$$L_X(\circ)(\partial_i, \partial_j) = \begin{cases} [X, \partial_{i+j}] - [X, \partial_i] \circ \partial_j - \partial_i \circ [X, \partial_j] & i+j \leq m-1 \\ -[X, \partial_i] \circ \partial_j - \partial_i \circ [X, \partial_j] & i+j \geq m. \end{cases}$$

In particular,

$$L_X(\circ)(\partial_0, \partial_0) = [\partial_0, X],$$
$$L_X(\circ)(\partial_1, \partial_{j-1}) = [X, \partial_j] - [X, \partial_1] \circ \partial_{j-1} - \partial_1 \circ [X, \partial_{j-1}], \quad 2 \leq j \leq m-1,$$
$$L_X(\circ)(\partial_1, \partial_{m-1}) = -[X, \partial_1] \circ \partial_{m-1} - \partial_1 \circ [X, \partial_{m-1}].$$

By induction, we obtain that the right hand side of these relations vanish if and only if the relations (6.1) hold. Moreover, if the relations (6.1) hold, then $L_X(\circ)(\partial_i, \partial_j) = 0$, for any i, j (easy check). $\qquad\square$

Proposition 6.3. *The system of vector fields* $\{Y_1, \cdots, Y_{m-1}\}$, *defined by*

$$Y_1 := (t^1 + 1)\partial_1 + \sum_{j=2}^{m-1} jt^j \partial_j, \quad Y_k := \partial_{k-1} \circ Y_1,\ 2 \leq k \leq m-1,$$

is a basis of the Lie algebra $\mathcal{L}$ *of infinitesimal symmetries of the standard model* $((\mathbb{C}^m, 0), \circ, e, E)$ *and*

$$[Y_i, Y_j] = \begin{cases} (i-j)Y_{i+j-1} & i+j \leq m \\ 0 & i+j > m. \end{cases} \qquad (6.2)$$

Proof. It is easy to check that Y_1 satisfies the relations (6.1) and $[E, Y_1] = 0$, *i.e.*, Y_1 belongs to $\mathcal{L}$. Using that $Y_1 \in \mathcal{L}$, $L_{\partial_k}(\circ) = 0$ and that E is an Euler field, we obtain: for any $k \geq 2$,

$$[E, Y_k] = [E, \partial_{k-1} \circ Y_1] = [E, \partial_{k-1}] \circ Y_1 + \partial_{k-1} \circ [E, Y_1] + \partial_{k-1} \circ Y_1 = 0$$
$$L_{Y_k}(\circ) = L_{\partial_{k-1} \circ Y_1}(\circ) = \partial_{k-1} \circ L_{Y_1}(\circ) + Y_1 \circ L_{\partial_{k-1}}(\circ) = 0.$$

We proved that $Y_k \in \mathcal{L}$, for any $k \geq 1$. Relation (6.2) can be checked directly.

Consider now an arbitrary vector field $X \in \mathcal{L}$. We write it as $X = f_0 \partial_0 + f_1 Y_1 + \cdots + f_{m-1} Y_{m-1}$, where f_k are functions. We will prove that $f_0 = 0$ and f_k are constant, for any $k \geq 1$. For any s (sufficiently close to 0), Φ_s^X is an automorphism of the F-manifold. Within the F-manifold, the hypersurfaces $\{t \mid t^0 + a = \text{const}\}$ are the submanifolds where the only eigenvalue of $\mathcal{U} = E\circ$, namely $t^0 + a$, is constant. As Φ_s^X is an automorphism which respects multiplication and Euler field, it does not change this eigenvalue. Therefore the flow of X respects the hypersurfaces $\{t \mid t^0 + a = \text{const}\}$ and we obtain that $f_0 = 0$. Let $Z := \sum_{k=1}^{n-1}(f_k - f_k(0))Y_k$. The flow of Θ_s^Z, for any s, fixes the point 0, so it is an automorphism of the germ $((\mathbb{C}^m, 0), \circ, e, E)$. By Lemma 4.3, $\Theta_s^Z = \text{Id}$. This implies $Z = 0$ and $X = \sum_{k=1}^{n-1} f_k(0)Y_k$. $\qquad\square$

Remark 6.4. The F-manifold in Theorem 2 i), $(\mathbb{C}^m, \circ, e, E)$ with $\circ$, e and E given there (with no index α), is regular and globally nilpotent on $\mathbb{C} \times (\mathbb{C} - \{1\}) \times \mathbb{C}^{m-2}$. By Theorem 2 ii), for any two values $t_1, t_2 \in \mathbb{C} \times (\mathbb{C} - \{1\}) \times \mathbb{C}^{m-2}$ with $t_1^0 = t_2^0$, the germs $((\mathbb{C}^m, t_1), \circ, e, E)$ and $((\mathbb{C}^m, t_2), \circ, e, E)$ are isomorphic, and the isomorphism is unique. It is of the form Φ_s^X, for suitably chosen $X \in \mathcal{L}$ and s.

7. Regular F-manifolds and meromorphic connections

Let ∇^0 be a meromorphic connection on the trivial vector bundle $V^0 = D \times \mathbb{C}^n \to D$ (where D is a small disc around the origin in $\mathbb{C}$), with connection form Ω^0 given by (2.9), in the standard trivialization of V^0. We assume that $B_0^o \in M_n(\mathbb{C})$ is regular. Let $M^{\mathrm{can}} = M^{\mathrm{can}}(B_0^o, B_\infty)$ be the parameter space of the Malgrange universal deformation ∇^{can} of ∇^0 (see Definition 2.11). Recall that it is the maximal integrable submanifold of the distribution $\mathcal{D}|_W$, defined by (2.10), passing through 0. The tangent bundle TM^{can} admits a natural multiplication $\circ_{\mathrm{can}}$: for any $\Gamma \in M^{\mathrm{can}}$, $(\circ_{\mathrm{can}})_\Gamma$, acting on $T_\Gamma M^{\mathrm{can}} = \mathcal{D}_\Gamma \subset M_n(\mathbb{C})$, is the multiplication of matrices (it preserves $\mathcal{D}_\Gamma$). It is clear that $\circ_{\mathrm{can}}$ is associative, commutative, with unit field $(\mathrm{Id}_{\mathrm{can}})_\Gamma = \mathrm{Id}$ (the identity matrix), for any $\Gamma \in M^{\mathrm{can}}$.

Proposition 7.1. i) *The multiplication $\circ_{\mathrm{can}}$ gives M^{can} the structure of a (regular) F-manifold, with Euler field*

$$(E_{\mathrm{can}})_\Gamma := -(B_0)_\Gamma = -B_0^o + \Gamma - [B_\infty, \Gamma], \quad \Gamma \in M^{\mathrm{can}}; \qquad (7.1)$$

ii) *Conversely, let $(M, \circ, e, E)$ be a regular F-manifold, $p \in M$, and $-B_0^o$ the representation of $\mathcal{U}_p : T_p M \to T_p M, \mathcal{U}_p(X) = X \circ E_p$, in a basis of $T_p M$. Let B_∞ be any matrix and $M^{\mathrm{can}} := M^{\mathrm{can}}(B_0^o, B_\infty)$. The germs $((M, p), \circ, e, E)$ and $((M^{\mathrm{can}}, 0), \circ_{\mathrm{can}}, \mathrm{Id}_{\mathrm{can}}, E_{\mathrm{can}})$ are isomorphic.*

Proof. Let $V = M^{\mathrm{can}} \times \mathbb{C}^n \to M^{\mathrm{can}}$ be the trivial bundle. Elements of V are pairs (Γ, v) where $\Gamma \in M^{\mathrm{can}}$ and $v \in \mathbb{C}^n$. We shall denote by $V_\Gamma = \mathbb{C}^n$ the fiber of V at $\Gamma \in M^{\mathrm{can}}$. From Proposition 2.9 and relation (2.12), ∇^{can} induces a Saito structure $(D^c, \Phi, B_0, -B_\infty)$ on V, as follows: D^c is the canonical flat connection (the constant sections of V are D^c-flat); $\Phi \in \Omega^1(M^{\mathrm{can}}, \mathrm{End}(V))$ is given by $\Phi_X = X \in M_n(\mathbb{C}) = \mathrm{End}(V_\Gamma)$, for any $X \in T_\Gamma(M^{\mathrm{can}}) \subset M_n(\mathbb{C}^n)$ (*i.e.*, for any $v \in V_\Gamma = \mathbb{C}^n$, $\Phi_X(v) = X(v)$ is the action of the matrix X on the vector v); $(B_0)_\Gamma, (B_\infty)_\Gamma \in \mathrm{End}(V_\Gamma)$ are given by

$$(B_0)_\Gamma = B_0^o - \Gamma + [B_\infty, \Gamma], \ (B_\infty)_\Gamma = B_\infty.$$

Let $v \in \mathbb{C}^n$ be a cyclic vector for B_0^o and $s \in \Gamma(V)$ the associated constant section. Thus, $s : M^{\mathrm{can}} \to V = M^{\mathrm{can}} \times \mathbb{C}^n, s(\Gamma) = (\Gamma, v)$, for any $\Gamma \in M^{\mathrm{can}}$. The map

$$I : TM^{\mathrm{can}} \to V, \ I(X) := \Phi_X(s) = (\Gamma, X(v)), \ X \in T_\Gamma M^{\mathrm{can}}$$

is an isomorphism. From the definition of $\circ_{\mathrm{can}}$ and E_{can}, $\Phi_{X \circ_{\mathrm{can}} Y}(s) = \Phi_X \Phi_Y(s)$ and $\Phi_{E_{\mathrm{can}}}(s) = -B_0(s)$, *i.e.*, $E_{\mathrm{can}} = -I^{-1}B_0(s)$. It follows that $(\circ_{\mathrm{can}}, E_{\mathrm{can}})$ is induced from the Saito bundle $(V, D^c, \Phi, B_0, -B_\infty)$, as in Proposition 2.8. In particular, $(M^{\mathrm{can}}, \circ_{\mathrm{can}}, \mathrm{Id}_{\mathrm{can}}, E_{\mathrm{can}})$ is a (regular) F-manifold, as required. This proves claim i).

For claim ii), let $\mathcal{U}_{\mathrm{can}} \in \mathrm{End}(TM^{\mathrm{can}})$ be defined by $\mathcal{U}_{\mathrm{can}}(X) := X \circ_{\mathrm{can}} E_{\mathrm{can}}$. From Proposition 2.8, $(\mathcal{U}_{\mathrm{can}})_0$ is conjugated to $-(B_0)_0 = -B_0^o$. Since $-B_0^o$ is the representation of $\mathcal{U}_p$ in a basis of $T_p M$, $(\mathcal{U}_{\mathrm{can}})_0$ and $\mathcal{U}_p$ belong to the same conjugacy class. We conclude with Theorem 1.3. $\square$

Corollary 7.2. *Any regular F-manifold* $(M, \circ, e, E)$ *is the parameter space of an integrable deformation of a meromorphic connection on* $V^0 = D \times \mathbb{C}^n \to D$, *in Birkhoff normal form, with a pole of Poincaré rank one in the origin.*

Proof. Trivial, from Proposition 7.1 ii). $\qquad\qquad\square$

8. Initial conditions for Frobenius metrics

In this section we prove an initial condition theorem for Frobenius metrics on regular F-manifolds (see Theorem 8.2 below). Our argument relies on Theorem 1.3 and the theory developed in [7]. A self contained proof for the existence of a Frobenius metric with given initial condition, which avoids the technicalities of [7], will be presented in Section A. The following remark justifies the properties of $\mathcal{V}_p$ from Theorem 8.2.

Remark 8.1. If $(M, \circ, e, E, g)$ is a Frobenius manifold and $L_E(g) = Dg$ then $\nabla^{\mathrm{LC}} E = \mathcal{V} + \frac{D}{2}\mathrm{Id}$, where ∇^{LC} is the Levi-Civita connection of g and $\mathcal{V}$ is the g skew-symmetric part of $\nabla^{\mathrm{LC}} E$. Using $[e, E] = e$ and $\nabla^{\mathrm{LC}}(e) = 0$, we obtain $\mathcal{V}(e) = (1 - \frac{D}{2})e$.

Our main result from this section is the following.

Theorem 8.2. *Let* $(M, \circ, e, E)$ *be a regular F-manifold and* $p \in M$. *Suppose that* $g_p \in S^2(T_p^*M)$ *and* $\mathcal{V}_p \in \mathrm{End}(T_pM)$ *are given, such that the following conditions are satisfied:*

i) g_p *is multiplication invariant and non-degenerate;*
ii) $\mathcal{V}_p$ *is* g_p *skew-symmetric and* $\mathcal{V}_p(e_p) = (1 - \frac{D}{2})e_p$, *for* $D \in \mathbb{C}$.

Then g_p *can be extended to a unique Frobenius metric g on the germ* $((M, p), \circ, e, E)$, *such that* $(\nabla^{\mathrm{LC}} E)|_{T_pM} = \mathcal{V}_p + \frac{D}{2}\mathrm{Id}$.

Proof. We consider the linear data $(T_pM, \mathcal{U}_p, \mathcal{V}_p, g_p)$ (as usual, $\mathcal{U}_p$ is the multiplication by E_p). From regularity, e_p together with $\mathcal{U}_p^k(e_p)$ ($k \geq 1$), generate T_pM. Therefore, we can apply [7, Theorem 4.5], with the Frobenius type structure reduced to the vector space $(T_pM, \mathcal{U}_p, \mathcal{V}_p, g_p)$ and $\tau := e_p$ (see also [7, Remark 4.6]). We obtain a germ of Frobenius manifolds $((\tilde{M}, \tilde{p}), \tilde{\circ}, \tilde{e}, \tilde{E}, \tilde{g})$, with $L_{\tilde{E}}(\tilde{g}) = D\tilde{g}$, and an isomorphism

$$j : (T_pM, e_p, \mathcal{U}_p, \mathcal{V}_p, g_p) \to \left(T_{\tilde{p}}\tilde{M}, \tilde{e}_{\tilde{p}}, \tilde{\mathcal{U}}_{\tilde{p}}, (\tilde{\nabla}^{\mathrm{LC}} \tilde{E})|_{T_{\tilde{p}}\tilde{M}} - \frac{D}{2}\mathrm{Id}, \tilde{g}_{\tilde{p}} \right) \quad (8.1)$$

(where $\tilde{\mathcal{U}}_{\tilde{p}}$ is the multiplication by $\tilde{E}_{\tilde{p}}$ and $\tilde{\nabla}^{\mathrm{LC}}$ is the Levi-Civita connection of $\tilde{g}$.) Since $j(e_p) = \tilde{e}_{\tilde{p}}$ and $j \circ \mathcal{U}_p = \tilde{\mathcal{U}}_{\tilde{p}} \circ j$, we obtain that $j(E_p^k) = \tilde{E}_{\tilde{p}}^k$, for any $k \geq 0$. Since $\mathcal{U}_p$ and $\tilde{\mathcal{U}}_{\tilde{p}}$ are conjugated, the germs $((M, p), \circ, e, E)$ and $((\tilde{M}, \tilde{p}), \tilde{\circ}, \tilde{e}, \tilde{E})$

are isomorphic (from Theorem 1.3). Let $f : ((M, p), \circ, e, E) \to ((\tilde{M}, \tilde{p}), \tilde{\circ}, \tilde{e}, \tilde{E})$ be an isomorphism and $g := f^* \tilde{g}$. The metric g is Frobenius on $((M, p), \circ, e, E)$. Since $f_*(e) = \tilde{e}$, $f_*(E) = \tilde{E}$ and f_* preserves multiplications, $f_*(E^k) = \tilde{E}^k$, for any $k \geq 0$. In particular, $(f_*)_p(E_p^k) = \tilde{E}_{\tilde{p}}^k$ and hence $(f_*)_p = j$. It follows that $g|_{T_p M \times T_p M} = j^*(\tilde{g}_{\tilde{p}}) = g_p$, i.e., g extends g_p. The Levi-Civita connections ∇^{LC} and $\tilde{\nabla}^{\mathrm{LC}}$ are related by

$$f_* \nabla_X^{\mathrm{LC}}(Y) = \tilde{\nabla}_{f_*(X)}^{\mathrm{LC}} f_*(Y), \quad X, Y \in \mathcal{T}_M.$$

Applying this relation to $Y := E$, using that $f_*(E) = \tilde{E}$, $(f_*)_p = j$ and

$$\left(\left(\tilde{\nabla}^{\mathrm{LC}} \tilde{E} \right) |_{T_{\tilde{p}} \tilde{M}} - \frac{D}{2} \mathrm{Id} \right) \circ j = j \circ \mathcal{V}_p$$

(from (8.1)), we obtain

$$\nabla_{X_p}^{\mathrm{LC}}(E) = j^{-1} \tilde{\nabla}_{j(X_p)}^{\mathrm{LC}}(\tilde{E}) = \mathcal{V}_p(X_p) + \frac{D}{2} X_p, \ X_p \in T_p M,$$

as required. The existence of the extension is proved.

The unicity follows also from [7, Theorem 4.5]. More precisely, from this theorem we know that any two extensions of $\tilde{g}_p$, with the required properties, are related by an isomorphism of the germ $((M, p), \circ, e, E)$. But any such isomorphism is the identity map (see Lemma 4.3). Our claim follows. $\square$

Remark 8.3. In [6, Chapter 3] it was asked whether there exist F-manifolds which do not admit, in the neighbourhood of any point, any Frobenius metric. There are F-manifolds for which the answer to this question is not known (*e.g.*, some generically semisimple F-manifolds near points where they are not semisimple). Below we describe two sources of examples for which the answer is negative.

a) [4, Proposition 5.32 and Remark 5.33] provide examples of germs $(M, 0)$ of generically semisimple F-manifolds such that $T_0 M$ is a local algebra, but not a Frobenius algebra, so it does not allow a nondegenerate multiplication invariant metric. In [4, Proposition 5.32] the F-manifolds are 3 dimensional, and $T_0 M$ is as an algebra isomorphic to $\mathbb{C}\{x, y\}/(x^2, xy, y^2)$.

b) There are examples of (globally nilpotent) F-manifolds which do not support any Frobenius metric. Such F-manifolds are described in [8, Sections 2.5.2 and 2.5.3]. Recall that an associative, commutative, with unit multiplication $\circ$ on the tangent bundle TM of a manifold M defines a (possible non-reduced) subvariety Y of $T^* M$, the spectral cover, by the ideal $I = (y^0 - 1, y^i y^j - \sum_k a_{ij}^k(x) y^k) \subset \mathcal{O}_{T^* M}$, where (x^i) are coordinates on M, with $\partial_0 = e$ the unit field, (x^i, y^j) are the induced coordinates on $T^* M$ and a_{ij}^k are defined by $\partial_i \circ \partial_j = a_{ij}^k \partial_k$. The integrability condition (1.1) from the definition of F-manifolds is equivalent to $\{I, I\} \subset I$, where $\{\cdot, \cdot\}$

is the canonical Poisson bracket of T^*M (see [8, Theorem 2.5]). The reduced variety Y_{red}, defined by $\sqrt{I}$, is the support of the Higgs bundle $(TM, \mathcal{C}_X(Y) = X \circ Y)$:

$$Y_{\mathrm{red}} = \cup_{x \in M} \left\{ \lambda \in T_x^* M, \quad \forall X \in T_x M, \ \ker(\mathcal{C}_X - \lambda(X)\mathrm{id} : T_x M \to T_x M) \neq 0 \right\}.$$

If the F-manifold can be enriched to a Frobenius manifold (even without Euler field), this induces on the pull back of T^*M to $\mathbb{C} \times M$ a (T)-structure (in the notation of [3]) respectively a holonomic $\mathcal{R}_{\mathcal{X}}$ module (in the notation of [16], where $X = M$). This is essentially the construction of the Saito bundle from the Frobenius manifold, but without the data from the Euler field. A result of Sabbah ([16, Proposition 1.2.5]) on holonomic $\mathcal{R}_{\mathcal{X}}$-modules says that the reduced variety Y_{red} is Lagrangian, or, equivalently, $\{\sqrt{I}, \sqrt{I}\} \subset \sqrt{I}$. The ideals defining the spectral covers in the examples of F-manifolds from [8], mentioned above, do not satisfy this last condition. Thus, these F-manifolds do not support any Frobenius metric.

Appendix

A. Proof of Theorem 8.2 revised

As promised in Section 8, we develop here an alternative argument for the existence of the extended metric in Theorem 8.2. Consider the setting from this theorem. Let B_0^o and $B_\infty \in M_n(\mathbb{C})$ be the matrix representations of $\mathcal{U}_p = (\mathcal{C}_E)_p$ and $\mathcal{V}_p$ in the basis $\mathcal{B} := \{e_p, E_p, \cdots, E_p^{n-1}\}$ of $T_p M$ (where $n := \dim(M)$):

$$\mathcal{U}_p(E_p^i) = (B_0^o)_{ji} E_p^j, \quad \mathcal{V}_p(E_p^i) = (B_\infty)_{ji} E_p^j.$$

For any $0 \leq i \leq n - 2$,

$$(B_0^o)_{i+1,i} = 1, \quad (B_0^o)_{ji} = 0, \ j \neq i + 1. \tag{A.1}$$

Since $\mathcal{V}_p(e_p) = (1 - \frac{D}{2})e_p$,

$$(B_\infty)_{j0} = \left(1 - \frac{D}{2}\right) \delta_{0j}, \ 0 \leq j \leq n - 1. \tag{A.2}$$

From the skew-symmetry of $\mathcal{V}_p$,

$$(B_\infty)_{ki} e_p^\flat(E_p^{k+j}) + (B_\infty)_{kj} e_p^\flat(E_p^{k+i}) = 0, \quad 0 \leq i, j \leq n - 1, \tag{A.3}$$

where $e_p^\flat(X) = g_p(e_p, X)$, for any $X \in T_p M$.

Let $M^{\text{can}} := M^{\text{can}}(-B_0^o, -B_\infty)$, with its F-manifold structure provided by Proposition 7.1. From this proposition, we know that there is an isomorphism

$$f : ((M, p), \circ, e, E) \to ((M^{\text{can}}, 0), \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}}). \tag{A.4}$$

Recall that the Euler field of M^{can} is given by

$$(E_{\text{can}})_\Gamma := B_0^o + \Gamma + [B_\infty, \Gamma], \ \ \Gamma \in M^{\text{can}}.$$

In particular, $(E_{\text{can}})_0 = B_0^o$ and, since $f_*(E^i) = (E_{\text{can}})^i$, we obtain that

$$(f_*)_p\left(E_p^i\right) = \left(B_0^o\right)^i, \quad i \geq 0. \tag{A.5}$$

Let

$$(g_{\text{can}})_0 : T_0 M^{\text{can}} \times T_0 M^{\text{can}} \to \mathbb{C}, \quad (g_{\text{can}})_0 := \left(f^{-1}\right)^*(g_p)$$

be the push-forward metric, given by

$$(g_{\text{can}})_0\left(\left(B_0^o\right)^i, \left(B_0^o\right)^j\right) = g_p\left(E_p^i, E_p^j\right) = e_p^\flat\left(E_p^{i+j}\right), \quad 0 \leq i, j \leq n - 1. \tag{A.6}$$

The endomorphism $(\mathcal{U}_{\text{can}})_0(X) = X \circ_{\text{can}} E_{\text{can}}$ of $T_0 M^{\text{can}}$ is the multiplication by $B_0^o \in M_n(\mathbb{C})$ on $T_0 M^{\text{can}} \subset M_n(\mathbb{C})$. It is $(g_{\text{can}})_0$ symmetric. The isomorphism (A.4) defines a bijection between the set of Frobenius metrics on its domain and target space and the existence part in Theorem 8.2 is a consequence of the following lemma.

Lemma A.1. *The metric $(g_{\text{can}})_0$ defined by (A.6) admits an extension to a Frobenius metric g_{can} on the germ $((M^{\text{can}}, 0), \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}})$, such that*

$$\left(D^{\text{LC}} E_{\text{can}}\right)_0 = (f_*)_p \circ \mathcal{V}_p \circ \left(f_*\right)_p^{-1} + \frac{D}{2}\text{Id}. \tag{A.7}$$

Above D^{LC} is the Levi-Civita connection of g_{can}.

Proof. We preserve the notation from the proof of Proposition 7.1. Let $V = M^{\text{can}} \times \mathbb{C}^n \to M^{\text{can}}$ be the trivial bundle over $M^{\text{can}} = M^{\text{can}}(-B_0^o, -B_\infty)$ and $s \in \Gamma(V)$ the constant section $s(\Gamma) = (\Gamma, v_0)$, where $v_0 := (1, 0, \cdots, 0) \in \mathbb{C}^n$. We denote by $v_1 := (0, 1, 0, \cdots, 0), v_2 := (0, 0, 1, 0, \cdots, 0), \ldots, v_{n-1} = (0, \cdots, 0, 1)$ the remaining standard vectors of $\mathbb{C}^n$. The F-manifold structure of M^{can} is obtained (as explained in Proposition 2.8) from the Saito bundle $(V, D^c, \Phi, B_0, B_\infty)$ (defined as in the proof of Proposition 7.1, with B_0^o replaced by $-B_0^o$ and B_∞ by $-B_\infty$), by means of the isomorphism

$$I : T M^{\text{can}} \to V, \ I(X) = \Phi_X(s) = (\Gamma, X(v_0)), \ X \in T_\Gamma M^{\text{can}} \subset M_n(\mathbb{C}). \tag{A.8}$$

We aim to construct on the Saito bundle $(V, D^c, \Phi, B_0, B_\infty)$ a Saito metric which will induce, via the isomorphism I, the Frobenius metric g_{can} we are looking for. From (A.1), v_0 is a cyclic vector for B_0^o and

$$I_0 : T_0 M^{\text{can}} \to V_0 = \mathbb{C}^n, \; I_0\left(\left(B_0^o\right)^i\right) = \left(B_0^o\right)^i(v_0) = v_i, \quad 0 \le i \le n-1. \quad \text{(A.9)}$$

Let $g_0 := (I_0^{-1})^*(g_{\text{can}})_0 \in S^2(V_0^*)$ be the push-forward of $(g_{\text{can}})_0 \in S^2(T_0^* M^{\text{can}})$:

$$g_0(v_i, v_j) := (g_{\text{can}})_0\left(I_0^{-1}(v_i), I_0^{-1}(v_j)\right) = (g_{\text{can}})_0\left(\left(B_0^o\right)^i, \left(B_0^o\right)^j\right) = e_p^\flat\left(E_p^{i+j}\right). \quad \text{(A.10)}$$

Since $(\mathcal{U}_{\text{can}})_0$ is $(g_{\text{can}})_0$ symmetric (as stated before the lemma), $I_0 \circ (\mathcal{U}_{\text{can}})_0 \circ I_0^{-1}$ is g_0 symmetric. But $I_0 \circ (\mathcal{U}_{\text{can}})_0 \circ I_0^{-1} = -(B_0)_0$ (see Proposition 2.8). Since $B_0 \in \text{End}(V)$ is given by

$$(B_0)_\Gamma = -(B_0^o + \Gamma + [B_\infty, \Gamma]), \quad \Gamma \in M^{\text{can}}, \quad \text{(A.11)}$$

we obtain that $-(B_0)_0 = B_0^o$. Therefore, $B_0^o \in M_n(\mathbb{C})$ is g_0 symmetric. From (A.3) and (A.10), $B_\infty \in M_n(\mathbb{C})$ is g_0 skew-symmetric. Since B_0^o is g_0 symmetric and B_∞ is g_0 skew-symmetric, $(B_0)_\Gamma$ is g_0 symmetric when Γ is so.

Let $M_n^{\text{sym}}(\mathbb{C})$ be the manifold of g_0 symmetric matrices. We claim that $(M^{\text{can}}, 0) \subset (M_n^{\text{sym}}(\mathbb{C}), 0)$. For this, we use the above observation (namely, $(B_0)_\Gamma \in M_n^{\text{sym}}(\mathbb{C})$ when $\Gamma \in M_n^{\text{sym}}(\mathbb{C})$) and the following general fact (which can be easily checked): if $\mathcal{D}$ is an integrable distribution on a manifold M, N is a submanifold of M such that $\mathcal{D}|_N \subset TN$ and I^{max} is the maximal integrable submanifold of $\mathcal{D}$, which contains $p \in N$, then there is a neighbourhood U of p in M, such that $I^{\text{max}} \cap U \subset N \cap U$. Applying this fact to $M := W$ (a small open neighbourhood of $0 \in M_n(\mathbb{C})$), $N := M_n^{\text{sym}}(\mathbb{C}) \cap W$ and the distribution $\mathcal{D}|_W$ whose maximal integrable submanifold is M^{can} (and whose fiber at $\Gamma \in W$ is the vector space of polynomials in $(B_0)_\Gamma$, with $(B_0)_\Gamma$ as in (A.11)), we obtain $(M^{\text{can}}, 0) \subset (M_n^{\text{sym}}(\mathbb{C}), 0)$, as needed.

Let $g_V \in S^2(V^*)$ be the constant extension of g_0 to the bundle V. It follows that $(V, D^c, \Phi, B_0, B_\infty, g_V)$ is a Saito bundle with metric (see Definition 2.7). The section s is primitive homogeneous, with $B_\infty(s) = (1 - \frac{D}{2})s$ (we use (A.2); recall that s is the constant section of V, determined by $v_0 = (1, 0, \cdots, 0) \in \mathbb{C}^m$). The metric $g_{\text{can}}(X, Y) := g_V(I(X), I(Y))$ extends $(g_{\text{can}})_0$ (from (A.10)). From Subsection 2.2, g_{can} is a Frobenius metric on $((M^{\text{can}}, 0), \circ_{\text{can}}, \text{Id}_{\text{can}}, E_{\text{can}})$ and

$$D^{\text{LC}} E_{\text{can}} = I^{-1} B_\infty I + \frac{D}{2} \text{Id}.$$

In order to conclude the proof, we need to check that

$$I_0^{-1} B_\infty I_0 = (f_*)_p \circ \mathcal{V}_p \circ (f_*)_p^{-1}. \quad \text{(A.12)}$$

From (A.9), the left hand side of (A.12), applied to $(B_0^o)^i \in T_0 M^{\mathrm{can}}$ (with $0 \le i \le n-1$), is given by

$$\left(I_0^{-1} B_\infty I_0\right)\left(B_0^o\right)^i = I_0^{-1} B_\infty(v_i) = (B_\infty)_{ji}\left(B_0^o\right)^j.$$

From (A.5), the right hand side of (A.12), applied to $(B_0^o)^i$, is given by

$$\begin{aligned}
\left((f_*)_p \circ \mathcal{V}_p \circ (f_*)_p^{-1}\right)\left(B_0^o\right)^i &= (f_*)_p \mathcal{V}_p\left(E_p^i\right) \\
&= (B_\infty)_{ji}(f_*)_p\left(E_p^j\right) = (B_\infty)_{ji}\left(B_0^o\right)^j.
\end{aligned}$$

Relation (A.12) follows. $\square$

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