Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 1031-1066

A gradient flow for open elastic curves with fixed length and clamped ends

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Abstract. We consider regular open curves in $\mathbb{R}^n$ with clamped ends subject to a fixed length constraint and moving according to the L^2-gradient flow of the elastic energy. For this flow we prove a long time existence result and subconvergence to critical points. In particular our result provides an alternative approach for finding equilibrium configurations of bending energy.

Mathematics Subject Classification (2010): 35K55 (primary); 35K35, 53C44 (secondary).

1. Introduction

The simplest model in elastic rod theory was originally proposed by Daniel Bernoulli to Leonhard Euler around 1743 [18]: there the shape of elastic rods in equilibrium was characterised by critical points of the elastic energy functional of curves (or the centerlines of rods). The elastic energy functional, which is also called bending energy functional (or Willmore energy), is defined by the integral of the squared curvature of a curve over its arc-length parameter (see (1.1) below). The equilibrium configuration is commonly called *elastica* or *elastic curve*. Besides classical rod theory, there are many other applications of elastic energies, *e.g.*, in the mechanical modeling of DNA [7], in edge completion problems of computer vision [15], and in the theory of nonlinear splines [10].

In this article, we consider a smooth map $f : \overline{I} \to \mathbb{R}^n$, $n \geq 2$, $I = (0, 1)$, together with the elastic energy

$$E(f) = \frac{1}{2} \int_I |\vec{\kappa}|^2 \, ds \,, \tag{1.1}$$

C.-C. L. would like to acknowledge the support from the National Science Council of Taiwan NSC 102-2115-M-003 -004, the National Center for Theoretical Sciences in Taipei, and the warm hospitality from Prof. Anna Dall'Acqua during the visit at Uni-Ulm in the summer of 2014. A.D. and C.-C. L. acknowledge support by the project "Ulm International Research Fellowship in Mathematics and Economics" of Ulm University.

Received November 25, 2015; accepted April 26, 2016.
Published online September 2017.

where $ds = |\partial_x f| dx$ denotes the arc-length element and $\vec{\kappa} = \partial_{ss} f$ the curvature vector. Since in many physical problems one has to consider boundary value problems, it is natural for variational problems in elasticity to prescribe the fixed end-point positions of the curves and fixed tangent vectors at these end-points under a prescribed length. Namely, given f_+, f_- vectors in $\mathbb{R}^n$ and τ_+, τ_- vectors in $\mathbb{S}^{n-1}$, we consider curves $f : \bar{I} \to \mathbb{R}^n$ satisfying the following clamped boundary conditions

$$f(0) = f_-, \quad f(1) = f_+, \quad \partial_s f(0) = \tau_-, \quad \partial_s f(1) = \tau_+ ,$$

with $\partial_s f = |\partial_x f|^{-1} \partial_x f$, and $\int_{\bar{I}} ds = L_0 > |f_+ - f_-|$. We refer to such curves as curves with *clamped* ends.

In this work we consider the elastic flow of curves with given length and clamped ends. Apart from its intrinsic interest, the study of a gradient flow in the setting of parabolic differential equations, is a very powerful method to prove existence of critical points of the considered energy functional. Indeed, starting from a smooth initial datum we follow a trajectory in the functional space along which the energy is decreasing; if this path exists for all time, then under suitable conditions the limit is a critical point.

In this article, we follow this Ansatz and consider the gradient flow equation,

$$\partial_t f = -\nabla_s^2 \vec{\kappa} - \frac{1}{2} |\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} , \tag{1.2}$$

with initial smooth datum

$$f(0, \cdot) = f_0(\cdot) \text{ in } \bar{I} = [0, 1] , \tag{1.3}$$

where $\nabla_s \phi = \partial_s \phi - \langle \partial_s \phi, \partial_s f \rangle \partial_s f$ denotes the normal component of the full derivative $\partial_s \phi$ of a vector field $\phi : I \to \mathbb{R}^n$ and

$$\lambda = \lambda(t) = \frac{\int_I \langle \nabla_s^2 \vec{\kappa} + \frac{1}{2} |\vec{\kappa}|^2 \vec{\kappa}, \vec{\kappa} \rangle \, ds}{\int_I |\vec{\kappa}|^2 \, ds} , \tag{1.4}$$

is the Lagrange-multiplier to keep the total length of the curves fixed along the flow (see (2.9) below). We look for a solution to (1.2), (1.3) subject to the clamped boundary conditions

$$f(t, 0) = f_-, \quad f(t, 1) = f_+ \text{ for all } t \geq 0 , \tag{1.5}$$

$$\partial_s f(t, 0) = \tau_-, \quad \partial_s f(t, 1) = \tau_+ \text{ for all } t \geq 0. \tag{1.6}$$

In the main Theorem 1.1 we prove that the above flow exists globally in time and that as the time goes to infinity the curves sub-converge, after reparametrizaton by arc-length, to open elasticae (see the statement of the theorem for a more precise formulation). The word "open" is here to be understood in the following sense: the prescribed two end-points positions f_- and f_+ are allowed to coincide, but in this case no periodicity across the glueing point is required (as in the case of the so-called closed curves).

The investigation of the elastic flow in the above mentioned spirit has by now a rich set of contributions by several authors. One usually differentiates between the case of open and closed curves, the conditions imposed at the boundary, and the way the length of the curve is controlled, namely by fixing it or by simply penalizing its growth (in this latter case λ is simply a positive fixed number).

The case of closed planar curves was first studied in [19] and [17], and successively in [6] in the case of arbitrary dimension. Further results in the setting of closed curves are presented in [21], where a Willmore-Helfrich type of energy functional is considered.

The case of open curves was tackled in far more recent years: in [11], the second author of this article obtained long-time existence of smooth solution for (1.2) with fixed positive constant λ, clamped ends (1.5), (1.6), and smooth initial data in $\mathbb{R}^n$. In [3] Dall'Acqua and Pozzi investigated a Willmore-Helfrich type of functional (which entails (1.1) as a special case) and investigated long-time existence for the evolution of open curves under natural boundary conditions and a fixed positive parameter λ. In [2], Dall'Acqua, Lin, and Pozzi extended the results in [11] to the case of flow (1.2) with hinged ends and fixed length. The generalization to the case of clamped boundary conditions is finally achieved in the present work. Before going into details of the difficulties presented by the treatment of clamped boundary conditions and the new contribution of this paper, let us briefly mention that in the literature one finds also several studies concerning flows that approach elasticae but are geometrically different: see for instance [9] and [8]. In the graph setting the stationary problem for the elastic energy of open curves subject to different boundary conditions is considered in [4,5], and [13]. Elastic motion of non-closed planar curve with infinite length is treated in [16]. Numerical simulations for the elastic flow of open and closed curves in $\mathbb{R}^n$ are presented in [1].

Let us finally briefly discuss our main result:

Theorem 1.1. *Let vectors $f_+, f_- \in \mathbb{R}^n$ and $\tau_+, \tau_- \in \mathbb{S}^{n-1}$ be given as well as a smooth regular curve $f_0 : \bar{I} \to \mathbb{R}^n$ satisfying*

$$\begin{aligned} f_0(0) &= f_-, & f_0(1) &= f_+, \\ \tau[f_0](0) &= \tau_-, & \tau[f_0](1) &= \tau_+, \end{aligned}$$

with $\tau[f_0]$ the unit tangent vector of f_0, together with suitable compatibility conditions. Let the length $\mathcal{L}(f_0) = L_0$ of the initial curve satisfy $L_0 > |f_+ - f_-|$. Then a smooth solution $f : [0, T) \times [0, 1] \to \mathbb{R}^n$ of the initial boundary value problem

$$\begin{cases} \partial_t f = -\nabla_s^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2\vec{\kappa} + \lambda\vec{\kappa} & \text{in } I \times (0, T) \\ f(0, x) = f_0(x) & \text{for } x \in [0, 1] \\ f(t, 0) = f_-, \ f(t, 1) = f_+ & \text{for } t \in [0, T) \\ \partial_s f(t, 0) = \tau_-, \ \partial_s f(t, 1) = \tau_+ & \text{for } t \in [0, T), \end{cases} \tag{1.7}$$

with

$$\lambda = \lambda(t) = \frac{\int_I \left\langle \nabla_s^2 \vec{\kappa} + \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa}, \vec{\kappa} \right\rangle ds}{\int_I |\vec{\kappa}|^2 \, ds}\,, \tag{1.8}$$

exists for all times, that is we may take $T = \infty$. Moreover, as $t_i \to \infty$ the curves $f(t_i, \cdot)$ subconverge, when reparametrized by arc-length, to a critical point of the elastic functional with clamped ends and subject to a fixed length constraint, that is to a solution of

$$\begin{cases} -\nabla_s^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} + \lambda \vec{\kappa} = 0 & \\ f(0) = f_- & f(1) = f_+ \\ \tau(0) = \tau_- & \tau(1) = \tau_+\,, \end{cases} \tag{1.9}$$

for some value $\lambda \in \mathbb{R}$.

We would like to point out that the long-time existence of smooth solutions is not generally expected for a fourth-order parabolic equation or system. For example, under the fourth-order curve diffusion flow $\partial_t f = -\nabla_s^2 \vec{\kappa}$ there exist initially immersed smooth curves that develop singularity in finite time (see [20]).

Similarly to many of the works mentioned above, our method of proof for Theorem 1.1 is based on L^2-estimates of the curvature by means of Gagliardo-Nirenberg-type inequalities. Under the assumption that the flow exists only up to some positive time $T > 0$, we obtain uniform bounds for the curvature and all its derivatives, so that a contradiction argument yields existence of the flow for all time. Some of the crucial ideas underlying our method of proof are extensively discussed in [3, Section 1], so we will not repeat all observations here. Let us just mention that a key starting point is Lemma 2.3, which provides an evolution equation for the L^2-norm of a general normal vector field $\vec{\phi} : \bar{I} \to \mathbb{R}^n$ along f. Mimicking the case of closed curves discussed in [6], one wishes to take $\vec{\phi} = \nabla_s^m \vec{\kappa}$ for any $m \in \mathbb{N}$ and obtain the aforementioned uniform bounds by means of interpolation inequalities, Gronwall arguments, and by exploiting the structure of the equation (see comments in [3, Section 1]). However, as soon as open curves are treated, one has to take care of boundary terms, a task that turns out to be rather tricky. In [2] the boundary terms disappear due to the special choice of hinged boundary conditions. In [11] the problem is avoided by considering $\vec{\phi} = \nabla_t^m f$: here the clamped boundary conditions make sure that the boundary terms do not interfere (*cf.* Lemma 2.4 below). In the present setting, where the parameter λ is now time-depending, such an approach is absolutely discouraging, since now $\nabla_t^m f$ contains also derivatives of λ up to order $m - 1$, which have to be controlled somehow.

Our idea is (after a special "initialization step" with $\vec{\phi} = \nabla_t f$) to "stick" to the choice of $\vec{\phi} = \nabla_s^m \vec{\kappa}$, but now considering only $m = 4j$ with $j \in \mathbb{N}$. In some sense and very roughly speaking, looking at derivatives in multiple of four is like looking at subsequent time derivatives of Equation (1.2). Yet, no derivatives of λ, appear in the term $\vec{\phi}$ and this is of great advantage. The derivatives of λ appear instead in the boundary terms and these have to be treated separately and with great precision. The successful analysis of the boundary terms exploits the structure of the considered PDE together with the given boundary conditions (see Lemma 2.6), strong interpolation inequalities (see Lemma 3.4), and a constant monitoring and improvement for the bounds obtained for the derivatives of λ. Last but not least the introduction of some smart notation helps in maintaining an overview of the rapidly growing number of terms in all equations. Let us note that our interpolation inequalities are more general than the ones used in the literature so far and therefore they are interesting in their own right.

Our work is organized as follows: in Section 2 we introduce notation and several useful geometrical results. All relevant facts about interpolation inequalities are collected in Section 3. The estimates for λ and its derivative are presented in Section 4: notice that here we also give the generalization of some previous results reported in [2] *without* using the information about the boundary conditions (see Lemma 4.3). Finally in Section 5 the proof of the Theorem 1.1 is discussed.

2. Preliminaries and notation

We consider a time dependent curve $f : [0, T) \times \bar{I} \to \mathbb{R}^n, n \geq 2, I = (0, 1)$. For a curve f let $ds = |\partial_x f| dx$ denote the arc-length element, $\tau = \partial_s f = \frac{\partial_x f}{|\partial_x f|}$ its unit tangent vector, and $\vec{\kappa} = \partial_{ss} f$ its curvature vector. For a vector-field $\phi : [0, 1] \to \mathbb{R}^n$ we set

$$\partial_s \phi := \frac{1}{|\partial_x f|} \partial_x \phi \quad \text{and} \quad \nabla_s \phi := \partial_s \phi - \langle \partial_s \phi, \partial_s f \rangle \partial_s f.$$

Notice that $\nabla_s \phi$ is the normal component of $\partial_s \phi$. Similar to ∇_s, let ∇_t be defined by

$$\nabla_t \eta := \partial_t \eta - \langle \partial_t \eta, \partial_s f \rangle \partial_s f$$

for any vector field η defined on the smooth family of curves f.

In the following vector fields with an arrow on the top are normal vector fields.

2.1. Geometrical lemmas

We start by recalling the variation of some geometrical quantities considering smooth solutions $f : [0, T) \times \bar{I} \to \mathbb{R}^n$ of an arbitrary entirely normal flow

$$\partial_t f = \vec{V}$$

with $\vec{V}$, $\langle \vec{V}, \tau \rangle \equiv 0$, the normal velocity. The following lemma is an immediate consequence of [3, Lemma 2.1] (or [6, Lemma 2.1]).

Lemma 2.1. *Given $\vec{\phi}$ any smooth normal field along f, the following formulas hold:*

$$\partial_t(ds) = -\langle\vec{\kappa}, \vec{V}\rangle\, ds\,, \tag{2.1}$$

$$\partial_t\partial_s - \partial_s\partial_t = \langle\vec{\kappa}, \vec{V}\rangle\partial_s\,, \tag{2.2}$$

$$\partial_t\tau = \nabla_s\vec{V}\,, \tag{2.3}$$

$$\partial_t\vec{\phi} = \nabla_t\vec{\phi} - \langle\nabla_s\vec{V}, \vec{\phi}\rangle\tau, \tag{2.4}$$

$$\partial_t\vec{\kappa} = \partial_s\nabla_s\vec{V} + \langle\vec{\kappa}, \vec{V}\rangle\vec{\kappa}\,, \tag{2.5}$$

$$\nabla_t\vec{\kappa} = \nabla_s^2\vec{V} + \langle\vec{\kappa}, \vec{V}\rangle\vec{\kappa}\,, \tag{2.6}$$

$$(\nabla_t\nabla_s - \nabla_s\nabla_t)\vec{\phi} = \langle\vec{\kappa}, \vec{V}\rangle\nabla_s\vec{\phi} + \Big[\langle\vec{\kappa}, \vec{\phi}\rangle\nabla_s\vec{V} - \langle\nabla_s\vec{V}, \vec{\phi}\rangle\vec{\kappa}\Big]\,. \tag{2.7}$$

Furthermore, if $\vec{V} = 0$ at the boundary then we have at the boundary $\partial_t\partial_s = \partial_s\partial_t$. If additionally $\vec{\phi} = 0$ or $\nabla_s\vec{V} = 0$ at the boundary, then

$$\nabla_t\nabla_s\vec{\phi} = \nabla_s\nabla_t\vec{\phi}\,. \tag{2.8}$$

Remark 2.2. Thanks to the previous lemma we can now show that the value of λ given in (1.4) ensures that the length of the curve $\mathcal{L}[f] = \int_I ds$ is preserved along the flow (1.2). Indeed, using (2.1) we infer that

$$\frac{d}{dt}\mathcal{L}[f] = \frac{d}{dt}\int_I ds = -\int_I \left\langle\vec{\kappa}, -\nabla_s^2\vec{\kappa} - \frac{1}{2}|\vec{\kappa}|^2\vec{\kappa} + \lambda\vec{\kappa}\right\rangle ds = 0\,. \tag{2.9}$$

Note also that the elastic energy (1.1) decreases along the flow, *i.e.*

$$\frac{d}{dt}E(f) = -\int_I |\partial_t f|^2\, ds \le 0\,. \tag{2.10}$$

This follows for instance from [3, Lemma A.1], (2.9) and Lemma 2.4 below.

Lemma 2.3. *Suppose $\partial_t f = \vec{V}$ on $(0, T)\times I$. Let $\vec{\phi}$ be a normal vector field along f and $Y = \nabla_t\vec{\phi} + \nabla_s^4\vec{\phi}$. Then*

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_I |\vec{\phi}|^2\, ds + \int_I |\nabla_s^2\vec{\phi}|^2\, ds = &-\Big[\langle\vec{\phi}, \nabla_s^3\vec{\phi}\rangle\Big]_0^1 + \Big[\langle\nabla_s\vec{\phi}, \nabla_s^2\vec{\phi}\rangle\Big]_0^1 \\ &+ \int_I \langle Y, \vec{\phi}\rangle\, ds - \frac{1}{2}\int_I |\vec{\phi}|^2\langle\vec{\kappa}, \vec{V}\rangle\, ds.\end{aligned} \tag{2.11}$$

Proof. The claim follows using (2.1) and integration by parts (*cf.* also [6, Lemma 2.2], [11, Lemma 3], [3, Lemma 2.3]). □

We will use Lemma 2.3 first with $\vec{\phi} = \nabla_t f$ and then with $\vec{\phi} = \nabla_s^m\vec{\kappa}$, $m \in \mathbb{N}$. In both cases we need to understand well the boundary terms and for this purpose the following lemma is crucial.

Lemma 2.4. *Let f be a smooth solution of* (1.2) *subject to* (1.5)*,* (1.6) *and* (1.4) *on* $[0, T) \times I$*. Then for all* $m \in \mathbb{N}$

$$\nabla_t^m f(t,x) = 0, \qquad \nabla_t^m \tau(t,x) = 0 \quad \textit{and} \quad \nabla_s \nabla_t^m f(t,x) = 0 \textit{ for } x \in \{0,1\}\,.$$

This result has already been observed and used in [3, Lemma 2.2 and Remark 2.5] and [11]. For the sake of readability we give here again the proof.

Proof. We need to prove only the second statement since the first two follow directly from (1.5) and (1.6). For $x \in \{0,1\}$ using that $\nabla_t f(t,x) = \partial_t f(t,x) = \vec{V} = 0$, (2.2) and (1.6) we have

$$\partial_s \nabla_t f(t,x) = \partial_s \partial_t f - \partial_s(\langle \partial_t f, \tau\rangle \tau) = \partial_t \partial_s f = 0\,.$$

In particular, $\nabla_s \nabla_t f(t,x) = 0$ and $\nabla_s \vec{V}(t,x) = 0$ for $x \in \{0,1\}$. For $m \geq 2$ the claim follows by induction from (2.8) as follows

$$\nabla_s \nabla_t^m f(t,x) = \nabla_s \nabla_t (\nabla_t^{m-1} f) = \nabla_t (\nabla_s \nabla_t^{m-1} f) = 0\,,$$

using in the last equation the induction hypothesis. □

2.2. Technical lemmas

In this section we compute the parabolic equations satisfied by $\nabla_s^k \vec{\kappa}$ and $\nabla_t^m f$. For this we need to introduce the following notation.

As in [6], for normal vector fields $\vec{\phi}_1, \ldots, \vec{\phi}_k$, the product $\vec{\phi}_1 * \cdots * \vec{\phi}_k$ defines for even k a function given by $\langle \vec{\phi}_1, \vec{\phi}_2\rangle \ldots \langle \vec{\phi}_{k-1}, \vec{\phi}_k\rangle$, while for k odd it defines a normal vector field $\langle \vec{\phi}_1, \vec{\phi}_2\rangle \ldots \langle \vec{\phi}_{k-2}, \vec{\phi}_{k-1}\rangle \vec{\phi}_k$.

Following the notation adopted in [3], for $\vec{\phi}$ a normal vector field, $P_b^{a,c}(\vec{\phi})$ denotes any linear combination of terms of type

$$\nabla_s^{i_1}\vec{\phi} * \cdots * \nabla_s^{i_b}\vec{\phi} \ \text{ with } \ i_1 + \cdots + i_b = a \ \text{ and } \ \max i_j \leq c\,,$$

with coefficients bounded by some universal constant. Notice that a gives the total number of derivatives, b gives the number of factors and c gives the highest number of derivatives falling on one factor. Notice also that, with a slight abuse of notation, $|P_b^{a,c}(\vec{\phi})|$ denotes any linear combination with non-negative coefficients of terms of type

$$|\nabla_s^{i_1}\vec{\phi}| \cdot |\nabla_s^{i_2}\vec{\phi}| \cdot \ldots \cdot |\nabla_s^{i_b}\vec{\phi}| \text{ with } i_1 + \cdots + i_b = a \text{ and } \max i_j \leq c\,.$$

Observe that for odd $b \in \mathbb{N}$ we have $\nabla_s P_b^{a,c}(\vec{\phi}) = P_b^{a+1,c+1}(\vec{\phi})$. Finally, for sums over a, b and c we set

$$\sum_{\substack{[[a,b]] \leq [[A,B]] \\ c \leq C}} P_b^{a,c}(\vec{\phi}) := \sum_{a=0}^{A} \sum_{b=1}^{2A+B-2a} \sum_{c=0}^{C} P_b^{a,c}(\vec{\phi})\,. \tag{2.12}$$

Similarly we set $\sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le C}} |P_b^{a,c}(\vec{\phi})|:=\sum_{a=0}^{A}\sum_{b=1}^{2A+B-2a}\sum_{c=0}^{C}|P_b^{a,c}(\vec{\phi})|$. As already mentioned in [3], the relation between A and B in (2.12) can be interpreted as follows: the more derivatives we take the less factors are present, or equivalently, if we take one derivative less, then we may allow for two factors more.

Derivatives with respect to time of the parameter λ will be denoted by

$$\lambda^{(j)}=\frac{d^j\lambda}{dt^j},$$

for all $j\in\mathbb{N}_0$. In particular $\lambda^{(0)}=\lambda$. Moreover, for all $i\in\mathbb{N}$, $\ell\in\mathbb{N}_0$, we let

$$Q_i(\lambda_\ell):=\sum_{\beta\in S_i^\ell} c_\beta\cdot\prod_{a=0}^{\ell}\left(\lambda^{(a)}\right)^{\beta_a},$$
$$\text{where } S_i^\ell:=\left\{\beta\in(\mathbb{N}_0)^{\ell+1}:\sum_{a=0}^{\ell}(4a+2)\beta_a=2i\right\},$$

with constant coefficients c_β bounded by some universal constant. Here, we set $(\lambda^{(a)})^0:=1$ for $a\in\mathbb{N}_0$; $Q_0(\lambda_\ell):=1$, $\forall\,\ell\in\mathbb{N}_0$; and $Q_i(\lambda_\ell):=0$, $\forall\,i\in\mathbb{N}_0$, and $\ell<0$. Note that the parameter ℓ indicates the highest order of derivative of λ possibly present in any of these polynomials, whereas the parameter i takes into account both the order and the power of the derivatives.

Lemma 2.5. *Suppose $f:[0,T)\times\bar{I}\to\mathbb{R}^n$ is a smooth regular solution to* (1.2) *in* $(0,T)\times I$. *Then, the following formulas hold on* $(0,T)\times I$:

(1) *For any $k\in\mathbb{N}$,*

$$\left[\nabla_t\nabla_s^k-\nabla_s^k\nabla_t\right]\vec{\kappa}=\sum_{\substack{[[a,b]]\le[[k+2,3]]\\ c\le k+2,\ b\text{ odd}}}P_b^{a,c}(\vec{\kappa})+\lambda\sum_{\substack{[[a,b]]\le[[k,3]]\\ c\le k,\ b\text{ odd}}}P_b^{a,c}(\vec{\kappa});\quad(2.13)$$

(2) *For any $\ell\in\mathbb{N}_0$, we have that*

$$\nabla_t\nabla_s^\ell\vec{\kappa}=-\nabla_s^{\ell+4}\vec{\kappa}+\lambda\nabla_s^{\ell+2}\vec{\kappa}+\sum_{\substack{[[a,b]]\le[[\ell+2,3]]\\ c\le\ell+2,\ b\text{ odd}}}P_b^{a,c}(\vec{\kappa})+\lambda\sum_{\substack{[[a,b]]\le[[\ell,3]]\\ c\le\ell,b\text{ odd}}}P_b^{a,c}(\vec{\kappa});\quad(2.14)$$

(3) *For any $A,C\in\mathbb{N}_0$, $B\in\mathbb{N}$,*

$$\nabla_t\sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le C,\ b\text{ odd}}}P_b^{a,c}(\vec{\kappa})=\sum_{\substack{[[a,b]]\le[[A+4,B]]\\ c\le C+4,\ b\text{ odd}}}P_b^{a,c}(\vec{\kappa})+\lambda\sum_{\substack{[[a,b]]\le[[A+2,B]]\\ c\le C+2,\ b\text{ odd}}}P_b^{a,c}(\vec{\kappa});\quad(2.15)$$

Similarly, for any $A,C\in\mathbb{N}_0$, $B\in\mathbb{N}$,

$$\partial_t\sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le C,\ b\text{ even}}}P_b^{a,c}(\vec{\kappa})=\sum_{\substack{[[a,b]]\le[[A+4,B]]\\ c\le C+4,\ b\text{ even}}}P_b^{a,c}(\vec{\kappa})+\lambda\sum_{\substack{[[a,b]]\le[[A+2,B]]\\ c\le C+2,\ b\text{ even}}}P_b^{a,c}(\vec{\kappa});\quad(2.16)$$

(4) *For any* $i, \ell \in \mathbb{N}_0$,

$$\frac{d}{dt} Q_i(\lambda_\ell) = Q_{i+2}(\lambda_{\ell+1}) \, ; \tag{2.17}$$

(5) *For any* $m \in \mathbb{N}$,

$$\begin{aligned} \nabla_t^m f = (-1)^m \nabla_s^{4m-2} \vec{\kappa} + \lambda^{(m-1)} \vec{\kappa} + \sum_{\substack{[[a,b]] \le [[4m-4,3]] \\ c \le 4m-4, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \\ + \sum_{i=1}^{2m-2} Q_i(\lambda_{m-2}) \sum_{\substack{[[a,b]] \le [[4m-2i-2,1]] \\ c \le 4m-2i-2, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \, ; \end{aligned} \tag{2.18}$$

(6) *For any* $m \in \mathbb{N}$,

$$\begin{aligned} \nabla_t^m \tau = (-1)^m \nabla_s^{4m-1} \vec{\kappa} + \lambda^{(m-1)} \nabla_s \vec{\kappa} + \sum_{\substack{[[a,b]] \le [[4m-3,3]] \\ c \le 4m-3, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \\ + \sum_{i=1}^{2m-2} Q_i(\lambda_{m-2}) \sum_{\substack{[[a,b]] \le [[4m-2i-1,1]] \\ c \le 4m-2i-1, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \, . \end{aligned} \tag{2.19}$$

Proof. Formula (2.13) is proven in [3, Lemma 3.1] or [11]. By (2.6) and (1.2), we have

$$\nabla_t \vec{\kappa} = -\nabla_s^4 \vec{\kappa} + \lambda \nabla_s^2 \vec{\kappa} + \sum_{\substack{[[a,b]] \le [[2,3]] \\ c \le 2, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]] \le [[0,3]] \\ c \le 0, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}), \tag{2.20}$$

that is (2.14) for $\ell = 0$. For $\ell \in \mathbb{N}$ we find by (2.13) and (2.20)

$$\begin{aligned} \nabla_t \nabla_s^\ell \vec{\kappa} = \nabla_s^\ell \Big[-\nabla_s^4 \vec{\kappa} + \lambda \nabla_s^2 \vec{\kappa} + \sum_{\substack{[[a,b]] \le [[2,3]] \\ c \le 2, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]] \le [[0,3]] \\ c \le 0, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) \Big] \\ + \sum_{\substack{[[a,b]] \le [[\ell+2,3]] \\ c \le \ell+2, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]] \le [[\ell,3]] \\ c \le \ell, \ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}), \end{aligned}$$

from which (2.14) follows. A proof of (2.15) can be found in [3, Lemma 3.1] or [11].

To prove (2.16) observe that $\partial_t(P^{\alpha,\zeta}_{\beta}(\vec{\kappa}))$ (with $\alpha,\zeta \in \mathbb{N}_0$, $\beta \in \mathbb{N}$ an even number) is given by a linear combination of terms of type

$$\left\langle \nabla_s^{i_1}\vec{\kappa}, \nabla_s^{i_2}\vec{\kappa}\right\rangle \cdots \left\langle \nabla_t\nabla_s^{i_j}\vec{\kappa}, \nabla_s^{i_{j+1}}\vec{\kappa}\right\rangle \cdots \left\langle \nabla_s^{i_{\beta-1}}\vec{\kappa}, \nabla_s^{i_\beta}\vec{\kappa}\right\rangle$$

where $i_1+\ldots+i_\beta=\alpha$, as well as $i_\ell \leq \zeta$ for all $\ell = 1,\ldots,\beta$. Using (2.14) one obtains

$$\begin{aligned}\partial_t(P^{\alpha,\zeta}_{\beta}(\vec{\kappa})) &= P^{\alpha+4,\zeta+4}_{\beta}(\vec{\kappa}) + \lambda P^{\alpha+2,\zeta+2}_{\beta}(\vec{\kappa}) \\ &\quad + \sum_{\substack{[[a,b]]\leq[[\alpha+2,\beta+2]]\\ c\leq\zeta+2,\ b \text{ even}}} P^{a,c}_b(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\leq[[\alpha,\beta+2]]\\ c\leq\zeta,\ b \text{ even}}} P^{a,c}_b(\vec{\kappa}) \\ &= \sum_{\substack{[[a,b]]\leq[[\alpha+4,\beta]]\\ c\leq\zeta+4,\ b \text{ even}}} P^{a,c}_b(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\leq[[\alpha+2,\beta]]\\ c\leq\zeta+2,\ b \text{ even}}} P^{a,c}_b(\vec{\kappa}).\end{aligned}$$

Equation (2.16) now follows.

Formula (2.17) follows since by the definition of $Q_i(\lambda_\ell)$ we get (formally)

$$\frac{d}{dt}Q_i(\lambda_\ell) = \sum_{k=0}^{\ell}\sum_{\beta\in S_i^\ell} c_\beta \cdot \prod_{a=0}^{\ell}(\lambda^{(a)})^{\beta_a}\cdot\beta_k\left(\frac{\lambda^{(k+1)}}{\lambda^{(k)}}\right) = Q_{i+2}(\lambda_{\ell+1}),$$

using that

$$\begin{aligned}2(i+2) &= \sum_{a=0}^{\ell}(4a+2)\beta_a + 4 \\ &= \sum_{\substack{a=0\\ a\neq k,k+1}}^{\ell}(4a+2)\beta_a + (4k+2)(\beta_k-1) + (4(k+1)+2)(\beta_{k+1}+1).\end{aligned}$$

Equation (2.18) for $m=1$ is (1.2). For general m we proceed by induction. For $m\geq 2$ by induction hypothesis, we have

$$\begin{aligned}\nabla_t^m f = \nabla_t \Bigg[&(-1)^{m-1}\nabla_s^{4m-6}\vec{\kappa} + \lambda^{(m-2)}\vec{\kappa} + \sum_{\substack{[[a,b]]\leq[[4m-8,3]]\\ c\leq 4m-8,\ b \text{ odd}}} P^{a,c}_b(\vec{\kappa}) \\ &+ \sum_{i=1}^{2m-4} Q_i(\lambda_{m-3}) \sum_{\substack{[[a,b]]\leq[[4m-2i-6,1]]\\ c\leq 4m-2i-6,\ b \text{ odd}}} P^{a,c}_b(\vec{\kappa})\Bigg].\end{aligned} \tag{2.21}$$

By (2.14), (2.15) and (2.17) we get

$$\begin{aligned}\nabla_t^m f &= (-1)^m \nabla_s^{4m-2}\vec{\kappa} + (-1)^{m-1}\lambda\nabla_s^{4m-4}\vec{\kappa}\\ &\quad + \sum_{\substack{[[a,b]]\le[[4m-4,3]]\\ c\le 4m-4,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\le[[4m-6,3]]\\ c\le 4m-6, b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\\ &\quad + \lambda^{(m-1)}\vec{\kappa} + \lambda^{(m-2)}\left(\sum_{\substack{[[a,b]]\le[[4,1]]\\ c\le 4,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\le[[2,1]]\\ c\le 2, b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\right)\\ &\quad + \sum_{i=1}^{2m-4} Q_{i+2}(\lambda_{m-2}) \sum_{\substack{[[a,b]]\le[[4m-2i-6,1]]\\ c\le 4m-2i-6,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\\ &\quad + \sum_{i=1}^{2m-4} Q_i(\lambda_{m-3})\left[\sum_{\substack{[[a,b]]\le[[4m-2i-2,1]]\\ c\le 4m-2i-2,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\le[[4m-2i-4,1]]\\ c\le 4m-2i-4,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\right],\end{aligned}$$

from which (2.18) follows since $\lambda = Q_1(\lambda_{m-2})$ and

$$\begin{aligned}\lambda^{(m-2)} = Q_{2m-3}(\lambda_{m-2}), \quad \lambda\lambda^{(m-2)} = Q_{2m-2}(\lambda_{m-2})\\ \text{and } \lambda Q_i(\lambda_{m-3}) = Q_{i+1}(\lambda_{m-3}).\end{aligned} \tag{2.22}$$

The proof of (2.19) is very similar to the proof of (2.18). For $m = 1$ the equation follows from (2.3), (1.2) and the definition of ∇_t. By induction we find for $m \ge 2$ using (2.14), (2.15) and (2.17)

$$\begin{aligned}\nabla_t^m \tau &= (-1)^m \nabla_s^{4m-1}\vec{\kappa} + (-1)^{m-1}\lambda\nabla_s^{4m-3}\vec{\kappa}\\ &\quad + \sum_{\substack{[[a,b]]\le[[4m-3,3]]\\ c\le 4m-3,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\le[[4m-5,3]]\\ c\le 4m-5, b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\\ &\quad + \lambda^{(m-1)}\nabla_s\vec{\kappa} + \lambda^{(m-2)}\left[\sum_{\substack{[[a,b]]\le[[5,1]]\\ c\le 5,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\le[[3,1]]\\ c\le 3, b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\right]\\ &\quad + \sum_{i=1}^{2m-4} Q_{i+2}(\lambda_{m-2}) \sum_{\substack{[[a,b]]\le[[4m-2i-5,1]]\\ c\le 4m-2i-5,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\\ &\quad + \sum_{i=1}^{2m-4} Q_i(\lambda_{m-3})\left[\sum_{\substack{[[a,b]]\le[[4m-2i-1,1]]\\ c\le 4m-2i-1,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\le[[4m-2i-3,1]]\\ c\le 4m-2i-3,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa})\right],\end{aligned}$$

from which (2.19) follows by (2.22). □

Lemma 2.4 gives us also some information about the derivatives of the curvature vector at the boundary. More precisely we see the following:

Lemma 2.6. *Suppose $f : [0,T) \times \bar{I} \to \mathbb{R}^n$ is a smooth regular solution to* (1.2) *in* $(0,T) \times I$. *At the boundary we have that for $m \in \mathbb{N}_0$,*

$$\begin{aligned}\nabla_s^{4m+2}\vec{\kappa} = &\sum_{\substack{[[a,b]]\le[[4m,3]]\\ c\le 4m,\ b\ \text{odd}}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{2m} Q_i(\lambda_{m-1}) \sum_{\substack{[[a,b]]\le[[4m-2i+2,1]]\\ c\le 4m-2i+2,\ b\ \text{odd}}} P_b^{a,c}(\vec{\kappa})\\ &+(-1)^m \lambda^{(m)}\vec{\kappa}\,,\\ \nabla_s^{4m+3}\vec{\kappa} = &\sum_{\substack{[[a,b]]\le[[4m+1,3]]\\ c\le 4m+1,\ b\ \text{odd}}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{2m} Q_i(\lambda_{m-1}) \sum_{\substack{[[a,b]]\le[[4m-2i+3,1]]\\ c\le 4m-2i+3,\ b\ \text{odd}}} P_b^{a,c}(\vec{\kappa})\\ &+(-1)^m \lambda^{(m)}\nabla_s\vec{\kappa}\,.\end{aligned}$$

Proof. The claim follows directly from Lemma 2.4, (2.18) and (2.19) replacing m by $m+1$ in these formulas. □

Note that from the above lemma we infer that some derivatives of the curvature at the boundary are actually of lower order than at first sight. We close this section by comparing $\nabla_s^m\vec{\kappa}$ with the full derivative $\partial_s^m\vec{\kappa}$. This will be needed in the main theorem.

Lemma 2.7. *We have the identities*

$$\begin{aligned}\partial_s\vec{\kappa} &= \nabla_s\vec{\kappa} - |\vec{\kappa}|^2\tau,\\ \partial_s^m\vec{\kappa} &= \nabla_s^m\vec{\kappa} + \tau \sum_{\substack{[[a,b]]\le[[m-1,2]]\\ c\le m-1,\ b\ \text{even}}} P_b^{a,c}(\vec{\kappa}) + \sum_{\substack{[[a,b]]\le[[m-2,3]]\\ c\le m-2\ b\ \text{odd}}} P_b^{a,c}(\vec{\kappa}) \quad \textit{for } m\ge 2\,.\end{aligned}$$

Proof. The proof can be found for instance in [3, Lemma 4.5]. The first claim is obtained directly using that

$$\partial_s\vec{\kappa} = \nabla_s\vec{\kappa} + \langle\partial_s\vec{\kappa},\tau\rangle\tau = \nabla_s\vec{\kappa} - |\vec{\kappa}|^2\tau\,.$$

The second claim follows by induction. □

3. Interpolation inequalities

Here we report briefly some fundamental interpolation inequalities which will be used repeatedly in the main proof. Note that all results stated in this section actually hold for closed and open curves (independently of the prescribed boundary conditions) without imposing any special constraint on the length.

As in [6] it is useful to introduce the following scale invariant norms for $k \in \mathbb{N}_0$ and $p \in [1, \infty)$

$$\|\vec{\kappa}\|_{k,p} := \sum_{i=0}^{k} \|\nabla_s^i \vec{\kappa}\|_p \quad \text{with} \quad \|\nabla_s^i \vec{\kappa}\|_p := \mathcal{L}[f]^{i+1-1/p} \left(\int_I |\nabla_s^i \vec{\kappa}|^p \, ds \right)^{1/p} ,$$

as opposed to

$$\|\nabla_s^i \vec{\kappa}\|_{L^p} := \left(\int_I |\nabla_s^i \vec{\kappa}|^p \, ds \right)^{1/p} .$$

The following Lemma 3.1 and Lemma 3.3 are adaptations to the present setting and notation of those used in [6] for closed curves and in [11] and [3] for open ones.

Lemma 3.1. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. Then for all $k \in \mathbb{N}$, $p \geq 2$ and $0 \leq i < k$ we have*

$$\|\nabla_s^i \vec{\kappa}\|_p \leq C \|\vec{\kappa}\|_2^{1-\alpha} \|\vec{\kappa}\|_{k,2}^{\alpha} ,$$

with $\alpha = (i + \frac{1}{2} - \frac{1}{p})/k$ and $C = C(n, k, p)$.

Proof. A proof of this fact is hinted at in [6, Lemma 2.4] and [11, Lemma 5]. All details are given in [3, Appendix]. □

Corollary 3.2. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. Then for all $k \in \mathbb{N}$ we have*

$$\|\vec{\kappa}\|_{k,2} \leq C \left(\|\nabla_s^k \vec{\kappa}\|_2 + \|\vec{\kappa}\|_2 \right) ,$$

with $C = C(n, k)$.

Proof. The claim follows by an induction argument: see [3, Corollary 4.2]. □

Lemma 3.3. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. For any $a, c \in \mathbb{N}_0$, $k, b \in \mathbb{N}$, $b \geq 2$, $c \leq k - 1$ we find*

$$\int_I \left| P_b^{a,c}(\vec{\kappa}) \right| \, ds \leq C \mathcal{L}[f]^{1-a-b} \|\vec{\kappa}\|_2^{b-\gamma} \|\vec{\kappa}\|_{k,2}^{\gamma} , \tag{3.1}$$

with $\gamma = (a + \frac{1}{2}b - 1)/k$ and $C = C(n, k, a, b)$.

Proof. See for instance [3, Lemma 4.3]. □

It turns out that inequality (3.1) can be generalized in some cases also to allow $c = k$ by applying directly Cauchy-Schwarz inequality. The sharper version of this inequality (as stated in Lemma 3.4 below) will be useful in many parts of the proof. In the next statement we “rename” the parameter k as $\ell + 2$.

Lemma 3.4. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. For any $a, c, \ell \in \mathbb{N}_0$, $b \in \mathbb{N}$, $b \geq 2$, $c \leq \ell + 2$ and $a < 2(\ell + 2)$ we find*

$$\int_I \left|P_b^{a,c}(\vec{\kappa})\right| \, ds \leq C\mathcal{L}[f]^{1-a-b} \|\vec{\kappa}\|_2^{b-\gamma} \|\vec{\kappa}\|_{\ell+2,2}^{\gamma}\,, \tag{3.2}$$

with $\gamma = (a + \frac{1}{2}b - 1)/(\ell + 2)$ and $C = C(n, \ell, a, b)$. Further if $a + \frac{1}{2}b < 2\ell + 5$, then for any $\varepsilon > 0$

$$\begin{aligned}\int_I \left|P_b^{a,c}(\vec{\kappa})\right| \, ds \leq \varepsilon \int_I \left|\nabla_s^{\ell+2}\vec{\kappa}\right|^2 \, ds + C\varepsilon^{-\frac{\gamma}{2-\gamma}} \left(\|\vec{\kappa}\|_{L^2}^2\right)^{\frac{b-\gamma}{2-\gamma}} \\ + C\mathcal{L}[f]^{1-a-\frac{b}{2}} \|\vec{\kappa}\|_{L^2}^b\,,\end{aligned} \tag{3.3}$$

with $C = C(n, \ell, a, b)$.

Proof. We start with the proof of (3.2). If $c < \ell + 2$, this is exactly (3.1) with $k = \ell + 2$. If $c = \ell + 2$, then since $a < 2(\ell + 2)$ there exists $\tilde{c} \leq \ell + 1$ such that

$$\int_I \left|P_b^{a,c}(\vec{\kappa})\right| \, ds = \int_I \left|\nabla_s^{\ell+2}\vec{\kappa}\right| \left|P_{b-1}^{a-\ell-2,\tilde{c}}(\vec{\kappa})\right| \, ds + \int_I \left|P_b^{a,\tilde{c}}(\vec{\kappa})\right| \, ds.$$

The treatment of the second integral is clear, hence we neglect it in the following computations. By Cauchy-Schwarz inequality, the definition of the scale invariant norms and (3.1) (with $k = \ell + 2$) we get

$$\begin{aligned}&\int_I |P_b^{a,c}(\vec{\kappa})| \, ds \leq \|\nabla_s^{\ell+2}\vec{\kappa}\|_{L^2} \left(\int_I |P_{2(b-1)}^{2(a-\ell-2),\tilde{c}}(\vec{\kappa})| \, ds\right)^{\frac{1}{2}} \\ &\leq C\|\nabla_s^{\ell+2}\vec{\kappa}\|_2 \mathcal{L}[f]^{-\ell-2-\frac{1}{2}} \left(\mathcal{L}[f]^{1-2(a-\ell-2)-2(b-1)} \|\vec{\kappa}\|_2^{2(b-1)-\tilde{\gamma}} \|\vec{\kappa}\|_{\ell+2,2}^{\tilde{\gamma}}\right)^{\frac{1}{2}}\,,\end{aligned}$$

with

$$\tilde{\gamma} = \frac{2(a - \ell - 2) + b - 2}{\ell + 2}\,.$$

Since $\gamma = 1 + \frac{1}{2}\tilde{\gamma}$, we get (3.2). Inequality (3.3) follows from Corollary 3.2, Young-inequality and $\gamma < 2$. □

Lemma 3.5. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve and $\ell \in \mathbb{N}_0$. If $A, B \in \mathbb{N}$ with $B \geq 2$ and $A + \frac{1}{2}B < 2\ell + 5$ then we have*

$$\begin{aligned}&\sum_{\substack{[[a,b]] \leq [[A,B]] \\ c \leq \ell+2,\, 2 \leq b}} \int_I |P_b^{a,c}(\vec{\kappa})| \, ds \\ &\leq C \min\{1, \mathcal{L}([f])\}^{1-2A-B} \max\{1, \|\vec{\kappa}\|_2\}^{2A+B} \max\{1, \|\vec{\kappa}\|_{\ell+2,2}\}^{\overline{\gamma}}\,,\end{aligned} \tag{3.4}$$

and for any $\varepsilon \in (0,1)$

$$\sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le\ell+2,2\le b}} \int_I \left|P_b^{a,c}(\vec{\kappa})\right| ds \le \varepsilon \int_I \left|\nabla_s^{\ell+2}\vec{\kappa}\right|^2 ds + C\varepsilon^{-\frac{\overline{\gamma}}{2-\overline{\gamma}}} \max\left\{1, \|\vec{\kappa}\|_{L^2}^2\right\}^{\frac{2A+B}{2-\overline{\gamma}}} + C\min\{1,\mathcal{L}[f]\}^{1-A-\frac{B}{2}} \max\left\{1, \|\vec{\kappa}\|_{L^2}\right\}^{2A+B}, \tag{3.5}$$

with $\overline{\gamma} = (A+\frac{1}{2}B-1)/(\ell+2)$ *and* $C = C(n,\ell,A,B)$.

Proof. Formulas (3.4) and (3.5) follow from (3.2) and (3.3) respectively using the fact that for each a, b in the sum we have $a+\frac{1}{2}b-1 \le A+\frac{B}{2}-1$ and b, $a+b \le 2A+B$ and $2 \le b \le \max\{2, 2A+B-a\}$. Moreover $a < 2(\ell+2)$ for all a in the sum since $a+1 \le a+b/2 \le a+(B+2A-2a)/2 = A+B/2 < 2\ell+5$. ☐

From the above estimates we easily infer the following bounds at the boundary.

Lemma 3.6. *Let* $f : I \to \mathbb{R}^n$ *be a smooth regular curve and* $\ell \in \mathbb{N}_0$. *If* $A, B \in \mathbb{N}$ *with* $B \ge 2$ *and* $A+\frac{1}{2}B < 2\ell+4$ *then we have*

$$\left| \sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le\ell+1,\, 2\le b\\ b \text{ even}}} P_b^{a,c}(\vec{\kappa})\Big|_0^1 \right| \le C\min\{1,\mathcal{L}([f])\}^{-1-2A-B} \max\{1,\|\vec{\kappa}\|_2\}^{2A+2+B} \max\{1,\|\vec{\kappa}\|_{\ell+2,2}\}^{\overline{\gamma}}, \tag{3.6}$$

and for any $\varepsilon \in (0,1)$

$$\left| \sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le\ell+1,\, 2\le b\\ b \text{ even}}} P_b^{a,c}(\vec{\kappa})\Big|_0^1 \right| \le \varepsilon \int_I \left|\nabla_s^{\ell+2}\vec{\kappa}\right|^2 ds + C\varepsilon^{-\frac{\overline{\gamma}}{2-\overline{\gamma}}} \max\left\{1, \|\vec{\kappa}\|_{L^2}^2\right\}^{\frac{2A+2+B}{2-\overline{\gamma}}} + C\min\{1,\mathcal{L}[f]\}^{-A-\frac{B}{2}} \max\left\{1,\|\vec{\kappa}\|_{L^2}\right\}^{2A+2+B}, \tag{3.7}$$

with $\overline{\gamma} = (A+\frac{1}{2}B)/(\ell+2)$ *and* $C = C(n,\ell,A,B)$.

Proof. Since each term in the sum has an even number of factors, we have

$$\begin{aligned}
\left| \sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq \ell+1,\, 2\leq b\\ b \text{ even}}} P_b^{a,c}(\vec{\kappa})\Big|_0^1 \right| &= \left| \sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq \ell+1,\, 2\leq b\\ b \text{ even}}} \int_I \partial_s (P_b^{a,c}(\vec{\kappa}))\, ds \right| \\
&= \left| \sum_{\substack{[[a,b]]\leq[[A,B]]\\ c\leq \ell+1,\, 2\leq b}} \int_I \left(P_b^{a+1,c+1}(\vec{\kappa}) \right) ds \right| \\
&\leq \sum_{\substack{[[a,b]]\leq[[A+1,B]]\\ c\leq \ell+2,\, 2\leq b}} \int_I \left| P_b^{a,c}(\vec{\kappa}) \right| ds\,.
\end{aligned}$$

The claim follows directly from Lemma 3.5. □

In our case, where the length is constant along the flow, we trivially have $\mathcal{L}[f] = L_0$.

In the proof of the main result, it will be convenient to derive pointwise estimates from bounds on the (scale invariant) Sobolev norms of the curvature . This is done in the spirit of the following results.

Lemma 3.7 ([3, Lemma C.1, C.2]). *Let $J \subset \mathbb{R}$ be a bounded open interval and $g : J \to \mathbb{R}^n$, $g(x)$, be a sufficiently smooth function. Then*

$$\|g\|_{C^0(\bar{J})} \leq c(n)\|\partial_x g\|_{L^1(J)} + \frac{c(n)}{|J|}\|g\|_{L^1(J)}\,,$$

and for any $\varepsilon \in (0, 1)$

$$\|g\|_{C^0(\bar{J})} \leq \varepsilon\|\partial_x g\|_{L^2(J)} + \frac{c}{\varepsilon}\|g\|_{L^2(J)}\,,$$

with $c = c(J, n)$.

Lemma 3.8. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve such that $\mathcal{L}[f] = L_0$ and $\|\vec{\kappa}\|_{m,2} \leq C_1$ for some $m \in \mathbb{N}$ and $C_1 > 0$. Then for $\ell \in \mathbb{N}_0$, $\ell \leq m-1$, we have*

$$\|\nabla_s^\ell \vec{\kappa}\|_{C^0(\bar{I})} \leq C_2\,,$$

with $C_2 = C_2(n, L_0, C_1)$.

Proof. We parametrize the curve such that $f : [0, L_0] \to \mathbb{R}^n$ and $|\partial_y f| = 1$ for all $y \in [0, L_0]$. In this way, $\partial_y = \partial_s$. By the first estimate in Lemma 3.7 we find for

$\ell \leq m-1$

$$\begin{aligned}\left\|\nabla_s^\ell \vec\kappa\right\|_{C^0(\bar I)} &\leq c(n)\left\|\partial_s|\nabla_s^\ell\vec\kappa|\right\|_{L^1([0,L_0])}+\frac{c(n)}{L_0}\left\|\nabla_s^\ell\vec\kappa\right\|_{L^1([0,L_0])}\\ &\leq c(n)L_0^{\frac12}\left\|\partial_s|\nabla_s^\ell\vec\kappa|\right\|_{L^2([0,L_0])}+\frac{c(n)}{L_0^{\frac12}}\left\|\nabla_s^\ell\vec\kappa\right\|_{L^2([0,L_0])}\\ &\leq c(n)L_0^{\frac12}\left\|\nabla_s^{\ell+1}\vec\kappa\right\|_{L^2([0,L_0])}+\frac{c(n)}{L_0^{\frac12}}\left\|\nabla_s^\ell\vec\kappa\right\|_{L^2([0,L_0])}\leq C\|\vec\kappa\|_{m,2}\leq C,\end{aligned}$$

using that $\partial_s|\vec\phi| \leq |\nabla_s\vec\phi|$ holds almost everywhere in $[0, L_0]$ for a normal vector field $\vec\phi$ (see [3, Lemma C.3]). □

Lemma 3.9. *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve such that $\mathcal{L}[f] = L_0$. Then for any $\ell \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$*

$$\left\|\nabla_s^\ell\vec\kappa\right\|_{C^0(\bar I)} \leq \varepsilon\left\|\nabla_s^{\ell+1}\vec\kappa\right\|_{L^2}+\frac{C_3}{\varepsilon}\left\|\nabla_s^\ell\vec\kappa\right\|_{L^2},$$

with $C_3 = C_3(n, L_0)$.

Proof. Using the same ideas as in the proof of Lemma 3.8 and the second estimate in Lemma 3.7 the claim follows directly. □

4. Estimates on the time-dependent parameter λ

The results we give in this section do *not* use any information regarding the conditions imposed at the boundary (except of course for (1.5)). Thus they generalize the results given in [2, Section 2], where analogous results were given for the special case of hinged boundary conditions. We give here two different estimates for the absolute value of λ.

Lemma 4.1. *Let f be a smooth solution of* (1.2) *subject to* (1.4), (1.3) *with $\mathcal{L}[f_0]= L_0$, and assume that there exists $\delta > 0$ such that $0 < \delta \leq \|\vec\kappa\|^2_{L^2} \leq \delta^{-1}$ for all times. Then for $\lambda = \lambda(t)$ we have that*

$$|\lambda| \leq C\|\vec\kappa\|_{m+2,2}^{\frac{2}{m+2}}+C \leq C\left(\left\|\nabla_s^{m+2}\vec\kappa\right\|_{L^2}^{\frac{2}{m+2}}+1\right) \text{ for all } m \in \mathbb{N}_0, \tag{4.1}$$

where $C = C(L_0, \delta, n, m)$.

Proof. Using (2.9) we obtain

$$\lambda(t)\int_I|\vec\kappa|^2\,ds = \int_I\langle\nabla_s^2\vec\kappa,\vec\kappa\rangle\,ds+\frac12\int_I|\vec\kappa|^4\,ds. \tag{4.2}$$

The claim follows using (3.2), $\mathcal{L}[f] = L_0$, the bounds on the elastic energy and Corollary 3.2 for the second estimate in the claim. □

As already observed in [2], in the main proof it will be important to have an estimate for the absolute value of λ, which is linear in $\|\partial_t f\|_2$. A straightforward computation, that does *not* use any information on the boundary conditions but only integration by parts and the fact that $\nabla_s^2\vec\kappa + \frac{1}{2}|\vec\kappa|^2\vec\kappa = \partial_s(\nabla_s\vec\kappa + \frac{1}{2}|\vec\kappa|^2\tau)$, gives

Lemma 4.2. *Suppose $f : [0,T)\times\bar I \to \mathbb{R}^n$ is a smooth regular solution to* (1.2) *in $(0,T)\times I$. For any smooth function $g : \bar I \to \mathbb{R}^n$, we have*

$$\int_I \langle \partial_t f, f-g\rangle\, ds = \left\langle \left(\lambda - \frac{1}{2}|\vec\kappa|^2\right)\tau - \nabla_s\vec\kappa,\, f-g\right\rangle\Big|_{\partial I} + \frac{1}{2}\int_I |\vec\kappa|^2\, ds - \lambda\int_I ds$$
$$- \int_I \left\langle \nabla_s\vec\kappa + \frac{1}{2}|\vec\kappa|^2\tau - \lambda\tau,\, \partial_s g\right\rangle ds\,.$$

Note that if the curve is closed in the sense that $f_- = f_+$ (however it does not have to be smooth across the point $f_- = f_+$), then we obtain the same result as in [6] by taking g to be the constant function $g = f_- = f_+ =: p$, namely

$$\int_I \langle \partial_t f, f-p\rangle\, ds = \frac{1}{2}\int_I |\vec\kappa|^2\, ds - \lambda\int_I ds.$$

If we allow for the more general situation where $f_- \neq f_+$, we can show still *without* using any boundary conditions (except for (1.5)) that the following holds.

Lemma 4.3. *Suppose $f : [0,T)\times\bar I \to \mathbb{R}^n$ is a smooth regular solution to* (1.2), (1.4) *in $(0,T)\times I$. Let $\int_I ds = L_0$ with $L_0 > |f_- - f_+|$,* (1.5)*, and $\|\vec\kappa\|_{L^2} \le \delta^{-1}$ hold along the flow. Then for all $m \in \mathbb{N}_0$*

$$|\lambda|(L_0 - |f_+ - f_-|) \le C\|\partial_t f\|_{L^2} + C + C\|\vec\kappa\|_{m+2,2}^{\frac{1}{m+2}}\,,$$
$$\le C\|\partial_t f\|_{L^2} + C + C\|\nabla_s^{m+2}\vec\kappa\|_{L^2}^{\frac{1}{m+2}}\,,$$

with constants depending only on L_0, n, m, δ^{-1} and $|f_+ - f_-|$.

Proof. Let $l : [0,T)\times[0,1] \to [0,L_0]$ be the parametrization of the line segment from f_- to f_+ given by

$$l(t,x) = f_- + \frac{\varphi(t,x)}{L_0}(f_+ - f_-)\,, \tag{4.3}$$

with $\varphi(t,\xi) = \int_0^\xi |\partial_x f|dx$ for all $\xi \in [0,1]$ and $t \in [0,T)$. Notice that the parametrization depends on t, $l(t,0) = f_-$, $l(t,1) = f_+$ for all $t \in [0,T)$ and $\partial_s l(t,\cdot) = \frac{1}{L_0}(f_+ - f_-)$. Hence, Lemma 4.2, with $g(\cdot) = l(t,\cdot)$ for $t \in (0,T)$ fixed, yields

$$\int_I \langle \partial_t f, f-l\rangle\, ds = \frac{1}{2}\int_I |\vec\kappa|^2\, ds - \lambda\int_I ds - \frac{1}{L_0}\int_I \left\langle \nabla_s\vec\kappa + \frac{1}{2}|\vec\kappa|^2\tau - \lambda\tau,\, f_+ - f_-\right\rangle ds\,,$$

which is equivalent to

$$\lambda \int_I \left(1 - \frac{1}{L_0}\langle \tau, f_+ - f_-\rangle\right) ds = \frac{1}{2}\int_I |\vec\kappa|^2\, ds - \int_I \langle \partial_t f, f - l\rangle\, ds - \frac{1}{L_0}\int_I \left\langle \nabla_s\vec\kappa + \frac{1}{2}|\vec\kappa|^2\tau, f_+ - f_-\right\rangle ds\,.$$

Since $|f_+ - f_-| < L_0$ and $L_0 = \int_I ds$ we find

$$|\lambda|(L_0 - |f_+ - f_-|) \le |\lambda| \int_I \left(1 - \frac{1}{L_0}\langle \tau, f_+ - f_-\rangle\right) ds$$
$$= \left|\frac{1}{2}\int_I |\vec\kappa|^2\, ds - \int_I \langle \partial_t f, f - l\rangle\, ds - \frac{1}{L_0}\int_I \left\langle \nabla_s\vec\kappa + \frac{1}{2}|\vec\kappa|^2\tau, f_+ - f_-\right\rangle ds\right|,$$

from which we infer

$$|\lambda|(L_0 - |f_+ - f_-|) \le \frac{1}{2}\int_I |\vec\kappa|^2\, ds + \|\partial_t f\|_{L^2}\|f - l\|_\infty L_0^{\frac{1}{2}} + \frac{|f_+ - f_-|}{2L_0}\int_I |\vec\kappa|^2\, ds + \frac{|f_+ - f_-|}{L_0} L_0^{\frac{1}{2}}\|\nabla_s\vec\kappa\|_{L^2}$$
$$\le C\|\partial_t f\|_{L^2} + C + C\|\vec\kappa\|_{m+2,2}^{\frac{1}{m+2}},$$

using (3.1) in the last inequality. □

4.1. Estimates for the derivatives of λ

Lemma 4.4. *For any $\ell \in \mathbb{N}$,*

$$\lambda^{(\ell)} \int_I |\vec\kappa|^2\, ds = (-1)^\ell \int_I \left\langle \vec\kappa, \nabla_s^{4\ell+2}\vec\kappa\right\rangle ds + \sum_{i=0}^{2\ell} Q_i(\lambda_{\ell-1}) \int_I \sum_{\substack{[[a,b]]\le[[4\ell+2-2i,2]]\\ c\le 4\ell,\ b\ge 2,\ b \text{ even}}} P_b^{a,c}(\vec\kappa)\, ds\,. \tag{4.4}$$

Proof. We prove the claim by induction on $\ell \in \mathbb{N}$. For this it is useful to notice first that by (1.2) and (2.1) we find

$$\frac{\partial}{\partial t}(ds) = \sum_{i=0}^{1} \lambda^i \sum_{\substack{[[a,b]]\le[[2-2i,2]]\\ c\le 2-2i,\ b \text{ even}}} P_b^{a,c}(\vec\kappa)\, ds$$

so that together with (2.16) we obtain

$$\partial_t \left(\sum_{\substack{[[a,b]]\le[[A,B]]\\ c\le C,\ b \text{ even}}} P_b^{a,c}(\vec\kappa)\, ds \right) = \sum_{i=0}^{1} \lambda^i \sum_{\substack{[[a,b]]\le[[A+4-2i,B]]\\ c\le C+4-2i,\ b \text{ even}}} P_b^{a,c}(\vec\kappa)\, ds\,.$$

Then differentiating (4.2) with respect to time we find (using also (2.6) and again (2.16))

$$\begin{aligned}\lambda'(t)\int_I |\vec{\kappa}|^2\,ds + \lambda\sum_{i=0}^{1}\lambda^i\int_I \sum_{\substack{[[a,b]]\le[[4-2i,2]]\\ c\le 4-2i,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds \\ = \int_I \left\langle \nabla_t\nabla_s^2\vec{\kappa},\vec{\kappa}\right\rangle ds + \sum_{i=0}^{1}\lambda^i\int_I \sum_{\substack{[[a,b]]\le[[6-2i,2]]\\ c\le 4-2i,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds\,,\end{aligned}$$

where $\lambda'(t) = \frac{d}{dt}\lambda(t) = \lambda^{(1)}$. Using (2.14) (with $\ell = 2$) we find

$$\lambda'(t)\int_I |\vec{\kappa}|^2\,ds = -\int_I \left\langle \nabla_s^6\vec{\kappa},\vec{\kappa}\right\rangle ds + \sum_{i=0}^{2}\lambda^i\int_I \sum_{\substack{[[a,b]]\le[[6-2i,2]]\\ c\le 4,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds\,.$$

Assume now that the claim is true up to some $\ell \in \mathbb{N}$. We prove it for $\ell + 1$. Differentiating (4.4) with respect to time, proceeding similarly as before and using now also (2.17) we find

$$\begin{aligned}&\lambda^{(\ell+1)}\int_I |\vec{\kappa}|^2\,ds + \lambda^{(\ell)}\sum_{i=0}^{1}\lambda^i\int_I \sum_{\substack{[[a,b]]\le[[4-2i,2]]\\ c\le 4-2i,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds \\ &= (-1)^\ell\int_I \left\langle \vec{\kappa},\nabla_t\nabla_s^{4\ell+2}\vec{\kappa}\right\rangle ds \\ &\quad + \sum_{i=0}^{1}\lambda^i\int_I \left\langle \sum_{\substack{[[a,b]]\le[[4-2i,1]]\\ c\le 4-2i,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa}),\,\nabla_s^{4\ell+2}\vec{\kappa}\right\rangle ds \\ &\quad + \sum_{i=0}^{2\ell} Q_{i+2}(\lambda_\ell)\int_I \sum_{\substack{[[a,b]]\le[[4\ell+2-2i,2]]\\ c\le 4\ell,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds \\ &\quad + \sum_{i=0}^{2\ell} Q_i(\lambda_{\ell-1})\int_I \sum_{j=0}^{1}\lambda^j \sum_{\substack{[[a,b]]\le[[4(\ell+1)+2-2i-2j,2]]\\ c\le 4(\ell+1)-2j,\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds\,.\end{aligned}$$

Since $\lambda^{(\ell)} = Q_{2\ell+1}(\lambda_\ell)$, $\lambda\lambda^{(\ell)} = Q_{2\ell+2}(\lambda_\ell)$ and $\lambda Q_i(\lambda_{\ell-1}) = Q_{i+1}(\lambda_{\ell-1})$, using (2.14) the previous formula simplifies to

$$\begin{aligned}\lambda^{(\ell+1)}\int_I |\vec{\kappa}|^2\,ds &= (-1)^{\ell+1}\int_I \left\langle \vec{\kappa},\nabla_s^{4(\ell+1)+2}\vec{\kappa}\right\rangle ds \\ &\quad + \sum_{i=0}^{2(\ell+1)} Q_i(\lambda_\ell)\int_I \sum_{\substack{[[a,b]]\le[[4(\ell+1)+2-2i,2]]\\ c\le 4(\ell+1),\ b\text{ even}}} P_b^{a,c}(\vec{\kappa})\,ds\,.\end{aligned}$$

The claim follows. □

Lemma 4.5. *Assume that there exists* $\delta > 0$ *such that* $0 < \delta \leq \|\vec{\kappa}\|_{L^2} \leq \delta^{-1}$ *holds along the flow. Then, for any* $m \in \mathbb{N}$ *and* $\ell \in \{0, 1, ..., m\}$,

$$|\lambda^{(\ell)}| \leq C\Big(\|\nabla_s^{4m+2}\vec{\kappa}\|_{L^2}^{(4\ell+2)/(4m+2)} + 1\Big),$$

where $C = C(L_0, \delta, n, m)$.

Proof. We prove the claim by induction on m. In the proof we denote by C a constant that might change at each inequality but that is allowed to depend only on L_0, δ, n and m. For $m = 1$, the bound for $\ell = 0$ is established in (4.1) taking in that estimate $m = 4$. For $m = 1$ and $\ell = 1$, by (4.4), the uniform bounds on $\|\vec{\kappa}\|_{L^2}$ together with (3.4) we find

$$\delta^2|\lambda'(t)| \leq \delta^{-1}\left\|\nabla_s^6\vec{\kappa}\right\|_{L^2} + \sum_{i=0}^{2}|Q_i(\lambda_0)|C\left(1 + \|\vec{\kappa}\|_{6,2}^{\frac{6-2i}{6}}\right).$$

Now since for $i \in \{0, 1, 2\}$ by definition $Q_i(\lambda_0) = c_i\lambda^i$, by the estimate already obtained for $\ell = 0$ and Corollary 3.2 it follows

$$\begin{aligned}\delta^2|\lambda'(t)| &\leq \delta^{-1}\left\|\nabla_s^6\vec{\kappa}\right\|_{L^2} + \sum_{i=0}^{2}\tilde{C}\left(\|\nabla_s^6\vec{\kappa}\|_{L^2}^{2/6} + 1\right)^i C\left(1 + \|\vec{\kappa}\|_{6,2}^{\frac{6-2i}{6}}\right)\\ &\leq C\left(1 + \|\nabla_s^6\vec{\kappa}\|_{L^2}\right).\end{aligned}$$

Assume the claim is true up to m and let's prove it for $m + 1$. If $\ell \in \{0, \dots, m\}$ then by the induction hypothesis, (3.1) and Corollary 3.2

$$\begin{aligned}|\lambda^{(\ell)}| &\leq C\left(\|\nabla_s^{4m+2}\vec{\kappa}\|_{L^2}^{(4\ell+2)/(4m+2)} + 1\right)\\ &\leq C\left(\|\nabla_s^{4(m+1)+2}\vec{\kappa}\|_{L^2}^{(4\ell+2)/(4(m+1)+2)} + 1\right).\end{aligned} \tag{4.5}$$

Hence it remains to prove the claim for $\ell = m + 1$. By (4.4), the uniform bounds on $\|\vec{\kappa}\|_{L^2}$, (3.4) and Corollary 3.2 we obtain

$$\begin{aligned}\delta^2\left|\lambda^{(m+1)}(t)\right| &\leq \delta^{-1}\left\|\nabla_s^{4(m+1)+2}\vec{\kappa}\right\|_{L^2}\\ &\quad + \sum_{i=0}^{2(m+1)}|Q_i(\lambda_m)|C\left(1 + \left\|\nabla_s^{4(m+1)+2}\vec{\kappa}\right\|_{L^2}^{\frac{4(m+1)+2-2i}{4(m+1)+2}}\right).\end{aligned} \tag{4.6}$$

By (4.5) and the definition of $Q_i(\lambda_m)$ we find

$$\begin{aligned}|Q_i(\lambda_m)| &\leq C\prod_{a=0}^{m}\left(\left\|\nabla_s^{4(m+1)+2}\vec{\kappa}\right\|_{L^2}^{(4a+2)/(4(m+1)+2)} + 1\right)^{\beta_a}\\ &\leq C\left(\left\|\nabla_s^{4(m+1)+2}\vec{\kappa}\right\|_{L^2}^{2i/(4(m+1)+2)} + 1\right).\end{aligned} \tag{4.7}$$

The claim follows by combining (4.6) and (4.7). ☐

The following lemma is important for the induction argument in the proof of the main result.

Lemma 4.6. *Assume that there exists $\delta > 0$ such that $0 < \delta \le \|\vec{\kappa}\|_{L^2} \le \delta^{-1}$ holds along the flow and also $\|\vec{\kappa}\|_{4m,2} \le M$ for some $m \in \mathbb{N}$ and a positive constant M. Then,*

$$\begin{gathered}\left|\lambda^{(\ell)}\right| \le C \ \textit{ for all } \ \ell \in \{0, \dots, m-1\}, \qquad \sum_{i=1}^{2m} |Q_i(\lambda_m)| \le C\,, \\ \textit{and } \ |Q_i(\lambda_m)| \le C\left|\lambda^{(m)}\right| \ \textit{ for } \ i = 2m+1, 2m+2\,,\end{gathered} \tag{4.8}$$

with $C = C(L_0, \delta, n, m, M)$. Moreover for any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon = C(\varepsilon, L_0, \delta, n, m, M)$ such that

$$\left|\lambda^{(m)}\right| \le \varepsilon \left\| \nabla_s^{4(m+1)+2} \vec{\kappa} \right\|_{L^2}^{(4m+2)/(4(m+1)+2)} + C_\varepsilon\,. \tag{4.9}$$

Proof. In the proof we denote by C a constant that might change at each inequality but that is allowed to depend only on L_0, δ, n, m and M.

From Lemma 4.5 and the assumption $\|\vec{\kappa}\|_{4m,2} \le M$ it follows directly for $\ell \in \{0, \dots, m-1\}$ that

$$|\lambda^{(\ell)}| \le C \cdot \left(\|\nabla_s^{4(m-1)+2} \vec{\kappa}\|_{L^2}^{(4\ell+2)/(4(m-1)+2)} + 1 \right) \le C\,.$$

In $Q_i(\lambda_m)$, λ and its derivatives with respect to time up to order m appear. The only one that is not already bounded is $\lambda^{(m)}$. Such a factor might appear in $Q_i(\lambda_m)$ when $i = 2m+1$ or $2m+2$ and, if it appears, only to the power one. These observations give the second and third estimate in (4.8).

It remains to prove (4.9). By (4.4), the uniform bound from below on $\|\vec{\kappa}\|_{L^2}$ and (4.8) it follows

$$\delta^2 \left|\lambda^{(m)}\right| \le \left| \int_I \left\langle \vec{\kappa}, \nabla_s^{4m+2} \vec{\kappa} \right\rangle ds \right| + C \left| \int_I \sum_{\substack{[[a,b]] \le [[4m+2,2]] \\ c \le 4m,\ b \ge 2}} P_b^{a,c}(\vec{\kappa})\, ds \right|. \tag{4.10}$$

The second term on the right hand side is uniformly bounded since by Lemma 3.5

$$\left| \int_I \sum_{\substack{[[a,b]] \le [[4m+2,2]] \\ c \le 4m,\ b \ge 2}} P_b^{a,c}(\vec{\kappa})\, ds \right| \le C \left(\|\vec{\kappa}\|_{4m,2}^{\frac{4m+2}{4m}} + 1 \right) \le C\,. \tag{4.11}$$

In the first term on the right hand side of (4.10) we gain an ε integrating by parts. Indeed, we find using $\|\vec{\kappa}\|_{C^0}, \|\nabla_s\vec{\kappa}\|_{C^0} \le C(L_0, M, n)$ (which follows from by Lemma 3.8) and Lemma 3.9

$$\begin{aligned}\left|\int_I \langle\vec{\kappa}, \nabla_s^{4m+2}\vec{\kappa}\rangle\, ds\right| &= \left|\left\langle\vec{\kappa}, \nabla_s^{4m+1}\vec{\kappa}\right\rangle\Big|_0^1 - \int_I \left\langle\nabla_s\vec{\kappa}, \nabla_s^{4m+1}\vec{\kappa}\right\rangle ds\right| \\ &\le C\left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{C^0} \le \frac{\varepsilon}{2}\delta^2\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2} + \frac{C}{\varepsilon}\left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{L^2}.\end{aligned}$$

On the other hand, by Lemma 3.3, Corollary 3.2 and Young's inequality

$$\begin{aligned}\left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{L^2} &\le C\|\vec{\kappa}\|_{4m+2,2}^{\frac{4m+1}{4m+2}} \\ &\le C\left(1 + \left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}^{\frac{4m+1}{4m+2}}\right) \le \frac{\varepsilon^2}{2C}\delta^2\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2} + C_\varepsilon.\end{aligned}$$

Combining the two inequalities above and using once again Lemma 3.3 and Corollary 3.2 we obtain

$$\begin{aligned}\left|\int_I \left\langle\vec{\kappa}, \nabla_s^{4m+2}\vec{\kappa}\right\rangle ds\right| &\le \varepsilon\delta^2\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2} + C(\varepsilon) \\ &\le \varepsilon\delta^2\left\|\nabla_s^{4(m+1)+2}\vec{\kappa}\right\|_{L^2}^{\frac{4m+2}{4(m+1)+2}} + C_\varepsilon.\end{aligned}$$

The claim follows from the inequality above, (4.10) and (4.11). □

5. Proof of the main result

We are ready to prove our main result. A proof of short time existence of smooth solutions for the problem (1.2), (1.3), (1.4), (1.5), (1.6) is standard but outside the scope of this work. A method of proof could be devised along the lines of [6, Section 3], where the case of closed curves is treated.

Proof of Theorem 1.1.
Part 1: ***Global Existence***. A short time existence result gives that a smooth solution exists in a small time interval. We assume by contradiction that the solution to (1.7) does not exist globally in time. Let $0 < T < \infty$ be the maximal time. From (2.10) a bound from above for the L^2-norm of the curvature is immediately derived. The bound from below on the elastic energy follows from the assumption $L_0 > |f_+ - f_-|$, as shown in [2, Theorem 3.1]. We repeat the reasoning for completeness. One has

$$L_0 = \int_I \langle\partial_s f, \partial_s(f - f_-)\rangle\, ds = \langle\tau(1), f_+ - f_-\rangle - \int_I \left\langle\partial_s^2 f, f - f_-\right\rangle ds\,,$$

and hence

$$L_0 - |f_+ - f_-| \leq \|\vec{\kappa}\|_{L^2} L_0^{\frac{3}{2}} ,$$

from which the uniform bound from below follows. Thus we can state

$$0 < \delta \leq \|\vec{\kappa}\|_{L^2} \leq \delta^{-1} , \tag{5.1}$$

with $\delta = \delta(L_0, f_-, f_+, E(f_0)) > 0$. In particular λ is well defined and the results collected in Section 4 can be used. We are going to prove that appropriate norms of the solution f are uniformly bounded in $t \in (0, T)$ and hence f can be smoothly extended to time bigger than T, yielding a contradiction.

In the following C denotes a positive constant that might change from line to line but depends only on $E(f_0)$, f_0, f_-, f_+, n, L_0. In particular, the constant is not allowed to depend on T.

First Step: Here we show that $\|\nabla_t f\|_{L^2} \leq C$, $\|\nabla_s^2 \vec{\kappa}\|_{L^2} \leq C$ and $|\lambda| \leq C$ for all $t \in (0, T)$.

Using (1.2), (2.14), (2.15) and (2.20) we can derive

$$\nabla_t \nabla_t f + \nabla_s^4 \nabla_t f = \sum_{i=0}^{2} \lambda^i \sum_{\substack{[[a,b]] \leq [[4-2i,3]] \\ c \leq 4-2i,\ b \text{ odd}}} P_b^{a,c}(\vec{\kappa}) + \lambda'(t)\vec{\kappa} - \lambda \nabla_s^4 \vec{\kappa} + \lambda^2 \nabla_s^2 \vec{\kappa} , \tag{5.2}$$

an equation which is slightly more precise than the expression derived in (2.18) (with $m = 2$) and serves better our purposes. Taking $\vec{\phi} = \nabla_t f$ in Lemma 2.3, using the fact that the boundary terms disappear (due to Lemma 2.4), (5.2), and the fact that $\int_I \langle \vec{\kappa}, \nabla_t f \rangle \, ds = 0$ (recall (2.9)), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_I |\nabla_t f|^2 \, ds + \int_I \left| \nabla_s^2 \nabla_t f \right|^2 ds \\ &= \sum_{i=0}^{2} \lambda^i \sum_{\substack{[[a,b]] \leq [[4-2i,3]] \\ c \leq 4-2i,\ b \text{ odd}}} \int_I \langle P_b^{a,c}(\vec{\kappa}), \nabla_t f \rangle \, ds - \lambda \int_I \left\langle \nabla_s^4 \vec{\kappa}, \nabla_t f \right\rangle ds \\ &\quad + \lambda^2 \int_I \left\langle \nabla_s^2 \vec{\kappa}, \nabla_t f \right\rangle ds - \frac{1}{2} \int_I |\nabla_t f|^2 \left\langle \vec{\kappa}, -\nabla_s^2 \vec{\kappa} - P_3^{0,0}(\vec{\kappa}) + \lambda \vec{\kappa} \right\rangle ds . \end{aligned} \tag{5.3}$$

First of all we rewrite in a more convenient way the terms in the expression above. From (1.2) it follows directly that

$$\sum_{i=0}^{2} \lambda^i \sum_{\substack{[[a,b]] \leq [[4-2i,3]] \\ c \leq 4,\ b \text{ odd}}} \int_I \langle P_b^{a,c}(\vec{\kappa}), \nabla_t f \rangle \, ds = \sum_{i=0}^{3} \lambda^i \sum_{\substack{[[a,b]] \leq [[6-2i,4]] \\ c \leq 4,\ b \text{ even}}} \int_I P_b^{a,c}(\vec{\kappa}) \, ds .$$

In the subsequent calculations the b-index appearing in the summation symbols will always be an even number, hence, for the sake of readability we omit this information. Since

$$|\nabla_t f|^2 = \sum_{i=0}^{2} \lambda^i \sum_{\substack{[[a,b]]\leq[[4-2i,2]] \\ c\leq 2}} P_b^{a,c}(\vec{\kappa}) \, ,$$

we observe that

$$\int_I |\nabla_t f|^2 \left\langle \vec{\kappa}, -\nabla_s^2\vec{\kappa} - P_3^{0,0}(\vec{\kappa}) + \lambda\vec{\kappa} \right\rangle ds = \sum_{i=0}^{3} \lambda^i \sum_{\substack{[[a,b]]\leq[[6-2i,4]] \\ c\leq 2}} \int_I P_b^{a,c}(\vec{\kappa})\, ds.$$

Now, in order to be able to absorb (later on) some terms on the left-hand side, we express $|\nabla_s^2 \nabla_t f|^2$ in terms of the curvature. Since $\nabla_s^2\nabla_t f = -\nabla_s^4\vec{\kappa} + P_3^{2,2}(\vec{\kappa}) + \lambda\nabla_s^2\vec{\kappa}$, we find

$$\begin{aligned}
\left|\nabla_s^2\nabla_t f\right|^2 &= \left|\nabla_s^4\vec{\kappa}\right|^2 + \lambda^2\left|\nabla_s^2\vec{\kappa}\right|^2 - 2\lambda\left\langle \nabla_s^2\vec{\kappa}, \nabla_s^4\vec{\kappa}\right\rangle + \sum_{i=0}^{1}\lambda^i \sum_{\substack{[[a,b]]\leq[[6-2i,4]] \\ c\leq 4}} P_b^{a,c}(\vec{\kappa}) \\
&= \left|\nabla_s^4\vec{\kappa}\right|^2 + \lambda^2\left|\nabla_s^2\vec{\kappa}\right|^2 + 2\lambda\left\langle \nabla_t f, \nabla_s^4\vec{\kappa}\right\rangle \\
&\quad - 2\lambda^2\left\langle \vec{\kappa}, \nabla_s^4\vec{\kappa}\right\rangle + \sum_{i=0}^{1}\lambda^i \sum_{\substack{[[a,b]]\leq[[6-2i,4]] \\ c\leq 4}} P_b^{a,c}(\vec{\kappa}) \, .
\end{aligned}$$

Thus, using the expressions above, adding on both sides $\frac{1}{2}\|\nabla_t f\|_{L^2}^2$ we can write (5.3) as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I |\nabla_t f|^2\, ds + \int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds + \frac{1}{2}\int_I |\nabla_t f|^2 + \lambda^2\int_I\left|\nabla_s^2\vec{\kappa}\right|^2 ds \\
&= \sum_{i=0}^{3}\lambda^i \sum_{\substack{[[a,b]]\leq[[6-2i,4]] \\ c\leq 4}} \int_I P_b^{a,c}(\vec{\kappa})\, ds - 3\lambda\int_I \left\langle \nabla_s^4\vec{\kappa}, \nabla_t f\right\rangle ds \\
&\quad + 2\lambda^2\int_I\left\langle \nabla_s^4\vec{\kappa}, \vec{\kappa}\right\rangle ds + \lambda^2\int_I\left\langle \nabla_s^2\vec{\kappa}, \nabla_t f\right\rangle ds \, .
\end{aligned} \tag{5.4}$$

Next we absorb the terms on the right-hand side of (5.4) using the bounds obtained for λ in Section 4 and the interpolation inequalities of Section 3. Since by Lemma 4.3

$$|\lambda| \leq C\|\nabla_t f\|_{L^2} + C + C\|\nabla_s^4\vec{\kappa}\|_{L^2}^{\frac{1}{4}} \, ,$$

we get using several times Young's inequality

$$
\begin{aligned}
\left|-3\lambda\int_I \left\langle \nabla_s^4\vec{\kappa}, \nabla_t f\right\rangle ds\right| &\leq \frac{\varepsilon}{2}\left\|\nabla_s^4\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon\lambda^2\|\nabla_t f\|_{L^2}^2 \\
&\leq \varepsilon\left\|\nabla_s^4\vec{\kappa}\right\|_{L^2}^2 + \tilde{C}_\varepsilon\|\nabla_t f\|_{L^2}^4 + \tilde{C}_\varepsilon\,, \\
\left|2\lambda^2\int_I \left\langle \nabla_s^4\vec{\kappa}, \vec{\kappa}\right\rangle ds\right| &\leq \frac{\varepsilon}{2}\left\|\nabla_s^4\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon\lambda^4 \\
&\leq \varepsilon\left\|\nabla_s^4\vec{\kappa}\right\|_{L^2}^2 + \tilde{C}_\varepsilon\|\nabla_t f\|_{L^2}^4 + \tilde{C}_\varepsilon\,, \\
\left|\lambda^2\int_I \left\langle \nabla_s^2\vec{\kappa}, \nabla_t f\right\rangle ds\right| &\leq \left\|\nabla_s^2\vec{\kappa}\right\|_{L^2}\lambda^2\|\nabla_t f\|_{L^2} \leq \frac{\varepsilon}{2}\left\|\nabla_s^2\vec{\kappa}\right\|_{L^2}^4 + C_\varepsilon\left(\lambda^2\|\nabla_t f\|_{L^2}\right)^{\frac{4}{3}} \\
&\leq \varepsilon\left\|\nabla_s^4\vec{\kappa}\right\|_{L^2}^2 + \tilde{C}_\varepsilon\|\nabla_t f\|_{L^2}^4 + \tilde{C}_\varepsilon\,,
\end{aligned}
$$

using Lemma 3.3 and Corollary 3.2 in the last inequality. The term with the sum on the right hand side of (5.4) can be estimated using that $|\lambda| \leq C\|\vec{\kappa}\|_{4,2}^{\frac{1}{2}} + C$ by (4.1), (3.4) (with $\ell = 2$) and (5.1) as follows

$$
\begin{aligned}
\sum_{i=0}^{3}|\lambda|^i \sum_{\substack{[[a,b]]\leq[[6-2i,4]] \\ c\leq 4,\ b \text{ even}}} \int_I \left|P_b^{a,c}(\vec{\kappa})\right| ds &\leq C\left(1+\|\vec{\kappa}\|_{4,2}^{\frac{i}{2}}\right)\left(1+\|\vec{\kappa}\|_{4,2}^{\frac{7-2i}{4}}\right) \\
&\leq C\|\vec{\kappa}\|_{4,2}^{\frac{7}{4}} + C \leq \varepsilon\int_I\left|\nabla_s^4\vec{\kappa}\right|^2 ds + C_\varepsilon\,,
\end{aligned}
$$

where we have used Corollary 3.2 and again (5.1) in the last inequality.

By the above estimates, an appropriate choice of ε and neglecting the positive term $\lambda^2\int_I |\nabla_s^2\vec{\kappa}|^2\,ds$ we obtain from (5.4)

$$
\frac{1}{2}\frac{d}{dt}\int_I |\nabla_t f|^2\,ds + \frac{1}{2}\int_I |\nabla_t f|^2\,ds \leq C + C\|\nabla_t f\|_{L^2}^2\int_I |\nabla_t f|^2\,ds\,. \tag{5.5}
$$

Finally, setting $\xi(t) := e^t\|\nabla_t f\|_{L^2}^2(t)$, we deduce

$$
\xi'(t) \leq C\left(e^t + \|\nabla_t f\|_{L^2}^2(t)\,\xi(t)\right).
$$

Since $\partial_t f = \nabla_t f$ we have by (2.10) for any time $v \in (0, T)$ that

$$
\int_0^v \|\nabla_t f\|_{L^2}^2\,dt = -\int_0^v \frac{d}{dt}E(f)\,dt \leq E(f_0)\,,
$$

we now use Gronwall's Lemma to infer that

$$\xi(t) \leq e^{CE(f_0)} \left(\xi(0) + C \int_0^t e^r \, dr \right).$$

The bound for $\|\nabla_t f\|_{L^2}$ follows, precisely

$$\|\nabla_t f\|_{L^2} \leq C \text{ for all } t \in (0, T). \tag{5.6}$$

Next we use (1.2), (5.1), Lemma 3.4, Lemma 4.3 (with $m = 0$), and Young-inequality to infer

$$\begin{aligned}
\int_I \left|\nabla_s^2 \vec{\kappa}\right|^2 ds &= \int_I \left| \partial_t f + \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa} - \lambda \vec{\kappa} \right|^2 ds \leq C\|\nabla_t f\|_{L^2}^2 + C \int_I \left| P_6^{0,0}(\vec{\kappa}) \right| ds + C\lambda^2 \\
&\leq C\|\nabla_t f\|_{L^2}^2 + \varepsilon \int_I \left|\nabla_s^2 \vec{\kappa}\right|^2 ds + C_\varepsilon + C\left(1 + \|\partial_t f\|_{L^2}^2 + \left\|\nabla_s^2 \vec{\kappa}\right\|_{L^2}\right) \\
&\leq C\|\nabla_t f\|_{L^2}^2 + 2\varepsilon \int_I \left|\nabla_s^2 \vec{\kappa}\right|^2 ds + C_\varepsilon.
\end{aligned}$$

An appropriate choice of ε and (5.6) yield

$$\left\|\nabla_s^2 \vec{\kappa}\right\|_{L^2} \leq C \text{ for all } t \in (0, T),$$

and then from (4.1), Corollary 3.2 and Lemma 3.8 we also get

$$|\lambda|, \|\vec{\kappa}\|_{2,2}, \|\vec{\kappa}\|_{C^0}, \|\nabla_s \vec{\kappa}\|_{C^0} \leq C \text{ for all } t \in (0, T). \tag{5.7}$$

Second Step: We show that $\|\nabla_s^4 \vec{\kappa}\|_{L^2} \leq C$ and $|\lambda'(t)| \leq \varepsilon \|\nabla_s^{10} \vec{\kappa}\|_{L^2}^{\frac{3}{5}} + C_\varepsilon$ for any $\varepsilon \in (0, 1)$ for all $t \in (0, T)$.

Taking $\vec{\phi} = \nabla_s^4 \vec{\kappa}$ in Lemma 2.3 we obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2} \int_I \left|\nabla_s^4 \vec{\kappa}\right|^2 ds + \int_I \left|\nabla_s^6 \vec{\kappa}\right|^2 ds + \frac{1}{2} \int_I \left|\nabla_s^4 \vec{\kappa}\right|^2 ds \\
&\quad = -\left[\left\langle \nabla_s^4 \vec{\kappa}, \nabla_s^7 \vec{\kappa} \right\rangle\right]_0^1 + \left[\left\langle \nabla_s^5 \vec{\kappa}, \nabla_s^6 \vec{\kappa} \right\rangle\right]_0^1 \\
&\qquad + \int_I \left\langle Y, \nabla_s^4 \vec{\kappa} \right\rangle ds - \frac{1}{2} \int_I \left|\nabla_s^4 \vec{\kappa}\right|^2 \left\langle \vec{\kappa}, \vec{V} \right\rangle ds + \frac{1}{2} \int_I \left|\nabla_s^4 \vec{\kappa}\right|^2 ds,
\end{aligned} \tag{5.8}$$

where $Y = (\nabla_t + \nabla_s^4)\nabla_s^4 \vec{\kappa}$.

Critical terms for interpolation techniques are given by some of the boundary terms. First of all we treat these ciritical terms. Using Lemma 2.6 we obtain

$$\begin{aligned}
&-\Big[\Big\langle \nabla_s^4\vec{\kappa}, \nabla_s^7\vec{\kappa}\Big\rangle\Big]_0^1 + \Big[\Big\langle \nabla_s^5\vec{\kappa}, \nabla_s^6\vec{\kappa}\Big\rangle\Big]_0^1 \\
&= \lambda' \Big[\Big\langle \nabla_s^4\vec{\kappa}, \nabla_s\vec{\kappa}\Big\rangle - \Big\langle \nabla_s^5\vec{\kappa}, \vec{\kappa}\Big\rangle\Big]_0^1 \\
&\quad + \left[\sum_{\substack{[[a,b]]\le[[9,4]]\\ c\le 5,\ b \text{ even}}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{2} Q_i(\lambda_0) \sum_{\substack{[[a,b]]\le[[11-2i,2]]\\ c\le 5,\ b \text{ even}}} P_b^{a,c}(\vec{\kappa})\rangle\right]_0^1 =: I + II\ ,
\end{aligned} \tag{5.9}$$

with $I := \lambda'[\langle \nabla_s^4\vec{\kappa}, \nabla_s\vec{\kappa}\rangle - \langle \nabla_s^5\vec{\kappa}, \vec{\kappa}\rangle]_0^1$. The term II turns out not to be critical. Indeed, using Lemma 3.6 (with $\ell = 4$) as well as the bound for λ obtained in (5.7) we obtain

$$|II| \le \varepsilon \int_I \left|\nabla_s^6\vec{\kappa}\right|^2 \, ds + C_\varepsilon\ . \tag{5.10}$$

The term I in (5.9) (*i.e.*, the term multiplying λ') has to be treated differently. Since $|\lambda'| \sim \|\nabla_s^6\vec{\kappa}\|_{L^2}$ by Lemma 4.5 (with $\ell = m = 1$) if we use again Lemma 3.6 we would get a $\|\nabla_s^6\vec{\kappa}\|_{L^2}^2$. We gain here an ε using Lemma 3.8 instead. We proceed as follows. By Lemma 4.5 (with $\ell = m = 1$), (5.7) and Lemma 3.9 we find

$$\begin{aligned}
|I| &\le C\left(1 + \|\nabla_s^6\vec{\kappa}\|_{L^2}\right)\left(\left\|\nabla_s^5\vec{\kappa}\right\|_{C^0} + \left\|\nabla_s^4\vec{\kappa}\right\|_{C^0}\right) \\
&\le C\left(1 + \left\|\nabla_s^6\vec{\kappa}\right\|_{L^2}\right)\left(\varepsilon\left\|\nabla_s^6\vec{\kappa}\right\|_{L^2} + \frac{C'}{\varepsilon}\left(\left\|\nabla_s^5\vec{\kappa}\right\|_{L^2} + \left\|\nabla_s^4\vec{\kappa}\right\|_{L^2}\right)\right) \\
&\le C\left(\varepsilon\left\|\nabla_s^6\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon\right),
\end{aligned}$$

since by (3.3)

$$\left\|\nabla_s^5\vec{\kappa}\right\|_{L^2}, \left\|\nabla_s^4\vec{\kappa}\right\|_{L^2} \le \epsilon^2 \left\|\nabla_s^6\vec{\kappa}\right\|_{L^2} + C_\epsilon. \tag{5.11}$$

Thus, after renaming ε, we can state

$$|I| \le \varepsilon \left\|\nabla_s^6\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon. \tag{5.12}$$

For the other terms on the right hand side of (5.8) from (1.2), (2.14) (with $\ell = 4$) we obtain using the bound on λ in (5.7) and Lemma 3.5 (with $\ell = 4$)

$$\left|\int_I \left\langle Y, \nabla_s^4 \vec{\kappa}\right\rangle ds - \frac{1}{2}\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 \langle \vec{\kappa}, \vec{V}\rangle ds + \frac{1}{2}\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds\right|$$
$$\leq \int_I \sum_{\substack{[[a,b]]\leq[[10,4]]\\ c\leq 6}} \left|P_b^{a,c}(\vec{\kappa})\right| ds + |\lambda| \int_I \sum_{\substack{[[a,b]]\leq[[10,2]]\\ c\leq 6}} \left|P_b^{a,c}(\vec{\kappa})\right| ds$$
$$\leq \varepsilon \left\|\nabla_s^6\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon\,.$$

From (5.9), (5.10), (5.12) and the estimate above we find

$$\frac{d}{dt}\frac{1}{2}\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds + \frac{1}{2}\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds + \int_I \left|\nabla_s^6\vec{\kappa}\right|^2 ds \leq 3\varepsilon \int_I \left|\nabla_s^6\vec{\kappa}\right|^2 ds + C_\varepsilon\,,$$

from which choosing ε appropriately it follows that

$$\frac{d}{dt}\frac{1}{2}\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds + \frac{1}{2}\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds \leq C\,.$$

Then Gronwall's Lemma gives

$$\int_I \left|\nabla_s^4\vec{\kappa}\right|^2 ds \leq C \quad \text{for all} \quad t \in (0, T)\,, \tag{5.13}$$

and then from Corollary 3.2 and Lemma 3.8 we also get

$$\|\vec{\kappa}\|_{4,2},\ \left\|\nabla_s^i\vec{\kappa}\right\|_{C^0(\bar{I})} \leq C \quad \text{for } i = 0, 1, 2, 3 \text{ and for all } t \in (0, T). \tag{5.14}$$

From these estimates we may now improve the estimate on λ'. Indeed, Lemma 4.4 (with $\ell = 1$), the uniform bound on λ and the bound from below on the elastic energy imply

$$\delta^2|\lambda'(t)| \leq \left|\int_I \left\langle \vec{\kappa}, \nabla_s^6\vec{\kappa}\right\rangle ds\right| + C\int_I \sum_{\substack{[[a,b]]\leq[[6,2]]\\ c\leq 4}} \left|P_b^{a,c}(\vec{\kappa})\right| ds\,.$$

By Lemma 3.5 (with $\ell = 2$) and (5.14) the second integral is bounded by a constant. Integrating by parts in the first integral and using (5.14), Lemma 3.9 and (5.11) we get for any $\varepsilon \in (0,1)$

$$\begin{aligned}\delta^2|\lambda'(t)| &\leq \left|\left[\left\langle \vec{\kappa}, \nabla_s^5\vec{\kappa}\right\rangle\right]_0^1\right| + \left|\int_I \left\langle \nabla_s\vec{\kappa}, \nabla_s^5\vec{\kappa}\right\rangle ds\right| + C \\ &\leq C \left\|\nabla_s^5\vec{\kappa}\right\|_{C^0} + C \leq C\left(\varepsilon\left\|\nabla_s^6\vec{\kappa}\right\|_{L^2} + \frac{C'}{\varepsilon}\left\|\nabla_s^5\vec{\kappa}\right\|_{L^2}\right) + C \\ &\leq C\left(\varepsilon\left\|\nabla_s^6\vec{\kappa}\right\|_{L^2} + C_\varepsilon\right) + C \\ &\leq C\varepsilon\left\|\nabla_s^{10}\vec{\kappa}\right\|_{L^2}^{\frac{3}{5}} + C_\varepsilon \text{ for all } t\in(0,T),\end{aligned}$$

with Lemma 3.3 and Corollary 3.2 in the last estimate.

Third Step: We prove by induction that:
For any $m \in \mathbb{N}$ there exist a constant $C = C(m, E(f_0), f_0, L_0, f_-, f_+, n)$ such that

$$\left\|\nabla_s^{4m}\vec{\kappa}\right\|_{L^2}, \left\|\nabla_s^{i}\vec{\kappa}\right\|_{C^0(\bar{I})} \leq C, \quad \text{and} \quad \left|\lambda^{(j)}\right| \leq C,$$

or $i \in \{0, .., 4m-1\}$, $j \in \{0, .., m-1\}$ and for all $t \in (0,T)$. Moreover, for any $\varepsilon \in (0,1)$ there exists a constant $C_\varepsilon = C(\varepsilon, m, E(f_0), f_0, L_0, f_-, f_+, n)$ such that

$$\left|\lambda^{(m)}\right| \leq \varepsilon\left\|\nabla_s^{4m+6}\vec{\kappa}\right\|_{L^2}^{\frac{4m+2}{4m+6}} + C_\varepsilon,$$

for all $t \in (0,T)$.

The initial step of the induction is proven in the first and second step. Let us assume that the claim is true up to $m-1$ (for $m \geq 2$). Taking $\phi = \nabla_s^{4m}\vec{\kappa}$ in Lemma 2.3 we obtain

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_I\left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds + \int_I\left|\nabla_s^{4m+2}\vec{\kappa}\right|^2 ds + \frac{1}{2}\int_I\left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds \\ &= -\left[\left\langle\nabla_s^{4m}\vec{\kappa}, \nabla_s^{4m+3}\vec{\kappa}\right\rangle\right]_0^1 + \left[\left\langle\nabla_s^{4m+1}\vec{\kappa}, \nabla_s^{4m+2}\vec{\kappa}\right\rangle\right]_0^1 \\ &\quad + \int_I\left\langle Y, \nabla_s^{4m}\vec{\kappa}\right\rangle ds - \frac{1}{2}\int_I\left|\nabla_s^{4m}\vec{\kappa}\right|^2\left\langle\vec{\kappa}, \vec{V}\right\rangle ds + \frac{1}{2}\int_I\left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds,\end{aligned} \tag{5.15}$$

where $Y = (\nabla_t + \nabla_s^4)\nabla_s^{4m}\vec{\kappa}$. Using the formula for $Y = (\nabla_t + \nabla_s^4)\nabla_s^{4m}\vec{\kappa}$ given in (2.14) and by a direct computation one sees that the last three terms in (5.15) can be written as

$$\begin{aligned}&\int_I\left\langle Y, \nabla_s^{4m}\vec{\kappa}\right\rangle ds - \frac{1}{2}\int_I\left|\nabla_s^{4m}\vec{\kappa}\right|^2\left\langle\vec{\kappa}, \vec{V}\right\rangle ds + \frac{1}{2}\int_I\left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds \\ &= \int_I \sum_{\substack{[[a,b]]\leq[[8m+2,4]] \\ c\leq 4m+2,\ b \text{ even}}} P_b^{a,c}(\vec{\kappa}) + \lambda \sum_{\substack{[[a,b]]\leq[[8m+2,2]] \\ c\leq 4m+2,\ b \text{ even}}} P_b^{a,c}(\vec{\kappa})\, ds.\end{aligned}$$

Since λ is uniformly bounded (as proven in the first step), by the interpolation inequality (3.5) (with $\ell = 4m$) and the bounds on $\|\vec{\kappa}\|_{L^2}$ we then find for any $\varepsilon \in (0,1)$

$$\begin{aligned}&\left|\int_I \left\langle Y, \nabla_s^{4m}\vec{\kappa}\right\rangle ds - \frac{1}{2}\int_I \left|\nabla_s^{4m}\vec{\kappa}\right|^2 \left\langle \vec{\kappa}, \vec{V}\right\rangle ds + \frac{1}{2}\int_I \left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds\right| \\ &\leq \varepsilon \left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon .\end{aligned} \tag{5.16}$$

Using Lemma 2.6 the boundary terms in (5.15) can be written as

$$\begin{aligned}&-\left[\left\langle \nabla_s^{4m}\vec{\kappa}, \nabla_s^{4m+3}\vec{\kappa}\right\rangle\right]_0^1 + \left[\left\langle \nabla_s^{4m+1}\vec{\kappa}, \nabla_s^{4m+2}\vec{\kappa}\right\rangle\right]_0^1 \\ &= (-1)^{m+1}\lambda^{(m)}\left(\left[\left\langle \nabla_s^{4m}\vec{\kappa}, \nabla_s\vec{\kappa}\right\rangle - \left\langle \nabla_s^{4m+1}\vec{\kappa}, \vec{\kappa}\right\rangle\right]_0^1\right) \\ &+ \left[\sum_{\substack{[[a,b]]\leq[[8m+1,4]] \\ c\leq 4m+1,\ b \text{ even}}} P_b^{a,c}(\vec{\kappa}) + \sum_{i=1}^{2m} Q_i(\lambda_{m-1}) \sum_{\substack{[[a,b]]\leq[[8m-2i+3,2]] \\ c\leq 4m+1,\ b \text{ even}}} P_b^{a,c}(\vec{\kappa})\right]_0^1 =: I + II ,\end{aligned} \tag{5.17}$$

where $I := (-1)^{m+1}\lambda^{(m)}\left[\langle \nabla_s^{4m}\vec{\kappa}, \nabla_s\vec{\kappa}\rangle - \langle \nabla_s^{4m+1}\vec{\kappa}, \vec{\kappa}\rangle\right]_0^1$. By choosing $\ell = m$ in Lemma 4.5, we derive $|\lambda^{(m)}| \leq C(\|\nabla_s^{4m+2}\vec{\kappa}\|_{L^2} + 1)$. Thus, together with (5.7), Lemma 3.9 and (3.5) (with $\ell = 4m$), we obtain for $\varepsilon \in (0,1)$

$$\begin{aligned}|I| &\leq C\left(1 + \left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}\right)\left(\left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{C^0} + \left\|\nabla_s^{4m}\vec{\kappa}\right\|_{C^0}\right) \\ &\leq C\left(1 + \left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}\right)\left(\varepsilon\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}\right. \\ &\qquad\qquad \left. + \frac{C'}{\varepsilon}\left(\left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{L^2} + C\left\|\nabla_s^{4m}\vec{\kappa}\right\|_{L^2}\right)\right) \\ &\leq C\left(\varepsilon\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon\right) ,\end{aligned} \tag{5.18}$$

where we have used

$$\left\|\nabla_s^{4m+1}\vec{\kappa}]\right\|_{L^2}, \left\|\nabla_s^{4m}\vec{\kappa}\right\|_{L^2} \leq \varepsilon^2\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2} + C_\varepsilon \tag{5.19}$$

(which follows from (3.3) with $\ell = 4m$). Thus we can state

$$|I| \leq \varepsilon\left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2}^2 + C_\varepsilon. \tag{5.20}$$

We now estimate II in (5.17). By (3.7) (with $\ell = 4m$), (4.8) (with $m-1$ instead of m), the induction hypothesis, (3.6) and Corollary 3.2 we find

$$\begin{aligned} |II| &\leq \frac{1}{4}\varepsilon \left\| \nabla_s^{4m+2}\vec{\kappa} \right\|_{L^2}^2 + C_\varepsilon + C \sum_{\substack{[[a,b]]\leq[[8m+1,2]]\\ c\leq 4m+1,\ b\ \text{even}}} \left| P_b^{a,c}(\vec{\kappa})|_0^1 \right| \\ &\quad + C\left|\lambda^{(m-1)}\right| \sum_{\substack{[[a,b]]\leq[[4m+5,2]]\\ c\leq 4m+1,\ b\ \text{even}}} \left| P_b^{a,c}(\vec{\kappa})|_0^1 \right| \\ &\leq \frac{1}{2}\varepsilon \left\| \nabla_s^{4m+2}\vec{\kappa} \right\|_{L^2}^2 + C_\varepsilon + C\left(\tilde{\varepsilon} \left\| \nabla_s^{4m+2}\vec{\kappa} \right\|_{L^2}^{\frac{4m-2}{4m+2}} + C_{\tilde{\varepsilon}} \right) \left(1 + \|\vec{\kappa}\|_{4m+2}^{\frac{4m+6}{4m+2}} \right) \\ &\leq \varepsilon \left\| \nabla_s^{4m+2}\vec{\kappa} \right\|_{L^2}^2 + C_\varepsilon\,, \end{aligned}$$

choosing an appropriate $\tilde{\varepsilon}$.

From (5.15), (5.16), (5.17), (5.20) and the estimate above we obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_I \left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds + \frac{1}{2}\int_I \left|\nabla_s^{4m}\vec{\kappa}\right|^2 ds + \int_I \left|\nabla_s^{4m+2}\vec{\kappa}\right|^2 ds \\ &\quad \leq 3\varepsilon \int_I \left|\nabla_s^{4m+2}\vec{\kappa}\right|^2 ds + C_\varepsilon\,. \end{aligned}$$

Thus, by choosing a sufficiently small $\varepsilon > 0$ and applying again Gronwall's lemma, we obtain the uniform bound

$$\left\| \nabla_s^{4m}\vec{\kappa} \right\|_{L^2} \leq C(m, E(f_0), f_0, L_0, f_-, f_+, n) \ \text{ for all } \ t \in (0,T)\,, \tag{5.21}$$

and then from Corollary 3.2, Lemma 3.8 and Lemma 4.5 we also get

$$\begin{aligned} &\|\vec{\kappa}\|_{4m,2},\ \left\|\nabla_s^i\vec{\kappa}\right\|_{C^0(\bar{I})},\ \left|\lambda^{(j)}\right| \\ &\quad \leq C(m, E(f_0), f_0, L_0, f_-, f_+, n) \ \text{ for all } \ t \in (0,T)\,, \end{aligned} \tag{5.22}$$

for $i = 0, 1, \dots, 4m-1$ and $j = 0, \dots, m-1$.

It remains to prove the estimate on $\lambda^{(m)}$. Lemma 4.4 and the bound from below on the elastic energy imply

$$\delta^2\left|\lambda^{(m)}\right| \leq \left| \int_I \langle\vec{\kappa}, \nabla_s^{4m+2}\vec{\kappa}\rangle\, ds \right| + \sum_{i=0}^{2m} |Q_i(\lambda_{m-1})| \int_I \sum_{\substack{[[a,b]]\leq[[4m+2-2i,2]]\\ c\leq 4m}} \left| P_b^{a,c}(\vec{\kappa}) \right| ds.$$

The estimates in (5.22) or Lemma 3.5 (with $\ell = 4m-2$) give that the second term is bounded by C. Integrating by parts in the first integral on the right hand side, using (5.22), Lemma 3.9, (5.19), Lemma 3.3 (with $k = 4m + 6$) and Corollary 3.2 we get for $\varepsilon \in (0, 1)$

$$\begin{aligned}\delta^2 \left|\lambda^{(m)}\right| &\leq \left|\left[\left\langle \vec{\kappa}, \nabla_s^{4m+1}\vec{\kappa}\right\rangle\right]_0^1\right| + \left|\int_I \left\langle \nabla_s\vec{\kappa}, \nabla_s^{4m+1}\vec{\kappa}\right\rangle\, ds\right| + C \\ &\leq C \left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{C^0} + C \leq C\left(\varepsilon \left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2} + \frac{C'}{\varepsilon}\left\|\nabla_s^{4m+1}\vec{\kappa}\right\|_{L^2}\right) + C \\ &\leq C\varepsilon \left\|\nabla_s^{4m+2}\vec{\kappa}\right\|_{L^2} + C_\varepsilon \leq C\varepsilon \left\|\nabla_s^{4m+6}\vec{\kappa}\right\|_{L^2}^{\frac{4m+2}{4m+6}} + C_\varepsilon \text{ for all } t \in (0, T)\,.\end{aligned}$$

Fourth Step: ***Long-time existence*****.** First of all we show

$$\|\partial_x^m \vec{\kappa}\|_{C^0(\bar{I})} \leq C(m, E(f_0), f_0, L_0, f_+, f_-, n, T) \text{ for all } t \in (0, T).$$

From the previous step, Lemma 2.7, and the fact that the length remains constant along the flow we can state that

$$\|\partial_s^m \vec{\kappa}\|_{C^0(\bar{I})}\,, \|\partial_s^m \vec{\kappa}\|_{L^2}\,, \left|\lambda^{(m)}\right| \leq C(m, E(f_0), f_0, L_0, f_+, f_-, n), \tag{5.23}$$

for any $m \in \mathbb{N}$. From now on the proof follows most of the arguments depicted in [3, Section 5 (Step seventh onwards)]. For the sake of completeness we sketch here again the main ideas. In the following let $\gamma := |\partial_x f|$. Then, $\partial_x = \gamma\,\partial_s$. By induction it can be proven that for any function $h : \bar{I} \to \mathbb{R}$ or vector field $h : \bar{I} \to \mathbb{R}^n$, and for any $m \in \mathbb{N}$

$$\partial_x^m h = \gamma^m \partial_s^m h + \sum_{j=1}^{m-1} P_{m-1}\left(\gamma, \ldots, \partial_x^{m-j}\gamma\right) \partial_s^j h\,, \tag{5.24}$$

with P_{m-1} a polynomial of degree at most $m - 1$. A bound on $\|\partial_x^\ell \vec{\kappa}\|_{C^0(\bar{I})}$ follows from (5.24) taking $h = \vec{\kappa}$ and from bounds on $\|\partial_s^\ell \vec{\kappa}\|_{C^0(\bar{I})}$ (see (5.23)) and on $\|\partial_x^\ell \gamma\|_{C^0(\bar{I})}$. Thus it remains to estimate $\|\partial_x^\ell \gamma\|_{C^0(\bar{I})}$ for $\ell \in \mathbb{N}_0$. We start by showing that $\gamma = |\partial_x f|$ is uniformly bounded from above and below. The function γ satisfies the following parabolic equation

$$\partial_t \gamma = \left\langle \tau, \partial_x \vec{V}\right\rangle = -\left\langle \vec{\kappa}, \vec{V}\right\rangle \gamma\,, \tag{5.25}$$

with $\vec{V} = \partial_t f$ as in (1.2). By regularity of the initial datum we have that $1/c_0 \leq \gamma(0) \leq c_0$ for some positive c_0. From the estimates given in (5.23) it follows that the coefficient $\|\langle \vec{\kappa}, \vec{V}\rangle\|_{C^0(\bar{I})}$ in (5.25) is uniformly bounded and hence we infer that $1/C \leq \gamma \leq C$, with C having the same dependencies as the constant in (5.23) as well as T. In order to prove bounds on $\partial_x^m \gamma$ we proceed by induction. Let us assume that we have shown

$$\left\|\partial_x^j \gamma\right\|_{C^0(\bar{I})} \leq C(m, E(f_0), L_0, f_+, f_-, f_0, n, T) \text{ for } 0 \leq j \leq m \tag{5.26}$$

and $m \in \mathbb{N}_0$. Choosing $h = \langle \vec{\kappa}, \vec{V} \rangle$ in (5.24), the induction assumption and (5.23) yields that

$$\left\| \partial_x^i \left\langle \vec{\kappa}, \vec{V} \right\rangle \right\|_{C^0(\bar{I})} \leq C(m, E(f_0), L_0, f_+, f_-, f_0, n, T) \tag{5.27}$$

for all $0 \leq i \leq m+1$. Differentiating (5.25) $(m+1)$-times with respect to x, we find

$$\partial_t \partial_x^{m+1} \gamma = -\left\langle \vec{\kappa}, \vec{V} \right\rangle \partial_x^{m+1} \gamma - \sum_{\substack{i+j=m+1 \\ j \leq m}} c(i,j,m) \partial_x^i \left(\left\langle \vec{\kappa}, \vec{V} \right\rangle \right) \partial_x^j \gamma \,,$$

for some coefficients $c(i, j, m)$. Together with (5.26), (5.27) we derive

$$\left| \sum_{\substack{i+j=m+1 \\ j \leq m}} c(i,j,m) \partial_x^i \left(\left\langle \vec{\kappa}, \vec{V} \right\rangle \right) \partial_x^j \gamma \right| \leq C\big(m, E(f_0), L_0, f_+, f_-, f_0, n, T\big) \,,$$

which implies

$$\left\| \partial_x^{m+1} \gamma \right\|_{C^0(\bar{I})} \leq C\big(m, E(f_0), L_0, f_+, f_-, f_0, n, T\big) \,.$$

Next note that (5.23) implies $\|\partial_s^m \vec{V}\|_{C^0(\bar{I})} \leq C(m, E(f_0), L_0, f_+, f_-, f_0, n)$, which in turns gives uniform estimates for $\|\partial_x^m \vec{V}\|_{C^0(\bar{I})}$ in view of (5.24) and the bounds for the length elements and its derivatives.

Finally, the uniform C^0-bounds on the curvature $\vec{\kappa}$, the velocity $\vec{V}$, γ, and all their derivatives, allow for a smooth extension of f up to $t = T$ and then by the short-time existence result even beyond. In view of this contradiction, the flow must exist globally.

Part 2: ***Subconvergence.*** This part of the proof is standard. From reparametrizing f by arc-length (in order to have a control on the parametrization), the fact that the length is fixed along the flow, and the uniform bounds

$$\|\partial_s^m \vec{\kappa}\|_{C^0\left([0,L_0]\right)} \leq C(m, E(f_0), f_0, L_0, f_+, f_-, n\big)$$

which follow directly from (5.23), it follows that there exist sequences of times $t_i \to \infty$ such that the curves $f(t_i, \cdot)$ converges smoothly to a smooth curve f_∞.

It remains to show that f_∞ is a critical point for the elastic energy, that is, a solution to $\vec{V} = 0$. We prove this by considering the function $u(t) := \|\vec{V}\|^2_{L^2(I)}(t)$ and showing that $\lim_{t \to \infty} u(t) = 0$. First observe that

$$\frac{d}{dt} u(t) = -\int_I \left| \vec{V} \right|^2 \left\langle \vec{\kappa}, \vec{V} \right\rangle ds + \int_I \left\langle \vec{V}, \nabla_t \vec{V} \right\rangle ds.$$

Since $\nabla_t \vec{V} = \nabla_t^2 f$ we infer from (5.2) and the bounds (5.22)

$$\left|\frac{d}{dt}u(t)\right| \leq C\big(m, E(f_0), f_0, L_0, f_+, f_-, n\big).$$

On the other hand from (2.10) it follows that $u \in L^1((0,\infty))$ and hence necessarily $u(t) \to 0$ for $t \to \infty$. □

Remark 5.1. The condition $L_0 > |f_+ - f_-|$ is automatically satisfied if $\tau_- \neq \tau_+$.

Remark 5.2. As a by-product of Theorem 1.1 we get the existence of elastic curves with clamped end points and given length. For related existence and qualitative results for planar open elasticae see for instance in [4,9,12], and [14].

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