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Dispersive estimates with loss of derivatives via the heat semigroup and the wave operator

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Abstract. In this paper our aim is to give a general (possibly compact or non-compact) analog of the Strichartz inequalities with loss of derivatives, obtained by Burq, Gérard and Tzvetkov [21] and Staffilani and Tataru [56]. Moreover we present a new approach, relying only on the heat semigroup, in order to understand the analytic connection between the heat semigroup and the unitary Schrödinger group (both related to a same self-adjoint operator). One of the novelties is to forget the endpoint L^1-L^∞ dispersive estimate and to look for a weaker H^1-BMO estimate (Hardy and BMO spaces both adapted to the heat semigroup). This new point of view allows us to give a general framework (infinite metric spaces, Riemannian manifolds with rough metric, manifolds with boundary, ...) where Strichartz inequalities with loss of derivatives can be reduced to microlocalized L^2-L^2 dispersive properties. We also use the link between the wave propagator and the unitary Schrödinger group to prove how short-time dispersion for waves implies dispersion for the Schrödinger group.

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1. Introduction

A powerful tool to study nonlinear Schrödinger equations is the family of so-called Strichartz estimates. Those estimates allow to control the size of solutions to a linear problem in terms of the size of the initial data. The "size" notion is usually given by a suitable functional space $L^p_t L^q_x$. Such inequalities were first introduced by Strichartz in [58] for Schrödinger waves on Euclidean space. They were then extended by Ginibre and Velo in [38] (and the endpoint is due to Keel and Tao in [47]) for the propagator operator associated with the linear Schrödinger equation

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in $\mathbb{R}^d$. So for an initial data u_0, we are interested in controlling $u(t,.) = e^{it\Delta}u_0$ which is the solution of the linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

It is well-known that the unitary group $e^{it\Delta}$ satisfies the following inequality:

$$\left\| e^{it\Delta}u_0 \right\|_{L^pL^q([-T,T]\times\mathbb{R}^d)} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}$$

for every pair (p,q) of admissible exponents which means: $2 \leq p,q \leq \infty$, $(p,q,d) \neq (2,\infty,2)$, and

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{1.1}$$

The Strichartz estimates can be deduced via a TT^* argument from the dispersive estimates

$$\left\| e^{it\Delta}u_0 \right\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^d)}. \tag{1.2}$$

If $\sup_{T>0} C_T < +\infty$, we will say that a global-in-time Strichartz estimate holds. Such a global-in-time estimate has been proved by Strichartz for the flat Laplacian on $\mathbb{R}^d$ while the local-in-time estimate is known in several geometric situation where the manifold is nontrapping (asymptotically Euclidean, conic, or hyperbolic, Heisenberg group); see [8, 18, 19, 41, 56] or for an equation with variable coefficients [52, 60]. The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.

The situation for compact manifolds presents a new difficulty, since considering the constant initial data $u_0 = 1$ yields a contradiction in (1.2) for large time.

Burq, Gérard and Tzvetkov [21] and Staffilani and Tataru [56] proved that Strichartz estimates hold on compact manifolds for finite time if one considers regular data $u_0 \in W^{1/p,2}$. Those are called "with a loss of derivatives":

$$\left\| e^{it\Delta}u_0 \right\|_{L^pL^q} \lesssim \|u_0\|_{W^{1/p,2}}.$$

An interesting problem is to determine for specific situations, which loss of derivatives is optimal (for example the work by Bourgain [20] on the flat torus and [59] by Takaoka and Tzvetkov).

Numerous recent works aim also to obtain such Strichartz estimates with a loss of derivatives in various situations, for example corresponding to a Laplacian operator on a smooth domain with boundary condition (Dirichlet or Neumann); see the works of Anton [3], Blair-Smith-Sogge [17] and Blair-Ford-Herr-Marzuola [16]. All these works are built on the approach for compact manifolds of [21]. Concerning noncompact manifolds, Strichartz estimates with the same loss of derivatives have been obtained in [22] by Burq-Gérard-Tzvetkov for the complement of a smooth and bounded domain in the Euclidean space. Let us precise that for exponents satisfying (1.1), the two approaches [21] (for the compact situation) and [22] (for the non-compact situation) are completely different, although they give exactly the same loss of derivatives. Indeed in [21] the loss of derivatives is due to the use of only the semi-classical dispersive inequality and in [22] the loss of derivatives is due to the use of Sobolev embeddings together with the local smoothing near the boundary. Let us point out that in [22], combining with a smoothing effect allows the authors to get around the loss of regularity up to restrict the range of the available Lebesgue exponents.

The case of unbounded manifolds (with boundary) with one trapped orbit was considered by Christianson in [30] where a larger loss of derivatives of $1/p+\varepsilon$ is obtained. There the author allows to perturb the Laplacian by a smooth potential.

We remark that, by Sobolev embedding, the loss of $2/p$ derivatives is straightforward. Indeed $W^{\frac{2}{p},2} \hookrightarrow L^q$ since $d(\frac{1}{2}-\frac{1}{q})=\frac{2}{p}$ so that

$$\left\| e^{it\Delta}u_0 \right\|_{L^q} \lesssim \left\| e^{it\Delta}u_0 \right\|_{W^{\frac{2}{p},2}} \leq \|u_0\|_{W^{\frac{2}{p},2}} \tag{1.3}$$

and taking the $L^p([-T,T])$ norm yields

$$\left\| e^{it\Delta}u_0 \right\|_{L^p([-T,T],L^q)} \leq C_T \|u_0\|_{W^{\frac{2}{p},2}}.$$

Therefore Strichartz estimates with loss of derivatives are interesting for a loss smaller than $2/p$.

The purpose of this article is multiple:

- To present a general result with loss of derivatives for a (possibly compact or noncompact) general setting (involving metric space with a self-adjoint operator);
- To try to understand the link between the heat semigroup and the unitary Schrödinger group, through the use of corresponding Hardy and BMO spaces. Such spaces allow us to get around the pointwise dispersive estimates and only to consider L^2-L^2 microlocalized estimates (in space and in frequency);
- To connect (short time) dispersive properties for Schrödinger group with dispersion for the wave propagator.

Let us set the general framework of our study. Let (X,d,μ) be a metric measured space of homogeneous type. That is d is a metric on X and μ a nonnegative σ-finite Borel measure satisfying the doubling property:

$$\forall x\in X, \forall r>0,\ \mu(B(x,2r)) \lesssim \mu(B(x,r)),$$

where $B(x,r)$ denote the open ball with center $x\in X$ and radius $r>0$. As a consequence, there exists a homogeneous dimension $d>0$, such that

$$\forall x \in X\,,\ \forall r>0\,,\ \forall t\geq 1\,,\ \mu(B(x,tr)) \lesssim t^d \mu(B(x,r)). \tag{1.4}$$

Thus we aim our result to apply in numerous cases of metric spaces such as open subsets of $\mathbb{R}^d$, smooth d-manifolds, some fractal sets, Lie groups, Heisenberg group, ...

Keeping in mind the canonical example of the Laplacian operator in $\mathbb{R}^d$: $\Delta = \sum_{1\leq j\leq d} \partial_j^2$, we will be more general in the following sense: we consider a nonnegative, self-adjoint operator H on $L^2 = L^2(X,\mu)$ densely defined, which means that its domain

$$\mathcal{D}(H) := \left\{ f \in L^2,\ Hf \in L^2 \right\}$$

is supposed to be dense in L^2. It is known that $-H$ is the generator of a L^2-holomorphic semigroup $(e^{-tH})_{t\geq 0}$ (see Definition 2.1 and [31]) and we assume that it satisfies L^2 Davies-Gaffney estimates: for every $t > 0$ and every subsets $E, F \subset X$:

$$\left\| e^{-tH} \right\|_{L^2(E)\to L^2(F)} \lesssim e^{-\frac{d(E,F)^2}{4t}} \tag{DG}$$

(with the restriction to $t \lesssim \operatorname{diam}(X)$ if X is bounded). Without losing generality (up to consider λH for some positive real $\lambda > 0$), we assumed that H satisfies the previous normalized estimates, which are equivalent to a finite speed propagation property of the associated wave propagator at speed 1 (see later (1.6)).

We will assume also that the heat semigroup $(e^{-tH})_{t\geq 0}$ satisfies the typical upper estimates (for a second order operator): for every $t > 0$ the operator e^{-tH} admits a kernel p_t with

$$0 \leq p_t(x,x) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))}, \quad \forall\, t > 0, \text{ a.e. } x \in X. \tag{DUE}$$

It is well-known that such on-diagonal pointwise estimates self-improve into the full pointwise Gaussian estimates (see [39, Theorem 1.1] or [28, Section 4.2], *e.g.*):

$$0 \leq p_t(x,y) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\left(-c\frac{d(x,y)^2}{t} \right), \quad \forall\, t > 0, \text{ a.e. } x, y \in X. \tag{UE}$$

Before we carry on, let us give some examples to point out that (DUE) is a quite common estimate:

- It is well known that on a Riemannian manifold [39, Theorem 1.1] or for the Laplacian on a subset with boundary conditions [40], under very weak conditions the heat kernel satisfies (DUE) and so (UE). It is also the case for the semigroup generated by a self-adjoint elliptic operator of divergence form $H = -\operatorname{div}(A\nabla)$ on the Euclidean space with a bounded and real valued elliptic matrix A (see [4, Theorem 4]);
- If $(X_1, \dots, X_n)$ is a family of vector fields satisfying Hörmander condition and if $H := -\sum_{i=1}^n X_i^2$, then, in the situation of Lie groups or Riemannian manifolds with bounded geometry, the heat semigroup satisfies Gaussian upper-bounds (UE) (see [53, Theorem 5.14] and [29, Section 3, Appendix 1]);

- When one considers an infinite volume Euclidean surface with conic singularities with H equals to its Laplacian, then it is proved in [16, Section 4] that the heat kernel satisfies Gaussian pointwise estimates (UE).

Let us now emphasize why we put such importance on those estimates and on the heat semigroup. The considered operator H is self-adjoint and so admits a C^∞-functional calculus, which allows us to control $\|\phi(H)\|_{L^p \to L^p}$ for some regular functions ϕ. Such estimates can be obtained as explained in the Appendix of [44] as a consequence of pointwise Gaussian estimates (UE) on the heat kernel. Moreover, we aim to use some extrapolation techniques (to go from localized $L^2 - L^2$ dispersive estimates to $L^p - L^{p'}$ estimates) which require local information, as off-diagonal estimates of some functional operators. Such local information could be transferred from those on the heat semigroup to some operators coming from a C^∞-calculus, (see [48] for example). However, it requires to deal with the whole class of C^∞ functions (compactly supported) with suitable norms…For an easier readability and a more intrinsic method, we prefer to work only with the semigroup and its time derivatives. We refer the reader to Remark 2.6 for the equivalence between the two points of view.

In the end, we would like to point out that we assume the operator H to be self-adjoint to guarantee some useful properties. However, the approach that we develop here could be extended to a non self-adjoint operator H, as soon as we can define the Schrödinger group $(e^{itH})_{t\in\mathbb{R}}$ as well as the wave propagator with a finite speed propagation property. In such a setting, a C^∞-calculus is not available and it will be important to only use a sectorial functional calculus as we do here.

Motivated by this program, we decide to only work with the holomorphic functional calculus associated with the operator H and more precisely, we will see that all of our study relies on a (sectorial) functional calculus only involving the heat semigroup and its time derivatives (and could also be written in terms of a C^∞-calculus, see Remark 2.6).

Moreover, we still keep in mind the following very general/interesting question: what assumptions on the heat semigroup $(e^{-tH})_{t\geq 0}$ could imply dispersive estimates and Strichartz estimates (possibly with a loss of derivatives) for the unitary Schrödinger group $(e^{itH})_{t\in\mathbb{R}}$? Such a question is natural, since the application $z \mapsto e^{-zH}$ is holomorphic on $\{z \in \mathbb{C},\ \mathrm{Re}(z) > 0\}$. Dispersive informations on Schrödinger group should be connected to some specific properties of the heat kernel.

In the first part of this work, we investigate this question, allowing loss of derivatives, as in [21], proposing a new approach, related to the heat semigroup and the use of Hardy-BMO spaces associated with the semigroup. We point out that our approach gives an "unified" way to prove Strichartz estimates with loss of derivatives for the compact and noncompact manifolds.

Let us briefly explain the study of Hardy and BMO spaces associated with such a heat semigroup. The classical Hardy space H^1 (also called of Coifman-Weiss [26]) and BMO ("Bounded Mean Oscillations", introduced by John and Nirenberg in [46]) naturally arise (from a point of view of Harmonic Analysis) as a "limit/extension" of the Lebesgue spaces scale $(L^p)_{1<p<\infty}$ when $p \to \infty$ (for BMO) and $p \to 1$ (for H^1). Indeed, these two spaces have many properties, which are very useful and which fail for the critical spaces L^1 and L^∞, as Fourier characterization, duality, boundedness of some maximal functions or Calderón-Zygmund operators, equivalence between several definitions... Even if BMO is strictly containing L^∞ and H^1 is strictly contained in L^1, these spaces still satisfy a very convenient interpolation results: indeed H^1 or BMO interpolates with Lebesgue spaces L^p, $1 < p < \infty$ and the intermediate spaces are the corresponding intermediate Lebesgue spaces.

However, there are situations where these spaces H^1 and BMO are not the right substitutes to L^1 or L^∞ (for example it can be shown that the Riesz transform may be not bounded from H^1 to L^1) and there has been recently numerous works whose goal is to define Hardy and BMO spaces adapted to the context of a semigroup (see [5,6,10,13,14,34,35,43]). In [10,13], Bernicot and Zhao have described a very abstract theory for Hardy spaces (built via atomic decomposition) and interpolation results with Lebesgue spaces. The main idea is to consider the oscillation given by the semigroup instead of the classical oscillation involving the average operators. Then these adapted Hardy and BMO spaces have been extensively studied these last years (see the previous references) and it is known that an interpolation property still holds. We refer the reader to Subsection 2.4 for precise definitions and refer to the previous references for more details on this theory.

We aim to prove a $H^1 - \mathrm{BMO}$ dispersive estimate and to use an interpolation result with the trivial $L^2 - L^2$ estimate. Even if we loose the endpoint ($L^1 - L^\infty$ estimate), by interpolation we know that we will at least recover the intermediate $L^p - L^{p'}$ dispersive estimates for all $p \in (1,2]$. Moreover, such an approach has the advantage to not require any pointwise estimates on the Schrödinger propagator. We first point out that such Hardy-BMO approach have already been used in [50,61] to obtain dispersive estimates, but there the authors considered the classical spaces and not the ones associated with the heat semigroup.

Up to our knowledge the combination of dispersive estimates and $H^1 - \mathrm{BMO}$ spaces associated with the heat semigroup is a new program. Due to the novelty of this approach, we first describe it in a very general setting, by introducing the following notion: we say that a L^2-bounded operator T satisfies Property $(H_m(A))$ for some integer $m \geq 0$ and constant A (which is intended to be $|t|^{-\frac{d}{2}}$ in the applications to dispersive estimates), if for every $r > 0$ (and $r \lesssim \mathrm{diam}(X)$ if X is bounded)

$$\left\| T\psi_m(r^2H) \right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim A\mu(B_r)^{\frac{1}{2}}\mu\big(\widetilde{B_r}\big)^{\frac{1}{2}} \qquad (H_m(A))$$

where B_r and $\widetilde{B_r}$ are any two balls of radius r and $\psi_m(x) := x^m e^{-x}$.

For the first result we assume an uniform lower control of the volume, namely: it exists $\nu > 0$ such that

$$\forall x \in X,\ \forall r \lesssim \min(1, \mathrm{diam}(X)), \quad r^\nu \lesssim \mu(B(x,r)). \tag{1.5}$$

We then prove:

Theorem 1.1. *Assume* (1.4), (1.5) *with* (DUE). *Consider a self-adjoint and L^2-bounded operator T (with $\|T\|_{L^2\to L^2}\lesssim 1$), which commutes with H and satisfies Property $(H_m(A))$ for some $m\geq \frac{d}{2}$. Then T is bounded from H^1 to BMO and from L^p to $L^{p'}$ for $p\in(1,2)$ with*

$$\|T\|_{H^1\to BMO}\lesssim A \qquad \textit{and} \qquad \|T\|_{L^p\to L^{p'}}\lesssim A^{\frac{1}{p}-\frac{1}{p'}}$$

if the ambient space X is unbounded and

$$\|T\|_{H^1\to BMO}\lesssim \max(A,1) \qquad \textit{and} \qquad \|T\|_{L^p\to L^{p'}}\lesssim \max\left(A^{\frac{1}{p}-\frac{1}{p'}},B\right)$$

if the ambient space X is bounded, and where, for the last inequality, we assumed that $\|T\|_{L^p\to L^2}\lesssim B$.

This theorem allows to reduce $L^p-L^{p'}$ dispersive estimates to microlocalized L^2-L^2 estimates (localized in the frequency through the operators $\psi(r^2H)$ and in the physical space through the balls B_r and $\widetilde{B_r}$, respecting the Heisenberg uncertainty principle).

Note that Property $(H_m(A))$ is weaker than (or necessary to have) a L^1-L^∞ estimate. Indeed if T is supposed to map L^1 to L^∞ with a bound lower than A then Cauchy-Schwarz inequality leads us to Property $(H_m(A))$: for every balls B_r, $\widetilde{B_r}$ and every function $f\in L^2(B_r)$:

$$\begin{aligned}\left\|T\psi(r^2H)f\right\|_{L^2(\widetilde{B_r})} &\leq A\left\|\psi\big(r^2H\big)f\right\|_{L^1}\mu(\widetilde{B_r})^{1/2}\\ &\lesssim A\mu\big(\widetilde{B_r}\big)^{\frac{1}{2}}\|f\|_{L^1(B_r)}\\ &\leq A\mu\big(\widetilde{B_r}\big)^{\frac{1}{2}}\mu(B_r)^{\frac{1}{2}}\|f\|_{L^2(B_r)}.\end{aligned}$$

For the L^1-L^1 continuity of $\psi(r^2H)$ see Corollary 2.4.

Our goal is to obtain the dispersive estimate

$$\|T_t(H)\|_{H^1\to \mathrm{BMO}}\lesssim |t|^{-\frac{d}{2}},$$

where $T_t(H)=e^{itH}\psi_m(h^2H)$, $h>0$. The case of the full range $|t|\leq 1$ (that is t independent of h) is the most difficult. However the case $|t|\leq h^2$ is straightforward: indeed for $m=0$ ($m\neq 0$ then deduces easily) we have $T_t(H)=e^{itH}e^{-h^2H}=e^{-zH}$ with $z=h^2-it$. The key observation is that $|z|=\sqrt{h^4+t^2}\lesssim h^2=\mathrm{Re}(z)$. Thus (the complex time z lives into a sector far away from the axis of imaginary complex numbers) by analyticity, Property (UE) can be extended to complex time semigroup and so

$$\|T_t(H)\|_{L^1\to L^\infty} \leq \frac{1}{\mathrm{Re}(z)^{\frac{d}{2}}} \lesssim \frac{1}{|z|^{\frac{d}{2}}} \leq \frac{1}{|t|^{\frac{d}{2}}}.$$

So the full $L^1 - L^\infty$ (and so Property $(H_m(A))$ which is weaker, as we have just seen) are obviously satisfied in the range $|t| \leq h^2$.

The intermediate case $h^2 \leq |t| \leq h$ is treated in the particular case of compact Riemannian manifolds in [21] together with the implicated Strichartz estimates with a nontrivial loss of derivatives. We will focus on this interesting situation and will describe in this very general setting how dispersive estimates imply these Strichartz estimates (see Theorem 4.5).

The second goal we have in mind, is also to emphasize the link between the heat semigroup and the wave operator. In the second part of the paper (from Section 5) we aim to study which dispersive properties on the wave equation would be sufficient to ensure our hypothesis $(H_m(A))$ for $T = e^{itH}\psi_m(h^2H)$, in order to regain the dispersive estimates and then Strichartz estimates for the Schrödinger group. Mainly, we are interested in the wave propagator $\cos(t\sqrt{H})$ which is defined as follows: for any $f \in L^2$, $u(t) := t \mapsto \cos(t\sqrt{H})f$ is the unique solution of the linear wave equation:

$$\begin{cases} \partial_t^2 u + Hu = 0 \\ u_{|t=0} = f \\ \partial_t u_{|t=0} = 0. \end{cases}$$

One can find the explicit solution of this problem in [37] for the Euclidean case and in [9] for the compact Riemannian manifold case through precise formula for the kernel of the wave propagator. The remarkable property of this operator comes from its finite speed propagation. We know that Davies-Gaffney estimates (DG) imply (and indeed are equivalent to [28, Theorem 3.4] to) the finite speed propagation property at a speed equal to 1: namely, for every disjoint open subsets $U_1, U_2 \subset X$, every function $f_i \in L^2(U_i)$, $i = 1, 2$, then

$$\left\langle \cos\left(t\sqrt{H}\right) f_1, f_2 \right\rangle = 0, \tag{1.6}$$

for all $0 < t < d(U_1, U_2)$. If $\cos(t\sqrt{H})$ is an integral operator with kernel K_t, then (1.6) simply means that K_t is supported in the "light cone"

$$\mathcal{D}_t := \left\{(x, y) \in X^2,\ d(x, y) \leq t\right\}.$$

To apply Theorem 1.1 in order to get dispersive estimates, we first have to prove that Schrödinger propagators satisfy Property $(H_m(A))$ for some suitable constant A. The following formula (see Section 5): for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$:

$$e^{-zH} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \cos\left(s\sqrt{H}\right) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{z}},$$

allows to describe the link between Schrödinger propagators and wave propagators.

We will need extra assumptions (deeper than just the finite speed propagation property), in order to be able to check Property ($H_m(A)$). More precisely, we need the following short time $L^2 - L^2$ dispersive estimate:

Assumption 1.2. There exist $\kappa \in (0, \infty]$ and an integer m_0 such that for every $s \in (0, \kappa)$ we have: for every $r > 0$, every balls B_r, $\widetilde{B_r}$ of radius r then

$$\left\| \cos(s\sqrt{H})\psi_{m_0}(r^2 H) \right\|_{L^2(B_r) \to L^2(\widetilde{B_r})} \lesssim \left(\frac{r}{s+r} \right)^{\frac{d-1}{2}} \left(1 + \frac{|L-s|}{r} \right)^{-\frac{d+1}{2}}$$

where $L = d(B_r, \widetilde{B_r})$.

To obtain our second result we will need more regularity on the measure than (1.5). For the next theorem assume that μ is Ahlfors regular, that is: there exist two absolute positive constants c and C such that for all $x \in X$ and $r > 0$:

$$cr^d \leq \mu(B(x, r)) \leq Cr^d. \tag{1.7}$$

Then our second main theorem is the following:

Theorem 1.3. *Suppose* (1.7) *with $d > 1$, (DUE) and Assumption* 1.2 *with $\kappa \in (0, \infty]$. Then for every integer $m \geq \max(\frac{d}{2}, m_0 + \lceil \frac{d-1}{2} \rceil)$ we have*

- *If $\kappa = \infty$: the propagator e^{itH} satisfies Property ($H_m(|t|^{-\frac{d}{2}})$) for every $t \in \mathbb{R}^*$ and so we have Strichartz estimates without loss of derivatives;*
- *If $\kappa < \infty$: for every $\varepsilon > 0$, every $0 < h \leq 1$ with $h^2 \leq |t| \leq h^{1+\varepsilon}$ and every integer $m' \geq 0$ the propagator $e^{itH}\psi_{m'}(h^2 H)$ satisfies Property ($H_m(|t|^{-\frac{d}{2}})$) and so we have Strichartz estimates with loss of $\frac{1+\varepsilon}{p}$ derivatives.*

It is worth noting that, in the proof, the same approach raises the two cases:

- $\kappa < +\infty$ which leads to Strichartz estimates with loss of derivatives;
- $\kappa = +\infty$ which leads to estimates without loss of derivatives.

The general setting we work with allows our result to apply in many situations. For instance Assumption 1.2 holds in the context of smooth compact Riemannian manifold with κ given by the injectivity radius and in the Euclidean situation with $\kappa = \infty$ (also with smooth perturbation and there $\kappa < \infty$).

As an example, note that in the case of the Euclidean space with an operator of the form $H = -\mathrm{div} A \nabla$, where A is a matrix with variable and $C^{1,1}$ coefficients, then Smith has built a short time parametrix [54] of the corresponding wave equation (see also the work of Blair [15]), which yields in particular our Assumption 1.2 for some $\kappa < \infty$. As a consequence, we deduce that the solutions of Schrödinger equation satisfies Strichartz estimates with loss of $\frac{1+\varepsilon}{p}$ derivatives for every $\varepsilon > 0$.

Moreover, properties of the spectral measure (more precisely the microlocalized dispersive estimates in [62, Proposition 3.3]) obtained by Zhang allow to recover Assumption 1.2 with $\kappa = +\infty$ and to have a new demonstration of the result of Hassel-Zhang in [42], namely global-in-time Strichartz estimates without loss of derivatives on non-trapping asymptotically conic Riemannian manifolds (except for $q = \infty$).

By this way, we have a unified approach to deal with compact or noncompact situations and we recover (up to a loss ε as small as we want) the Strichartz estimates with loss of derivatives for a compact smooth manifold due to [21,56] and full Strichartz estimates for the Euclidean or non-trapping setting. However even if the obtained estimates do not improve the ones already known in the literature for specific situations (some of them optimal), our results take place in a much more general framework and are derived from a new method relying on the heat semigroup. Moreover, the link between dispersion for waves and dispersive estimates for Schrödinger equation was unknown in such a general framework.

The plan of this article is as follow: In Section 2 we first set the notation and definitions used throughout the paper. Then we describe the assumptions required on the heat semigroup $(e^{-tH})_{t\geq 0}$ together with some basic properties about Hardy-BMO spaces and functional calculus associated with H. Theorem 1.1 is proved in Section 3 and we apply it in Section 4 to prove Strichartz estimates (with a possible loss of derivatives). Section 5 shows Theorem 1.3, and Section 6 how the hypothesis $(H_m(A))$ can be derived from the small time parametrix of the associated wave operator.

2. Definitions and preliminaries

2.1. Notation

For $B(x,r)$ a ball ($x \in X$ and $r > 0$) and any parameter $\lambda > 0$, we denote $\lambda B(x,r) := B(x, \lambda r)$ the dilated and concentric ball. As a consequence of the doubling property, a ball $B(x, \lambda r)$ can be covered by $C\lambda^d$ balls of radius r, uniformly in $x \in X, r > 0$ and $\lambda > 1$ (C is a constant only dependent on the ambient space). Moreover, the volume of the balls satisfies the following behavior:

$$\mu(B(y,r)) \lesssim \left(1 + \frac{d(x,y)}{r}\right)^d \mu(B(x,r)) \tag{2.1}$$

uniformly for all $x, y \in X$ and $r > 0$.

For a ball Q, and an integer $i \geq 1$, we denote $C_i(Q)$ the ith dyadic corona around Q:

$$C_i(Q) := 2^i Q \backslash 2^{i-1} Q.$$

We also set $C_0(Q) = Q$.

If no confusion arises, we will note L^p instead of $L^p(X,\mu)$ for $p \in [1, \infty]$. For a set Ω, we say that $f \in L^p(\Omega)$ if f is supported in Ω and if

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} < +\infty.$$

Notice that if $f \in L^p$, we can compute $\|f\|_{L^p(\Omega)}$ even if f is not supported in Ω.

We will use $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \leq Cv$ and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$. If Ω is a set, $\mathbb{1}_\Omega$ is the characteristic function of Ω, defined by

$$\mathbb{1}_\Omega(x) = \begin{cases} 1 \text{ if } x \in \Omega \\ 0 \text{ if } x \notin \Omega. \end{cases}$$

The Hardy-Littlewood maximal operator is denoted by $\mathcal{M}$ and is given for every $x \in X$ and function $f \in L^1_{\text{loc}}$ by:

$$\mathcal{M}(f)(x) := \sup_{\substack{B \text{ ball} \\ x \in B}} \left(\frac{1}{\mu(B)} \int_B |f| d\mu \right) = \sup_{\substack{B \text{ ball} \\ x \in B}} ⨍_B |f| d\mu.$$

Since the space is of homogeneous type, it is well-known that this maximal operator is bounded in any L^p spaces, for $p \in (1, \infty]$.

2.2. The heat semigroup and associated functional calculus

We recall the definition of a L^2-holomorphic semigroup:

Definition 2.1. A family of operators $(S(z))_{\text{Re}(z) \geq 0}$ on $\mathcal{L}(L^2)$ is said to be a holomorphic semigroup on L^2 if (with $\Gamma := \{z \in \mathbb{C},\ \text{Re}(z) \geq 0\}$):

(1) $S(0) = id$;
(2) $\forall z_1, z_2 \in \Gamma\ ,\ S(z_1 + z_2) = S(z_1) \circ S(z_2)$;
(3) $\forall f \in L^2\ ,\ \lim\limits_{\substack{z \to 0 \\ z \in \Gamma}} \|S(z)f - f\|_{L^2} = 0$;
(4) $\forall f, g \in L^2$, the map $z \mapsto \langle S(z)f, g\rangle$ is holomorphic on the interior $\text{Int}(\Gamma)$.

We recall the bounded functional calculus theorem from [51]:

Theorem 2.2. *Since H is a nonnegative self-adjoint operator, it admits a L^∞-functional calculus: if $\rho \in L^\infty(\mathbb{R}_+)$, then we may consider the operator $\rho(H)$ as a L^2-bounded operator and*

$$\|\rho(H)\|_{L^2 \to L^2} \leq \|\rho\|_{L^\infty}.$$

For any integer $m \geq 1$ and real $n > 0$, we set $\psi_{m,n}(x) = x^m e^{-nx}$ and $\psi_m := \psi_{m,1}$. These smooth functions $\psi_{m,n} \in \mathcal{C}^\infty(\mathbb{R}^+)$, vanish at 0 and at infinity; moreover $\|\psi_{m,n}\|_{L^\infty(\mathbb{R}_+)} \lesssim 1$. The previous theorem allows us to define the operators $\psi_{m,n}(tH)$ for any $t \geq 0$ and $m \in \mathbb{N}, n > 0$.

From the Gaussian estimates of the heat kernel (*UE*) and the analyticity of the semigroup (see [31, Corollary 5] or [25]) it comes that for every integer $m \in \mathbb{N}$ and

$n > 0$ the operator $\psi_{m,n}(tH)$ has a kernel $p_{m,n,t}$ also satisfying upper Gaussian estimates:

$$|p_{m,n,t}(x,y)| \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\left(-c\frac{d(x,y)^2}{t}\right), \quad \forall\, t > 0, \text{ a.e. } x, y \in X. \tag{2.2}$$

We now give some basic results about the semigroup thanks to our assumptions.

Proposition 2.3. *Under* (1.4) *and* (UE), *the heat semigroup is pointwisely bounded by the Hardy-Littlewood maximal operator and is uniformly bounded in every* L^p*-spaces for* $p \in [1,\infty]$*: for every locally integrable function* f *and every* $x_0 \in X$*, we have*

$$\sup_{t>0} \left\| e^{-tH} f \right\|_{L^\infty(B(x_0,\sqrt{t}))} \lesssim \mathcal{M}(f)(x_0) \quad \textit{and} \quad \sup_{t>0} \|e^{-tH} f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Proof. The pointwise boundedness by the maximal function is an easy consequence of (UE) with the doubling property (1.4). As a consequence, the L^p-boundedness of the maximal operator yields the uniform L^p-boundedness of the heat semigroup, for $p > 1$. Let us now check the L^1-boundedness. By (UE), we have:

$$\int_{x\in X} |e^{-tH} f(x)| d\mu(x) \lesssim \int_{x\in X}\int_{y\in X} \frac{1}{\mu(B(x,\sqrt{t}))} e^{-c\frac{d(x,y)^2}{t}} |f(y)| d\mu(y) d\mu(x)$$
$$\lesssim \int_{y\in X} |f(y)| \frac{1}{\mu(B(y,\sqrt{t}))} \int_{x\in X} \left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^d e^{-c\frac{d(x,y)^2}{t}} d\mu(x) d\mu(y).$$

A decomposition in coronas around $B(y,\sqrt{t})$ allows us to control the integral over x:

$$\begin{aligned}
&\int_{B(y,\sqrt{t})} \left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^d e^{-c\frac{d(x,y)^2}{t}} d\mu(x) \\
&\quad + \sum_{j\geq 1} \int_{C_j(B(y,\sqrt{t}))} \left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^d e^{-\frac{d(x,y)^2}{t}} d\mu(x) \\
&\leq 2^d \mu(B(y,\sqrt{t})) + \sum_{j\geq 1} \left(1 + 2^j\right)^d e^{-c2^{2j}} \mu(B(y, 2^j\sqrt{t})) \\
&\lesssim \left(2^d + \sum_{j\geq 1} (1+2^j)^d 2^{jd} e^{-c2^{2j}}\right) \mu(B(y,\sqrt{t})) \lesssim \mu(B(y,\sqrt{t})),
\end{aligned}$$

where the last line results from the doubling property of μ. Hence, uniformly in $t > 0$

$$\|e^{-tH}\|_{L^1\to L^1} \lesssim 1. \qquad \square$$

Corollary 2.4. *For $m \in \mathbb{N}$ and $n > 0$, since $\psi_{m,n}(tH)$ satisfies (UE), we deduce that the operators $\psi_{m,n}(tH)$ also satisfy the same estimates.*

Since H is a self-adjoint operator on L^2, it admits a bounded Borel functional calculus on L^2. Under the additional assumption of (1.4) and (UE), it is known that H can be extended to an unbounded operator acting on L^p, for $p \in (1, +\infty)$, with a bounded H^∞ functional calculus on L^p as shown in [33, Theorem 3.1]. It also admits a bounded Hörmander-type functional calculus on L^p, see [33] and [32, Theorem 3.1]. We refer to [1] and references in [1] for more details on functional calculus. In the sequel, we will mostly make use of H^∞ functional calculus rather than Hörmander-type functional calculus.

Let us now give some basic properties about the functions $\psi_{m,n}$, which are a consequence of the L^p holomorphic functional calculus:

Proposition 2.5. *Under* (1.4) *and* (UE), *we have:*

(a) *For all $k \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $n > 0$ then $\psi_{km,kn} = \left(\psi_{m,n}\right)^k$;*
(b) *For all $m, m' \in \mathbb{N}$ and $n, n', u, v > 0$ then:*

$$\psi_{m,n}(u\cdot)\psi_{m',n'}(v\cdot) = \frac{u^m v^{m'}}{(nu+n'v)^{m+m'}}\psi_{m+m',1}((nu+n'v)\cdot);$$

(c) *For every $r > 0$ and every $f \in L^2$ then:*

$$(1-e^{-r^2H})f = \int_0^{r^2} He^{-sH} f ds = \int_0^{r^2} \psi_{1,1}(sH)f\frac{ds}{s};$$

(d) *For $m \in \mathbb{N}^*$, $n > 0$ and $f \in L^2$, then $\left(\int_0^{+\infty} \|\psi_{m,n}(vH)f\|_{L^2}^2 \frac{dv}{v}\right)^{\frac{1}{2}} + \|P_{N(H)}f\|_{L^2} \lesssim \|f\|_{L^2}$; where $P_{N(H)}$ is the projector on the kernel of H: $N(H) := \{f \in L^2 \cap \mathcal{D}(H),\ Hf = 0\}$;*
(e) *For $m \in \mathbb{N}^*$, $n > 0$, up to a constant $c_{m,n}$, we have the decomposition (usually called Calderón reproducing formula):*

$$Id = c_{m,n}\int_0^{+\infty} \psi_{m,n}(sH)\frac{ds}{s} + P_{N(H)}.$$

Proof. (a), (b), (c) and (e) are straightforward and refer the reader to [11, Proposition 2.11]. (d) is classical and a direct application of (e) with the almost-orthogonality of $\psi_{m,n}(vH)$ operators: for every $u, v > 0$

$$\|\psi_{m,n}(uH)\psi_{m,n}(vH)\|_{L^2\to L^2} \lesssim \min\left(\frac{u}{v}, \frac{v}{u}\right)^m;$$

for which we refer to , *e.g.*, [7] □

It is crucial to keep in mind that by the holomorphic functional calculus, item (e) gives a decomposition of the identity, the following Calderón reproducing formula:

$$Id = c_{m,n} \int_0^\infty \psi_{m,n}(sH) \frac{ds}{s} + P_{N(H)},$$

which has to be seen/thought as a smooth version of the spectral decomposition. Indeed the operator $\psi_{m,n}(sH)$ plays the role of a regularized version of the projector $\mathbb{1}_{[s^{-1},2s^{-1}]}(H)$.

Remark 2.6. We would like to emphasize that the use of $\psi_{m,n}$ functions is exactly equivalent to the use of smooth compactly supported cut-off functions. Indeed, it is easy by a smooth partition of the unity to build $\psi_{m,n}$ by an absolutely convergent serie of smooth and compactly supported cut-off functions. From functional calculus, we also know how we can build a smooth and compactly supported function by the resolvent of H (using the semigroup) and so the $\psi_{m,n}$ functions (see [44, Appendix] or [48], *e.g.*).

We have chosen to work with $\psi_{m,n}$ functions to enlighten the connection between dispersive estimates and heat semigroup and also to get around the different norms that we have to consider on the C^∞ space.

2.3. Quadratic functionals associated with the heat semigroup and Sobolev spaces

Let us define some tools for the next theorem, for all $\lambda > 0$:

$$\varphi(\lambda) := \int_\lambda^{+\infty} \psi_{m,n}(u) \frac{du}{u},$$

$$\tilde{\varphi}(\lambda) := \int_0^\lambda \psi_{m,n}(v) \frac{dv}{v} = \int_0^1 \psi_{m,n}(\lambda u) \frac{du}{u}.$$

Remark 2.7. Notice that φ is, by integration by parts, a finite linear combination of functions $\psi_{k,\ell}$ for $k \in \{0, .., m\}$ and $\ell > 0$. Moreover for every $\lambda \in \mathbb{R}_+$,

$$\tilde{\varphi}(\lambda) + \varphi(\lambda) = \int_0^{+\infty} u^{m-1} e^{-nu} du = \frac{\Gamma(m)}{n^m} := c_{m,n}.$$

The following theorem will be useful to estimate the L^p-norm through the heat semigroup:

Theorem 2.8. *Assume* (1.4) *and* (DUE). *For every integer $m \geq 1$, real number $n > 0$ and all $p \in (1, \infty)$, we have*

$$\|f\|_{L^p} \simeq \|\varphi(H)f\|_{L^p} + \left\| \left(\int_0^1 |\psi_{m,n}(uH)f|^2 \frac{du}{u} \right)^{\frac{1}{2}} \right\|_{L^p}.$$

So if $q \geq 2$

$$\|f\|_{L^q} \lesssim \|\varphi(H)f\|_{L^q} + \left(\int_0^1 \|\psi_{m,n}(uH)f\|_{L^q}^2 \frac{du}{u}\right)^{\frac{1}{2}}.$$

Such a result can be seen as a semigroup version of the Littlewood-Paley characterization of Lebesgue spaces.

Proof. We give the sketch of the proof (and refer to [5, Chapter 6, Theorem 6.1] and [7, Proposition 2.12] for more details where it is proved that such inequalities hold for every exponent p belonging to the range dictated by the heat semigroup $(e^{-tH})_{t\geq 0}$; here $(1,\infty)$). We aim to study the boundedness of the quadratic functional

$$T : f \mapsto \left(\int_0^1 \left|\psi_{m,n}(s^2 H) f\right|^2 \frac{ds}{s}\right)^{\frac{1}{2}}.$$

Indeed T is a horizontal square function (or Littlewood-Paley-Stein g-function), and its L^p-boundedness is well-known by functional calculus theory (see [49, 57] and references therein) when the semigroup is Submarkovian and conservative.

We aim here to quickly explain another approach (more analytic) of its boundedness, which does not require submarkovian property and conservativeness but relies on Gaussian estimates (UE). We are looking to apply extrapolation result [5, Theorem 1.1] or [13] to T with $p_0 = 1$. To keep the notation of [5] we recall that

$$A_r := Id - \left(Id - e^{-r^2 H}\right)^M \quad \text{and} \quad B_r := Id - A_r = \left(Id - e^{-r^2 H}\right)^M,$$

with M a large enough integer. First by L^2 holomorphic functional calculus, it is known that T is bounded on L^2 (see [7], *e.g.*). We now have to check the two main hypothesis of [5, Theorem 1.1].

By expanding, A_r behaves like $e^{-r^2 H}$, in the sense that it admits a kernel satisfying the Gaussian upper estimates at the scale r. Note B a ball of radius r, and f supported in B. For all $j \geq 1$ and for all $x \in C_j(B)$, we have

$$\begin{aligned} |A_r f(x)| &\lesssim \int_B \frac{1}{\mu(B(x,r))} e^{-c\frac{d(x,y)^2}{r^2}} |f(y)| d\mu(y) \\ &\lesssim \frac{1}{\mu(B(x,r))} \int_B e^{-c2^{2j}} |f| d\mu. \end{aligned}$$

If z denotes the center of B, then by the doubling property (2.1) of the measure it comes

$$\mu(B) \lesssim \left(1 + \frac{d(z,x)}{r}\right)^d \mu(B(x,r)),$$

so that

$$\mu(B(x,r))^{-1} \lesssim \left(1+\frac{d(z,x)}{r}\right)^d \mu(B)^{-1} \lesssim 2^{jd}\mu(B)^{-1},$$

where we used that $z \in B$ and $x \in C_j(B)$ so $d(z,x) \lesssim 2^j r$. Hence,

$$\left(\frac{1}{\mu(2^{j+1}B)}\int_{C_j(B)} |A_r f|^2 d\mu\right)^{\frac{1}{2}} \lesssim g(j)\frac{1}{\mu(B)}\int_B |f| d\mu,$$

with $g(j) \lesssim e^{-c4^j} 2^{jd}$ satisfying

$$\sum_j g(j) 2^{dj} < +\infty.$$

That is the first assumption required in [5, Theorem 1.1].

Then the second (and last) assumption of [5, Theorem 1.1] has been weakened in [13] and we only have to check that for all $j \geq 2$:

$$\left(\frac{1}{\mu(2^{j+1}B)}\int_{C_j(B)} \left|T(B_r f)\right|^2 d\mu\right)^{\frac{1}{2}} \leq g(j)\left(\frac{1}{\mu(B)}\int_B |f|^2 d\mu\right)^{1/2}.$$

We refer the reader to [5, Step 3, item 1, Theorem 6.1] and also [27] and [6], where such inequalities are proved, and the arguments only rely on the Davies-Gaffney estimates (DG) for $\psi_{m,n}(tH)$.

By this way, we may apply [5, Theorem 1.1] and deduce that the square function T is bounded on L^p for every $p \in (1,2]$. For $p > 2$, we have to apply [5, Theorem 1.2] and this is also detailed in [5, Step 2, Theorem 6.1]. Thus, if $p \in (1,\infty)$ then

$$\left\| \left(\int_0^1 \left|\psi_{m,n}\big(s^2 H\big) f\right|^2 \frac{ds}{s}\right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{L^p}. \tag{2.3}$$

It remains to check the reverse inequalities. We proceed by duality to finish the proof. Since $\varphi(x) + \int_0^1 \psi_{m,n}(tx)\frac{dt}{t} = c_{m,n}$ is a constant independent of x, then:

$$\begin{aligned} c_{m,n}\langle f, g\rangle &= \left\langle f, \varphi(H)g + \int_0^1 \psi_{m,n}(tH) g \frac{dt}{t}\right\rangle \\ &= \langle \varphi(H) f, g\rangle + \int_0^1 \left\langle \psi_{\frac{m}{2},\frac{n}{2}}(tH) f, \psi_{\frac{m}{2},\frac{n}{2}}(tH) g\right\rangle \frac{dt}{t}. \end{aligned}$$

We should decompose $m = m_1 + m_2$ with 2 integers m_1, m_2 comparable to $\frac{m}{2}$. For simplicity we take $m_1 = m_2 = \frac{m}{2}$ and assume they are integers. We let to the reader the minor modifications. The Cauchy-Schwarz inequality for the scalar product $(u, v) = \int_0^1 u(t)v(t)\frac{dt}{t}$ gives then:

$$
\begin{aligned}
|\langle f, g\rangle| &\lesssim |\langle\varphi(H)f, g\rangle| + \int \left(\int_0^1 \left|\psi_{\frac{m}{2},\frac{n}{2}}(tH)f\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_0^1 \left|\psi_{\frac{m}{2},\frac{n}{2}}(tH)g\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} d\mu \\
&\leq \|\varphi(H)f\|_{L^p}\|g\|_{L^{p'}} + \left\|\left(\int_0^1 \left|\psi_{\frac{m}{2},\frac{n}{2}}(tH)f\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p} \left\|\left(\int_0^1 \left|\psi_{\frac{m}{2},\frac{n}{2}}(tH)g\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^{p'}} \\
&\lesssim \|\varphi(H)f\|_{L^p}\|g\|_{L^{p'}} + \left\|\left(\int_0^1 \left|\psi_{\frac{m}{2},\frac{n}{2}}(tH)f\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p} \|g\|_{L^{p'}},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and we used (2.3) for p'.

Thus, by duality

$$
\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} | < f, g > | \lesssim \|\varphi(H)f\|_{L^p} + \left\|\left(\int_0^1 |\psi_{\frac{m}{2},\frac{n}{2}}(tH)f|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p}.
$$

That concludes the proof of the characterization of the Lebesgue norms, via these square functionals.

Then in particular for $q \in [2, +\infty)$, Minkowski generalized inequality finally gives

$$
\|f\|_{L^q(M)} \lesssim \|\varphi(H)f\|_{L^q} + \left(\int_0^1 \|\psi_{m,n}(uH)f\|^2_{L^q(M)} \frac{du}{u}\right)^{\frac{1}{2}}. \qquad \square
$$

We will also work with the inhomogeneous Sobolev spaces associated with H, defined in terms of Bessel type: for $s \geq 0$ and $p \in (1, \infty)$, $W^{s,p}_H$ which will be noted $W^{s,p}$ is the Sobolev space of order s associated/equipped with the norm

$$
\|f\|_{W^{s,p}} := \left\|(1 + H)^{\frac{s}{2}} f\right\|_{L^p} \simeq \|f\|_{L^p} + \left\|H^{\frac{s}{2}} f\right\|_{L^p}.
$$

The equivalence of the norms are a direct consequence of the L^p holomorphic functional calculu, applied to the functions

$$
z \mapsto (1 + z)^{-s/2}, \quad z \mapsto [z(1 + z)]^{-s/2} \quad \text{and} \quad z \mapsto (1 + z)^{s/2}/\left(1 + z^{s/2}\right)
$$

which are holomorphic and bounded in some small conical neighborhood (in $\mathbb{C}$) of $(0, \infty)$ and so generate L^p-bounded operators.

Following the previous result, it comes

$$\begin{aligned}\|f\|_{W^{s,p}} &\simeq \|\varphi(H)f\|_{L^p} + \left\| \left(\sum_{k=1}^{+\infty} 2^{2ks} |\psi_{m,n}(2^{-2k}H)f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\simeq \|\varphi(H)f\|_{L^p} + \left\| \left(\int_0^1 u^{-2s} |\psi_{m,n}(u^2 H)f|^2 \frac{du}{u} \right)^{\frac{1}{2}} \right\|_{L^p}.\end{aligned}$$

We refer the reader to [7] for more details about such Sobolev spaces. We can move from the discrete to the continuous case of those partitions of "Littlewood-Paley" writing:

$$\begin{aligned}\sum_{k=1}^{+\infty} \int_1^2 \psi_{m,n}\big(2^{-2k}u^2\lambda\big)\frac{du}{u} &= \sum_{k=1}^{+\infty} \int_{2^{-k}}^{2^{-(k-1)}} \psi_{m,n}\big(\lambda v^2\big)\frac{dv}{v} = \int_0^1 \psi_{m,n}\big(\lambda v^2\big)\frac{dv}{v} \\ &= \int_0^{\lambda} \psi_{m,n}(u)\frac{du}{2u} = \int_0^1 \psi(\lambda u)\frac{du}{2u}.\end{aligned}$$

Remark 2.9. The left hand integral works over $u \in [1,2]$, so we can pass from information on the discrete case to the same information on the continuous case.

2.4. Hardy and BMO spaces

We define now atomic Hardy spaces adapted to our situation (dictated by a semi-group on a doubling space) using the construction introduced in [13]. Let $\mathcal{Q}$ be the family of all balls of X:

$$\mathcal{Q} := \{B(x,r)\,,\ x \in X\,,\ r > 0\}.$$

We define $\big(B_Q\big)_{Q\in\mathcal{Q}}$ the family of :

$$\forall Q \in \mathcal{Q}\,,\ B_Q := \left(1 - e^{-r^2 H}\right)^M,$$

where r is the radius of the ball Q and M is an integer (large enough: $M \geq \min(\frac{3}{4} + \frac{3d}{8}, 3)$ is sufficient). Those operators are bounded on L^2 uniformly in r. Indeed, by expanding, B_Q is a finite linear combination of operators e^{-kr^2H} with $k \in \{0,\dots,M\}$ and Theorem 2.2 gives

$$\left\| e^{-kr^2H} \right\|_{L^2\to L^2} \leq \left\| x \mapsto e^{-kr^2x} \right\|_{L^\infty(\mathbb{R}_+)} \leq 1,$$

because H is nonnegative.

Remark 2.10. $M \geq \frac{3}{4} + \frac{3d}{8}$ ensures that $\frac{4M}{3} - \frac{d}{2} \geq 1$ so there exists an integer $m \in [\frac{d}{2}, \frac{4M}{3}]$. This property will be needed in Section 3.

Definition 2.11. A function $a \in L^1_{\mathrm{loc}}$ is an *atom* associated with the ball Q if there exists a function f_Q whose support is included in Q such that $a = B_Q(f_Q)$, with

$$\|f_Q\|_{L^2(Q)} \leq (\mu(Q))^{-\frac{1}{2}}.$$

That last condition allows us to normalize f_Q in L^1. Indeed by the Cauchy-Schwarz inequality

$$\|f_Q\|_{L^1} \leq \|f_Q\|_{L^2(Q)} \mu(Q)^{\frac{1}{2}} \leq 1.$$

Moreover, B_Q is bounded on L^1 so every atom is in L^1 and they are also normalized in L^1:

$$\sup_a \|a\|_{L^1} \lesssim 1, \tag{2.4}$$

where we take the supremum over all the atoms. Indeed, consider an atom $a = B_Q(f_Q) = (1 - e^{-r^2 H})^M f_Q$ with suitable function f_Q supported on a ball Q. By the binomial theorem, B_Q behaves like $e^{-kr^2 H}$. So Proposition 2.3 gives

$$\|a\|_{L^1(X)} = \|B_Q(f_Q)\|_{L^1(X)} \leq \sum_{k=1}^{M} \binom{M}{k} \|e^{-kr^2 H} f_Q\|_{L^1} \lesssim \|f_Q\|_{L^1} \lesssim 1.$$

We may now define the Hardy space by atomic decomposition:

Definition 2.12. A measurable function h belongs to the *atomic Hardy space* H^1_{ato}, which will be denoted H^1, if there exists a decomposition

$$h = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \mu - \text{a.e.}$$

where a_i are atoms and λ_i real numbers satisfying:

$$\sum_{i \in \mathbb{N}} |\lambda_i| < +\infty.$$

We equip the space H^1 with the norm:

$$\|h\|_{H^1} := \inf_{h = \sum_i \lambda_i a i} \sum_{i \in \mathbb{N}} |\lambda_i|,$$

where we take the infimum over all the atomic decompositions.

For a more general definition and some properties about atomic spaces we refer to [10,13], and the references therein. From (2.4), we deduce:

Corollary 2.13. *The Hardy space is continuously embedded into L^1:*

$$\|f\|_{L^1} \lesssim \|f\|_{H^1}.$$

From [13, *Corollary* 7.2]*, the Hardy space H^1 is also a Banach space.*

We refer the reader to [13, Section 8], for details about the problem of identifying the dual space $(H^1)^*$ with a BMO space. For a L^∞-function, we may define the BMO norm

$$\|f\|_{\mathrm{BMO}} := \sup_Q \left(\fint_Q |B_Q(f)|^2 \, d\mu \right)^{1/2},$$

where the supremum is taken over all the balls. If $f \in L^\infty$ then $B_Q(f)$ is also uniformly bounded (with respect to the ball Q), since the heat semigroup is uniformly bounded in L^∞ (see Proposition 2.3) and so $\|f\|_{\mathrm{BMO}}$ is finite.

Definition 2.14. The *functional space* BMO is defined as the closure

$$\mathrm{BMO} := \overline{\left\{ f \in L^\infty + L^2,\ \|f\|_{\mathrm{BMO}} < \infty \right\}},$$

for the BMO norm.

Remark 2.15. The following characterization of the BMO norm will be useful: for $f \in L^2$ then

$$\|f\|_{\mathrm{BMO}} = \sup_{a \text{ atom}} |\langle f, a\rangle|, \tag{2.5}$$

and f belongs to BMO if and only if the right hand side is finite. Indeed if $f \in L^2$ then for all ball Q

$$\begin{aligned} \mu(Q)^{-\frac{1}{2}} \|B_Q(f)\|_{L^2(Q)} &= \mu(Q)^{-\frac{1}{2}} \sup_{\substack{g \in L^2(Q) \\ \|g\|_{L^2(Q)} \leq 1}} |\langle B_Q(f), g\rangle| \\ &= \sup_{\substack{g \in L^2(Q) \\ \|g\|_{L^2(Q)} \leq 1}} \left| \left\langle f, B_Q\left(\mu(Q)^{-\frac{1}{2}} g\right)\right\rangle \right|, \end{aligned}$$

where we used that B_Q is self-adjoint. One can check that the collection of atoms exactly corresponds to the collection of functions of type $B_Q(\mu(Q)^{-\frac{1}{2}} g)$ with $g \in L^2(Q)$ and $\|g\|_{L^2} \leq 1$.

Following [13, Section 8], it comes that BMO is continuously embedded into the dual space $(H^1)^*$ and contains L^∞:

$$L^\infty \hookrightarrow \mathrm{BMO} \hookrightarrow \left(H^1\right)^*.$$

Hence

$$\|T\|_{H^1 \to (H^1)^*} \lesssim \|T\|_{H^1 \to \mathrm{BMO}}, \tag{2.6}$$

and

$$\forall \theta \in (0,1), \quad \left(L^2, \mathrm{BMO}\right)_\theta \hookrightarrow \left(L^2, \left(H^1\right)^*\right)_\theta. \tag{2.7}$$

The following interpolation theorem between Hardy spaces and Lebesgue spaces is the key of our study:

Theorem 2.16. *For all $\theta \in (0,1)$, consider the exponent $p \in (1,2)$ and $q = p' \in (2,\infty)$ given by*

$$\frac{1}{p} = \frac{1-\theta}{2} + \theta \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{2}.$$

Then (using the interpolation notation), we have

$$\left(L^2, H^1\right)_\theta = L^p \quad \textit{and} \quad \left(L^2, \left(H^1\right)^*\right)_\theta \hookrightarrow L^q,$$

if the ambient space X is non-bounded and

$$L^p \hookrightarrow L^2 + \left(L^2, H^1\right)_\theta \quad \textit{and} \quad L^2 \cap \left(L^2, \left(H^1\right)^*\right)_\theta \hookrightarrow L^q,$$

if the space X is bounded.

The same results hold replacing $(H^1)^*$ by BMO thanks to (2.7).

Proof. The result follows directly from [10, Theorems 4 and 5] (and we keep here its notation). To ensure that it applies in our setting, we have to check that $H^1 \hookrightarrow L^1$ (which we knew from Corollary 2.13), and that the maximal function M_∞ is bounded by $\mathcal{M}$, where we recall that

$$M_\infty(f)(x) = \sup_{Q \ni x} \left\| A_Q^*(f) \right\|_{L^\infty(Q)},$$

with

$$A_Q = Id - \left(Id - e^{-r^2 H}\right)^M \text{ is self-adjoint and } r \text{ denotes the radius of } Q.$$

The binomial theorem shows that A_Q is a linear combination of operators $e^{-kr^2 H}$ for $k \in \{1, \dots, M\}$. So the property that M_∞ is pointwisely controlled by $\mathcal{M}$ is a direct consequence of Proposition 2.3. □

In the situation of bounded space (with a finite measure), interpolation is more delicate since the previous result does not give a complete characterization of L^p as an intermediate space. We have the following:

Theorem 2.17. *Assume that the space is bounded (or equivalently that $\mu(X) < +\infty$) and consider a self-adjoint operator T satisfying the following boundedness:*

$$\begin{cases} \|T\|_{L^2 \to L^2} \lesssim 1 \\ \|T\|_{H^1 \to (H^1)^*} \lesssim A < +\infty \\ \|T\|_{L^p \to L^2} \lesssim B < +\infty \quad \textit{for } p \in (1,2) \end{cases},$$

then T is bounded from L^p to $L^{p'}$ with

$$\|T\|_{L^p\to L^{p'}} \lesssim B + A^{\frac{1}{p}-\frac{1}{p'}}.$$

The same result holds with BMO instead of $(H^1)^*$ by (2.7).

Proof. Let $p \in (1,2)$. We aim to apply Theorem 2.16 to T. Pick $\theta \in (0,1)$ such that $\frac{1-\theta}{2} = 1 - \frac{1}{p}$. Then $\theta = \frac{1}{p} - \frac{1}{p'}$. Let $f \in L^p \hookrightarrow L^2 + (L^2, H^1)_\theta$. We choose a decomposition $f = a + b$ with $a \in L^2$ and $b \in (L^2, H^1)_\theta$ such that

$$\|a\|_{L^2} + \|b\|_{(L^2,H^1)_\theta} \lesssim \|f\|_{L^p}.$$

Since T is self-adjoint, T is bounded from L^2 to $L^{p'}$ with a norm at most B. Thus

$$\|Ta\|_{L^{p'}} \lesssim B\|a\|_{L^2}.$$

Similarly, by Theorem 2.16:

$$\|Tb\|_{L^{p'}} \lesssim \|Tb\|_{L^2} + \|Tb\|_{(L^2,(H^1)^*)_\theta} \lesssim B\|b\|_{L^p} + A^{\frac{1}{p}-\frac{1}{p'}}\|b\|_{(L^2,H^1)_\theta}.$$

Moreover $H^1 \hookrightarrow L^1$ so $(L^2, H^1)_\theta \hookrightarrow (L^2, L^1)_\theta = L^p$. Consequently,

$$\|Tb\|_{L^{p'}} \lesssim \left(B + A^{\frac{1}{p}-\frac{1}{p'}}\right)\|b\|_{(L^2,H^1)_\theta}.$$

Hence

$$\|Tf\|_{L^{p'}} \lesssim B\|a\|_{L^2} + \left(B + A^{\frac{1}{p}-\frac{1}{p'}}\right)\|b\|_{(L^2,H^1)_\theta} \lesssim \left(B + A^{\frac{1}{p}-\frac{1}{p'}}\right)\|f\|_{L^p}. \qquad \square$$

2.5. On the hypothesis $(H_m(A))$

We aim here to study the behavior of Assumption $(H_{m,n}(A))$ with respect to the parameters m, n.

Consider a fixed operator T, a positive real $A > 0$ and let us define property $(H_{m,n}(A))$ for $m \in \mathbb{N}$ and $n > 0$:

$$\left\|T\psi_{m,n}(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim A\mu(B_r)^{\frac{1}{2}}\mu\big(\widetilde{B_r}\big)^{\frac{1}{2}}, \tag{$H_{m,n}(A)$}$$

where B_r and $\widetilde{B_r}$ are any two balls of radius $r > 0$.

Proposition 2.18. *For all integer $m \geq 0$ and $n > 0$:*

$$(H_{m,1}(A)) \Rightarrow (H_{m,n}(A)).$$

Proof. Assume Property $(H_{m,1}(A))$. Since

$$\psi_{m,n}(x) = x^m e^{-nx} = (nx)^m e^{-nx} n^{-m} = n^{-m}\psi_{m,1}(nx),$$

it comes

$$T\psi_{m,n}\big(r^2H\big) = n^{-m}T\psi_{m,1}\big(nr^2H\big).$$

If $n \geq 1$ then $B_r \subset \sqrt{n}B_r$ and $\widetilde{B_r} \subset \sqrt{n}\widetilde{B_r}$. Hence, using the doubling property we get

$$\begin{aligned}\|T\psi_{m,n}\big(r^2H\big)\|_{L^2(B_r)\to L^2(\widetilde{B_r})} &= n^{-m}\left\|T\psi_{m,1}\big(nr^2H\big)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \\ &\leq n^{-m}\left\|T\psi_{m,1}\big(nr^2H\big)\right\|_{L^2(\sqrt{n}B_r)\to L^2(\sqrt{n}\widetilde{B_r})} \\ &\leq n^{-m}A\mu\left(\sqrt{n}B_r\right)^{\frac{1}{2}}\mu\big(\sqrt{n}\widetilde{B_r}\big)^{\frac{1}{2}} \lesssim A\mu(B_r)^{\frac{1}{2}}\mu\big(\widetilde{B_r}\big)^{\frac{1}{2}}.\end{aligned}$$

If $n \leq 1$ then $\sqrt{n}\widetilde{B_r} \subset \widetilde{B_r}$. We cover $\widetilde{B_r}$ by $N \simeq \left(\frac{r}{\sqrt{nr}}\right)^d = n^{-\frac{d}{2}}$ balls $\widetilde{B_j}$ of radius $\sqrt{n}r$ and B_r by N balls B_k of radius $\sqrt{n}r$ (satisfying the bounded overlap property). Thus

$$\left\|T\Big(\psi_{m,n}\big(r^2H\big)f\Big)\right\|_{L^2(\widetilde{B_r})} \leq \sum_j\sum_k \left\|T\Big(\psi_{m,n}\big(r^2H\big)f.\mathbb{1}_{B_k}\Big)\right\|_{L^2(\widetilde{B_j})}.$$

Finally:

$$\begin{aligned}\left\|T\psi_{m,n}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} &\leq \sum_j\sum_k n^{-m}\left\|T\psi_{m,1}\big(nr^2H\big)\right\|_{L^2(B_k)\to L^2(\widetilde{B_j})} \\ &\lesssim \sum_j\sum_k n^{-m}A\mu(B_k)^{\frac{1}{2}}\mu\big(\widetilde{B_j}\big)^{\frac{1}{2}} \\ &\lesssim A\left(\sum_j \mu(\widetilde{B_j})\right)^{\frac{1}{2}}\left(\sum_k \mu(B_k)\right)^{\frac{1}{2}} \\ &\lesssim A\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}}.\end{aligned}$$

□

We will now be able to focus on $(H_{m,1}(A))$ and functions $\psi_{m,1} = \psi_m$ rather than keeping the dependence in the parameter n.

Proposition 2.19. *If $m' > m \geq 0$ are two integers then*

$$(H_{m,1}(A)) \Rightarrow (H_{m',1}(A)).$$

Proof. Assume $(H_{m,1}(A))$ is satisfied. Then, by Proposition 2.18, $(H_{m,n}(A))$ is also true for all $n > 0$. First we remark that

$$T\psi_{m',1}\big(r^2H\big) = T\psi_{m,\frac{1}{2}}\big(r^2H\big)\psi_{m'-m,\frac{1}{2}}\big(r^2H\big).$$

Hence, decomposing X in dyadic coronas around B_r:

$$\begin{aligned}&\left\|T\psi_{m',1}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\\ &\quad\leq \sum_{j=0}^{+\infty}\left\|T\psi_{m,\frac{1}{2}}\big(r^2H\big)\right\|_{L^2(C_j(B_r))\to L^2(\widetilde{B_r})}\left\|\psi_{m'-m,\frac{1}{2}}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(C_j(B_r))}.\end{aligned}$$

Let $f\in L^2(B_r)$. We treat the case $j=0$ with Proposition 2.3 and $(H_{m,\frac{1}{2}}(A))$:

$$\left\|T\psi_{m,\frac{1}{2}}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\left\|\psi_{m'-m,\frac{1}{2}}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(B_r)}\lesssim A\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}}.$$

Assume now that $j\geq 1$. If $x\in C_j(B_r)$, then by Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\psi_{m'-m,\frac{1}{2}}\big(r^2H\big)f(x)\right| &\leq \int_{B_r}\frac{1}{\mu(B(x,r))}e^{-c\frac{d(x,y)^2}{r^2}}|f(y)|d\mu(y)\\ &\leq \frac{e^{-c2^{2j}}\mu(B_r)^{\frac{1}{2}}}{\mu(B(x,r))}\|f\|_{L^2(B_r)}.\end{aligned}$$

By (2.1), we have already seen that for every $x\in C_j(B_r)$

$$\mu(B_r)\lesssim 2^{jd}\mu(B(x,r)),$$

which yields

$$\left|\psi_{m'-m,\frac{1}{2}}\big(r^2H\big)f(x)\right|\lesssim e^{-c2^{2j}}2^{jd}\mu(B_r)^{-\frac{1}{2}}\|f\|_{L^2(B_r)}.$$

Hence, by the doubling property:

$$\left\|\psi_{m'-m,\frac{1}{2}}\big(r^2H\big)f\right\|_{L^2(B_r)\to L^2(C_j(B_r))}\lesssim e^{-c2^{2j}}2^{\frac{3jd}{2}}. \tag{2.8}$$

Consider $(B_k)_{k=0,\dots,K}$ a collection of balls of radius r (with a bounded overlap property so $K\lesssim 2^{jd}$) which covers $C_j(B_r)$ with, by the doubling property: $\mu(B_k)\lesssim 2^{jd}\mu(B_r)$. From $(H_{m,\frac{1}{2}})$ it follows

$$\begin{aligned}\left\|T\psi_{m,\frac{1}{2}}\big(r^2H\big)\right\|_{L^2(C_j(B_r))\to L^2(\widetilde{B_r})} &\leq \sum_{k=0}^{K}\left\|T\psi_{m,\frac{1}{2}}\big(r^2H\big)\right\|_{L^2(B_k)\to L^2(\widetilde{B_r})}\\ &\leq \sum_{k=0}^{K}A\mu(B_k)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}}\\ &\lesssim A2^{\frac{3}{2}jd}\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}}.\end{aligned} \tag{2.9}$$

Thus, combining (2.8) and (2.9), it comes

$$\begin{aligned}\left\| T\psi_{m',1}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} &\lesssim \left(1+\sum_{j\geq 1} e^{-c2^{2j}}2^{3jd}\right) A\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}} \\ &\lesssim A\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}},\end{aligned}$$

which ends the proof of Property $(H_{m',1}(A))$. □

We sum up Propositions 2.18 and 2.19 in:

Theorem 2.20. *Assume* (1.4). *For every integer $m \geq 0$, Property $(H_{m,n}(A))$ is independent on $n > 0$. So let us call $(H_m(A))$ this property. It is "increasing in m", since for two integers $m' > m \geq 0$*

$$(H_m(A)) \Rightarrow (H_{m'}(A)).$$

3. Dispersion inequality from Property $(H_m(A))$

The aim of this section is to show Theorem 1.1, more precisely that Property $(H_m(A))$ implies a $H^1 - BMO$ and $L^p - L^{p'}$ dispersive estimates. The main idea is first to prove boundedness of the operator on atoms, then to deduce boundedness on the whole Hardy space H^1, and finally to interpolate with the L^2-boundedness.

In all this section, we fix a large enough integer $M \geq \max(3, \frac{3}{4} + \frac{3d}{8})$, which allows us to consider the notions of *atoms* and Hardy spaces H^1, built with this parameter. As pointed out in Remark 2.10, that also allows us to find an integer $m \in [\frac{d}{2}, \frac{4M}{3}]$.

3.1. Boundedness on atoms

Theorem 3.1. *Assume* (1.4) *and* (DUE). *Let T be a L^2-bounded operator, which commutes with H. If T satisfies Property $(H_m(A))$ for a certain integer $m \leq \frac{4M}{3}$, then one gets*

$$\sup_{a,b} |\langle Ta, b\rangle| \lesssim A,$$

where the supremum is taken over all atoms a, b.

Proof. Let a and b be two atoms. By definition, there exists B_1 and B_2 two balls with radii r_1 and r_2 respectively, and $f \in L^2(B_1)$, $g \in L^2(B_2)$, such that

$$\begin{cases} a = \big(1-e^{-r_1^2H}\big)^M f & \text{with } \|f\|_{L^2(B_1)} \leq \mu(B_1)^{-\frac{1}{2}} \\ b = \big(1-e^{-r_2^2H}\big)^M g & \text{with } \|g\|_{L^2(B_2)} \leq \mu(B_2)^{-\frac{1}{2}}. \end{cases}$$

We first remark by (c) of Proposition 2.5 that:

$$
\begin{aligned}
a &= \left(\int_0^{r_1^2} He^{-sH} ds\right)^M f = \int_0^{r_1^2} \dots \int_0^{r_1^2} H^M e^{-(s_1+\dots+s_M)H} f ds_1 \dots ds_M \\
&= \int_0^{Mr_1^2} \underbrace{\left(\int_{\substack{s_1+\dots+s_M=u \\ 0\le s_i \le r_1^2}} ds_1 \dots ds_{M-1} \right)}_{=I_M(u)} H^M e^{-uH} f du.
\end{aligned}
$$

As $s_i \geq 0$ for all $i \in \{1, \dots, M\}$ with $s_1 + \dots + s_M = u$, we have: $0 \leq s_i \leq u$. Hence

$$I_M(u) \leq u^{M-1}.$$

Thus

$$\langle Ta, b\rangle = \int_0^{Mr_1^2} \int_0^{Mr_2^2} I_M(u) I_M(v) \langle T\psi_M(uH)f, \psi_M(vH)g\rangle \frac{dv}{v^M} \frac{du}{u^M}.$$

Moreover ψ_M is continuous and H is self-adjoint, so $\psi_M(uH)$ and $\psi_M(vH)$ are also self-adjoint. Using (a) and (b) of Proposition 2.5 and the fact that T commutes with H (and so with every operator $\psi_{m,n}(H)$), we get:

$$
\begin{aligned}
&|\langle Ta, b\rangle| \\
&\leq \int_0^{Mr_1^2} \int_0^{Mr_2^2} \left|\left\langle T\psi_{M,1}(uH)\psi_{\frac{M}{3},\frac{1}{3}}(vH)\psi_{\frac{M}{3},\frac{1}{3}}(vH)f, \psi_{\frac{M}{3},\frac{1}{3}}(vh)g\right\rangle\right| \frac{du}{u}\frac{dv}{v} \\
&= \iint \left|\left\langle T\frac{(uv^{\frac{1}{3}})^M}{(u+\frac{v}{3})^{\frac{4M}{3}}} \psi_{\frac{4M}{3},1}\left(\left(u+\frac{v}{3}\right)H\right)\psi_{\frac{M}{3},\frac{1}{3}}(vH)f, \psi_{\frac{M}{3},\frac{1}{3}}(vh)g\right\rangle\right| \frac{du}{u}\frac{dv}{v}.
\end{aligned}
$$

Here we have decomposed $\psi_{M,1}$ in three terms involving $\psi_{\frac{M}{3},\frac{1}{3}}$. We aim to use in particular the Gaussian estimates (2.2), which hold only if $\frac{M}{3}$ is an integer. We should decompose $M = M_1 + M_2 + M_3$ with 3 integers M_1, M_2, M_3 which are comparable to $M/3$ (that is why we picked $M \geq 3$). For simplicity we take $M_1 = M_2 = M_3 = M/3$ and assume that they are integers. We let to the reader the minor modifications.

Without loss of generality because the problem is symmetric in u and v, we can assume that $u \leq v$ so that $\frac{v}{3} \leq u + \frac{v}{3} \leq \frac{4v}{3}$. Hence $\frac{uv^{\frac{1}{3}}}{(u+\frac{v}{3})^{\frac{4}{3}}} \simeq \frac{u}{v}$. We cover the whole space X by balls B_j and B_k of radius $\sqrt{u+\frac{v}{3}}$. The covering satisfies the bounded overlap property. We use Cauchy-Schwarz inequality and Property

$(H_m(A))$ to obtain:

$$
\begin{aligned}
&|\langle Ta, b\rangle| \\
&\lesssim \iint \sum_{j,k} \left(\frac{u}{v}\right)^M \left|\left\langle \mathbb{1}_{B_k} T \psi_{\frac{4M}{3},1}\left(\left(u+\frac{v}{3}\right)H\right) \mathbb{1}_{B_j} \psi_{\frac{M}{3},\frac{1}{3}}(vH) f,\right.\right. \\
&\qquad\qquad\qquad \left.\left. \mathbb{1}_{B_k} \psi_{\frac{M}{3},\frac{1}{3}}(vH) g\right\rangle\right| \frac{du}{u}\frac{dv}{v} \\
&\lesssim \iint \sum_{j,k} \left(\frac{u}{v}\right)^M \left\| T \psi_{\frac{4M}{3},1}\left(\left(u+\frac{v}{3}\right)H\right) \mathbb{1}_{B_j} \psi_{\frac{M}{3},\frac{1}{3}}(vH) f\right\|_{L^2(B_k)} \\
&\quad \cdot \left\|\psi_{\frac{M}{3},\frac{1}{3}}(vH) g\right\|_{L^2(B_k)} \frac{du}{u}\frac{dv}{v} \\
&\lesssim \iint \sum_{j,k} \left(\frac{u}{v}\right)^M A\mu(B_k)^{\frac{1}{2}} \mu(B_j)^{\frac{1}{2}} \left\|\psi_{\frac{M}{3},\frac{1}{3}}(vH) f\right\|_{L^2(B_j)} \\
&\quad \cdot \left\|\psi_{\frac{M}{3},\frac{1}{3}}(vH) g\right\|_{L^2(B_k)} \frac{du}{u}\frac{dv}{v},
\end{aligned}
$$

where we have used that T satisfies Property $(H_{4M/3})$. Indeed T satisfies property $(H_m(A))$ for $m \leq 4M/3$ (so T satisfies also $(H_{4M/3})$ by Theorem 2.20). To simplify the notation we will now note $\psi_{\frac{M}{3},\frac{1}{3}} = \psi$. We use a decomposition in dyadic coronas around B_1:

$$
\sum_{j\in J} \mu\big(B_j\big)^{\frac{1}{2}} \|\psi(vH) f\|_{L^2(B_j)} \leq \sum_{j\in J} \sum_{l=0}^{+\infty} \mu\big(B_j\big)^{\frac{1}{2}} \|\mathbb{1}_{C_l(B_1)} \psi(vH) f\|_{L^2(B_j)}.
$$

We study the terms $l = 0$ and $l \geq 1$ separately.

First when $l = 0$:

$$
\begin{aligned}
&\sum_{j\in J} \mu\big(B_j\big)^{\frac{1}{2}} \|\mathbb{1}_{C_0(B_1)} \psi(vH) f\|_{L^2(B_j)} \\
&\leq \left(\sum_J \mu(B_j)\right)^{\frac{1}{2}} \left(\sum_J \|\mathbb{1}_{B_1} \psi(vH) f\|^2_{L^2(B_j)}\right)^{\frac{1}{2}} \\
&\lesssim \mu(B_1)^{\frac{1}{2}} \left(\sum_J \int_{B_j} |\mathbb{1}_{B_1}(x) \psi(vH) f(x)|^2 d\mu(x)\right)^{\frac{1}{2}} \\
&\lesssim \mu(B_1)^{\frac{1}{2}} \|\psi(vH) f\|_{L^2}.
\end{aligned}
$$

Now when $l \geq 1$ the number of indices in J for which the sum is nonzero is equivalent to the number of balls of radius $\sqrt{u+\frac{v}{3}}$ we need to cover $C_l(B_1)$, that is $|J| \simeq \left(\frac{2^l r_1}{\sqrt{u+\frac{v}{3}}}\right)^d$.

Now, remark that by the doubling property of the measure and (2.1), since B_j is a ball of radius $\sqrt{u+v/3} \simeq \sqrt{v}$ we deduce that for $x \in B_j \cap C_l(B_1)$ then

$$\mu\Big(B(x,\sqrt{v})\Big) \simeq \mu\big(B_j\big).$$

By (2.2), we have:

$$\begin{aligned}
&\sum_{j\in J}\sum_{l=1}^{+\infty} \mu\big(B_j\big)^{\frac{1}{2}} \|\mathbb{1}_{C_l(B_1)}\psi(vH)f\|_{L^2(B_j)} \\
&\lesssim \sum_{j\in J}\sum_{l=1}^{+\infty} \mu\big(B_j\big)^{\frac{1}{2}} \left\| \frac{\mathbb{1}_{C_l(B_1)}(x)}{\mu\Big(B(x,\sqrt{v})\Big)} e^{-\frac{2^{2l}r_1^2}{v}} \|f\|_{L^1(B_1)} \right\|_{L^2_x(B_j)} \\
&\lesssim \sum_{j\in J}\sum_{l=1}^{+\infty} \mu\big(B_j\big)^{\frac{1}{2}} \mu\big(B_j\big)^{\frac{1}{2}} \frac{1}{\mu\big(B_j\big)} e^{-\frac{2^{2l}r_1^2}{v}} \\
&\lesssim \sum_{l\geq 1} \left(\frac{2^l r_1}{\sqrt{u+\frac{v}{3}}}\right)^d e^{-\frac{2^{2l}r_1^2}{v}} \lesssim \sum_{l\geq 1} \left(\frac{2^l r_1}{\sqrt{v}}\right)^d e^{-\frac{2^{2l}r_1^2}{v}},
\end{aligned}$$

where we have used the L^2-normalization of f, which yields that $\|f\|_{L^1} \lesssim 1$.

We then refer the reader to Lemma 3.2 to estimate the sum and it comes

$$\sum_{j\in J}\sum_{l=1}^{+\infty} \mu\big(B_j\big)^{\frac{1}{2}} \|\mathbf{1}_{C_l(B_1)}\psi(vH)f\|_{L^2(B_j)} \lesssim \left(\frac{r_1}{\sqrt{v}}\right)^{-1}.$$

Thus

$$\sum_{j\in J} \mu\big(B_j\big)^{\frac{1}{2}} \|\psi(vH)f\|_{L^2(B_j)} \leq \mu(B_1)^{\frac{1}{2}} \|\psi(vH)f\|_{L^2} + \frac{\sqrt{v}}{r_1}.$$

Similarly for B_2 and the sum over $k \in K$:

$$\sum_{k\in K} \mu(B_k)^{\frac{1}{2}} \|\psi(vH)g\|_{L^2(B_k)} \leq \mu(B_2)^{\frac{1}{2}} \|\psi(vH)g\|_{L^2} + \frac{\sqrt{v}}{r_2}.$$

Hence, one concludes:

$$\begin{aligned}
|\langle Ta, b\rangle| \lesssim A \iint \left(\frac{u}{v}\right)^M &\left(\mu(B_1)^{\frac{1}{2}} \|\psi(vH)f\|_{L^2} + \frac{\sqrt{v}}{r_1}\right) \\
&\cdot \left(\mu(B_2)^{\frac{1}{2}} \|\psi(vH)g\|_{L^2} + \frac{\sqrt{v}}{r_2}\right) \frac{du}{u}\frac{dv}{v}.
\end{aligned}$$

We then develop the product to split the problem into four different terms:

$$\begin{aligned} I &= \iint \left(\frac{u}{v}\right)^M \mu(B_1)^{\frac12}\mu(B_2)^{\frac12}\|\psi(vH)f\|_{L^2}\|\psi(vH)g\|_{L^2}\frac{du}{u}\frac{dv}{v}, \\ II &= \iint \left(\frac{u}{v}\right)^M \mu(B_1)^{\frac12}\|\psi(vH)f\|_{L^2}\frac{\sqrt{v}}{r_2}\frac{du}{u}\frac{dv}{v}, \\ III &= \iint \left(\frac{u}{v}\right)^M \mu(B_2)^{\frac12}\|\psi(vH)g\|_{L^2}\frac{\sqrt{v}}{r_1}\frac{du}{u}\frac{dv}{v}, \\ IV &= \iint \left(\frac{u}{v}\right)^M \frac{\sqrt{v}}{r_1}\frac{\sqrt{v}}{r_2}\frac{du}{u}\frac{dv}{v}. \end{aligned}$$

We discern now two cases:

Case 1: $0 \le u \le v \le R = \min(Mr_1^2, Mr_2^2)$
Then Item (d) of Proposition 2.5 yields

$$\begin{aligned} I &= \int_{v=0}^{R}\int_{u=0}^{v} \left(\frac{u}{v}\right)^M \mu(B_1)^{\frac12}\mu(B_2)^{\frac12}\|\psi(vH)f\|_{L^2}\|\psi(vH)g\|_{L^2}\frac{du}{u}\frac{dv}{v} \\ &\simeq \mu(B_1)^{\frac12}\mu(B_2)^{\frac12}\int_0^R \|\psi(vH)f\|_{L^2}\|\psi(vH)g\|_{L^2}\frac{dv}{v} \\ &\le \mu(B_1)^{\frac12}\mu(B_2)^{\frac12}\left(\int_0^{+\infty}\|\psi(vH)f\|_{L^2}^2\frac{dv}{v}\right)^{\frac12}\left(\int_0^{+\infty}\|\psi(vH)g\|_{L^2}^2\frac{dv}{v}\right)^{\frac12} \\ &\le \mu(B_1)^{\frac12}\|f\|_{L^2}\mu(B_2)^{\frac12}\|g\|_{L^2} \lesssim 1, \end{aligned}$$

similarly for the second term,

$$\begin{aligned} II =& \mu(B_1)^{\frac12}\frac{1}{r_2}\int_0^R \|\psi(vH)f\|_{L^2}\sqrt{v}\frac{dv}{v} \\ &\le \mu(B_1)^{\frac12}\frac{1}{r_2}\|f\|_{L^2}\left(\int_0^R v\frac{dv}{v}\right)^{\frac12} \le \frac{R^{\frac12}}{r_2} \le \frac{\sqrt{Mr_2^2}}{r_2} \lesssim 1. \end{aligned}$$

The third term is treated the same way. The fourth term gives:

$$IV = \frac{1}{r_1r_2}\int_0^R \sqrt{v}\sqrt{v}\frac{dv}{v} = \frac{R}{r_1r_2} \le \frac{\min(Mr_1^2, Mr_2^2)}{\sqrt{\min(r_1^2, r_2^2)}\sqrt{\min(r_1^2, r_2^2)}} \lesssim 1.$$

So in this first case, we obtain

$$I + II + III + IV \lesssim 1. \tag{3.1}$$

Case 2: $0 \leq u \leq Mr_1^2 \leq v \leq Mr_2^2$

Similarly we get:

$$\begin{aligned}
I &= \mu(B_1)^{\frac{1}{2}}\mu(B_2)^{\frac{1}{2}} \int_{v=Mr_1^2}^{Mr_2^2}\int_{u=0}^{Mr_1^2} \left(\frac{u}{v}\right)^M \|\psi(vH)f\|_{L^2}\|\psi(vH)g\|_{L^2}\frac{du}{u}\frac{dv}{v} \\
&\leq \mu(B_1)^{\frac{1}{2}}\mu(B_2)^{\frac{1}{2}} \int_{v=Mr_1^2}^{Mr_2^2}\int_{u=0}^{Mr_1^2} \frac{uv^{M-1}}{v^M} \|\psi(vH)f\|_{L^2}\|\psi(vH)g\|_{L^2}\frac{du}{u}\frac{dv}{v} \\
&= \mu(B_1)^{\frac{1}{2}}\mu(B_2)^{\frac{1}{2}} \int_{v=Mr_1^2}^{Mr_2^2} \frac{Mr_1^2}{v} \|\psi(vH)f\|_{L^2}\|\psi(vH)g\|_{L^2}\frac{dv}{v} \\
&\leq \mu(B_1)^{\frac{1}{2}}\mu(B_2)^{\frac{1}{2}} Mr_1^2 \sup_{v\in[Mr_1^2, Mr_2^2]} \frac{1}{v}\left(\int_0^{Mr_2^2} \|\psi(vH)f\|_{L^2}^2\frac{dv}{v}\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_0^{Mr_2^2} \|\psi(vH)g\|_{L^2}^2\frac{dv}{v}\right)^{\frac{1}{2}} \\
&\leq \frac{Mr_1^2}{Mr_1^2} = 1.
\end{aligned}$$

For the second term, since $r_1 \leq r_2$:

$$II \leq \mu(B_1)^{\frac{1}{2}} \frac{Mr_1^2}{r_2}\left(\int_0^{Mr_1^2} \|\psi(vH)f\|_{L^2}^2\frac{dv}{v}\right)^{\frac{1}{2}}\left(\int_{Mr_1^2}^{+\infty} \frac{1}{v}\frac{dv}{v}\right)^{\frac{1}{2}} \lesssim \frac{r_1^2}{r_2 r_1} \leq 1.$$

The third term is similar:

$$III \leq \mu(B_2)^{\frac{1}{2}} Mr_1^2 \int_{Mr_1^2}^{Mr_2^2} \frac{1}{v}\|\psi(vH)g\|_{L^2}\frac{\sqrt{v}}{r_1}\frac{dv}{v} \lesssim r_1\left(\int_{Mr_1^2}^{+\infty}\frac{1}{v}\frac{dv}{v}\right)^{\frac{1}{2}} \lesssim \frac{r_1}{r_1} = 1.$$

Finally we treat the last term:

$$IV \leq \int_{Mr_1^2}^{Mr_2^2} \frac{Mr_1^2}{v}\frac{\sqrt{v}}{r_1}\frac{\sqrt{v}}{r_2}\frac{dv}{v} \lesssim \frac{r_1}{r_2}\int_{Mr_1^2}^{Mr_2^2}\frac{1}{v}\frac{dv}{v} = 2\frac{r_1}{r_2}\ln\left(\frac{r_2}{r_1}\right) \lesssim 1,$$

because $x \mapsto \frac{\ln x}{x}$ is continuous if $x \geq 1$, equals 0 if $x = 1$ and tends to 0 as x tends to $+\infty$, so is bounded uniformly in $x \geq 1$ (here $\frac{r_2}{r_1} \geq 1$).

Thus, in this second case, we also conclude that

$$I + II + III + IV \lesssim 1. \tag{3.2}$$

Since $u \leq v$ (which was assumed at the beginning by symmetry), cases 1 and 2 cover all the possible situations. Consequently, we deduce that for all atoms a and b, we have

$$|\langle Ta, b\rangle| \lesssim A,$$

where the implicit constant does not depend on the atoms (but maybe on the parameters M and m). □

We used the following lemma with $N = 1$ and $x = \frac{r}{\sqrt{v}}$:

Lemma 3.2. *Let $x > 0$ and $d \in \mathbb{N}$. For all $N \in \mathbb{N}^*$:*

$$\sum_{l=0}^{+\infty} \left(2^l x\right)^d e^{-(2^l x)^2} \lesssim x^{-N}.$$

Proof. We remark that $\int_{2^l}^{2^{l+1}} \frac{dt}{t} = \ln\left(\frac{2^{l+1}}{2^l}\right) = \ln 2$. Thus:

$$\sum_{l=0}^{+\infty} \left(2^l x\right)^d e^{-(2^l x)^2} = \frac{1}{\ln 2} \sum_{l=0}^{+\infty} \left(2^l x\right)^d e^{-(2^l x)^2} \int_{2^l}^{2^{l+1}} \frac{dt}{t}.$$

$2^l \leq t \leq 2^{l+1}$ yields $(2^l x)^d \leq (tx)^d$ and $e^{-(tx)^2} \geq e^{-(2^{l+1}x)^2}$. So we have

$$e^{-(2^l x)^2} \leq e^{-\frac{(tx)^2}{4}}.$$

Hence:

$$\begin{aligned}\sum_{l=0}^{+\infty} \left(2^l x\right)^d e^{-(2^l x)^2} &\lesssim \int_1^{+\infty} (tx)^d e^{-\frac{(tx)^2}{4}} \frac{dt}{t} = \int_{\frac{x}{2}}^{+\infty} (2u)^d e^{-u^2} \frac{du}{u} \\ &\lesssim \int_{\frac{x}{2}}^{+\infty} \frac{1}{u^N} \frac{du}{u} = \left[\frac{u^{-N}}{-N}\right]_{\frac{x}{2}}^{+\infty} \lesssim x^{-N}\end{aligned}$$

for $N \in \mathbb{N}^*$ as large as we want. □

3.2. Boundedness on Hardy space

After having proved that the operator T (of Theorem 1.1) is bounded on atoms, we now aim to show that T is bounded from the Hardy space H^1 to its dual $(H^1)^*$ (and more precisely to BMO) with a norm controlled by A. If $f \in H^1$ then there exists an atomic decomposition $f = \sum_{i=0}^{+\infty} \lambda_i a_i$ where a_i are atoms and $\sum_{i=0}^{+\infty} |\lambda_i| <$

$2\|f\|_{H^1}$. We know how to bound the operator on atoms, we would like to extend it passing to the limit in

$$T\left(\sum_{i=0}^{N}\lambda_i a_i\right)=\sum_{i=0}^{N}\lambda_i T a_i,$$

in order to apply Theorem 3.1. As N goes to $+\infty$, that last equality may not be true. Indeed, one can find in [24] an example (due to Meyer) of a linear form bounded on atoms, which is not bounded on the whole Hardy space. So to rigorously check this step, we need to prove it using specificities of our situation. Aiming that, we are going to use an approximation of the identity well suited to our frame: $(e^{-sH})_{s>0}$.

We start by showing that Te^{-sH} (the regularized version of T) satisfies the same estimate as the one in Theorem 3.1:

Theorem 3.3. *Assume* (1.4) *and* (DUE). *Consider a fixed operator T, L^2-bounded, commuting with H and satisfying Property* $(H_m(A))$ *for some integer* $m\in[\frac{d}{2},\frac{4M}{3}]$. *Then uniformly with respect to* $s>0$, *the operator* Te^{-sH} *satisfies Property* $(H_m(A))$ *and so by Theorem* 3.1*:*

$$\sup_{s>0}\sup_{a,b}\left|\left\langle Te^{-sH}a,b\right\rangle\right|\lesssim A,$$

where the supremum is taken over all the atoms a, b.

Proof. Set $U_s:=Te^{-sH}$. It suffices to check that U_s satisfies Property $(H_m(A))$ uniformly in s, which is

$$\left\|U_s\psi_{m,1}(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\lesssim A\mu(B_r)^{\frac12}\mu(\widetilde{B_r})^{\frac12},$$

for any two balls B_r and $\widetilde{B_r}$ with radius $r>0$. First, remark that

$$U_s\psi_{m,1}(r^2H)=Te^{-sH}(r^2H)^m e^{-r^2H}=T\psi_{m,1}((r^2+s)H)\left(\frac{r^2}{r^2+s}\right)^m,$$

so that

$$\left\|U_s\psi_{m,1}(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}=\left(\frac{r^2}{r^2+s}\right)^m\left\|T\psi_{m,1}((r^2+s)H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}.$$

As $r^2<r^2+s$, the balls of radius r are included into the balls with same centers and radius $\sqrt{r^2+s}$ denoted $B_{\sqrt{r^2+s}}$ and $\widetilde{B_{\sqrt{r^2+s}}}$. Then it comes (with Property

$(H_m(A))$ for T and the doubling property)

$$\begin{aligned}
&\left\| U_s \psi_{m,1}\big(r^2 H\big) \right\|_{L^2(B_r) \to L^2(\widetilde{B_r})} \\
&\leq \left(\frac{r^2}{r^2+s} \right)^m \left\| T\psi_{m,1}\big((r^2+s)H\big) \right\|_{L^2\left(B_{\sqrt{r^2+s}}\right) \to L^2\left(\widetilde{B_{\sqrt{r^2+s}}}\right)} \\
&\leq \left(\frac{r^2}{r^2+s} \right)^m A\mu\left(B_{\sqrt{r^2+s}}\right)^{\frac{1}{2}} \mu\left(\widetilde{B_{\sqrt{r^2+s}}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{r^2}{r^2+s} \right)^m A\sqrt{\frac{r^2+s}{r^2}}^{\frac{d}{2}} \mu(B_r)^{\frac{1}{2}} \sqrt{\frac{r^2+s}{r^2}}^{\frac{d}{2}} \mu(\widetilde{B_r})^{\frac{1}{2}} \\
&\leq A \left(\frac{r^2}{r^2+s} \right)^{m-\frac{d}{2}} \mu(B_r)^{\frac{1}{2}} \mu(\widetilde{B_r})^{\frac{1}{2}} \leq A\mu(B_r)^{\frac{1}{2}} \mu(\widetilde{B_r})^{\frac{1}{2}},
\end{aligned}$$

where the last inequality comes from $m \geq \frac{d}{2}$. That concludes the proof of Property $(H_m(A))$ for the operator U_s and all the estimates are uniform with respect to $s > 0$. $\square$

In order to prove that we can pass to the limit as N goes to $+\infty$ in

$$Te^{-sH}\left(\sum_{i=0}^{N} \lambda_i a_i\right) = \sum_{i=0}^{N} \lambda_i Te^{-sH} a_i$$

for atoms a_i, we have to show some continuity of the operator Te^{-sH}.

Theorem 3.4. *If T is a L^2-bounded operator which commutes with H and the ambient space X satisfies the uniform control of the volume* (1.5)*, then for all $s > 0$: Te^{-sH} maps L^1 to L^∞ and*

$$\left\| Te^{-sH} \right\|_{L^1 \to L^\infty} \lesssim s^{-\frac{\nu}{2}}.$$

Proof. By the commutativity property, we write $Te^{-sH} = e^{-sH/2} Te^{-sH/2}$. Hence

$$\left\| Te^{-sH} \right\|_{L^1 \to L^\infty} \leq \left\| e^{-\frac{s}{2}H} \right\|_{L^1 \to L^2} \|T\|_{L^2 \to L^2} \left\| e^{-\frac{s}{2}H} \right\|_{L^2 \to L^\infty}.$$

Using the Gaussian pointwise estimates of the heat kernel and (1.5), we deduce by a T^*T argument that

$$\begin{aligned}
\left\| e^{-\frac{s}{2}H} \right\|_{L^1 \to L^2}^2 &= \left\| e^{-sH} \right\|_{L^1 \to L^\infty} \\
&= \sup_{x,y} p_s(x,y) \lesssim s^{-\frac{\nu}{2}},
\end{aligned}$$

and by duality

$$\|e^{-\frac{s}{2}H}\|_{L^1\to L^2} = \|e^{-\frac{s}{2}H}\|_{L^2\to L^\infty} \lesssim s^{-\frac{\nu}{4}}.$$

As a consequence, we deduce the desired estimate. □

We are now able to establish the result on the whole Hardy space H^1:

Theorem 3.5. *Assume* (1.4), (1.5) *and* (DUE). *Consider a L^2-bounded operator T, which commutes with H and which satisfies Property* $(H_m(A))$ *for some integer $m \in [\frac{d}{2}, \frac{4M}{3}]$. Then T and Te^{-sH}, for all $s > 0$, can be continuously extended from H^1 to BMO* (*and so in particular to its dual* $(H^1)^*$) *and we have*

$$\|T\|_{H^1\to BMO} + \sup_{s>0} \|Te^{-sH}\|_{H^1\to BMO} \lesssim A.$$

Proof. Let $f \in H^1$ and consider an atomic decomposition. The atoms are uniformly bounded in L^1 so the limit

$$f = \sum_{i=0}^{+\infty} \lambda_i a_i = \lim_{N\to+\infty} \sum_{i=0}^{N} \lambda_i a_i,$$

takes place in L^1.

Moreover $a_i \in L^1$ implies $Te^{-sH}(a_i) \in L^\infty$ due to Theorem 3.4. Hence the limit

$$Te^{-sH}\left(\lim_{N\to+\infty} \sum_{i=0}^{N} \lambda_i a_i\right) = \lim_{N\to+\infty} Te^{-sH}\left(\sum_{i=0}^{N} \lambda_i a_i\right) = \lim_{N\to+\infty} \sum_{i=0}^{N} \lambda_i Te^{-sH}(a_i),$$

is valid and takes place in L^∞ for every $s > 0$ fixed. Thus

$$Te^{-sH}(f) = \sum_{i=0}^{+\infty} \lambda_i Te^{-sH}(a_i).$$

Let $f \in H^1$. There exists a decomposition $f = \sum_i \lambda_i a_i$ with a_i atoms, $\sum_i |\lambda_i| < +\infty$ and $\sum_i |\lambda_i| \le 2\|f\|_{H^1}$. We want to estimate

$$\|Te^{-sH} f\|_{\mathrm{BMO}} = \sup_b \left|\langle Te^{-sH} f, b\rangle\right|$$

where the supremum is taken over all atoms b (see Remark 2.15). By Theorem 3.3, and what we just proved, we have:

$$\begin{aligned}\left|\left\langle Te^{-sH} \sum_i \lambda_i a_i, b\right\rangle\right| &\le \sum_i |\lambda_i| \left|\langle Te^{-sH} a_i, b\rangle\right| \\ &\lesssim \sum_i |\lambda_i| A \lesssim A\|f\|_{H^1}.\end{aligned}$$

Hence
$$\left\| Te^{-sH} \right\|_{H^1 \to \mathrm{BMO}} \lesssim A,$$
and the implicit constant is uniform in $s > 0$.

Let us now consider the boundedness of the operator T. We know (see [13], *e.g.*) that $H^1 \cap L^2$ is dense in H^1 (since every atoms are L^2 functions). Moreover $(e^{-sH})_{s \geq 0}$ is a strongly continuous semigroup on $\mathcal{L}(L^2)$ so:
$$\forall f \in L^2, \ \lim_{s \to 0} \left\| e^{-sH} f - f \right\|_{L^2} = 0.$$

Let $f \in H^1 \cap L^2$ so that $Tf \in L^2$ and let a be an atom. We also have
$$\left| \left\langle Te^{-sH} f - Tf , \ a \right\rangle \right| \leq \left\| e^{-sH} Tf - Tf \right\|_{L^2} \|a\|_{L^2} \underset{s \to 0}{\to} 0.$$

Consequently, uniformly with respect to the atom a, we have
$$|\langle Tf , \ a \rangle| = \lim_{s \to 0} \left| \left\langle Te^{-sH} f , \ a \right\rangle \right| \lesssim A \|f\|_{H^1}.$$

Then for all $f \in H^1 \cap L^2$:
$$\|Tf\|_{\mathrm{BMO}} \lesssim A \|f\|_{H^1}.$$

As BMO is a Banach space, T admits an extension (still denoted T) which is bounded from H^1 to BMO and then from H^1 to $(H^1)^*$ because $\mathrm{BMO} \hookrightarrow (H^1)^*$. □

3.3. Interpolation

Having obtained a bound on the Hardy space, we now aim to use interpolation to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Consider a L^2-bounded operator T, which commutes with H and satisfies Property $(H_m(A))$ for some $m \in [\frac{d}{2}, \frac{4M}{3}]$. Then Theorem 3.5 shows that T admits a continuous extension from H^1 to $(H^1)^*$. So we aim now to interpolate the two following continuities:
$$\begin{cases} \|T\|_{L^2 \to L^2} \lesssim 1 \\ \|T\|_{H^1 \to (H^1)^*} \lesssim A. \end{cases}$$

Let p be fixed in $(1, 2)$. Then by choosing $\theta = \frac{2}{p} - 1 \in (0, 1)$ and $\frac{1}{q} = 1 - \frac{1}{p}$ (that is $q = p'$) in Theorem 2.16, if $\mu(X) = +\infty$, we have
$$T\colon \left(L^2, H^1\right)_\theta = L^p \to \left(L^2, \left(H^1\right)^*\right)_\theta \hookrightarrow L^q.$$

It follows the boundedness of T from L^p to $L^{p'}$. More precisely, if the space X is unbounded then

$$\|T\|_{L^p\to L^{p'}} \lesssim \|T\|^{\theta}_{H^1\to (H^1)^*}\|T\|^{1-\theta}_{L^2\to L^2} \lesssim A^{\theta} = A^{\frac{1}{p}-\frac{1}{p'}}.$$

If the space X is bounded, then Theorem 2.17 shows

$$\|T\|_{L^p\to L^{p'}} \lesssim A^{\frac{1}{p}-\frac{1}{p'}} + B,$$

provided that $\|T\|_{L^p\to L^2} \lesssim B$. $\square$

4. Application to Strichartz estimates

In this section we aim to take advantage of the dispersive estimates previously obtained in the particular situation where T is given by the Schrödinger propagator, to deduce some Strichartz estimates with loss of derivatives, as introduced in [21].

In particular, we are looking to dispersive estimates $L^p - L^{p'}$ with polynomial bound. It is also natural to work in the setting of an Ahlfors regular space (and not only in the doubling situation). The space X of homogeneous type is said Ahlfors regular if there exist two absolute positive constants c and C such that for all $x \in X$ and $r > 0$:

$$cr^d \leq \mu(B(x,r)) \leq Cr^d. \tag{4.1}$$

From now on, we will assume this property.

To establish Strichartz estimates from dispersive inequalities we adapt the result by Keel-Tao in [47], namely:
Consider $(U(t))_{t\in\mathbb{R}}$ a collection of uniformly L^2-bounded operators, *i.e.*

$$\sup_{t\in\mathbb{R}} \|U(t)\|_{L^2\to L^2} \lesssim 1 \tag{4.2}$$

and such that for a certain $\sigma > 0$

$$\forall t \neq s \in \mathbb{R}, \|U(s)U(t)^*\|_{L^1\to L^\infty} \lesssim |t-s|^{-\sigma}. \tag{4.3}$$

Then in [47], it is proved that for all pair of exponents (p,q) satisfying $\frac{1}{p}+\frac{\sigma}{q}=\frac{\sigma}{2}$, we have

$$\|U(t)f\|_{L^p_t L^q_x} \lesssim \|f\|_{L^2}.$$

By the exact same proof, we have the following:

Theorem 4.1. *Suppose that the collection* $(U(t))_t$ *satisfies* (4.2) *and for some* $\sigma > 0$

$$\forall t \neq s \in \mathbb{R}, \|U(s)U(t)^*\|_{H^1\to (H^1)^*} \lesssim |t-s|^{-\sigma}. \tag{4.4}$$

Then for all pair (p, q) satisfying $\frac{1}{p} + \frac{\sigma}{q} = \frac{\sigma}{2}$ with $q \neq +\infty$, we have

$$\|U(t)f\|_{L^p_t L^q_x} \lesssim \|f\|_{L^2},$$

where we assume in addition that

$$\forall t \neq s \in \mathbb{R}, \|U(s)U(t)^*\|_{L^{p'} \to L^2} \lesssim |t-s|^{-\sigma\left(\frac{1}{p'}-\frac{1}{p}\right)} \tag{4.5}$$

if X is bounded.

We do not give a proof of this result, since it is exactly the same as the one in [47] by replacing the space L^1 with the Hardy space H^1. The proof relies on interpolating the two boundedness (4.2) and (4.4), which is still possible with the Hardy space, due to Theorem 2.16.

We are now in position to prove the following result:

Theorem 4.2. *Assume* (4.1) *with* (DUE). *Consider an integer $\ell \geq 0$. Assume that the operator $T_t(H) := e^{itH}\psi_{2\ell}(h^2H)$ satisfies Property $(H_m(|t|^{-\frac{d}{2}}))$ for some $m \geq \frac{d}{2}$ and every $t \in [-1,1]$, $t \neq 0$. Then for all pair (p, q) satisfying* (1.1) *with $q \neq +\infty$ we have:*

$$\left(\int_{-1}^{1} \left\|e^{itH}\psi_{2\ell}\big(h^2H\big)f\right\|_{L^q}^p dt\right)^{\frac{1}{p}} \lesssim \left\|\psi_{\ell,\frac{1}{2}}\big(h^2H\big)f\right\|_{L^2}.$$

We also have the "semi-classical" version, involving a loss of derivatives:

Theorem 4.3. *Assume* (4.1) *with* (DUE). *Consider an integer $\ell \geq 0$. Assume that for some $h_0 > 0$ and $\gamma \in [0,2)$ (or $\gamma \in [1,2)$ if X is bounded) the operator $T_t(H) := e^{itH}\psi_{2\ell}(h^2H)$ satisfies Property $(H_m(|t|^{-\frac{d}{2}}))$ for some $m \geq \frac{d}{2}$ and every t satisfying*

$$h^2 \leq |t| \leq h^\gamma \qquad \text{and} \quad h \leq h_0.$$

Then for all pair (p, q) satisfying (1.1) *with $q \neq +\infty$ we have*

$$\left(\int_{-1}^{1} \left\|e^{itH}\psi_{2\ell}\big(h^2H\big)f\right\|_{L^q}^p dt\right)^{\frac{1}{p}} \lesssim h^{-\frac{\gamma}{p}} \left\|\psi_{\ell,\frac{1}{2}}\big(h^2H\big)f\right\|_{L^2}.$$

Remark 4.4.

(1) Following the arguments of Proposition 2.19, if $e^{itH}\psi_{2\ell}(h^2H)$ satisfies Property $(H_m(|t|^{-\frac{d}{2}}))$ for some integer $\ell \geq 0$ then $e^{itH}\psi_{2\ell'}(h^2H)$ also satisfies Property $(H_m(|t|^{-\frac{d}{2}}))$ for every integer $\ell' \geq \ell$;

(2) The case $\gamma \geq 2$ is easy (as explained in the Introduction). When X is bounded, one cannot expect $\gamma = 0$ because of the example of a constant initial data (see Introduction).

Proof of Theorems 4.2 *and* 4.3. We only detail the proof of Theorem 4.3 which is slightly more technical, we let the minor modifications to the reader to prove Theorem 4.2.

Fix an interval $J \subset [-1,1]$ of length $|J| = h^\gamma$ and consider

$$U(t) = \mathbb{1}_J(t) e^{itH} \psi_{\ell,\frac{1}{2}}\big(h^2 H\big).$$

We aim to apply Theorem 4.1 with $\sigma = \frac{d}{2}$ and a suitable large enough integer M (defining the Hardy space). So fix this integer $M \geq \frac{3m}{4}$ large enough which allows us to consider atoms and Hardy space, related to this parameter and we have $m \in [\frac{d}{2}, \frac{4M}{3}]$ as required in Theorem 3.5.

Since $x \mapsto e^{itx}\psi_{\ell,\frac{1}{2}}(x) \in L^\infty(\mathbb{R}_+)$ is uniformly bounded, with respect to t, then Theorem 2.2 yields that

$$\sup_{t>0} \|U(t)f\|_{L^2} = \Big\| \mathbb{1}_J(t) e^{itH} \psi_{\ell,\frac{1}{2}}\big(h^2 H\big) f \Big\|_{L^2} \lesssim \|f\|_{L^2},$$

which is (4.2).

Then let us check (4.4). We have

$$\begin{aligned} U(t)U(s)^* &= \mathbb{1}_J(t)\mathbb{1}_J(s) e^{itH} \psi_{\ell,\frac{1}{2}}\big(h^2 H\big) \Big(e^{isH} \psi_{\ell,\frac{1}{2}}\big(h^2 H\big)\Big)^* \\ &= \mathbb{1}_J(t)\mathbb{1}_J(s) T_{t-s}(H), \end{aligned}$$

where we used that H is self-adjoint and $|\psi_{\ell,\frac{1}{2}}|^2 = \psi_{2\ell}$. Since J has a length equal to h^γ then $U(t)U(s)^*$ is vanishing or else $|t-s| \leq h^\gamma$. In this last case, $U(t)U(s)^*$ satisfies Property $(H_m(|t-s|^{-\frac{d}{2}}))$, by assumption. Hence, by Theorem 3.5, we deduce that

$$\|U(t)U(s)^* f\|_{(H^1)^*} \lesssim \frac{1}{|t-s|^{\frac{d}{2}}} \|f\|_{H^1},$$

which is (4.4). Let us check (4.5) in case X is bounded: similarly since the Schrödinger propagators are unitary in L^2, we have

$$\|U(t)U(s)^*\|_{L^{p'} \to L^2} \leq \Big\| \psi\big(h^2 H\big) \Big\|_{L^{p'} \to L^2}$$

with $|t-s| \leq h^\gamma \lesssim 1$. Recall that for all $p' \in [1,2)$:

$$\Big\| \psi\big(h^2 H\big) f \Big\|_{L^p} \lesssim h^{-d\left(\frac{1}{p'} - \frac{1}{p}\right)} \|f\|_{L^{p'}}.$$

By a TT^* argument we have:

$$\Big\| \psi\big(h^2 H\big) \Big\|_{L^{p'} \to L^p} = \Big\| \psi\big(h^2 H\big) \Big\|^2_{L^{p'} \to L^2}.$$

Hence

$$\|U(t)U(s)^*\|_{L^{p'}\to L^2} \lesssim h^{-\frac{d}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)} \le |t-s|^{-\frac{d}{2\gamma}\left(\frac{1}{p'}-\frac{1}{p}\right)} \lesssim |t-s|^{-\frac{d}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)}$$

as soon as $\gamma \ge 1$.

Thus we can apply Theorem 4.1. For all pair (p,q) satisfying (1.1) with $q \neq +\infty$, then

$$\left(\int_{\mathbb{R}} \|U(t)g\|_{L^q}^p dt\right)^{\frac{1}{p}} \lesssim \|g\|_{L^2}.$$

That is

$$\left(\int_J \left\|e^{itH}\psi_{\ell,\frac{1}{2}}\big(h^2H\big)g\right\|_{L^q}^p dt\right)^{\frac{1}{p}} \lesssim \|g\|_{L^2}.$$

Take $g=\psi_{\ell,\frac{1}{2}}(h^2H)f$ then $\psi_{\ell,\frac{1}{2}}(h^2H)g=\psi_{2\ell}(h^2H)f$ and so

$$\left(\int_J \left\|e^{itH}\psi_{2\ell}\big(h^2H\big)f\right\|_{L^q}^p dt\right)^{\frac{1}{p}} \lesssim \left\|\psi_{\ell,\frac{1}{2}}\big(h^2H\big)f\right\|_{L^2}. \tag{4.6}$$

We write $[-1,1]=\bigcup_{k=1}^N J_k$, where J_k are disjoint intervals with a length smaller than h^γ, so the number of intervals satisfies $N \lesssim \frac{1}{h^\gamma}$.

Hence, by (4.6)

$$\int_{-1}^1 \left\|e^{itH}\psi_{2\ell}\big(h^2H\big)f\right\|_{L^q}^p dt \lesssim \sum_{k=1}^N \int_{J_k} \left\|e^{itH}\psi_{2\ell}\big(h^2H\big)f\right\|_{L^q}^p dt \lesssim N \left\|\psi_{\ell,\frac{1}{2}}\big(h^2H\big)f\right\|_{L^2}^p,$$

and so

$$\left(\int_{-1}^1 \left\|e^{itH}\psi_{2\ell}\big(h^2H\big)f\right\|_{L^q}^p dt\right)^{\frac{1}{p}} \lesssim \frac{1}{h^{\frac{\gamma}{p}}} \left\|\psi_{\ell,\frac{1}{2}}\big(h^2H\big)f\right\|_{L^2}. \qquad \square$$

We can now prove the main result of this section: How Property $(H_m(|t|^{-\frac{d}{2}}))$ implies Strichartz estimates with loss of $\frac{\gamma}{p}$ derivatives:

Theorem 4.5. *Assume* (4.1) *with* (DUE). *Consider an integer* $\ell \ge 0$. *Assume that for some* $h_0 > 0$ *and* $\gamma \in [0,2)$ *the operator* $T_t(H) := e^{itH}\psi_{2\ell}(h^2H)$ *satisfies Property* $(H_m(|t|^{-\frac{d}{2}}))$ *for some* $m \ge \frac{d}{2}$ *and every* t *satisfying*

$$h^2 \le |t| \le h^\gamma \qquad \text{and} \quad h \le h_0.$$

Then for all pair (p,q) *satisfying* (1.1) *with* $q \neq +\infty$, *every solution* $u=e^{itH}u_0$ *of the problem*

$$\begin{cases} i\partial_t u + Hu = 0 \\ u_{|t=0} = u_0 \end{cases}$$

satisfies

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{\gamma}{p},2}}.$$

Remark 4.6. We can consider more regular initial data, in the sense that if for some $\delta > 0$

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - \delta,$$

then we have

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\delta+\frac{\gamma}{p},2}}.$$

Proof. Apply Theorem 2.8 to $u(t) = e^{itH}u_0$

$$\|u(t)\|_{L^q} \lesssim \|\varphi(H)u(t)\|_{L^q} + \left\| \left(\int_0^{h_0} |\psi_{2\ell}(s^2H)u(t)|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^q}.$$

The function φ is also depending of the parameters h_0, ℓ. We omit this dependence. Take the $L^p([-1,1])$ norm in time of that expression. Minkowski inequality leads to

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \underbrace{\|\varphi(H)u\|_{L^p([-1,1],L^q)}}_{=I} + \underbrace{\left\| \left(\int_0^{h_0} \|\psi_{2\ell}(s^2H)u\|_{L^q}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^p([-1,1])}}_{=II}.$$

Then (UE) with (4.1) yields that $\varphi(H)$ has a kernel satisfying Gaussian pointwise estimate (2.2) at the scale 1 (or more precisely h_0 but we forget this dependence) so is in particular bounded from L^2 to L^q (since $q \geq 2$) and so

$$I \lesssim \left\| e^{itH}u_0 \right\|_{L^p([-1,1],L^2)} \lesssim \|u_0\|_{L^2} \lesssim \|u_0\|_{W^{\frac{\gamma}{p},2}},$$

because the Schrödinger group is an isometry on L^2.

Since $p \geq 2$, generalized Minkowski inequality and Theorem 4.3 yield

$$\begin{aligned} II &\leq \left(\int_0^{h_0} \left\| \psi_{2\ell}(s^2H)u \right\|_{L^p([-1,1],L^q)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^{h_0} s^{-\frac{2\gamma}{p}} \left\| \psi_{\ell,\frac{1}{2}}(s^2H)u_0 \right\|_{L^2}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\lesssim \left\| \left(\int_0^{h_0} s^{-\frac{2\gamma}{p}} \left| \psi_{\ell,\frac{1}{2}}(s^2H)u_0 \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^2} \lesssim \|u_0\|_{W^{\frac{\gamma}{p},2}}, \end{aligned}$$

where we used $\ell > \frac{\gamma}{p}$ (since $\ell \geq 1, \gamma \in [0,2)$ and $p \geq 2$) and the fact that

$$s^{-\frac{\gamma}{p}} \psi_{\ell,\frac{1}{2}}\big(s^2 H\big) = \psi_{\ell-\frac{\gamma}{2p},\frac{1}{2}}\big(s^2 H\big) H^{\frac{\gamma}{2p}}$$

with Theorem 2.8. Finally, we get

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{\gamma}{p},2}}. \qquad \square$$

5. Dispersive estimates for the Schrödinger operator through wave operator

In this section, we show that a suitable dispersive property on the wave propagator imply dispersive estimates for the Schrödinger unitary group. Namely, a refined version of the finite speed propagation property for waves implies Property $(H_m(A))$.

5.1. Dispersive estimates from wave to Schrödinger propagators

We recall that we want to obtain

$$\|T_t(H)\|_{H^1 \to \mathrm{BMO}} \lesssim |t|^{-\frac{d}{2}},$$

where $T_t(H) = e^{itH}\psi_{2\ell}(h^2 H)$ for t belonging to an interval, as large as possible. In regard of the previous section, it suffices to check that $e^{itH}\psi_{2\ell}(h^2 H)$ satisfies Property $(H_m(|t|^{-\frac{d}{2}}))$ (for some parameters ℓ, m, γ, h_0), which may be written with (4.1) as: for every balls B_r, $\widetilde{B_r}$

$$\left\| e^{itH}\psi_{2\ell}\big(h^2 H\big)\psi_m\big(r^2 H\big) \right\|_{L^2(B_r) \to L^2(\widetilde{B_r})} \lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}}. \tag{5.1}$$

We aim to use the Hadamard formula, which describes how the Schrödinger propagator may be built using the wave propagator. Let us quickly recall it: the Cauchy formula gives that for any $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$

$$a^{-\frac{1}{2}} e^{-\frac{\xi^2}{2a}} = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} e^{-\frac{ax^2}{2}}\, dx.$$

Using imparity and noting $z = \frac{1}{2a}$, we get

$$e^{-z\xi^2} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \cos(s\xi) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{z}}.$$

Since H is a self-adjoint nonnegative operator admitting a L^∞-functional calculus, one deduces the Hadamard transmutation formula:

$$e^{-zH} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \cos\left(s\sqrt{H}\right) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{z}}. \tag{5.2}$$

We now give a suitable condition on the wave propagators, under which (5.1) can be proved through (5.2). The next section aims to check that this assumption is satisfied in well-known situations as Euclidean space or smooth Riemannian manifolds.

Assumption 5.1. There exists $\kappa \in (0,\infty]$ and an integer m_0 such that for every $s \in (0,\kappa)$ we have: for every $r > 0$, every balls B_r, $\widetilde{B_r}$ of radius r then

$$\left\| \cos\left(s\sqrt{H}\right)\psi_{m_0}\left(r^2 H\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim \left(\frac{r}{s+r}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}},$$

where $L = d(B_r, \widetilde{B_r})$.

Remark 5.2. Using the same arguments as in Proposition 2.19, one can show that if Assumption 5.1 is true for an integer m_0 then it also holds for every integer $m \geq m_0$.

The main result of this section is the following:

Theorem 5.3. *Suppose* (4.1) *with* $d > 1$, (*DUE*) *and Assumption* 5.1 *with* $\kappa = \infty$. *Then for every integer* $m \geq \max(\frac{d}{2}, m_0 + \lceil\frac{d-1}{2}\rceil)$ (*where the integer* m_0 *is the one given by Assumption* 5.1) *we have for every* $t \in \mathbb{R}^*$

$$\left\| e^{itH}\psi_m\left(r^2 H\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}}, \tag{5.3}$$

where the implicit constant only depends on integers m, m'. *Consequently,* e^{itH} *satisfies Property* $(H_m(|t|^{-\frac{d}{2}}))$ *for every* $t \in \mathbb{R}^*$.

Theorem 5.4. *Suppose* (4.1) *with* $d > 1$, (*DUE*) *and Assumption* 5.1 *with* $\kappa \in (0,\infty)$. *Then for every* $\varepsilon > 0$, *every* $h > 0$ *with* $h^2 \leq |t| \leq h^{1+\varepsilon}$, *and for every integers* $m' \geq 0$ *and* $m \geq \max(\frac{d}{2}, m_0 + \lceil\frac{d-1}{2}\rceil)$ (*where the integer* m_0 *is the one given by Assumption* 5.1) *we have*

$$\left\| e^{itH}\psi_{m'}\left(h^2 H\right)\psi_m\left(r^2 H\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}}, \tag{5.4}$$

where the implicit constant only depends on $\varepsilon > 0$ *and integers* m, m'. *Consequently,* $e^{itH}\psi_{m'}(h^2H)$ *satisfies Property* $(H_m(|t|^{-\frac{d}{2}}))$ *for every* $h^2 \leq |t| \leq h^{1+\varepsilon}$ *and every* $\varepsilon > 0$.

Proof of Theorems 5.3 *and* 5.4. We only prove Theorem 5.4, which is more difficult and let the reader to check that the exact same proof allows us to get Theorem 5.3, which is indeed easier since the quantity I_κ (defined later in the proof) is vanishing.

Step 1: Some easy reductions

Remark that the case $r \geq \sqrt{|t|}$ is easy via the bounded functional calculus, indeed

$$\begin{aligned}&\left\| e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\\ &\leq \left\| e^{it\cdot}\psi_{m'}(h^2\cdot)\psi_m(r^2\cdot)\right\|_{L^\infty(\mathbb{R}_+)} \lesssim 1 \lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}}.\end{aligned}$$

So now we only restrict our attention and assume that $r^2 \leq |t|$.

Then assume that (5.4) is proved for every $h \in (0,r]$. We aim to check that it also holds for $h > r$. So fix balls B_r and $\widetilde{B_r}$ of radius $r < h$. It comes

$$\begin{aligned}&\left\| e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\\ &\lesssim \frac{r^{2m}}{\left(\frac{h^2}{2}+r^2\right)^m}\left\| e^{itH}\psi_{m'}\left(\frac{h^2}{2}H\right)\psi_m\left(\left(\frac{h^2}{2}+r^2\right)H\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\\ &\lesssim \left(\frac{r}{h}\right)^{2m}\left\| e^{itH}\psi_{m'}\left(\frac{h^2}{2}H\right)\psi_m\left(\left(\frac{h^2}{2}+r^2\right)H\right)\right\|_{L^2(B_\rho)\to L^2(\widetilde{B_\rho})},\end{aligned}$$

where $\rho = \sqrt{\frac{h^2}{2}+r^2} \geq r$, $\rho \simeq h$ and we write $B_\rho = \frac{\rho}{r}B_r$ the dilated ball (similar notation for $\widetilde{B_\rho}$). Using (5.4) at the scale ρ (since $\rho \geq h/\sqrt{2}$) yields

$$\begin{aligned}\left\| e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} &\lesssim \left(\frac{r}{h}\right)^{2m}\left(\frac{\rho^2}{|t|}\right)^{\frac{d}{2}}\\ &\lesssim \left(\frac{r}{h}\right)^{2m}\left(\frac{h^2}{|t|}\right)^{\frac{d}{2}}\\ &\lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}},\end{aligned}$$

where we have used that $m \geq d/2$ and (since $r < h$)

$$\frac{r^{2m}}{h^{2m}}h^d = r^d\frac{r^{2m-d}}{h^{2m-d}} \leq r^d.$$

So as soon as (5.4) will be proved for $h \leq r$, then the other case immediately follows.

Consequently, we can restrict our study to $h \leq r$ and $r^2 \leq |t|$, that we now assume for the sequel.

For an integer $m' \neq 0$, we have

$$e^{itH}\psi_{m'}\big(h^2H\big)\psi_m\big(r^2H\big) = \left(\frac{h^2}{r^2}\right)^{m'} e^{itH}e^{-h^2H}\psi_{m'+m}\big(r^2H\big).$$

Using $h \leq r$, it comes

$$\begin{aligned}&\left\| e^{itH}\psi_{m'}\big(h^2H\big)\psi_m(r^2H)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\\ &\lesssim \left\| e^{(it-h^2)H}\psi_{m'+m}\big(r^2H\big)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}.\end{aligned}$$

So if (5.4) is proved for $m' = 0$ and some integer m then by Theorem 2.20, it also holds for $m' = 0$ and any integer $m'' \geq m$. Hence, by the previous observation, (5.4) will hold for every $m' = m'' - m \geq 0$.

Finally, we can restrict our attention to prove (5.4) for $m' = 0$ with $h \leq r$ and $r^2 \leq |t|$, which is now supposed for the rest of the proof.

Step 2: Decomposition into three regimes

We fix the parameter h and consider $e^{itH}e^{-h^2H} = e^{-zH}$ with $z = h^2 - it$. By the representation (5.2), it comes

$$e^{-zH} = \int_0^\infty \cos(s\sqrt{H})e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{\pi z}}.$$

We split this integral into three ranges. Let us consider a smooth cut-off function $\chi \in C^\infty(\mathbb{R}_+)$ such that $\begin{cases} 0 \leq \chi \leq 1 \\ \chi(x) = 1 \text{ if } x \in [0, \frac{|t|}{r}] \\ \chi(x) = 0 \text{ if } x \in [\frac{2|t|}{r}, +\infty] \end{cases}$, with $\forall n \in \mathbb{N}, \|\chi^{(n)}\|_{L^\infty} \lesssim \left(\frac{r}{|t|}\right)^n$.

We split the integral into three terms

$$e^{-zH} = \int \chi(s)\cos\big(s\sqrt{H}\big)e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{\pi z}} + \int_{\frac{|t|}{r}}^{\kappa}(1-\chi(s))\cos\big(s\sqrt{H}\big)e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{\pi z}} + I_\kappa(H),$$

where $I_\kappa = 0$ if $\kappa = \infty$ and else

$$I_\kappa(H) := \int_\kappa^\infty (1-\chi(s))\cos\big(s\sqrt{H}\big)e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{\pi z}}.$$

Step 3: The two last regimes

The second term is estimated using Assumption 5.1 as follows (we recall that $z = h^2 - it$ so that $|z| \simeq |t|$):

$$\begin{aligned}
&\left\| \int_{\frac{|t|}{r}}^{\kappa} (1-\chi(s)) \cos\left(s\sqrt{H}\right) \psi_m\left(r^2 H\right) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{z}} \right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \\
&\lesssim \int_{\frac{|t|}{r}}^{\kappa} \left(\frac{r}{s}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}} \frac{ds}{\sqrt{|t|}} \\
&\lesssim \int_{\frac{|t|}{r}}^{\kappa} \left(\frac{r}{\frac{|t|}{r}}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}} \frac{ds}{\sqrt{|t|}} \\
&\lesssim \int_{0}^{\infty} \left(\frac{r^2}{|t|}\right)^{\frac{d-1}{2}} (1+u)^{-\frac{d+1}{2}} \frac{r\,du}{\sqrt{|t|}} \\
&\lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}}.
\end{aligned}$$

The last term $I_\kappa(H)$ is estimated by only using the L^2-boundedness of the wave propagator:

$$\begin{aligned}
&\|I_\kappa(H)\psi_m\left(r^2H\right)\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \\
&\lesssim \int_{\kappa}^{+\infty} \left\| \cos\left(s\sqrt{H}\right)\psi_m\left(r^2H\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} e^{-\frac{s^2}{4}\operatorname{Re}(\frac{1}{z})} \frac{ds}{\sqrt{|z|}} \\
&\lesssim \int_{\kappa\sqrt{\operatorname{Re}(\frac{1}{4z})}}^{+\infty} e^{-u^2} \frac{du}{\sqrt{\operatorname{Re}\left(\frac{1}{z}\right)}\sqrt{|z|}} \\
&\lesssim \left(\int_0^{+\infty} e^{-\frac{u^2}{2}}\,du\right) e^{-\frac{\kappa^2 \operatorname{Re}(\frac{1}{4z})}{2}} \left(\sqrt{\operatorname{Re}\left(\frac{1}{z}\right)}\sqrt{|t|}\right)^{-1}.
\end{aligned}$$

Given that $\operatorname{Re}(\frac{1}{z}) = \frac{h^2}{h^4+t^2} \gtrsim \frac{h^2}{t^2}$ (since we assumed $|t| \geq h^2$, see Step 1), we get

$$\left\| I_\kappa(H)\psi_m\left(r^2H\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim \frac{|t|^{\frac{1}{2}}}{h}\left(\frac{h}{|t|}\right)^{-k},$$

for $k > 0$ as large as we want because $\frac{h}{|t|} \gtrsim 1$ (indeed $|t| \leq h^{1+\varepsilon} \leq h$). Note that the implicit constant here may depend on κ.

Since we have reduced the situation to $h \leq r$, it comes

$$\frac{|t|^{\frac{1}{2}}}{h}\left(\frac{|t|}{h}\right)^{k} \lesssim \left(\frac{h}{\sqrt{|t|}}\right)^{d} \lesssim \left(\frac{r}{\sqrt{|t|}}\right)^{d},$$

as soon as $|t|^{\frac{1}{2}+k+\frac{d}{2}} \leq h^{1+k+d}$, *i.e.*, $|t| \leq h^{\frac{1+k+d}{\frac{1}{2}+k+\frac{d}{2}}} \leq h^{1+\frac{\frac{1}{2}+\frac{d}{2}}{\frac{1}{2}+k+\frac{d}{2}}}$ which is true for k large enough since $|t| \leq h^{1+\varepsilon}$.

So we have obtained the desired bound for the two last terms. It remains to study the first and more difficult one.

Step 4: The first regime

We aim to use integration by parts in s. For all integer $n \geq 0$, all $s > 0$ and $\mathrm{Re}(z) > 0$, we have

$$\partial_s^n\left(e^{-\frac{s^2}{4z}}\right) = e^{-\frac{s^2}{4z}}\left(c_n\frac{s^n}{z^n} + c_{n-1}\frac{s^{n-2}}{z^{n-1}} + \ldots + c_{n-2\lfloor\frac{n}{2}\rfloor}\frac{s^{n-2\lfloor\frac{n}{2}\rfloor}}{z^{n-\lfloor\frac{n}{2}\rfloor}}\right),$$

where $(c_j)_j$ are numerical constants. Making $2n$ integrations by parts gives (as soon as $m \geq n$)

$$\begin{aligned}
&\int_0^\infty \cos\left(s\sqrt{H}\right)\psi_m\left(r^2H\right)\chi(s)e^{-\frac{s^2}{4z}}\,ds\\
&= \int_0^\infty \frac{\cos\left(s\sqrt{H}\right)}{H^n}\psi_m\left(r^2H\right)\partial_s^{2n}\left[\chi(s)e^{-\frac{s^2}{4z}}\right]ds\\
&= \int_0^\infty \cos\left(s\sqrt{H}\right)r^{2n}\psi_{m-n}\left(r^2H\right)\sum_{k=0}^{2n} c_k\chi^{(2n-k)}(s)\partial_s^k\left(e^{-\frac{s^2}{4z}}\right)ds\\
&= \int_0^\infty \cos\left(s\sqrt{H}\right)r^{2n}\psi_{m-n}\left(r^2H\right)\sum_{k=0}^{2n}\chi^{(2n-k)}(s)e^{-\frac{s^2}{4z}}\\
&\quad\cdot\left(c_k\frac{s^k}{z^k} + \ldots + c_{n-2\lfloor\frac{n}{2}\rfloor}\frac{s^{k-2\lfloor\frac{k}{2}\rfloor}}{z^{k-\lfloor\frac{k}{2}\rfloor}}\right)ds,
\end{aligned}$$

where c_j always denotes some numerical constants, possibly changing from line to line. The behaviour of the sum over k is governed by its two extremal terms (that is $k = 0$ and $k = 2n$ where we only keep the first and last terms of the sum) which leads us to (since $|z| \simeq |t|$)

$$\begin{aligned}
&\left\|\int_0^{+\infty}\cos\left(s\sqrt{H}\right)\psi_m\left(r^2H\right)\chi(s)e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{z}}\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\\
&\lesssim \int_0^{2\frac{|t|}{r}}\|\cos\left(s\sqrt{H}\right)\psi_{m-n}\left(r^2H\right)\|_{L^2(B_r)\to L^2(\widetilde{B_r})}r^{2n}\\
&\quad\cdot\left[\left(\frac{r}{|t|}\right)^{2n} + \left(\frac{s}{|t|}\right)^{2n} + \frac{1}{|t|^n}\right]\frac{ds}{\sqrt{|t|}}\\
&\lesssim \int_0^{2\frac{|t|}{r}}\left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}\left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}r^{2n}\left[\left(\frac{r}{|t|}\right)^{2n} + \left(\frac{s}{|t|}\right)^{2n} + \frac{1}{|t|^n}\right]\frac{ds}{\sqrt{|t|}},
\end{aligned}$$

where we used Assumption 5.1 (this is allowed if $m-n \geq m_0$) and $L := d(B_r, \widetilde{B_r})$. If $n = \lceil \frac{d-1}{2} \rceil$, then firstly

$$\begin{aligned}
&\int_0^{2\frac{|t|}{r}} \left(\frac{r}{s+r}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}} \left(\frac{r^2}{|t|}\right)^{2n} \frac{ds}{\sqrt{|t|}} \\
&\leq \int_0^{+\infty} (1+u)^{-\frac{d+1}{2}} \frac{r\,du}{\sqrt{|t|}} \left(\frac{r^2}{|t|}\right)^{2n} \\
&\lesssim \left(\frac{r^2}{|t|}\right)^{2n+\frac{1}{2}} \leq \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}},
\end{aligned}$$

since $d > 1$ and $2n + \frac{1}{2} \geq \frac{d}{2}$. For the second term, we have

$$\begin{aligned}
&\int_0^{\frac{|t|}{r}} \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}} \left(\frac{rs}{|t|}\right)^{2n} \frac{ds}{\sqrt{|t|}} \\
&\leq \frac{r^{\frac{d-1}{2}+2n}}{|t|^{2n}} \int_0^{\frac{|t|}{r}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}} s^{2n-\frac{d-1}{2}} \frac{ds}{\sqrt{|t|}} \\
&\lesssim \frac{r^{\frac{d-1}{2}+2n}}{|t|^{2n}} \left(\frac{|t|}{r}\right)^{2n-\frac{d-1}{2}} \int_0^{+\infty} (1+u)^{-\frac{d+1}{2}} \frac{r\,du}{\sqrt{|t|}} \lesssim \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}},
\end{aligned}$$

since $2n - \frac{d-1}{2} \geq 0$. And for the third and last term, it comes

$$\begin{aligned}
\int_0^{\frac{|t|}{r}} \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}} \frac{r^{2n}}{|t|^n} \frac{ds}{\sqrt{|t|}} &\leq \left(\frac{r^2}{|t|}\right)^{n} \int_0^{+\infty} (1+u)^{-\frac{d+1}{2}} \frac{r\,du}{\sqrt{|t|}} \\
&\lesssim \left(\frac{r^2}{|t|}\right)^{n+\frac{1}{2}} \leq \left(\frac{r^2}{|t|}\right)^{\frac{d}{2}},
\end{aligned}$$

since $n + \frac{1}{2} \geq \frac{d}{2}$. The intermediate terms in the integrations by parts have an intermediate behavior. We point out that these last computations required $m - n \geq m_0$ which is true, since $m \geq m_0 + \left\lceil \frac{d-1}{2} \right\rceil$ and $n = \lceil \frac{d-1}{2} \rceil$.

That concludes the proof, since each of the three terms have a satisfying bound. $\square$

5.2. A digression about these dispersive properties and the spectral measure

Let us assume Assumption 5.1 for $\kappa = 1$.

Following the same reasoning as in Sections 3 and 4, it comes that the assumed inequality

$$\left\| \cos\left(s\sqrt{H}\right)\psi_{m_0}(r^2 H)\right\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{s+r}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}$$

allows us to prove that $\cos(s\sqrt{H})$ is bounded from the Hardy space H^1 to BMO (built with some parameter M sufficiently large) with

$$\left\| \cos(s\sqrt{H})\psi_1\left(r^2 H\right)\right\|_{H^1\to \mathrm{BMO}} \lesssim r^{-\frac{d+1}{2}}(s+r)^{-\frac{d-1}{2}}, \quad \forall |s|\leq 1.$$

That corresponds to the $H^1 \to$ BMO counterpart of more classical $L^1 \to L^\infty$ dispersive estimates. Following interpolation and Keel-Tao's argument (as detailed previously) for the wave propagator, it allows us to deduce Strichartz estimates for the wave equations: for exponents p, q *wave-admissible* and $\delta \geq 0$ satisfying

$$\frac{1}{p}+\frac{d}{q}=\frac{d}{2}-\delta,$$

every solution $u = \cos(t\sqrt{H})u_0$ of the problem

$$\begin{cases} \partial_{tt}^2 u + Hu = 0 \\ u_{|t=0} = u_0 \\ \partial_t u_{|t=0} = 0, \end{cases}$$

satisfies:

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\delta,2}}. \tag{5.5}$$

Such Strichartz estimates for the wave equation, allow us to deduce some sharp $L^2 - L^q$ estimates for the spectral projector (introduced by Sogge [55]), as detailed by Smith in [54]. Without details, we just sketch the proof of [54] to check that it can be adapted to this very general setting.

Indeed, consider $\lambda > 0$ and the spectral projector

$$\Pi_\lambda = \mathbb{1}_{[\lambda,\lambda+1)}\left(\sqrt{H}\right).$$

Define the function

$$\rho_\lambda(x) := \int_{-1}^{1} e^{-it\lambda}\cos(tx)dt$$

which a direct computation gives

$$\rho_\lambda(x) = \frac{\sin(\lambda-x)}{\lambda-x}+\frac{\sin(\lambda+x)}{\lambda+x}.$$

So we observe that $\rho_\lambda(x) \in [\frac{1}{2}, 2]$ if $x \in [\lambda, \lambda+1)$. As a consequence, by bounded L^2-functional calculus, we deduce that for $f \in L^2$

$$\Pi_\lambda(f) = \int_{-1}^{1} e^{-it\lambda} \cos\left(t\sqrt{H}\right) \left[\rho_\lambda(\sqrt{H})^{-1} \Pi_\lambda f\right] dt,$$

with $\rho_\lambda(\sqrt{H})^{-1}\Pi_\lambda$ a uniformly L^2-bounded operator (and also in any L^2 Sobolev space since it commutes with H).

By applying (5.5), we deduce that for $q \in [\frac{2(d+1)}{d-1}, \infty)$

$$\begin{aligned}\|\Pi_\lambda(f)\|_{L^q} &\lesssim \left\| \cos\left(t\sqrt{H}\right) \left[\rho_\lambda\left(\sqrt{H}\right)^{-1} \Pi_\lambda f\right] \right\|_{L^2([-1,1],L^q)} \\ &\lesssim \left\| \rho_\lambda\left(\sqrt{H}\right)^{-1} \Pi_\lambda f \right\|_{W^{\delta(q),2}} \\ &\lesssim \|\Pi_\lambda f\|_{W^{\delta(q),2}} \lesssim \lambda^{\delta(q)} \|f\|_{L^2},\end{aligned}$$

where $\delta(q)$ is given by

$$\frac{1}{2} + \frac{d}{q} = \frac{d}{2} - \delta(q).$$

If in addition, we assume the so-called "square function estimates" (see [54]), then by interpolating with the trivial $L^2 - L^2$ bound, Strichartz estimates yield (as explained in [54]):

$$\|\Pi_\lambda\|_{L^2 \to L^q} \lesssim \begin{cases} \lambda^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} & \text{if } 2 \leq q \leq 2\dfrac{d+1}{d-1} \\ \lambda^{d\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2}} & \text{if } q \geq 2\dfrac{d+1}{d-1}. \end{cases} \tag{5.6}$$

Let us point out that if now we only assume Assumption 5.1 for $\kappa = \infty$, then by combining Theorems 4.5 and 5.3 we get free dispersive estimates without loss of derivatives: for $p \in (1, 2]$ then

$$\left\|e^{itH}\right\|_{L^p \to L^{p'}} \lesssim |t|^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{p'}\right)},$$

uniformly with respect to $t \in \mathbb{R}$. Then if the operator H (or $\sqrt{H}$) has a spectral measure with a Radon-Nicodym derivative, then following [12, Corollary 3.3], we know that *Restriction estimates* hold which are:

$$\left\| \frac{dE_H(\lambda)}{d\lambda} \right\|_{L^p \to L^{p'}} \lesssim \lambda^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})-1},$$

where $E_H(\lambda)$ is the spectral measure of H and $p \in [1, \frac{2d}{d+2})$. We also have other estimates for higher order derivatives and we refer to [12] for more details. Such estimates give in particular for $\lambda \geq 1$

$$\begin{aligned} \|\Pi_\lambda\|_{L^p \to L^{p'}} &\lesssim \int_{\lambda^2}^{(\lambda+1)^2} \left\| \frac{dE_H(s)}{ds} \right\|_{L^p \to L^{p'}} ds \\ &\lesssim \int_{\lambda^2}^{(\lambda+1)^2} s^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} \frac{ds}{s} \\ &\lesssim \lambda^{d(\frac{1}{p}-\frac{1}{p'})-1} \lesssim \lambda^{2d(\frac{1}{p}-\frac{1}{2})-1}. \end{aligned} \tag{5.7}$$

Thus, Assumption 5.1 alone allows to recover the estimate in (5.6) but with a smaller range for $q = p'$ (than the one obtained by assuming the "square function estimates"). Indeed the range in (5.6) is given by the sharp critical exponent $1 \leq p \leq 2\frac{d+1}{d+3}$.

6. The Euclidean and Riemannian cases

To enhance the legitimacy of Assumption 5.1, we check its validity for the Laplace-Beltrami operator $H = -\Delta$ in four situations:

- The Euclidean space $X = \mathbb{R}^d$ with $\kappa = \infty$;
- Any smooth compact Riemannian manifold of dimension d and κ is given by the injectivity radius;
- Any smooth noncompact Riemannian manifold of dimension d, with C_b^∞-geometry and κ given by the geometry;
- Smooth perturbation of the Euclidean space $X = \mathbb{R}^d$, $H = -\frac{1}{\rho}\nabla \cdot (A\nabla \cdot)$ (for uniformly nondegenerate function ρ and matrix A, with bounded derivatives) which is a self-adjoint operator on $\mathbb{R}^d$, equipped with the measure $d\mu = \rho dx$, with $\kappa < \infty$ (given by A and ρ).

Proposition 6.1. *In these four previous cases, Assumption* 5.1 *is satisfied.*

The proof is based on the following properties (which are a refinement of the finite speed propagation property): for B, $\widetilde{B}$ two balls of radius r, then with $L = d(B, \widetilde{B})$ and $s \in (0, \kappa)$:

- If $L > s + 2r$ then the finite speed propagation property occurs

$$\|\cos(s\sqrt{H})\|_{L^2(B) \to L^2(\widetilde{B})} = 0; \tag{6.1}$$

- If $L \leq s - 10r$ then

$$\|\cos(s\sqrt{H})\|_{L^2(B) \to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(1 + \frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}. \tag{6.2}$$

We refer the reader to the introduction for more details about the finite speed propagation property, which yields (6.1). Property (6.2) is quite standard, see for example [9] for the case of a compact Riemannian manifold (where a short time parametrix is detailed) and Appendix A where we detail computations in the Euclidean situation.

In particular, we partly recover the results of [21,56] (up to a loss $\varepsilon > 0$ as small as we want). Indeed, by combining Proposition 6.1 with Theorems 4.5 and 5.4 (with $\gamma = 1 + \varepsilon$), we have the following:

Corollary 6.2. *Any smooth compact Riemannian manifold or non-compact Riemannian manifold with a C_b^∞ geometry (or as previously for a smooth perturbation of the Euclidean setting with suitable functions ρ, A) satisfy Strichartz estimates with a loss of derivatives $\frac{1}{p} + \varepsilon$, for every $\varepsilon > 0$.*

As a conclusion, we have obtained that as soon as we have suitable (short time) $L^2 - L^2$ microlocalized dispersive properties on the wave equation (Assumption 5.1) then we can obtain their Strichartz estimates and dispersive estimates for Schrödinger equation (with an eventual loss of derivatives if $\kappa < \infty$). We just point out that in the case of a convex subset of the Euclidean space with a boundary, then wave operators for the Dirichlet Laplacian do not satisfy Assumption 5.1 (since there is a loss of $1/4$ in the main exponent), see [45] by Ivanovici, Lebeau and Planchon.

Proof of Proposition 6.1. We detail the proof in the Euclidean case with $\kappa = \infty$. We let the reader to check that everything still holds (up to some change of notation) for a compact Riemannian manifold with κ given by the injectivity radius. Indeed, the proof relies on (6.2) and a precise formulation of the wave kernel around the light cone, which is obtained by the Hadamard parametrix (and has the same form as in the Euclidean case), see [9]. So let us focus on the Euclidean situation.

First, if $s \leq 10r$ then by the finite speed propagation property and Davies-Gaffney estimates, we have

$$\begin{aligned}
&\left\| \cos\left(s\sqrt{H}\right)\psi\left(r^2 H\right) \right\|_{L^2(B)\to L^2(\widetilde{B})} \leq \left\| \psi\left(r^2 H\right) \right\|_{L^2(B)\to L^2(10\widetilde{B})} \\
&\lesssim e^{-\frac{d(B,10\widetilde{B})^2}{4r^2}} \lesssim \left(1 + \frac{d(B,\widetilde{B})}{r}\right)^{-\frac{d+1}{2}} = \left(1 + \frac{L-s+s}{r}\right)^{-\frac{d+1}{2}} \\
&\lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(1 + \frac{|L-s|}{r}\right)^{-\frac{d+1}{2}},
\end{aligned}$$

since $s \lesssim r$, which is the desired estimate.

So we now only focus in the situation where $s \geq 10r$ and consider $(B_k)_k$ a bounded covering of X, by balls of radius r. Let χ_{B_k} be a smooth partition of the unity, adapted to this covering: so χ_{B_k} is supported in $2B_k$, takes values in $[0, 1]$ and satisfies for all $n \in \mathbb{N}$

$$\|\nabla^n \chi_{B_k}\|_{L^\infty} \leq \frac{1}{r^n}. \tag{6.3}$$

We decompose

$$\left\|\cos\left(s\sqrt{H}\right)\psi\left(r^2H\right)\right\|_{L^2(B)\to L^2(\widetilde{B})} \leq \sum_{B_k}\left\|\cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi\left(r^2H\right)\right)\right\|_{L^2(B)\to L^2(\widetilde{B})}.$$

Due to (6.1), the sum is restricted to balls B_k such that $d(B_k, \widetilde{B}) \leq s + 2r$.

Step 1: The case $d(B_k, \widetilde{B}) \leq s - 10r$

Using (6.2) and Davies-Gaffney estimates, it comes

$$\begin{aligned}
&\sum_{d(B_k,\widetilde{B})\leq s-10r}\left\|\cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi\left(r^2H\right)\right)\right\|_{L^2(B)\to L^2(\widetilde{B})} \\
&\leq \sum_{d(B_k,\widetilde{B})\leq s-10r}\left\|\cos\left(s\sqrt{H}\right)\right\|_{L^2(B_k)\to L^2(\widetilde{B})}\left\|\psi\left(r^2H\right)\right\|_{L^2(B)\to L^2(B_k)} \\
&\lesssim \sum_{d(B_k,\widetilde{B})\leq s-10r}\left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}\left(1+\frac{s-d(B_k,\widetilde{B})}{r}\right)^{-\frac{d+1}{2}} e^{-\frac{d(B,B_k)^2}{4r^2}}.
\end{aligned}$$

Note that $s - d(B_k, \widetilde{B}) \geq 10r \geq 0$.

We can evaluate the following sum

$$\sum_k e^{-\frac{d(B,B_k)^2}{4r^2}} \leq \sum_{l=0}^{+\infty} e^{-2^{2l}}\sharp\left\{k,\ \frac{d(B,B_k)}{2r}\sim 2^l\right\} \lesssim \sum_{l=0}^{+\infty}2^{ld}e^{-2^{2l}} < +\infty. \quad (6.4)$$

We distinguish two cases. If $s - d(\widetilde{B}, B_k) \geq \frac{1}{2}|s - d(B, \widetilde{B})| = \frac{1}{2}|s - L|$ then

$$\begin{aligned}
&\sum_{d(B_k,\widetilde{B})\leq s-10r}\left(\frac{r}{s}\right)^{\frac{d-1}{2}}\left(1+\frac{s-d(B_k,\widetilde{B})}{r}\right)^{-\frac{d+1}{2}} e^{-\frac{d(B,B_k)^2}{4r^2}} \\
&\qquad \leq \sum_k\left(\frac{r}{s}\right)^{\frac{d-1}{2}}\left(1+\frac{|s-L|}{2r}\right)^{-\frac{d+1}{2}} e^{-\frac{d(B,B_k)^2}{4r^2}} \\
&\qquad \lesssim \left(\frac{r}{s}\right)^{\frac{d-1}{2}}\left(1+\frac{|s-L|}{r}\right)^{-\frac{d+1}{2}}.
\end{aligned}$$

If $s - d(\widetilde{B}, B_k) \leq \frac{1}{2}|s - d(B, \widetilde{B})|$ then

$$d(B, B_k) \geq \left|d\left(B_k, \widetilde{B}\right) - d\left(B, \widetilde{B}\right)\right| = \left|(s-L) - \left(s - d\left(\widetilde{B}, B_k\right)\right)\right| \geq \frac{1}{2}|s-L|.$$

Hence

$$\begin{aligned}
&\sum_{d(B_k,\widetilde{B})\le s-10r} \left(\frac{r}{s}\right)^{\frac{d-1}{2}} \left(1+\frac{s-d(B_k,\widetilde{B})}{r}\right)^{-\frac{d+1}{2}} e^{-\frac{d(B,B_k)^2}{4r^2}} \\
&\lesssim \left(\frac{r}{s}\right)^{\frac{d-1}{2}} \sum_{d(B_k,\widetilde{B})\le s-10r} \left(1+\frac{10r}{r}\right)^{-\frac{d+1}{2}} e^{-\frac{d(B,B_k)^2}{8r^2}} \underbrace{e^{-\frac{d(B,B_k)^2}{8r^2}}}_{\le e^{-\frac{|s-L|^2}{16r^2}}} \\
&\lesssim \left(\frac{r}{s}\right)^{\frac{d-1}{2}} \left(1+\frac{|s-L|}{r}\right)^{-\frac{d+1}{2}},
\end{aligned}$$

because for every $x \geq 0, e^{-x^2} \lesssim (1+x)^{-\alpha}$ for all $\alpha > 1$.

Step 2: The case $s - 10r \leq d(B_k, \widetilde{B}) \leq s + 2r$ with an odd dimension $d \geq 3$

In this case, we have to use a sharp expression of the kernel of the wave propagator. It is known that the behavior of the kernel is different according to the parity of the dimension. Let us start with the case of an odd dimension $d \geq 3$. In the Euclidean situation, we have an exact representation of the kernel (see [37], *e.g.*): for every $s \geq 0$ and every sufficiently smooth function g

$$\begin{aligned}
\cos\left(s\sqrt{H}\right)g(x) &= \partial_s \left(\frac{1}{s}\partial_s\right)^{\frac{d-3}{2}} \left(s^{d-2}\int_{|y|=1} g(x+sy)dy\right) \\
&= \sum_{n=0}^{\frac{d-1}{2}} c_n s^n \int_{|y|=1} \partial_s^n (g(x+sy))dy,
\end{aligned}$$

where c_n are some numerical constants.

Consider $g = \chi_{B_k}\psi(r^2 H)f$; then it satisfies the following regularity estimates (with a slight abuse of notation): for every integer $n \geq 0$

$$\left|\partial_s^n\left(\chi_{B_k}(x+sy)\psi\left(r^2 H\right)f(x+sy)\right)\right| \lesssim \frac{1}{r^n}\widetilde{\chi}_{B_k}(x+sy)\widetilde{\psi}\left(r^2 H\right)f(x+sy).$$

Let us explain this point. Indeed, we can control the derivatives of χ_{B_k} by (6.3). It remains to explain the behavior of the derivatives of $\psi(r^2 H)f(x+sy)$. The kernel of the heat semigroup, for $t > 0$, is

$$p_t(x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

Thus for all $r > 0$:

$$\partial_s(p_{r^2}(x+sy,z)) = \frac{1}{(4\pi r^2)^{\frac{d}{2}}} e^{-\frac{|x+sy-z|^2}{4r^2}} \frac{(x+sy-z)y}{2r^2}.$$

Hence

$$
\begin{aligned}
|\partial_s(p_{r^2}(x+sy,z))| &\lesssim \frac{1}{(4\pi r^2)^{\frac{d}{2}}}\frac{1}{2r^2}e^{-\frac{|x+sy-z|^2}{4r^2}}|x+sy-z| \\
&= \frac{1}{r}\frac{1}{(4\pi r^2)^{\frac{d}{2}}}\frac{|x+sy-z|}{2r}e^{-\left(\frac{|x+sy-z|}{2r}\right)^2} \\
&\lesssim \frac{1}{r}\frac{1}{(4\pi r^2)^{\frac{d}{2}}}e^{-\frac{|x+sy-z|^2}{8r^2}},
\end{aligned}
$$

which means that, up to some numerical constants, the n-th derivative of $\psi(r^2H)f(x+sy)$ behaves as $\frac{1}{r^n}\psi(r^2H)f(x+sy)$ in the sense that their kernels have both similar Gaussian pointwise decays. Such a property also holds on a compact smooth Riemannian manifold.

So we have for $f\in L^2(B)$ a function supported on B,

$$
\begin{aligned}
&\left\|\cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi(r^2H)f\right)\right\|_{L^2(\widetilde{B})} \\
&\lesssim \sum_{n=0}^{\frac{d-1}{2}}\left(\frac{s}{r}\right)^n\int_{|y|=1}\|\widetilde{\chi}_{B_k}(x+sy)\widetilde{\psi}(r^2H)f(\cdot+sy)\|_{L^2(\widetilde{B})}dy \\
&\lesssim \sum_{n=0}^{\frac{d-1}{2}}\left(\frac{s}{r}\right)^n\int_{S(0,1)\cap A}\left\|\widetilde{\psi}(r^2H)f\right\|_{L^2(B_k)}dy,
\end{aligned}
$$

where $S(0,1)$ is the unit sphere and $A=\frac{1}{s}(B_k-\widetilde{B})$. Hence from the exponential decay of the kernel of $\widetilde{\psi}(r^2H)$, we get

$$
\begin{aligned}
\left\|\cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi\left(r^2H\right)f\right)\right\|_{L^2(\widetilde{B})} &\lesssim \sum_{n=0}^{\frac{d-1}{2}}\left(\frac{s}{r}\right)^n|S(0,1)\cap A|e^{-c\frac{d(B,B_k)^2}{r^2}}\|f\|_{L^2(B)} \\
&\lesssim \|f\|_{L^2(B)}\sum_{n=0}^{\frac{d-1}{2}}\left(\frac{r}{s}\right)^{d-1-n}e^{-c\frac{d(B,B_k)^2}{r^2}} \\
&\lesssim \|f\|_{L^2(B)}\left(\frac{r}{s}\right)^{\frac{d-1}{2}}e^{-c\frac{d(B,B_k)^2}{r^2}},
\end{aligned}
$$

where we have used that the $(d-1)$-dimensional volume of $S(0,1)\cap A=S(0,1)\cap\frac{1}{s}(B_k-\widetilde{B})$ is equivalent to $\left(\frac{r}{s}\right)^{d-1}$ and $\left(\frac{r}{s}\right)^{d-1-n}\leq\left(\frac{r}{s}\right)^{\frac{d-1}{2}}$. Hence, it remains to evaluate the sum

$$
\sum_{s-10r\leq d(\widetilde{B},B_k)\leq s+2r} e^{-c\frac{d(B,B_k)^2}{r^2}}.
$$

Since
$$d(B,B_k)\geq |d\big(B,\widetilde{B}\big)-d\big(\widetilde{B},B_k\big)|-2r\geq |L-s|-4r.$$
Then
$$|L-s|^2\leq 2\big(d(B,B_k)^2+16r^2\big)$$
that is
$$d(B,B_k)^2\geq \frac{|L-s|^2}{2}-16r^2.$$
Thus, we deduce
$$\begin{aligned}\sum_{\substack{B_k\\ s-10r\leq d(\widetilde{B},B_k)\leq s+2r}} e^{-c\frac{d(B,B_k)^2}{r^2}} &\leq \sum_{B_k} e^{-c\frac{d(B,B_k)^2}{2r^2}}e^{-c\frac{|L-s|^2}{2r^2}}\\ &\lesssim \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}.\end{aligned}$$
In the end, we have obtained that
$$\begin{aligned}&\sum_{s-10r\leq d(\widetilde{B},B_k)\leq s+2r}\left\|\cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi\left(r^2H\right)f\right)\right\|_{L^2(\widetilde{B})}\\ &\lesssim \left(\frac{r}{s}\right)^{\frac{d-1}{2}}\left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}\|f\|_{L^2(B)}\end{aligned}$$
which gives the desired estimate (for an odd dimension).

Step 3: The case $s-10r\leq d(B_k,\widetilde{B})\leq s+2r$ with an even dimension $d\geq 2$
In this case the wave propagator is given by
$$\begin{aligned}\cos\left(s\sqrt{H}\right)g(x)&=\partial_s\left(\frac{1}{s}\partial_s\right)^{\frac{d-2}{2}}\left(s^{d-1}\int_{|y|<1}g(x+sy)\frac{dy}{\sqrt{1-|y|^2}}\right)\\ &=\sum_{n=0}^{\frac{d}{2}}c_ns^n\int_{|y|<1}\partial_s(g(x+sy))\frac{dy}{\sqrt{1-|y|^2}},\end{aligned}$$
with some numerical constants c_n. The same arguments as above give
$$\begin{aligned}&\left\|\cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi\left(r^2H\right)f\right)\right\|_{L^2(\widetilde{B})}\\ &\lesssim \|f\|_{L^2(B)}\sum_{n=0}^{\frac{d}{2}}\left(\frac{s}{r}\right)^n e^{-c\frac{d(B_k,B)^2}{r^2}}\int_{y\in A\cap B(0,1)}\frac{dy}{\sqrt{1-|y|^2}},\end{aligned}$$

where $A = \frac{1}{s}(B_k - \widetilde{B})$. Moreover

$$\begin{aligned}
\int_{A\cap B(0,1)} \frac{dy}{\sqrt{1-|y|^2}} &\leq \int_{A\cap B(0,1-\frac{r}{s})} \frac{dy}{\sqrt{1-|y|^2}} + \int_{1-\frac{r}{s}}^{1} |S(0,\rho)\cap A| \frac{d\rho}{\sqrt{1-\rho^2}} \\
&\leq \int_{A\cap B(0,1)} \left(\frac{s}{r}\right)^{\frac{1}{2}} dy + \left(\frac{r}{s}\right)^{d-1} \int_{1-\frac{r}{s}}^{1} \frac{d\rho}{\sqrt{1-\rho}} \\
&\lesssim \left(\frac{r}{s}\right)^{d-\frac{1}{2}} + \left(\frac{r}{s}\right)^{d-1} \left[\sqrt{1-\rho}\right]_{1-\frac{r}{s}}^{1} \lesssim \left(\frac{r}{s}\right)^{d-\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{s-10r\leq d(\widetilde{B},B_k)\leq s+2r} \left\| \cos\left(s\sqrt{H}\right)\left(\chi_{B_k}.\psi\left(r^2H\right)f\right)\right\|_{L^2(\widetilde{B})} \\
&\lesssim \sum_{B_k} \left(\frac{r}{s}\right)^{\frac{d-1}{2}} e^{-c\frac{d(B,B_k)^2}{r^2}} \|f\|_{L^2(B)} \\
&\lesssim \left(\frac{r}{s}\right)^{\frac{d-1}{2}} \left(1+\frac{|s-L|}{r}\right)^{-\frac{d+1}{2}} \|f\|_{L^2(B)},
\end{aligned}$$

which gives the desired estimate.

Note that since $r \lesssim s$ we have $\frac{r}{s} \lesssim \frac{r}{r+s}$ so in any dimension $d > 1$:

$$\left\|\cos\left(s\sqrt{H}\right)\psi\left(r^2H\right)\right\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}. \quad \square$$

Appendix

A. Wave propagation in the Euclidean setting

In this appendix we aim to check (6.2) in the Euclidean situation, from the exact and global formula giving the wave operators. Let us consider the Euclidean space $X = \mathbb{R}^d$, equipped with its canonical structure and $H = -\Delta$.

Proposition A.1. *For every balls B_r, $\widetilde{B_r}$ of radius $r > 0$ and every $s > 0$, if $L := d(B_r, \widetilde{B_r}) \leq s - 10r$ then*

$$\left\|\cos\left(s\sqrt{H}\right)\right\|_{L^2(B_r)\to L^2(\widetilde{B_r})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(1+\frac{|L-s|}{r}\right)^{-\frac{d+1}{2}}.$$

Proof. Let $f \in L^2(B_r)$. If $d \geq 3$ is odd then the wave propagator is given by

$$\cos\left(s\sqrt{H}\right)f(x) = \sum_{n=0}^{\frac{d-1}{2}} c_n s^n \int_{|y|=1} \partial_s(f(x+sy))\,dy,$$

for some numerical constants c_n. If $x \in \widetilde{B_r}$ and $x + sy \in B_r$ then $y = \frac{x+sy-x}{s} \in \frac{B-\widetilde{B}}{s}$ hence

$$|y| \leq \frac{d(B, \widetilde{B}) + 2r}{s} \leq \frac{s - 8r}{s} < 1. \tag{A.1}$$

Thus

$$\cos\left(s\sqrt{H}\right) f(x) = 0.$$

If $d \geq 2$ is even then the wave propagator is given by

$$\cos\left(s\sqrt{H}\right) f(x) = \partial_s \left(\frac{1}{s}\partial_s\right)^{\frac{d-2}{2}} \left(s^{d-1} \int_{|y|<1} f(x+sy) \frac{dy}{\sqrt{1-|y|^2}}\right).$$

Set

$$I_{n,m} := \int_{|y|<1} f(x+sy) \frac{|y|^{2m}}{(1-|y|^2)^n} \, dy.$$

Since

$$\cos\left(s\sqrt{H}\right) f(x) = \partial_s \left(\frac{1}{s}\partial_s\right)^{\frac{d-2}{2}} \left(s^{d-1} I_{\frac{1}{2},0}\right),$$

we want to evaluate

$$\partial_s I_{n,m} = \int_{|y|<1} \nabla f(x+sy).y \frac{|y|^{2m}}{(1-|y|^2)^n} \, dy.$$

By (A.1) the boundary term in Green's formula vanishes and so

$$\partial_s I_{n,m} = -\int_{|y|<1} \frac{f(x+sy)}{s} \nabla \cdot \left(\frac{y|y|^{2m}}{(1-|y|^2)^n}\right) dy.$$

Consequently, it comes with numerical constants $\alpha_{n,m}, \alpha_{n+1,m+1}$

$$\partial_s I_{n,m} = \frac{1}{s}(\alpha_{n,m} I_{n,m} + \alpha_{n+1,m+1} I_{n+1,m+1}).$$

It follows that (with other coefficients but for simplicity we keep the same notation)

$$\left(\frac{1}{s}\partial_s\right)(s^{d-1} I_{n,m}) = s^{d-3}(\alpha_{n,m} I_{n,m} + \alpha_{n+1,m+1} I_{n+1,m+1}).$$

By iterating, we deduce that for $n = \frac{1}{2}$ and $m = 0$

$$\left(\frac{1}{s}\partial_s\right)^{\frac{d-2}{2}} \left(s^{d-1} I_{\frac{1}{2},0}\right) = s^{d-1-(d-2)} \left(\alpha_{\frac{1}{2},0} I_{\frac{1}{2},0} + \cdots + \alpha_{\frac{1}{2}+\frac{d-2}{2},\frac{d-2}{2}} I_{\frac{1}{2}+\frac{d-2}{2},\frac{d-2}{2}}\right).$$

Hence,

$$\cos\left(s\sqrt{H}\right)f(x) = \alpha_{\frac{1}{2},0}I_{\frac{1}{2},0} + \cdots + \alpha_{\frac{d+1}{2},\frac{d}{2}}I_{\frac{d+1}{2},\frac{d}{2}},$$

where coefficients $\alpha_{n,m}$ are some numerical constants, possibly changing from line to line.

Since $\frac{1}{1-|y|^2} \geq 1$ and $|y| \leq 1$ we have:

$$\begin{aligned}\left\|\cos\left(s\sqrt{H}\right)f\right\|_{L^2(\widetilde{B_r})} &\lesssim \left\|\int_{|y|<1} |f(x+sy)| \frac{dy}{(1-|y|^2)^{\frac{d+1}{2}}}\right\|_{L^2(\widetilde{B_r})} \\ &\lesssim \int_{B(0,1)\cap A} \|f(\cdot+sy)\|_{L^2(\widetilde{B_r})} \frac{dy}{((1+|y|)(1-|y|))^{\frac{d+1}{2}}},\end{aligned}$$

where $A := \frac{1}{s}(B_r - \widetilde{B_r})$ so that $|y| \geq \frac{d(B_r,\widetilde{B_r})-2r}{s}$. Moreover

$$\|f(\cdot+sy)\|_{L^2(\widetilde{B_r})} \leq \|f\|_{L^2(B_r)}.$$

Hence:

$$\begin{aligned}&\left\|\cos\left(s\sqrt{H}\right)\right\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \frac{1}{\left(1-\frac{L-2r}{s}\right)^{\frac{d+1}{2}}}|B(0,1)\cap A| \\ &\lesssim \left(\frac{s-L+2r}{s}\right)^{-\frac{d+1}{2}}\left(\frac{r}{s}\right)^d \lesssim \left(1+\frac{|s-L|}{r}\right)^{-\frac{d+1}{2}}\left(\frac{r}{s}\right)^{\frac{d-1}{2}} \\ &\lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}\left(1+\frac{|s-L|}{r}\right)^{-\frac{d+1}{2}},\end{aligned}$$

where the last inequality holds if $r \leq s$. If $s \leq r$ then use $|B(0,1)\cap A| \leq |B(0,1)| \lesssim 1$ to get the same estimation with $1 \lesssim \frac{r}{r+s}$ instead of $\frac{r}{s}$. □

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