Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 931-947

On the b-ary expansions of $\log\left(1+\frac{1}{a}\right)$ and e

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Abstract. Let $b \geq 2$ be an integer and ξ be an irrational real number. We prove that, if the irrationality exponent of ξ is equal to 2 or slightly greater than 2, then the b-ary expansion of ξ cannot be "too simple", in a suitable sense. Our result applies to, among other classical numbers, to badly approximable numbers, non-zero rational powers of e, and $\log(1+\frac{1}{a})$, provided that the integer a is sufficiently large. It establishes an unexpected connection between the irrationality exponent of a real number and its b-ary expansion.

Mathematics Subject Classification (2010): 11A63 (primary); 11J82, 68R15 (secondary).

1. Introduction and main result

Throughout the present paper, b always denotes an integer greater than or equal to 2 and ξ a real number. There exists a unique infinite sequence $(a_j)_{j\geq 1}$ of integers from $\{0, 1, \ldots, b-1\}$, called the b-ary expansion of ξ, such that

$$\xi = \lfloor \xi \rfloor + \sum_{j\geq 1} \frac{a_j}{b^j} \tag{1.1}$$

and $a_j \neq b-1$ for infinitely many indices j. Here, $\lfloor \cdot \rfloor$ denotes the integer part function. Clearly, the sequence $(a_j)_{j\geq 1}$ is ultimately periodic if, and only if, ξ is rational.

The real number ξ is called *normal to base b* if, for any positive integer k, each one of the b^k blocks of k digits from $\{0, 1, \ldots, b-1\}$ occurs in the b-ary expansion $a_1 a_2 \ldots$ of ξ with the same frequency $1/b^k$. The first explicit example of a number normal to base 10, namely the number

$$0.1234567891011121314\ldots,$$

Dong Han Kim was supported by the National Research Foundation of Korea (NRF-2015R1A2A2A01007090).

Received March 2, 2016; accepted in revised form March 22, 2016.
Published online September 2017.

whose sequence of digits is the concatenation of all positive integers ranged in increasing order, was given in 1933 by Champernowne [15] (see the monograph [13] for further results). Almost all real numbers (here and below, "almost all" always refers to the Lebesgue measure) are normal to every base b, but proving that a specific number, like e, π, $\sqrt{2}$ or $\log 2$ is normal to some base remains a challenging open problem, which seems to be completely out of reach.

In the present paper, we focus our attention to apparently simpler questions. We take a point of view from combinatorics on words. Let $\mathcal{A}$ be a finite set called an alphabet and denote by $|\mathcal{A}|$ its cardinality. A word over $\mathcal{A}$ is a finite or infinite sequence of elements of $\mathcal{A}$. For a (finite or infinite) word $\mathbf{x} = x_1 x_2 \ldots$ written over $\mathcal{A}$, let $n \mapsto p(n, \mathbf{x})$ denotes its subword complexity function which counts the number of different subwords of length n occurring in $\mathbf{x}$, that is,

$$p(n, \mathbf{x}) = \#\big\{x_{j+1} x_{j+2} \ldots x_{j+n} : j \geq 0\big\}, \quad n \geq 1.$$

Clearly, we have

$$1 \leq p(n, \mathbf{x}) \leq |\mathcal{A}|^n, \quad n \geq 1.$$

If $\mathbf{x}$ is ultimately periodic, then there exists an integer C such that $p(n, \mathbf{x}) \leq C$ for $n \geq 1$. Otherwise, we have

$$p(n+1, \mathbf{x}) \geq p(n, \mathbf{x}) + 1, \quad n \geq 1, \tag{1.2}$$

thus $p(n, \mathbf{x}) \geq n + 1$ for $n \geq 1$. There exist uncountably many infinite words $\mathbf{s}$ over $\{0, 1\}$ such that $p(n, \mathbf{s}) = n + 1$ for $n \geq 1$. These words are called Sturmian words. Classical references on combinatorics on words and on Sturmian sequences include [9,19,23].

A natural way to measure the complexity of the real number ξ written in base b as in (1.1) is to count the number of distinct blocks of given length in the infinite word $\mathbf{a} = a_1 a_2 \ldots$. Thus, for $n \geq 1$, we set $p(n, \xi, b) = p(n, \mathbf{a})$. Obviously, we have

$$p(n, \xi, b) = \#\big\{a_{j+1} a_{j+2} \ldots a_{j+n} : j \geq 0\big\}, \quad n \geq 1,$$

and

$$1 \leq p(n, \xi, b) \leq b^n, \quad n \geq 1,$$

where both inequalities are sharp.

If ξ is normal to base b, then $p(n, \xi, b) = b^n$ for every positive integer n. Clearly, the converse does not always hold. To establish a good lower bound for $p(n, \xi, b)$ is a first step towards the confirmation that the real number ξ is normal to base b. This point of view has been taken by Ferenczi and Mauduit [17, 1997]. It follows from their approach (see also [8]) that we have

$$\lim_{n \to +\infty} \big(p(n, \xi, b) - n\big) = +\infty,$$

for every algebraic irrational number ξ and every integer $b \geq 2$. Subsequently, by means of a new transcendence criterion established in [6], their result was improved in [4] as follows.

Theorem 1.1. *For every integer $b \geq 2$, every algebraic irrational number ξ satisfies*

$$\lim_{n\to+\infty} \frac{p(n,\xi,b)}{n} = +\infty.$$

Much less is known for specific transcendental numbers. The only result available so far was obtained in [3]; see also [13, Section 8.5]. Before stating it, we recall a basic notion from Diophantine approximation.

Definition 1.2. The irrationality exponent $\mu(\xi)$ of an irrational real number ξ is the supremum of the real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{\mu}}$$

has infinitely many solutions in rational numbers $\frac{p}{q}$.

The theory of continued fraction implies that every irrational real number ξ satisfies $\mu(\xi) \geq 2$. Combined with an easy covering argument, we get that the irrationality exponent of almost every real number is equal to 2. [3, Theorem 1], reproduced below as Theorem 1.3, extends the result of Ferenczi and Mauduit mentioned above to real numbers whose irrationality exponent is equal to 2 (recall that, by Roth's theorem [24], the irrationality exponent of every real algebraic irrational number is equal to 2).

Theorem 1.3. *For every integer $b \geq 2$, every irrational real number ξ whose irrationality exponent is equal to 2 satisfies*

$$\lim_{n\to+\infty} \big(p(n,\xi,b) - n\big) = +\infty.$$

Theorem 1.3 is an almost immediate consequence of two combinatorial statements established in [11] and [5] on the structure of Sturmian words. It applies to a wide class of classical numbers, including non-zero rational powers of e, badly approximable numbers, $\tan\frac{1}{a}$, where a is a positive integer, etc. Further examples of real numbers whose irrationality exponent is known to be equal to 2 are listed in [3].

Theorem 1.3 covers all what is known at present on the b-ary expansion of transcendental numbers. The main result of the present paper is the following considerable improvement of Theorem 1.3.

Theorem 1.4. *Let $b \geq 2$ be an integer and ξ an irrational real number. If μ denotes the irrationality exponent of ξ, then*

$$\liminf_{n\to+\infty} \frac{p(n,\xi,b)}{n} \geq 1 + \frac{1-2\mu(\mu-1)(\mu-2)}{\mu^3(\mu-1)}, \tag{1.3}$$

and

$$\limsup_{n\to+\infty} \frac{p(n,\xi,b)}{n} \geq 1 + \frac{1-2\mu(\mu-1)(\mu-2)}{3\mu^3-6\mu^2+4\mu-1}. \tag{1.4}$$

In particular, every irrational real number ξ whose irrationality exponent is equal to 2 *satisfies*

$$\liminf_{n\to+\infty} \frac{p(n,\xi,b)}{n} \geq \frac{9}{8} \quad \textit{and} \quad \limsup_{n\to+\infty} \frac{p(n,\xi,b)}{n} \geq \frac{8}{7}. \tag{1.5}$$

We display an immediate consequence of Theorem 1.4.

Theorem 1.5. *For any integer $b \geq 2$ we have*

$$\liminf_{n\to+\infty} \frac{p(n,\mathrm{e},b)}{n} \geq \frac{9}{8} \quad \textit{and} \quad \limsup_{n\to+\infty} \frac{p(n,\mathrm{e},b)}{n} \geq \frac{8}{7}.$$

Theorem 1.4 establishes an unexpected connection between the irrationality exponent of a real number and its b-ary expansion. It gives a non-trivial result on the b-ary expansion of a real number ξ when $2 \leq \mu(\xi) < 2.1914\ldots$ It applies to a much wider class of classical numbers than Theorem 1.3, which includes in particular the transcendental number $\log(1+\frac{1}{a})$, where a is a large positive integer. More examples are given in Section 2. Theorem 1.4 is sharp up to the values of the numerical constants occurring in (1.3) to (1.5).

The present paper illustrates the fruitful interplay between combinatorics on words and Diophantine approximation, which has already led to several recent progress. The proof of Theorem 1.4, given in Section 3, is mostly combinatorial and essentially self-contained. The main ingredient is the study of two new combinatorial exponents which measure the repetitions occurring at the beginning of an infinite word $\mathbf{x} = x_1x_2x_3\ldots$: a (classical) exponent rep and a uniform exponent Rep. These exponents are defined as follows (see Definition 3.4): for every positive real number ε, for arbitrary large integers n (respectively, for every sufficiently large integer n), the prefix of $\mathbf{x}$ of length $\lfloor n(\mathrm{rep}(\mathbf{x})+\varepsilon)\rfloor$ (respectively, $\lfloor n(\mathrm{Rep}(\mathbf{x})+\varepsilon)\rfloor$) contains two (possibly overlapping) distinct occurrences of a word of length n, and this is not anymore true if $\mathrm{rep}(\mathbf{x})$ and $\mathrm{Rep}(\mathbf{x})$ are replaced by smaller numbers. The key point of the proof of Theorem 1.4 is Theorem 3.5 below, where we establish that $\mathrm{Rep}(\mathbf{x})$ is always greater than $\mathrm{rep}(\mathbf{x})$ (provided that, of course, $\mathrm{rep}(\mathbf{x})$ is finite). Combined with the fact that $n \mapsto p(n,\mathbf{x})$ is increasing (when $\mathbf{x}$ is not ultimately periodic) and with an easy relationship, given in Lemma 3.6, between $\mathrm{rep}(\mathbf{x})$ and the irrationality exponent of the real number whose digits in some integer base are $x_1, x_2, x_3, \ldots$, we get Theorem 1.4.

ACKNOWLEDGEMENTS. We are very grateful to the referees for several remarks which helped us to improve the presentation of the paper.

2. A further result, comments, and examples

A key ingredient for the proof of Theorem 1.4 is the study of a complexity function which takes into account the smallest return time of a factor of an infinite word. For an infinite word $\mathbf{x} = x_1x_2\dots$ and an integer $n \geq 1$, set

$$r(n,\mathbf{x}) = \min\{m \geq 1 : x_i \dots x_{i+n-1} = x_{m-n+1} \dots x_m \text{ for some } i \text{ with } 1 \leq i \leq m-n\}.$$

Said differently, $r(n,\mathbf{x})$ denotes the length of the smallest prefix of $\mathbf{x}$ containing two (possibly overlapping) occurrences of some word of length n. The function $n \mapsto r(n,\mathbf{x})$ has been introduced and studied in [14], where the following two assertions are established. For every infinite word $\mathbf{x}$ which is not ultimately periodic, there exist arbitrarily large integers n such that $r(n,\mathbf{x}) \geq 2n+1$. The only infinite words $\mathbf{x}$ such that $r(n,\mathbf{x}) \leq 2n+1$ for $n \geq 1$ and which are not ultimately periodic are the Sturmian words.

Let ξ be an irrational real number and $b \geq 2$ be an integer. Write ξ in base b as in (1.1) and set $\mathbf{a} = a_1a_2\dots$ For $n \geq 1$, set $r(n,\xi,b) = r(n,\mathbf{a})$. The following result asserts that, if the irrationality exponent of ξ is not too large, then the function $n \mapsto r(n,\xi,b)$ cannot increase too slowly.

Theorem 2.1. *Let $b \geq 2$ be an integer and ξ an irrational real number. If μ denotes the irrationality exponent of ξ, then*

$$\limsup_{n\to+\infty} \frac{r(n,\xi,b)}{n} \geq 2 + \frac{1-2\mu(\mu-1)(\mu-2)}{3\mu^3-6\mu^2+4\mu-1}. \tag{2.1}$$

In particular, every irrational real number ξ whose irrationality exponent is equal to 2 *satisfies*

$$\limsup_{n\to+\infty} \frac{r(n,\xi,b)}{n} \geq \frac{15}{7}. \tag{2.2}$$

By Lemma 3.1 below, $p(n,\xi,b) \geq r(n,\xi,b) - n$ holds for all integers $n \geq 1, b \geq 2$ and every irrational real number ξ. Thus, (1.4) and the second assertion of (1.5) are immediate consequences of (2.1) and (2.2), respectively.

We will establish Theorems 1.4 and 2.1 simultaneously in Section 3. Our key ingredient is a purely combinatorial auxiliary result, stated as Theorem 3.5 below.

We stress that, even for real numbers whose irrationality exponent is equal to 2, Theorem 1.4 improves Theorem 1.3. Indeed, Aberkane [2] proved the existence of infinite words $\mathbf{x}$ with the property that

$$\lim_{n\to+\infty} p(n,\mathbf{x}) - n = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{p(n,\mathbf{x})}{n} = 1.$$

Furthermore, he established in [1] that, for any real number δ with $\delta > 1$, there are infinite words $\mathbf{x}$ satisfying

$$1 < \liminf_{n\to+\infty} \frac{p(n,\mathbf{x})}{n} < \limsup_{n\to+\infty} \frac{p(n,\mathbf{x})}{n} \leq \delta.$$

See also Heinis [20,21] for further results on words with small subword complexity.

Independently, Kmošek [22] and Shallit [25] (see also [13, Section 7.6]) established that the real number $\xi_{\mathrm{KS}} := \sum_{k\ge 1} 2^{-2^k}$ has a bounded continued fraction expansion. In particular, it satisfies $\mu(\xi_{\mathrm{KS}}) = 2$. Since

$$\limsup_{n\to+\infty} \frac{r(n, \xi_{\mathrm{KS}}, 2)}{n} = \frac{5}{2} \quad \text{and} \quad \liminf_{n\to+\infty} \frac{p(n, \xi_{\mathrm{KS}}, 2)}{n} = \frac{3}{2},$$

this shows that the value $\frac{15}{7}$ in (2.2) cannot be replaced by a real number greater than $\frac{5}{2}$. Also, the value $\frac{9}{8}$ in (1.5) cannot be replaced by a real number greater than $\frac{3}{2}$. Actually, with some additional effort and a case-by-case analysis, it is possible to replace the value $\frac{15}{7}$ in (2.2) and $\frac{9}{8}$ in (1.5) by slightly larger numbers; see the additional comments at the end of Section 3. However, we have chosen to present an elegant, short proof of Theorem 2.1, rather than a more complicated proof of a slightly sharper version of it.

It has been proved in [12] (see also [13, Section 7.6]) that, for every real number $\mu \ge 2$, the irrationality exponent of $\xi_\mu := \sum_{k\ge 1} 2^{-\lfloor \mu^k \rfloor}$ is equal to μ. Since $p(n, \xi_\mu, 2) = O(n)$, this shows that Theorems 1.4 and 2.1 are best possible up to the values of the numerical constants.

Any real number whose sequence of partial quotients is bounded has its irrationality exponent equal to 2, thus it satisfies (1.5) and (2.2), and its expansion in an integer base b cannot be "too simple".

Theorems 1.4 and 2.1 give non-trivial results on the b-ary expansion of a real number ξ when $2 \le \mu(\xi) < 2.1914\ldots$ By means of a specific analysis of repetitions in Sturmian words, we were able in [14] to extend Theorem 1.3 to real numbers whose irrationality exponent is less than or equal to $\frac{5}{2}$. Note that if $\mathbf{f} = f_1 f_2 \ldots$ denotes the Fibonacci word $\mathbf{f} = 01001010\ldots$ (that is, the fixed point of the substitution $0 \mapsto 01$, $1 \mapsto 0$; this is a Sturmian word), then the real number $\xi_{\mathbf{f}} := \sum_{k\ge 1} 2^{-f_k}$ satisfies $\mu(\xi_{\mathbf{f}}) = \frac{3+\sqrt{5}}{2} = 2.618\ldots$ and $p(n, \xi_{\mathbf{f}}, 2) = n + 1$ for $n \ge 1$.

An important feature of Theorems 1.4 and 2.1 is that they apply not only to real numbers whose irrationality exponent is equal to 2, but also to real numbers whose irrationality exponent is slightly larger than 2. To prove that the irrationality exponent of a given real number is equal to 2 is often a very difficult problem, while it is sometimes possible to bound its value from above. For example, Alladi and Robinson [7] (who improved earlier results of A. Baker [10]) and Danilov [16] established that, for any positive integer s, the irrationality exponents of $\log(1 + \frac{s}{t})$ and $\sqrt{t^2 - s^2} \arcsin \frac{s}{t}$ are bounded from above by a function of t which tends to 2 as the integer t tends to infinity. The next statement then follows at once from Theorem 1.4.

Theorem 2.2. *Let ε be a positive real number. For any positive integer s, there exists an integer t_0 such that, for any integer $t > t_0$, we have*

$$\liminf_{n\to+\infty} \frac{p\big(n, \log\big(1 + \frac{s}{t}\big), b\big)}{n} \ge \frac{9}{8} - \varepsilon$$

and

$$\liminf_{n\to+\infty} \frac{p\big(n, \sqrt{t^2-s^2}\arcsin\frac{s}{t}, b\big)}{n} \geq \frac{9}{8} - \varepsilon.$$

Using the results from [7, 16], it is easy to give a suitable explicit value for t_0 in terms of s and ε. In particular, there exists an absolute positive constant c such that, if s, t are integers with $s \geq 2$ and $t \geq s^c$, then

$$\liminf_{n\to+\infty} \frac{p\big(n, \log\big(1+\frac{s}{t}\big), b\big)}{n} \geq \frac{9}{8} - 4\,\frac{\log s}{\log t}.$$

Up to now, not a single result was known on the b-ary expansion of the transcendental real number $\log(1+\frac{1}{a})$.

3. Proofs

We start with establishing a relationship between the subword complexity function of an infinite word $\mathbf{x}$ and the function $n \mapsto r(n, \mathbf{x})$.

Here and below, for integers i, j with $1 \leq i \leq j$, we write x_i^j for the factor $x_i x_{i+1} \ldots x_j$ of the word $\mathbf{x} = x_1 x_2 \ldots$

Lemma 3.1. *For any infinite word* $\mathbf{x}$ *and any positive integer* n*, we have*

$$p(n, \mathbf{x}) \geq r(n, \mathbf{x}) - n.$$

Proof. It follows from the definition of $r(n, \mathbf{x})$ that the $r(n, \mathbf{x}) - 1 - (n-1)$ factors of length n of $x_1^{r(n,\mathbf{x})-1}$ are all distinct. Since $x_{r(n,\mathbf{x})-n+1}^{r(n,\mathbf{x})}$ is a factor of $x_1^{r(n,\mathbf{x})-1}$, we have

$$p(n, \mathbf{x}) \geq p\left(n, x_1^{r(n,\mathbf{x})-1}\right) = p\left(n, x_1^{r(n,\mathbf{x})}\right) = r(n, \mathbf{x}) - n. \qquad \square$$

We stress that there is no analogue lower bound for the subword complexity function of $\mathbf{x}$ in terms of $n \mapsto r(n, \mathbf{x})$.

In the course of the proof of Theorem 3.5, we need the following auxiliary lemma, which is [14, Lemma 5.4].

Lemma 3.2. *Let* $\mathbf{x}$ *be an infinite word and* n *a positive integer. If* $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 2$*, then* $r(n+1, \mathbf{x}) \geq 2n + 3$.

For sake of completeness, we give an alternative proof of Lemma 3.2, based on a theorem of Fine and Wilf [18, Theorem 1].

Proof. For shorten the notation, we simply write $r(\cdot)$ for $r(\cdot, \mathbf{x})$. Let n be a positive integer and assume that $r(n+1) \geq r(n) + 2$ and $r(n+1) \leq 2(n+1)$. There exist integers s, t with

$$1 \leq s \leq r(n) - n, \quad 1 \leq t \leq r(n+1) - n - 1,$$

such that

$$x_s^{s+n-1} = x_{r(n)-n+1}^{r(n)}, \quad x_{s+n} \neq x_{r(n)+1}, \quad \text{and} \quad x_t^{t+n} = x_{r(n+1)-n}^{r(n+1)}.$$

Observe that $t \le r(n+1) - n - 1 \le n + 1 \le r(n)$. Thus, the non-empty word $x_{\max\{s,t\}}^{r(n)}$ is periodic with period $r(n) - s - n + 1$ and $r(n+1) - t - n$, but $x_s^{r(n)+1}$ is not periodic with period $r(n) - s - n + 1$. A theorem of Fine and Wilf [18, Theorem 1] asserts that if $(f_n)_{n\ge1}$, $(g_n)_{n\ge1}$ are periodic sequences of periods h, k, respectively, such that $f_n = g_n$ for $h + k - \gcd(h, k)$ consecutive integers n, then $f_n = g_n$ for every $n \ge 1$. By applying this theorem we get that, if

$$\begin{aligned} |x_{\max\{s,t\}}^{r(n)}| \ge r(n) - s - n + 1 + r(n+1) - t - n \\ - \gcd(r(n) - s - n + 1, r(n+1) - t - n), \end{aligned}$$

then $x_{\min\{s,t\}}^{r(n+1)}$ is periodic with period $r(n) - s - n + 1$, which is a contradiction. Therefore we get

$$\begin{aligned} |x_{\max\{s,t\}}^{r(n)}| &= r(n) - \max\{s, t\} + 1 \\ &\le r(n) + r(n+1) - s - t - 2n \\ &\quad - \gcd(r(n) - s - n + 1, r(n+1) - t - n) \\ &\le r(n) + r(n+1) - s - t - 2n - 1 \\ &\le r(n) - s - t + 1. \end{aligned}$$

Hence, we obtain

$$\max\{s, t\} \ge s + t,$$

which is a contradiction since s and t are positive. This completes the proof of the lemma. □

We also need the following easy result, already established in [14].

Lemma 3.3. *Let* $\mathbf{x}$ *be an infinite word such that* $r(n+1, \mathbf{x}) = r(n, \mathbf{x}) + 1$ *for every sufficiently large integer* n. *Then* $\mathbf{x}$ *is ultimately periodic.*

Proof. Let n_0 be an integer such that $r(n+1, \mathbf{x}) = r(n, \mathbf{x}) + 1$ for every $n \ge n_0$. Let j be the integer satisfying $1 \le j \le r(n_0, \mathbf{x}) - n_0$ and $x_j^{j+n_0-1} = x_{r(n_0,\mathbf{x})-n_0+1}^{r(n_0,\mathbf{x})}$. Let h be a positive integer. Since $r(n_0 + h, \mathbf{x}) = r(n_0, \mathbf{x}) + h$, we deduce that $x_j^{j+n_0+h-1} = x_{r(n_0,\mathbf{x})-n_0+1}^{r(n_0+h,\mathbf{x})}$ and conclude that $\mathbf{x}$ is ultimately periodic of period $r(n_0, \mathbf{x}) - n_0 - j + 1$. □

For our combinatorial analysis, it is convenient to introduce two combinatorial exponents which measure the repetitions in an infinite word.

Definition 3.4. Let $\mathbf{x}$ be an infinite word. The exponent of repetition of $\mathbf{x}$, denoted by $\mathrm{rep}(\mathbf{x})$, is the quantity

$$\mathrm{rep}(\mathbf{x}) = \liminf_{n\to+\infty} \frac{r(n,\mathbf{x})}{n}.$$

The uniform exponent of repetition of $\mathbf{x}$, denoted by $\mathrm{Rep}(\mathbf{x})$, is the quantity

$$\mathrm{Rep}(\mathbf{x}) = \limsup_{n\to+\infty} \frac{r(n,\mathbf{x})}{n}.$$

The key ingredient for the proof of Theorem 2.1 is the following combinatorial theorem.

Theorem 3.5. *Every infinite word* $\mathbf{x}$ *which is not ultimately periodic satisfies* $\mathrm{Rep}(\mathbf{x}) \ge 2$,

$$\mathrm{Rep}(\mathbf{x}) \ge \mathrm{rep}(\mathbf{x}) + \frac{1}{1+\mathrm{rep}(\mathbf{x}) + (\mathrm{rep}(\mathbf{x}))^2}, \tag{3.1}$$

and

$$\liminf_{n\to+\infty} \frac{p(n,\mathbf{x})}{n} \ge \mathrm{rep}(\mathbf{x}) - 1 + \frac{1}{(\mathrm{rep}(\mathbf{x}))^3}. \tag{3.2}$$

Proof. Let $\mathbf{x}$ be an infinite word satisfying $\mathrm{Rep}(\mathbf{x}) < 2$. It follows from Lemma 3.2 that $r(n+1,\mathbf{x}) = r(n,\mathbf{x})+1$, for every sufficiently large integer n. By Lemma 3.3, this shows that $\mathbf{x}$ is ultimately periodic and proves the first assertion of the theorem.

Let $\mathbf{x} = x_1x_2\ldots$ be an infinite word which is not ultimately periodic. Without any loss of generality, we may assume that $\mathrm{rep}(\mathbf{x})$ is finite. Set $\rho = \mathrm{rep}(\mathbf{x})$. Since $\mathrm{Rep}(\mathbf{x}) \ge 2$ and $p(n,x) \ge n+1$ for $n \ge 1$, inequalities (3.1) and (3.2) hold for $\rho \le \frac{8}{5}$. Therefore, we may assume that $\rho > \frac{8}{5}$.

Let ε be a positive real number with $\varepsilon < \frac{1}{10}$ and $n_0 \ge 3\frac{\rho^2}{\varepsilon}$ be such that

$$(\rho-\varepsilon)n \le r(n,\mathbf{x}), \quad \text{for } n \ge \tfrac{n_0}{8\rho}.$$

By Lemma 3.3, there are arbitrarily large integers n such that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+2$. Let $n > n_0$ be an integer such that $r(n+1,\mathbf{x}) > r(n,\mathbf{x})+1$ and define α by setting $r(n,\mathbf{x}) = \alpha n$. This implies that the word $x_{(\alpha-1)n+1}^{\alpha n}$ of length n has two occurrences in $x_1^{\alpha n}$ and that these two occurrences are not followed by the same letter. Let m_1 be the index at which the first occurrence of $x_{(\alpha-1)n+1}^{\alpha n}$ starts. We have $m_1+n-1 < \alpha n$ and the letters x_{m_1+n} and $x_{\alpha n+1}$ are different.

Let β be such that $r(n+1,\mathbf{x}) = \beta(n+1)$. Since $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+2$, we have $\beta(n+1) \ge \alpha n+2$, that is $1+(\beta-1)(n+1) \ge (\alpha-1)n+2$. Then, the word $x_{(\beta-1)(n+1)+1}^{\beta(n+1)}$ of length $n+1$ has two occurrences in $x_1^{\beta(n+1)}$. Let m_2 be the index at which its first occurrence starts. We have $m_2 < (\beta-1)(n+1)+1$.

We assume that $\alpha < \rho+2$ and

$$\frac{1-\beta+\alpha-\varepsilon}{\beta-1} > \frac{1+\rho}{(\rho-\varepsilon)^2} \tag{3.3}$$

and we will get a contradiction.

Consider the word $V_n := x_{(\beta-1)(n+1)+1}^{\alpha n}$ of length

$$v_n = (1-\beta+\alpha)n - \beta + 1.$$

Observe that $\rho - \varepsilon > \frac{8}{5} - \frac{1}{10} \geq \frac{3}{2}$ implies that $\beta \geq \frac{3}{2}$ and check that, by (3.3),

$$v_n \geq (\beta-1)\frac{1+\rho}{\rho^2}n - (\beta-1) \geq \frac{1}{2}\left(\frac{n}{\rho} - 1\right) \geq \frac{n}{4\rho},$$

since $n \geq 2\rho$.

The word V_n is a proper suffix of $x_{(\alpha-1)n+1}^{\alpha n}$ and we have

$$V_n = x_{(\beta-1)(n+1)+1}^{\alpha n} = x_{m_2}^{m_2+v_n-1} = x_{m_1+n-v_n}^{m_1+n-1}.$$

If $m_2 = m_1 + n - v_n$, then $x_{m_2+v_n} = x_{m_1+n}$ and we deduce from $x_{m_2+v_n} = x_{\alpha n+1}$ that $x_{m_1+n} = x_{\alpha n+1}$, a contradiction with our choice of n. Consequently, the word V_n has (at least) three occurrences in $x_1^{\alpha n}$. Set

$$j_3 = (\beta-1)(n+1)+1. \tag{3.4}$$

Let j_1, j_2 with $j_1 < j_2 < j_3$ be the indices at which the two other occurrences of $x_{j_3}^{\alpha n}$ start. In particular, the letters $x_{j_1+v_n}$ and $x_{j_2+v_n}$ must be different.

The proof decomposes into five steps. We show that j_2 and j_1 cannot be too small and that the three occurrences of V_n in $x_1^{\alpha n}$ overlap. We conclude in Step 5 that the letters $x_{j_1+v_n}$ and $x_{j_2+v_n}$ must be the same. This contradiction shows that (3.3) cannot hold.

For a finite word W and a real number $t > 1$, we denote by $(W)^t$ the word equal to the concatenation of $\lfloor t \rfloor$ copies of the word W followed by the prefix of W of length $\lceil (t - \lfloor t \rfloor)|W| \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. We say that $(W)^t$ is the t-th power of W.

Step 1. Since $v_n \geq \frac{n}{4\rho}$, our choice of n_0 implies that

$$(\rho-\varepsilon)v_n \leq r(v_n, \mathbf{x}) \leq j_2 + v_n - 1,$$

thus we get

$$j_2 \geq (\rho - 1 - \varepsilon)v_n + 1. \tag{3.5}$$

We have established that j_2 cannot be too small.

Step 2. Using the fact that j_2 is not too small, we want to deduce that the subwords $x_{j_3}^{\alpha n} = x_{j_3}^{j_3+v_n-1}$ and $x_{j_2}^{j_2+v_n-1}$ (which are both equal to V_n) have a quite big overlap.

By [9, Theorem 1.5.2], the word V_n is the t-th power with

$$t := \frac{v_n}{j_3 - j_2}$$

of some word U_n of length $u_n := j_3 - j_2$ and $x_{j_2}^{j_3+v_n-1} = (U_n)^{1+t}$. By (3.4) and (3.5) we get

$$t \ge \frac{v_n}{(\beta-1)(n+1)-(\rho-1-\varepsilon)v_n}.$$

Observe that $n+1 > n_0 \ge \frac{3\rho^2}{\varepsilon} > \frac{\rho+2}{\varepsilon} > \frac{\alpha}{\varepsilon}$, thus $v_n \ge (1-\beta+\alpha-\varepsilon)(n+1)$ and, by (3.3),

$$t \ge \frac{1+\rho}{(\rho-\varepsilon)^2-(\rho-1-\varepsilon)(1+\rho)} \ge \frac{1+\rho}{1+\varepsilon+\varepsilon^2}.$$

Recalling that $\rho \ge \frac{8}{5}$ and $\varepsilon \le \frac{1}{10}$, we have established that $t \ge \frac{9}{4}$.

Step 3. Let W_n be the word such that $V_n = U_n W_n$ and let w_n denote its length. Observe that

$$w_n = \frac{t-1}{t} v_n = v_n - j_3 + j_2 \tag{3.6}$$

and, recalling that $v_n \ge \frac{n}{4\rho}$ and $t \ge \frac{9}{4}$,

$$w_n = \frac{t-1}{t} v_n \ge \frac{5}{9} \cdot \frac{n}{4\rho} \ge \frac{n}{8\rho}.$$

Since $V_n = (U_n)^t$ and $t > 2$, the word $W_n = (U_n)^{t-1}$ is a prefix of V_n. Thus, W_n has two occurrences in V_n and also in the prefix of $\mathbf{x}$ of length $j_1 + v_n - 1$. It then follows from our choice of n_0 that

$$(\rho-\varepsilon)w_n \le r(w_n, \mathbf{x}) \le j_1 + v_n - 1.$$

Combined with (3.6), this gives

$$j_1 \ge (\rho-1-\varepsilon)v_n - (\rho-\varepsilon)(j_3-j_2) + 1. \tag{3.7}$$

We have established that j_1 cannot be too small.

Step 4. Observe first that (3.3) is equivalent to the inequality

$$(\rho-\varepsilon)^2(1-\beta+\alpha-\varepsilon) > (\beta-1)(\rho+1).$$

This gives

$$\begin{aligned}(\rho-\varepsilon)^2(1-\beta+\alpha)n - (\rho+1-\varepsilon)(\beta-1)n &> n\varepsilon(\beta-1)\\ &> (\beta-1)\big[(\rho-\varepsilon)^2+\rho+1-\varepsilon\big],\end{aligned}$$

since $n\varepsilon > n_0\varepsilon \ge 3\rho^2$. Consequently, we get

$$(\rho-\varepsilon)^2 v_n > (\rho+1-\varepsilon)(\beta-1)(n+1) = (\rho+1-\varepsilon)(j_3-1). \tag{3.8}$$

We deduce from (3.5) that

$$(\rho-\varepsilon)^2 v_n \le (\rho-\varepsilon)v_n + (\rho-\varepsilon)(j_2-1),$$

which, combined with (3.8), gives

$$\begin{aligned}(\rho-\varepsilon)v_n &\ge (\rho+1-\varepsilon)(j_3-1) - (\rho-\varepsilon)(j_2-1)\\ &= (\rho-\varepsilon)(j_3-j_2) + j_3 - 1.\end{aligned}$$

We conclude by (3.7) that

$$v_n > j_3 - j_1. \tag{3.9}$$

Thus, the subwords $x_{j_1}^{j_1+v_n-1}$ and $x_{j_3}^{j_3+v_n-1}$, which are both equal to V_n, overlap.

Step 5. It follows from (3.9) that

$$v_n - (j_2 - j_1) > j_3 - j_2 = u_n,$$

which means that the length of the overlap between the subwords $x_{j_1}^{j_1+v_n-1}$ and $x_{j_2}^{j_2+v_n-1}$ exceeds the length u_n of U_n. We show that this implies that $x_{j_1}^{\alpha n} = x_{j_1}^{j_3+v_n-1}$ is equal to a (large) power of some word. To do this, we distinguish two cases.

If there exists an integer h such that $j_2 = j_1 + hu_n$, then we have

$$x_{j_1}^{j_3+v_n-1} = x_{j_1}^{j_2-1}\, x_{j_2}^{j_3+v_n-1} = (U_n)^{h+1+t}$$

and the letters $x_{j_1+v_n}$ and $x_{j_2+v_n}$ are the same, since j_1+v_n and j_2+v_n are congruent modulo the length u_n of U_n. This is a contradiction.

If $j_2 - j_1$ is not an integer multiple of u_n, then let h be the smallest integer such that $j_1 + hu_n > j_2$. The word $Z_n := x_{j_2}^{j_1+hu_n-1}$ is a suffix of U_n and the word $Z_n' := x_{j_1+hu_n}^{j_2+u_n-1} = x_{j_1+hu_n}^{j_3-1}$ is a prefix of U_n. They satisfy

$$U_n = Z_n Z_n' = Z_n' Z_n.$$

By [9, Theorem 1.5.3], the words Z_n and Z_n' are integer powers of a same word. Thus, there exist a word T_n of length t_n and positive integers k, ℓ such that

$$Z_n = (T_n)^k \quad \text{and} \quad Z_n' = (T_n)^\ell.$$

Consequently, there exists an integer q such that $j_2 = j_1 + qt_n$ and we have

$$x_{j_1}^{j_3+v_n-1} = x_{j_1}^{j_2-1}\, x_{j_2}^{j_3+v_n-1} = (T_n)^{q+(1+t)(k+\ell)}.$$

As above, the letters $x_{j_1+v_n}$ and $x_{j_2+v_n}$ are the same, since j_1+v_n and j_2+v_n are congruent modulo the length t_n of T_n. This is a contradiction.

We have shown that (3.3) does not hold and we are in position to complete the proof of the theorem.

Let $(n_k)_{k\ge 1}$ denote the increasing sequence comprising all the integers n such that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+2$. For $k \ge 1$, define α_k and β_k by setting

$$r(n_k,\mathbf{x}) = \alpha_k n_k \quad \text{and} \quad r(n_k+1,\mathbf{x}) = \beta_k(n_k+1).$$

Let ε be a positive real number with $\varepsilon < \frac{1}{10}$. Let k_0 be an integer such that $r(n_\ell,\mathbf{x}) \ge (\rho-\varepsilon)n_\ell$ for $\ell \ge k_0$. For every integer k greater than k_0 and large enough in terms of ε, we have established that $\alpha_k \ge \rho+2$ or

$$\frac{1-\beta_k+\alpha_k-\varepsilon}{\beta_k-1} \le \frac{1+\rho}{(\rho-\varepsilon)^2}.$$

If $\alpha_k \ge \rho+2$, then $\beta_k(n+1) \ge (\rho+2)n_k+2$ and we deduce that $\beta_k \ge \rho+1$ since $n_k > n_0 > \rho+1$. Thus, we get

$$\beta_k \ge \min\left\{\rho+1, \frac{(\rho-\varepsilon)^2(\rho+1-2\varepsilon)+\rho+1}{1+\rho+(\rho-\varepsilon)^2}\right\},$$

by using that $\alpha_k \ge \rho-\varepsilon$. Since ε can be taken arbitrarily small, this gives

$$\limsup_{n\to+\infty} \frac{r(n,\mathbf{x})}{n} \ge \min\left\{\rho+1, \frac{(\rho+1)(\rho^2+1)}{1+\rho+\rho^2}\right\},$$

and we have established (3.1).

Observe that, by definition of the sequence $(n_k)_{k\ge 1}$,

$$r(n_{k+1},\mathbf{x}) = r(n_k+1,\mathbf{x}) + n_{k+1} - n_k - 1 \ge (\rho-\varepsilon)n_{k+1}.$$

Consequently,

$$n_{k+1} \le \frac{r(n_k+1,\mathbf{x})-n_k-1}{\rho-1-\varepsilon}.$$

Let n be an integer with $n_k+1 \le n \le n_{k+1}$. By (1.2) and Lemma 3.1 we have

$$p(n,\mathbf{x}) \ge p(n_k+1,\mathbf{x}) + n - n_k - 1 \ge r(n_k+1,\mathbf{x}) + n - 2n_k - 2,$$

thus

$$\frac{p(n,\mathbf{x})}{n} \ge 1 + \frac{r(n_k+1,\mathbf{x})-2n_k-2}{n} \ge 1 + \frac{r(n_k+1,\mathbf{x})-2n_k-2}{n_{k+1}},$$

giving that

$$\begin{aligned}\frac{p(n,\mathbf{x})}{n} &\ge 1 + (\rho-1-\varepsilon)\,\frac{r(n_k+1,\mathbf{x})-2n_k-2}{r(n_k+1,\mathbf{x})-n_k-1}\\ &\ge \rho-\varepsilon-(\rho-1-\varepsilon)\,\frac{1}{\beta_k-1}.\end{aligned}$$

Since ε can be taken arbitrarily small, we conclude that

$$\liminf_{n\to+\infty} \frac{p(n,\mathbf{x})}{n} \geq \min\left\{\rho - 1 + \frac{1}{\rho},\ \rho - 1 + \frac{1}{\rho^3}\right\}.$$

This proves (3.2) and completes the proof of the theorem. □

Let $b \geq 2$ be an integer. Our last auxiliary result establishes a close connection between the exponent of repetition of an infinite word $\mathbf{x}$ written over $\{0, 1, \dots, b-1\}$ and the irrationality exponent (see Definition 1.2) of the real number whose b-ary expansion is given by $\mathbf{x}$.

Lemma 3.6. *Let $b \geq 2$ be an integer and $\mathbf{x} = x_1 x_2 \dots$ an infinite word over $\{0, 1, \dots, b-1\}$, which is not ultimately periodic. Then, the irrationality exponent of the irrational number $\sum_{k\geq 1} \frac{x_k}{b^k}$ satisfies*

$$\mu\left(\sum_{k\geq 1} \frac{x_k}{b^k}\right) \geq \frac{\mathrm{rep}(\mathbf{x})}{\mathrm{rep}(\mathbf{x}) - 1},$$

where the right hand side is infinite if $\mathrm{rep}(\mathbf{x}) = 1$.

Proof. Since the irrationality exponent of an irrational real number is at least equal to 2, we can assume that $\mathrm{rep}(\mathbf{x}) < 2$. Let n and C be positive integers such that $1 < C < 2$ and $r(n,\mathbf{x}) \leq Cn$. By [9, Theorem 1.5.2], this implies that there are finite words W, U, V and a positive integer t (we do not indicate the dependence on n) such that $|(UV)^t U| = n$ (here and below, $|\cdot|$ denotes the length of a finite word) and $W(UV)^{t+1}U$ is a prefix of $\mathbf{x}$ of length at most Cn. Observe that

$$|W(UV)^{t+1}U| \leq Cn \leq C|(UV)^t U|,$$

thus $|WUV| \leq (C-1)|(UV)^t U|$. Setting $\xi = \sum_{k\geq 1} \frac{x_k}{b^k}$, there exists an integer p such that

$$\left|\xi - \frac{p}{b^{|W|}(b^{|UV|} - 1)}\right| \leq \frac{1}{b^{|W(UV)^{t+1}U|}} \leq \frac{1}{b^{|WUV|} b^{|WUV|/(C-1)}}.$$

Consequently, if there are arbitrarily large integers n with $r(n,\mathbf{x}) \leq Cn$, then $\mu(\xi) \geq 1 + \frac{1}{C-1}$. Here, we have used the fact that $|WUV|$ tends to infinity when n tends to infinity: this follows from the assumption that $\mathbf{x}$ is not ultimately periodic. By choosing C arbitrarily close to $\mathrm{rep}(\mathbf{x})$, we complete the proof. □

Lemma 3.6 shows that, when the exponent of repetition of an infinite word is less than 2, then the irrationality exponent of the associated real number exceeds 2. We are in position to complete the proof of Theorems 1.4 and 2.1.

Proof of Theorems 1.4 *and* 2.1. Let $b \geq 2$ be an integer and ξ an irrational real number. Write ξ in base b as in (1.1) and put $\mathbf{a} = a_1 a_2 \ldots$. Lemma 3.6 asserts that

$$\mathrm{rep}(\mathbf{a}) \geq \frac{\mu(\xi)}{\mu(\xi) - 1}.$$

Combined with Theorem 3.5, this gives

$$\mathrm{Rep}(\mathbf{a}) \geq 1 + \frac{(\mathrm{rep}(\mathbf{a}))^3}{1 + \mathrm{rep}(\mathbf{a}) + (\mathrm{rep}(\mathbf{a}))^2} \geq 1 + \frac{\mu^3}{3\mu^3 - 6\mu^2 + 4\mu - 1},$$

where μ denotes the irrationality exponent of ξ. As well, we obtain

$$\begin{aligned}\liminf_{n\to+\infty} \frac{p(n,\mathbf{a})}{n} &\geq \min\left\{1,\ \mathrm{rep}(\mathbf{a}) - 1 + \frac{1}{(\mathrm{rep}(\mathbf{a}))^3}\right\} \\ &\geq \frac{\mu^4 - 3\mu^3 + 6\mu^2 - 4\mu + 1}{\mu^3(\mu - 1)}.\end{aligned}$$

We have established (1.3) and (2.1) and thereby completed the proofs of Theorems 1.4 and 2.1. □

Additional comments

We can slightly improve Theorem 3.5 (and, consequently, Theorems 1.4 and 2.1) by means of a refined case-by-case analysis. With the notation used in the proof of Theorem 3.5, the two cases to distinguish are:

(i) $j_1 = m_2$ and $j_2 = m_1 + n - v_n$ (that is, $m_2 < m_1 + n - v_n$);
(ii) $j_1 = m_1 + n - v_n$ and $j_2 = m_2$ (that is, $m_2 > m_1 + n - v_n$).

Then, (3.1) can be replaced by the stronger inequality which holds for Case (i)

$$\mathrm{Rep}(\mathbf{x}) \geq \mathrm{rep}(\mathbf{x}) + \frac{1}{\mathrm{rep}(\mathbf{x}) + (\mathrm{rep}(\mathbf{x}))^2} \tag{3.10}$$

and (2.1) by

$$\limsup_{n\to+\infty} \frac{r(n,\xi,b)}{n} \geq 2 + \frac{2\mu^2 + \mu - 1 - \mu^3}{\mu(\mu-1)(2\mu-1)}.$$

Furthermore, we may also see that, under a slightly weaker assumption than (3.10), Case (i) cannot occur for two consecutive integers n such that $r(n+1,\mathbf{x}) \geq r(n,\mathbf{x}) + 2$. Hence, a further small improvement can be obtained.

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