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A topological invariant of line arrangements

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Abstract. We define a new topological invariant of line arrangements in the complex projective plane. This invariant is a root of unity defined under some combinatorial restrictions for arrangements endowed with some special torsion character on the fundamental group of their complement. It is derived from the peripheral structure on the group induced by the inclusion map of the boundary of a tubular neigborhood in the exterior of the arrangement. By similarity with knot theory, it can be viewed as an analogue of linking numbers. This is an orientation-preserving invariant for ordered arrangements. We give an explicit method to compute the invariant from the equations of the arrangement, using the wiring diagrams introduced by Arvola, that encode the braid monodromy. Moreover, this invariant is a crucial ingredient to compute the depth of a character satisfying some resonant conditions, and completes the existent methods by Libgober and the first author. Finally, we compute the invariant for extended MacLane arrangements with an additional line and observe that it takes different values for the deformation classes.

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Introduction

The influence of the combinatorial data on the topology of a projective line arrangement is not at present well understood. In the eighties, Orlik and Solomon [16] showed that the cohomology ring of the complement of an arrangement is determined by the description of the incidence relations between the multiple points. This is not true for the deformation classes, as it was shown for MacLane combinatorics [6,15]. Rybnikov [4,17] constructed a pair of complex line arrangements with the same combinatorics but whose fundamental groups are not isomorphic. This il-

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lustrates that the combinatorics of an arrangement $\mathcal{A}$ does not determine in general the homeomorphism type of the pair $(\mathbb{CP}^2, \mathcal{A})$. Other examples were exhibited by the first author, Carmona, Cogolludo and Marco by using the braid monodromy [3].

One of the strongest invariant of the topology of an arrangement is the fundamental group of its complement, which can be computed using Zariski-van Kampen method, see also the specific approach by Arvola [5]. Even when the fundamental group can be computed, it is very difficult to handle directly with it. This is why a number of invariants derived from the group have emerged, as the Alexander invariant and the characteristic varieties. These invariants can be computed from the fundamental group, but the task can be endless for present computers for most arrangements. A weaker invariant is the one-variable Alexander polynomial (also known as characteristic polynomial of the monodromy of the Milnor fiber) which can be computed directly from the fundamental group or by a general method available for any projective curve. Though there are some partial results, it is still unknown whether characteristic varieties or Alexander polynomial are combinatorially determined.

Using a method by Ligboger [14] it is possible to compute *most* irreducible components of characteristic varieties (only some isolated points may fail to be found). A method to compute these extra components can be found in [2].

The *boundary manifold* $B_{\mathcal{A}}$, defined as the common boundary of a regular neighbourhood of $\mathcal{A}$ and its exterior $E_{\mathcal{A}}$, is a graph 3-manifold whose structure is determined by the combinatorics of $\mathcal{A}$. In the present paper, we construct a new topological invariant of line arrangements derived from the *peripheral structure* $\pi_1(B_{\mathcal{A}}) \to \pi_1(E_{\mathcal{A}})$, induced by the inclusion map, and previously studied in [11]. This invariant is a root of unity defined for triplets composed of an arrangement, a torsion character in $\mathbb{C}^*$ and a cycle in the incidence graph of the arrangement; there are combinatorial restrictions for the availability of this invariant, in particular the cycle must be non trivial and satisfy some resonant conditions. As it was remarked in [12], it is in fact extracted from the homological reduction $i_* : H_1(B_{\mathcal{A}}) \to H_1(E_{\mathcal{A}})$, which is a more tractable object, and corresponds to the value of the character on certain homology classes of the boundary manifold, viewed in $E_{\mathcal{A}}$ through the inclusion map. This construction has similarities with knot theory, and the invariant is a sort of analogue of linking numbers. As we state in the first main result, Theorem 2.1, it is an *ordered-oriented* topological invariant.

As a second main contribution of this work, we give an explicit method to compute the invariant, in terms of braid monodromy. We use *braided wiring diagrams* introduced by Arvola [5] (see also Suciu-Cohen [7]) to encode the braid monodromy relative to a generic projection of the arrangement. Note that this invariant is most probably of algebraic nature, even though our computations are topological.

It appears that this invariant is crucial for the computation of the *quasi-projective* depth of a (torsion) character in [2]. The knowledge of depths of all characters is equivalent to the knowledge of characteristic varieties; the depth can be decomposed into a *projective term* and a *quasi-projective term*, vanishing for characters that ramify along all the lines. An algorithm to compute the projective part was given by Libgober (see [14] and also [2] for details). An explicit way to compute the quasi-projective depth of resonant (torsion) characters is given in [2], and it happens that the invariant in this paper is crucial for that method. Hence, our invariant may help to find examples of combinatorially equivalent arrangements with different structure for their characteristic varieties, though we have failed till now in finding such arrangements. But as we show in this paper, this invariant is interesting in its own.

We compute the invariant for MacLane arrangements with an additional line, and observe that it takes different values for the two deformation classes. This shows that it provides information on their topologies, not contained in the combinatorics. In particular, there is no ordered-oriented homeomorphism between both realizations; note that this fact is a consequence of the same result for MacLane arrangements (as shown by Rybnikov).

In [13], the third author presents new examples of *Zariski pairs* of arrangements with 12 lines (of different nature of previous examples), where this invariant is used to show that they have not the same topological type.

In Section 1 we introduce the combinatorial and topological objects to be used in the paper, namely the concept of inner-cyclic triplets. In Section 2, we define the invariant of the realization of an inner-cyclic triplet and we prove its topological invariance. In Section 3, the relationship of the invariant with characteristic varieties is described. The topological computation of the invariant via wiring diagrams and the results of [11] is in Section 4. Finally, Section 5 computes the invariant for the two ordered realizations of the extended MacLane combinatorics.

1. Inner cyclic triplets

1.1. Combinatorics

In what follows, a brief reminder on line combinatorics is given, see [3] for details. We also introduce the notion of *inner cyclic*.

Definition 1.1. A *combinatorial type*, or simply a (*line*) *combinatorics*, is a couple $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, where $\mathcal{L}$ is a finite set and $\mathcal{P} \subset \mathcal{P}(\mathcal{L})$, satisfying that:

- For all $p \in \mathcal{P}$, $\sharp p \geq 2$;
- For any $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, $\exists! p \in \mathcal{P}$ such that $\ell_1, \ell_2 \in p$.

An *ordered combinatorics* $\mathcal{C}$ is a combinatorics where $\mathcal{L}$ is an ordered set.

This notion encodes the intersection pattern of a collection of lines in a projective planes, see Subsection 1.2, where the relation $\in$ corresponds to the dual plane. There are several ways to encode a line combinatorics.

Definition 1.2. The *incidence graph* $\Gamma_{\mathcal{C}}$ of a line combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ is a non-oriented bipartite graph where the set of vertices $V(\mathcal{C})$ decomposes as $V_P(\mathcal{C}) \amalg V_L(\mathcal{C})$, with:

$$V_P(\mathcal{C}) = \{v_p \mid p \in \mathcal{P}\}, \quad \text{and} \quad V_L(\mathcal{C}) = \{v_\ell \mid \ell \in \mathcal{L}\}.$$

The vertices of $V_P(\mathcal{C})$ are called *point-vertices* and those of $V_L(\mathcal{C})$ are called *line-vertices*. An edge of $\Gamma_\mathcal{C}$ joins v_ℓ to v_p if and only if $\ell \in p$. Such an edge is denoted by $e(\ell, p)$.

For a line arrangement in the projective plane the incidence graph is the dual graph of the divisor obtained by the preimage of the line arrangement in the blowing-up of the projective plane along the set of multiples points of the arrangement, see Subsection 1.2.

Definition 1.3. A *character* on a line combinatorics $(\mathcal{L}, \mathcal{P})$ is a map $\xi : \mathcal{L} \to \mathbb{C}^*$ such that

$$\prod_{\ell \in \mathcal{L}} \xi(\ell) = 1. \tag{1.1}$$

A *torsion character* on a line combinatorics $(\mathcal{L}, \mathcal{P})$ is a character ξ where for all $\ell \in \mathcal{L}$, $\xi(\ell)$ is a root of unity.

Namely, we are associating a non-zero complex number to each element of $\mathcal{L}$, such that the product of all of them equals 1. These characters have a cohomological meaning for line arrangements in the complex projective plane.

Definition 1.4. Let ξ be a character on a line combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$. For each $p \in \mathcal{P}$, we define $\xi(p) := \prod_{\ell \in p} \xi(\ell)$.

A *cycle* of $\Gamma_\mathcal{C}$ is an element of $H_1(\Gamma_\mathcal{C})$. Finally, we introduce the main object of this work.

Definition 1.5. An *inner cyclic triplet* $(\mathcal{C}, \xi, \gamma)$ is a line combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, a torsion character ξ on $\mathcal{L}$ and a cycle γ of $\Gamma_\mathcal{C}$ such that:

(1) For all line-vertex v_ℓ of γ, $\xi(\ell) = 1$;
(2) For all point-vertex v_p of γ, and for all $\ell \in p$, $\xi(\ell) = 1$;
(3) For all $p \in \ell$, with $v_\ell \in \gamma$, $\xi(p) = 1$.

The above conditions can be understood in a shorter way: all the vertices of γ and all their neighbours come from elements $m \in V(\mathcal{C})$ such that $\xi(m) = 1$.

1.2. Realization

Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2 \equiv \mathbb{P}^2$ and let $\mathcal{P}_\mathcal{A}$ be the set of multiple points of $\mathcal{A}$; then $\mathcal{C}_\mathcal{A} := (\mathcal{A}, \mathcal{P}_\mathcal{A})$ is the *combinatorics* of $\mathcal{A}$. Given a combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$, a *complex realization* of $\mathcal{C}$ is a line arrangement $\mathcal{A}$ in $\mathbb{P}^2$ such that its combinatorics agrees with $\mathcal{C}$. An *ordered realization* of an ordered combinatorics is defined accordingly. The existence of realizations of a combinatorics depends on the field $\mathbb{K}$.

Given a line arrangement $\mathcal{A}$ in $\mathbb{P}^2$, or more generally a set of irreducible curves $\mathcal{A}$ in a projective surface X, we will denote $\bigcup \mathcal{A}$ the union of those curves.

Remark 1.6. Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ with combinatorics $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ (note that $\mathcal{L} = \mathcal{A}$). For $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$ the notion $\ell \in p$ can be understood in the dual plane $\check{\mathbb{P}}^2$. For convenience we also use the notation $p \in \ell$. Let $\pi : \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the composition of the blow-ups of the points in $\mathcal{P}$; then $\widetilde{\mathcal{A}} := \pi^{-1}(\mathcal{A})$ defines a normal crossing divisor in $\widehat{\mathbb{P}}^2$ whose dual graph is exactly $\Gamma_{\mathcal{C}}$. Note also that $M_{\mathcal{A}} := \mathbb{P}^2 \setminus \bigcup \mathcal{A} = \widehat{\mathbb{P}}^2 \setminus \bigcup \widetilde{\mathcal{A}}$. Let us identify each $v_\ell \in V_L(\mathcal{C})$ with a meridian of ℓ as an element in $H_1(M_{\mathcal{A}}; \mathbb{Z})$. Note that $V_L(\mathcal{C})$ generates $H_1(M_{\mathcal{A}}; \mathbb{Z})$ and the only relation satisfied by them is (1.1). Moreover, if $v_p \in V_P(\mathcal{C})$ is identified with a meridian of $\pi^{-1}(P)$ as element of $H_1(M_{\mathcal{A}}; \mathbb{Z})$, then, the equality

$$v_p = \prod_{p \in \ell} v_\ell = \prod_{\ell \in p} v_\ell \tag{1.2}$$

holds. Via this identification the space of characters of $\mathcal{C}$ coincides with

$$H^1\big(M_{\mathcal{A}}; \mathbb{C}^*\big) = \operatorname{Hom}\big(H_1(M_{\mathcal{A}}; \mathbb{Z}), \mathbb{C}^*\big) \cong \big(\mathbb{C}^*\big)^{\sharp\mathcal{A}-1}.$$

Equation (1.2) agrees with Definition 1.4.

The space $M_{\mathcal{A}}$ is not compact and this may cause some trouble. To avoid this complication, let $\operatorname{Tub}(\mathcal{A})$ be a compact regular neighbourhood of $\mathcal{A}$ in $\mathbb{P}^2$, and let $E_{\mathcal{A}} := \overline{\mathbb{P}^2 \setminus \operatorname{Tub}(\mathcal{A})}$ be the *exterior* of the arrangement. This is an oriented 4-manifold with boundary such that the inclusion $E_{\mathcal{A}} \hookrightarrow M_{\mathcal{A}}$ is a homotopy equivalence and in particular $H_1(E_{\mathcal{A}}; \mathbb{Z}) \equiv H_1(M_{\mathcal{A}}; \mathbb{Z})$. As we have seen, if $\sharp\mathcal{A} = n+1$, then $H_1(E_{\mathcal{A}}; \mathbb{Z}) \cong \mathbb{Z}^n$ and it is freely generated by the meridians of any subset of n lines in $\mathcal{A}$.

Notation 1.7. We use the notation $\Gamma_{\mathcal{A}}$ for the incidence graph of $\Gamma_{\mathcal{C}}$.

Definition 1.8. Let $(\mathcal{C}, \xi, \gamma)$ be a triplet, where $\mathcal{C}$ is an ordered combinatorics, ξ a character and γ a circular cycle of $\Gamma_{\mathcal{C}}$. A *realization* of $(\mathcal{C}, \xi, \gamma)$ is a triplet $(\mathcal{A}, \xi_{\mathcal{A}}, \gamma_{\mathcal{A}})$:

- An ordered realization $\mathcal{A}$ of $\mathcal{C}$;
- A character $\xi_{\mathcal{A}} : \mathrm{H}_1(E_{\mathcal{A}}; \mathbb{Z}) \to \mathbb{C}^*$ such that $\xi_{\mathcal{A}}(v_\ell) = \xi(\ell)$ under the identification of Remark 1.6;
- A cycle $\gamma_{\mathcal{A}}$ in $\Gamma_{\mathcal{A}}$ which coincides with γ via the natural identification $\Gamma_{\mathcal{A}} \equiv \Gamma_{\mathcal{C}}$.

Due to this natural identifications we usually drop the subindex $\mathcal{A}$. If $(\mathcal{C}, \xi, \gamma)$ is an inner cyclic triplet, then the realization $(\mathcal{A}, \xi, \gamma)$ is *inner cyclic*.

1.3. Regular neighbourhoods

Let $\mathcal{A}$ be a line arrangement with combinatorics $\mathcal{C}$. Let us describe how to construct a compact regular neighbourhood $\operatorname{Tub}(\mathcal{A})$. There are several ways to define it (see [9,10]) and they produce isotopic results.

For each $\ell \in \mathcal{A}$ we consider a tubular neighbourhood $\operatorname{Tub}(\ell)$ of $\ell \subset \mathbb{P}^2$ and for each $p \in \mathcal{P}$ we consider a closed 4-ball $\mathbb{B}_p$ centered at p.

Definition 1.9. We say that the set $\{\mathrm{Tub}(\ell) \mid \ell \in \mathcal{A}\} \cup \{\mathbb{B}_p \mid p \in \mathcal{P}\}$ is a *compatible system of neighbourhoods in* $\mathcal{A}$ if $\forall \ell_1 \neq \ell_2$ we have

$$\mathrm{Tub}(\ell_1) \cap \mathrm{Tub}(\ell_2) = \mathring{\mathbb{B}}_p, \quad p = \ell_1 \cap \ell_2$$

and the balls are pairwise disjoint. The union of these neighbourhoods is a *regular neighbourhood of* $\mathcal{A}$. Given such a system, for each ℓ we define the *holed neighbourhood*

$$\mathcal{N}(\ell) := \overline{\mathrm{Tub}(\ell) \setminus \bigcup_{p \in \ell} \mathbb{B}_p}.$$

For each $\ell \in \mathcal{A}$ we denote by

$$\check{\ell} := \overline{\ell \setminus \bigcup_{p \in \ell} \mathbb{B}_p}.$$

This is a punctured sphere (which has as many punctures as the number of multiple points in ℓ) and the space $\mathcal{N}(\ell)$ is (non-naturally) homeomorphic to $\check{\ell} \times D^2$, where D^2 is a closed disk in $\mathbb{C}$.

Let us consider now $\pi : \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$. For each $p \in \mathcal{P}$, $\pi^{-1}(\mathbb{B}_p)$ is a tubular neighbourhood of the rational curve $E_p := \pi^{-1}(p)$; this is a locally trivial D^2-bundle, but not trivial. Note that $\pi^{-1}(\mathcal{N}(\ell))$ is naturally isomorphic to $\mathcal{N}(\ell)$ and that $\mathrm{Tub}(\widetilde{\mathcal{A}}) := \pi^{-1}(\mathrm{Tub}(\mathcal{A}))$ is a regular neighbourhood of $\widetilde{\mathcal{A}}$ obtained by plumbing the tubular neighbourhood of its irreducible components.

If $p \in \ell$ then $V_{\ell,p} := \mathcal{N}(\ell) \cap \mathbb{B}_p$ is a tubular neighbourhood of the trivial knot $\ell \cap \partial\mathbb{B}_p \subset \partial\mathbb{B}_p$, *i.e.*, a solid torus.

The manifolds $E_{\mathcal{A}}$ and $\mathrm{Tub}(\mathcal{A})$ share their boundaries $B_{\mathcal{A}} = E_{\mathcal{A}} \cap \mathrm{Tub}(\mathcal{A})$ and note also that $B_{\mathcal{A}}$ can be identified with the boundary of $\mathrm{Tub}(\widetilde{\mathcal{A}})$. This manifold is a graph manifold obtained by gluing the following pieces:

$$B_\ell := \overline{\partial\mathcal{N}(\ell) \setminus \bigcup_{p \in \ell} V_{\ell,p}}, \qquad B_p := \overline{\partial\mathbb{B}_p \setminus \bigcup_{\ell \ni p} V_{\ell,p}}.$$

Note also that $B_{\mathcal{A}} \hookrightarrow \mathrm{Tub}(\mathcal{A}) \setminus \bigcup \mathcal{A}$ is a homotopy equivalence.

1.4. Nearby cycles

Let $\mathcal{A}$ be an arrangement, and γ be a circular cycle of $\Gamma_{\mathcal{A}}$. The *support* of γ is defined as:

$$\mathrm{supp}(\gamma) = \{\ell \in \mathcal{A} \mid v_\ell \in \gamma\} = \{\ell_1, \dots, \ell_r\},$$

with cyclic order $\ell_1 < \cdots < \ell_r$ and $\ell_{r+1} := \ell_1$. Let $p_j := \ell_j \cap \ell_{j+1}$, $j = 1, \dots, r$.

Definition 1.10. An *embedding* of γ in $\mathcal{A}$ is a simple closed loop $r(\gamma) \subset \bigcup \mathcal{A}$ defined as follows. Take a point $q_j \in \check{\ell}_j$ ($q_{r+1} := q_1$), $j = 1, \dots, r$. We denote by p_j^α a point in $\ell_j \cap \mathbb{B}_{p_j}$ and by p_j^ω a point in $\ell_{j+1} \cap \mathbb{B}_{p_j}$. Let ρ_j^α be a radius in ℓ_j from

p_j to p_j^α and ρ_j^ω be a radius in ℓ_{j+1} from p_j to p_j^ω. Pick up arbitrary simple paths α_j from q_j to p_j^α in $\check{\ell}_j$, $j = 1, \ldots, r$, and ω_j from p_{j-1}^ω to q_j, $j = 2, \ldots, r+1$. Then:

$$r(\gamma) := \alpha_1 \cdot (\rho_1^\alpha)^{-1} \cdot \rho_1^\omega \cdot \omega_2 \cdot \alpha_2 \cdot \ldots \cdot (\rho_r^\alpha)^{-1} \cdot \rho_r^\omega \cdot \omega_{r+1}.$$

Definition 1.11. A *nearby cycle* $\widetilde{\gamma}$ associated with γ is a smooth path in $B_{\mathcal{A}}$, homologous to an embedding $r(\gamma)$ of γ in $\operatorname{Tub}(\mathcal{A})$, lying in

$$\left(\bigcup_{j=1}^{r} \mathcal{N}(\ell_j) \cup \bigcup_{j=1}^{r} \mathbb{B}_{p_j}\right) \setminus \bigcup \mathcal{A}.$$

Remark 1.12. Note that if $p \neq p_j$ a nearby cycle $\widetilde{\gamma}$ can intersect $\mathbb{B}_p$ only at some $V_{\ell_k,p}$ for some k; there are always nearby cycles which do not intersect $\mathbb{B}_p$ for $p \neq p_j$.

2. Invariant

Let $(\mathcal{C}, \xi, \gamma)$ be a inner-cyclic triplet; and suppose that $(\mathcal{A}, \xi, \gamma)$ is a realization. Denote by i the inclusion map of the boundary manifold in the exterior, *i.e.*, $i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$. We consider the following composition map:

$$\chi_{(\mathcal{A},\xi)} : \mathrm{H}_1(B_{\mathcal{A}}) \xrightarrow{i_*} \mathrm{H}_1(E_{\mathcal{A}}) \xrightarrow{\xi} \mathbb{C}^*.$$

Let $\widetilde{\gamma}$ be a nearby cycle associated with γ. As we are going to prove in Lemma 2.2, we can define

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) := \chi_{(\mathcal{A},\xi)}(\widetilde{\gamma}).$$

Our main goal is to prove that this is a topological invariant, as we state here.

Theorem 2.1 (Main result). *If $(\mathcal{A}, \xi, \gamma)$ and $(\mathcal{A}', \xi, \gamma)$ are two inner-cyclic realizations of $(\mathcal{C}, \xi, \gamma)$ with the same (oriented and ordered) topological type, then*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) = \mathcal{I}(\mathcal{A}', \xi, \gamma).$$

The aim of this section is to prove Theorem 2.1. The first step is to prove the following Lemma which shows that for a given inner-cyclic realization $(\mathcal{A}, \xi, \gamma)$, the image of a nearby cycle $\widetilde{\gamma}$ by $\chi_{(\mathcal{A},\xi)}$ depends only on γ.

Lemma 2.2. *Let $(\mathcal{A}, \xi, \gamma)$ be an inner-cyclic realization. If $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are nearby cycles associated with γ, then:*

$$\chi_{(\mathcal{A},\xi)}(\widetilde{\gamma}) = \chi_{(\mathcal{A},\xi)}(\widetilde{\gamma}').$$

Proof. The choice of a nearby cycle in $B_{\mathcal{A}}$ associated with γ depends first on the embedding $r(\gamma)$ in $\mathcal{A}$, see Definition 1.10 and use the notation inside. There is some freedom in the choice of $r(\gamma)$; in $r(\gamma) \cap \ell_j$ we can add some meridians around the multiple points of $\mathcal{A}$ in ℓ_j; recall that for any such multiple point p we have $\xi(p) = 1$ by Definition 1.5 (3).

Once $r(\gamma)$ has been chosen, $\widetilde{\gamma}$ lies in $B_{\mathcal{A}}$, and is homologous to $r(\gamma)$ in $\operatorname{Tub}(\mathcal{A})$. Two different representants will differ by the meridians v_{ℓ_j} and by cycles in $\mathbb{B}_{p_j} \setminus \mathcal{A}$; these ones are linear combinations of meridians v_ℓ for $\ell \in p_j$.

Summarizing the difference of two nearby cycles is a linear combination of meridians v_ℓ, where either $\ell \in \operatorname{supp}(\gamma)$ or $\ell \in p_j$, and meridians v_p, $p \in \ell_j$. By Definition 1.5 all these meridians are in $\ker \xi$ and the result follows. □

We have proved with this lemma that $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ is well-defined for a particular $\mathcal{A}$. The following result studies the behavior of this invariant under orientation and order preserving homeomorphisms. We prove that there are nearby cycles of the first realization which are sent to nearby cycles of the second realization; note that we do not need the stronger result that would say that the image of *any* nearby cycle is a nearby cycle.

Lemma 2.3. *Let $(\mathcal{C}, \xi, \gamma)$ be an ordered inner-cyclic triplet. Suppose that $(\mathcal{A}, \xi, \gamma)$ and $(\mathcal{A}', \xi, \gamma)$ are two ordered realizations of $(\mathcal{C}, \xi, \gamma)$ such that there exists a homeomorphism $\phi : (\mathbb{P}^2, \mathcal{A}) \to (\mathbb{P}^2, \mathcal{A}')$ preserving orders and orientations of the lines. Then there is a nearby cycle $\widetilde{\gamma}$ associated to γ in $B_{\mathcal{A}}$, such that $\phi(\widetilde{\gamma})$ is isotopic to a nearby cycle associated with γ in $B_{\mathcal{A}'}$.*

Proof. If ℓ is a line in $\mathcal{A}$, ℓ' is the corresponding line in $\mathcal{A}'$; we use the same convention for multiple points.

We start fixing a regular neighbourhood $\operatorname{Tub}(\mathcal{A}')$ obtained as a union of $\mathbb{B}_{p'}$ and $\mathcal{N}(\ell')$. Since ϕ is a homeomorphism we can construct a regular neighbourhood $\operatorname{Tub}(\mathcal{A})$ obtained as a union of $\mathbb{B}_p$ and $\mathcal{N}(\ell)$, such that $\phi(\mathbb{B}_p) \subset \mathbb{B}_{p'}$ and $\phi(\mathcal{N}(\ell)) \subset \mathcal{N}(\ell')$.

Let us choose an embedding $r(\gamma)$ in $\mathcal{A}$ such that $r(\gamma)$ is disjoint from $\mathbb{B}_p$ for $p \neq p_1, \dots, p_r$ which is always possible. Moreover we can construct a continuous family of cycles $\widetilde{\gamma}_t$, $t \in (0, 1]$ with the following properties:

- $\{\widetilde{\gamma}_t\}_{t \in [0,1]}$ defines an embedding of $\gamma \times [0, 1]$ in $\operatorname{Tub}(\mathcal{A})$;
- $\widetilde{\gamma}_0 = \gamma$;
- $\{\widetilde{\gamma}_t\}_{t \in (0,1)}$ is in $\operatorname{Tub}(\mathcal{A}) \setminus (\mathcal{A} \cup B_{\mathcal{A}})$;
- $\widetilde{\gamma}_1$ is a nearby cycle;
- This family is still disjoint with $\mathbb{B}_p$ for $p \neq p_1, \dots, p_r$.

Then, $\phi(\widetilde{\gamma}_1)$ is a nearby cycle associated with γ which can be deformed to a nearby cycle in $B_{\mathcal{A}'}$. □

Remark 2.4. The nearby cycles used in Section 4 are different of the one used in this proof. Those cycles may intersect the boundary of $\partial\mathcal{N}$, and this is important for the computation of the invariant, but it does not affect our goal.

Proof of Theorem 2.1. Suppose that there exists $\phi : (\mathbb{P}^2, \mathcal{A}) \to (\mathbb{P}^2, \mathcal{A}')$ preserving orders and orientations of the lines. Hence ϕ induces an isomorphism $\phi_* : H_1(\mathbb{P}^2 \setminus \mathcal{A}) \to H_1(\mathbb{P}^2 \setminus \mathcal{A}')$ preserving ξ. Let $\widetilde{\gamma}$ be an arbitrary nearby cycle associated with γ. By Lemma 2.2, $\mathcal{I}(\mathcal{A}, \xi, \gamma) = \chi_{(\mathcal{A},\xi)}(\widetilde{\gamma})$ depends only on γ. We can then assume that $\widetilde{\gamma}$ is one of the particular nearby cycle considered in Lemma 2.3. But this lemma implies that $\phi(\widetilde{\gamma})$ is isotopic to a nearby cycle $\widetilde{\gamma}'$ associated to γ in $B_{\mathcal{A}'}$. It follows that $\chi_{(\mathcal{A},\xi)}(\widetilde{\gamma}') = \chi_{(\mathcal{A},\xi)}(\widetilde{\gamma})$, and we conclude using Lemma 2.2. $\square$

Proposition 2.5. *Let $(\mathcal{A}, \xi, \gamma)$ and $(\overline{\mathcal{A}}, \xi, \gamma)$ be two inner cyclic complex conjugated realization arrangements. We have then:*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma)^{-1} = \mathcal{I}\big(\overline{\mathcal{A}}, \xi, \gamma\big).$$

Proof. The complex conjugation $(\mathbb{P}^2, \mathcal{A}) \to (\mathbb{P}^2, \overline{\mathcal{A}})$ is an ordered homeomorphism which preserves the orientation of $\mathbb{P}^2$ but exchanges the orientation of the line. For this reason the action of the homeomorphism sends ξ to ξ^{-1} (since the conjugation sends a meridian v_ℓ to $(v_{\bar{\ell}}^{-1}$, with complex notation). The result follows from the fact that a nearby cycle for $(\overline{\mathcal{A}}, \xi, \gamma)$ it is also a nearby cycle for $(\overline{\mathcal{A}}, \xi^{-1}, \gamma)$. $\square$

Corollary 2.6. *If $\mathcal{A}$ is a complexified real arrangement, that is the complexification of a line arrangement defined on $\mathbb{RP}^2$, then:*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) \in \{-1, 1\}.$$

The same result occurs if there is a continuous path of combinatorially equivalent line arrangements connecting $\mathcal{A}$ and $\overline{\mathcal{A}}$.

Example 2.7. Let $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ be the combinatorics defined by $\mathcal{L} = \{L_0, \cdots, L_6\}$ and

$$\begin{aligned}\mathcal{P} = \Big\{ &\{L_0, L_1, L_2\}, \{L_0, L_3\}, \{L_0, L_4, L_5\}, \{L_0, L_6\}, \{L_1, L_3, L_5\}, \\ &\{L_1, L_4, L_6\}, \{L_2, L_3, L_4\}, \{L_2, L_5, L_6\}, \{L_3, L_6\} \Big\}.\end{aligned}$$

The following triplet $(\mathcal{C}, \xi, \gamma)$ is inner-cyclic:

- ξ is the character on $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ defined by

$$(L_0, L_1, \cdots, L_6) \longmapsto (1, -1, -1, 1, -1, -1, 1);$$

- γ is the cycle of $\Gamma_{\mathcal{C}}$ defined by

$$v_{L_0} \to v_{P_{0,3}} \to v_{L_3} \to v_{P_{3,6}} \to v_{L_6} \to v_{P_{0,6}} \to v_{L_0}.$$

The arrangement $\mathscr{C}_7$ pictured in Figure 2.1 is –up to projective transformation– the only one realization of the combinatorics $\mathcal{C}$. There is a nearby cycle $\widetilde{\gamma}$ associated with the cycle γ such that its image by the map $i_* : \mathrm{H}_1(B_{\mathscr{C}_7}) \to \mathrm{H}_1(E_{\mathscr{C}_7})$ is $-v_5$ (an algorithm to compute is given in Section 4). Then we have:

$$\mathcal{I}(\mathscr{C}_7, \xi, \gamma) = \xi(-v_5) = -1.$$

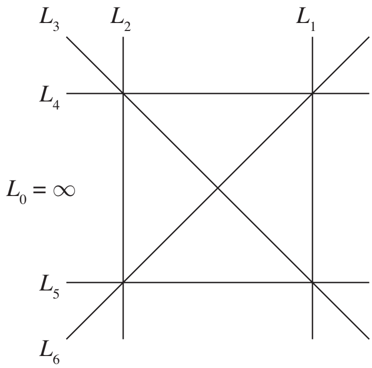

Figure 2.1. The Ceva-7 arrangement $\mathscr{C}_7$.

3. Characteristic varieties

The characteristic varieties of an arrangement can be defined as the jumping loci of twisted cohomology of the rank-one be the local system. They only depend on its fundamental group. For $\xi \in H^1(M_{\mathcal{A}}; \mathbb{C}^*)$, let $\mathcal{L}_\xi$ be the local system of coefficients defined by ξ.

Definition 3.1. The *characteristic varieties* of an arrangement $\mathcal{A}$ are:

$$\mathcal{V}_k(\mathcal{A}) = \left\{ \xi \in \mathbb{T}(\mathcal{A}) \mid \dim_{\mathbb{C}} \left(\mathrm{H}^1(M_{\mathcal{A}}; \mathcal{L}_\xi)\right) \geq k \right\}.$$

Definition 3.2. The *depth* of a character $\xi \in \mathbb{T}(\mathcal{A})$ is:

$$\operatorname{depth}(\xi) = \max\{k \in \mathbb{N} \mid \xi \in \mathcal{V}_k\} = \dim_{\mathbb{C}} \mathrm{H}^1\left(M_{\mathcal{A}}; \mathcal{L}_\xi\right).$$

Hence for the study of characteristic varieties, we need to be able to compute the twisted cohomology spaces of $M_{\mathcal{A}}$.

3.1. Geometric interpretation of the notion of inner cyclic

Let us recall that $\pi : \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ over the points of $\mathcal{P}$; the main goal of the construction of $\widehat{\mathbb{P}}^2$ is to obtain $\widehat{\mathcal{A}} := \pi^{-1}(\mathcal{A})$ as a normal crossing divisor (in fact, we need only to blow up the points of multiplicity at least three, but it is harmless to do extra blow-ups). By construction, we have that $\widehat{\mathbb{P}}^2 \setminus \bigcup \widehat{\mathcal{A}} \equiv E_{\mathcal{A}}$, then a character ξ on $\pi_1(E_{\mathcal{A}})$ can be view as a character on $\pi_1(\widehat{\mathbb{P}}^2 \setminus \bigcup \widehat{\mathcal{A}})$ (also noted ξ). Let $\widehat{\Gamma}_{\mathcal{A}}$ be the dual graph of $\widehat{\mathcal{A}}$.

Definition 3.3. A component $H \in \widehat{\mathcal{A}}$ is *unramified* for the character ξ if $\xi(v_H) = 1$. It is *inner unramified* for ξ if it is unramified and all its neighbours (in $\widehat{\Gamma}_{\mathcal{A}}$) too. The set of all the inner unramified components of $\widehat{\mathcal{A}}$ is denoted by $\mathcal{U}_\xi \subset \widehat{\mathcal{A}}$. The dual graph of $\widehat{\mathcal{A}}$ of inner unramified components is denoted by $\widehat{\Gamma}_{\mathcal{U}_\xi}$.

Definition 3.4. Let $\mathcal{A}$ be an arrangement. A character ξ is *inner-cyclic* if it is torsion and $b_1(\widehat{\Gamma}_{\mathcal{U}_\xi}) > 0$.

Proposition 3.5. *An* inner-cyclic arrangement *is the data of a triplet* $(\mathcal{A}, \xi, \gamma)$*, where* ξ *is an inner-cyclic character on* $E_{\mathcal{A}}$ *and* γ *a cycle of* $\widehat{\Gamma}_{\mathcal{U}_\xi}$.

Remark 3.6. With this point of view, a nearby cycle is a cycle leaving in the boundary of a regular neighbourhood of the union of inner unramified components.

3.2. Quasi-projective depth

The depth of a torsion character is decomposed in two terms, the projective and quasi-projective depth, see [2, 12]. Summarizing, if ξ is a torsion character of order n, then there is an n-fold unbranched ramified quasi-projective cover $\rho : E_{\mathcal{A}}^{\xi} \to E_{\mathcal{A}}$ where $\sigma : E_{\mathcal{A}}^{\xi} \to E_{\mathcal{A}}^{\xi}$ generates the deck group of the cover. There is a natural isomorphism from $H^1(E_{\mathcal{A}}; \mathcal{L}_\xi)$ to the eigenspace of $H^1(E_{\mathcal{A}}^{\xi}; \mathbb{C})$ for σ with eigenvalue $e^{\frac{2i\pi}{n}}$.

Let $\bar{\rho} : X^{\xi} \to \widehat{\mathbb{P}}^2$ be a smooth model of the projectivization of ρ, where $\bar{\sigma}$ generates the deck group. The inclusion $j_N : E_{\mathcal{A}}^{\xi} \hookrightarrow X^{\xi}$ induces an injection

$$j_N^* : \mathrm{H}^1\left(X^{\xi}; \mathbb{C}\right) \to \mathrm{H}^1\left(E_{\mathcal{A}}^{\xi}; \mathbb{C}\right).$$

We denote by $j_{N,\xi}^*$ the restriction of j_N^* to the eigenspaces for $\sigma, \bar{\sigma}$ with eigenvalue $e^{\frac{2i\pi}{n}}$.

Definition 3.7. Let ξ be a torsion character on $\pi_1(E_{\mathcal{A}})$. The *projective depth* of ξ is $\dim \operatorname{Im} j_{N,\xi}^*$ while the *quasi-projective depth* of ξ is:

$$\overline{\operatorname{depth}}(\xi) = \dim \operatorname{coker}\left(j_{N,\xi}^*\right).$$

Note that there are known formulas for the computation of the projective depth given by Libgober [14], see [1] for details. Moreover, there is a *finite-time* algorithm to compute this projective depth for any character. We explain now the method of [1] to compute the quasi-projective depth.

We are going to construct a matrix for a *twisted* hermitian intersection form $\cdot$ in a vector space having as basis the elements of $\mathcal{U}_\xi$; for an arbitrary order of this basis we will consider a square matrix A_ξ of size $\#\mathcal{U}_\xi$, where coefficients are indexed by elements of $\mathcal{U}_\xi$. This matrix will depend on the choice of a maximal tree $\mathcal{T}_{\mathcal{U}_\xi}$ of $\widehat{\Gamma}_{\mathcal{U}_\xi}$ (maybe a maximal forest, since $\widehat{\Gamma}_{\mathcal{U}_\xi}$ is non necessarily connected). For each (oriented) edge e not in $\mathcal{T}_{\mathcal{U}_\xi}$ we consider a cycle γ_e consisting on e and a linear chain of $\widehat{\Gamma}_{\mathcal{U}_\xi}$ connecting the final end of e with its starting point. Let us denote:

$$\chi(e) := \begin{cases} \mathcal{I}(\mathcal{A}, \xi, \gamma_e) & \text{if } e \notin \mathcal{T}_{\mathcal{U}_\xi} \\ 1 & \text{otherwise.} \end{cases}$$

The coefficient associated to two components E and F is:

$$\begin{cases} E \cdot F & \text{if } E = F \\ \sum_e \chi(e) & \text{if } E \neq F, \end{cases}$$

where the sum runs along all the oriented edges from E to F. Note that since there is at most one edge between E and F, then the sum $\sum_e \chi(e)$ is either void (hence it vanishes) or consists of a single summand.

Theorem 3.8 ([2]). *Let $\mathcal{A}$ be an arrangement, and let ξ be a torsion character on $\pi_1(E_{\mathcal{A}})$ then:*

$$\overline{\text{depth}}(\xi) = \text{corank}(A_\xi).$$

The description of the inclusion map done in [11] (see also Section 4) and the result on the invariant obtain in Section 2 allow to compute explicitly the quasi-projective depth of any torsion character.

Example 3.9. In the case of the arrangement $\mathscr{C}_7$ with the character ξ defined in Example 2.7, the matrix A_ξ is:

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \chi_{(\mathscr{C}_7,\xi)}(\gamma) \\ 1 & \chi_{(\mathscr{C}_7,\xi)}(\gamma)^{-1} & -1 \end{pmatrix}.$$

Example 2.7 implies that $\overline{\text{depth}}(\xi) = 2$. Remark that this result is in harmony with the one obtained by D. Cohen and A. Suciu in [8].

4. Computation of the invariant

The definition of $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ is quite clear, and most probably it may have a more algebraic description. Nevertheless, its actual definition is topological, and we need concrete models of the topology of $\mathcal{A}$ and more specifically of the embedding $B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$ at a homology level, see [11,12].

In this section we compute the invariant $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ from a *wiring diagram* of $\mathcal{A}$. As we can see below, the wiring diagram is a (3, 1)-dimensional model of the pair ($\mathbb{P}^2$, $\mathcal{A}$) which contains all the multiple points of $\mathcal{A}$. Given a cycle γ in $\Gamma_{\mathcal{A}}$, we will use the diagram to construct a specific nearby cycle $\widetilde{\gamma}$ and calculate its value by the character ξ. This gives a general method to compute the invariant from the equations of $\mathcal{A}$.

Let us fix an arbitrary line $\ell_0 \in \mathcal{A}$ which will be considered as the line at infinity. Consider the affine arrangement $\mathcal{A}_0 := \mathcal{A} \setminus \ell_0$ of $\mathbb{C}^2 \equiv \mathbb{P}^2 \setminus \ell_0$; for a projective line $\ell \in \mathcal{A}$, $\ell \neq \ell_0$ we will denote by L the corresponding affine line $\ell \setminus \ell_0$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection, *generic* with respect to $\mathcal{A}$ in the sense that for all $i \in \{1, \ldots, n\}$, the restriction of $\pi_{|L_i}$ is a homeomorphism. We choose the coordinates in $\mathbb{C}^2$ such that π is the first projection. Suppose that the multiple points of $\mathcal{A}$ lie in different fibers of π and that their images x_i satisify $\Re(x_1) < \cdots < \Re(x_k)$.

Consider a smooth path $\nu : [0,1] \to \mathbb{C}$ whose image starts from a regular value $x_0 \in \pi(\mathrm{Tub}(L_0))$ and passes through $x_1, \dots, x_k$ in order with $\nu(t_j) = x_j$. The *braided wiring diagram* $W_{\mathcal{A}}$ associated to ν (see [5]) is defined by:

$$W_{\mathcal{A}} = \left\{(t, y) \in [0,1] \times \mathbb{C} \mid \ (\nu(t), y) \in \bigcup \mathcal{A}_0\right\}.$$

The space $W_{\mathcal{A}} \subset [0,1] \times \mathbb{C}$ is a singular braid with n strings labelled according to the lines, whose singular points correspond to the multiple points of $\mathcal{A}$. For $u = 0, \dots, k-1$, the wiring diagram over $\nu(t)$, for $t \in (t_u, t_{u+1})$, is identified with a regular braid β_u in the braid group B_n with n strands, see Figure 4.1.

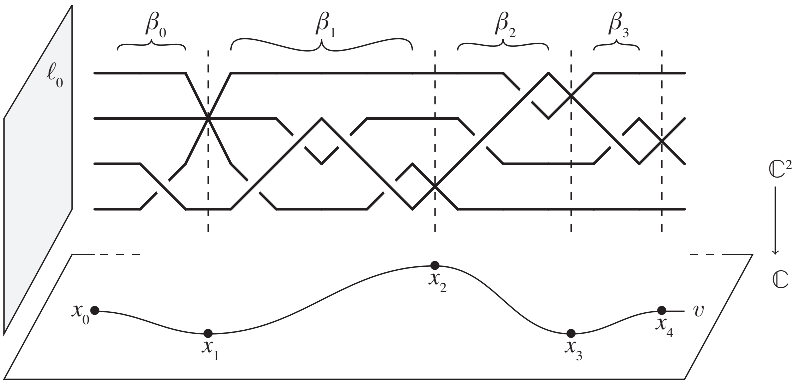

Figure 4.1. Decomposition of a wiring diagram.

Associated to any singular fiber $\pi^{-1}(x_u)$, containing the multiple point $P_u = L_{p_1} \cap \cdots \cap L_{p_{m(u)}}$ (where $m(u)$ is the multiplicity of P_u), let τ_u be the local positive half-twist between the strings associated with the lines $L_{p_1}, \dots, L_{p_{m(u)}}$ (note that those strings are consecutive) and leaving straight the other strings.

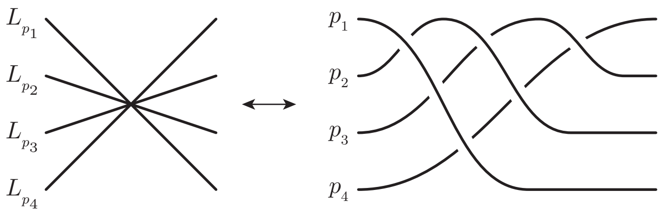

Figure 4.2. The half-twist associated to a singular point of multiplicity 4.

For all $u, v \in \{0, \dots, k\}$ with $u \neq v$, define

$$\beta_{u,v} := \begin{cases} \beta_{v-1} \cdot \tau_{v-1} \cdot \beta_{v-2} \cdot \dots \cdot \beta_{u+1} \cdot \tau_{u+1} \cdot \beta_u & u < v \\ (\beta_{v,u})^{-1} & v < u. \end{cases} \tag{4.1}$$

The braid $\beta_{u,v}$ is obtained from $W_{\mathcal{A}}$ by taking the sub-singular braid bounded by the singular points P_u and P_v and replacing the singular crossings by the corresponding local half-twist τ. It should be noted that the braided wiring diagram encodes in fact the braid monodromy of $\mathcal{A}$ relative to the projection π. The operation of replacing singular crossings by half twists corresponds to a particular choice of a geometric basis of $\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_k\})$, obtained from perturbations of the path ν.

For any braid β obtained from the wiring diagram, and all $k, l = 1, \ldots, n$ with $k \neq l$, let $a_{k,l}(\beta)$ be defined as follows. If p is a crossing of β between the strings associated with the lines L_k and L_l, let $c_{k,l}(p) = 1$ if the string of L_k goes over the string of L_l and 0 otherwise. Let $w(p)$ be the sign of the crossing. Hence

$$a_{k,l}(\beta) := \left(\sum_p w(p) c_{k,l}(p) \right) \cdot v_k \in H_1(E_{\mathcal{A}}),$$

where the sum is taken over all the crossings of β between the strings associated with L_k and L_l.

Remark 4.1. The coefficient of $a_{k,l}(\beta)$ is the algebraic sum of crossings in β between the strings associated with L_k and L_l where the one of L_k goes over the one of L_l. Note that if β, $\widetilde{\beta}$ are two *consecutive* braids in the diagram, then

$$a_{k,l}(\beta \cdot \widetilde{\beta}) = a_{k,l}(\beta) + a_{k,l}(\widetilde{\beta}).$$

The cycle γ of $\Gamma_{\mathcal{A}}$ can be described by a cyclically ordered sequence of line-vertices $v_{\ell_{i_1}}, \cdots, v_{\ell_{i_r}}$. Up to re-ordering the lines, suppose that $i_q \neq 0$ for all $q \in \{1, \cdots, r\}$ (this will simplify the computation of the invariant, though it is not necessary). Let j_q be the sequence of the indices of the point-vertices of γ, such that $L_{i_q} \cap L_{i_{q+1}} = P_{j_q}$ $(i_{r+1} \equiv i_1, j_r \equiv j_1)$.

Remark 4.2. If the assumption $i_q \neq 0$ for all $q \in \{1, \cdots, r\}$ is not possible then γ passes through all the lines of the arrangement and the invariant is always trivial, since the character must be trivial.

Through the wiring diagram $W_{\mathcal{A}}$, the cycle γ can be seen directly in $\mathcal{A}$, in a unique way, by identifying its vertices to their corresponding wires between two multiple points. Since γ does not pass through the line at infinity, we obtain a cycle in $W_{\mathcal{A}}$, uniquely defined. Indeed, one starts from the singular point in $\pi^{-1}(x_{i_1})$, along the string of L_{i_1}, we go to the singular point containing both L_{i_1} and L_{i_2}; then along the string of L_{i_2}, we go to the singular point containing both L_{i_2} and L_{i_3} and so on. We finish by going back to the singular point P_{i_1} along the string of L_{i_r}.

This process *almost* produces an embedding $r(\gamma)$ as in Definition 1.10. Such an embedding can be obtained by avoiding the singular points in the wires corresponding to points p not in the cycle. In order to construct the nearby cycle $\widetilde{\gamma}$ we follow the procedure in [11, Section 4] where the *pushed* cycle in the boundary manifold $B_{\mathcal{A}}$ is called a *framed cycle*.

If the wire of some L_{i_j} passes through a point $p \neq P_{j_q}$, then the part of framed cycle in $\partial\mathbb{B}_p$ is in the boundary of a regular neighbourhood of the trivial knot $L_{i_j} \cap \partial\mathbb{B}_p$ in $\partial\mathbb{B}_p$. Hence the framed cycle $\widetilde{\gamma}$ is a nearby cycle.

It is worth noticing that the way to push γ off $\mathcal{A}$ depends on conventions, in particular around the multiple points of $\mathcal{A}$ different from P_{j_q}. Different conventions might give another nearby cycle, but its image by the character ξ depends only on γ.

The next step is to compute the image of $\widetilde{\gamma}$ by i_* induced by the inclusion map $B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}$ at homology level. We use an Abelian version of [11, Theorem 4.3], in the following Lemma 4.3.

Lemma 4.3. *Let $s, t \in \{1, \cdots, n\}$ with $s \neq t$, and $\mathfrak{e}_{s,t}$ be the cycle of $\Gamma_{\mathcal{A}}$ defined by v_{L_0}, v_{L_s} and v_{L_t}. Suppose that the intersection $L_s \cap L_t$ lies in the fiber $\pi^{-1}(x_u)$. There exists a nearby cycle $\widetilde{\mathfrak{e}}_{s,t}$ associated with $\mathfrak{e}_{s,t}$ such that the image by $i_* : H_1(B_{\mathcal{A}}) \to H_1(E_{\mathcal{A}})$ induced by the inclusion is*

$$i_*(\widetilde{\mathfrak{e}}_{s,t}) = \sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}(\beta_{0,u}) + \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{k,t}(\beta_{u,0}) \in \mathrm{H}_1(E_{\mathcal{A}}).$$

We postpone the proof of Lemma 4.3 to the end of the section. For all $i, j = 1, \cdots, n$ with $i \neq j$, the cycles $\mathfrak{e}_{i,j}$ generate $H_1(\Gamma_{\mathcal{A}})$ and γ can be decomposed as $\gamma = \mathfrak{e}_{i_1,i_2} + \mathfrak{e}_{i_2,i_3} + \cdots + \mathfrak{e}_{i_{r-1},i_r}$. Since γ does not pass through L_0, then $i_r = i_1$. Let us recall that for $q \in \{1, \cdots, r-1\}$, P_{j_q} is the intersection point of L_{i_q} and $L_{i_{q+1}}$, it is contained in $\pi^{-1}(x_{j_q})$. To simplify the notation, we identify j_0 with j_{r-1}. As a direct consequence of the Lemma 4.3, the image of $\widetilde{\gamma}$ by the character ξ is given in the next Proposition.

Proposition 4.4. *Let $\widetilde{\gamma}$ be a nearby cycle for an inner cyclic realization $(\mathcal{A}, \xi, \gamma)$. With the previous notations, we have:*

$$\chi_{(\mathcal{A},\xi)}(\widetilde{\gamma}) = \xi\left(\sum_{q=1}^{r-1}\sum_{\substack{k=1,\\ k\neq i_q}}^{n} a_{k,i_q}(\beta_{j_{q-1},j_q})\right).$$

Proof. For any choice of nearby cycles $\widetilde{\mathfrak{e}}_{i,j}$ associated with $\mathfrak{e}_{i,j}$, the sum $\widetilde{\gamma} = \widetilde{\mathfrak{e}}_{i_1,i_2} + \cdots + \widetilde{\mathfrak{e}}_{i_{r-1},i_r}$ is a nearby cycle associated with γ. Remark that Lemma 2.2 implies that the image of $i_*(\widetilde{\gamma})$ by ξ does not depend on the choice of $\widetilde{\gamma}$. From Lemma 4.3, we have:

$$\xi \circ i_*\big(\widetilde{\mathfrak{e}}_{i_q,i_{q+1}}\big) = \xi\big(A_{i_q,j_q} + B_{i_{q+1},j_q}\big)$$
$$A_{i_q,j_q} := \sum_{\substack{k=1,\\ k\neq i_q}}^{n} a_{k,i_q}\big(\beta_{0,j_q}\big), \qquad B_{i_{q+1},j_q} := \sum_{\substack{k=1,\\ k\neq i_{q+1}}}^{n} a_{k,i_{q+1}}\big(\beta_{j_q,0}\big).$$

For $q \in \{1, \cdots, r-1\}$, using (4.1) we have

$$\beta_{j_q,0} = \beta_{j_{q+1},0} \cdot \tau_{j_{q+1}}^{-1} \cdot \beta_{j_q,j_{q+1}} \qquad \text{if } j_q > j_{q+1},$$
$$\beta_{0,j_{q+1}} = \beta_{j_q,j_{q+1}} \cdot \tau_{j_q} \cdot \beta_{0,j_q} \qquad \text{if } j_{q+1} > j_q,$$

where j_r is identified with j_1. Then we have:

$$\begin{aligned} B_{i_{q+1},j_q} + A_{i_{q+1},j_{q+1}} &= \sum_{\substack{k=1,\\ k\neq i_{q+1}}}^{n} \Big(a_{k,i_{q+1}}\big(\beta_{j_q,0}\big) + a_{k,i_{q+1}}\big(\beta_{0,j_{q+1}}\big)\Big) \\ &= \sum_{\substack{k=1,\\ k\neq i_{q+1}}}^{n} a_{k,i_{q+1}}\big(\beta_{j_q,j_{q+1}}\big) + \theta_{i_q,i_{q+1}}, \end{aligned}$$

where $\theta_{i_q,i_{q+1}}$ is the *linking cycle* of γ with the strands of the braid $\tau_{j_{q+1}}$ or τ_{j_q} (depending of the values of j_q and j_{q+1}). More precisely, if we denote $J_q := \min(j_q, j_{q+1})$ then

$$\theta_{i_q,i_{q+1}} = \sum_{\substack{k=1,\\ k\neq i_q}}^{n} a_{k,i_q}\big(\tau_{J_q}^{\varepsilon_q}\big), \qquad \varepsilon_q := \begin{cases} -1 & \text{if } J_q = j_{q+1} \\ 1 & \text{if } J_q = j_q. \end{cases}$$

Since $(\mathcal{A}, \xi, \gamma)$ is an inner cyclic realization, the meridians of the lines involved in τ_{J_q} are in $\ker \xi$. Finally, with a re-indexation of q, we have:

$$i_*(\widetilde{\gamma}) = \sum_{q=1}^{r-1} \sum_{\substack{k=1,\\ k\neq i_q}}^{n} a_{k,i_q}\big(\beta_{j_{q-1},j_q}\big) + \Theta,$$

where Θ is an element of $\ker(\xi)$. □

Remark 4.5. If the support of γ contains only three lines, it is easier to assume that one of them is the line at infinity L_0 and use Lemma 4.3 for the computation.

Proof of Lemma 4.3. As mentioned above, we *push* a model of $\mathfrak{e}_{s,t}$ from $\mathcal{A}$ to the boundary manifold $B_{\mathcal{A}}$. The cycle obtained is a *framed* cycle. To compute the image of this particular nearby cycle, we express it as a product of meridians and a *geometric* cycle $\mathcal{E}_{s,t}$.

By construction, geometric cycles bound 2-cells with holes in the exterior $E_{\mathcal{A}}$ and counting intersections allows a direct computation of $i_* : H_1(B_{\mathcal{A}}) \to H_1(E_{\mathcal{A}})$ in this basis. The previous expression can be obtained, by [11, Proposition 4.1], using the unknotting map δ (sending framed cycle on the corresponding geometric cycle). The contribution of each singular point P correspond to the contribution of the strings overcrossing the string s and the string t in τ_P. An Abelian computation shows that:

$$\delta(\widetilde{\mathfrak{e}}_{s,t}) = \mathcal{E}_{s,t} = -\sum_{q=1}^{m-1} \left(\sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}(\tau_{P_q}) \right) + \widetilde{\mathfrak{e}}_{s,t} + \sum_{q=1}^{m-1} \left(\sum_{\substack{k=1,\\ k\neq t}}^{n} a_{k,t}(\tau_{P_q}) \right).$$

The model of $\mathfrak{e}_{s,t}$ in $W_{\mathcal{A}}$ is composed of two parts:

- A string labelled s from $\pi^{-1}(x_0)$ to $L_s \cap L_t \in \pi^{-1}(x_u)$, and
- A string labelled t from $\pi^{-1}(x_u)$ to $\pi^{-1}(x_0)$.

Let S_s (respectively S_t) be the set of arcs that go over the string s (respectively t) between $\pi^{-1}(x_0)$ and $\pi^{-1}(x_u)$. For each arc ς in S_s or S_t, let $\operatorname{sgn}(\varsigma) \in \{\pm 1\}$ be the sign of the basis $\{\varsigma, \mathfrak{e}_{s,t}\}$ (ς is oriented from left to right in $W_{\mathcal{A}}$) where ς and $\mathfrak{e}_{s,t}$ are considered as vectors in $\mathbb{R}^2$. Let $v_\varsigma \in H_1(E_{\mathcal{A}})$ be the meridian of the line corresponding to ς. Once again, an Abelian computation shows that:

$$
\begin{aligned}
i_*(\mathcal{E}_{s,t}) = \mu_{s,t} &= \sum_{\varsigma \in S_s} \operatorname{sgn}(\varsigma) v_\varsigma + \sum_{\varsigma \in S_t} \operatorname{sgn}(\varsigma) v_\varsigma \\
&= \sum_{q=1}^{u} \sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}(\beta_{q-1,q}) + \sum_{q=1}^{u} \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{k,t}\big(\beta_{q-1,q}^{-1}\big).
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&i_*(\widetilde{\mathfrak{e}}_{s,t}) \\
&= \sum_{q=1}^{u-1} \left(\sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}\big(\tau_{P_q}\big) \right) + \sum_{q=1}^{u} \left(\sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}\big(\beta_{q-1,q}\big) \right) + \sum_{q=1}^{u} \left(\sum_{\substack{k=1,\\ k\neq t}}^{n} a_{k,t}\big(\beta_{q-1,q}^{-1}\big) \right) \\
&\quad + \sum_{q=1}^{u-1} \left(\sum_{\substack{k=1,\\ k\neq t}}^{n} a_{k,t}\big(\tau_{P_q}^{-1}\big) \right) \\
&= \sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}\left(\beta_{0,1} \cdot \tau_{P_1} \cdot \ldots \cdot \tau_{P_{u-1}} \cdot \beta_{u-1,u}\right) + \sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,t}\left(\beta_{u-1,u}^{-1} \cdot \tau_{P_{u-1}}^{-1} \cdot \ldots \cdot \tau_{P_1}^{-1} \cdot \beta_{i_0,i_1}^{-1}\right) \\
&= \sum_{\substack{k=1,\\ k\neq s}}^{n} a_{k,s}(\beta_{0,u}) + \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{k,t}(\beta_{u,0}). \qquad \square
\end{aligned}
$$

5. Example

The MacLane arrangements are two conjugated arrangements coming from MacLane's matroid [15]. It is the arrangement with a minimal number of lines such that the combinatorics admits a realization over $\mathbb{C}$ but not over $\mathbb{R}$ (see [6, Example 6.6.2(3)]). These arrangements are constructed as follows. Let us consider the 2-dimensional vector space on the field $\mathbb{F}_3$ of three elements. Such a plane contains 9 points and 12 lines, 4 of them pass through the origin. Let $\mathcal{L}$ be $\mathbb{F}_3^2 \setminus \{(0,0)\}$ and $\mathcal{P}$ the set of lines in $\mathbb{F}_3^2$. This provides a line combinatorics $(\mathcal{L}, \mathcal{P}, \Subset)$, where for all $\ell \in \mathcal{L}$, $P \in \mathcal{P}$, we have $P \Subset \ell \Leftrightarrow (\ell \in P, \text{ in } \mathbb{F}_3^2)$. Figure 5.1 represents the ordered MacLane's combinatorics viewed in $\mathbb{F}_3^2$. As an ordered combinatorics, it admits two ordered complex realizations.

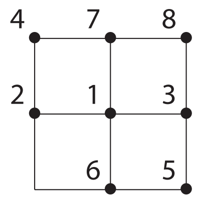

Figure 5.1. Ordered MacLane combinatorics: lines are points in $\mathbb{F}_3^2 \setminus \{(0,0)\}$ and multiple points are affine lines $\mathbb{F}_3^2$.

We can give equations to the realizations:

$$L_1 : y - \zeta z = 0,\ L_2 : y - z = 0,\ L_3 : y - \bar{\zeta} z = 0,\ L_4 : x - z = 0,$$
$$L_5 : x - \bar{\zeta} y = 0,\ L_6 : x - \zeta y = 0,\ L_7 : x - \bar{\zeta} z = 0,\ L_8 : x - \zeta z = 0,$$

where ζ is a primitive cubic root of unity (its choice determines the realization). Add to MacLane arrangements a line L_0 passing through the intersection points: $L_1 \cap L_2 \cap L_3$ and $L_4 \cap L_7 \cap L_8$, *i.e.*, $L_0 : z = 0$ in the above equations. We obtain two ordered realization denoted by $\mathcal{M}^+$ and $\mathcal{M}^-$ and called respectively positive and negative extended MacLane arrangement. The wiring diagram of $\mathcal{M}^+$ is pictured in Figure 5.2; note that this diagram is not generic; by a slight counterclockwise rotation of the vertical lines, the obtained diagram is generic, see Figure 5.3.

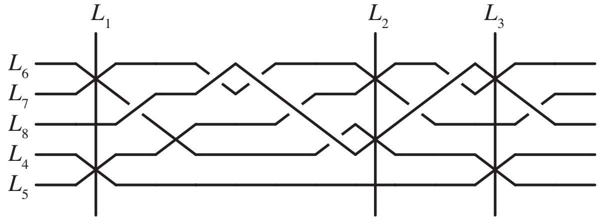

Figure 5.2. Wiring diagram of extended MacLane arrangement of $\mathcal{M}^+$.

It is not hard to see that the only inner-cyclic characters are ξ and ξ^{-1} where ξ is defined by:

$$(v_0, v_1, \ldots, v_8) \longmapsto \left(1, \zeta, \zeta, \zeta, \zeta^2, 1, 1, \zeta^2, \zeta^2\right),$$

with corresponding cycle $\gamma : v_{L_0} \to v_{P_{0,6}} \to v_{L_6} \to v_{P_{5,6}} \to v_{L_5} \to v_{P_{0,5}} \to v_{L_0}$ in $\Gamma_{\mathcal{M}^+}$. With the notation of Section 4, the cycle $\widetilde{\mathfrak{e}}_{6,5}^{\pm}$ is a nearby cycle associated with γ. In the positive case, the braid $\beta^+ \in \mathrm{B}_8$ associated with $\widetilde{\mathfrak{e}}_{6,5}^{+}$ is

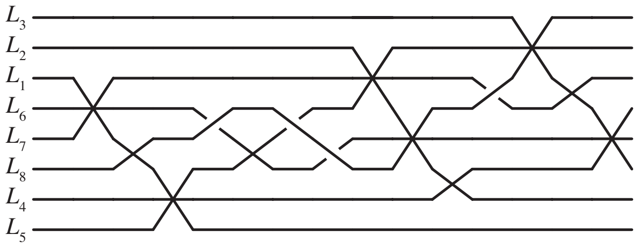

Figure 5.3. Simplified generic wiring diagram of $\mathcal{M}^+$.

$(\sigma_5\sigma_4\sigma_5)\sigma_3(\sigma_2\sigma_1\sigma_2)\sigma_4^{-1}$, while in the negative $\beta^- = (\sigma_5\sigma_4\sigma_5)\sigma_3(\sigma_2\sigma_1\sigma_2)\sigma_4$. Then using Remark 4.5 and Lemma 4.3, the images in the complement of the nearby cycle in both cases are:

$$i_*(\tilde{\mathfrak{e}}^+_{6,5}) = \sum_{\substack{k=1,\\ k\neq 6}}^{8} a_{k,6}(\beta^+) + \sum_{\substack{k=1,\\ k\neq 5}}^{8} a_{k,5}\left(\left(\beta^+\right)^{-1}\right) = (v_1 - v_8) + (-v_4 - v_1) = -v_8 - v_4,$$

$$i_*(\tilde{\mathfrak{e}}^-_{6,5}) = \sum_{\substack{k=1,\\ k\neq 6}}^{8} a_{k,6}(\beta^-) + \sum_{\substack{k=1,\\ k\neq 5}}^{8} a_{k,5}(\left(\beta^-\right)^{-1}) = v_1 + (-v_4 - v_1) = -v_4.$$

Finally, we obtain that:

$$\mathcal{I}(\mathcal{M}^+, \xi, \gamma) = \zeta^2, \qquad \mathcal{I}(\mathcal{M}^-, \xi, \gamma)(\gamma) = \zeta.$$

As a consequence of this paper there is no homeomorphism $\phi : (\mathbb{P}^2, \mathcal{M}^+) \to (\mathbb{P}^2, \mathcal{M}^-)$ preserving orders and orientations of the lines. Note that it is a consequence that this result is already true for the MacLane arrangements; the fact that MacLane arrangements satisfy this property is done using the techniques in [4,17] and the invariant in this paper does not provide an obstruction.

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