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# Higher codimension isoperimetric problems

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**Abstract.** We consider a variational problem for submanifolds $Q \subset M$ with nonempty boundary $\partial Q = K$. We propose the definition that the boundary $K$ of any critical point $Q$ have constant mean curvature, which seems to be a new perspective when $\dim Q < \dim M$. We then construct small nearly-spherical solutions of this higher codimension CMC problem; these concentrate near the critical points of a certain curvature function.

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## 1. Introduction and statement of the result

Constant mean curvature (CMC) hypersurfaces are critical points of the area functional subject to a volume constraint. Examples include sufficiently smooth solutions to the isoperimetric problem. If $K$ is an embedded submanifold in a Riemannian manifold $(M^{m+1}, g)$, then its mean curvature vector $H_K$ is the trace of its second fundamental form. When $K$ is a hypersurface, then we say that $K$ has CMC if this vector has constant length, and this is the only sensible definition in this case. However, when $\operatorname{codim} K > 1$, it is less obvious how to formulate the CMC condition, since there is more than one way one might regard the mean curvature vector as being constant. One definition that has perhaps received the most attention is to require that $H_K$ be parallel. This is quite restrictive, and for that reason, not very satisfactory.

We propose a different, and directly variational definition building on the ideas of F. Almgren [1]. The classical isoperimetric problem amounts to find $m$-dimensional hypersurfaces $K$ of least $m$-dimensional volume enclosing a region of prescribed $m+1$ dimensional volume. F. Almgren generalized the isoperimetric problem in higher codimension by defining the volume enclosed by $S$ as the infimum of volumes of $(m+1)$-dimensional submanifolds $Q$ with $\partial Q = S$.

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Extending the standard characterization of CMC hypersurfaces, we propose to define constant mean curvature submanifolds to be boundaries of submanifolds which are critical for a certain energy functional. Roughly speaking, we say that $K$ has constant mean curvature if $K = \partial Q$ where $Q$ is minimal, $K$ has CMC in $Q$, and $H_K$ has no component orthogonal to $Q$.

Our goal is to show that generic metrics on any compact manifold admit "small" CMC submanifolds in this sense. The result proved here is a generalization of the theorem by R. Ye [11] which proves the existence of families of CMC hypersurfaces that are small perturbations of geodesic spheres centered at nondegenerate critical points of the scalar curvature function $\mathcal{R}$ of the ambient manifold $M$. The more recent paper [8] by F. Pacard and X. Xu obtains such families of CMC hypersurfaces when the scalar curvature is not a Morse function; in that case, these hypersurfaces are centered near critical points of a different curvature invariant.

Let us now introduce the relevant curvature function. For any $(k+1)$-dimensional subspace $\Pi_p \subset T_pM$, define the partial scalar curvature

$$\mathcal{R}_{k+1}(\Pi_p) := -\sum_{i,j=1}^{k+1} g\big(R(E_i, E_j)E_i, E_j\big),$$

where $E_1, \ldots, E_{k+1}$ is any orthonormal basis for $\Pi_p$. Note that $\mathcal{R}_{m+1}(T_pM)$ is the standard scalar curvature at $p$, while $\mathcal{R}_2(\Pi_p)$ is twice the sectional curvature of the 2-plane $\Pi_p$. The Grassmann bundle $G_{k+1}(TM)$ is the fiber bundle over $M$ with fiber at $p \in M$ the Grassmannian of all $(k+1)$-planes in $T_pM$. We regard $\mathcal{R}_{k+1}$ as a smooth function on $G_{k+1}(TM)$.

We denote by $\mathcal{S}^k_\varepsilon(\Pi_p)$ and $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$ the images of the sphere and ball of radius $\varepsilon$ in $\Pi_p$ under the exponential map $\exp_p$, $p \in M$. We can now state our main result.

**Theorem 1.1.** *If $\Pi_p$ is a nondegenerate critical point of $\mathcal{R}_{k+1}$, then for all $\varepsilon$ sufficiently small, there exists a CMC submanifold $K_\varepsilon(\Pi_p)$ which is a normal graph over $\mathcal{S}^k_\varepsilon(\tilde{\Pi}_{\tilde{p}})$ by some section with $\mathcal{C}^{2,\alpha}$ norm bounded by $C\varepsilon^3$ and* $\operatorname{dist}(\tilde{\Pi}_{\tilde{p}}, \Pi_p) \leq c\varepsilon^2$.

Our construction of CMC submanifolds generalizes the method introduced in [8], and can also be carried out in certain cases when the partial scalar curvature has degenerate critical points, for example when $(M, g)$ has constant partial scalar curvature.

**Theorem 1.2.** *There exists $\varepsilon_0 > 0$ and a smooth function*

$$\Psi : G_{k+1}(TM) \times (0, \varepsilon_0) \longrightarrow \mathbb{R},$$

*defined in* (4.5) *below, such that if $\varepsilon \in (0, \varepsilon_0)$, and $\Pi_p$ is a critical point of $\Psi(\cdot, \varepsilon)$, then there exists an embedded $k$-dimensional submanifold $K_\varepsilon(\Pi_p)$ with constant mean curvature equal to $k/\varepsilon$. This submanifold is a normal graph over the geodesic sphere $\mathcal{S}^k_\varepsilon(\Pi_p)$ of a vector field the $\mathcal{C}^{2,\alpha}$ norm of which is bounded by $c\,\varepsilon^3$. Moreover $\Psi(\cdot, \varepsilon) = \mathcal{R}_{m+1} + \mathcal{O}(\varepsilon^2)$ in the smooth topology.*

The function $\Psi$ is essentially just the associated energy functional restricted to a particular finite dimensional set of approximately CMC submanifolds.

Existence of CMC submanifolds also follows from the work of F. Morgan and M.C. Salavessa [6] as smooth solutions to the higher codimension isoperimetric problem defined by F. Almgren. Observe that these solutions should correspond to points where $\mathcal{R}_{k+1}$ has a maximum as in [7].

## 2. Outline of the paper

The outline of this paper is as follows. We first give a more careful description of our proposed definition of constant mean curvature and its relationship to the associated energy functional. We introduce the linearization and the second variation of this energy, then compute these operators in detail for the round sphere $S^k \subset \mathbb{R}^{m+1}$, $k \leq m$. The construction of "small" solutions of the CMC problem concentrating around critical points of the function $\Psi$ proceeds in stages. We construct a family of approximate solutions, then solve the problem up to a finite dimensional defect. This defect depends on certain parameters in the approximate solution, and in the last step we employ a variational argument to choose the parameters appropriately to solve the exact problem. Certain long technical calculations are relegated to the appendices.

## 3. Preliminaries

In this section we begin by setting notations and recalling some standard formulæ. This is followed by the introduction of a variational notion of constant mean curvature for closed submanifolds of arbitrary codimension. We compute the first and the second variations of the associated energy functional, and then explain what these look like for round spheres (of arbitrary codimension) in $\mathbb{R}^{m+1}$.

### 3.1. Mean curvature vector

Let $(M^{m+1}, g)$ be a compact smooth Riemannian manifold. We write $\nabla^\Sigma$ for the induced connection on any embedded submanifold $\Sigma$, and reserve $\nabla$ for the full Levi-Civita connection on $M$.

The second fundamental form of $\Sigma$ is the symmetric bilinear form on $T\Sigma$ taking values in the normal bundle $N\Sigma$ defined by

$$h_\Sigma(X, Y) := \nabla_X Y - \nabla_X^\Sigma Y = \pi_{N\Sigma} \nabla_X Y, \quad X, Y \in T\Sigma;$$

here $\pi_{N\Sigma}$ is the fibrewise orthogonal projection $T_\Sigma M \to N\Sigma$. The trace of $h_\Sigma$ is a section of $N\Sigma$, and is called the mean curvature vector field

$$H_\Sigma := \operatorname{tr}^g h_\Sigma = \sum_{i=1}^{\dim \Sigma} h_\Sigma(E_i, E_i),$$

where $\{E_i\}$ is any orthonormal basis for $T\Sigma$. By definition, $\Sigma$ is minimal provided $H_\Sigma \equiv 0$.

### 3.2. Constant mean curvature in higher codimension

Let us now specialize to the case where $Q^{k+1} \subset M$ is a smooth, compact submanifold with boundary, and $\partial Q =: K$. The normal bundle $NK$ decomposes as an orthogonal direct sum

$$NK = NK^\perp \oplus NK^\parallel,$$

where $NK^\parallel = NK \cap TQ$ has rank 1, and $NK^\perp = NK \cap NQ$ has rank $m-k$. We shall write $n$ for the inward pointing unit normal to $K$ in $Q$. Thus if $\Phi \in NK$, then

$$\Phi = [\Phi]^\perp + [\Phi]^\parallel = [\Phi]^\perp + \phi\, n$$

for some scalar function $\phi$.

**Definition 3.1.** The closed submanifold $K \subset M$ is said to have *constant* mean curvature if $K = \partial Q$ where $Q$ is minimal in $M$, $K$ has constant mean curvature in $Q$ and the $Q$-normal component $[H_K]^\perp \in NK^\perp$ vanishes.

A key motivation is that this definition is variational, where the relevant energy is given by

$$\mathcal{E}_{h_0}(Q) := \mathrm{Vol}_k(\partial Q) - h_0\, \mathrm{Vol}_{k+1}(Q), \tag{3.1}$$

where $h_0$ is a constant.

**Proposition 3.2.** *The submanifold $K = \partial Q$ has constant mean curvature $h_0$ (in the sense of Definition* 3.1*) if and only if*

$$D\mathcal{E}_{h_0}\big|_Q = 0.$$

The meaning of the differential here is the usual one. Let $\Xi$ be a smooth vector field on $M$ and denote by $\xi$ its associated flow. For $t$ small, write $Q_t = \xi(Q, t)$ and $K_t := \partial Q_t = \xi(K, t)$. The requirement in the Proposition is then that for any smooth vector field $\Xi$,

$$\frac{d}{dt}\mathcal{E}_{h_0}(Q_t)\bigg|_{t=0} = 0.$$

The proof is standard. The classical first variation formula (see Appendix 1) states that

$$\frac{d}{dt}\mathrm{Vol}(K_t)\bigg|_{t=0} = -\int_K g(H_K, \Xi)\, \mathrm{dvol}_K,$$

and

$$\frac{d}{dt}\,\mathrm{Vol}(Q_t)\bigg|_{t=0} = -\int_Q g(H_Q, \Xi)\, \mathrm{dvol}_Q - \int_K g(n, \Xi)\, \mathrm{dvol}_K\,.$$

It follows directly from this that

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}_{h_0}(Q_t) = 0,$$

for all vector fields $\Xi$ if and only if $H_K = h_0\, n$ and $H_Q \equiv 0$, as claimed.

The definition above coincides with the standard meaning of CMC when $K$ is a hypersurface in $M$ which is the boundary of a region $Q$. In particular, if $K^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{m+1}$ and $K$ has CMC as a hypersurface in $\mathbb{R}^{k+1}$, then it has CMC in the sense of Definition 3.1. In particular, any round sphere $S^k \subset \mathbb{R}^{m+1}$ has CMC in this sense.

A similar result has been obtained in [6] for stationary submanifolds for the isoperimetric problem in higher codimension.

## 3.3. Jacobi operator

Let us now study the differential of the mean curvature operator, which is known as the Jacobi operator. For this subsection, we revert to considering an arbitrary submanifold $\Sigma$, either closed or with boundary, and shall now recall the expression for this operator.

The Jacobi operator $J_\Sigma$ is the differential of the mean curvature vector field with respect to normal perturbations of $\Sigma$. To describe this more carefully, consider the exponential map exp from an $\varepsilon$-neighborhood of the zero section in $T_\Sigma M$ into $M$. Since $\exp_*\big|_{\{v=0\}} = \mathrm{Id}$, if $\Phi \in \mathcal{C}^2(\Sigma; N\Sigma)$ has $||\Phi||_{\mathcal{C}^0}$ sufficiently small, then

$$\Sigma_\Phi := \left\{\exp_q(\Phi(q)) : q \in \Sigma\right\}$$

is an embedded submanifold. We shall denote the family of submanifolds $\Sigma_{s\Phi}$ by $\Sigma_s$, and their mean curvature vector fields by $H_s$. We also write $F_s : \Sigma \to \Sigma_s$ for the map $q \mapsto \exp_q(s\Phi(q))$. By definition,

$$J_\Sigma(\Phi) = \nabla_{\partial/\partial s} H_s\big|_{s=0}.$$

When $\partial\Sigma \neq \emptyset$, we also require that $\Phi = 0$ on $\partial\Sigma$. The operator $\pi_{N\Sigma} \circ J_\Sigma$ will be denoted $J_\Sigma^N$. We recall in Appendix 1 the proof of the standard formula

$$J_\Sigma^N = -\Delta_\Sigma^N + \mathrm{Ric}_\Sigma^N + \mathfrak{H}_\Sigma^{(2)}, \tag{3.2}$$

where $\Delta_\Sigma^N$ is the (positive definite) connection Laplacian on sections of $N\Sigma$,

$$\forall \Phi \in N\Sigma, \quad \Delta_\Sigma^N \Phi = \sum_{i=1}^{\dim(\Sigma)} \nabla_{E_i}^N \nabla_{E_i}^N \Phi - \nabla^N_{\nabla^\Sigma_{E_i} E_i} \Phi,$$

where $\nabla_X^N Y = \pi_{N\Sigma} \circ \nabla_X Y$ and the other two terms are the following symmetric endomorphisms of $N\Sigma$ :

(i) The orthogonal projection $\mathrm{Ric}_\Sigma^N = \pi_{N_\Sigma} \circ \mathrm{Ric}_\Sigma$ on the normal bundle of $\Sigma$ of the partial Ricci curvature $\mathrm{Ric}_\Sigma$, defined by

$$\begin{aligned} g\big(\mathrm{Ric}_\Sigma X, Y\big) &:= -\mathrm{tr}^g\, g\big(R(\cdot, X)\cdot, Y)\big) \\ &= -\sum_{i=1}^{\dim\Sigma} g\big(R(E_i, X)E_i, Y\big), \quad \text{for all} \quad X, Y \in TM, \end{aligned} \tag{3.3}$$

(note that the curvature tensor appearing on the right is the one on all of $M$, and is not the curvature tensor for $\Sigma$);

(ii) the square of the shape operator, defined by

$$\mathfrak{H}_\Sigma^{(2)}(X) := \sum_{i,j=1}^{\dim \Sigma} g\big(h(E_i, E_j), X\big)\, h(E_i, E_j), \quad \text{for all} \quad X \in TM. \tag{3.4}$$

In general, $J_\Sigma(\Phi) \neq J_\Sigma^N(\Phi)$ since $J_\Sigma(\Phi)$ has a nontrivial component $J_\Sigma^T(\Phi)$ which is parallel to $\Sigma$; as we show later, that part is canceled in our final formula so we do not need to make it explicit. Note, however, that $J_\Sigma^T(\Phi)$ vanishes when $\Sigma$ is minimal. Indeed, writing the mean curvature vector field to $\Sigma_{s\Phi}$ in the form

$$H_s = \sum_\nu g\big(H_s, N_\nu(s)\big)\, N_\nu(s),$$

where $N_\nu(s)$, $\nu = \dim \Sigma + 1, \ldots, m+1$ is a local orthonormal frame for $N\Sigma_{s\Phi}$ we find

$$\begin{aligned}
[J_\Sigma(\Phi)]^T &= \sum_\nu \Big[\Big(g\left(\nabla_{\partial/\partial s} H_s\big|_{s=0}, N_\nu(0)\right) + g\left(H_\Sigma, \nabla_{\partial/\partial s} N_\nu(s)\big|_{s=0}\right)\Big) N_\nu(0) \\
&\qquad + g\left(H_\Sigma, N_\nu(0)\right) \nabla_{\partial/\partial s} N_\nu\big|_{s=0}\Big]^T \\
&= \sum_\nu g\left(H_\Sigma, N_\nu(0)\right) \left[\nabla_{\partial/\partial s} N_\nu(s)\big|_{s=0}\right]^T,
\end{aligned}$$

and if $H_\Sigma = 0$, we have $J_\Sigma^T = 0$.

### 3.4. Linearization about a constant mean curvature submanifold

Let $Q$ be a smooth compact minimal submanifold with a boundary $K$ such that

$$H_K = h_0\, n$$

where $n$ is a unit normal to $K$ in $Q$ and $h_0$ is a constant. We set

$$\mathcal{C}_0^{2,\alpha}(NQ) := \left\{V \in \mathcal{C}^{2,\alpha}(NQ) : V|_K = 0\right\}.$$

With this notation in mind, we have the :

**Definition 3.3.** The minimal submanifold $Q$ is *nondegenerate* if

$$J_Q : \mathcal{C}_0^{2,\alpha}(NQ) \longrightarrow \mathcal{C}^{0,\alpha}(NQ),$$

is invertible.

**Lemma 3.4.** *If $Q$ is nondegenerate, then there is a smooth mapping $\Phi \mapsto Q_\Phi$ from a neighborhood of* $0$ *in $\mathcal{C}^{2,\alpha}(NK)$ into the space of $(k+1)$-dimensional minimal submanifolds of $M$ with $\mathcal{C}^{2,\alpha}$ boundary, such that $Q_0$ is the initial submanifold $Q$ and $\partial Q_\Phi = K_\Phi$.*

*Proof.* Fix a continuous linear extension operator

$$\mathcal{C}^{2,\alpha}(NK) \ni \Phi \mapsto V_\Phi \in \mathcal{C}^{2,\alpha}(T_Q M).$$

Thus $V_\Phi$ is a vector field along $Q$ which restricts to $\Phi$ on $K$. Without loss of generality, we can assume that $V_\Phi \in TQ$ if $[\Phi]^\perp = 0$ and $V_\Phi \in NQ$ when $[\Phi]^\parallel = 0$. Next, let $W$ be a $\mathcal{C}^{2,\alpha}$ section of $NQ$ which vanishes on $K$. If both $||\Phi||_{2,\alpha}$ and $||W||_{2,\alpha}$ are sufficiently small, then $\exp_Q(V_\Phi + W)$ is an embedded $\mathcal{C}^{2,\alpha}$ submanifold $Q_U$ with $U = V_\Phi + W$, and $K_\Phi := \partial Q_U$.

Denoting the mean curvature vector of $Q_U$ by $H(\Phi, W)$, we find

$$D_W H|_{(0,0)}(W) = J_Q W.$$

Since $Q$ is minimal, $D_W H|_{(0,0)}(W)$ takes values in $NQ$, whereas $H(\Phi, W) \in NQ_U \subset T_{Q_U} M$, so we cannot directly apply the implicit function theorem. To remedy this, first let $\widetilde{H}(\Phi, W)$ be the parallel transport of $H(\Phi, W)$ along the geodesic $s \mapsto \exp_q(sU(q))$, from $s = 1$ to $s = 0$. Parallel transport preserves regularity (this reduces to the standard result on smooth dependence on initial conditions for the solutions of a family of ODE's), so $\widetilde{H}(\Phi, W)$ is a $\mathcal{C}^{0,\alpha}$ section of $T_Q M$. Now define

$$\widehat{H}(\Phi, W) := \pi_{NQ} \circ \widetilde{H}(\Phi, W),$$

where $\pi_{NQ} : T_Q M \to NQ$ is the orthogonal projection. Since $H(\Phi, W) \in N_{Q_U} M$ and since $||U||_{\mathcal{C}^1}$ is small, $\widetilde{H}(\Phi, W)$ lies in the nullspace of $\pi_{NQ}$ at any $q \in Q$ if and only if it actually vanishes. Thus it is enough to look for solutions of $\widehat{H}(\Phi, W) = 0$. Notice that $D_W \widehat{H}|_{(0,0)} = J_Q$. We can now apply the implicit function theorem to conclude the existence of a $\mathcal{C}^{2,\alpha}$ map $\Phi \mapsto W(\Phi)$ such that $H(\Phi, W(\Phi)) = \widehat{H}(\Phi, W(\Phi)) \equiv 0$ for all small $\Phi$. □

We henceforth denote by $Q_\Phi$ the minimal submanifold $\exp_Q(V_\Phi + W(\Phi))$. Observe that when $[\Phi]^\perp = 0$, the submanifold parametrized by $\exp_Q(V_{[\Phi]^\parallel})$ is $\mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$ close to $Q_\Phi$; this is easy to check when $\Phi := \phi n$ where $\phi$ is small. Therefore, in this 'tangential' case, we conclude that

$$U_\Phi = V_{[\Phi]^\parallel} + \mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}}).$$

Next, when $[\Phi]^\parallel = 0$, we define $Z_{[\Phi]^\perp}$ as the solution of

$$J_Q Z_{[\Phi]^\perp} = 0, \qquad Z_{[\Phi]^\perp}\big|_K = \Phi,$$

and it is easy to check that the submanifold parametrized by $\exp_Q(Z_{[\Phi]^\perp})$ is also $\mathcal{O}(\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$ close to $Q_\Phi$. We summarize all this in the

**Lemma 3.5.** *When $\|\Phi\|_{C^{2,\alpha}}$ is small, we have the decomposition*

$$U_\Phi = V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp} + \mathcal{O}\left(\|\Phi\|^2_{C^{2,\alpha}}\right).$$

Now consider the energy $\mathcal{E}_{h_0}$ along a one-parameter family $s \mapsto Q_s := Q_{s\Phi}$ of minimal submanifolds with boundaries $K_s := \partial Q_s = K_{s\Phi}$. By the formulæ of the last subsection,

$$\frac{d}{ds}\mathcal{E}_{h_0}(Q_s) = -\int_{K_s} g(H_s - h_0\, n_s, \partial/\partial s)\, \mathrm{dvol}_{K_s},$$

where $H_s$ is the mean curvature of $K_s$ and $n_s$ is the inward pointing unit normal to $K_s$ in $Q_s$. Note that this first variation of energy is localized to the boundary; the interior terms vanish because of the minimality of the $Q_s$. Our task is to compute

$$\left.\frac{d^2}{ds^2}\mathcal{E}_{h_0}(Q_s)\right|_{s=0},$$

when $Q$ is critical for $\mathcal{E}_{h_0}$.

Parametrize both $K_s$ and $Q_s$ by $y \mapsto F_s(y) := \exp_y(U_{s\Phi}(y))$ (with $y \in K$ or $y \in Q$, respectively). As before, choose a smooth local orthonormal frame $E_\alpha$ for $TK$, so that $(F_s)_* E_\alpha = E_\alpha(s)$ is a local (non-orthonormal) frame for $TK_{s\Phi}$. We then include $n_s$, the unit inward normal to $K_s$ in $Q_s$. Moreover, we extend $n_s$ to a vector $\bar{n}_s \in TQ_s$ so that it satisfies $\nabla^{Q_s}_{\bar{n}_s}\bar{n}_s = 0$. We supplement this to a complete local frame for $T_{Q_s}M$ (at least near points of $K_s$) by adding a local orthonormal frame $N_\mu(s) \in NQ_s$. Here we let the indexes $\alpha, \beta, \ldots$ run from 1 to $k$ while $\mu, \nu, \ldots$ run from $k+2$ to $m+1$ .

**Notation 3.6.** Set $\mathcal{H}(s) = H(K_s) - h_0\, n_s$. We also write

$$L_Q = \nabla_{\partial/\partial s}\mathcal{H}_s\big|_{s=0}.$$

Note that we can decompose $\mathcal{H}'(0)$ into $\mathcal{H}'(0)^{N_K} + \mathcal{H}'(0)^{T_K}$, its components perpendicular and parallel to $K$. Since $\mathcal{H}(s) \perp K_s$, we have that $\langle \mathcal{H}(s), E_\alpha(s)\rangle = 0$, so

$$\langle \mathcal{H}'(0), E_\alpha\rangle + \langle \mathcal{H}(0), E'_\alpha(0)\rangle = 0.$$

Since $\mathcal{H}(0) = 0$, we obtain $\pi_{TK} \circ L_Q = 0$.

Next decompose $\Phi = [\Phi]^\perp + \phi\, n$ into parts perpendicular and parallel to $Q$ (along $K$). Note that we can choose the vector field $U_\Phi$ extending $\Phi$ and defined in Lemma 3.4 so that its component tangent to $Q$ lies in the span of $\bar{n}$. More precisely, we have a decomposition $U_\Phi = [U_\Phi]^\perp + u_\phi\, \bar{n}$ locally near $K_\Phi$, where $[U_\Phi]^\perp\big|_K = [\Phi]^\perp$ and $u_\phi\big|_K = \phi$.

To see that $E'_\alpha(0) = \nabla_{E_\alpha}\Phi$, choose a curve $c(t)$ in $K$ with $c(0) = p$, $c'(0) = E_\alpha$ and define $G(t,s) = \exp_{c(t)}(s\Phi(c(t)))$; we then obtain that

$$\nabla_{\partial/\partial s} E_\alpha\big|_{s=0} = \nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\big|_{s=t=0} G(t,s) = \nabla_{\partial/\partial t}\Phi(c(t))\big|_{t=0} = \nabla_{E_\alpha}\Phi,$$

as claimed. To compute $n'(0)$, observe that $(F_s)_*(n(0))$ is always tangent to $Q_s$ and transverse, but not necessarily a unit normal, to $K_s$. We can adjust it, using the Gram-Schmidt process, to get that

$$n_s = \Big((F_s)_*(n(0)) - \sum c_\alpha E_\alpha(s)\Big) \,/\, \Big|((F_s)_*(n(0)) - \sum c_\alpha E_\alpha(s)\Big|,$$

where

$$c_\alpha(s) = \langle E_\alpha(s), (F_s)_* n(0)\rangle / |E_\alpha(s)|^2.$$

Arguing as before, take a curve $d(t)$ in $Q$ such that $d(0) = p$ and $d'(0) = n$ and define $\tilde{G}(t,s) = \exp_{d(t)}(U_{s\Phi}(d(t)))$. Note that $U_{s\Phi} = s(V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp}) + \mathcal{O}(s^2\|\Phi\|^2_{\mathcal{C}^{2,\alpha}})$. We get

$$\nabla_{\partial/\partial s}(F_s)_* n(0)\big|_{s=0} = \nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\tilde{G}(t,s)\Big|_{t=s=0} = \nabla_n(V_{[\Phi]^\parallel} + Z_{[\Phi]^\perp})$$

and since $c_\alpha(0) = 0$, we obtain

$$[n'(0)]^\perp = \big[\nabla_n V_{[\Phi]^\parallel} + \nabla_n Z_{[\Phi]^\perp}\big]\big|_K^\perp = \Big[\nabla_n^\perp Z_{\Phi^\perp} + \phi\,\nabla_n^\perp \bar{n}\Big]\Big|_K.$$

Finally, the component $[n'(0)]^\parallel = 0$. Combining these calculations gives the

**Proposition 3.7.** *If $Q$ is critical for $\mathcal{E}_{h_0}$, then*

$$L_Q\,\Phi = \big(\pi_{NK}\circ J_K - h_0\, D_Q\big)\ \Phi,$$

*where*

$$D_Q\Phi = \Big[\nabla_n^\perp Z_\Phi + \phi\,\nabla_n^\perp \bar{n}\Big]\Big|_K.$$

### 3.5. Linearization about the Euclidean sphere

We conclude this section by discussing the precise form of this linearization, and its nullspace, when

$$K = S^k \times \{0\} \subset Q = B^{k+1} \times \{0\} \subset \mathbb{R}^{m+1},$$

since this is our basic model later. It is easy to see that $B^{k+1}$ is critical for $\mathcal{E}_k$.

The unit inward normal to $S^k$ in $B^{k+1}$ is $n_{S^k}(\Theta) = -\Theta$. If $\Phi \in \mathcal{C}^{2,\alpha}(NS^k)$, then

$$\Phi = [\Phi]^\perp - \phi\,\Theta,$$

where the first term on the right is perpendicular to $B^{k+1}$. The operator $J^N_{S^k}$ acts on these two components separately, via $J^\perp_{S^k}$ and $J^\parallel_{S^k}$, respectively.

The first of these operators acts on sections of the trivial bundle of rank $m-k$. Obviously, $\mathrm{Ric}^N_{S^k} = 0$, *cf.* (3.3), and $(\mathfrak{H}^{(2)}_{S^k})^\perp = 0$ as well, so

$$J^\perp_{S^k} = \Delta_{S^k}$$

acting on $(m-k)$-tuples of functions. Its eigenvalues are $\ell(k+\ell-1)$. The operator $D_{B^{k+1}}$ also acts on sections of the trivial bundle $NB^{k+1}\big|_{S^k}$. In fact, since $J_{B^{k+1}} = \Delta_{B^{k+1}}$, this operator is simply the standard Dirichlet-to-Neumann operator for the Laplacian (acting on $\mathbb{R}^{m-k}$-valued functions). Its eigenfunctions are the restrictions to $r = 1$ of the homogeneous harmonic polynomials $P(x)$, $x = r\Theta$, $\Theta \in S^k$. If $P$ is homogeneous of order $\ell$, then $P(x) = r^\ell P(\Theta)$, so $D_{B^{k+1}} P(\Theta) = -\ell P(\Theta)$ (recall we are using the inward-pointing normal). Combining these two operators, we see that $\Delta_{S^k} - kD_{B^{k+1}}$ has eigenvalues $-\ell(k+\ell-1)+k\ell = -\ell(\ell-1)$, hence

$$\left(J^{\perp}_{S^k} - kD_{B^{k+1}}\right)[\Phi]^{\perp} = 0 \Rightarrow \ [\Phi]^{\perp} \in \operatorname{span}\left\{ \left(b_\mu + c_{j\mu}\Theta^j\right) E_\mu \right\},$$

$j = 1, \ldots, k+1$, $\mu = k+2, \ldots, m+1$, where $E_\mu$ form an orthonormal basis for $NB^{k+1} = \mathbb{R}^{m-k}$.

The remaining part is

$$J^{\parallel}_{S^k} = \Delta_{S^k} + k,$$

since $\mathrm{Ric}_{S^k} = 0$ and $\mathfrak{H}^{(2)}_{S^k} = k\,\mathrm{Id}$. Thus

$$J^{\parallel}_{S^k}(\phi\,\Theta) = J^{\parallel}_{S^k}(\phi)\,\Theta = 0 \ \Rightarrow\ \phi \in \operatorname{span}\left\{\Theta^1, \ldots, \Theta^{k+1}\right\}.$$

We have now shown that the nullspace $\mathcal{K}$ of $L_{B^{k+1}}$ splits as $\mathcal{K}^{\perp} \oplus \mathcal{K}^{\parallel}$. The first of these summands is comprised by infinitesimal translations in $\mathbb{R}^{m-k}$ and infinitesimal rotations in the $j\mu$ planes (now $j \le k+1$); the second summand corresponds to infinitesimal translations in $\mathbb{R}^{k+1}$.

## 4. Construction of constant mean curvature submanifolds

We now turn to the main task of this paper, which is to construct small constant mean curvature submanifolds concentrated near the critical points of $\mathcal{R}_{k+1}$. The first step is to define a family of approximate solutions, *i.e.* a family of pairs $(Q_\varepsilon, K_\varepsilon)$ where $Q_\varepsilon$ is minimal and has nearly CMC boundary. We then use a variational argument to perturb this to a minimal submanifold with exactly CMC boundary.

### 4.1. Approximate solutions

We adopt all the notations used earlier. Thus we fix $\Pi_p \in G_{k+1}(TM)$ and an orthonormal basis $E_i$, $1 \le i \le m+1$ of $T_pM$, where $E_j$, $1 \le j \le k+1$ span $\Pi_p$ and $E_\mu$, $\mu \ge k+2$, span $\Pi_p^{\perp}$. This induces a Riemann normal coordinate system $(x^1, \ldots, x^{m+1})$ near $p$, and it is standard that

$$g_{ij}(x) = g(\partial_{x^i}, \partial_{x^j}) = \delta_{ij} + \frac{1}{3}\sum_{k,\ell} \left(R_p\right)_{ikj\ell}\, x^k x^\ell + \mathcal{O}\left(|x|^3\right), \tag{4.1}$$

where $\delta$ is the Euclidean metric.

**4.1.1.** *Rescaling*

In terms of the map $F_\varepsilon : T_pM \to M$, $F_\varepsilon(v) = \exp_p(\varepsilon v)$, used earlier, define the metric

$$g_\varepsilon = \varepsilon^{-2} F_\varepsilon^* g$$

on $T_pM$, or equivalently, work in the rescaled coordinates $y^j = x^j/\varepsilon$. In either case,

$$g_\varepsilon = |dy|^2 + \varepsilon^2 h_\varepsilon(y, dy), \tag{4.2}$$

where $h_\varepsilon$ is family of smooth symmetric two-tensors depending smoothly on $\varepsilon \in [0, \varepsilon_0]$. The mean curvature vectors $H^g$ and $H^{g_\varepsilon}$ with respect to $g$ and $g_\varepsilon$ satisfy

$$\varepsilon^2 H^g = (F_\varepsilon)_* H^{g_\varepsilon}, \quad \text{and} \quad \|H^{g_\varepsilon}\|_{g_\varepsilon} = \varepsilon \, \|H^g\|_g .$$

Let $B^{k+1} = B^{k+1}(\Pi_p) \subset \Pi_p$ be the unit ball and $S^k = S^k(\Pi_p) = \partial B^{k+1}$, and denote their images under $F_\varepsilon$ by $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ and $\mathcal{S}_\varepsilon^{k+1}(\Pi_p)$. These have parametrizations

$$S^k \ni \Theta \longmapsto \exp_p^g\left(\varepsilon \sum_{j=1}^{k+1} \Theta^j E_j\right), \quad B^{k+1} \ni y \longmapsto \exp_p^g\left(\varepsilon \sum_{j=1}^{k+1} y^j E_j\right).$$

In the Lemmas 4.2 and 4.3 below we give the expansion of the mean curvature of $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ and $\mathcal{S}_\varepsilon^k(\Pi_p)$ in terms of $\varepsilon$. To this end we introduce two supplementary curvature invariants which are restrictions of the Ricci curvature of the ambient manifold $M$:

**Notation 4.1.**

$$\mathcal{R}ic_{k+1}(\Pi_p)(v_1, v_2) = -\sum_{i=1}^{k+1} g_p(R_p(E_i, v_1)E_i, v_2), \qquad v_1, v_2 \in \Pi_p,$$

$$\mathcal{R}ic_{k+1}^{\perp}(\Pi_p)(v, N) = -\sum_{i=1}^{k+1} g_p(R_p(E_i, v)E_i, N), \qquad v \in \Pi_p,\ N \in \Pi_p^{\perp}.$$

Note that

$$\mathcal{R}ic_{k+1}^{\perp}(\Pi_p) = \left[\mathrm{Ric}^N_{\mathcal{B}_\varepsilon^{k+1}(\Pi_p)}\right]_p .$$

Moreover, here and below, we write $\mathcal{O}(\varepsilon^k)$ for a function with $\mathcal{C}^{0,\alpha}$ norm bounded by $C\varepsilon^k$ for a constant $C > 0$ independent of $\varepsilon$.

**Lemma 4.2.** *The mean curvature of the geodesic ball* $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$

$$H^g(\mathcal{B}_\varepsilon^{k+1}(\Pi_p))(y) = \sum_{\mu=k+2}^{m+1} \left(\frac{2\,\varepsilon}{3}\, \mathcal{R}ic_{k+1}^{\perp}(\Pi_p)(y, E_\mu) + \mathcal{O}(\varepsilon^2)\right) \mathcal{N}_\mu,$$

*where* $y \in B^{k+1}$ *and* $\mathcal{N}_\mu$, $k+2 \le \mu \le m+1$ *is an orthonormal basis of* $N\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$.

*Proof.* Recall that

$$H^g\left(\mathcal{B}_\varepsilon^{k+1}(\Pi_p)\right) = \frac{1}{\varepsilon^2}\,(F_\varepsilon)_*\,H^{g_\varepsilon}\left(B^{k+1}\right).$$

We denote $\mathcal{N}_\mu^\varepsilon, k+1 < \mu < m+1$ the orthonormal basis of the normal bundle of $B^{k+1}$ with respect to the metric $g_\varepsilon$ obtained by applying the Gram-Schmidt process to the vectors $E_i(p), 1 \le i \le m+1$. Remark that

$$g_\varepsilon(\mathcal{N}_\mu^\varepsilon, E_\nu) = \delta_{\mu\nu} + \mathcal{O}(\varepsilon^2), \quad \mu = k+2, \dots, m+1,$$

while the vector fields $\mathcal{N}_\mu = \frac{1}{\varepsilon}(F_\varepsilon)_*(\mathcal{N}_\mu^\varepsilon)$ form an orthonormal basis of $N\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ with respect to the metric $g$.

The Christoffel symbols corresponding to the metric $g_\varepsilon$ are:

$$\begin{aligned}
(\Gamma^{g_\varepsilon})_{ij}^\ell(y) &= \frac{1}{2}\,g_\varepsilon^{\ell q}\left(\partial_{y^j}(g_\varepsilon)_{iq} + \partial_{y^i}(g_\varepsilon)_{jq} - \partial_{y^q}(g_\varepsilon)_{ij}\right)\\
&= \delta^{\ell q}\,\frac{\varepsilon^2}{6}\,y^p\,\left(R_{ijqp} + R_{ipqj} + R_{jiqp} + R_{jpqi} - R_{iqjp} - R_{ipjq}\right) + \mathcal{O}(\varepsilon^3)\\
&= -\frac{\varepsilon^2}{3}\left(R_{ipj\ell} + R_{i\ell jp}\right)\,y^p + \mathcal{O}(\varepsilon^3),
\end{aligned}$$

whence

$$g_\varepsilon\left(\nabla^{g_\varepsilon}_{\partial_{y^i}}\partial_{y^j}, \mathcal{N}_\mu^\varepsilon\right) = \left(\Gamma^{g_\varepsilon}\right)_{ij}^\mu + \mathcal{O}\left(\varepsilon^4\right).$$

Taking the trace in the indexes $i, j = 1, \dots, k+1$ with respect to $g_\varepsilon$ gives the result. □

**Lemma 4.3.** *The mean curvature of the geodesic sphere $\mathcal{S}_\varepsilon^k(\Pi_p)$ satisfies*

$$\begin{aligned}
H^g(\mathcal{S}_\varepsilon^k(\Pi_p)) &= \left(\frac{k}{\varepsilon} - \frac{\varepsilon}{3}\,\mathcal{R}ic_{k+1}\left(\Pi_p\right)(\Theta,\Theta) + \mathcal{O}\left(\varepsilon^2\right)\right) n_{\mathcal{S}}\\
&\quad + \sum_{\mu=k+2}^{m+1}\left(\frac{2\,\varepsilon}{3}\mathcal{R}ic_{k+1}^\perp\left(\Pi_p\right)\left(\Theta, E_\mu\right) + \mathcal{O}\left(\varepsilon^2\right)\right)\mathcal{N}_\mu, \qquad \Theta \in S^k,
\end{aligned}$$

*where $n_{\mathcal{S}}$ is a unit normal vector field to $\mathcal{S}_\varepsilon^k(\Pi_p)$ in $\mathcal{B}_\varepsilon^{k+1}(\Pi_p)$ with respect to the metric $g$.*

*Proof.* The proof is similar to that of the previous lemma, but with several changes. Let $u^1, \dots, u^k \mapsto \Theta(u^1, \dots, u^k)$ be a local parametrization of $S^k \subset \Pi_p$. The tangent bundle $TS^k$ is spanned by the vector fields $\Theta_\alpha = \partial_{u^\alpha}\Theta, \alpha = 1, \dots, k$. We remark that

$$H^g\left(\mathcal{S}_\varepsilon^k(\Pi_p)\right) = \frac{1}{\varepsilon^2}\,(F_\varepsilon)_*\,H_\varepsilon^g\left(S^k\right).$$

By Gauss's lemma,

$$g\big((F_\varepsilon)_*\Theta_\alpha, (F_\varepsilon)_*\Theta\big)\big(F_\varepsilon(\Theta)\big) = g_p(\Theta_\alpha, \Theta) = 0,$$

for $\alpha = 1, \ldots, k$, hence, we put $n_{\mathcal{S}} := -\frac{1}{\varepsilon}\,(F_\varepsilon)_*\Theta$. We have

$$\nabla^{g_\varepsilon}_{\partial_{u^\alpha}} \partial_{u^\beta} = \partial_{u^\alpha}\,\partial_{u^\beta}\Theta + \left(\Gamma^{g_\varepsilon}\right)^{\ell}_{ij}\,(\Theta_\alpha)^i\,\left(\Theta_\beta\right)^j\; E_\ell,$$

where $\alpha, \beta = 1, \ldots, k;\; i, j, \ell = 1, \ldots, m+1$. Since the vector field $\partial_{u^\alpha}\,\partial_{u^\beta}\Theta$ is tangent to $B^{k+1}(\Theta)$, we find

$$g_\varepsilon\left(\nabla^{g_\varepsilon}_{\partial_{u^\alpha}} \partial_{u^\beta}, \mathcal{N}^\varepsilon_\mu\right) = \left(\Gamma^{g_\varepsilon}\right)^{\mu}_{ab}\,(\Theta_\alpha)^a\,\left(\Theta_\beta\right)^b + \mathcal{O}\left(\varepsilon^3\right).$$

Taking trace in the indexes $\alpha, \beta$ with respect to the metric induced on $S^k$ from $g_\varepsilon$ we get

$$g_\varepsilon\left(H^{g_\varepsilon}(S^k), \mathcal{N}^\varepsilon_\mu\right) = \frac{2\,\varepsilon^2}{3}\,\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\left(\Theta, E_\mu\right) + \mathcal{O}\left(\varepsilon^3\right).$$

In order to find $[H^{g_\varepsilon}(S^k(\Pi_p))]^{||}$, recall the standard fact that if $\Sigma \subset M$ is an oriented hypersurface with unit inward pointing normal $N_\Sigma$, and if $\Sigma_z$ is the family of hypersurfaces defined by

$$\Sigma \times \mathbb{R}(q, z) \mapsto \exp_q\left(zN_\Sigma(q)\right) \in \Sigma_z,$$

with induced metric $g_z$, then

$$|H_\Sigma| = -\frac{d}{dz}\;\log\sqrt{\det g_z}.$$

In our case, considering $S^k = \partial B^{k+1}$ with metric $g_\varepsilon$, let $g_{\varepsilon z}$ be the induced metrics on the Euclidean sphere of radius $1 - z$. Then,

$$\det g_{\varepsilon z} = (1-z)^{2k}\det g^S\left(1 - \frac{\varepsilon^2(1-z)^2}{3}\mathcal{R}ic_{S^k}\left(\Pi_p\right)(\Theta, \Theta) + \mathcal{O}\left(\varepsilon^3\right)\right),$$

where $g^S$ is the standard spherical metric on $S^k(\Pi_p)$. From this we deduce that

$$g_\varepsilon\left(H^{g_\varepsilon}(S^k), -\Theta\right) = k - \frac{\varepsilon^2}{3}\,\mathcal{R}ic_{k+1}\left(\Pi_p\right)(\Theta, \Theta) + \mathcal{O}\left(\varepsilon^3\right).$$

This completes the proof. □

**Proposition 4.4.** *Fix $\Pi_p \in G_{k+1}(TM)$. Then for $\varepsilon > 0$ small enough, there exists a minimal submanifold $Q_\varepsilon(\Pi_p)$ which is a small perturbation of $\mathcal{B}^{k+1}_\varepsilon(\Pi_p)$, whose boundary $K_\varepsilon(\Pi_p) = \partial Q_\varepsilon(\Pi_p)$ is a normal graph over $S^k_\varepsilon(\Pi_p)$ and whose mean curvature vector field satisfies*

$$H^g\left(K_\varepsilon(\Pi_p)\right)(\Theta) - \frac{k}{\varepsilon}\, n_K = \langle \vec{a}, \Theta\rangle\, n_K + \sum_{\mu=k+2}^{m+1} \left(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\right) N_\mu, \tag{4.3}$$

*for some constant vectors $\vec{a} = \vec{a}(\varepsilon, \Pi_p)$, $\vec{c}_\mu = \vec{c}_\mu(\varepsilon, \Pi_p) \in \mathbb{R}^{k+1}$ and constants $d_\mu = d_\mu(\varepsilon, \Pi_p) \in \mathbb{R}$ and where by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbb{R}^{m+1}$. Here $n_K$ is a normal vector field to $K_\varepsilon(\Pi_p)$ in $Q_\varepsilon(\Pi_p)$ and $N_\mu$, $\mu = k+2, \ldots, m+1$ form an orthonormal basis of $\left[N K_\varepsilon(\Pi_p)\right]^\perp$.*

*Proof.* Take a vector field $\Phi \in \mathcal{C}^{2,\alpha}(T_pM)$ defined along the unit sphere $S^k(\Pi_p)$, such that

$$\Phi(\Theta) = -\phi(\Theta)\,\Theta + \sum_{\mu=k+2}^{m+1} \Phi^\mu(\Theta)\, E_\mu,$$

and write

$$S^k_\Phi = \left\{\Theta + \Phi(\Theta),\ \Theta \in S^k\right\}.$$

Then there exists a submanifold $B^{k+1}_{\varepsilon,\Phi}$ such that $\partial B^{k+1}_{\varepsilon,\Phi} = S^k_\Phi$ and which is minimal with respect to $g_\varepsilon$. The proof of this fact is almost the same as the proof of the Lemma 3.4; the only difference is that we use a "perturbed" metric and the starting submanifold is no longer minimal. Let $V_\Phi$ be a linear extension of $\Phi$ in $B^{k+1}$ and take

$$W \in \mathcal{C}^{2,\alpha}(T_pM), \quad W = \sum_{\mu=k+2}^{m+1} W^\mu\, E_\mu, \quad W|_{S^k} = 0.$$

We put $U(y) := V_\Phi(y) + W(y)$ and let $H(\varepsilon, \Phi, W)$ denote the mean curvature with respect to the metric $g_\varepsilon$ of the submanifold

$$B^{k+1}_U := \left\{y + U(y),\ y \in B^{k+1}\right\}.$$

Note that $H(0, 0, 0) = 0$ and

$$D_3 H|_{(0,0,0)} = J_{B^{k+1}} = \Delta_{B^{k+1}}.$$

We can then apply the implicit function theorem to $\hat{H}(\varepsilon, \Phi, W) = \pi \circ H(\varepsilon, \Phi, W)$, where $\pi$ is the orthogonal projection onto the subspace of $T_pM$ spanned by $E_\mu$, $k+2 \le \mu \le m+1$. Then for $\varepsilon$ and $\|\Phi\|_{\mathcal{C}^{2,\alpha}}$ small enough, there exists a mapping $(\varepsilon, \Phi) \mapsto W(\varepsilon, \Phi)$ such that

$$\hat{H}(\varepsilon, \Phi, W(\varepsilon, \Phi)) = 0 \quad \text{and} \quad H(\varepsilon, \Phi, W(\varepsilon, \Phi)) = 0.$$

Moreover, we can write

$$U_{\varepsilon,\Phi} = V_\Phi + W(\varepsilon, \Phi) = V_\phi + Z_\Phi + W_\varepsilon + \mathcal{O}\left(\|\varepsilon^3\|\right) + \mathcal{O}\left(\varepsilon^2 \| \Phi\|\right) + \mathcal{O}\left(\|\Phi^2\|\right),$$

where $V_\phi(y) = -\phi(y/\|y\|)\, y$, the vector field $Z_\Phi$ is the harmonic extension of $\Phi$ in $B^{k+1}$ and $W_\varepsilon$ satisfies

$$\Delta_{B^{k+1}} W_\varepsilon^\mu = -\frac{2\,\varepsilon^2}{3}\, \mathcal{R}ic^\perp_{k+1}\left(\Pi_p\right)\left(y, E_\mu\right) \quad \text{in} \quad B^{k+1}, \qquad W_\varepsilon|_{S^k} = 0.$$

**Remark 4.5.** A simple calculation shows that

$$W_\varepsilon(y) = \frac{\varepsilon^2}{3}\, \frac{1}{k+3}\, \left(1 - |y|^2\right) \sum_{\mu=k+2}^{m+1} \mathcal{R}ic^\perp_{k+1}\left(\Pi_p\right)\left(y, E_\mu\right)\, E_\mu.$$

As a next step, we calculate the mean curvature of $S^k_\Phi$ with respect to the metric $g_\varepsilon$. First note that the vector fields

$$\tau_\alpha = (1-\phi)\,\Theta_\alpha - \partial_{u_\alpha}\phi\,\Theta + \sum_{\mu=k+2}^{m+1} \partial_{u_\alpha}\Phi^\mu\, E_\mu,$$

locally frame $T\,S^k_\Phi$, while

$$\Theta_\Phi = \Theta + \frac{1}{1-\phi}\,\nabla_{S^k}\phi, \quad \text{and} \quad (E_\mu)_\Phi = E_\mu - \frac{1}{1-\phi}\,\nabla_{S^k}\Phi^\mu$$

are a local basis for the normal bundle of $S^k_\Phi$ with respect to the Euclidean metric. Applying the Gram-Schmidt process with respect to the metric $g_\varepsilon$ to these local frames we obtain the unit normal $n^\varepsilon_\Phi$ to $S^k_\Phi$ in $B^{k+1}_{\varepsilon,\Phi}$ and an orthonormal frame $(\mathcal{N}_\Phi)^\varepsilon_\mu$ for the normal bundle of $B^{k+1}_{\varepsilon,\Phi}$ along $S^k_\Phi$ with respect to $g_\varepsilon$. It is clear that

$$\left\langle n^\varepsilon_\Phi, -\Theta_\Phi/|\Theta_\Phi|_{g_{\text{eucl}}}\right\rangle_{g_\varepsilon} = 1 + \mathcal{O}(\varepsilon^2),$$

$$\left\langle (\mathcal{N}_\mu)^\varepsilon_\Phi, (E_\mu)_\Phi/|(E_\mu)_\Phi|_{g_{\text{eucl}}}\right\rangle_{g_\varepsilon} = 1 + \mathcal{O}(\varepsilon^2),$$

and $n^\varepsilon_0 = -\Theta$ and $(\mathcal{N}_\mu)^\varepsilon_0 = \mathcal{N}^\varepsilon_\mu$. We can then write

$$H^{g_\varepsilon}(S^k_\Phi) - k\, n^\varepsilon_\Phi = \left(g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), n^\varepsilon_\Phi\right) - k\right)\, n^\varepsilon_\Phi + \sum_{\mu=k+2}^{m+1} g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), (\mathcal{N}_\Phi)^\varepsilon_\mu\right)\, (\mathcal{N}_\Phi)^\varepsilon_\mu.$$

**Notation 4.6.** We let $\mathcal{L}_{\Pi_p}(\Phi)$ denote any second order linear differential operator acting on $\Phi$. The coefficients of $\mathcal{L}_{\Pi_p}(\Phi)$ may depend on $\Pi_p \in G_{k+1}(TM)$ and $\varepsilon \in (0,1)$, but for all $j \in \mathbb{N}$ there exists a constant $C_j > 0$ independent of $\Pi_p$ and $\varepsilon$ such that

$$\|\mathcal{L}_{\Pi_p}(\Phi)\|_{\mathcal{C}^{j,\alpha}(S^k)} \leq C_j \, \|\Phi\|_{\mathcal{C}^{j+2,\alpha}(NS^k)}.$$

Similarly, for $\ell \in \mathbb{N}$, $\mathcal{Q}^\ell_{\Pi_p}(\Phi)$ denotes some nonlinear operator in $\Phi$, depending also on $\Pi_p$ and $\varepsilon$, such that $\mathcal{Q}^\ell_{\Pi_p}(0) = 0$ and which has the following properties. The coefficients of the Taylor expansion of $\mathcal{Q}^\ell_{\Pi_p}(\Phi)$ in powers of the components of $\Phi$ and its derivatives satisfy that for any $j \geq 0$, there exists a constant $C_j > 0$, independent of $\Pi_p \in G_{k+1}(TM)$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \|\mathcal{Q}^\ell_{\Pi_p}(\Phi_1) - \mathcal{Q}^\ell_{\Pi_p}(\Phi_2)\|_{\mathcal{C}^{j,\alpha}(S^k)} \leq C_j \, &\left(\|\Phi_1\|_{\mathcal{C}^{j+2,\alpha}(NS^k)} + \|\Phi_2\|_{\mathcal{C}^{j+k,\alpha}(NS^k)}\right)^{\ell-1} \\ &\times \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{j+k,\alpha}(NS^k)} \end{aligned}$$

provided $\|\Phi_i\|_{\mathcal{C}^1(NS^k)} \leq 1$, $i = 1, 2$.

Using the fact that the Christoffel symbols associated to the metric $g_\varepsilon$ are of order $\mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned} g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), n^\varepsilon_\Phi\right) - k &= -\frac{\varepsilon^2}{3}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + J^{\|}_{S^k}\phi \\ &\quad + \mathcal{O}(\varepsilon^3) + \varepsilon^2\,\mathcal{L}_{\Pi_p}(\Phi) + \mathcal{Q}^2_{\Pi_p}(\Phi), \end{aligned}$$

$$\begin{aligned} g_\varepsilon\left(H^{g_\varepsilon}(S^k_\Phi), (\mathcal{N}_\Phi)^\varepsilon_\mu\right) &= \frac{2\varepsilon^2}{3}\,\mathcal{R}ic^\perp_{k+1}(\Pi_p)(\Theta, E_\mu) + L^\perp_{B^{k+1}}\,\Phi^\mu \\ &\quad + \mathcal{O}(\varepsilon^3) + \varepsilon^2\,\mathcal{L}_{\Pi_p}(\Phi) + \mathcal{Q}^2_{\Pi_p}(\Phi). \end{aligned}$$

As before, we let $\mathcal{K}^{\|}$ and $\mathcal{K}^\perp$ be the null-spaces of the operators

$$J^{\|}_{S^k} = \Delta_{S^k} + k \quad \text{and} \quad L^\perp_{B^{k+1}} = \Delta_{S^k} - D_{B^{k+1}},$$

and write $\mathcal{P}^{\|}$ and $\mathcal{P}^\perp$ for the $L^2$ orthogonal complements of $\mathcal{K}^{\|}$ and $\mathcal{K}^\perp$ in $\mathcal{C}^{2,\alpha}(S^k)$. Define the space

$$\mathfrak{E} := \mathbb{R}^{k+1} \times \left(\mathbb{R}^{k+1} \oplus \mathbb{R}\right)^{m-k} \times \mathcal{P}^{\|} \times \left(\mathcal{P}^\perp\right)^{m-k}. \tag{4.4}$$

There exists an operator

$$\mathcal{G} : \left(\mathcal{C}^{0,\alpha}\left(S^k\right)\right)^{m-k} \longrightarrow \mathfrak{E},$$

such that

$$\mathcal{G}(f_0, f_1, \ldots, f_{m-k}) = \Big(\vec{a}(\Pi_p, f), \vec{c}_\mu(\Pi_p, f), d_\mu(\Pi_p, f), \phi(\Pi_p, f), \Phi^\perp(\Pi_p, f)\Big)$$

is the solution to

$$\begin{cases} J^{\parallel}_{S^k}\, \phi = \langle \vec{a}, \Theta\rangle + f_0 \\ L^{\perp}_{B^{k+1}}\, \Phi^\mu = \langle \vec{c}_\mu, \Theta\rangle + d_\mu + f_{\mu-k}. \end{cases}$$

Applying a standard fixed point theorem for contraction mappings, we find that there exist $c > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $\Pi_p \in G_{k+1}(TM)$ there is a unique element

$$\Big(\vec{a}(\varepsilon, \Pi_p), \vec{c}_\mu(\varepsilon, \Pi_p), d_\mu(\varepsilon, \Pi_p), \phi(\varepsilon, \Pi_p), \Phi^\perp(\varepsilon, \Pi_p)\Big),$$

in a closed ball of radius $c\,\varepsilon^2$ centered at 0 in $\mathfrak{E}$ (for some constant $c > 0$) such that

$$H^{g_\varepsilon}(S^k_\Phi) = k\, n^\varepsilon_\Phi + \langle \vec{a}, \Theta\rangle\, n^\varepsilon_\Phi + \sum_{\mu=k+2}^{m+1} \big(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\big)\, (\mathcal{N}_\Phi)^\varepsilon_\mu.$$

Finally, to finish the proof we put

$$n_K = \frac{1}{\varepsilon}\,(F_\varepsilon)_*\, n^\varepsilon_\Phi \quad \text{and} \quad N_\mu = \frac{1}{\varepsilon}\,(F_\varepsilon)_*\, (\mathcal{N}_\mu)^\varepsilon_\Phi$$

and $K_\varepsilon(\Pi_p) := F_\varepsilon(S^k_{\Phi(\varepsilon,\Pi_p)}),\ Q_\varepsilon(\Pi_p) := F_\varepsilon(B^{k+1}_{\varepsilon,\Phi(\varepsilon,\Pi_p)})$. □

**Remark 4.7.** Using the fact that

$$\mathcal{R}ic_{k+1}(\Pi_p)(\Theta, \Theta) \in \mathcal{P}^{\parallel} \quad \text{and} \quad \mathcal{R}ic^{\perp}_{k+1}(\Pi_p)(\Theta, E_\mu) \in \mathcal{K}^\perp,$$

and decomposing

$$\mathcal{R}ic_{k+1}(\Pi_p)(\Theta, \Theta) = \sum_{a=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)_{aa}\, (\Theta^a)^2 + \sum_{a\neq b=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)_{ab}\, \Theta^a\, \Theta^b,$$

one can easily verify that the vector field $\Phi_{\varepsilon,\Pi_p}$ obtained in Proposition 4.4, satisfies

$$\phi_{\varepsilon,\Pi_p} = \frac{\varepsilon^2}{3}\left(\frac{2}{k(k+2)}\,\mathcal{R}_{k+1}(\Pi_p) - \frac{1}{k+2}\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta, \Theta)\right) + \mathcal{O}(\varepsilon^3),$$

$$[\Phi]^{\perp}_{\varepsilon,\Pi_p} = \mathcal{O}(\varepsilon^3).$$

### 4.2. Variational argument

We now employ a variational argument to prove that one can choose $\Pi_p \in G_k(M)$ in such a way that the submanifold $K_\varepsilon(\Pi_p)$ obtained in the previous Proposition has constant mean curvature.

To state our result, we introduce the following restrictions of the Riemann tensor of $M$:

**Notation 4.8.**

$$R_{k+1}(\Pi_p)(v_1, v_2, v_3, v_4) = g_p(R_p(v_1, v_2)v_3, v_4), \qquad v_1, v_2, v_3, v_4 \in \Pi_p,$$
$$R^\perp_{k+1}(\Pi_p)(v_1, v_2, v_3, N) = g_p(R_p(v_1, v_2)v_3, N), \qquad v_1, v_2, v_3 \in \Pi_p, \quad N \in \Pi_p^\perp,$$

Finally, we introduce the function $\mathbf{r}$ on $G_{k+1}(TM)$:

$$\begin{aligned}\mathbf{r}(\Pi_p) = \tfrac{1}{36(k+5)} \Big[ & 8\, \|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 - 18 \sum_{i,j,\ell=1}^{k+1} \nabla_{E_i} \nabla_{E_i}\, g(R(E_j, E_\ell)E_j, E_\ell)\big|_p \\ & -3\, \|R_{k+1}(\Pi_p)\|^2 + 5\, \mathcal{R}_{k+1}(\Pi_p)^2 + 24\, \tfrac{k+1}{k+3}\, \|\mathcal{R}ic^\perp_{k+1}(\Pi_p)\|^2 \\ & +12\, \|R^\perp_{k+1}(\Pi_p)\|^2 \Big] \\ & + \frac{\varepsilon^4}{18} \frac{1}{(k+2)(k+3)} \Big[ \tfrac{k+6}{k}\, \mathcal{R}^2_{k+1}(\Pi_p) - 2\, \|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 \Big].\end{aligned}$$

Now consider the energy $\mathcal{E}_\varepsilon$ restricted to this finite dimensional space of submanifolds,

$$\mathcal{E}_\varepsilon(\Pi_p) := \mathrm{Vol}_k\left(K_\varepsilon(\Pi_p)\right) - \frac{k}{\varepsilon}\, \mathrm{Vol}_{k+1}\left(Q_\varepsilon(\Pi_p)\right),$$

which is a function on $G_{k+1}(TM)$. Tracing through the construction of $K_\varepsilon(\Pi_p)$ one obtains the relationship of this function to the curvature functions defined above.

**Lemma 4.9.** *There is an expansion*

$$\frac{(k+1)\, \mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k\, \mathrm{Vol}(S^k)} = \left(1 - \frac{\varepsilon^2}{2(k+3)}\, \mathcal{R}_{k+1}\left(\Pi_p\right) + \frac{\varepsilon^4}{2(k+3)}\, \mathbf{r}(\Pi_p) + \mathcal{O}\left(\varepsilon^5\right)\right).$$

*Proof.* The proof is a technical calculation, contained in the Appendix. □

The main result of this section is the following proposition

**Proposition 4.10.** *If $\Pi_p$ is a critical point of $\mathcal{E}_\varepsilon$, then $K_\varepsilon(\Pi_p)$ has constant mean curvature.*

**Remark 4.11.** Theorems 1.1 and 1.2 are corollaries of Proposition 4.10. Indeed, if we define

$$\Psi(\varepsilon, \Pi_p) = 2\,\varepsilon^{-2}\,(k+3)\left(1-(k+1)\frac{\mathcal{E}_\varepsilon(\Pi_p)}{\varepsilon^k \mathrm{Vol}(S^k)}\right); \tag{4.5}$$

then for any $j \geq 0$, there exists a constant $C_j$ which is independent of $\varepsilon$ such that

$$\|\Psi(\varepsilon,\cdot) - \mathcal{R}_{k+1}(\cdot) + \varepsilon^2 \mathbf{r}(\cdot)\|_{\mathcal{C}^j(G_{k+1}(TM))} \leq C_j\,\varepsilon^3.$$

*Proof of the Proposition* 4.10. Let $\Pi_p$ be a critical point of $\mathcal{E}_\varepsilon$. We show that the parameters $\vec{a}$, $\vec{c}$ and $d$ must then necessarily vanish. We do this by considering various types of perturbations of $\Pi_p$.

First consider the perturbations in $G_{k+1}(M)$ which correspond to parallel translations of $\Pi_p$. In other words, we suppose that the family of planes $\Pi_{\exp_p(t\xi)}$ in $G_{k+1}(M)$ are parallel translates of $\Pi_p$ along the geodesic $\exp_p(t\xi)$.

The submanifold $K_\varepsilon(\Pi_{\exp_p(t\xi)})$ is a normal graph over $K_\varepsilon(\Pi_p)$ by a vector field $\Psi_{\varepsilon,\Pi_p,\xi,t}$ which depends smoothly on $t$. This defines a vector field on $K_\varepsilon(\Pi_p)$ by

$$Z_{\varepsilon,\Pi_p,\xi} = \partial_t \Psi_{\varepsilon,\Pi_p,\xi,t}\big|_{t=0}.$$

The first variation of the volume formula yields

$$\begin{aligned} 0 &= D\mathcal{E}_\varepsilon|_{\Pi_p}(\xi) \\ &= \int_{K_\varepsilon(\Pi_p)} \left( g\left(H\left(K_\varepsilon\left(\Pi_p\right)\right), Z_{\varepsilon,\Pi_p,\xi}\right) - \frac{k}{\varepsilon}\, g\left(n, Z_{\varepsilon,\Pi_p,\xi}\right)\right) \mathrm{dvol}_{K_\varepsilon(\Pi_p)} \\ &\quad -\frac{k}{\varepsilon}\int_{Q_\varepsilon(\Pi_p)} g\left(H\left(Q_\varepsilon\left(\Pi_p\right)\right), Z_{\varepsilon,\Pi_p,\xi}\right)\,\mathrm{dvol}_{Q_\varepsilon(\Pi_p)}, \end{aligned} \tag{4.6}$$

and then the construction of $Q_\varepsilon(\Pi_p)$ and $K_\varepsilon(\Pi_p)$ gives that

$$\int_{K_\varepsilon(\Pi_p)} \left( \langle \vec{a}, \Theta\rangle g\left(n, Z_{\varepsilon,\Pi_p,\xi}\right) + \sum_{\mu=k+2}^{m+1} \left(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\right) g\left(Z_{\varepsilon,\Pi_p,\xi}, N_\mu\right)\right) = 0.$$

Let $\Xi$ be the vector field obtained by parallel transport of $\xi$ along geodesics issuing from $p$, and suppose that $c$ is a constant independent of $\varepsilon$ and $\xi$. Then

$$\|Z_{\varepsilon,\Pi_p,\xi} - \Xi\|_g \leq c\,\varepsilon^2\,\|\xi\|.$$

By construction of $K_\varepsilon(\Pi_p)$, we have

$$\|n + \frac{1}{\varepsilon}\,(F_\varepsilon)_*\Theta\|_g \leq c\,\varepsilon^2, \quad \text{and} \quad \|N_\mu - \frac{1}{\varepsilon}\,(F_\varepsilon)_* E_\mu\|_g \leq c\,\varepsilon^2.$$

Now take $\xi \in \Pi_p \subset TM_p$, so that

$$g(n, Z_{\varepsilon,\Pi_p,\xi}) = g\left(-\frac{1}{\varepsilon}\,(F_\varepsilon)_*\Theta + \left(n + \frac{1}{\varepsilon}\,(F)_*\Theta\right),\ \Xi + \left(Z_{\varepsilon,\Pi_p,\xi} - \Xi\right)\right),$$

and

$$g(N_\mu, Z_{\varepsilon,\Pi_p,\xi}) = g\left(\frac{1}{\varepsilon}\,(F_\varepsilon)_* E_\mu + \left(N_\mu - \frac{1}{\varepsilon}\,(F_\varepsilon)_*\vec{E}_\mu,\ \Xi + \left(Z_{\varepsilon,\Pi_p,\xi} - \Xi\right)\right)\right).$$

We conclude that

$$\left|g(n, Z_{\varepsilon,\Pi_p,\xi}) + g_p(\xi,\Theta)\right| \le c\,\varepsilon^2\|\xi\|, \quad \text{and} \quad \left|g(N_\mu, Z_{\varepsilon,\Pi_p,\xi})\right| \le c\,\varepsilon^2\|\xi\|,$$

hence

$$\begin{aligned}\int_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle\, g_p(\xi,\Theta) &\le \Bigg| \int_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle\, g_p(\xi,\Theta) + \int_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle\, g(Z_{\varepsilon,\Pi_p,\xi}, n) \\ &\qquad + \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} \left(\langle\vec{c}_\mu,\Theta\rangle + d_\mu\right) g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu)\Bigg| \\ &\le c\,\varepsilon^2\,\|\xi\|\left(\int_{K_\varepsilon(\Pi_p)} |\langle\vec{a},\Theta\rangle| + \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} \left|\langle\vec{c}_\mu,\Theta\rangle + d_\mu\right|\right).\end{aligned}$$

Now taking $\xi = \sum_{i=1}^{k+1} a^i\, E_i$ we obtain

$$\int_{K_\varepsilon(\Pi_p)} \langle\vec{a},\Theta\rangle^2 \le c\,\varepsilon^2\,\|\vec{a}\|\left(\int_{K_\varepsilon(\Pi_p)} |\langle\vec{a},\Theta\rangle| + \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} \left|\langle\vec{c}_\mu,\Theta\rangle + d_\mu\right|\right).$$

In Euclidean space, we have the equality

$$\mathrm{Vol}_k(S^k)\|v\|^2 = (k+1)\int_{S^k} \langle v,\Theta\rangle^2, \quad \text{for all } v \in \mathbb{R}^{k+1}.$$

By the expansion of the induced metric, we obtain for $\varepsilon$ small enough

$$\frac{1}{2}\,\mathrm{Vol}_k\left(S^k\right)\,\varepsilon^k\,\|v\|^2 \le (k+1)\int_{K_\varepsilon(\Pi_p)} \langle v,\Theta\rangle^2.$$

Also, since $\mathrm{Vol}_k(K_\varepsilon(\Pi_p)) = \mathcal{O}(\varepsilon^k)$, we deduce

$$\|\vec{a}\| \leq c\,\varepsilon^2\Big(\|\vec{a}\| + \sum_{\mu=k+2}^{m+1}\big(\|\vec{c}_\mu\| + |d_\mu|\big)\Big). \tag{4.7}$$

Now move $p$ in the direction of a vector $\xi \in \Pi_p^\perp$ to get

$$\big|g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) - g_p(\xi, E_\mu)\big| \leq c\,\varepsilon^2\|\xi\|, \quad \text{and} \quad |g(n, Z_{\varepsilon,\Pi_p,\xi})| \leq c\,\varepsilon^2\|\xi\|.$$

We can write

$$\begin{aligned}
&\sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} \big(\langle\vec{c}_\mu,\Theta\rangle + d_\mu\big)\, g_p(\xi, E_\mu)\\
&\quad\leq \Big|\sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} \big(\langle\vec{c}_\mu,\Theta\rangle + d_\mu\big)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu)\\
&\qquad - \sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)} \big(\langle\vec{c}_\mu,\Theta\rangle + d_\mu\big)\, g_p(\xi, E_\mu)\\
&\qquad + \int_{K_\varepsilon(\Pi_p)}\langle\vec{a},\Theta\rangle\, g(Z_{\varepsilon,\Pi_p,\xi}, n)\Big|\\
&\quad\leq c\,\varepsilon^2\,\|\xi\|\int_{K_\varepsilon(\Pi_p)}\Big(|\langle\vec{a},\Theta\rangle| + \sum_{\mu=k+2}^{m+1}|\langle\vec{c}_\mu,\Theta\rangle + d_\mu|\Big).
\end{aligned}$$

Taking $\xi = d_\nu\, E_\nu$ gives

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} d_\nu\,\langle\vec{c}_\nu,\Theta\rangle + {d_\nu}^2 \leq c\,\varepsilon^2\,|d_\nu|\Bigg(&\int_{K_\varepsilon(\Pi_p)}|\langle\vec{a},\Theta\rangle|\\
&+ \sum_{\mu=k+2}^{m+1}\int_{K_\varepsilon(\Pi_p)}|\langle\vec{c}_\mu,\Theta\rangle + d_\mu|\Bigg).
\end{aligned} \tag{4.8}$$

Next consider a perturbation of $\Pi_p$ by a one-parameter family of rotations of $\Pi_p$ in $T_pM$ generated by an $(m+1)\times(m+1)$ skew matrix $A$. Then

$$D\mathcal{E}_\varepsilon|_{\Pi_p}(A) = \frac{d}{dt}\bigg|_{t=0}\mathcal{E}_\varepsilon\Big(\Big(I + tA + O\big(t^2\big)\Big)\Pi_p\Big) = \frac{d}{dt}\bigg|_{t=0}\mathcal{E}\left(A_t\left(K_\varepsilon\left(\Pi_p\right)\right)\right),$$

where, in geodesic normal coordinates

$$A_t(x) = x + tAx + \mathcal{O}\left(t^2\right).$$

The coordinates of the vector field associated to this flow are

$$Z_{\varepsilon,\Pi_p,\xi}(x) = \frac{d}{dt}\bigg|_{t=0} A_t(x) = Ax.$$

Considering only matrices $A \in \mathfrak{o}(m)$ such that $A : \Pi_p \to \Pi_p^\perp$, we obtain

$$\left|g(Z_{\varepsilon,\Pi_p,\xi}, n)\right| \le c\,\varepsilon^2 \|A\Theta\|, \quad \text{and} \quad \left|g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) - \langle A\Theta, E_\mu\rangle\right| \le c\,\varepsilon^2 \|A\Theta\|.$$

This gives the

$$\begin{aligned}
&\sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} \big(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\big)\, \langle A\Theta, E_\mu\rangle \\
&\qquad \le \Big| \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} \big(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\big)\, g(Z_{\varepsilon,\Pi_p,\xi}, N_\mu) \\
&\qquad\quad - \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} \big(\langle \vec{c}_\mu, \Theta\rangle + d_\mu\big)\, \langle A\Theta, E_\mu\rangle \\
&\qquad\quad + \int_{K_\varepsilon(\Pi_p)} \langle \vec{a}, \Theta\rangle\, g(Z_{\varepsilon,\Pi_p,\xi}, n) \Big| \\
&\qquad \le c\,\varepsilon^2 \int_{K_\varepsilon(\Pi_p)} \left( \|A\Theta\|\, |\langle \vec{a}, \Theta\rangle| + \sum_{\mu=k+2}^{m+1} \|A\Theta\|\, |\langle \vec{c}_\mu, \Theta\rangle + d_\mu| \right).
\end{aligned}$$

Let $C_\nu$ be the $(m-k)\times(k+1)$ matrix with column $\nu$ equal to the vector $\vec{c}_\nu \in \mathbb{R}^{k+1}$, and all other columns equal to 0. Then if

$$A = \begin{pmatrix} 0 & -C_\nu^T \\ C_\nu & 0 \end{pmatrix},$$

we get

$$\begin{aligned}
\int_{K_\varepsilon(\Pi_p)} \langle \vec{c}_\nu, \rangle^2 + \langle \vec{c}_\nu, \Theta\rangle d_\nu \;\le C\,\varepsilon^2 \Bigg( &\int_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\nu, \Theta\rangle|\, |g_p(\vec{a}, \Theta)| \\
&+ \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\nu, \Theta\rangle|\, \big|\langle \vec{c}_\mu, \Theta\rangle + d_\mu\big| \Bigg)
\end{aligned} \tag{4.9}$$

Adding (4.8) and (4.9) now gives

$$\int_{K_\varepsilon(\Pi_p)} |d_\nu + \langle \vec{c}_\nu, \Theta\rangle|^2 \le c\,\varepsilon^2 \left( \int_{K_\varepsilon(\Pi_p)} (|d_\nu| + |\langle \vec{c}_\nu, \Theta\rangle|)\,|\langle \vec{a}, \Theta\rangle| + \sum_{\mu=k+2}^{m+1} (|d_\nu| + |\langle \vec{c}_\nu, \Theta\rangle|)\,\left|\langle \vec{c}_\mu, \Theta\rangle + d_\mu\right| \right).$$

In Euclidean space, if $v \in \mathbb{R}^{k+1}$ and $\alpha \in \mathbb{R}$ are arbitrary, then

$$\int_{S^k} |\alpha + \langle v, \Theta\rangle|^2 = \left(\alpha^2 + \frac{1}{k+1}\|v\|^2\right) \mathrm{Vol}_k(S^k).$$

Using once again the decomposition of the induced metric on $K_\varepsilon(\Pi_p)$ we find for $\varepsilon$ small enough

$$\frac{1}{2(k+1)}\varepsilon^k\,\mathrm{Vol}_k(S^k)\left(\alpha^2 + \|v\|^2\right) \le \int_{K_\varepsilon(\Pi_p)} |\alpha + \langle v, \Theta\rangle|^2 . \tag{4.10}$$

which gives

$$\varepsilon^{2-k}\,\|\vec{c}_\nu\|^2 + |d_\nu|^2 \le c\,(\|\vec{c}_\nu\| + |d_\nu|) \left( \int_{K_\varepsilon(\Pi_p)} |\langle \vec{a}, \Theta\rangle| + \sum_{\mu=k+2}^{m+1} \int_{K_\varepsilon(\Pi_p)} |\langle \vec{c}_\mu, \Theta\rangle + d_\mu| \right).$$

Since $\mathrm{Vol}_k(K_\varepsilon(\Pi_p)) = \mathcal{O}(\varepsilon^k)$, we get

$$\|\vec{c}_\nu\| + |d_\nu| \le c\,\varepsilon^2 \left( \|\vec{a}\| + \sum_{\mu=k+2}^{m+1} \left(\|\vec{c}_\mu\| + |d_\mu|\right) \right). \tag{4.11}$$

Adding (4.7) and (4.11) gives

$$\left( \|\vec{a}\| + \sum_{\mu=k+2}^{m+1} \left(\|\vec{c}_\mu\| + |d_\mu|\right) \right) \le c\,\varepsilon^2 \left( \|\vec{a}\| + \sum_{\mu=k+2}^{m+1} \left(\|\vec{c}_\mu\| + |d_\mu|\right) \right),$$

which implies finally that $\|\vec{a}\| = 0$, $\|\vec{c}_\mu\| = 0$ and $|d_\mu| = 0$, $k+1 \le \mu$.

We conclude that if $\Pi_p$ is a critical point of the functional $\mathcal{E}_\varepsilon$, then the manifold $K_\varepsilon(\Pi_p)$ is a constant mean curvature submanifold of $M$. □

## Appendix

## A. Mean curvature of submanifolds

Let $\Sigma^k \subset M^{m+1}$ be an embedded submanifold. Let $x^1, \ldots, x^k$ be local coordinates on $\Sigma$ and

$$E_\alpha = \partial_{x_\alpha},$$

the corresponding coordinate vector fields. Suppose that $E_{k+1}, \ldots, E_{m+1}$ is a local frame for $N\Sigma$. This gives local coordinates transverse to $\Sigma$ by

$$p \in \Sigma \longmapsto \exp_p \left( \sum_{j=k+1}^{m+1} x^j \, E_j \right).$$

We make the convention that Greek indexes run from 1 to $k$, while Latin indexes run from $k+1$ to $m+1$. The induced metric on $\Sigma$ has coefficients $\bar{g}_{\alpha\beta}$, while

$$\bar{h}^i_{\alpha\beta} := \Gamma^i_{\alpha\beta} = g(\nabla_{E_\alpha} E_\beta, E_i),$$

are the coefficients of the second fundamental form. We also record the Christoffel symbols

$$\Gamma^j_{\alpha i} = g(\nabla_{E_\alpha} E_i, E_j).$$

The following result is standard, *cf.* [5] for a proof.

**Lemma A.1.** *If* $X = \sum_{j=k+1}^{m+1} x^j \, E_j$, *then*

$$\begin{aligned} g_{\alpha\beta} &= \bar{g}_{\alpha\beta} - 2\, \bar{g}(\bar{h}_{\alpha\beta}, X) + g\left(R(E_\alpha, X)E_\beta, X\right) + g(\nabla_{E_\alpha} X, \nabla_{E_\beta} X) + \mathcal{O}\left(|x|^3\right) \\ &= \bar{g}_{\alpha\beta} - 2\, \bar{h}^i_{\alpha\beta}\, x^i + \left( g\left(R(E_\alpha, E_i)E_\beta, E_j\right) + g^{\gamma\gamma'} \bar{h}^i_{\alpha\gamma}\, \bar{h}^j_{\gamma'\beta} + \Gamma^i_{\alpha\ell}\, \Gamma^j_{\ell\beta} \right) x^i\, x^j \\ &\quad + \mathcal{O}\left(|x|^3\right) \\ g_{\alpha j} &= -\Gamma^i_{\alpha j}\, x^i + \mathcal{O}\left(|x|^2\right) \\ g_{ij} &= \delta_{ij} + \frac{1}{3}\, g\left(R(E_i, E_\ell)E_j, E_{\ell'}\right) x^\ell\, x^{\ell'} + \mathcal{O}\left(|x|^3\right). \end{aligned}$$

Let $\Phi$ be a smooth section of $N\Sigma$ and consider the normal graph $\Sigma_\Phi = \{\exp_p(\Phi(p)) : p \in \Sigma\}$. Now let us use the previous lemma to expand the metric and volume form on $\Sigma_\Phi$. To state this result properly, introduce $\nabla^N$, the induced connection on $N\Sigma$,

$$\nabla^N \Phi = \pi_{N\Sigma} \circ \nabla\Phi.$$

Using the definitions of Section 2, we find that

**Lemma A.2.**

$$\begin{aligned} \mathrm{Vol_k}(\Sigma_\Phi) \;=\; & \mathrm{Vol}_k(\Sigma) - \int_\Sigma g(H(\Sigma), \Phi)\, \mathrm{dvol}_\Sigma \\ & + \frac{1}{2} \int_\Sigma \left( |\nabla^N \Phi|^2_g - g\big((\mathrm{Ric}_\Sigma + \mathfrak{H}^2_\Sigma)\, \Phi, \Phi\big) \right) \mathrm{dvol}_\Sigma \\ & + \frac{1}{2} \int_\Sigma \left( g\left(H(\Sigma), \Phi\right) \right)^2 \mathrm{dvol}_\Sigma + \ldots \end{aligned}$$

*Proof*. First of all we expand the induced metric on $\Sigma_\Phi$. Using the result of the previous Lemma, we find

$$\begin{aligned}(\bar g_\Phi)_{\alpha\beta} &= \bar g_{\alpha\beta} - 2\,g\left(\bar h_{\alpha\beta},\Phi\right) + g\left(R\left(E_\alpha,\Phi\right)\,E_\beta,\Phi\right) + g\left(\nabla_{E_\alpha}\Phi,\nabla_{E_\beta}\Phi\right) + \dots\\ &= \bar g_{\alpha\beta} - 2\,g\left(\bar h_{\alpha\beta},\Phi\right) + g\left(R\left(E_\alpha,\Phi\right)\,E_\beta,\Phi\right)\\ &\quad + \bar g^{\gamma\gamma'} g\left(\bar h_{\alpha\gamma},\Phi\right)\,g\left(\bar h_{\gamma\beta},\Phi\right) + g\left(\nabla^N_{E_\alpha}\Phi,\nabla^N_{E_\beta}\Phi\right) + \dots\end{aligned}$$

Next we use the well known expansion

$$\sqrt{\det(I+A)} = 1 + \frac{1}{2}\mathrm{Tr}\,A + \frac{1}{8}\,(\mathrm{Tr}A)^2 - \frac{1}{4}\left(\mathrm{Tr}\left(A^2\right)\right) + \dots$$

to find

$$\begin{aligned}\sqrt{\det \bar g_\Phi} = \Bigg(1 - g(H(\Sigma),\Phi) + \frac{1}{2}\Big(|\nabla^N\Phi|^2_g\; - g\left(\left(\mathrm{Ric}_\Sigma + (\mathfrak{H})^2_\Sigma\right)\,\Phi,\Phi\right)\\ +\;(g\left(H\left(\Sigma\right),\Phi\right))^2\Big) + \dots\Bigg)\,\sqrt{\det \bar g}.\end{aligned}$$

This completes the proof. □

From this we obtain the first and second variations of the volume functional,

$$D_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_\Phi\Psi = -\int_\Sigma g(H(\Sigma_\Phi),\Psi)\,\mathrm{dvol}_{\Sigma_\Phi}, \tag{A.1}$$

and

$$\begin{aligned}D^2_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_{\Phi=0}(\Psi,\Psi) \;=\; &\int_\Sigma \left(|\nabla^N\Psi|^2 - g\left(\left(\mathrm{Ric}_\Sigma + \mathfrak{H}^2_\Sigma\right)\,\Psi,\Psi\right)\right)\,\mathrm{dvol}_\Sigma\\ &+\int_\Sigma \left(g\left(H\left(\Sigma\right),\Psi\right)\right)^2\,\mathrm{dvol}_\Sigma.\end{aligned}$$

On the other hand, differentiating (A.1) once more gives

$$\begin{aligned}D^2_\Phi \mathrm{Vol_k}(\Sigma_\Phi)|_{\Phi=0}(\Psi,\Psi) \;=\; &-\int_\Sigma g\left(D_\Phi H\left(\Sigma_\Phi\right)|_{\Phi=0}\Psi,\Psi\right)\,\mathrm{dvol}_\Sigma\\ &+\int_\Sigma \left(g\left(H\left(\Sigma\right),\Psi\right)\right)^2\,\mathrm{dvol}_K.\end{aligned}$$

Comparing the two formulæ implies that the orthogonal projection of the Jacobi operator to $N\Sigma$ equals

$$J^N_\Sigma := D_\Phi H(\Sigma_\Phi)|_{\Phi=0} = \Delta^N_g + \mathrm{Ric}^N_\Sigma + \mathfrak{H}^2_\Sigma\,.$$

## B. Appendix

We give here the proof of Lemma 4.9, namely the proof of the formula

$$\frac{(k+1)\,\mathcal{E}_\varepsilon\left(\Pi_p\right)}{\varepsilon^k\,\mathrm{Vol}(S^k)} = \left(1 - \frac{\varepsilon^2}{2(k+3)}\,\mathcal{R}_{k+1}\left(\Pi_p\right) + \frac{\varepsilon^4}{2(k+3)}\,\mathbf{r}\left(\Pi_p\right) + \mathcal{O}\left(\varepsilon^5\right)\right)$$

and find the expression of the function $r$. Let $K_\varepsilon(\Pi_p)$ be the constant mean curvature submanifold constructed in Proposition 4.4 and consider the mapping

$$F : \mathbb{R}^{m+1} \to M, \quad F(v) = \exp_p\left(\sum_{i=1}^{m+1} v^i\, E_i\right),$$

where $E_i, i = 1, \ldots, m+1$ is an orthonormal basis of $T_pM$. Recall that

$$K_\varepsilon(\Pi_p) = F(S^k_{\varepsilon,\Phi}),$$

where $S^k_{\varepsilon,\Phi} \subset \mathbb{R}^{m+1}$ is parametrized by $\left\{\varepsilon\,(1-\phi)\,\Theta + \varepsilon\,\Phi^\perp,\ \Theta \in S^k\right\}$. It follows from the proof of that proposition that

$$\phi(\Theta) = \frac{\varepsilon^2}{3}\left(\frac{2}{k(k+2)}\mathcal{R}_{k+1}(\Pi_p) - \frac{1}{k+2}\mathcal{R}ic(\Pi_p)(\Theta,\Theta)\right) + \mathcal{O}(\varepsilon^3),$$

$$\Phi^\perp = \mathcal{O}\left(\varepsilon^3\right).$$

Next, consider the minimal submanifold

$$Q_\varepsilon(\Pi_p) = F\left(B^{k+1}_{\varepsilon,\Phi}\right),$$

where $B^{k+1}_{\varepsilon,\Phi} = \left\{\varepsilon\, y + \varepsilon\, U_\Phi(y),\ y \in B^{k+1}\right\}$ and recall that

$$U_\Phi(y) = \phi\left(y/\|y\|\right) + W(y) + \mathcal{O}\left(\varepsilon^3\right),$$

$$W(y) = \frac{\varepsilon^2}{(k+3)} \sum_{\mu=k+2}^{m+1} \sum_{i=1}^{k+1} \mathcal{R}ic^\perp(\Pi_p)_{i\mu}\left(|y|^2 - 1\right)\, y^i\, E_\mu.$$

We shall calculate the volume forms of $S^k_{\varepsilon,\Phi}$ and $B^{k+1}_{\varepsilon,\Phi}$ with respect to $F^*g$. Recall that in the neighborhood of $x = 0$ we have

$$\begin{aligned}(F^*g)_{ij} = \delta_{ij} &+ \frac{1}{3}\, g_p\left(R_p(x,E_i)x, E_j\right) + \frac{1}{6}\, g_p\left(\nabla_x R_p(x,E_i)x, E_j\right)\\ &+ \frac{1}{20}\, g_p\left(\nabla_x\nabla_x R_p(x,E_i)x, E_j\right)\\ &+ \sum_{\ell=1}^{m+1} \frac{2}{45}\, g_p\left(R_p(x,E_i)x, E_\ell\right)\, g_p\left(R_p(x,E_j)x, E_\ell\right) + \mathcal{O}_p\left(|x|^5\right),\end{aligned}$$

where $R_p$ is the curvature tensor of $M$ at the point $p$, *cf.* [9].

**Volume of the CMC sphere**

We first find the expansion of the metric induced on $S^k_{\varepsilon,\Phi}$. To this end we express the tangent vector fields to $S^k_{\varepsilon,\Phi}$ in terms of the vector fields $\Theta_\alpha,\ \alpha = 1, \dots, k$ tangent to the unit sphere $S^k$ :

$$\tau_\alpha = \varepsilon\,(1-\phi(\Theta))\,\Theta_\alpha - \varepsilon\,\partial_\alpha\,\phi\,\Theta + \sum_{\mu=k+2}^{m+1} \varepsilon\,\partial_\alpha \Phi^\mu\, E_\mu, \quad \alpha = 1, \dots, k.$$

The metric coefficients then satisfy

$$\begin{aligned} g^K_{\alpha\beta} &= \varepsilon^2(1-\phi)^2 g^S_{\alpha\beta} + \varepsilon^2 \partial_\alpha\phi\,\partial_\beta\phi + \frac{\varepsilon^4}{3}(1-\phi)^4 g_p\big(R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta\big) \\ &+ \frac{\varepsilon^5}{6} g_p\big(\nabla_\Theta R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta\big) + \frac{\varepsilon^6}{20} g_p\big(\nabla_\Theta\nabla_\Theta R_p(\Theta,\Theta_\alpha)\Theta,\Theta_\beta\big) \\ &+ \sum_{l=1}^{k+1}\frac{2\varepsilon^6}{45} g_p\big(R_p(\Theta,\Theta_\alpha)\Theta,E_l\big)\, g_p\big(R_p\big(\Theta,\Theta_\beta\big)\Theta,E_l\big) \\ &+ \sum_{\mu=k+2}^{m+1}\frac{2\varepsilon^6}{45} g_p\big(R_p(\Theta,\Theta_\alpha)\Theta,E_\mu\big)\, g_p\big(R_p\big(\Theta,\Theta_\beta\big)\Theta,E_\mu\big) + \mathcal{O}\Big(\varepsilon^7\Big). \end{aligned}$$

Using

$$\sqrt{\det(I+A)} = 1 + \frac{1}{2}\mathrm{tr}A + \frac{1}{8}(\mathrm{tr}A)^2 - \frac{1}{4}\mathrm{tr}(A^2) + \mathcal{O}(|A|^3),$$

we get

$$\begin{aligned} \varepsilon^{-k}\frac{\sqrt{\det g^K}}{\sqrt{\det g^S}} &= 1 - k\phi + \frac{k(k-1)}{2}\phi^2 + \frac{1}{2}|\nabla_{S^k}\phi|^2 \\ &- \frac{\varepsilon^2}{6}(1-(k+2)\phi)\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \\ &- \frac{\varepsilon^3}{12}\nabla_\Theta \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) \\ &- \frac{\varepsilon^4}{40}\nabla^2_\Theta \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \frac{\varepsilon^4}{72}\big(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\big)^2 \\ &- \frac{\varepsilon^4}{180}\sum_{i,j=1}^{k+1} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 \\ &+ \frac{\varepsilon^4}{45}\sum_{i=1}^{k+1}\sum_{\mu=k+2}^{m+1} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2 + \mathcal{O}_p\big(\varepsilon^5\big). \end{aligned}$$

**Volume of the minimal ball**

Now let us calculate the volume element of $Q_\varepsilon(\Pi_p)$. Take $u(y) = \phi(y/|y|)$. The tangent vectors to $B^{k+1}_{\varepsilon,\Phi}$ are given by

$$T_i(y) = \varepsilon\,(1-u(y))\,E_i + \varepsilon\,\partial_{y^i}\,u(y)\,y + \varepsilon \sum_{\mu=k+2}^{m+1} \partial_{y^i}\,W^\mu(y)\,E_\mu + \mathcal{O}_p\big(\varepsilon^4\big).$$

The corresponding expansion of metric coefficients is

$$\begin{aligned}
\varepsilon^{-2} g^Q_{ij} =& (1-u)^2\delta_{ij} + (1-u)\big(\partial_{y^i} u\, y_j + \partial_{y^j} u\, y^i\big) + |y|^2 \partial_{y^i} u\, \partial_{y^j} u \\
&+ \sum_{\mu=k+2}^{m+1} \partial_{y_i} W^\mu \partial_{y_j} W^\mu + \frac{\varepsilon^2}{3}(1-u)^4 g_p(R_p(y,E_i)y,E_j) \\
&+ \frac{\varepsilon^2}{3} \sum_{\mu=k+2}^{m+1} \Big[ W^\mu g_p(R_p(E_\mu,E_i)y,E_j) + W^\mu g_p(R_p(y,E_i)E_\mu,E_j) \\
&\qquad + \partial_{y^i} W^\mu g_p(R_p(y,E_\mu)y,E_j) + \partial_{y^j} W^\mu g_p(R_p(y,E_i)y,E_\mu)\Big] \\
&+ \frac{\varepsilon^3}{6} g_p(\nabla_y R_p(y,E_i)y,E_j) + \frac{\varepsilon^4}{20} g_p(\nabla_y\nabla_y R_p(y,E_i)y,E_j) \\
&+ \frac{2\varepsilon^4}{45} \sum_{l=1}^{k+1} g_p(R_p(y,E_i)y,E_l)\, g_p(R_p(y,E_i)y,E_l) \\
&+ \frac{2\varepsilon^4}{45} \sum_{\mu=k+2}^{m+1} g_p(R_p(y,E_i)y,E_\mu)\, g_p(R_p(y,E_i)y,E_\mu) + \mathcal{O}\big(\varepsilon^5\big).
\end{aligned}$$

Using the fact $\langle \nabla u, y\rangle = 0$ and the fact that for the matrix $A_{ij} = y^i\,\partial_{y^j} u + y^j\,\partial_{y^i} u$ we have $\frac{1}{4}\operatorname{tr}(A^2) = \frac{1}{2}\,|y|^2\,|\nabla u|^2$, we calculate the volume element of $Q_\varepsilon(\Pi_p)$:

$$\begin{aligned}
\varepsilon^{-(k+1)}\sqrt{\det g^Q} =& 1 - (k+1)u + \frac{k(k+1)}{2} u^2 + \sum_{\mu=k+2}^{m+1} \frac{1}{2} |\nabla_{S^k} W^\mu|^2 \\
&- \frac{\varepsilon^2}{6}(1-(k+3)u)\mathcal{R}ic_{k+1}(\Pi_p)(y,y) \\
&+ \frac{\varepsilon^2}{3} \sum_{i=1}^{k+1} \sum_{\mu=k+2}^{m+1} \Big[ W^\mu g_p(R_p(y,E_i,E_\mu,E_i) \\
&- \frac{\varepsilon^3}{12} \nabla_y \mathcal{R}ic_{k+1}(\Pi_p)(y,y) - \frac{\varepsilon^4}{40} \nabla^2_y \mathcal{R}ic_{k+1}(\Pi_p)(y,y) \\
&+ \frac{\varepsilon^4}{72}\big(\mathcal{R}ic_{k+1}(\Pi_p)(y,y)\big)^2 - \frac{\varepsilon^4}{180} \sum_{i,j=1}^{k+1} g_p(R_p(y,E_i)y,E_j)^2 \\
&+ \frac{\varepsilon^4}{45} \sum_{i=1}^{k+1} \sum_{\mu=k+2}^{m+1} g_p(R_p(y,E_i)y,E_\mu)^2 + \mathcal{O}_p(\varepsilon^5).
\end{aligned}$$

**Expansion of the energy functional**

Collecting the results obtained above, we find

$$
\begin{aligned}
&\varepsilon^{-k}\Big(\mathrm{Vol}(K_\varepsilon(\Pi_p)) - \frac{k}{\varepsilon}\,\mathrm{Vol}(Q_\varepsilon(\Pi_p))\Big) = \frac{1}{k+1}\,\mathrm{Vol}(S^k)\\
&- \frac{\varepsilon^2}{2(k+3)}\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\,d\sigma + \frac{\varepsilon^2}{3}\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\,\phi\,d\sigma\\
&+ \frac{5\,\varepsilon^4}{k+5}\int_{S^k}\Big[-\frac{1}{40}\,\nabla^2_\Theta\,\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta) + \frac{1}{72}\,\big(\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)\big)^2\\
&- \frac{1}{180}\sum_{i,j=1}^{k+1} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 + \frac{1}{45}\sum_{i=1}^{k+1}\sum_{\mu=k+2}^{m+1} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2\Big]d\sigma\\
&- \frac{\varepsilon^2\,k}{3}\sum_{i=1}^{k+1}\Big(W^\mu\,g_p(R_p(\Theta,E_i,E_\mu,E_i)) + \partial_{y_i}W^\mu\,R_p(\Theta,E_i,\Theta,E_\mu)\Big)\,dy\\
&+ \sum_{\mu=k+2}^{m+1}\int_{B^{k+1}}\Big[\frac{k}{2}\,W^\mu\,\Delta_{B^{k+1}}\,W^\mu\Big] - \frac{1}{2}\int_{S^k}\phi\,\Delta_{S^k}\,\phi - \frac{k}{2}\int_{S^k}\phi^2\,d\sigma + \mathcal{O}(\varepsilon^5).
\end{aligned}
$$

We now recall some identities:

$$
\begin{aligned}
&\int_{S^k}(\Theta^i)^2\,d\sigma = \frac{1}{k+1}\,\mathrm{Vol}(S^k),\\
&\int_{S^k}(\Theta^i)^4\,d\sigma = 3\int_{S^k}(\Theta^i\,\Theta^j)^2\,d\sigma = \frac{3}{(k+1)(k+3)}\,\mathrm{Vol}(S^k);
\end{aligned}
$$

and if $a_{ijpq}\in\mathbb{R}\ i,j,p,q = 1,\ldots,k+1$, then

$$
\begin{aligned}
&\sum_{p,q,l,n=1}^{k+1}\int_{S^k} a_{pqln}\,\Theta^p\,\Theta^q\,\Theta^l\,\Theta^n\,d\sigma\\
&\qquad = \frac{3}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{i=1}^{k+1} a_{pppp}\\
&\qquad + \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{q\neq p=1}^{k+1}\big(a_{ppqq} + a_{pqpq} + a_{pqqp}\big)\\
&\qquad = \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\sum_{p,q=1}^{k+1}\big(a_{ppqq} + a_{pqpq} + a_{pqqp}\big)\,;
\end{aligned}
$$

and develop each term:

$$\int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta)d\sigma = \sum_{i,j=1}^{k+1} \int_{S^k} \mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_j)\Theta^k\Theta^l d\sigma$$

$$= \sum_{i=1}^{k+1} \mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_i)(\Theta^i)^2 d\sigma = \frac{1}{k+1}\mathrm{Vol}(S^k)\mathcal{R}_{k+1}(\Pi_p);$$

$$\int_{S^k} (\mathcal{R}ic_{k+1}(\Pi_p)(\Theta,\Theta))^2 d\sigma$$

$$= \frac{1}{(k+1)(k+3)}\mathrm{Vol}(S^k)\Big[2\sum_{i,j=1}^{k+1}(\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_j))^2$$

$$+ \sum_{i,j=1}^{k+1}\mathcal{R}ic_{k+1}(\Pi_p)(E_i,E_i)\mathcal{R}ic_{k+1}(\Pi_p)(E_j,E_j)\Big]$$

$$= \frac{1}{(k+1)(k+3)}\mathrm{Vol}(S^k)\Big(2\left\|\mathcal{R}ic_{k+1}(\Pi_p)\right\|^2 + \mathcal{R}_{k+1}(\Pi_p)^2\Big);$$

$$\sum_{i,j=1}^{k+1}\int_{S^k} g_p(R_p(\Theta,E_i)\Theta,E_j)^2 d\sigma$$

$$= \frac{1}{(k+1)(k+3)}\mathrm{Vol}(S^k)\sum_{i,j,p,q=1}^{k+1}\Big(R^2_{ipjq} + R_{ipjp}R_{iqjq} + R_{ipjq}R_{iqjp}\Big)$$

$$= \frac{1}{(k+1)(k+3)}\mathrm{Vol}(S^k)\left(\left\|\mathcal{R}ic_{k+1}(\Pi_p)\right\|^2 + \frac{3}{2}\left\|R_{k+1}(\Pi_p)\right\|^2\right);$$

(we use here that $R^2_{ijpq} = (R_{ipjq} - R_{iqjp})^2 = R^2_{ipjq} + R^2_{iqjp} - 2\,R_{ipjq}\,R_{iqjp}$);

$$\sum_{i=1}^{k+1}\sum_{\mu=k+2}^{m+1}\int_{S^k} g_p(R_p(\Theta,E_i)\Theta,E_\mu)^2\,d\sigma$$

$$= \frac{1}{(k+1)(k+3)}\,\mathrm{Vol}(S^k)\left(\left\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\right\|^2 + \frac{3}{2}\left\|R^{\perp}_{k+1}(\Pi_p)\right\|^2\right);$$

$$\int_{S^k}\nabla^2_{\Theta}\mathcal{R}ic_{k+1}(\Theta,\Theta)d\sigma$$

$$= \frac{1}{(k+1)(k+3)}\sum_{i,j=1}^{k+1}\Big(\nabla^2_{E_i}\mathcal{R}ic_{k+1}(\Pi_p)(E_j,E_j) + 2\nabla_{E_i}\nabla_{E_j}\mathcal{R}ic_{k+1}(E_i,E_j)\Big)$$

$$= \frac{2}{(k+1)(k+3)}\mathrm{Vol}(S^k)\sum_{i,j,\ell=1}^{k+1}\nabla_{E_i}\nabla_{E_i}g(R(E_k,E_l)E_k,E_l)\big|_p;$$

$$\sum_{\mu=k+2}^{m+1}\int_{B^{k+1}} W^\mu\,\Delta_{B^{k+1}} W^\mu\,dy$$
$$= -\frac{2\,\varepsilon^4}{9}\,\frac{1}{k+3}\sum_{\mu=k+2}^{m+1}\int_{B^{k+1}}\sum_{j=1}^{k+1}\Big(\mathcal{R}ic^\perp_{k+1}(E_j,E_\mu)\Big)^2\,(y^j)^2\,(1-|y|^2)\,dy$$
$$= -\frac{2\,\varepsilon^4}{9}\,\frac{1}{(k+3)(k+1)}\,\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^\perp_{k+1}\|^2\left(\frac{1}{k+3}-\frac{1}{k+5}\right)$$
$$= -\frac{\varepsilon^4}{9}\,\frac{4}{(k+1)(k+3)^2(k+5)}\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^\perp_{k+1}\|^2;$$

$$\sum_{\mu=k+2}^{m+1}\int_{B^{k+1}} W^\mu\sum_{i,p=1}^{k+1} R_{ipi\mu}\,y^p\,dy$$
$$= -\frac{\varepsilon^2}{3}\,\frac{1}{(k+3)}\int_{B^{k+1}}\sum_{\mu=k+2}^{m+1}\sum_{j=1}^{k+1}\Big(\mathcal{R}ic^\perp_{k+1}(E_j,E_\mu)\Big)^2\,(y^j)^2(1-|y|^2)\,dy$$
$$= -\frac{\varepsilon^2}{3}\,\frac{2}{(k+1)(k+3)^2(k+5)}\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^\perp_{k+1}\|^2;$$

and

$$\sum_{\mu=k+2}^{m+1}\int_{B^{k+1}}\partial_{y^i}\,W^\mu R_{piq\mu}\,y^p\,y^q\,dy$$
$$= \frac{\varepsilon^2}{3}\,\frac{1}{(k+3)}\sum_{\mu=k+2}^{m+1}\int_{B^{k+1}}\sum_{i,p,q=1}^{k+1}\Big(\mathcal{R}ic(\Pi_p)^\perp_{i\mu}\,R_{piq\mu}\,y^p\,y^q\,(1-|y|^2)$$
$$-\;2\sum_{j=1}^{k+1}\mathcal{R}ic(\Pi_p)^\perp_{j\mu}\,R_{piq\mu}\,y^j\,y^i\,y^p\,y^q\Big)\,dy$$
$$= \frac{\varepsilon^2}{3}\,\frac{2}{(k+1)(k+3)^2(k+5)}\mathrm{Vol}(S^k)\Big[-\|\mathcal{R}ic^\perp_{k+1}\|^2$$
$$-\sum_{p,q=1}^{k+1}\sum_{\mu=k+2}^{m+1}\Big(\mathcal{R}ic(\Pi_p)^\perp_{p\mu}\,R_{qpq\mu}+\mathcal{R}ic(\Pi_p)^\perp_{q\mu}\,R_{ppq\mu}$$
$$+\,\mathcal{R}ic(\Pi_p)^\perp_{q\mu}\,R_{qpp\mu}\Big)\Big]$$
$$= -\frac{2\,\varepsilon^2}{3}\,\frac{1}{(k+1)(k+3)^2(k+5)}\,\mathrm{Vol}(S^k)\,\|\mathcal{R}ic^\perp_{k+1}\|^2.$$

This finally gives

$$
\begin{aligned}
\frac{(k+1)\,\mathcal{E}(\Pi_p)}{\varepsilon^k\,\mathrm{Vol}(S^k)} &= 1 - \frac{\varepsilon^2}{2}\,\frac{1}{k+3}\,\mathcal{R}_{k+1}(\Pi_p) \\
&\quad + \frac{\varepsilon^4}{72}\,\frac{1}{(k+3)(k+5)}\Big[8\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2 \\
&\quad - 18\sum_{i,j,\ell=1}^{k+1} \nabla_{E_i}\nabla_{E_i}\, g(R(E_j,E_\ell)E_j,E_\ell)\big|_p \\
&\quad - 3\,\|R_{k+1}(\Pi_p)\|^2 + 5\,\mathcal{R}_{k+1}(\Pi_p)^2 \\
&\quad + 24\,\frac{k+1}{k+3}\,\|\mathcal{R}ic^{\perp}_{k+1}(\Pi_p)\|^2 + 12\,\|R^{\perp}_{k+1}(\Pi_p)\|^2\Big] \\
&\quad + \frac{\varepsilon^4}{18}\,\frac{1}{(k+2)(k+3)}\left[\frac{k+6}{k}\,\mathcal{R}^2_{k+1}(\Pi_p) - 2\,\|\mathcal{R}ic_{k+1}(\Pi_p)\|^2\right] \\
&\quad + \mathcal{O}\big(\varepsilon^5\big) \\
&= 1 - \frac{\varepsilon^2}{2(k+3)}\,\mathcal{R}_{k+1}(\Pi_p) + \frac{\varepsilon^4}{2(k+3)}\,\mathbf{r}(\Pi_p) + \mathcal{O}\big(\varepsilon^5\big).
\end{aligned}
$$

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