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Errata corrige. Invertible harmonic mappings, beyond Kneser

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Abstract. We amend the proof of the main theorem in the authors' paper *Invertible harmonic mappings, beyond Kneser*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (5) VIII (2009), 451-468,

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The proof of [1, Lemma 3.11], which is instrumental to the proof of the main Theorem 1.3, contains a gap. In fact the monotonicity of the function g with respect to ψ is incorrectly deduced from Lemma 3.8.

Furthermore the proof of Theorem 7.3 is also flawed, because $\mathrm{WN}(\Phi(\partial \mathrm{B}))=1$ does not imply the injectivity of Φ.

We provide here a new proof of Theorem 1.3. Unfortunately we have not been able to amend the proof of Theorem 7.3. Consequently, Remark 1.5, Theorem 7.3 and the following Corollary 7.4 should be expunged. For the sake of clarity we rewrite almost completely the content of Section 3 and the initial part of Section 4 of [1], in order to provide a correct proof of Theorem 1.3. In this Erratum formulas and statements are single-numbered: double numbering (section.number) is used to refer to items in [1].

What follows replaces the content of [1, from page 455, 5$^{\text{th}}$ line from the bottom, to page 459, 7$^{\text{th}}$ line from the top].

Proof of Proposition 3.2. Obviously $\nabla u_\alpha \neq 0$ everywhere on ∂B for every $\alpha \in [0, 2\pi]$. By the argument principle for holomorphic functions

$$M_\alpha = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\left(\frac{\partial u_\alpha}{\partial z}\right), \quad \text{for every } \alpha \in [0, 2\pi].$$

We shall show that $M_\alpha = M_0$ for every $\alpha \in [0, 2\pi]$. It is clear that it suffices to consider $\alpha \in (0, \pi)$. We set

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and we have

$$\nabla u_\alpha \cdot J\nabla u = \sin(\alpha)\det DU > 0, \quad \text{on } \partial B,$$

hence $\left|\arg\left(\frac{\partial u_\alpha}{\partial z}\right) - \arg\left(i\frac{\partial u}{\partial z}\right)\right| < \pi$. We conclude that

$$M_\alpha = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\left(\frac{\partial u_\alpha}{\partial z}\right) = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\left(i\frac{\partial u}{\partial z}\right) = M_0. \qquad \square$$

Proof of Corollary 3.3. Let us assume that for a given $\alpha \in [0, 2\pi]$, we have $M_\alpha = 0$. By Proposition 3.2 one has $M_\alpha = 0$ for every $\alpha \in [0, 2\pi]$. Hence, for every $P \in B$, the vectors $\nabla u(P)$ and $\nabla v(P)$ are linearly independent, that is $\det DU(P) \neq 0$. Being $\det DU > 0$ on ∂B, by continuity we have $\det DU > 0$ everywhere in B. The reverse implication is trivial. $\square$

Definition 1. Given a closed curve γ, parameterized by $\Phi \in C^1([0, 2\pi]; \mathbb{R}^2)$ and such that

$$\frac{d\Phi}{d\theta} \neq 0, \text{ for every } \theta \in [0, 2\pi],$$

we define the *winding number* of γ as the following integer

$$\mathrm{WN}(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\arg\left(\frac{d\Phi}{d\theta}\right).$$

Remark 2. Denoting $\beta = \beta(\theta) = \frac{d}{d\theta}\Phi(\theta)$, it is well-known that

$$\mathrm{WN}(\gamma) = \frac{1}{2\pi i}\int_0^{2\pi} d\log(\beta(\theta)) = \frac{1}{2\pi i}\int_\beta \frac{dz}{z}.$$

That is, $\mathrm{WN}(\gamma)$ is the index (degree) of the curve β with respect to the origin. In view of the fact that the 1-form $\frac{dz}{z}$ is closed on $\mathbb{C} \setminus \{0\}$, it is also well-known that the winding number is invariant under homotopy, see for instance [3, Theorem 1].

Definition 3. Let u be a harmonic function in B. We denote by $\tilde{u}$ its conjugate harmonic function and we set

$$f = u + i\tilde{u}.$$

Note that if, in addition, $u \in C^1(\overline{B})$ and $\nabla u \neq 0$ on ∂B, then $f\big|_{\partial B}$ gives us a regular C^1 parametrization of a closed curve.

Proposition 4. *Let $u \in C^1(\overline{B})$ be harmonic in B. If $\nabla u \neq 0$ on ∂B, then*

$$M = \mathrm{WN}(f(\partial B)) - 1,$$

with M as in Definition 3.1.

Proof. The proof is elementary, and we claim no novelty in this case. We have

$$\begin{aligned}\mathrm{WN}(f(\partial B)) &= \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}\right) = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\left[\arg\left(\frac{\partial f}{\partial z}\right)+\theta\right] \\ &= \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\left(\frac{\partial u}{\partial z}\right)+1 = M+1. \qquad \square\end{aligned}$$

Remark 5. Let us emphasize that, if the harmonic mapping $U=(u,v)\in C^1(\overline{B};\mathbb{R}^2)$ is such that $\det DU > 0$ on ∂B, then we also have $|\frac{\partial f}{\partial z}| = |\nabla u| > 0$ on ∂B. Moreover, for any $P \in \partial B$, the mapping U is a diffeomorphism near P.

We are now ready to state a theorem which contains the main elements towards a proof of Theorem 1.3.

Theorem 6. Let $U \in C^1(\overline{B};\mathbb{R}^2)$ be harmonic in B and let $\Phi = U\big|_{\partial B}$. If $\det DU >$ 0 on ∂B, then we have

$$\mathrm{WN}\big(f(\partial B)\big) = \mathrm{WN}\big(\Phi(\partial B)\big). \tag{0.1}$$

Proof. Denote

$$U_t = (1-t)f + tU, \text{ for every } t \in [0,1],$$

and

$$\beta_t(\theta) = \frac{d}{d\theta}U_t\left(e^{i\theta}\right), \text{ for every } t \in [0,1],\ \theta \in [0,2\pi].$$

We have

$$U_t = (u, (1-t)\tilde{u} + tv), \text{ for every } t \in [0,1],$$

and therefore

$$\det DU_t = (1-t)|\nabla u|^2 + t\det DU > 0, \text{ on } \partial B, \text{ for every } t \in [0,1],$$

hence, for every $t \in [0,1]$, $\theta \in [0,2\pi]$, $\beta_t(\theta) \neq 0$. Therefore

$$\frac{d}{d\theta}U \text{ and } \frac{d}{d\theta}f,$$

are homotopically equivalent closed curves in $\mathbb{C}\setminus\{0\}$. The thesis follows. $\square$

Proof of Theorem 1.3. Let us assume that (1.2) holds. By assumption Φ is one–to–one and sense-preserving. Then it is well-known that

$$\mathrm{WN}\Big(\Phi\big(\partial B\big)\Big) = 1,$$

see for instance Hopf [2, page 53] and Whitney [3, Theorem 2]. Hence, by Theorem 6

$$\mathrm{WN}\big(f(\partial B)\big) = \mathrm{WN}\big(\Phi(\partial B)\big) = 1.$$

By Proposition 3.6, ∇u never vanishes in B. By Corollary 3.3, $\det DU > 0$ everywhere in $\overline{B}$. By Theorem 2.1, $U : \overline{B} \to \overline{D}$ is a diffeomorphism. The reverse implication is obvious. $\square$

The remaining part of Section 4 and Sections 5–6 need no changes.

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