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The rational sectional category of certain maps

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Abstract. We give a simple algebraic characterisation of the sectional category of rational maps admitting a homotopy retraction. As a particular case we get the Felix-Halperin theorem for the rational Lusternik-Schnirelmann category and we prove the conjecture of Jessup-Murillo-Parent on rational topological complexity. We also give a characterisation for relative categories in the sense of Doeraene-El Haouari.

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1. Introduction

Throughout this work we consider all spaces to be of the homotopy type of simply connected CW-complexes of finite type and we use the standard rational homotopy techniques which are explained in the excellent text [13]. The sectional category is an invariant of the homotopy type of maps introduced by Schwarz in [21]. If $f\colon X \to Y$ is a continuous map, its *sectional category* is the smallest m for which there are $m+1$ local homotopy sections for f whose sources form an open cover of Y. Its most studied particular case is the well-known Lusternik-Schnirelmann (LS) category of a space X introduced in [18] as a lower bound for the number of critical points on any smooth map defined on a smooth manifold X. Namely, the LS category of a pointed space X, $\operatorname{cat}(X)$, is the sectional category of the base point inclusion map, $* \hookrightarrow X$.

A remarkable theorem of Félix-Halperin [12, Theorem 4.7] gives an algebraic characterisation of the LS category of rational spaces in terms of their Sullivan models. Explicitly, if X is a space modelled by $(\Lambda V, d)$ and X_0 is its rationalisation (see [13,23]) then $\operatorname{cat}(X_0)$ is the smallest m for which the commutative differential

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graded algebra (cdga) projection

$$\rho_m\colon (\Lambda V, d) \to \left(\frac{\Lambda V}{\Lambda^{>m} V}, \overline{d}\right)$$

admits a homotopy retraction, that is, a strict retraction for a relative Sullivan model for ρ_m.

Let $f\colon X \to Y$ be a map such that its rationalisation f_0 admits a homotopy retraction r. Then, through standard rational homotopy techniques f can be modelled by a retraction $\varphi\colon (B \otimes \Lambda W, D) \to (B, d)$ of a relative Sullivan algebra $(B, d) \rightarrowtail (B \otimes \Lambda W, D)$ modeling r. For simplicity in the notation, write $\varphi\colon A \to B$ and call it from now on an *s-model* of f.

Theorem 1.1. *The sectional category of the rationalisation of f,* $\mathrm{secat}(f_0)$*, is the smallest m for which the cdga projection*

$$A \to \frac{A}{(\ker\varphi)^{m+1}}$$

admits a homotopy retraction.

Observe that, choosing $\varphi\colon (\Lambda V, d) \to \mathbb{Q}$, this theorem reduces to the Félix-Halperin theorem for rational LS-category. On the other hand, it also generalises the Murillo-Jessup-Parent conjecture on rational topological complexity [16]. Indeed, in his famous paper [9] M. Farber introduced the concept of *topological complexity* of a space X, $\mathrm{TC}(X)$, which can be seen as the sectional category of the diagonal map $\Delta\colon X \to X \times X$. This invariant is used to estimate the *motion planning complexity* of a mechanical system and also has applications to other fields of mathematics [10]. As a direct generalisation of this invariant, Rudyak introduced in [20] the concept of higher topological n-complexity of a space, $\mathrm{TC}_n(X)$, as the sectional category of the n-diagonal map $\Delta_n\colon X \to X^n$. Several explicit computations of topological complexity of rational spaces have been performed in [2, 15–17]. Inspired by the Félix-Halperin theorem, Jessup, Murillo and Parent, conjectured that $\mathrm{TC}(X_0)$ is the smallest m such that the projection

$$(\Lambda V \otimes \Lambda V, d) \longrightarrow \left(\frac{\Lambda V \otimes \Lambda V}{K^{m+1}}, \overline{d}\right)$$

admits a homotopy retraction, where K denotes the kernel of the multiplication morphism $\mu_2\colon \Lambda V \otimes \Lambda V \to \Lambda V$.

Theorem 1.1 applied to higher topological complexity is a bit more general than the Murillo-Jessup-Parent conjecture. Namely, if A is *any* cdga model for a space X, then Δ_n admits an s-model of the form $\varphi = (\mathrm{Id}_A, \eta, \dots, \eta)\colon A \otimes (\Lambda V)^{\otimes n-1} \to A$ where $\eta\colon \Lambda V \xrightarrow{\simeq} A$ is a Sullivan model for A. From Theorem 1.1 we immediately deduce:

Theorem 1.2. *Let X be a topological space. Then* $\mathrm{TC}_n(X_0)$ *is the smallest m such that the projection*

$$A \otimes (\Lambda V)^{\otimes n-1} \longrightarrow \frac{A \otimes (\Lambda V)^{\otimes n-1}}{(\ker\ \varphi)^{m+1}}$$

admits a homotopy retraction.

We remark that, since $\mathrm{secat}(f_0) \leq \mathrm{secat}(f)$ [2], we get algebraic lower bounds for *integral* sectional category which are better than $\mathrm{nil}\ker f^*$. Some of the ideas in this paper come from [4].

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2. Fibrewise pointed cdgas and relative nilpotency

In this section we develop some technical tools that will be needed later on. Let $\mathcal{C}$ be a J-category in the sense of Doeraene [5, 6] or a closed model category in the sense of Quillen [19] and fix an object B of $\mathcal{C}$. Consider the *fibrewise pointed category* over B [1, page 30], denoted by $\mathcal{C}(B)$, whose objects are factorisations of Id_B, $B \xrightarrow{s_X} X \xrightarrow{p_X} B$, and whose morphism are morphisms in $\mathcal{C}$, $f\colon X \to Y$, such that $f \circ s_X = s_Y$ and $p_Y \circ f = p_X$. Such a morphism is said to be a *fibration* ($\twoheadrightarrow$), *cofibration* ($\rightarrowtail$) or *weak equivalence* ($\xrightarrow{\simeq}$) if the underlying morphism f is such in $\mathcal{C}$. With these definitions $\mathcal{C}(B)$ is also either a J-category or a closed model category (note that this structure is not the same as that of [14]). We denote by $[X, Y]_B$ the homotopy classes of morphism in $\mathcal{C}(B)$ from the fibrant-cofibrant object X into Y.

Now, and for the rest of the paper, we particularise on $\mathcal{C} = \mathbf{cdga}$. Remark that the fibrant-cofibrant objects of $\mathbf{cdga}(B)$ are precisely the relative Sullivan algebras $(B \otimes \Lambda W, D)$ with the natural inclusion $B \hookrightarrow (B \otimes \Lambda W, D)$ and endowed with a given retraction. In this context, the general property [19] by which weak equivalences induce bijections on homotopy classes reads:

Lemma 2.1. *Suppose $\theta\colon A \to C$ is a quasi-isomorphism in* $\mathbf{cdga}(B)$ *and $(B \otimes \Lambda V, D)$ a fibrant-cofibrant object of* $\mathbf{cdga}(B)$*, then composition with θ induces a bijection $\theta_\#\colon [B \otimes \Lambda V, A]_B \to [B \otimes \Lambda V, C]_B$.*

Definition 2.2. Let $A \in \mathbf{cdga}(B)$, its *relative nilpotency index*, $\mathrm{nil}_B(A)$, is the nilpotency index $\mathrm{nil}\ker p_A$ of the ideal $\ker p_A$.

The following lemma is crucial. It tells us that we can control the relative nilpotency index of certain homotopy pullbacks of $\mathbf{cdga}(B)$.

Lemma 2.3. *Let $i\colon C \rightarrowtail (C \otimes \Lambda V, D)$ be a cofibration in* $\mathbf{cdga}(B)$ *such that $D(V) \subset (\ker p_C) \oplus (C \otimes \Lambda^+ V)$ and $p_{C\otimes\Lambda V}(V) = 0$. Then there is an object $N \in \mathbf{cdga}(B)$ weakly equivalent to the homotopy pullback of i and $s_{C\otimes\Lambda V}$ for which* $\mathrm{nil}_B\, N = \mathrm{nil}_B\, C + 1$.

Proof. In $\mathbf{cdga}(B)$, factor $s_{C\otimes\Lambda V}$ as $B \overset{\alpha}{\hookrightarrow} S \overset{h}{\twoheadrightarrow} C \otimes \Lambda V$ where

$$S = B \oplus \big(C \otimes \Lambda V \otimes \Lambda^+(t, dt)\big),$$

in which t has degree 0, $b(c\otimes v\otimes\xi) = s_C(b)c\otimes v\otimes\xi$, and $h(c\otimes v\otimes t) = c\otimes v$. As $C \otimes \Lambda V \otimes \Lambda^+(t, dt)$ is acyclic, α is a quasi-isomorphism and thus, the homotopy pullback of i and $s_{C\otimes\Lambda V}$ is the pullback

$$\begin{array}{ccc} M' & \xrightarrow{\ \overline{h}\ } & C \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ S & \underset{h}{\twoheadrightarrow} & C \otimes \Lambda V \end{array}$$

of i and h. This is in fact a pullback in $\mathbf{cdga}(B)$ by choosing $p_{M'} = p_C \circ \overline{h}$ and $s_{M'} = (\alpha, s_C)$. To finish, we will construct an object N of $\mathbf{cdga}(B)$ weakly equivalent to M' with $\mathrm{nil}_B\, N = \mathrm{nil}_B\, C + 1$.

Write $K_\epsilon = \ker\epsilon$ where $\epsilon\colon \Lambda^+(t, dt) \to \mathbb{Q}$ is the augmentation sending t to 1, and consider the $\mathbf{cdga}(B)$ isomorphism

$$\eta\colon M \xrightarrow{\cong} M',$$

in which:

$$M = B \oplus \big(C \otimes \Lambda^+(t, dt)\big) \oplus \big(C \otimes \Lambda^+ V \otimes K_\epsilon\big),$$

$$s_M(b) = b, \quad p_M(b) = b, \quad p_M(c\otimes\xi) = p_C(c)\epsilon(\xi), \quad p_M(c\otimes v\otimes\omega) = 0,$$

$$\eta(b) = (b, s_C(b)), \quad \eta(c\otimes\xi) = (c\otimes 1\otimes\xi, c\epsilon(\xi)), \quad \eta(c\otimes v\otimes\omega) = (c\otimes v\otimes\omega, 0),$$

with $b \in B$, $c \in C$, $\xi \in \Lambda^+(t, dt)$, $v \in V$ and $\omega \in K_\epsilon$.

Next, write $C = \ker p_C \oplus R$ and consider

$$N = B \oplus \big(\ker p_C \otimes \Lambda^+(t, dt)\big) \oplus \big(\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon\big) \oplus \big(R \otimes \Lambda^+ V \otimes dt\big)$$

which, since $D(V) \subset (\ker p_C) \oplus (C \otimes \Lambda^+ V)$, is a sub $\mathbf{cdga}(B)$ of M. Moreover, the inclusion $N \hookrightarrow M$ is a weak equivalence in $\mathbf{cdga}(B)$ as the subcomplexes $\big(\ker p_C \otimes \Lambda^+(t, dt)\big) \oplus \big(\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon\big)$ and $\big(C \otimes \Lambda^+(t, dt)\big) \oplus \big(\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon\big)$ are quasi-isomorphic and the inclusion of quotient complexes $B \oplus \big(R \otimes \Lambda^+ V \otimes dt\big) \hookrightarrow B \oplus \big(R \otimes \Lambda^+ V \otimes K_\epsilon\big)$ is a quasi-isomorphism.

Finally, we have that

$$\ker p_N = \big(\ker p_C \otimes \Lambda^+(t, dt)\big) \oplus \big(\ker p_C \otimes \Lambda^+ V \otimes K_\epsilon\big) \oplus \big(R \otimes \Lambda^+ V \otimes dt\big),$$

and thus, a non-trivial product of maximal length in this ideal is given by $z(1 \otimes v \otimes dt)$ where z is a non-trivial product of maximal length in $\ker p_C \otimes \Lambda^+(t)$. This proves that $\mathrm{nil}_B N = \mathrm{nil}_B C + 1$. □

3. The main result

Let $f\colon X \to Y$ be a continuous map. Recall from [1,3,11] that, by iterated joins, one can construct an m-Ganea map for f, $G_m(f)$, fitting into a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\iota}\swarrow & & \searrow{\scriptstyle f} \\ P^m(f) & \xrightarrow[G_m(f)]{} & Y, \end{array} \qquad (3.1)$$

and that $\mathrm{secat}(f) \le m$ if and only if $G_m(f)$ admits a homotopy section. Also, if $\varphi\colon A \twoheadrightarrow B$ is a surjective model for f, then Diagram (3.1) can be modelled by a diagram

$$\begin{array}{ccc} A & \xrightarrow{\kappa_m} & C_m \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle p_m} \\ & B & \end{array},$$

where κ_m models $G_m(f)$ and can be constructed inductively by taking the homotopy pullback of the induced maps by the homotopy pushout of φ and any model, $g\colon A \to D$, of $G_{m-1}(f)$. Standard arguments show that $\mathrm{secat}(f_0) \le m$ if and only if κ_m admits a homotopy retraction. One can extend this to:

Definition 3.1. Let $f\colon X \to Y$ be a continuous map. Then:

(i) $\mathrm{msecat}(f) \le m$ if and only if κ_m admits a homotopy retraction as A-module;
(ii) $\mathrm{Hsecat}(f) \le m$ if and only if κ_m is homology injective.

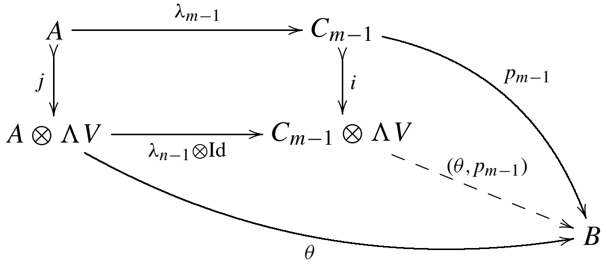
We now give the key model for the m-Ganea map $G_m(f)$:

Proposition 3.2. *Let f be a map such that f_0 admits a homotopy retraction and let $\varphi\colon A \twoheadrightarrow B$ be an s-model for f. Then there is a model λ_m for $G_m(f)$ which is a morphism in* **cdga**(B),

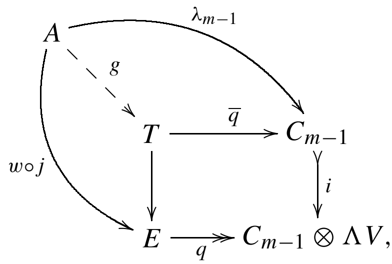
$$\begin{array}{ccc} & B & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle s_m} \\ A & \xrightarrow{\lambda_m} & C_m \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle p_m} \\ & B & \end{array},$$

with $\mathrm{nil}_B\, C_m = m$.

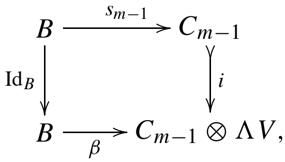
Proof. We will proceed by induction. For $m = 0$ the assertion holds since $p_0 = \mathrm{Id}_B$. Suppose λ_{m-1} exists. Since φ is surjective, one can take a relative Sullivan model for φ, $\theta\colon (A \otimes \Lambda V, D) \xrightarrow{\simeq} B$, such that $D(V) \subset (\ker\varphi) \oplus \big(A \otimes \Lambda^+ V\big)$ and $\theta(V) = 0$. Now take the homotopy pushout



and factor $\lambda_{n-1} \otimes \mathrm{Id}$ as $q \circ w$ with $q\colon E \twoheadrightarrow C_{m-1} \otimes \Lambda V$ a surjective cdga morphism and $w\colon A \otimes \Lambda V \xrightarrow{\simeq} E$ a weak equivalence. Then the pullback's universal property



gives a model g for $G_m(f)$ which can be seen as a morphism in **cdga**(B) by taking $p_T = p_{m-1} \circ \overline{q}$ and $s_T = g \circ s$. Now, define $\beta = i \circ s_{m-1}\colon B \to C_{m-1} \otimes \Lambda V$ and consider the factorisation of $\beta = q \circ (w \circ j \circ s)$ as a quasi-isomorphism followed by a fibration. On the other hand, consider also the factorisation of $\beta = h \circ \alpha$ as in the proof of Lemma 2.3. Applying [6, Lemma 1.8] to previous factorisations and the following commutative square in **cdga**(B),



we get quasi-isomorphisms $M' \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} T$. Now, applying Lemma 2.3 to $C_{m-1} \rightarrowtail C_{m-1} \otimes \Lambda V$, with $s_{C_{m-1}\otimes\Lambda V} = \beta$ and $p_{C_{m-1}\otimes\Lambda V} = (\theta, p_{m-1})$ we get an object C_m of **cdga**(B), with $\mathrm{nil}_B\, C_m = m$, which is weakly equivalent M'. Observe that we cannot use the pullback's universal property to get a model of $G_m(f)$ because, in general, $\beta \circ \varphi$ does not coincide with $i \circ \lambda_{m-1}$. We get then a diagram in **cdga**(B)

$$A \xrightarrow{g} T \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} M' \xleftarrow{\simeq} C_m.$$

Since A is a fibrant-cofibrant object of **cdga**(B), we can apply Lemma 2.1 to get a model for $G_m(f)$ in **cdga**(B), $\lambda_m\colon A \to C_m$, with $\mathrm{nil}_B\, C_m = m$. □

Given $\varphi\colon A \twoheadrightarrow B$ a surjective cdga morphism, consider, for each $m \geq 0$, the cdga projection

$$\rho_m\colon A \to \frac{A}{(\ker \varphi)^{m+1}}.$$

Then Theorem 1.1 is just statement (i) in the following:

Theorem 3.3. *Let $\varphi\colon A \twoheadrightarrow B$ be an s-model for a map f such that f_0 admits a homotopy retraction. Then:*

(i) $\operatorname{secat}(f_0)$ *is the smallest m for which ρ_m admits a homotopy retraction;*
(ii) $\operatorname{msecat}(f_0)$ *is the smallest m for which ρ_m admits a homotopy retraction as A-module;*
(iii) $\operatorname{Hsecat}(f)$ *is the smallest m such that $H(\rho_m)$ is injective.*

Proof. Take from Proposition 3.2 a morphism of **cdga**(B), $\lambda_m\colon A \to C_m$, modelling $G_m(f)$ with $\operatorname{nil}_B C_m = m$. Since $\lambda_m((\ker\ \varphi)^{m+1}) = 0$ we get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda_m} & C_m \\ {\scriptstyle\rho_m}\downarrow & \nearrow{\scriptstyle\overline{\lambda_m}} & \downarrow{\scriptstyle p_m} \\ \dfrac{A}{(\ker\ \varphi)^{m+1}} & \xrightarrow[\overline{\varphi}]{} & B \end{array}$$

and the result follows by standard rational homotopy techniques and [2, Proposition 12]. ☐

Observe that [2, Example 10] shows that the hypothesis s is a cofibration is necessary.

In [22] D. Stanley gives an example of a map f for which f_0 does not admit a homotopy retraction and $\operatorname{msecat}(f) < \operatorname{secat}(f_0)$. Here we state:

Conjecture 3.4. If f is a map and f_0 admits a homotopy retraction, then

$$\operatorname{msecat}(f) = \operatorname{secat}(f_0).$$

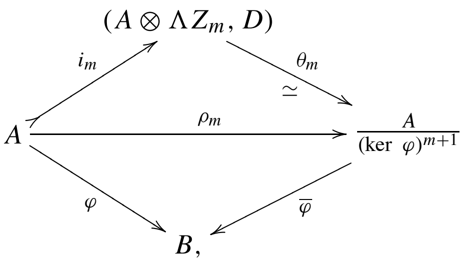
Concerning the rational topological complexity of a given space X and, with the notation in Theorem 1.2, we may define $\operatorname{mTC}(X)$ as the smallest integer m for which the projection

$$A \otimes \Lambda V \to \frac{A \otimes \Lambda V}{(\ker \varphi)^{m+1}}$$

admits a homotopy retraction as $A \otimes \Lambda V$-module. Then Theorem 3.3 (ii) combined with [16, Theorem 1.6] gives the Ganea conjecture for mTC.

Theorem 3.5. *Given any space X then* $\operatorname{mTC}(X \times S^n) = \operatorname{mTC}(X) + \operatorname{mTC}(S^n)$.

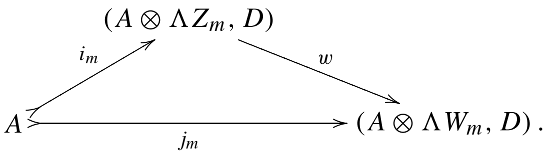
We finish by presenting, via Theorem 3.3, an algebraic description of the rational relative category. Recall [7] that the relative category, relcat$f$, of a map f is the smallest m for which $G_m(f)$ of Diagram (3.1) admits a homotopy section s such that $s \circ f \simeq \iota$. Also, in [7], Doeraene and El Haouari proved that secat(f) and relcat(f) differ at most by one and conjectured in [8] that they agree on maps admitting a homotopy retraction. Consider then such a map f and $\varphi\colon A \to B$ and s-model for f. This gives a diagram



where i_m is a relative Sullivan model for ρ_m.

Theorem 3.6. *With the previous notation,* relcat(f_0) *is the smallest m such that i_m admits a retraction r verifying $\varphi \circ r \simeq \overline{\varphi} \circ \theta_m$ rel A.*

Proof. Consider the commutative diagram in the proof of Theorem 3.3, where p_m is a model for ι in Diagram (3.1). Taking j_m a relative model of λ_m and applying Lemma 2.1 we get a diagram in **cdga**(B)



If j_m admits a retraction r' such that $\varphi \circ r' \simeq p_m$ rel A then i_m admits a retraction $r := r' \circ w$ such that $\varphi \circ r = \varphi \circ r' \circ w \simeq p_m \circ \omega = \overline{\varphi} \circ \theta_m$ rel A. $\square$

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