Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 685-711

A new approach to the L^p-theory of $-\Delta + b \cdot \nabla$, and its applications to Feller processes with general drifts

DAMIR KINZEBULATOV

Abstract. We develop a detailed regularity theory of $-\Delta+b\cdot\nabla$ in $L^p(\mathbb{R}^d)$, for a wide class of vector fields. The L^p-theory allows us to construct associated strong Feller process in $C_\infty(\mathbb{R}^d)$. Our starting object is an operator-valued function, which, we prove, determines the resolvent of an operator realization of $-\Delta+b\cdot\nabla$, the generator of a holomorphic C_0-semigroup on $L^p(\mathbb{R}^d)$. Then the very form of the operator-valued function yields crucial information about smoothness of the domain of the generator.

Mathematics Subject Classification (2010): 35J15 (primary); 47D07, 35J75 (secondary).

1. Introduction

Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d, \mathcal{L}^d)$ and $W^{1,p} = W^{1,p}(\mathbb{R}^d, \mathcal{L}^d)$ the standard (complex) Lebesgue and Sobolev spaces, $C^{0,\gamma} = C^{0,\gamma}(\mathbb{R}^d)$ the space of Hölder continuous functions $(0 < \gamma < 1)$, $C_b = C_b(\mathbb{R}^d)$ the space of bounded continuous functions endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity, $\mathcal{W}^{\alpha,p}$, $\alpha > 0$, the Bessel space endowed with norm $\|u\|_{p,\alpha} := \|g\|_p$, $u = (1-\Delta)^{-\frac{\alpha}{2}} g$, $g \in L^p$, and $\mathcal{W}^{-\alpha,p'}$, $p' = p/(p-1)$, the anti-dual of $\mathcal{W}^{\alpha,p}$. $\mathcal{W}^{\alpha,p}_{\text{loc}}$ denotes the class of (distributions) u such that $(1-\Delta)^{\frac{\alpha}{2}}(u\varphi) \in L^p$ for any $\varphi \in C_c^\infty$. We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between complex Banach spaces $X \to Y$, endowed with operator norm $\|\cdot\|_{X\to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Set $\|\cdot\|_{p\to q} := \|\cdot\|_{L^p \to L^q}$.

For each $p \geqslant 1$, by $\langle u, v\rangle$ we denote the $(L^p, L^{p'})$ pairing, so that

$$\langle u, v\rangle = \langle u\bar{v}\rangle := \int_{\mathbb{R}^d} u\bar{v}\,d\mathcal{L}^d \qquad (u \in L^p, v \in L^{p'}).$$

Received October 20, 2015; accepted in revised form February 3, 2016.
Published online June 2017.

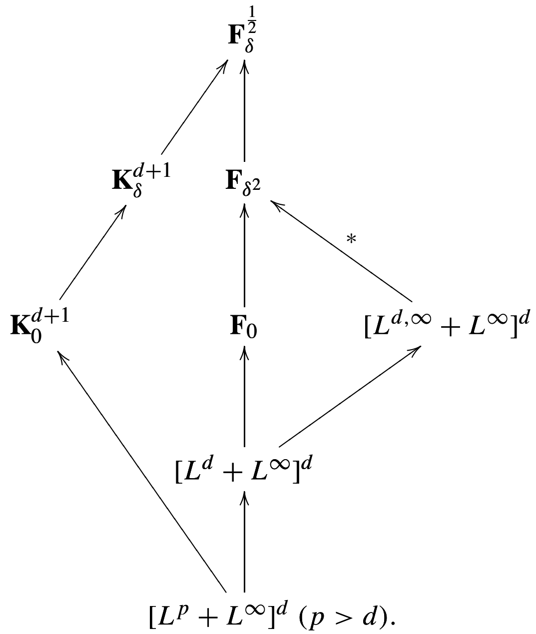

Figure 1.1. General classes of vector fields $b : \mathbb{R}^d \to \mathbb{R}^d$ studied in the literature in connection with the operator $-\Delta + b\cdot\nabla$. Here $\rightarrow$ stands for strict inclusion, and $\overset{*}{\rightarrow}$ reads "if $b = b_1 + b_2 \in [L^{d,\infty} + L^\infty]^d$, then $b \in \mathbf{F}_{\delta^2}$ with $\delta > 0$ determined by the value of the $L^{d,\infty}$-norm of $|b_1|$", see Remark 1.2 below for details, $\mathbf{K}_0^{d+1} := \bigcap_{\delta>0} \mathbf{K}_\delta^{d+1}$, $\mathbf{F}_0 := \bigcap_{\delta>0} \mathbf{F}_\delta$.

Let $d \geqslant 3$. Consider the following classes of vector fields:

(1) We say that a $b : \mathbb{R}^d \to \mathbb{C}^d$ belongs to the Kato class $\mathbf{K}_\delta^{d+1}$, and write $b \in \mathbf{K}_\delta^{d+1}$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta;$$

(2) We say that a $b : \mathbb{R}^d \to \mathbb{C}^d$ belongs to $\mathbf{F}_\delta$, the class of form-bounded vector fields, and write $b \in \mathbf{F}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \sqrt{\delta};$$

(3) We say that a $b : \mathbb{R}^d \to \mathbb{C}^d$ belongs to $\mathbf{F}_\delta^{\frac{1}{2}}$, the class of *weakly* form-bounded vector fields, and write $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda =$

$\lambda_\delta > 0$ such that

$$\| |b|^{\frac{1}{2}} (\lambda - \Delta)^{-\frac{1}{4}} \|_{2\to 2} \leqslant \sqrt{\delta}.$$

Simple examples show:

$$\mathbf{F}_{\delta_1} - \mathbf{K}_\delta^{d+1} \neq \varnothing, \quad \text{and} \quad \mathbf{K}_0^{d+1} - \mathbf{F}_\delta \neq \varnothing \quad \text{for any } \delta, \delta_1 > 0,$$

for instance,

1) by the Hardy inequality, $b(x) := \sqrt{\delta_1}\frac{d-2}{2}x|x|^{-2} \in \mathbf{F}_{\delta_1} - \mathbf{K}_\delta^{d+1}$ for any $\delta, \delta_1 > 0$;
2) $b(x) := e\mathbf{1}_{|x_1|<1}|x_1|^{s-1}$, where $\frac{1}{2} < s < 1$, $e = (1, \dots, 1) \in \mathbb{R}^d$, $x = (x_1, \dots, x_d)$, is in $\mathbf{K}_0^{d+1} - \mathbf{F}_\delta$, for any $\delta > 0$. (An example of a $b \in \mathbf{K}_\delta^{d+1} - \mathbf{K}_0^{d+1}$ can be obtained, *e.g.*, by modifying [1, Example 1, page 250].)

The classes $\mathbf{F}_{\delta_1}$, $\mathbf{K}_\delta^{d+1}$ cover singularities of b of critical order[1], at isolated points or along hypersurfaces, respectively. The classes $\mathbf{F}_0$ and $\mathbf{K}_0^{d+1}$ do not contain vector fields having critical order singularities.

Remark 1.1. The classes $\mathbf{F}_\delta$ and $\mathbf{K}_\delta^{d+1}$ have been intensely studied in the literature: after 1996, the Kato class $\mathbf{K}_\delta^{d+1}$, with $\delta > 0$ sufficiently small (yet allowed to be non-zero), has been recognized as 'the right' class for the Gaussian upper and lower bounds on the fundamental solution of $-\Delta + b \cdot \nabla$, see [14], which, in turn, allow to construct an associated Feller process (in C_b). The class $\mathbf{F}_\delta$, $\delta < 4$, is responsible for dissipativity of $\Delta - b \cdot \nabla$ in L^p, $p \geqslant \frac{2}{2-\sqrt{\delta}}$, needed to run the iterative procedure of [8] (taking $p \to \infty$, assuming additionally $\delta < \min\{4/(d-2)^2, 1\}$), which produces an associated Feller process. We emphasize that, in general, the Gaussian bounds are not valid if $b \in \mathbf{F}_\delta$, while $b \in \mathbf{K}_\delta^{d+1}$, in general, destroys L^p-dissipativity.

The class $\mathbf{F}_\delta^{\frac{1}{2}}$ combines critical point and critical hypersurface singularities:

$$\begin{gathered} \mathbf{K}_\delta^{d+1} \subsetneq \mathbf{F}_\delta^{\frac{1}{2}}, \qquad \mathbf{F}_{\delta_1} \subsetneq \mathbf{F}_\delta^{\frac{1}{2}} \quad \text{for } \delta = \sqrt{\delta_1}, \\ \left(b \in \mathbf{F}_{\delta_1} \text{ and } \mathsf{f} \in \mathbf{K}_{\delta_2}^{d+1}\right) \Longrightarrow \left(b + \mathsf{f} \in \mathbf{F}_\delta^{\frac{1}{2}},\ \sqrt{\delta} = \sqrt[4]{\delta_1} + \sqrt{\delta_2}\right) \end{gathered} \tag{1.1}$$

(for the proof, if needed, see Appendix B).

Remark 1.2. The inclusion $|b| \in L^d \Rightarrow b \in \mathbf{F}_0$ (*cf.* the diagram above) follows by the Sobolev embedding theorem. For $|b| \in L^{d,\infty}$, we can verify, using [7,

[1] In particular, the uniqueness of solution of Cauchy problem for $-\Delta + b \cdot \nabla$ can fail if $b \in \mathbf{F}_\delta$ is replaced with cb ($\in \mathbf{F}_{c^2\delta}$), for a sufficiently large constant c, *cf.* [8, Example 5].

Proposition 2.5, 2.6, Corollary 2.9]:

$$
\begin{aligned}
b \in \mathbf{F}_\delta^{\frac{1}{2}}, \quad \text{with } \sqrt{\delta} &= \||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \|(|b|^*)^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \\
&\leqslant \left(\|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\right)^{\frac{1}{2}} \||x|^{-\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \\
&= \left(\|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\right)^{\frac{1}{2}} 2^{-\frac{1}{2}} \frac{\Gamma\left(\frac{d-1}{4}\right)}{\Gamma\left(\frac{d+1}{4}\right)},
\end{aligned}
$$

where $\Omega_d = \pi^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)$, and $|b|^*$ is the symmetric decreasing rearrangement of $|b|$. Similarly,

$$
\begin{aligned}
b \in \mathbf{F}_{\delta_1}, \quad \text{with } \sqrt{\delta_1} &= \||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \\
&\leqslant \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}} \||x|^{-1}(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \\
&\leqslant \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}} 2^{-1} \frac{\Gamma\left(\frac{d-2}{4}\right)}{\Gamma\left(\frac{d+2}{4}\right)} = \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}} \frac{2}{d-2}.
\end{aligned}
$$

In particular, using [7, Corollary 2.9],

$$
\begin{aligned}
x|x|^{-2} \in \mathbf{F}_\delta^{\frac{1}{2}}, \quad & \sqrt{\delta} = 2^{-\frac{1}{2}} \frac{\Gamma\left(\frac{d-1}{4}\right)}{\Gamma\left(\frac{d+1}{4}\right)}, \\
x|x|^{-2} \in \mathbf{F}_{\delta_1}, \quad & \sqrt{\delta_1} = \frac{2}{d-2},
\end{aligned}
$$

and so $\delta < \sqrt{\delta_1}$ (*cf.* (1.1)).

Denote

$$
m_d := \pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}} d^{\frac{d}{2}} (d-1)^{\frac{1-d}{2}}, \qquad c_p := pp'/4.
$$

The following two theorems are the main results of our paper.

Theorem 1.3 (**L^p-theory**). *Let $d \geqslant 3$ and $b : \mathbb{R}^d \to \mathbb{C}^d$. Assume that $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $m_d\delta < 1$. Then, for every*

$$
p \in \mathcal{I} := \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right),
$$

there exists a C_0-semigroup $e^{-t\Lambda_p(b)}$ in L^p such that:

(i) *The resolvent set $\rho(-\Lambda_p(b))$ contains the half-plane $\mathcal{O} := \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \geqslant \kappa_d\lambda_\delta\}$, $\kappa_d := \frac{d}{d-1}$, and the resolvent admits the representation:*

$$
(\zeta + \Lambda_p(b))^{-1} = \Theta_p(\zeta, b), \quad \zeta \in \mathcal{O},
$$

where

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - Q_p(1 + T_p)^{-1} G_p, \tag{1.2}$$

the operators $Q_p, G_p, T_p \in \mathcal{B}(L^p)$,

$$\|G_p\|_{p \to p} \leqslant C_1 |\zeta|^{-\frac{1}{2p'}}, \ \|Q_p\|_{p \to p} \leqslant C_2 |\zeta|^{-\frac{1}{2} - \frac{1}{2p}}, \ \|T_p\|_{p \to p} \leqslant m_d c_p \delta < 1,$$

$$G_p \equiv G_p(\zeta, b) := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}, \quad b^{\frac{1}{p}} := |b|^{\frac{1}{p} - 1} b,$$

Q_p, T_p *are the extensions by continuity of densely defined (on* $\mathcal{E} := \bigcup_{\epsilon > 0} e^{-\epsilon |b|} L^p$*) operators*

$$Q_p|_{\mathcal{E}} \equiv Q_p(\zeta, b)|_{\mathcal{E}} := (\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}},$$

$$T_p|_{\mathcal{E}} \equiv T_p(\zeta, b)|_{\mathcal{E}} := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}};$$

(ii) *It follows from* (i) *that* $e^{-t\Lambda_p(b)}$ *is holomorphic: there is a constant* C_p *such that*

$$\|(\zeta + \Lambda_p(b))^{-1}\|_{p \to p} \leqslant C_p |\zeta|^{-1}, \quad \zeta \in \mathcal{O};$$

(iii) *For each* $1 \leqslant r < p < q$ *and* $\zeta \in \mathcal{O}$, *define*

$$G_p(r) \equiv G_p(r, \zeta, b) := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2r}}, \quad G_p(r) \in \mathcal{B}(L^p),$$

$$Q_p(q) \equiv Q_p(q, \zeta, b) := (\zeta - \Delta)^{-\frac{1}{2q'}} |b|^{\frac{1}{p'}} \text{ on } \mathcal{E}.$$

The extension of $Q_p(q)$ *by continuity we denote again by* $Q_p(q)$. *Then, for each* $\zeta \in \mathcal{O}$,

$$\Theta_p(\zeta, b) = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} Q_p(q)(1 + T_p)^{-1} G_p(r)(\zeta - \Delta)^{-\frac{1}{2r'}};$$

$\Theta_p(\zeta, b)$ *extends by continuity to an operator in* $\mathcal{B}\big(\mathcal{W}^{-\frac{1}{r'}, p}, \ \mathcal{W}^{1 + \frac{1}{q}, p}\big)$;

(iv) *By* (i) *and* (iii), $D\big(\Lambda_p(b)\big) \subset \mathcal{W}^{1 + \frac{1}{q}, p}$ $(q > p)$. *In particular, if* $m_d \delta < 4\frac{d-2}{(d-1)^2}$, *there exists* $p \in \mathcal{I}$, $p > d - 1$, *so* $D\big(\Lambda_p(b)\big) \subset C^{0,\gamma}$, $\gamma < 1 - \frac{d-1}{p}$;

(v) *Let* $u \in D(\Lambda_p(b))$. *Then*

$$\langle \Lambda_p(b) u, v \rangle = \langle u, -\Delta v \rangle + \langle b \cdot \nabla u, v \rangle, \qquad v \in C_c^\infty(\mathbb{R}^d);$$

$$u \in \mathcal{W}^{2,1}_{\text{loc}};$$

(vi) $e^{-t\Lambda_p(b_n)} \overset{s}{\to} e^{-t\Lambda_p(b)}$ *in* L^p, $\quad t > 0$, *where* $b_n := b$ *if* $|b| \leqslant n$, $b_n := n|b|^{-1} b$ *if* $|b| > n$, *and* $\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_p(b_n)) = \mathcal{W}^{2,p}$;

(vii) *If* b *is real-valued, then* $e^{-t\Lambda_p(b)}$ *is positivity preserving;*

(viii) *By Theorem* 3(b) *below,* $\|e^{-t\Lambda_p(b)}\|_{p \to r} \leqslant c_{p,r} t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{r})}$, $0 < t \leqslant 1$, $p < r$.

Remark 1.4. Theorem 1.3 provides a complete description of $\Lambda_p(b)$, an operator realization of $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, generating a holomorphic C_0-semigroup on L^p.

Let

$$\eta(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1, \end{cases}$$

where c is adjusted to $\int_{\mathbb{R}^d} \eta(x)dx = 1$. Define the standard mollifier

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \quad x \in \mathbb{R}^d.$$

Theorem 1.5 (**C_∞-theory**). *Let $d \geqslant 3$. Assume that*

$$b : \mathbb{R}^d \to \mathbb{R}^d, \quad b \in \mathbf{F}_\delta^{\frac{1}{2}}, \quad m_d\delta < 4\frac{d-2}{(d-1)^2}.$$

Then for every $\tilde{\delta} > \delta$ satisfying $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$ there exists $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$, such that

$$\tilde{b}_n := \eta_{\varepsilon_n} * b_n \in C^\infty\left(\mathbb{R}^d, \mathbb{R}^d\right) \cap \mathbf{F}_{\tilde{\delta}}^{\frac{1}{2}}, \quad n = 1, 2, \dots,$$

and

(i) $e^{-t\Lambda_{C_\infty}(b)} := s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(\tilde{b}_n)}$, $t > 0$, *determines a positivity-preserving contraction C_0-semigroup on C_∞, where the b_n's were defined in Theorem* 1.3, $\Lambda_{C_\infty}(\tilde{b}_n) := -\Delta + \tilde{b}_n \cdot \nabla$, $D(\Lambda_{C_\infty}(\tilde{b}_n)) = (1-\Delta)^{-1}C_\infty$;

(ii) *(L^p-strong Feller property)* $(\mu + \Lambda_{C_\infty}(b))^{-1}[L^p \cap C_\infty] \subset C^{0,\alpha}$, $\mu > 0$, $p \in \left(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\right)$, $\alpha < 1 - \frac{d-1}{p}$;

(iii) *The integral kernel $e^{-t\Lambda_{C_\infty}(b)}(x,y)$ $(x, y \in \mathbb{R}^d)$ of $e^{-t\Lambda_{C_\infty}(b)}$ determines the (sub-Markov) transition probability function of a strong Feller process.*

Remark 1.6.

1. In the proof of Theorem 1.5, we define

$$(\mu + \Lambda_{C_\infty}(b))^{-1}|_{\mathcal{S}} := s\text{-}C_\infty\text{-}\lim_n\big((\mu + \Lambda_p(\tilde{b}_n))^{-1}|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d\lambda,$$
$$p \in \left(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\right),$$

appealing to Theorem 1.3(iv), which allows us to move the proof of convergence in C_∞ to L^p, $p > d-1$, a space having much weaker topology (locally). Earlier proofs for a smaller class $\mathbf{K}_0^{d+1}$ verified convergence in C_∞ (in fact, in C_b) directly.

2. The problem of constructing a Feller process associated with $-\Delta + b\cdot\nabla$, for an unbounded $b:\mathbb{R}^d\to\mathbb{R}^d$ ("a diffusion with drift b"), has been thoroughly studied in the literature, see [9] and references therein, motivated by applications, as well as by the search for the *maximal* general class of vector fields b such that the associated process exists. To the author's knowledge, Theorem 1.5 is the first result on diffusion processes with drifts combining different kinds of singularities, *e.g.*, $\big||x|-1\big|^{-\beta}$, $\beta<1$, and $|x|^{-1}$ (originally, the main motivation of this work).

1.1. On the existing results prior to our work

First, it had been known for a long time, see [KS], that, for $b:\mathbb{R}^d\to\mathbb{R}^d$, $d\geqslant 3$, and $b\in\mathbf{F}_\delta$,

(i) *(The basic fact)* $D(\Lambda_p(b))\subset W^{1,jp}$ *for every* $p\in(d-2,2/\sqrt{\delta})$, $j=\frac{d}{d-2}$, provided that $0<\delta<\min\{1,(\frac{2}{d-2})^2\}$;

(ii) *If, in addition to the assumptions in (i),* $|b|\in L^2+L^\infty$, *then*

$$s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(b_n)}$$

exists uniformly in each finite interval of $t\geqslant 0$, *and hence determines a strongly Feller semigroup on* C_∞.

Remark 1.7. The additional (to $|b|\in L^2_{\rm loc}$) assumption $|b|\in L^2+L^\infty$ in (ii) was removed in [6] (albeit at expense of imposing a more restrictive assumption on the maximal admissible value of $\delta>0$).

Theorem 1.8 (Yu. A. Semenov). *Let* $b:\mathbb{R}^d\to\mathbb{R}^d$, $d\geqslant 3$.

a) [12] *If* $b\in\mathbf{K}^{d+1}_\delta$, $m_d\delta<1$, *then, for each* $p\in[1,\infty)$, s-L^p-$\lim_n e^{-t\Lambda_p(b_n)}$ *exists uniformly on each finite interval of* $t\geqslant 0$, *and hence determines a* C_0-*semigroup* $e^{-t\Lambda_p(b)}$.
$e^{-t\Lambda_p(b)}$ *is a quasi-bounded positivity preserving* L^∞-*contraction* C_0- *semigroup;*

$$\|e^{-t\Lambda_r(b)}\|_{r\to q}\leqslant c_{d,\delta}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}\ \textit{for all}\ 0<t\leqslant 1,\ 1\leqslant r<q\leqslant\infty;$$

The resolvent set $\rho(-\Lambda_p(b))$ *contains the half-plane* $\mathcal{O}$,

$$\big(\zeta+\Lambda_p(b)\big)^{-1}=\Theta_p(\zeta,b),\quad \zeta\in\mathcal{O},$$

$$\Theta_p(\zeta,b):=(\zeta-\Delta)^{-1}-(\zeta-\Delta)^{-\frac{1}{2}}S_p(1+T_p)^{-1}G_p,$$

$$S_p:=(\zeta-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}},\ G_p:=b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1},\ T_p:=b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}};$$

$$\Theta_p(\zeta,b)\in\mathcal{B}(L^p,\mathcal{W}^{1,p});$$

$D(\Lambda_p(b))\subset\mathcal{W}^{1,p}$. *In particular, for* $p>d$, $D(\Lambda_p(b))\subset C^{0,\alpha}$, $\alpha=1-\dfrac{d}{p}$;

$$\langle\Lambda_p(b)f,g\rangle=\langle\nabla f,\nabla g\rangle+\langle b\cdot\nabla f,g\rangle,\qquad f\in D(\Lambda_p(b)),\ g\in C_c^\infty(\mathbb{R}^d).$$

b) [13, Theorem 5.1] *If* $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, *then, for each* $p \in [2, \infty)$, s-L^p-$\lim_n e^{-t\Lambda_p(b_n)}$ *exists uniformly on each finite interval of* $t \geqslant 0$, *and hence determines a* C_0*-semigroup* $e^{-t\Lambda_p(b)}$.

$e^{-t\Lambda_p(b)}$ *is a quasi-bounded positivity preserving* L^∞*-contraction* C_0*- semigroup.*

$\|e^{-t\Lambda_r(b)}\|_{r\to q} \leqslant c_{d,\delta}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}$ *for all* $0 < t \leqslant 1,\ 2 \leqslant r < q \leqslant \infty$.

$D(\Lambda_2(b)) \subset W^{\frac{3}{2},2}$.

$\langle \Lambda_2(b) f, g\rangle = \langle \nabla f, \nabla g\rangle + \langle b\cdot\nabla f, g\rangle, \qquad f \in D(\Lambda_2(b)),\ g \in C_c^\infty(\mathbb{R}^d)$.

Remark 1.9. The additional (to $|b| \in L^1_{\text{loc}}$) assumption $|b| \in L^1 + L^\infty$ in [13, Theorem 5.1] is not essential for the proof, and can be eliminated.

For the sake of completeness, we now outline the proof of Theorem 1.8, with permission of its author.

Proof. a) Indeed, for all ζ with Re $\zeta > 0$,

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant m_d(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}}(x,y) \text{ pointwise on } \mathbb{R}^d \times \mathbb{R}^d$$

(see (A.2) in the Appendix). Therefore, for $b \in \mathbf{K}_\delta^{d+1}$,

$$\|b\cdot\nabla(\zeta-\Delta)^{-1}\|_{1\to 1} \leqslant m_d\delta, \qquad \mathrm{Re}\ \zeta \geqslant \kappa_d\lambda,$$

and so by the Miyadera perturbation theorem, the operator $-\Lambda_1(b) := \Delta - b\cdot\nabla$ of domain $D(\Lambda_1(b)) = \mathcal{W}^{2,1}$ is the generator of a quasi-bounded C_0 semigroup on L^1 *whenever* $m_d\delta < 1$.

Clearly $b_n \in \mathbf{K}_\delta^{d+1}$, $\|b_n\cdot\nabla(\zeta-\Delta)^{-1}\|_{1\to1} \leqslant m_d\delta$, and, for $m_d\delta < 1$ and every $f \in D(\Lambda_1(b))$, $\Lambda_1(b_n)f \overset{s}{\to} \Lambda_1(b)f$ by the Dominated Convergence Theorem. (See, if needed, (A.1).) The latter easily implies the strong resolvent and the semigroup convergence of $\Lambda_1(b_n)$ to $\Lambda_1(b)$.

Then, for each $n = 1, 2, \ldots$, the semigroups $e^{-t\Lambda_1(b_n)}$, $t > 0$, are positivity preserving L^∞-contractions, and so is $e^{-t\Lambda_1(b)}$. The bounds

$$\|e^{-t\Lambda_1(b)}\|_{1\to1} \leqslant Me^{t\omega},\ \omega = \kappa_d\lambda, \text{ and } \|e^{-t\Lambda_1(b)}f\|_\infty \leqslant \|f\|_\infty,\ f \in L^1\cap L^\infty,$$

yield via the Riesz interpolation theorem

$$\|e^{-t\Lambda_1(b)}f\|_p \leqslant M^{1/p}e^{t\omega/p}\|f\|_p, \quad f \in L^1\cap L^\infty.$$

Therefore, we obtain a family $\{e^{-t\Lambda_p(b)}\}_{1\leqslant p<\infty}$ of consistent C_0-semigroups by setting $e^{-t\Lambda_p(b)} :=$ the extension by continuity in L^p of $e^{-t\Lambda_1(b)} \upharpoonright L^1\cap L^\infty$.

Next, for each $p \in [1,\infty)$ and all $f \in \mathcal{E} := \bigcup_{\epsilon>0} e^{-\epsilon|b|}L^p$, the inequality

$$\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}f\|_p \leqslant \delta\|f\|_p$$

as well as the inequality

$$\big\| \big(|b| + \sqrt{\lambda}\big)^{\frac{1}{p}} (\lambda - \Delta)^{-\frac{1}{2}} \big(|b| + \sqrt{\lambda}\big)^{\frac{1}{p'}} f \big\|_p \leqslant (1+\delta) \big\| f \big\|_p$$

follow from the very definition of $\mathbf{K}_\delta^{d+1}$ (*e.g.*, by interpolating between $\|(|b| + \sqrt{\lambda})(\lambda - \Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant 1+\delta$ and (by duality) $\|(\lambda - \Delta)^{-\frac{1}{2}}(|b| + \sqrt{\lambda})\|_\infty \leqslant 1+\delta$). The latter implies that

$$\Big\| |b|^{\frac{1}{p}} (\lambda - \Delta)^{-\frac{1}{2}} \Big\|_{p \to p} \leqslant (1+\delta)\lambda^{-\frac{1}{2p'}},$$

and the first inequality implies that, for every $\zeta \in \mathcal{O}$, $p \in [1, \infty)$ and all $f \in \mathcal{E}$,

$$\Big\| b^{\frac{1}{p}} \cdot \nabla (\zeta - \Delta)^{-1} \, |b|^{\frac{1}{p'}} f \Big\|_p \leqslant m_d \Big\| |b|^{\frac{1}{p}} (\lambda - \Delta)^{-\frac{1}{2}} |b|^{\frac{1}{p'}} |f| \Big\|_p \leqslant m_d \delta \|f\|_p.$$

Now, it is seen that for every $p \in [1, \infty)$ and $\zeta \in \mathcal{O}$ the operator G_p is bounded:

$$\|G_p\|_{p \to p} \leqslant m_d \Big\| b^{\frac{1}{p}} (\lambda - \Delta)^{-\frac{1}{2}} \Big\|_{p \to p} \leqslant m_d (1+\delta) \lambda^{-\frac{1}{2p'}}.$$

S_p and T_p are densely defined (on $\mathcal{E}$) and, for all $f \in \mathcal{E}$,

$$\|S_p f\|_p \leqslant (1+\delta)^{-1} \lambda^{-\frac{1}{2p}} \|f\|_p \text{ and } \|T_p f\|_p \leqslant m_d \delta \|f\|_p.$$

Now, we denote again by S_p, T_p their extensions by continuity.

Next, we define an operator function $\Theta_p(\zeta, b)$ in L^p by

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2}} S_p \, (1 + T_p)^{-1} \, G_p \qquad \zeta \in \mathcal{O}.$$

Obviously,

$$\Theta_p(\zeta, b) \in \mathcal{B}(L^p) \text{ and } \Theta_p(\zeta, b) \in \mathcal{B}(L^p, W^{1,p}).$$

It is also seen that

$$(\zeta + \Lambda_1(b))^{-1} = \Theta_1(\zeta, b), \quad (\zeta + \Lambda_p(b))^{-1} \mid L^1 \cap L^p = \Theta_p(\zeta, b) \mid L^1 \cap L^p,$$

and so

$$\big(\zeta + \Lambda_p(b)\big)^{-1} = \Theta_p(\zeta, b), \qquad \zeta \in \mathcal{O}.$$

The latter implies that $D(\Lambda_p(b)) \subset W^{1,p}$, for all $p \in [1, \infty)$. The main assertion is proved.

b) Let $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$. Define $H = |b|^{\frac{1}{2}}(\zeta - \Delta)^{-\frac{1}{4}}$, $S = b^{\frac{1}{2}} \cdot \nabla(\zeta - \Delta)^{-\frac{3}{4}}$ and

$$\begin{aligned} \Theta_2(\zeta, b) :=& (\zeta - \Delta)^{-\frac{3}{4}} (1 + H^* S)^{-1} (\zeta - \Delta)^{-\frac{1}{4}} \\ =& (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{3}{4}} H^* (1 + S H^*)^{-1} S (\zeta - \Delta)^{-\frac{1}{4}}, \quad \operatorname{Re} \zeta \geqslant \lambda. \end{aligned} \tag{$*$}$$

We represent $S = \hat{H}\nabla(\zeta - \Delta)^{-\frac{1}{2}}$, where the operator $\hat{H}$ defined by $\hat{H}h := b^{\frac{1}{2}} \cdot (\zeta - \Delta)^{-\frac{1}{4}}h, h : \mathbb{R}^d \to \mathbb{R}^d$, with $(\zeta - \Delta)^{-\frac{1}{4}}$ acting on h component-wise, clearly satisfies $\|\hat{H}h\|_2 \leqslant \||b|^{\frac{1}{2}}(\operatorname{Re}\zeta - \Delta)^{-\frac{1}{4}}|h|\|_2 \leqslant \sqrt{\delta}\|h\|_2, \operatorname{Re}\zeta \geqslant \lambda$. Therefore,

$$\begin{aligned}\|H^*S\|_{2\to2} &\leqslant \|H\|_{2\to2}\|S\|_{2\to2} \\ &\leqslant \|H\|_{2\to2}\|\hat{H}\|_{2\to2}\|\nabla(\zeta - \Delta)^{-\frac{1}{2}}\|_{2\to2} \leqslant \delta,\end{aligned}$$

and

$$\|\Theta_2(\zeta, b)\|_{2\to2} \leqslant (1-\delta)^{-1}|\zeta|^{-1}.$$

Note that $D(\Lambda_2(b_n)) = W^{2,2}$ and, for all Re $\zeta \geqslant \lambda$, by the first representation of $\Theta_2(\zeta, b_n)$,

$$\Theta_2(\zeta, b_n)^{-1}|W^{2,2} = (\zeta + \Lambda_2(b_n))|W^{2,2}, \quad \Theta_2(\zeta, b_n) = (\zeta + \Lambda_2(b_n))^{-1},$$

$$\zeta\Theta_2(\zeta, b_n) \overset{s}{\to} 1 \text{ as } \zeta \uparrow \infty \text{ by the second representation of } \Theta_2(\zeta, b_n).$$

Therefore, $\Theta_2(\zeta, b_n)$ is the resolvent of $-\Lambda_2(b_n)$.

Since $\|\Theta_2(\zeta, b_n)\|_{2\to2} \leqslant (1-\delta)^{-1}|\zeta|^{-1}$, the semigroups $e^{-t\Lambda_2(b_n)}$ are holomorphic and equi-bounded.

Finally, it is seen that $\Theta_2(\zeta, b_n) \overset{s}{\to} \Theta_2(\zeta, b)$ in L^2 on Re $\zeta \geqslant \lambda$, and $\mu\ \Theta_2(\mu, b_n) \overset{s}{\to} 1$ in L^2 as $\mu \uparrow \infty$ uniformly in n. Therefore, by the Trotter approximation theorem s-L^2-$\lim_n e^{-t\Lambda_2(b_n)}$ exists and determines a C_0-semigroup in L^2. It is also clear that this semigroup is holomorphic and L^∞-contractive. □

1.2. Comments

1. The fact that $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{K}_\delta^{d+1}$ or $\mathbf{F}_\delta$ allows us to construct operator realizations of the formal differential operator $-\Delta + b \cdot \nabla$ as (minus) generators of strongly continuous semigroups in L^p for some or all $p \in [1, \infty)$, C_∞ and/or C_b, by means of general tools of the standard perturbation theory (*e.g.*, theorems of Miyadera [15] or Phillips [11], respectively);
2. Concerning the class $\mathbf{F}_\delta^{\frac{1}{2}}$ one can not appeal to the standard perturbation theory (in contrast to $\mathbf{K}_\delta^{d+1}$ and $\mathbf{F}_\delta$) in order to properly characterize the domain of the generator $\Lambda_p(b)$. Indeed, the arguments in [13, p. 413–416] (repeated above in the proof of Theorem 1.8b) say nothing about $\mathcal{W}^{\alpha,p}$-smoothness of $D(\Lambda_p(b))$ for $p \neq 2$. The natural analogue of $(*)$ in L^p is valid only for a smaller class of vector fields: $|b| \in L^{d,\infty}$;
3. For $|b| \in L^{d,\infty}$, the assertion of Theorem 1.3(iv) can be strengthened:

$$|b| \in L^{d,\infty} \quad \Rightarrow \quad D(\Lambda_p(b)) \subset \mathcal{W}^{1+\frac{1}{p},s}, \quad s < dp. \tag{1.3}$$

Indeed, arguing as in Remark 1.2 (*i.e.*, appealing to [7, Proposition 2.5, 2.6, Corollary 2.9]), we can estimate, using (A.2), for every $f \in \mathcal{E}$,

$$\|b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2}-\frac{1}{2p}} f\|_s \leqslant c_1 \|f\|_s, \qquad c_1 := m_d(\Omega_d^{-\frac{1}{d}} \|b\|_{d,\infty})^{\frac{1}{p}} c(p,d),$$
$$\|(\zeta - \Delta)^{-\frac{1}{2p'}} |b|^{\frac{1}{p'}} f\|_s \leqslant c_2 \|f\|_s, \qquad c_2 := (\Omega_d^{-\frac{1}{d}} \|b\|_{d,\infty})^{\frac{1}{p'}} c(p',d),$$

where $c(p,d) := 2^{-\frac{1}{p}} \frac{\Gamma\left(\frac{d}{2p'}\right)}{\Gamma\left(\frac{d}{2p}\right)} \frac{\Gamma\left(\frac{d-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}+\frac{d}{2p'}\right)}$, so

$$\|b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}} f\|_s \leqslant c_3 \|f\|_s, \quad c_3 := m_d \Omega_d^{-\frac{1}{d}} \|b\|_{d,\infty} c(p,d) c(p',d).$$

Now, we can estimate in Theorem 1.3(iii):

$$\|Q_p(p)\|_{s\to s}, \|G_p(p)\|_{s\to s}, \|T_p\|_{s\to s} < \infty,$$

to conclude that $\|\Theta_p(\zeta, b)\|_{s\to s} < \infty$, $1 < s < dp$. In view of Theorem 1.3(i), the last estimate implies the required;

4. Theorem 1.8 can be obtained as a side product of the proof of Theorem 1.3. Indeed, the constraints on p and δ in Theorem 1.3 come solely from the estimate on $\|T_p\|_{p\to p}$. Now, if $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, then (representing $S = \hat{H}\nabla(\zeta - \Delta)^{-\frac{1}{2}}$)
$$\|T_2\|_{2\to 2} \leqslant \|\hat{H}\|_{2\to 2} \|H^*\|_{2\to 2} \|\nabla(\zeta - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \delta < 1.$$
And if $b \in \mathbf{K}_\delta^{d+1}$, $m_d\delta < 1$, then $\|T_p\|_{p\to p} < 1$ for *all* $p \in [1,\infty)$, so that the interval $\mathcal{I} \ni p$ transforms into $[1,\infty)$, and a possible causal dependence of the properties of $D(\Lambda_p(b))$ on δ gets lost. The latter indicates the smallness of $\mathbf{K}_\delta^{d+1}$ as a subclass of $\mathbf{F}_\delta^{\frac{1}{2}}$;
5. Both proofs of Theorem 1.3 and Theorem 1.8 are based on similar operator-valued functions, although the arguments involved differ considerably;
6. Note that for $b \in \mathbf{K}_\delta^{d+1}$, $m_d\delta < 1$, $D(\Lambda_1(b)) = \mathcal{W}^{2,1}$; for $b \in \mathbf{F}_\delta$, $\delta < 1$, $D(\Lambda_2(b)) = W^{2,2}$, while for $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, $D(\Lambda_2(b)) \subset \mathcal{W}^{2,1}_{\text{loc}}$;
7. Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $m_d\delta < 1$. Theorem 1.3(i), (vi) and the argument in the proof of Theorem 1.8a (using the Riesz interpolation theorem) yield a consistent family of positivity preserving quasi-bounded C_0-semigroups $e^{-t\Lambda_p(b)}$ on L^p, for all $p \in \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \infty\right)$;
8. The author considers the assertion (iv) of Theorem 1.3 (the $\mathcal{W}^{1+\frac{1}{q},p}$-smoothness) as the main result of the paper. Theorem 1.3, compared to [8] and Theorem 1.8a, covers the larger class of vector fields, and at the same time establishes stronger smoothness properties of $D(\Lambda_p(b))$: $D(\Lambda_p(b)) \subset \mathcal{W}^{1+\frac{1}{q},p}$, $p \in \mathcal{I}$ ($q > p$), while in [8] $D(\Lambda_p(b)) \subset W^{1,jp}$, $jp \in (d, 2j/\sqrt{\delta})$, and in Theorem 1.8a $D(\Lambda_p(b)) \subset \mathcal{W}^{1,p}$, $p \in [1,\infty)$;

9. The C_∞-theory of operator $-\Delta + b\cdot\nabla$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$ (Theorem 1.5) follows almost automatically from the L^p-theory (Theorem 1.3) (with $p > d-1$), in contrast to [8], where the C_∞-theory is obtained from the L^p-theory by running a specifically tailored iterative procedure (see also [6]).

ACKNOWLEDGEMENTS. I am deeply grateful to Yu. A. Semenov for many important suggestions, and constant attention throughout this work. I am also thankful to the anonymous referee for a number of valuable comments that helped to improve the presentation.

2. Proof of Theorem 1.3

The method of the proof. We start with an operator-valued function

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - Q_p(1+T_p)^{-1}G_p, \quad \zeta \in \mathcal{O},$$

defined in L^p for each p from the interval

$$\mathcal{I} := \left]\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right[, \quad m_d\delta < 1,$$

and step by step prove that, for $n = 1, 2, \dots,$

$$\|\Theta_p(\zeta, b_n)\|_{p\to p},\ \|\Theta_p(\zeta, b)\|_{p\to p} \leqslant c|\zeta|^{-1};$$
$$\Theta_p(\zeta, b_n) \text{ is a pseudo-resolvent;}$$
$$\Theta_p(\zeta, b_n) \text{ coincides with the resolvent } R(\zeta, -\Lambda_p(b_n)) = (\zeta + \Lambda_p(b_n))^{-1} \text{ on } \mathcal{O};$$
$$\Theta_p(\zeta, b_n) \overset{s}{\to} \Theta_p(\zeta, b) \text{ in } L^p \text{ as } n \uparrow \infty;$$
$$\mu\, \Theta_p(\mu, b_n) \overset{s}{\to} 1 \text{ as } \mu \uparrow \infty \text{ in } L^p \text{ uniformly in } n.$$

All this combined leads to the conclusion: for each $p \in \mathcal{I}$ there is a holomorphic semigroup $e^{-t\Lambda_p(b)}$ in L^p such that the resolvent $R(\zeta, -\Lambda_p(b))$ on $\zeta \in \mathcal{O}$ has the representation $\Theta_p(\zeta, b)$;

$\Theta_p(\zeta, b)$ can be written as $(\zeta - \Delta)^{-1} + ABC$, where $C \in \mathcal{B}(\mathcal{W}^{-\frac{1}{r'}, p}, L^p)$, $B \in \mathcal{B}(L^p)$, $A \in \mathcal{B}(L^p, \mathcal{W}^{1+\frac{1}{q}, p})$, $r < p < q$, $r' = r/(r-1)$.

Propositions 2.1-2.6 below constitute the core of the proof of Theorem 1.3.

Proposition 2.1. *Let $p \in \mathcal{I}$.*

(i) *For every $1 \leqslant r < p < q \leqslant \infty$ and $\zeta \in \mathcal{O}$ $(= \{\zeta \in \mathbb{C} : \text{Re } \zeta \geqslant \kappa_d \lambda\},\ \lambda = \lambda_\delta)$ define operators on L^p*

$$Q_p(q) = (\zeta - \Delta)^{-\frac{1}{2q'}} |b|^{\frac{1}{p'}}, \quad G_p(r) = b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2r}},$$
$$T_p = b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}}.$$

Then $G_p(r)$ is bounded: $\|G_p(r)\|_{p \to p} \leqslant K_{1,r}$. $Q_p(q)$ and T_p are densely defined (on $\mathcal{E}$), and for all $f \in \mathcal{E}$,

$$\|Q_p(q) f\|_p \leqslant K_{2,q} \|f\|_p,$$

$$\|T_p f\|_p \leqslant m_d c_p \delta \|f\|_p, \quad m_d c_p \delta < 1, \quad c_p = pp'/4. \tag{2.1}$$

Their extensions by continuity we denote again by $Q_p(q)$, T_p.

(ii) *Set $G_p = b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}$, $Q_p = (\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}}$, $P_p = |b|^{\frac{1}{p}} (\zeta - \Delta)^{-1}$. The operator Q_p is densely defined on $\mathcal{E}$. There exist constants C_i, $i = 1, 2, 3$, such that*

$$\begin{aligned} &\|G_p\|_{p \to p} \leqslant C_1 |\zeta|^{-\frac{1}{2p'}}, \quad \|P_p\|_{p \to p} \leqslant C_3 |\zeta|^{-\frac{1}{2} - \frac{1}{2p'}}, \\ &\|Q_p f\|_p \leqslant C_2 |\zeta|^{-\frac{1}{2} - \frac{1}{2p}} \|f\|_p \quad (f \in \mathcal{E}), \quad \zeta \in \mathcal{O}. \end{aligned} \tag{2.2}$$

We denote again by Q_p the extension of Q_p by continuity.

Remark 2.2. The proof of Proposition 2.1 uses ideas from [2, 10], and appeals to the L^p-inequalities between the operator $(\lambda - \Delta)^{\frac{1}{2}}$ and the "potential" $|b|$.

Proof. (i) Let $r \in (1, \infty)$. Then

(a) $\quad \mu \geqslant \lambda \ \Rightarrow\ \||b|^{\frac{1}{r}} (\mu - \Delta)^{-\frac{1}{2}}\|_{r \to r} \leqslant C_{r,\delta} \mu^{-\frac{1}{2r'}}, C_{r,\delta} = (c_r \delta)^{\frac{1}{r}}, c_r = rr'/4.$

Indeed, define in L^2 $A = (\mu - \Delta)^{\frac{1}{2}}$, $D(A) = W^{1,2}$. Then $-A + \mu^{\frac{1}{2}}$ is a symmetric Markov generator. Therefore (see, *e.g.*, [10, Theorem 2.1]), for any $r \in (1, \infty)$,

$$0 \leqslant u \in D(A_r) \Rightarrow v := u^{\frac{r}{2}} \in D(A^{\frac{1}{2}}) \text{ and } c_r^{-1} \left\| A^{\frac{1}{2}} v \right\|_2^2 \leqslant \langle A_r u, u^{r-1} \rangle.$$

Now let u be the solution of $A_r u = |f|$, with $f \in L^r$. Note that $\|u\|_r \leqslant \mu^{-\frac{1}{2}} \|f\|_r$ (using $\|(\mu - \Delta)^{-1}\|_{r \to r} \leqslant \mu^{-1}$ in (A.6) with $\alpha = \frac{1}{2}$).

Since $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, we have

$$(c_r \delta)^{-1} \left\| |b|^{\frac{1}{2}} v \right\|_2^2 \leqslant \langle A_r u, u^{r-1} \rangle,$$

and so $\||b|^{\frac{1}{r}}u\|_r^r \leqslant c_r\delta\|f\|_r\|u\|_r^{r-1}$, $\||b|^{\frac{1}{r}}A_r^{-1}|f|\|_r^r \leqslant c_r\delta\mu^{-\frac{r-1}{2}}\|f\|_r^r$. Therefore (**a**) is proved.

(**b**) $\quad \mu \geqslant \lambda \;\Rightarrow\; \||b|^{\frac{1}{r}}(\mu-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{r'}}f\|_r \leqslant c_r\delta\|f\|_r,\ f \in \mathcal{E}.$

Indeed, let u be the solution of $Au = |b|^{\frac{1}{r'}}|f|$, $f \in \mathcal{E}$. Then, arguing as in (**a**), we have

$$\left\||b|^{\frac{1}{r}}u\right\|_r^r \leqslant c_r\delta\|f\|_r \left\||b|^{\frac{1}{r}}u\right\|_r^{r-1},$$

or $\||b|^{\frac{1}{r}}u\|_r \leqslant c_r\delta\|f\|_r$. So (**b**) is proved.

(**c**) $\quad \mu \geqslant \lambda \;\Rightarrow\; \|(\mu-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{r'}}f\|_r \leqslant C_{r',\delta}\,\mu^{-\frac{1}{2r}}\|f\|_r,\ f \in \mathcal{E}.$

Indeed, (**c**) follows from (**a**) by duality.

Let us prove (2.1). Let $\zeta \in \mathcal{O}$. Using (A.2) + (**b**) with $r = p \in \mathcal{I}$, $\mu = \lambda$, we obtain:

$$\|T_p f\|_p \leqslant m_d \left\| b^{\frac{1}{p}}(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\right\|_p \leqslant m_d c_p\delta\|f\|_p, \quad f \in \mathcal{E}.$$

$m_d c_p\delta < 1$ since $p \in \mathcal{I}$.

Next, we estimate $\|Q_p(q)\|_{p\to p}$, $\|G_p(r)\|_{p\to p}$. Let $\mathrm{Re}\,\zeta \geqslant \lambda$, $p < q$. We obtain:

$$\begin{aligned}
\|Q_p(q)f\|_p &\leqslant \left\|(\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}|f|\right\|_p \\
&\leqslant \left\|(\lambda - \Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}|f|\right\|_p \\
&\quad \text{(here we are using (A.6) with } \alpha = 1/2q'\text{)} \\
&\leqslant k_{q'}\int_0^\infty t^{-\frac{1}{2q'}}\left\|(t+\lambda-\Delta)^{-1}|b|^{\frac{1}{p'}}|f|\right\|_p dt \quad \left(k_{q'} := \frac{\sin\frac{\pi}{2q'}}{\pi}\right) \\
&\leqslant k_{q'}\int_0^\infty t^{-\frac{1}{2q'}}(t+\lambda)^{-\frac{1}{2}}\left\|(t+\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\right\|_p dt \\
&\quad \text{(here we are using (c) with } r = p \in \mathcal{I}, \mu = t+\lambda\text{)} \\
&\leqslant k_{q'}C_{p',\delta}\int_0^\infty t^{-\frac{1}{2q'}}(t+\lambda)^{-\frac{1}{2}-\frac{1}{2p}}dt\ \|f\|_p = K_{2,q}\|f\|_p,\ f \in \mathcal{E},
\end{aligned}$$

where, clearly, $K_{2,q} < \infty$ because $q > p$.

Let $\zeta \in \mathcal{O}$, $1 \leqslant r < p$. Using (A.3), we obtain

$$\|G_p(r)f\|_p \leqslant m_{r,d}\left\||b|^{\frac{1}{p}}(\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{1}{2r}}|f|\right\|_p$$

$$\leqslant m_{r,d}\left\||b|^{\frac{1}{p}}(\lambda - \Delta)^{-\frac{1}{2r}}|f|\right\|_p$$

(here we are using (A.6) with $\alpha = 1/2r$)

$$\leqslant m_{r,d}k_r\int_0^\infty t^{-\frac{1}{2r}}\||b|^{\frac{1}{p}}(t+\lambda-\Delta)^{-1}|f|\|_p dt$$

$$\leqslant m_{r,d}k_r\int_0^\infty t^{-\frac{1}{2r}}\left\||b|^{\frac{1}{p}}(t+\lambda-\Delta)^{-\frac{1}{2}}\right\|_{p\to p}\left\|(t+\lambda-\Delta)^{-\frac{1}{2}}|f|\right\|_p dt$$

(here we are using (**a**) with $r = p \in \mathcal{I}$, $\mu = t+\lambda$)

$$\leqslant m_{r,d}k_rC_{p,\delta}\int_0^\infty t^{-\frac{1}{2r}}(t+\lambda)^{-\frac{1}{2p'}-\frac{1}{2}}dt\ \|f\|_p = K_{1,r}\|f\|_p,\ f\in\mathcal{E},$$

where, clearly, $K_{1,r} < \infty$ because $r < p$.

The proof of (i) is complete.

(ii) Let Re $\zeta \geqslant \lambda$. We have

$$\|Q_p(2\zeta,b)f\|_p \leqslant \left\|(2\zeta-\Delta)^{-\frac{1}{2}}\right\|_{p\to p}\left\|(2\zeta-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}f\right\|_p$$

(here we are applying (A.5) twice + (**c**) with $r=p\in\mathcal{I}$, $\mu=|\zeta|$)

$$\leqslant 2^{\frac{d}{4}+\frac{1}{4}}2^{-\frac{1}{2}}|\zeta|^{-\frac{1}{2}}C_{p',\delta}2^{\frac{d}{4}+\frac{1}{4}}|\zeta|^{-\frac{1}{2p}}\|f\|_p,\quad f\in\mathcal{E}.$$

Now, using the identity $(\zeta-\Delta)^{-1} = \big(1+\zeta(\zeta-\Delta)^{-1}\big)(2\zeta-\Delta)^{-1}$, we obtain:

$$\|Q_p(\zeta,b)f\|_p \leqslant \|1+\zeta(\zeta-\Delta)^{-1}\|_{p\to p}\|Q_p(2\zeta,b)f\|_p$$

$$\leqslant 2^{\frac{1}{2}}|\zeta|^{-\frac{1}{2}}C_{p',\delta}2^{\frac{d}{2}+\frac{1}{2}}|\zeta|^{-\frac{1}{2p}}\|f\|_p$$

$$= C_2|\zeta|^{-\frac{1}{2}-\frac{1}{2p}}\|f\|_p,\quad f\in\mathcal{E}.$$

Let Re $\zeta \geqslant \lambda$. We have:

$$\|P_p(2\zeta,b)\|_{p\to p} \leqslant \left\||b|^{\frac{1}{p}}(2\zeta-\Delta)^{-\frac{1}{2}}\right\|_{p\to p}\left\|(2\zeta-\Delta)^{-\frac{1}{2}}\right\|_{p\to p}$$

(here we are applying (A.5) twice)

$$\leqslant 2^{\frac{d}{2}+\frac{1}{2}}\||b|^{\frac{1}{p}}(|\zeta|-\Delta)^{-\frac{1}{2}}\|_{p\to p}|\zeta|^{-\frac{1}{2}}$$

(here we are using (**a**) with $r = p\in\mathcal{I}$, $\mu = |\zeta|$)

$$\leqslant C_{p,\delta}2^{\frac{d}{2}+\frac{1}{2}}|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}.$$

Now, using the identity $(\zeta-\Delta)^{-1}=(2\zeta-\Delta)^{-1}\big(1+\zeta(\zeta-\Delta)^{-1}\big)$, we obtain:

$$\begin{aligned}\|P_p(\zeta,b)\|_{p\to p}&\leqslant 2C_{p,\delta}2^{\frac{d}{2}+\frac{1}{2}}|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}\\&=C_3|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}.\end{aligned}$$

Let $\zeta\in\mathcal{O}$. Using (A.4) + (**a**) with $r=p\in\mathcal{I}$, $\mu=|\zeta|$, we obtain:

$$\|G_p(2\kappa_d\zeta,b)\|_{p\to p}\leqslant m_dC_{p,\delta}2^{\frac{d}{4}}|\zeta|^{-\frac{1}{2p'}}.$$

Now, using the identity $(\zeta-\Delta)^{-1}=(2\kappa_d\zeta-\Delta)^{-1}\big(1+(2\kappa_d-1)\zeta(\zeta-\Delta)^{-1}\big)$, we obtain:

$$\begin{aligned}\|G_p(\zeta,b)\|_{p\to p}&\leqslant 2\kappa_dm_dC_{p,\delta}2^{\frac{d}{4}}|\zeta|^{-\frac{1}{2p'}}\\&=C_1|\zeta|^{-\frac{1}{2p'}}.\end{aligned}$$

The proof of (ii) is complete. □

Remark 2.3. Since $|b_n|\leqslant|b|$ a.e., Proposition 2.1 is valid for b_n, $n=1,2,\dots$, with the same constants.

Proposition 2.4. *For every $p\in\mathcal{I}$, and $n=1,2,\dots$, the operator-valued function $\Theta_p(\zeta,b_n)$ is a pseudo-resolvent on $\mathcal{O}$,* i.e.*,*

$$\Theta_p(\zeta,b_n)-\Theta_p(\eta,b_n)=(\eta-\zeta)\Theta_p(\zeta,b_n)\Theta_p(\eta,b_n),\quad \zeta,\eta\in\mathcal{O}.$$

Proof. Define $S_\zeta^k:=(-1)^k(\zeta-\Delta)^{-1}b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots b_n\cdot\nabla(\zeta-\Delta)^{-1}$, $k:=\#\,b_n$'s. Obviously,

$$\begin{aligned}\Theta_p(\zeta,b_n)&:=(\zeta-\Delta)^{-1}-Q\left(1+T\right)^{-1}G\\&=(\zeta-\Delta)^{-1}-Q\sum_{k=0}^{\infty}(-1)^kT^k\,G\\&=\sum_{k=0}^{\infty}S_\zeta^k\quad\text{(absolutely convergent in }L^p),\end{aligned}$$

$$\Theta_p(\zeta,b_n)\Theta_p(\eta,b_n)=\sum_{\ell=0}^{\infty}\sum_{i=0}^{\ell}S_\zeta^iS_\eta^{\ell-i},\quad \zeta,\eta\in\mathcal{O}. \tag{2.3}$$

Define

$$\begin{aligned}I_{j,m}^k(\zeta,\eta):=(\zeta-\Delta)^{-1}b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots b_n\cdot\nabla(\zeta-\Delta)^{-1}\\ b_n\cdot\nabla(\eta-\Delta)^{-1}b_n\cdot\nabla(\eta-\Delta)^{-1}\dots b_n\cdot\nabla(\eta-\Delta)^{-1},\\ j:=\#\zeta\text{'s},\quad m:=\#\eta\text{'s},\quad k:=\#b_n\text{'s}.\end{aligned}$$

Substituting the identity $(\zeta-\Delta)^{-1}(\eta-\Delta)^{-1} = (\eta-\zeta)^{-1}\big((\zeta-\Delta)^{-1}-(\eta-\Delta)^{-1}\big)$ inside the product

$$S_\zeta^k S_\eta^j = (-1)^{k+j}(\zeta-\Delta)^{-1}b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots\, b_n \cdot\nabla\underbrace{(\zeta-\Delta)^{-1}(\eta-\Delta)^{-1}}_{(\eta-\zeta)^{-1}((\zeta-\Delta)^{-1}-(\eta-\Delta)^{-1})} b_n\cdot\nabla(\eta-\Delta)^{-1}\dots\, b_n\cdot\nabla(\eta-\Delta)^{-1},$$

we obtain $S_\zeta^k S_\eta^j = (\eta-\zeta)^{-1}(-1)^{k+j}\big[I_{k+1,j}^{k+j} - I_{k,j+1}^{k+j}\big]$. Therefore,

$$\begin{aligned}\sum_{i=0}^{\ell} S_\zeta^i S_\eta^{\ell-i} &= (\eta-\zeta)^{-1}(-1)^\ell\Big[I_{1,\ell}^\ell - I_{0,\ell+1}^\ell + I_{2,\ell-1}^\ell - I_{1,\ell}^\ell + \dots + I_{\ell+1,0}^\ell - I_{\ell,1}^\ell\Big]\\ &= (\eta-\zeta)^{-1}(-1)^\ell\big(I_{\ell+1,0}^\ell - I_{0,\ell+1}^\ell\big).\end{aligned}$$

Substituting the last identity in the right-hand side of (2.3), we obtain

$$\begin{aligned}\Theta_p(\zeta,b_n)\Theta(\eta,b_n) &= (\eta-\zeta)^{-1}\sum_{\ell=0}^{\infty}(-1)^\ell\big(I_{\ell+1,0}^\ell - I_{0,\ell+1}^\ell\big)\\ &= (\eta-\zeta)^{-1}\big(\Theta_p(\zeta,b_n)h - \Theta_p(\eta,b_n)\big).\end{aligned}$$ □

Proposition 2.5. *For every $p\in\mathcal{I}$, and $n=1,2,\dots,$*

(i) $\|\Theta_p(\zeta,b_n)\|_{p\to p}\leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$, *for a constant C_p independent of n;*
(ii) $\mu\Theta_p(\mu,b_n)\overset{s}{\to} 1$ *in L^p as $\mu\uparrow\infty$ (uniformly in n).*

Proof of (i). Put $Q_p\equiv Q_p(\zeta,b_n)$, $T_p\equiv T_p(\zeta,b_n)$, $G_p\equiv G_p(\zeta,b_n)$. By the definition of $\Theta_p(\zeta,b_n)$, see (1.2), for every $\zeta\in\mathcal{O}$,

$$\begin{aligned}\|\Theta_p(\zeta,b_n)\|_{p\to p} &\leqslant \|(\zeta-\Delta)^{-1}\|_{p\to p} + \|Q_p\|_{p\to p}\|(1+T_p)^{-1}\|_{p\to p}\|G_p\|_{p\to p}\\ &\qquad\text{(here we are using (2.1), (2.2) in Proposition 2.1)}\\ &\leqslant |\zeta|^{-1} + C_2|\zeta|^{-\frac12-\frac{1}{2p'}}(1-m_dc_p\delta)^{-1}C_1|\zeta|^{-\frac{1}{2p}}\\ &\leqslant C_p|\zeta|^{-1},\quad C_p := 1 + C_1C_2(1-m_dc_p\delta)^{-1}.\end{aligned}$$

Proof of (ii). Put $\Theta_p\equiv\Theta_p(\mu,b_n)$, $Q_p\equiv Q_p(\mu,b_n)$, $T_p\equiv T_p(\mu,b_n)$, $P_p\equiv P_p(\mu,b_n)$. Since $\mu(\mu-\Delta)^{-1}\overset{s}{\to}1$, it suffices to show that $\mu\Theta_p-\mu(\mu-\Delta)^{-1}\overset{s}{\to}0$ in L^p. Since by (i) $\mu\Theta_p$ is uniformly (in μ) bounded in $\mathcal{B}(L^p)$, and C_c^∞ is dense in L^p, it suffices to show that $\mu\Theta_p h - \mu(\mu-\Delta)^{-1}h\to 0$ in L^p for every $h\in C_c^\infty$. Write

$$\Theta_p h - (\mu-\Delta)^{-1}h = -Q_p\big(1+T_p\big)^{-1} b_n^{\frac1p}\cdot(\mu-\Delta)^{-1}\nabla h.$$

By (2.1), $\|(1+T_p)^{-1}\|_{p\to p} \leqslant \frac{1}{1-\|T_p\|_{p\to p}} \leqslant \frac{1}{1-m_d c_p \delta} < \infty$, by (2.2), $\|Q_p\|_{p\to p} \leqslant C_2\mu^{-\frac{1}{2}-\frac{1}{2p}}$. Again, by (2.2),

$$\left\| b_n^{\frac{1}{p}} \cdot (\mu-\Delta)^{-1}\nabla h \right\|_p \leqslant \left\| |b_n|^{\frac{1}{p}} (\mu-\Delta)^{-1} |\nabla h| \right\|_p$$
$$\leqslant \|P_p\|_{p\to p} \|\nabla h\|_p$$
$$\leqslant C_3 \mu^{-\frac{1}{2}-\frac{1}{2p'}} \|\nabla h\|_p.$$

Therefore,

$$\|\Theta_p h - (\mu-\Delta)^{-1} h\|_p \leqslant \|Q_p\|_{p\to p} \left\| (1+T_p)^{-1} \right\|_{p\to p} \left\| b_n^{\frac{1}{p}} \cdot (\mu-\Delta)^{-1}\nabla h \right\|_p$$
$$\leqslant C_0 \mu^{-\frac{3}{2}} \|\nabla h\|_p$$

for some $C_0 < \infty$ independent of n, which clearly implies (ii). □

Proposition 2.6. *For every $p \in \mathcal{I}$, and $n = 1, 2, \ldots$, we have $\mathcal{O} \subset \rho(-\Lambda_p(b_n))$, the resolvent set of $-\Lambda_p(b_n)$. The operator-valued function $\Theta_p(\zeta, b_n)$ is the resolvent of $-\Lambda_p(b_n)$:*

$$\Theta_p(\zeta, b_n) = (\zeta + \Lambda_p(b_n))^{-1}, \quad \zeta \in \mathcal{O},$$

and

$$\|(\zeta + \Lambda_p(b_n))^{-1}\|_{p\to p} \leqslant C_p |\zeta|^{-1}, \quad \zeta \in \mathcal{O}.$$

Proof. By definition, we need to verify that, for every $\zeta \in \mathcal{O}$, $\Theta_p(\zeta, b_n)$ has dense image, and is the left and the right inverse of $\zeta + \Lambda_p(b_n)$. Indeed, Proposition 2.5(ii) implies that $\Theta_p(\zeta, b_n)$ has dense image. $\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_p(b_n)) = W^{2,p}$, is the generator of a C_0-semigroup $e^{-t\Lambda_p(b_n)}$ on L^p. Clearly, $\Theta_p(\zeta_n, b_n) = (\zeta_n + \Lambda_p(b_n))^{-1}$ for all sufficiently large ζ_n $(= \zeta(\|b_n\|_\infty))$, therefore, by Proposition 2.4,

$$\Theta_p(\zeta, b_n) = (\zeta_n + \Lambda_p(b_n))^{-1}\big(1 + (\zeta_n - \zeta)\Theta_p(\zeta, b_n)\big), \quad \zeta \in \mathcal{O},$$

so $\Theta_p(\zeta, b_n) L^p \subset D(\Lambda_p(b_n)) = W^{2,p}$, and $(\zeta + \Lambda_p(b_n))\Theta_p(\zeta, b_n) g = g$, $g \in L^p$, *i.e.*, $\Theta_p(\zeta, b_n)$ is the right inverse of $\zeta + \Lambda_p(b_n)$ on $\mathcal{O}$. Similarly, it is seen that $\Theta(\zeta, b_n)$ is the left inverse of $\zeta + \Lambda_p(b_n)$ on $\mathcal{O}$.

Remark 2.7. Alternatively, we could verify conditions of the Kato theorem [5]: in the reflexive space L^p, the pseudo-resolvent $\Theta_p(\zeta, b_n)$ (see Proposition 2.4) satisfying $\mu\Theta_p(\mu, b_n) \overset{s}{\to} 1$ in L^p as $\mu \uparrow \infty$ (see Proposition 2.5(ii)) is the resolvent of a densely defined closed operator on L^p. This operator coincides with $-\Lambda_p(b_n)$ (since $\Theta_p(\zeta_n, b_n) = (\zeta_n + \Lambda_p(b_n))^{-1}$ for all large ζ_n).

Now, $\|(\zeta + \Lambda_p(b_n))^{-1}\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$, follows from Proposition 2.5(i). □

Proposition 2.8. *For every $\zeta \in \mathcal{O}$ and $p \in \mathcal{I}$,*

$$\Theta_p(\zeta, b_n) \overset{s}{\to} \Theta_p(\zeta, b) \text{ in } L^p.$$

Proof. Put $\Theta_p(b) \equiv \Theta_p(\zeta, b)$, $Q_p(b) \equiv Q_p(\zeta, b)$, $T_p(b) \equiv T_p(\zeta, b)$, $G_p(b) \equiv G_p(\zeta, b)$ (similarly for b_n's). It suffices to prove that

$$Q_p(b_n)(1 + T(b_n))^{-1}G_p(b_n) \overset{s}{\to} Q_p(b)(1 + T_p(b))^{-1}G_p(b).$$

Thus it suffices to prove consecutively that

$$G_p(b_n) \overset{s}{\to} G_p(b),\ (1 + T_p(b_n))^{-1} \overset{s}{\to} (1 + T_p(b))^{-1},\ Q_p(b_n) \overset{s}{\to} Q_p(b).$$

In turn, since $(1+T_p(b_n))^{-1}-(1+T_p(b))^{-1} = (1+T_p(b_n))^{-1}(T_p(b)-T_p(b_n))(1+T_p(b))^{-1}$, it suffices to prove that $T_p(b_n) \overset{s}{\to} T_p(b)$. Finally,

$$T_p(b_n) - T_p(b) = T_p(b_n) - b_n^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}} + b_n^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}} - T_p(b),$$

and hence we have to prove that

$$b_n^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}} - T_p(b) := J_n^{(1)} \overset{s}{\to} 0$$

and

$$T_p(b_n) - b_n^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}} := J_n^{(2)} \overset{s}{\to} 0.$$

Now, by the Dominated Convergence Theorem (*cf*. the argument in the proof of (A.1)), $G_p(b_n) \overset{s}{\to} G_p(b)$, $J_n^{(1)}|_{\mathcal{E}} \overset{s}{\to} 0$. Also

$$\begin{aligned}
\|J_n^{(2)} f\|_p &= \left\| G_p(b_n)\left(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}}\right) f \right\|_p \\
&\leqslant \|G_p(b_n)\|_{p\to p} \left\| \left(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}}\right) f \right\|_p \\
&\leqslant m_d(1+\delta)|\zeta|^{-\frac{1}{2p'}} \left\| \left(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}}\right) f \right\|_p, \quad (f \in \mathcal{E}).
\end{aligned}$$

Thus, $J_n^{(2)}|_{\mathcal{E}} \overset{s}{\to} 0$. Since $\|J_n^{(2)}\|_{p\to p}, \|J_n^{(1)}\|_{p\to p} \leqslant m_d\delta$, we conclude that $T_p(b_n) \overset{s}{\to} T_p(b)$. It is clear now that $Q_p(b_n) \overset{s}{\to} Q_p(b)$. □

Now we are going to prove Theorem 1.3 using the Trotter approximation theorem [4, IX.2.5]. Recall its conditions (in terms of $\Theta_p(\zeta, b_n)$ on the base of Proposition 2.6):

1) $\sup_{n\geqslant 1} \|\Theta_p(\zeta, b_n)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$;

2) $\mu\Theta_p(\mu, b_n) \overset{s}{\to} 1$ in L^p as $\mu \uparrow \infty$ uniformly in n;

3) There exists s-L^p-$\lim_n \Theta_p(\zeta, b_n)$ for some $\zeta \in \mathcal{O}$.

Now, 1) is the content of Proposition 2.5(i). 2) is Proposition 2.5(ii). Proposition 2.8 implies 3).

Therefore, by the Trotter approximation theorem, $\Theta_p(\zeta, b) = (\zeta + \Lambda_p(b))^{-1}$, $\zeta \in \mathcal{O}$, where $\Lambda_p(b)$ is the generator of the holomorphic C_0-semigroup $e^{-t\Lambda_p(b)}$ on L^p. (Note that, by Proposition 2.8, $\|\Theta_p(\zeta, b)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$. Hence, $\Theta_p(\zeta, b)$ can be extended to $\mathcal{O} \cup \{\zeta \in \mathbb{C} : |\mathrm{Arg}\,\zeta| < \frac{\pi}{2} + \varepsilon, |\zeta| > R\}$, $\varepsilon > 0$, for a sufficiently large $R > 0$, where it satisfies $\|\Theta_p(\zeta, b)\|_{p\to p} \leqslant C'_p|\zeta|^{-1}$, see the corresponding argument in [16, IX.10].)

Hence, the assertions (i), (vi) of Theorem 1.3 follow. (ii) follows from Proposition 2.5(i) and Proposition 2.8. (iii) is obvious from the definitions of the operators involved, *cf.* Proposition 2.1.

(iii) $\Rightarrow$ (iv). In particular, if $p > d - 1$, given a $0 < \gamma < 1 - \frac{d-1}{p}$, we can select $q > p$ sufficiently close to p so that by the Sobolev embedding theorem the Bessel space $\mathcal{W}^{1+\frac{1}{q},p}$ is embedded into $C^{0,\gamma}$.

(v) Let $\zeta \in \mathcal{O}$. By Proposition 2.8, $\Lambda_p(b_n)(\zeta + \Lambda_p(b_n))^{-1} \overset{s}{\to} \Lambda_p(b)(\zeta + \Lambda_p(b))^{-1}$ in L^p. Put $Q_p(b) \equiv Q_p(\zeta, b)$, $T_p(b) \equiv T_p(\zeta, b)$, $G_p(b) \equiv G_p(\zeta, b)$ (similarly for b_n's). Since $(\zeta+\Lambda_p(b))^{-1} = (\zeta-\Delta)^{-1} - Q_p(b)(1+T_p(b))^{-1}G_p(b)$, we have

$$b^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b))^{-1} = G_p(b) - T_p(b)(1 + T_p(b))^{-1}G_p(b)$$

(similarly for the b_n's). Since $G_p(b_n) \overset{s}{\to} G_p(b)$, $T_p(b_n) \overset{s}{\to} T_p(b)$ in L^p (see the proof of Proposition 2.8),

$$b_n^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b_n))^{-1} \overset{s}{\to} b^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b))^{-1} \text{ in } L^p. \qquad (**)$$

Clearly, $|b|^{\frac{1}{p'}} \in L^{p'}_{\mathrm{loc}}$, for $|b| \in L^1_{\mathrm{loc}}$ by the definition of class $\mathbf{F}^{\frac{1}{2}}_{\delta}$. Now, given $u \in D(\Lambda_p(b))$, we have $u = (\zeta + \Lambda_p(b))^{-1}g$ for some $g \in L^p$, and so, for every

$v \in C_c^\infty$,

$$
\begin{aligned}
\langle \Lambda_p(b)u, v\rangle &= \langle \Lambda_p(b)(\zeta + \Lambda_p(b))^{-1}g, v\rangle \\
&= \lim_n \langle \Lambda_p(b_n)(\zeta + \Lambda_p(b_n))^{-1}g, v\rangle \\
&= \lim_n \langle (\zeta + \Lambda_p(b_n))^{-1}g, -\Delta v\rangle \\
&\quad + \lim_n \langle b_n^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b_n))^{-1}g, |b_n|^{\frac{1}{p'}} v\rangle \\
&\quad \text{(here we are using (}**\text{) and the fact that } |b_n|^{\frac{1}{p'}} v \to |b|^{\frac{1}{p'}} v \text{ in } L^{p'}) \\
&= \langle (\zeta + \Lambda_p(b))^{-1}g, -\Delta v\rangle + \langle b^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b))^{-1}g, |b|^{\frac{1}{p'}} v\rangle \\
&= \langle u, -\Delta v\rangle + \langle b^{\frac{1}{p}} \cdot \nabla u, |b|^{\frac{1}{p'}} v\rangle.
\end{aligned}
$$

Next, since for $u \in D(\Lambda_p(b))$, $b^{\frac{1}{p}} \cdot \nabla u \in L^p$, it follows that $b \cdot \nabla u = |b|^{\frac{1}{p'}}\, b^{\frac{1}{p}} \cdot \nabla u \in L^1_{\text{loc}}$. Also, $\Lambda_p(b)u \in L^p$, and hence $\langle \Lambda_p(b)u, v\rangle = \langle u, -\Delta v\rangle + \langle b \cdot \nabla u, v\rangle$. Therefore, Δu (understood in the sense of distributions) $= -\Lambda_p(b)u + b \cdot \nabla u \in L^1_{\text{loc}}$, *i.e.* $u \in \mathcal{W}^{2,1}_{\text{loc}}$. The proof of ($v$) is completed.

For the proof of (*viii*) see the argument in [13, p. 415-416].

The proof of Theorem 1.3 is complete.

3. Proof of Theorem 1.5

It is easily seen that, due to the strict inequality $m_d\delta < 4\frac{d-2}{(d-1)^2}$, for every $\tilde{\delta} > \delta$ such that $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$ there exists $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$, such that

$$
\tilde{b}_n := \eta_{\varepsilon_n} * b_n \in \mathbf{F}^{\frac{1}{2}}_{\tilde{\delta}}, \quad n = 1, 2, \dots
$$

(i) We verify conditions of the Trotter approximation theorem:

1°) $\sup_n \|(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}\|_{\infty\to\infty} \leqslant \mu^{-1}$, $\mu \geqslant \kappa_d\lambda$.

2°) $\mu(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1} \to 1$ in C_∞ as $\mu \uparrow \infty$ uniformly in n.

3°) There exists s-C_∞-$\lim_n(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}$ for some $\mu \geqslant \kappa_d\lambda$.

The condition 1°) is immediate. In view of 1°), it suffices to verify 2°), 3°) on $\mathcal{S}$, the L. Schwartz space of test functions. Fix $p \in \mathcal{I}$, $p > d-1$ (such p exists since $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$).

Proposition 3.1. *For every $\mu \geqslant \kappa_d\lambda$, $n = 1, 2, \dots$, $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$, and*

$$
(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}|_{\mathcal{S}} = \Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}.
$$

Proof. The inclusion $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$ is obvious. Also, $\Theta_p(\mu_n, \tilde{b}_n)|_{\mathcal{S}} = (\mu_n + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}|_{\mathcal{S}}$ for all sufficiently large μ_n $(= \mu(\|\tilde{b}_n\|_\infty))$. By $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$ and Proposition 2.4, $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ satisfies the resolvent identity on $\mu \geqslant \kappa_d\lambda$,

$$\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}} = (\mu_n + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}\big(1 + (\mu_n - \mu)\Theta_p(\mu, \tilde{b}_n)\big)|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d\lambda,$$

so $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ is the right inverse of $\mu + \Lambda_{C_\infty}(\tilde{b}_n)|_{\mathcal{S}}$ on $\mu \geqslant \kappa_d\lambda$. Similarly, it is seen that $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ is the left inverse of $\mu + \Lambda_{C_\infty}(\tilde{b}_n)|_{\mathcal{S}}$ on $\mu \geqslant \kappa_d\lambda$. ☐

Proposition 3.2. *For every $\mu \geqslant \kappa_d\lambda$, $\Theta_p(\mu, b)\mathcal{S} \subset C_\infty$, and*

$$\Theta_p(\mu, \tilde{b}_n) \overset{s}{\to} \Theta_p(\mu, b) \text{ in } C_\infty.$$

Proof. By Theorem 1.3(iv), since $p > d - 1$, $\Theta_p(\mu, b)L^p \subset C_\infty$. Put

$$Q_p(q, b) \equiv Q_p(q, \mu, b), \quad T_p(b) \equiv T_p(\mu, b), \quad G_p(b) \equiv G_p(\mu, b).$$

To establish the required convergence, it suffices to prove that

$$\begin{aligned}&(\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}} Q_p(q, \tilde{b}_n)(1 + T_p(\tilde{b}_n))^{-1} G_p(\tilde{b}_n)\\ &\overset{s}{\to} (\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}} Q_p(q, b)(1 + T_p(b))^{-1} G_p(b) \quad \text{in } C_\infty.\end{aligned}$$

We choose q $(> p)$ close to $d-1$ so that $(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}} L^p \hookrightarrow C_\infty$. Thus it suffices to prove that

$$G_p(\tilde{b}_n) \overset{s}{\to} G_p(b),\ (1+T_p(\tilde{b}_n))^{-1} \overset{s}{\to} (1+T_p(b))^{-1},\ Q_p(q,\tilde{b}_n) \overset{s}{\to} Q_p(q,b) \text{ in } L^p,$$

which can be done by repeating the arguments in the proof of Proposition 2.8. ☐

Proposition 3.3.

$$\mu\Theta_p(\mu, \tilde{b}_n) \overset{s}{\to} 1 \text{ as } \mu \uparrow \infty \text{ in } C_\infty \text{ uniformly in } n. \tag{3.1}$$

Proof. Put $\Theta_p \equiv \Theta_p(\mu, \tilde{b}_n)$, $T_p \equiv T_p(\mu, \tilde{b}_n)$. Since $\mu(\mu - \Delta)^{-1} \overset{s}{\to} 1$ in C_∞, and $\mathcal{S}$ is dense in C_∞, it suffices to show that $\|\mu\Theta_p f - \mu(\mu - \Delta)^{-1} f\|_\infty \to 0$ for every $f \in \mathcal{S}$. For each $f \in \mathcal{S}$ there is $h \in \mathcal{S}$ such that $f = (\lambda - \Delta)^{-\frac{1}{2}} h$, where $\lambda = \lambda_\delta > 0$. Let $q > p$. Write

$$\Theta_p f - (\mu-\Delta)^{-1} f = -(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}} Q_p(q)\big(1+T_p\big)^{-1} b^{\frac{1}{p}} (\lambda-\Delta)^{-\frac{1}{2}} \cdot (\mu-\Delta)^{-1}\nabla h.$$

Now, arguing as in the proof of Proposition 2.5(i), but using the estimates

$$\|(\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}}\|_{p\to\infty} \leqslant c\mu^{-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}}, \quad c < \infty,$$

and

$$\|Q_p(q)\|_{p\to p} \leqslant \tilde{K}_{2,q} < \infty \quad \text{(see (2.2))},$$

we obtain

$$\|\Theta_p f - (\mu - \Delta)^{-1} f\|_\infty \leqslant C\mu^{-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}} \mu^{-1} \|\nabla h\|_p.$$

Since $p > d-1$, choosing q sufficiently close to p, we obtain

$$-\frac{1}{2} + \frac{d}{2p} - \frac{1}{2q} - 1 < -1,$$

so $\mu\Theta_p - \mu(\mu - \Delta)^{-1} \overset{s}{\to} 0$ in C_∞, as needed. □

Now, Proposition 3.2 verifies condition 3°), and Proposition 3.3 verifies condition 2°). Assertion (i) of Theorem 1.5 now follows from the Trotter approximation theorem.

Assertion (ii) of Theorem 1.5 follows from Theorem 1.3(iii).

The proof of assertion (iii) is standard, and is omitted.

Remark 3.4. We could construct $e^{-t\Lambda_{C_\infty}(b)}$ alternatively as follows:

$$e^{-t\Lambda_{C_\infty}(b)} := \left(e^{-t\Lambda_p(b)}|_{C_\infty \cap L^p}\right)^{\text{clos}}_{C_\infty}$$
$$\text{(after a change on a set of measure zero)}, \quad t > 0,$$

where $p \in \left(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\right)$.

A. Appendix

Define $I_n := \|(b - b_n) \cdot \nabla(\zeta - \Delta)^{-1} f\|_1$.

1. *Let $b \in \mathbf{K}^{d+1}_\delta$. For every $f \in L^1$ and* Re $\zeta \geqslant \kappa_d \lambda$,

$$I_n \to 0 \text{ as } n \uparrow \infty. \tag{A.1}$$

Proof of (A.1). Since $I_n \leqslant 2m_d \||b|(\lambda - \Delta)^{-\frac{1}{2}}|f|\|_1 \leqslant 2m_d\delta\|f\|_1$, it suffices to prove (A.1) for each $f \in L^1 \cap L^\infty$. Let $f \in L^1 \cap L^\infty$, $\lambda > 0$ and b be fixed. Since $|b|(\lambda - \Delta)^{-\frac{1}{2}}|f| \in L^1$, for a given $\epsilon > 0$, there exists $\mathcal{K}$, a compact, such that

$$\left\|(\mathbf{1} - \mathbf{1}_\mathcal{K})|b|(\lambda - \Delta)^{-\frac{1}{2}}|f|\right\|_1 \leqslant \epsilon,$$

where $\mathbf{1}_\mathcal{K}$ is the characteristic function of $\mathcal{K}$. Define $I_{\mathcal{K},n} := \|\mathbf{1}_\mathcal{K}|b - b_n|(\lambda - \Delta)^{-\frac{1}{2}}|f|\|_1$. Clearly,

$$I_{\mathcal{K},n} \leqslant \lambda^{-\frac{1}{2}}\|f\|_\infty \|\mathbf{1}_\mathcal{K}|b - b_n|\|_1.$$

Since $|b| \in L^1_{\mathrm{loc}}$ and $\mathcal{K}$ independent of $n = 1, 2, \dots,$

$$\|\mathbf{1}_{\mathcal{K}}|b - b_n|\|_1 \leqslant \|\mathbf{1}_{|b|\geqslant n}(\mathbf{1}_{\mathcal{K}}|b|)\|_1 \to 0 \text{ as } n \uparrow \infty.$$

Therefore, for a given ϵ, there exists $n_0 = n_0(\epsilon) \geqslant 1$, such that $I_{\mathcal{K},n} \leqslant \epsilon$ whenever $n \geqslant n_0$, and so

$$I_n \leqslant 3m_d\epsilon \qquad \forall n \geqslant n_0. \qquad \square$$

We use the following pointwise estimates ($x, y \in \mathbb{R}^d, x \neq y$).

2. *For every* $\mathrm{Re}\,\zeta > 0$,

$$|\nabla(\zeta - \Delta)^{-1}(x, y)| \leqslant m_d(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}}(x, y), \tag{A.2}$$

where $m_d^2 := \pi(2e)^{-1}d^d(d-1)^{1-d}$, $\kappa_d := \frac{d}{d-1}$.

For every $r \in (1, \infty]$ *there exists a constant* $m_{r,d} < \infty$ *such that for all* $\mathrm{Re}\,\zeta > 0$,

$$|\nabla(\zeta - \Delta)^{-1+\frac{1}{2r}}(x, y)| \leqslant m_{r,d}(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}+\frac{1}{2r}}(x, y). \tag{A.3}$$

3. *For every* $\mathrm{Re}\,\zeta > 0$,

$$|\nabla(\zeta - \Delta)^{-1}(x, y)| \leqslant 2^{\frac{d}{4}}m_d\left(\kappa_d^{-1}2^{-1}|\zeta| - \Delta\right)^{-\frac{1}{2}}(x, y), \tag{A.4}$$

$$|(\zeta - \Delta)^{-\frac{1}{2}}(x, y)| \leqslant 2^{\frac{d}{4}+\frac{1}{4}}\left(2^{-1}|\zeta| - \Delta\right)^{-\frac{1}{2}}(x, y). \tag{A.5}$$

Proof of (A.2). Let $\alpha \in (0, 1)$. Set

$$c(\alpha) := \sup_{\xi > 0} \xi e^{-(1-\alpha)\xi^2} \left(= \frac{1}{\sqrt{2}}(1-\alpha)^{-\frac{1}{2}}e^{-\frac{1}{2}}\right),$$

so that

$$\xi e^{-\xi^2} \leqslant c(\alpha)e^{-\alpha\xi^2} \quad \text{for all } \xi > 0. \tag{$\star$}$$

We use the well known formula

$$(\zeta - \Delta)^{-\frac{\gamma}{2}}(x, y) = \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)}\int_0^\infty e^{-\zeta t}t^{\frac{\gamma}{2}-1}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt, \quad 0 < \gamma \leqslant 2,$$

first with $\gamma = 2$, and then with $\gamma = 1$, to obtain:

$$
\begin{aligned}
|\nabla(\zeta-\Delta)^{-1}(x,y)| &\leqslant \int_0^\infty e^{-t\operatorname{Re}\zeta}(4\pi t)^{-\frac{d}{2}}\frac{|x-y|}{2t}e^{-\frac{|x-y|^2}{4t}}dt \\
&\leqslant c(\alpha)\int_0^\infty e^{-t\operatorname{Re}\zeta}t^{-\frac12}(4\pi t)^{-\frac{d}{2}}e^{-\alpha\frac{|x-y|^2}{4t}}dt \\
&\qquad\left(\text{By } (\star) \text{ with } \xi := \frac{|x-y|}{2\sqrt{t}}\right) \\
&\leqslant c(\alpha)\alpha^{-\frac12-\frac{d}{2}+1}\int_0^\infty e^{-(\operatorname{Re}\zeta)\alpha t}t^{-\frac12}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \\
&\qquad\left(\text{change } t/\alpha \text{ to } t\right) \\
&= c(\alpha)\alpha^{\frac12-\frac{d}{2}}\Gamma\left(\frac12\right)\left(\alpha\operatorname{Re}\zeta-\Delta\right)^{-\frac12}(x,y).
\end{aligned}
$$

Now, we minimize $c(\alpha)\alpha^{\frac12-\frac{d}{2}}\Gamma(\frac12)$ in $\alpha \in (0,1)$. The minimum is attained at $\alpha_d = \frac{d-1}{d}$ $(=: \kappa_d^{-1})$, and is equal to m_d.

The proof of (A.3) is similar. □

Proof of (A.4). First, suppose that Im $\zeta \leqslant 0$. By the Cauchy theorem,

$$
\begin{aligned}
(\zeta-\Delta)^{-1}(x,y) &= \int_0^\infty e^{-\zeta t}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \\
&= \int_0^\infty e^{-\zeta re^{i\frac{\pi}{4}}}e^{-i\frac{\pi}{4}\frac{d}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}e^{i\frac{\pi}{4}}dr,
\end{aligned}
$$

(*i.e.*, we have changed the contour of integration from $\{t : t \geqslant 0\}$ to $\{re^{i\frac{\pi}{4}} : r \geqslant 0\}$). Thus,

$$
|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant \int_0^\infty \left|e^{-\zeta re^{i\frac{\pi}{4}}}\right|(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|\left|e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}\right|dr.
$$

We have

$$
|e^{-\zeta re^{i\frac{\pi}{4}}}| \leqslant e^{-r\frac{1}{\sqrt2}(\operatorname{Re}\zeta-\operatorname{Im}\zeta)}, \quad |e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}| \leqslant e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt2}}, \quad \operatorname{Re}\zeta-\operatorname{Im}\zeta \geqslant |\zeta|.
$$

Therefore,

$$
\begin{aligned}
|\nabla(\zeta-\Delta)^{-1}(x,y)| &\leqslant \int_0^\infty e^{-r\frac{1}{\sqrt{2}}|\zeta|}(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right| e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}dr \\
&\quad \big(\text{change } r\sqrt{2} \text{ to } r\big) \\
&= 2^{\frac{d}{4}}\int_0^\infty e^{-r\frac{1}{2}|\zeta|}(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right| e^{-\frac{|x-y|^2}{4r}}dr \\
&\leqslant \frac{2^{\frac{d}{4}}m_d}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty e^{-r\kappa_d^{-1}\frac{1}{2}|\zeta|}(4\pi r)^{-\frac{d}{2}}r^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{4r}}dr \\
&\quad \big(\textit{cf.}\text{ proof of (A.2)}\big) \\
&= 2^{\frac{d}{4}}m_d\left(\kappa_d^{-1}2^{-1}|\zeta|-\Delta\right)^{-\frac{1}{2}}(x,y)
\end{aligned}
$$

which yields (A.4) for Im $\zeta \leqslant 0$. The case Im $\zeta > 0$ is treated analogously. □

Proof of (A.5). First, suppose that Im $\zeta \leqslant 0$. By the Cauchy theorem,

$$
\begin{aligned}
(\zeta-\Delta)^{-\frac{1}{2}}(x,y) &= \int_0^\infty e^{-\zeta t}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \\
&= \int_0^\infty e^{-\zeta r e^{i\frac{\pi}{4}}}r^{-\frac{1}{2}}e^{-i\frac{\pi}{8}}e^{-i\frac{\pi}{4}\frac{d}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}e^{i\frac{\pi}{4}}dr,
\end{aligned}
$$

so we estimate as above:

$$
\begin{aligned}
\left|(\zeta-\Delta)^{-\frac{1}{2}}(x,y)\right| &\leqslant \int_0^\infty e^{-r\frac{1}{\sqrt{2}}|\zeta|}r^{-\frac{1}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}dr \\
&= 2^{\frac{d}{4}+\frac{1}{4}}\left(2^{-1}|\zeta|-\Delta\right)^{-\frac{1}{2}}(x,y).
\end{aligned}
$$

The case Im $\zeta > 0$ is treated analogously. □

4. *In the proof of Proposition* 2.1 *we need the following formula: for every* $0 < \alpha < 1$, Re $\zeta > 0$,

$$
(\zeta-\Delta)^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+\zeta-\Delta)^{-1}dt. \tag{A.6}
$$

B. Appendix

Proof of (1.1). Let $b \in \mathbf{K}_\delta^{d+1}$, *i.e.*, $\||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to1} \leqslant \delta$, $\|(\lambda-\Delta)^{-\frac{1}{2}}|b|\|_\infty \leqslant \delta$ (by duality). Then, using, *e.g.*, interpolation, $\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{2}}\|_{2\to2} \leqslant \delta$, *i.e.*, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$. The first inclusion is proved. (Here, the proof depends crucially of the fact that $(\lambda-\Delta)^{-\frac{1}{2}}$ is an integral operator with a symmetric kernel.)

The second inclusion $\mathbf{F}_{\delta_1} \subsetneq \mathbf{F}_{\delta}^{\frac{1}{2}}$, $\delta = \sqrt{\delta_1}$, follows, *e.g.*, by Heinz inequality [3]. The last assertion now follows from

$$b \in \mathbf{F}^{\frac{1}{2}}_{\sqrt{\delta_1}}, \mathsf{f} \in \mathbf{F}^{\frac{1}{2}}_{\delta_2} \Rightarrow b + \mathsf{f} \in \mathbf{F}^{\frac{1}{2}}_{\delta},$$

where we have used $(|b| + |\mathsf{f}|)^{\frac{1}{2}} \leqslant |b|^{\frac{1}{2}} + |\mathsf{f}|^{\frac{1}{2}}$. □

References

[1] M. AIZENMAN and B. SIMON, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure. Appl. Math. **35** (1982), 209–273.

[2] A. G. BELYI and YU. A. SEMENOV, *On the L^p-theory of Schrödinger semigroups. II*, Sibirsk. Math. J. **31** (1990), 16–26; English transl. in Siberian Math. J. **31** (1991), 540–549.

[3] E. HEINZ, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951) 415–438.

[4] T. KATO, "Perturbation Theory for Linear Operators", Springer-Verlag, Berlin, Heidelberg, 1995.

[5] T. KATO, *Remarks on pseudo-resolvents and infinitesimal generators*, Proc. Japan. Acad. **35** (1959), 467–468.

[6] D. KINZEBULATOV, *Feller evolution families and parabolic equations with form-bounded vector fields*, Osaka J. Math., to appear

[7] V. F. KOVALENKO, M. A. PERELMUTER, and YU. A. SEMENOV, *Schrödinger operators with $L_W^{1/2}(R^l)$-potentials*, J. Math. Phys. **22** (1981), 1033–1044.

[8] V. F. KOVALENKO and YU. A. SEMENOV, *C_0-semigroups in $L^p(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$ spaces generated by differential expression $\Delta + b \cdot \nabla$*, (Russian) Teor. Veroyatn. Primen. **35** (1990), 449–458; translation in Theory Probab. Appl. **35** (1991), 443–453.

[9] N. V. KRYLOV and M. RÖCKNER, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Related Fields **131** (2005), 154–196.

[10] V. A. LISKEVICH and YU. A. SEMENOV, *Some problems on Markov semigroups*, In: "Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras" M. Demuth *et al*. (eds.), Mathematical Topics: Advances in Partial Differential Equations, Vol. 11, Akademie Verlag, Berlin, 1996, 163–217.

[11] R. S. PHILLIPS, *Semigroups of positive contraction operators*, Czechoslovak Math. J. **12** (1962), 294–313.

[12] YU. A. SEMENOV, unpublished.

[13] YU. A. SEMENOV, *Regularity theorems for parabolic equations*, J. Funct. Anal. **231** (2006), 375–417.

[14] YU. A. SEMENOV, *On perturbation theory for linear elliptic and parabolic operators; the method of Nash*, Contemp. Math. **221** (1999), 217–284.

[15] J. VOIGT, *On the perturbation theory for strongly continuous semigroups*, Math. Ann. **229** (1977), 163–171.

[16] K. YOSIDA, "Functional Analysis", Springer-Verlag, Berlin, Heidelberg, 1980.

Department of Mathematics
University of Toronto
40 St. George Str.
Toronto, ON, M5S2E4, Canada
damir.kinzebulatov@utoronto.ca