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Frequency of Sobolev dimension distortion of horizontal subgroups in Heisenberg groups

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In memory of Frederick W. Gehring (1925–2012)

Abstract. We study the behavior of Sobolev mappings defined on the sub-Riemannian Heisenberg groups with respect to foliations by left cosets of a horizontal homogeneous subgroup. Our main result provides a quantitative estimate, in terms of Hausdorff dimension, of the size of the set of cosets whose dimension is raised under such mappings. Our approach unifies ideas of Gehring and Mostow about the absolute continuity of quasiconformal mappings with Mattila's projection and slicing machinery.

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1. Introduction

Bounds on the amount of change in Hausdorff dimension give important information on the regularity of mappings within specific classes. The following result states that there is a universal bound on the amount by which a super-critical Sobolev mapping f can increase the dimension of a subset.

Theorem 1.1. *Let $f : \mathbb{R}^n \to Y$ be a continuous mapping to a metric space Y that lies in the Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; Y)$, with $p > n$. For any subset $E \subseteq \mathbb{R}^n$ that is of σ-finite s-dimensional Hausdorff measure for some s, with $0 \le s < n$, it holds that $\mathcal{H}^\alpha(f(E)) = 0$, where*

$$\alpha = \frac{ps}{p-(n-s)}. \tag{1.1}$$

Moreover, if $\mathcal{H}^n_{\mathbb{R}^n}(E) = 0$, then $\mathcal{H}^n(f(E)) = 0$ as well.

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Here and below, we denote by $\mathcal{H}^\alpha_X$ the α-dimensional Hausdorff measure in a metric space X, and by $\dim_X(A)$ the Hausdorff dimension of a subset A of the space X; when it will not cause confusion, we may suppress the reference to the ambient space X.

Estimates such as these appeared previously for quasiconformal maps in the work of Gehring–Väisälä [14] and Astala [1]. A proof of the above theorem may be found, for example, in [21].

Notice that the dimension distortion bound

$$\dim_Y(f(E)) \leq \frac{p \dim_{\mathbb{R}^n}(E)}{p - (n - \dim_{\mathbb{R}^n}(E))}$$

given by Theorem 1.1 is strictly better than the estimate provided by the $(1 - \frac{n}{p})$-Hölder continuity of such mappings arising from the Sobolev-Morrey embedding theorem, and, as shown by Kaufman in [21, Theorem 2], it is sharp. The Sobolev-Morrey embedding theorem also shows that the *a priori* assumption of continuity is mild. However, that assumption is crucial, as both Theorem 1.1 and Morrey's inequality (on which it is based) deal with sets that may have measure zero. Those results may fail if an inappropriate representative of a given element of $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; Y)$ is chosen. Similar considerations apply to the rest of the results in this paper as well.

While Theorem 1.1 concerns a fixed set E, one can ask about dimension distortion within a family of sets $(E_a)_a$ where a takes values in some parameter set $a \in \Lambda$. In this sense we could ask if it is possible to bound the "quantity" $a \in \Lambda$ of sets E_a of a given dimension that can be *simultaneously distorted* by a Sobolev mapping. One way to formulate this statement precisely is in terms of foliations. Balogh, Monti, and Tyson [5] considered the foliation of $\mathbb{R}^n$ by translates of a linear subspace $E = V$ of the form $E_a = a + V$ and estimated the size of the set of translates in $a \in V^\perp$ that are mapped by a supercritical Sobolev mapping onto sets of prespecified Hausdorff dimension between $m = \dim V$ and the universal bound given by (1.1):

Theorem 1.2 (Balogh–Monti–Tyson). *Let f be a continuous map in the Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; Y)$, with $p > n$. Given a vector subspace V of $\mathbb{R}^n$ of dimension m, with $1 \leq m \leq n$, and given*

$$\alpha \in \left[m, \frac{pm}{p - (n - m)}\right],$$

it holds that

$$\dim_{\mathbb{R}^n}\{a \in V^\perp : \dim_Y f(a + V) \geq \alpha\} \leq \beta(p, m, \alpha), \tag{1.2}$$

where

$$\beta(p, m, \alpha) = (n - m) - p\left(1 - \frac{m}{\alpha}\right). \tag{1.3}$$

In fact, a slightly different result was stated in [5]: excluding the endpoint $\alpha = m$, the authors conclude that

$$\mathcal{H}^{\beta}_{\mathbb{R}^n}\Big(\{a \in V^{\perp} : \dim f(a+V) > \alpha\}\Big) = 0,$$

which recovers the universal estimate given in Theorem 1.1 as a special case. Theorem 1.2 is also sharp as is demonstrated by examples in [5].

Theorem 1.2 has recently been extended in several ways: The sub-critical case ($p < n$) was treated by Hencl–Honzík [19], who also obtained analogous results for the class of Orlicz-Sobolev maps [20]. On the other hand, one can also inquire about the sharpness of such results when the Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; Y)$ is replaced by the class of quasiconformal mappings of $\mathbb{R}^n$. This was done in dimension two by Bishop, Hakobyan, and Williams [10], and later in all dimensions by the authors [9]. One can also consider more complicated parameter spaces than the ones generating transversal foliations. In the recent work [4], Balogh, Mattila, and Tyson considered the case when the parameter space $V^{\perp}$ is replaced by the entire Grassmannian manifold $\Lambda = G(n, m)$ of m dimensional subspaces of $\mathbb{R}^n$.

The goal of the present paper is to consider the dimension distortion problem in the case of foliations by *left cosets of horizontal subgroups* of the Heisenberg group $\mathbb{H}^n$, which we equip with a sub-Riemannian metric. These foliations arise naturally in the theory of quasiconformal mappings in the Heisenberg groups and are of considerable importance in a variety of settings.

In order to put our results in context we recall that a result in this direction in the setting of general metric measure spaces X has been already obtained by the authors of the present paper in [8]. More precisely, an analog of Theorem 1.2 was proven for spaces carrying a “regular foliation” as introduced by David and Semmes in [12]. As in the Euclidean setting, these foliations can be defined by using projections $\pi : X \to W$ onto a parameterizing metric space W. The projection defines a David–Semmes s-regular foliation, $s > 0$, if it is a locally Lipschitz mapping and the truncated preimage of a ball of radius r in W can be covered by approximately r^{-s} balls of radius r in X. Sets of the form $\pi^{-1}(a) \subset X$ are the *leaves* of the foliation, parametrized by elements $a \in W$. We recall the following theorem from [8]:

Theorem 1.3. *Let $Q \geq 1$ and $0 < s < Q < p$. Let (X, d_X, μ) be a metric measure space that is proper, locally homogeneous of dimension at most Q, supports a local Q-Poincaré inequality, and is equipped with a locally David–Semmes s-regular foliation (X, W, π). Let Y be an arbitrary metric space and $f : X \to Y$ a continuous mapping with an upper gradient in $\mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathrm{X})$. For $\alpha \in \left(s, \frac{ps}{p-Q+s}\right]$, it holds that*

$$\dim_W\{a \in W : \dim f(\pi^{-1}(a)) \geq \alpha\} \leq (Q - s) - p\left(1 - \frac{s}{\alpha}\right).$$

Foliations of the Heisenberg group by *right* cosets of horizontal or vertical subgroups fit nicely into the framework of Theorem 1.3, as described in [8]. However, in the case of foliations by *left* cosets of horizontal subgroups, Theorem 1.3 does not yield good results, as indicated in Section 6 of [8]. The difficulty is caused by the intricate nature of the sub-Riemaniann geometry of the Heisenberg group. Here, this is manifested by the presence of a certain twisting effect of the left projection on Heisenberg balls. This generates a discrepancy between the regularity constant s and the dimension of the leaves of the foliation. In this paper we overcome these issues and provide an analog of Theorem 1.2 for Sobolev mappings acting on left coset foliations of the Heisenberg groups $\mathbb{H}^n$; our proof technique differs substantially from that of [5] and its descendants.

To begin, we recall that the universal bound on dimension distortion by supercritical Sobolev maps remains valid in the Heisenberg groups, as shown in our previous work [8, Theorem 4.1].

Theorem 1.4. *Let f be a continuous mapping in the Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{H}^n; Y)$, with $p > 2n+2$. For any subset $E \subseteq \mathbb{H}^n$ with σ-finite $\mathcal{H}^s_{\mathbb{H}^n}$ measure, it holds that $\mathcal{H}^\alpha(f(E)) = 0$, where*

$$\alpha = \frac{ps}{p - (2n + 2 - s)}. \tag{1.4}$$

Moreover, if $\mathcal{H}^{2n+2}_{\mathbb{H}^n}(E) = 0$, then $\mathcal{H}^n(f(E)) = 0$ as well.

Theorem 1.4 is sharp. Indeed, the set of such mappings that distort a given subset by the maximal amount is prevalent [8, Theorems 1.3 and 1.4].

We consider foliations of $\mathbb{H}^n$ by left cosets of arbitrary homogeneous subgroups that are tangent to the horizontal distribution. Such a subgroup $\mathbb{V}$ is called *horizontal* and may be identified with an isotropic subspace V of $\mathbb{R}^{2n}$; its dimension, now denoted by m, satisfies $1 \le m \le n$. The set of leaves of this foliation is parameterized by the *vertical complement* $\mathbb{V}^\perp = V^\perp \times \mathbb{R}$, where $V^\perp$ is the Euclidean orthogonal complement of V in $\mathbb{R}^{2n}$ and the additional copy of $\mathbb{R}$ corresponds to the vertical axis in $\mathbb{H}^n$. This yields the semidirect decomposition $\mathbb{H}^n = \mathbb{V}^\perp \ltimes \mathbb{V}$, and allows us to define a mapping

$$\pi_{\mathbb{V}^\perp} : \mathbb{H}^n \to \mathbb{V}^\perp. \tag{1.5}$$

The preimage of a point $a \in \mathbb{V}^\perp$ is the left coset $a * \mathbb{V}$, *i.e.*, a leaf of the foliation under consideration. In case $m = 1$ such leaves are precisely the integral curves of a horizontal vector field defining the one-dimensional horizontal subspace $\mathbb{V}$, which are used in the standard definition of the ACL property on the Heisenberg group [22]. These concepts are discussed further in Section 2 below as well as in [3].

A key point in this work is to deal with the basic fact of life that $\pi_{\mathbb{V}^\perp}$ fails to be Lipschitz on compact sets when the target is metrized as a subset of $\mathbb{H}^n$. However, $\pi_{\mathbb{V}^\perp}$ *is* Lipschitz on compact sets when $\mathbb{V}^\perp$ is metrized as a subset of the *Euclidean* space $\mathbb{R}^{2n+1}$. In our main result, for a given target dimension α between m and the universal upper bound given in (1.4), we quantitatively estimate, in terms of Hausdorff dimension in $\mathbb{V}^\perp$ *equipped with the Euclidean metric from* $\mathbb{R}^{2n+1}$, the size of the set of left cosets of $\mathbb{V}$ that are mapped onto a set of dimension at least α by an arbitrary supercritical Sobolev mapping.

We note that

$$\dim_{\mathbb{H}^n} \mathbb{V}^\perp = 2n + 2 - m,$$

while

$$\dim_{\mathbb{R}^{2n+1}} \mathbb{V}^\perp = 2n + 1 - m.$$

Using the Euclidean Hausdorff dimension to measure the size of the set of left-cosets of a horizontal supgroup that are distorted by a Sobolev mapping allows us to state the main result of the paper cleanly; see Figure 1.1.

Theorem 1.5. *Let f be a continuous mapping in the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{H}^n; Y)$, with $p > 2n+2$. Given a horizontal subgroup $\mathbb{V}$ of $\mathbb{H}^n$ of dimension m, with $1 \le m \le n$, and*

$$\alpha \in \left[m, \frac{pm}{p-(2n+2-m)}\right],$$

it holds that

$$\dim_{\mathbb{R}^{2n+1}} \{a \in \mathbb{V}^\perp : \dim f(a * \mathbb{V}) \ge \alpha\} \le \beta(p, m, \alpha),$$

where

$$\beta(p,m,\alpha) = \begin{cases} (2n+1-m) - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[m, \frac{pm}{p-2}\right] \\ (2n+2-m) - p\left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[\frac{pm}{p-2}, \frac{pm}{p-(2n+2-m)}\right]. \end{cases} \tag{1.6}$$

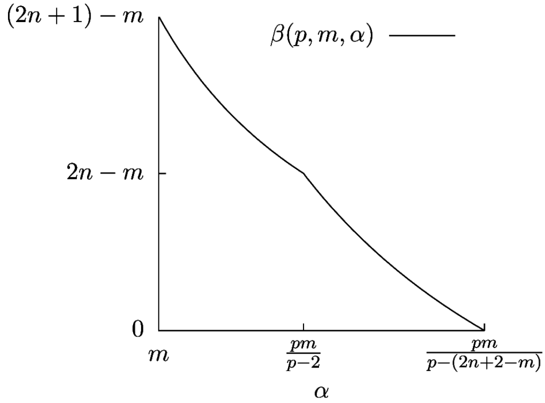

Figure 1.1. The quantity $\beta(p, m, \alpha)$ of Theorem 1.5 as a function of α.

Remark 1.6. The bifurcated nature of the conclusion of Theorem 1.5 is typical in the geometric measure theory of the Heisenberg and more general Carnot groups. Such phenomena appear in a natural way in comparison results between Hausdorff dimensions in terms of Euclidean and Carnot-Carathéodory metrics [7]. This is also indicated by recent results about the behavior of Hausdorff dimension under generic projections and slicing in Heisenberg groups [2,3]. In the present case, the bifurcation of the two different formulas occurs when

$$\beta\left(p, m, \frac{pm}{p-2}\right) = 2n - m = \dim_{\mathbb{R}^{2n}} V^{\perp}.$$

Remark 1.7. Using the Dimension Comparison Theorem of [6] and [7], the estimate

$$\dim_{\mathbb{R}^{2n+1}}\{a \in \mathbb{V}^{\perp} : \dim f(a * \mathbb{V}) \geq \alpha\} \leq \beta(p, m, \alpha)$$

for the Euclidean Hausdorff dimension of the exceptional set can be converted into an estimate of its Heisenberg Hausdorff dimension:

$$\dim_{\mathbb{H}^n}\left\{a \in \mathbb{V}^{\perp} : \dim f(a * \mathbb{V}) \geq \alpha\right\}$$
$$\leq \begin{cases} (2n+2-m) - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[m, \frac{pm}{p-2}\right] \\ (2n+3-m) - p\left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[\frac{pm}{p-2}, \frac{pm}{p-(2n+1-m)}\right] \\ 2\left((2n+2-m) - p\left(1 - \frac{m}{\alpha}\right)\right) & \alpha \in \left[\frac{pm}{p-(2n+1-m)}, \frac{pm}{p-(2n+2-m)}\right]. \end{cases}$$

It seems unlikely that this estimate is sharp. It would be very interesting to know if the more natural estimate

$$\dim_{\mathbb{H}^n}\{a \in \mathbb{V}^{\perp} : \dim f(a * \mathbb{V}) \geq \alpha\} \leq (2n+2-m) - p\left(1 - \frac{m}{\alpha}\right)$$

is valid. This estimate would have followed from Theorem 1.3 if $\pi_{\mathbb{V}^{\perp}}$ were David-Semmes m-regular, which it is not.

Remark 1.8. Theorem 1.5 provides the expected sharp estimates at the endpoints $\alpha = m$ and $\alpha = \frac{pm}{p-((2n+2)-m)}$. The estimate when $\alpha = m$ is trivial to verify, and taking $f : \mathbb{H}^n \to \mathbb{H}^n$ to be the identity mapping shows that it cannot be improved. On the other hand, when α is the universal upper bound provided by Theorem 1.4, we have

$$\beta\left(p, m, \frac{pm}{p-((2n+2)-m)}\right) = 0,$$

as expected. However, we do not recover Theorem 1.4 as a special case as the conclusion of Theorem 1.5 does not assert that the exceptional set $\{a \in \mathbb{V}^{\perp} : \dim f(a * \mathbb{V}) > \alpha\}$ has $\mathcal{H}^{\beta}_{\mathbb{R}^{2n+1}}$ measure zero.

The sharpness of Theorem 1.5 when $\alpha \in (m, \frac{pm}{p-((2n+2)-m)})$ is still unclear. However, we are able to construct examples of Sobolev mappings that distort the dimension of a large set of leaves by a small amount. Our construction is based on a similar example given in [5].

For the convenience of the reader we formulate this result only for the case of the first Heisenberg group $\mathbb{H} = \mathbb{H}^1$ and the foliation by left cosets of the x-axis $\mathbb{V}_x$. In this setting, Theorem 1.5 yields

$$\dim_{\mathbb{R}^3}\{a=(0,y,t)\in\mathbb{H} : \dim f(a*\mathbb{V}_x)\geq\alpha\}\leq \begin{cases} 2-\frac{p}{2}\left(1-\frac{1}{\alpha}\right) & \alpha\in\left[1,\frac{p}{p-2}\right] \\ 3-p\left(1-\frac{1}{\alpha}\right) & \alpha\in\left[\frac{p}{p-2},\frac{p}{p-3}\right]. \end{cases}$$

Theorem 1.9. *Let $\mathbb{V}_x$ denote the horizontal subgroup defined by the x-axis in $\mathbb{H}$, and let $p > 4$. For each*

$$\alpha \in \left(1, \frac{p}{p-2}\right)$$

there is a compact set $E_\alpha \subseteq \mathbb{V}_x^\perp$ and a continuous mapping $f \in \mathrm{W}^{1,p}(\mathbb{H};\mathbb{R}^2)$ such that

$$0 < \mathcal{H}_{\mathbb{R}^3}^{2-p\left(1-\frac{1}{\alpha}\right)}(E_\alpha) < \infty$$

and $\dim f(a*\mathbb{V}) \geq \alpha$ *for every $a \in E_\alpha$.*

We note that the dimension of E_α given above is strictly smaller that the corresponding estimate given by $\beta(p,1,\alpha)$ in Theorem 1.5. In particular, the dimension of E_α tends to 0 when α tends to $\frac{p}{p-2}$, whereas the estimate given by Theorem 1.5 in this setting is

$$\beta\left(p,1,\frac{p}{p-2}\right) = 1.$$

See Figure 1.2. It would be very interesting to improve this situation.

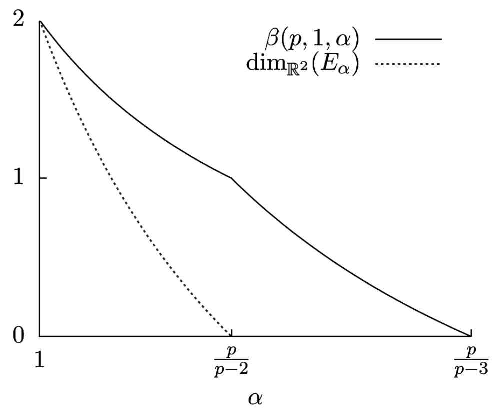

Figure 1.2. The quantity $\beta(p,1,\alpha)$ of Theorem 1.5 and the dimension of the set E_α of Theorem 1.9.

The main idea behind Theorem 1.5 is inspired by Gehring's proof [13] of the ACL property of Euclidean quasiconformal mappings. As mentioned above, for our problem, the first major difficulty in applying this idea arises from the fact that in Heisenberg groups, the projection $\pi_{\mathbb{V}^\perp}$ defined above is *not* Lipschitz on compact sets. In his amended proof of the ACL property for Heisenberg quasiconformal maps, Mostow overcame this complication with a clever argument involving projection to a lower dimensional subspace. We review this complication and Mostow's amended proof [22,27] for the ACL property of Heisenberg quasiconformal maps in Section 3. In our setting, a second major difficulty arises due to the fact that projection into such lower dimensional subspaces can in principle eliminate the desired exceptional set. This difficulty was not present in Mostow's original argument, which used a Fubini-type argument for the Lebesgue measure on the image subspace. We overcome this second difficulty by coupling projection and slicing techniques of Mattila [24] with both Gehring's original argument in the Euclidean setting and Mostow's amended proof in the Heisenberg setting. The eventual argument retains many of the key ideas and ingredients due to Gehring and Mostow, but goes significantly beyond these by taking advantage of more recent developments of geometric measure theory.

The paper is organized as follows. In Section 2, we establish notation and conventions for the Heisenberg groups and their homogeneous subgroups. Section 3 contains the proof of Theorem 1.5. Outlines of Gehring's and Mostow's proofs of the ACL property of quasiconformal mappings are included to clarify the structure of our argument. Section 4 contains the example asserted in Theorem 1.9.

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2. Notation and properties of the Heisenberg group

We employ standard notation for metric spaces. Given a metric space (X, d), a point $x \in X$ and a radius $r > 0$, we denote

$$B_X(x,r) = \{y \in X : d(x,y) < r\} \text{ and } \overline{B}_X(x,r) = \{y \in X : d(x,y) \leq r\}.$$

The open r-neighborhood of a set $A \subseteq X$ is denoted

$$\mathcal{N}_X(A,r) = \bigcup_{x \in A} B_X(x,r). \tag{2.1}$$

Where it will not cause confusion, we will suppress reference to the ambient space (X, d), and a similar convention will hold for all quantities that depend implicitly on (X, d).

We write $A \lesssim B$, respectively $A \gtrsim B$ to indicate that the inequality $A \le CB$, respectively $B \le CA$ holds, where C is a constant depending only on suitable data (which will be indicated in practice or clear from context). We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

2.1. Basic properties and notation

The Heisenberg group $\mathbb{H}^n$, $n \in \mathbb{N}$, is the unique step two nilpotent stratified Lie group with topological dimension $2n+1$ and one dimensional center. We denote $\mathbb{H}^1 = \mathbb{H}$. As a set, we identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ equipped with coordinate system $(x_1, y_1, \dots, x_n, y_n, t)$, which we also denote by (z, t). Given points $a = (z, t)$ and $a_0 = (z', t')$, the group law on $\mathbb{H}^n$ is defined by

$$a * a' = (z + z', t + t' + 2\omega(z, z'))$$

where $\omega(z, z') = \sum_{i=1}^{n}(x_i y_i' - x_i' y_i)$ is the standard symplectic form on $\mathbb{R}^{2n}$. The group $\mathbb{H}^n$ is equipped with a left-invariant metric $d_{\mathbb{H}^n}(a, a') = ||a^{-1} * a'||_{\mathbb{H}^n}$ via the *Korányi norm*

$$||a||_{\mathbb{H}^n} = (||z||_{\mathbb{R}^{2n}}^4 + |t|^2)^{1/4}.$$

The metric space $(\mathbb{H}^n, d_{\mathbb{H}^n})$ is proper and Ahlfors $(2n+2)$-regular when equipped with its Haar measure (which agrees up to constants with both the Lebesgue measure in the underlying Euclidean space $\mathbb{R}^{2n+1}$ and the $(2n+2)$-dimensional Hausdorff measure $\mathcal{H}_{\mathbb{H}^n}^{2n+2}$ in the Korányi metric $d_{\mathbb{H}^n}$). It is known that the metric measure space $(\mathbb{H}^n, d_{\mathbb{H}^n}, \mathcal{H}_{\mathbb{H}^n}^{2n+2})$ supports a p-Poincaré inequality for every $1 \le p < \infty$; see [15, Chapter 11] and the references therein.

The Heisenberg group $\mathbb{H}^n$ admits a one-parameter family of *intrinsic dilations* $\{\delta_r : \mathbb{H}^n \to \mathbb{H}^n\}_{r>0}$ defined, for a point $a = (z, t) \in \mathbb{H}^n$, by

$$\delta_r(a) = (rz, r^2 t).$$

These dilations commute with the group law and are homogeneous of order one with respect to the Korányi norm, *i.e.*,

$$\delta_r(a) * \delta_r(a') = \delta_r(a * a') \text{ and } ||\delta_r(a)||_{\mathbb{H}^n} = r||a||_{\mathbb{H}^n}.$$

2.2. Homogeneous subgroups of $\mathbb{H}^n$

A subgroup of $\mathbb{H}^n$ is *homogeneous* if it is invariant under intrinsic dilations. Homogeneous subgroups come in two types. A homogeneous subgroup is called *horizontal* if it is of the form $V \times \{0\}$ for an isotropic subspace V of the symplectic space $\mathbb{R}^{2n}$; recall that V is *isotropic* if $\omega|_V = 0$. Every homogeneous subgroup that is not horizontal contains the t-axis; these subgroups are called *vertical*. Any horizontal subgroup $\mathbb{V} = V \times \{0\}$ defines a semidirect decomposition $\mathbb{H}^n = \mathbb{V}^\perp \ltimes \mathbb{V}$ where $\mathbb{V}^\perp = V^\perp \times \mathbb{R}$ is the *vertical complement* of $\mathbb{V}$; here $V^\perp$ denotes the usual orthogonal complement of V in $\mathbb{R}^{2n}$.

Since ω vanishes on isotropic subgroups, the restriction of the Korányi metric to horizontal subgroups coincides with the Euclidean metric. Consequently,

$$\dim_{\mathbb{H}^n} \mathbb{V} = \dim_{\mathbb{R}^{2n+1}} \mathbb{V} = \dim V$$

for each horizontal homogeneous subgroup; we write $\dim \mathbb{V}$ without any subscript in this case. In the remainder of the paper, we will be working with a fixed horizontal homogeneous subgroup $\mathbb{V}$, and so

Unless otherwise noted, we will denote $\dim \mathbb{V}$ *by the letter* m. (2.2)

On the other hand, the Heisenberg metric on the vertical complement $\mathbb{V}^\perp$ differs drastically from the Euclidean metric on $(V^\perp \times \mathbb{R}) \subseteq \mathbb{R}^{2n+1}$; recall that the t-axis has Hausdorff dimension 2 in the Heisenberg metric. Since the integer $n \geq 1$ will be fixed throughout, given a subset $A \subseteq \mathbb{V}^\perp$, we write

$$\dim_{\mathbb{H}} A := \dim_{\mathbb{H}^n} A, \text{ and } \dim_{\mathbb{R}} A := \dim_{\mathbb{R}^{2n+1}} A.$$

We note that this applies to arbitrary subsets, not just vector subspaces. A similar convention will be used for all notions that depend on the choice of Euclidean or Heisenberg metric.

We make the further convention that

Unless otherwise noted, we will denote $\dim_{\mathbb{R}} \mathbb{V}^\perp$ *by* w. (2.3)

In this notation,

$$\dim_{\mathbb{R}} \mathbb{V}^\perp = w = (2n+1) - m, \text{ and } \dim_{\mathbb{H}} \mathbb{V}^\perp = w + 1 = (2n+2) - m. \quad (2.4)$$

As mentioned in the introduction, the semidirect product decomposition $\mathbb{H}^n = \mathbb{V}^\perp \ltimes \mathbb{V}$ defines maps

$$\pi_{\mathbb{V}} \colon \mathbb{H}^n \to \mathbb{V} \text{ and } \pi_{\mathbb{V}^\perp} \colon \mathbb{H}^n \to \mathbb{V}^\perp$$

by the formulas $\pi_{\mathbb{V}}(a) = a_{\mathbb{V}}$ and $\pi_{\mathbb{V}^\perp}(a) = a_{\mathbb{V}^\perp}$, where $a = a_{\mathbb{V}^\perp} * a_{\mathbb{V}}$. The map $\pi_{\mathbb{V}^\perp}$ is not Lipschitz on compact sets when $\mathbb{V}^\perp$ is equipped with the Heisenberg metric, but it is Lipschitz on compact sets when $\mathbb{V}^\perp$ is equipped with the Euclidean metric inherited from $\mathbb{R}^{2n+1}$. The proof of this fact and further information about the metric and measure-theoretic properties of these projection maps can be found in [3] and [8, Section 6.3].

2.3. Sobolev mappings on $\mathbb{H}^n$

All results stated in this paper are given for globally defined mappings only for convenience. The methods in use are local in nature and pass without difficulty to mappings defined on open subsets of the ambient space.

We now discuss the definition of a continuous Sobolev mapping defined on $\mathbb{H}^n$ and taking values in a metric space Y. A standard definition of a p-Sobolev mapping, $1 \leq p \leq \infty$, has two requirements: the mapping itself should be p-integrable, and the norm of the weak differential of the mapping should be p-integrable. As we consider only continuous mappings and our results are local in nature, only the second requirement is relevant.

There are many ways to define the class of continuous p-Sobolev mappings between metric spaces. We adopt the simplest and say that a continuous mapping $f : \mathbb{H} \to Y$ is in the *local Sobolev class* $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{H}; Y)$ if there is an *upper gradient* $g : \mathbb{H}^n \to [0, \infty]$ that is locally p-integrable with respect to $\mathcal{H}^{2n+2}_{\mathbb{H}}$, *i.e.*, a Borel function $g \in \mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathbb{H}^{\mathrm{n}}, \mathcal{H}^{2\mathrm{n}+2}_{\mathbb{H}})$ such that for every rectifiable path $\gamma : [0, 1] \to \mathbb{H}^n$,

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma g\, ds,$$

where ds refers to integration with respect to arclength. The upper gradient approach, developed by Cheeger [11], Heinonen and Koskela [17], and Shanmugalingam [28], is suitable in the general setting of doubling metric measure spaces that support a Poincaré inequality. For a thorough discussion of this and other possible approaches to the class of Sobolev mappings between metric spaces, see [18].

Aside from the definition, we shall only need one property of Sobolev mappings on $\mathbb{H}^n$. Namely, if $f : \mathbb{H}^n \to Y$ is a continuous mapping in the Sobolev class $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{H}^n; Y)$ with $p > 2n + 2$, then *Morrey's estimate* holds: there is a constant $c \geq 1$ and a dilation factor $\sigma \geq 1$, both depending only on n and p, such that for any Heisenberg ball $B \subseteq \mathbb{H}^n$,

$$\operatorname{diam} f(B) \leq c(n, p)(\operatorname{diam} B)^{1-\frac{2n+2}{p}} \left(\int_{\sigma B} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}} \right)^{1/p}. \tag{2.5}$$

For a proof of (2.5), see [15] and [18]. As mentioned in the introduction, every supercritical Sobolev mappings in this setting has a Hölder continuous representative, and so our a priori assumption of continuity, while necessary for the validity of our results, is not onerous.

3. The proof of Theorem 1.5

The kernel of the proof of Theorem 1.5 can be traced back to Gehring's proof of the ACL property of metrically defined quasiconformal mappings [13, Lemma 9]. This argument was at first incorrectly applied to the Heisenberg setting by Mostow [26], who mistakenly asserted that the vertical projection map $\pi_{\mathbb{V}^\perp}$ is Lipschitz in the Heisenberg metric. This error was ingeniously overcome by Mostow [27], and a simple presentation of the correct proof can be found in [22].

Roughly speaking, adapting Gehring's original proof to the setting of Theorem 1.5 will provide the claimed value of $\beta(p, m, \alpha)$ when α is large, and adapting

Mostow's correct proof will provide the claimed value when α is small. There are significant obstacles to adapting Mostow's proof methods to the question of frequency of dimension distortion, as discussed in Section 3.4 below. The main tool in overcoming these obstacles is the slicing and projection machinery developed by Mattila [23,24].

3.1. Gehring's method

To motivate and organize our proof, we first give a brief outline of the proof of the fact that a quasiconformal homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ has the ACL property.

Using the coordinate system (x, y, t) for $\mathbb{R}^3$, for the moment we denote the x-axis by V_x, and set $\pi : \mathbb{R}^3 \to V_x^\perp$ to be the standard Euclidean orthogonal projection.

We will show that for a closed ball K containing the origin,

$$\mathcal{H}^2_{\mathbb{R}^3}\Big(\{a \in V_x^\perp \cap K : \mathcal{H}^1_{\mathbb{R}^3}(f(a + V_x)) = \infty\}\Big) = 0. \tag{3.1}$$

This is not quite enough to show the ACL property, but provides sufficient intuition for our purposes.

We outline the three key steps of the proof that (3.1) holds when f is quasiconformal.

(i) Given $a = V_x^\perp \cap K$, if

$$\liminf_{r\to 0} \frac{\mathcal{H}^3_{\mathbb{R}^3}(f(\mathcal{N}_{\mathbb{R}^3}(a + V_x, r) \cap K))}{r^2} < \infty,$$

then $\mathcal{H}^1_{\mathbb{R}^3}(f((a + V_x) \cap K)) < \infty$. Recall the notation for neighborhoods from (2.1). This relationship between Minkowski content and Hausdorff measure can be seen in this quasiconformal case by using the standard distortion estimate

$$\operatorname{diam} f(B) \simeq \Big(\mathcal{H}^3_{\mathbb{R}^3}(f(B))\Big)^{\frac{1}{3}},$$

which holds for any ball $B \subseteq \mathbb{R}^3$, along with a covering argument and Hölder's inequality;

(ii) Define a Radon measure m on $V_x^\perp \cap K$ so that for each $a \in V_x^\perp \cap K$,

$$m(B_{V_x^\perp}(a, r) \cap K) = \mathcal{H}^3_{\mathbb{R}^3}(f(\pi^{-1}(B_{V_x^\perp}(a, r)) \cap K));$$

(iii) By the differentiation theorem for Radon measures [24, Theorem 2.12], the derivative of this measure with respect to two-dimensional Hausdorff measure on $V_x^\perp \cap K$ exists and is finite $\mathcal{H}^2_{V_x^\perp \cap K}$-almost everywhere. Thus

$$\lim_{r\to 0} \frac{\mathcal{H}^3_{\mathbb{R}^3}(f(\pi^{-1}(B_{V_x^\perp}(a, r)) \cap K)))}{r^2} < \infty$$

for $\mathcal{H}^2_{V_x^\perp \cap K}$-almost every point $a \in V_x^\perp \cap K$. Applying Step (i) now completes the proof, as in this setting we have

$$\pi^{-1}(B_{V_x^\perp}(a, r)) = \mathcal{N}_{\mathbb{R}^3}(a + V_x, r). \tag{3.2}$$

3.2. Adapting Gehring's method to the Heisenberg groups

We remind the reader of our notational conventions for the real and Heiseneberg dimensions for the fixed horizontal subgroup $\mathbb{V}$ and its complementary subspace $\mathbb{V}^\perp$ given in (2.2), (2.3), and (2.4).

The goal of this section is to prove the following statement, which in particular gives the desired estimate in Theorem 1.5 when

$$\alpha \in \left[\frac{pm}{p-2}, \frac{pm}{p-(w+1)}\right].$$

Proposition 3.1. *Let Y be an arbitrary metric space and let $f : \mathbb{H}^n \to Y$ be a mapping in the Sobolev space $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{H}; Y)$ for some $p > 2n+2$. Given a horizontal subgroup $\mathbb{V}$ of $\mathbb{H}^n$ of dimension $1 \le m \le n$, and*

$$\alpha \in \left[m, \frac{pm}{p-(w+1)}\right],$$

it holds that

$$\dim_{\mathbb{R}}\{a \in \mathbb{V}^\perp : \dim f(a * \mathbb{V}) \ge \alpha\} \le (w+1) - p\left(1 - \frac{m}{\alpha}\right).$$

For the remainder of this subsection, we assume the hypotheses of Proposition 3.1. In addition, we denote by K the closure of an arbitrary bounded neighborhood of the origin in $\mathbb{H}^n$, and let K' be the closure of a bounded neighborhood of the origin in $\mathbb{H}^n$ that contains K in its interior.

As we wish to estimate the Euclidean dimension of a subset of $\mathbb{V}^\perp$, we will consider Euclidean balls in $\mathbb{V}^\perp$. For ease of notation, we define W to be the metric space $(\mathbb{V}^\perp, d_{\mathbb{R}^{2n+1}})$, so that for $a \in \mathbb{V}^\perp$ and $r > 0$

$$B_W(a, r) = \left\{a' \in \mathbb{V}^\perp : d_{\mathbb{R}^{2n+1}}(a, a') < r\right\}. \tag{3.3}$$

The estimate in Proposition 3.1 is trivially true when $\alpha = m$, so we need only consider the case that $\alpha > m$. Define

$$E_\alpha = \left\{a \in \mathbb{V}^\perp : \mathcal{H}^\alpha(f((a * \mathbb{V}) \cap K)) > 0\right\}.$$

By basic properties of Hausdorff measure and dimension, it suffices to show

$$\dim_{\mathbb{R}} E_\alpha \le (w+1) - p\left(1 - \frac{m}{\alpha}\right). \tag{3.4}$$

By the countable additivity of Hausdorff measure, the following lemma allows us to assume without loss of generality that E_α is compact.

Lemma 3.2. *The set E_α is a countable union of compact sets.*

Proof. As closed and bounded sets in $\mathbb{R}^{2n+1}$ are compact, it suffices to show that E_α is a countable union of closed sets, which may then be decomposed into countably many closed and bounded parts. Since the α-dimensional Hausdorff measure and the α-dimensional Hausdorff content $\mathcal{H}^\alpha_\infty$ have the same null sets, it suffices to show that for each $n \in \mathbb{N}$, the set

$$E_\alpha(n) = \left\{ a \in \mathbb{V}^\perp : \mathcal{H}^\alpha_\infty\big(f((a * \mathbb{V}) \cap K)\big) \geq \frac{1}{n} \right\}$$

is closed. Let $\{a_j\}_{j \in \mathbb{N}} \subseteq E_\alpha(n)$ be a sequence converging to a point $a \in \mathbb{V}^\perp$. Since f and $\pi_{\mathbb{V}^\perp}$ are continuous, for every $\epsilon > 0$, there is an index $j(\epsilon) \in \mathbb{N}$ such that if $j \geq j(\epsilon)$, then

$$f((a_j * \mathbb{V}) \cap K)) \subseteq \mathcal{N}_Y\big(f((a * \mathbb{V}) \cap K)), \epsilon\big).$$

If $a \notin E_\alpha(n)$, then there is a cover $\{B_Y(y_i, r_i)\}_{i \in \mathbb{N}}$ of $f((a * \mathbb{V}) \cap K)$ by open balls such that

$$\sum_{i \in \mathbb{N}} r_i^\alpha < \frac{1}{n}.$$

Since $f((a * \mathbb{V}) \cap K)$ is compact, we may find $\epsilon > 0$ such that the neighborhood $\mathcal{N}_Y(f((a * \mathbb{V}) \cap K), \epsilon)$ is also covered by $\{B_Y(y_i, r_i)\}_{i \in \mathbb{N}}$. This implies that

$$\mathcal{H}^\alpha_\infty(f((a_j * \mathbb{V}) \cap K)) < \frac{1}{n}$$

for all $j \geq j(\epsilon)$, which yields the desired contradiction. □

We now establish a version of Step (i) in Gehring's method, which provides a sufficient condition for the desired bound on the dimension of the image of a line segment under f. It is only in the proof of this statement that we use the Morrey estimate.

Proposition 3.3. *Let $m \leq \alpha \leq p$. If*

$$\liminf_{r \to 0} \frac{\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V}, r) \cap K'} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}}}{r^{(w+1)-p(1-\frac{m}{\alpha})}} < \infty,$$

then $\mathcal{H}^\alpha(f(((a * \mathbb{V}) \cap K)) < \infty$.

Proof of Proposition 3.3. Fix $\epsilon > 0$. By the Morrey estimate (2.5), there is a constant $\sigma \geq 1$ such that for $q \in \mathbb{H}^n$ and $r > 0$, the Heisenberg ball $B_{\mathbb{H}}(q, r)$ satisfies

$$\operatorname{diam} f(B_{\mathbb{H}}(q, r)) \lesssim r^{1-\frac{2n+2}{p}} \left(\int_{B_{\mathbb{H}}(q, \sigma r)} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}} \right)^{1/p}. \tag{3.5}$$

Here the constant of comparability depends only on n and p.

By the uniform continuity of f on compact sets, we may find $\epsilon' > 0$ so that if B is a Heisenberg ball that intersects K and has radius no greater than ϵ', then $B \subseteq K'$ and $\operatorname{diam} f(B) < \epsilon$ Let $r < \epsilon'$. Since $a * \mathbb{V} \subseteq \mathbb{H}^n$ is isometric to $\mathbb{R}^m$ equipped with the Euclidean metric, we may find a cover of $(a * \mathbb{V}) \cap K$ by Heisenberg balls $B_1, \ldots, B_N$ of radius r/σ centered on $a * \mathbb{V}$, where $N \leq Mr^{-m}$ and M depends only on K and σ. We may moreover assume that there is a number $D \geq 1$, depending only on σ, such that no point of the Heisenberg group lies in more than D of the dilated balls $\sigma B_1, \ldots, \sigma B_N$. Denoting by $\mathcal{H}^{\alpha,\epsilon}$ the α-dimensional Hausdorff pre-measure calculated by considering coverings by sets of diameter no greater than $\epsilon > 0$, we see that

$$\mathcal{H}^{\alpha,\epsilon}\big(f((a * \mathbb{V}) \cap K)\big) \leq \sum_{i=1}^{N} \big(\operatorname{diam} f(B_i)\big)^{\alpha}.$$

Hence, by (3.5), Hölder's inequality, the bounded overlap property of the cover $\{\sigma B_i\}_{i=1}^{N}$, and the estimate $N \leq Mr^{-m}$ yield

$$\begin{aligned}\mathcal{H}^{\alpha,\epsilon}\big(f((a * \mathbb{V}) \cap K)\big) &\leq r^{(1-((2n+2)/p))\alpha} \sum_{i=1}^{N} \left(\int_{\sigma B_i} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}\right)^{\alpha/p} \\ &\lesssim r^{(1-((2n+2)/p))\alpha} N^{1-(\alpha/p)} \left(\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}\right)^{\alpha/p} \\ &\lesssim r^{(\alpha-m)-(w+1)\alpha/p} \left(\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}\right)^{\alpha/p}.\end{aligned}$$

Above, the constants of comparability now depend on n, p, and the compact set K.

The hypothesis implies that there is a number $c > 0$, depending on a, such that if r is sufficiently small, then

$$\left(\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}\right)^{\alpha/p} \leq c r^{(w+1)\alpha/p-(\alpha-m)}.$$

Thus there is a quantity $c' > 0$, independent of ϵ, such that $\mathcal{H}^{\alpha,\epsilon}(f(a * \mathbb{V})) \leq c'$. Letting ϵ tend to zero yields the desired result. □

Now, we establish a version of Step (ii) in Gehring's method. We define a measure Φ on $\mathbb{V}^{\perp}$, depending on g and K, by the following Carathéodory construction. For $\epsilon > 0$ and $E \subseteq \mathbb{V}^{\perp}$, set

$$\Phi_\epsilon(E) = \inf \left\{ \sum_{i \in \mathbb{N}} \int_{\pi_{\mathbb{V}^{\perp}}^{-1}(B_W(a_i,r_i))\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2} \right\}$$

where the infimum is taken over all countable covers $\{B_W(a_i, r_i)\}$ of E by Euclidean balls centered in $\mathbb{V}^{\perp}$ of radius less than ϵ (see (3.3)). Then set

$$\Phi(E) = \lim_{\epsilon \to 0} \Phi_\epsilon(E).$$

Since $g^p \in L^1(\mathbb{H}^n)$, the hypotheses of [24, Theorem 4.2] apply and the set function Φ defines a Borel regular measure on W. It follows from the sub-additivity of the integral that given $a \in \mathbb{V}^\perp$ and $r > 0$,

$$\Phi(B_W(a,r)) = \int_{\pi_{\mathbb{V}^\perp}^{-1}(B_W(a,r))\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2} .$$

Hence Φ is a Radon measure on W. Note that if f is a smooth diffeomorphism of $\mathbb{H}^n$ to itself, and g is the norm of the differential of f, then $\Phi(B_W(a,r))$ is comparable to the $(2n+2)$-dimensional Hausdorff measure of the image $f(\pi_{\mathbb{V}^\perp}^{-1}(B_W(a,r))\cap K)$, in analogy to the quasiconformal setting.

We now give the analog of Step (iii) in Gehring's method, thereby completing the proof of Proposition 3.1.

Proof of Proposition 3.1. As mentioned above, it suffices to prove the estimate (3.4). Suppose, by way of contradiction, that there exists a number t such that

$$(w+1) - p\left(1 - \frac{m}{\alpha}\right) < t < \dim_W E_\alpha.$$

Then $\mathcal{H}^t(E_\alpha) = \infty$. By Lemma 3.2 we may reduce to the case that E_α is compact, and then find a compact subset $E \subseteq E_\alpha$ such that $0 < \mathcal{H}^t(E) < \infty$ [24, Theorem 8.19]. By Frostman's lemma [24, Theorem 8.17], there exists a nonzero Radon measure m supported on E with the property that $m(B_W(a,r)) \le r^t$ for all $a \in W$ and all sufficiently small $r > 0$.

By applying the differentiation theorem for Radon measures [24, Theorem 2.12] to Ψ and m, we see that for m-almost every point $a \in W$,

$$\lim_{r\to 0} \frac{\int_{\pi_{\mathbb{V}^\perp}^{-1}(B_W(a,r))\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}}{m(B_W(a,r))} < \infty. \tag{3.6}$$

Since $\pi_{\mathbb{V}^\perp} : \mathbb{H}^n \to \mathbb{V}^\perp$ is Lipschitz on compact sets when $\mathbb{V}^\perp$ is equipped with the Euclidean metric, there is some $L \ge 1$ such that

$$\mathcal{N}_{\mathbb{H}}(a * \mathbb{V}, r) \cap K' \subseteq \pi_{\mathbb{V}^\perp}^{-1}(B_W(a, Lr))$$

for all $a \in \mathbb{V}^\perp$. This fact, the Frostman estimate on m, and (3.6) imply that for m-almost every $a \in W$,

$$\lim_{r\to 0} \frac{\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}}{r^t} < \infty.$$

Since $t > \beta$, we may find a number $\alpha' < \alpha$ such that

$$t = (w+1) - p\left(1 - \frac{m}{\alpha'}\right).$$

As we have assumed $t < \dim_{\mathbb{R}} E_\alpha$, and clearly $\dim_{\mathbb{R}} E_\alpha < w + 1$, we may also assume that $\alpha' > m$. Proposition 3.3 now implies that for m-almost every point $a \in E$, it holds that $\mathcal{H}^{\alpha'}(f((a * \mathbb{V}) \cap K)) < \infty$. However, since the non-zero measure m is supported on E, which is a subset of E_α, and $\alpha' < \alpha$, this is a contradiction. □

Remark 3.4. The argument given in this section generalizes to the metric space setting without substantial changes. We record this in the following statement and refer to [8] for the relevant definitions.

Theorem 3.5. *Let n be a positive integer. Assume that (X, d, μ) is a proper metric measure space that is locally Q-homogeneous and supports a local Q-Poincaré inequality, $Q \geq n$. Let $\pi : X \to \mathbb{R}^n$ be a Lipschitz map such that for each point $a \in \mathbb{R}^n$, the preimage $\pi^{-1}(a)$ is locally s-homogeneous, $0 \leq s < Q$. If $p > Q$ and $f : X \to Y$ is a continuous mapping into a metric space with an upper gradient in $\mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathrm{X})$, then for each $\alpha \in (s, ps/(p - Q + s)$,*

$$\dim\{a \in \mathbb{R}^n : \mathcal{H}^\alpha(f(\pi^{-1}(a))) > 0\} \leq (Q - s) - p\left(1 - \frac{s}{\alpha}\right).$$

We note that Theorem 3.5 implies Theorem 1.3 in the case that the parameterizing space of the David–Semmes foliation is Euclidean.

3.3. Mostow's method

We now outline how Mostow adjusted Gehring's method to show that a quasiconformal homeomorphism $f : \mathbb{H} \to \mathbb{H}$ has the ACL property.

Using the coordinate system (x, y, t) for $\mathbb{H}$, for the moment we denote the x-axis by $\mathbb{V}_x$, set W_x to be the metric space $(\mathbb{V}_x^\perp, d_{\mathbb{R}^2})$, which is isometric to the Euclidean plane, and denote by $\pi_{\mathbb{V}_x^\perp} : \mathbb{H} \to W_x$ the Heisenberg projection mapping defined by the splitting $\mathbb{H} = \mathbb{V}_x^\perp \ltimes \mathbb{V}_x$; we emphasize here that the target W_x is equipped with the Euclidean metric.

As null sets for $\mathcal{H}^2_{\mathbb{R}} \llcorner W_x$ coincide with null sets for $\mathcal{H}^3_{\mathbb{H}} \llcorner \mathbb{V}_x^\perp$, we will show that for a closed ball K in $\mathbb{H}$ containing the origin,

$$\mathcal{H}^2_{\mathbb{R}}\Big(\{a \in W_x \cap K : \mathcal{H}^1_{\mathbb{H}}(f(a * \mathbb{V}_x)) = \infty\}\Big) = 0. \tag{3.7}$$

Again, this is not quite enough to show the ACL property, but provides sufficient intuition for our purposes.

We now describe three steps in the proof that f has the ACL property.

$\widetilde{\text{(i)}}$ The first step of Mostow's method is basically same as step (i) of Gehring's method, and Proposition 3.3 has already accomplished its analog in the general setting of Theorem 1.5. Given $a \in W_x \cap K$, if

$$\liminf_{r \to 0} \frac{\mathcal{H}^4_{\mathbb{H}}(f(\mathcal{N}_{\mathbb{H}}(a * \mathbb{V}_x, r) \cap K))}{r^3} < \infty, \tag{3.8}$$

then $\mathcal{H}^1(f((a * \mathbb{V}_x) \cap K)) < \infty$. As before, this relationship between Minkowski content and Hausdorff measure can be seen in this quasiconformal case by using the standard distortion estimate

$$\operatorname{diam} f(B) \simeq \left(\mathcal{H}^4_{\mathbb{H}}(f(B))\right)^{\frac{1}{4}},$$

which holds for any ball $B \subseteq \mathbb{H}$, along with a covering argument and Hölder's inequality;

$(\widetilde{\text{ii}})$ We now diverge from Gehring's method, as the denominator appearing in (3.8) is r^3, and not r^2. We produce a measure, not on W_x as in Gehring's method, but instead on the y-axis $W_{x,t} = \{(0, y, 0) : y \in \mathbb{R}\}$ inside of W_x. Let $\pi_{W_{x,t}} : W_x \to W_{x,t}$ denote the standard Euclidean orthogonal projection. We define a measure m so that for each $y_0 \in W_{x,t}$

$$m(B_{W_{x,t}}(y_0, r) \cap K) = \mathcal{H}^4_{\mathbb{H}}(f \circ \pi^{-1}_{\mathbb{V}_x^\perp} \circ \pi^{-1}_{W_{x,t}}(B_{W_{x,t}}(y_0, r) \cap K));$$

$(\widetilde{\text{iii}})$ As in Gehring's method, we may apply the differentiation theorem for Radon measures [24, Theorem 2.12] to the measure m, but with respect to linear measure on $W_{x,t}$. Thus, for $\mathcal{H}^1$-almost every y_0 in $W_{x,t}$,

$$\lim_{r \to 0} \frac{\mathcal{H}^4_{\mathbb{H}}(f \circ \pi^{-1}_{\mathbb{V}_x^\perp} \circ \pi^{-1}_{W_{x,t}}(B_{W_{x,t}}(y_0, r) \cap K))}{r} < \infty. \tag{3.9}$$

We now claim that if (3.8) fails to hold for a set $\mathcal{C} \subseteq W_x \cap K$ of positive $\mathcal{H}^2$-measure, then (3.9) will fail on $\pi_{W_{x,t}}(\mathcal{C})$, which has positive $\mathcal{H}^1$-measure by Fubini's theorem, yielding a contradiction. The key point in the proof of this claim is the relationship between the sets

$$\pi^{-1}_{\mathbb{V}_x^\perp} \circ \pi^{-1}_{W_{x,t}}(B_{W_{x,t}}(y_0, r)) \text{ and } \mathcal{N}_{\mathbb{H}}((a * \mathbb{V}_x), r).$$

This relationship is clarified by a geometric statement: given $y_0 \in W_{x,t}$, points $a = (0, y_0, t)$ and $a' = (0, y_0, t') \in \pi^{-1}_{W_{x,t}}(y_0)$, and $r > 0$, then

$$\mathcal{N}_{\mathbb{H}}(a * \mathbb{V}_x, r) \cap \mathcal{N}_{\mathbb{H}}(a_0 * \mathbb{V}_x, r) \neq \emptyset$$

implies that

$$|t - t'| \lesssim r^2.$$

The claim now follows by a packing argument, completing the proof.

3.4. Adapting Mostow's method to the Heisenberg groups

We resume the notation of Section 3.1, but now additionally assume that $\alpha \in (m, mp/(p-2))$. We will show that

$$\dim_{\mathbb{R}}\{a \in \mathbb{V}^{\perp} : \mathcal{H}^{\alpha}(f(a * \mathbb{V} \cap K)) > 0\} \leq w - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right).$$

It suffices to show that for $m < \alpha' < \alpha$,

$$\dim_{\mathbb{R}}\{a \in \mathbb{V}^{\perp} : \mathcal{H}^{\alpha'}(f((a * \mathbb{V}) \cap K)) = \infty\} \leq w - \frac{p}{2}\left(1 - \frac{m}{\alpha'}\right). \tag{3.10}$$

As mentioned above, the analog of Step $\widetilde{\text{(i)}}$ in the preceding outline is accomplished in Proposition 3.3. Creating a measure as in Step $\widetilde{\text{(ii)}}$ is complicated by the following issue that arises in application of Fubini's theorem in Step $\widetilde{\text{(iii)}}$. For simplicity, we explain the complication only in the case $n = m = 1$. The right hand side of (3.10) is less than 2, and the projection $\pi_{W_{x,t}}$ maps some sets of dimension less than 2 onto sets of zero $\mathcal{H}^1_{W_{x,t}}$-measure, so no simple application of Fubini's theorem will suffice. We overcome this problem by applying the projection and slicing machinery of Mattila [24] to conclude that for almost every co-dimension 1 subspace of W_x, the corresponding projection of a set of dimension $t > 1$ has positive $\mathcal{H}^{t-1}_{W_x}$-measure.

The following proposition, combined with Proposition 3.3, quickly implies (3.10) and hence completes the proof of Theorem 1.5. Its proof, given Lemma 3.7, implements the strategy given in the previous paragraph.

Proposition 3.6. *Suppose that $m < \alpha < mp/(p-2)$. Set*

$$\mathcal{C}_{\alpha} = \left\{ a \in \mathbb{V}^{\perp} \cap K : \liminf_{r \to 0} \frac{\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}}}{r^{(w+1)-p(1-m/\alpha)}} = \infty \right\}.$$

Then

$$\dim_{\mathbb{R}} \mathcal{C}_{\alpha} \leq w - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right).$$

Before proving Proposition 3.6, we establish some more notation related to the strategy outlined above. Denote

$$\mathbb{S}^{w-1} = \left\{ \theta \in \mathbb{V}^{\perp} : ||\theta||_{\mathbb{R}} = 1 \right\}.$$

For $\theta \in \mathbb{S}^{w-1}$, denote by Θ the one-dimensional subspace of $\mathbb{V}^{\perp}$ generated by θ, and by $\Theta^{\perp}$ its Euclidean orthogonal complement in $\mathbb{V}^{\perp}$. We denote the Euclidean orthogonal projection map by $\pi_{\Theta^{\perp}} : \mathbb{V}^{\perp} \to \Theta^{\perp}$, so that for $\hat{a} \in \Theta^{\perp}$,

$$\pi^{-1}_{\Theta^{\perp}}(\hat{a}) = \{\hat{a} + \tau\theta : \tau \in \mathbb{R}\} \subseteq \mathbb{V}^{\perp}.$$

Note that if θ is parallel to the t-axis, then $\Theta^\perp = V^\perp \times \{0\}$. In what follows we will consider only those θ that lie in a small neighborhood of the t-axis, meaning that $\Theta^\perp$ should be thought of as a small perturbation of $(V^\perp \times \{0\}) \subseteq \mathbb{V}^\perp$.

We equip $\Theta^\perp$ with the restriction of the Euclidean metric on $\mathbb{V}^\perp$, so that for each $\hat{a} \in \Theta^\perp$ and $r > 0$,

$$B_{\Theta^\perp}(\hat{a}, r) = B_W(\hat{a}, r) \cap \Theta^\perp. \tag{3.11}$$

We also associate the $(w-1)$-dimensional Hausdorff measure in the Euclidean metric to $\Theta^\perp$.

We use the projection and slicing machinery of [24] to conclude Proposition 3.6 from the following lemma.

Lemma 3.7. *Suppose that $m < \alpha < mp/(p-2)$. There is a subset $S \subseteq \mathbb{S}^{w-1}$ with $\mathcal{H}^{w-1}(S) > 0$ such that if $\theta \in S$, then the following implication is true for $\mathcal{H}^{w-1}$-almost every point $\hat{a} \in \Theta^\perp$. If $\mathcal{C} \subseteq \pi_{\Theta^\perp}^{-1}(\hat{a}) \subseteq \mathbb{V}^\perp$ has the property*

- *For every $k \in \mathbb{N}$, there is a number $\epsilon(k) > 0$ such that for all $0 < r < \epsilon(k)$ and $a \in \mathcal{C} \cap K$*
$$\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2} \geq k r^{(w+1)-p(1-\frac{m}{\alpha})},$$

then $\mathcal{H}_{\mathbb{R}}^{1-\frac{p}{2}(1-\frac{m}{\alpha})}(\mathcal{C} \cap K) = 0$.

Assuming Lemma 3.7 we complete the proof of Proposition 3.6.

Proof of Proposition 3.6. Recall that

$$\mathcal{C}_\alpha = \left\{ a \in \mathbb{V}^\perp \cap K : \liminf_{r\to 0} \frac{\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}}{r^{(w+1)-p(1-m/\alpha)}} = \infty \right\}.$$

As $\alpha < mp/(p-2)$, it holds that

$$w - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right) > w - 1.$$

Fix any number

$$\beta > w - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right),$$

and towards a contradiction assume that $\mathcal{H}_{\mathbb{R}}^{\beta}(\mathcal{C}_\alpha) > 0$. One can check that $\mathcal{C}_\alpha$ is a Borel set, and so [24, Theorem 8.13] implies that after passing to a subset, we may assume that $\mathcal{H}_{\mathbb{R}}^{\beta}(\mathcal{C}_\alpha) < \infty$ as well.

Let $k \in \mathbb{N}$. For each $a \in \mathcal{C}_\alpha$, there exists a quantity $\epsilon(a,k) > 0$ such that for each $0 < r < \epsilon(a,k)$,

$$\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2} \geq k r^{(w+1)-p(1-m/\alpha)}.$$

For $l \in \mathbb{N}$, define

$$\mathcal{C}_{k,l} = \{a \in \mathcal{C}_\alpha : \epsilon(a,k) \geq 1/l\}.$$

As

$$\mathcal{C}_{k,1} \subseteq \mathcal{C}_{k,2} \subseteq \ldots \subseteq \bigcup_{l \in \mathbb{N}} \mathcal{C}_{k,l} = \mathcal{C}_\alpha,$$

we may choose natural numbers $\{l(k)\}_{k\in\mathbb{N}}$ such that

$$\mathcal{H}^\beta_{\mathbb{R}}(\mathcal{C}_{k,l(k)}) > \left(1 - 2^{-(k+2)}\right) \mathcal{H}^\beta_{\mathbb{R}}(\mathcal{C}_\alpha).$$

It follows that the set

$$\mathcal{C} = \bigcap_{k \in \mathbb{N}} \mathcal{C}_{k,l(k)}$$

also satisfies $0 < \mathcal{H}^\beta_{\mathbb{R}}(\mathcal{C}) < \infty$. Moreover, for each index $k \in \mathbb{N}$, radius $0 < r \leq l(k)^{-1}$, and point $a \in \mathcal{C}$, it holds that

$$\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}} \geq k r^{(w+1)-p(1-m/\alpha)}.$$

Let S be the positive measure subset of $\mathbb{S}^{w-1}$ guaranteed by Lemma 3.7. If $\theta \in S$, then for $\mathcal{H}^{w-1}$-almost every $\hat{a} \in \Theta^\perp$,

$$\mathcal{H}_{\mathbb{R}}^{1-\frac{p}{2}(1-\frac{m}{\alpha})}(\pi^{-1}_{\Theta^\perp}(\hat{a}) \cap \mathcal{C}) = 0. \tag{3.12}$$

However, because $\beta > w-1$, it follows from [24, Theorem 8.9 and Corollary 9.8] that for $\mathcal{H}^{w-1}$-almost every $\theta \in \mathbb{S}^{w-1}$ the Euclidean orthogonal projection $\pi_{\Theta^\perp}(\mathcal{C}) \subseteq \Theta^\perp$ has positive (and finite) $\mathcal{H}^{w-1}$-measure. Hence, by [24, Theorem 10.10], for $\mathcal{H}^{w-1}$-almost every $\theta \in \mathbb{S}^{w-1}$, there is a set $A \in \Theta^\perp$ of positive $\mathcal{H}^{w-1}$-measure such that if $\hat{a} \in A$, then

$$\dim_{\mathbb{R}} \pi^{-1}_{\Theta^\perp}(\hat{a}) \cap \mathcal{C} = \beta - (w-1) > 1 - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right). \tag{3.13}$$

In particular, we may choose $\theta \in S$ and $\hat{a} \in \Theta^\perp$ such that (3.13) and (3.12) hold, which is a contradiction. □

The proof of Lemma 3.7 roughly corresponds to Step $\widetilde{\text{(ii)}}$ in Mostow's method, with the slight perturbation of $\mathbb{V}^\perp$ taken into account. Given $\theta \in \mathbb{S}^{w-1}$, we define an appropriate measure on $\Theta^\perp$. For $\hat{a} \in \Theta^\perp$ and $r > 0$, we define a "tilted slab" in $\mathbb{H}^n$ by

$$\mathrm{Sl}(\hat{a}, r) = \pi^{-1}_{\mathbb{V}^\perp} \circ \pi^{-1}_{\Theta^\perp}(B_{\Theta^\perp}(\hat{a}, r)).$$

We define a measure Ψ on $\Theta^\perp$ by the following Carathéodory construction. For $\epsilon > 0$ and $E \subseteq \Theta^\perp$, set

$$\Psi_\epsilon(E) = \inf \left\{ \sum_{i \in \mathbb{N}} \int_{\mathrm{Sl}(\hat{a}_i, r))\cap K} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}} \right\}$$

where the infimum is taken over all countable covers $\{B_{\Theta^\perp}(\hat{a}_i, r_i)\}$ of E by Euclidean balls centered in $\Theta^\perp$ of radius less than ϵ (see (3.11)). Then set

$$\Psi(E) = \lim_{\epsilon\to 0} \Psi_\epsilon(E).$$

Since $g^p \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, the hypotheses of [24, Theorem 4.2] apply, and so the set function Ψ defines a Borel regular measure on $\Theta^\perp$. It follows from the sub-additivity of the integral that given $\hat{a} \in \Theta^\perp$ and $r > 0$,

$$\Psi(B_{\Theta^\perp}(\hat{a}, r)) = \int_{\mathrm{Sl}(\hat{a},r)\cap K} g^p \, d\mathcal{H}^{2n+2}.$$

Hence Ψ is a Radon measure on $\Theta^\perp$.

We remind the reader that $\Theta^\perp$ is a linear subspace equipped with the Euclidean metric (see (3.11). Hence, by applying the differentiation theorem for Radon measures [24, Theorem 2.12] to Ψ and $\mathcal{H}^{w-1}$, we see that for $\mathcal{H}^{w-1}$-almost every point $\hat{a} \in \Theta^\perp$,

$$\lim_{r\to 0} \frac{\int_{\mathrm{Sl}(\hat{a},r))\cap K} g^p \, d\mathcal{H}^{2n+2}_{\mathbb{H}}}{r^{w-1}} < \infty. \tag{3.14}$$

The following geometric lemma is the key point of the proof of Lemma 3.7; it corresponds to Step $(\widetilde{\mathrm{iii}})$ in Mostow's method.

Lemma 3.8. *There is a set $S \subseteq \mathbb{S}^{w-1}$ of positive $\mathcal{H}^{w-1}$-measure, depending only on K, with the following property. Given $\theta \in S$, $\hat{a} \in \Theta^\perp$, and*

$$a_1 = \hat{a} + t_1\theta \in \pi^{-1}_{\Theta^\perp}(\hat{a}) \subseteq \mathbb{V}^\perp$$
$$a_2 = \hat{a} + t_2\theta \in \pi^{-1}_{\Theta^\perp}(\hat{a}) \subseteq \mathbb{V}^\perp,$$

where $t_1, t_2 \in \mathbb{R}$, if there is $r > 0$ such that

$$\mathcal{N}_{\mathbb{H}}(a_1 * \mathbb{V}, r) \cap \mathcal{N}_{\mathbb{H}}(a_2 * \mathbb{V}, r) \cap K \neq \emptyset, \tag{3.15}$$

then $|t_1 - t_2| \le 8r^2$.

Proof. Assuming that (3.15) holds, we may find elements v_1 and v_2 in $\mathbb{V}$ such that

$$||(a_1 * v_1)^{-1} * (a_2 * v_2)||_{\mathbb{H}} < 2r. \tag{3.16}$$

We denote the Euclidean orthogonal projection of $\mathbb{R}^{2n+1}$ onto $\mathbb{R}^{2n}$ by $\pi_{\mathbb{R}^{2n}}$, and the Euclidean orthogonal projection of $\mathbb{R}^{2n+1}$ onto the t-axis (*i.e.*, the last coordinate of $\mathbb{R}^{2n+1}$) by π_t. For ease of notation, we omit reference to $\pi_{\mathbb{R}^{2n}}$ in the arguments of the symplectic form ω, so that for points $a, a' \in \mathbb{H}^n$, we write

$$\omega(a, a') := \omega\big(\pi_{\mathbb{R}^{2n}}(a), \pi_{\mathbb{R}^{2n}}(a')\big).$$

We note first that by the linearity of $\pi_{\mathbb{R}^{2n}}$ and the fact that ω is bi-linear and anti-symmetric,

$$\begin{aligned} a_1^{-1} * a_2 &= \big((t_2 - t_1)\pi_{\mathbb{R}^{2n}}(\theta), (t_2 - t_1)\pi_t(\theta) + 2\omega(\hat{a} + t_2\theta, \hat{a} + t_1\theta)\big) \\ &= (t_2 - t_1)\left(\pi_{\mathbb{R}^{2n}}(\theta), \pi_t(\theta) + 2\omega(\theta, \hat{a})\right). \end{aligned}$$

Define τ to be the t-component of $(a_1 * v_1)^{-1} * (a_2 * v_2)$. Using the above equation and the fact that both ω and π_t vanish on $\mathbb{V}$, we now compute that

$$\tau = (t_2 - t_1)\left(\pi_t(\theta) + 2\omega(\theta, \hat{a} + v_1 + v_2)\right).$$

As $\hat{a}$, v_1, and v_2 may all be assumed to lie in a fixed compact set depending only on K, whenever θ is in a sufficiently small neighborhood $S \subseteq \mathbb{S}^{w-1}$ of the unit vector in the t-direction,

$$\left|\pi_t(\theta) + 2\omega(\theta, \hat{a} + v_1 + v_2)\right| \geq |\pi_t(\theta)| - \left|2\omega(\theta, \hat{a} + v_1 + v_2)\right| \geq \frac{1}{2}$$

and hence $2|\tau| \geq |t_2 - t_1|$. The definition of the Korányi norm on $\mathbb{H}^n$ and (3.16) now yield

$$4r^2 \geq \frac{|t_2 - t_1|}{2},$$

as desired. □

We now provide the proof of Lemma 3.7, and so complete the proof of Theorem 1.5.

Proof of Lemma 3.7. Let $S \subseteq \mathbb{S}^{w-1}$ be the set provided by Lemma 3.8, and let $\theta \in S$. Recall that $\mathcal{H}^{w-1}$-almost every point $\hat{a} \in \Theta^\perp$ satisfies the differentiation estimate (3.14). Let $\hat{a}$ be such a point, and suppose that $\mathcal{C} \subseteq \pi_{\Theta^\perp}^{-1}(\hat{a}) \subseteq \mathbb{V}^\perp$ has the property that for every $k \in \mathbb{N}$, there is a number $\epsilon(k) > 0$ such that

$$\int_{\mathcal{N}_{\mathbb{H}}(a*\mathbb{V},r)\cap K'} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2} \geq k r^{(w+1)-p(1-\frac{m}{\alpha})} \tag{3.17}$$

for all $0 < r < \epsilon(k)$ and $a \in \mathcal{C} \cap \mathcal{K}$. Working towards a contradiction, we assume that $\mathcal{H}_{\mathbb{R}}^{\gamma}(\mathcal{C} \cap \mathcal{K}) > 0$ where

$$\gamma = 1 - \frac{p}{2}\left(1 - \frac{m}{\alpha}\right).$$

Since $\pi_{\Theta^\perp} \circ \pi_{\mathbb{V}^\perp} : \mathbb{H}^n \to \Theta^\perp$ is Lipschitz on compact sets (recall that we have equipped $\Theta^\perp$ with the Euclidean metric), there is a number $\kappa \geq 1$, depending only on K such that

$$\mathcal{N}_{\mathbb{H}}(a * \mathbb{V}, r/\kappa) \cap K \subseteq \mathrm{Sl}(\hat{a}, r)$$

for all $a \in \pi_{\Theta^\perp}^{-1}(\hat{a})$ provided $r > 0$ is sufficiently small.

Consider a maximal $8(r/\kappa)^2$-separated set $\{a_i\}_{i=1}^{N_r} \subseteq \mathcal{C} \cap K$; as usual we use the Euclidean metric on $\mathcal{C} \subseteq \mathbb{V}^\perp$. Lemma 3.8 implies that the corresponding family $\{\mathcal{N}_{\mathbb{H}}(a_i * \mathbb{V}, r/\kappa) \cap K\}_{i=1}^{N_r}$ is disjoint. Hence, the above statements and (3.17) imply that for sufficiently small $r > 0$,

$$\begin{aligned} N_r k \left(\frac{r}{\kappa}\right)^{(w+1)-p(1-\frac{m}{\alpha})} &\leq \sum_{i=1}^{N_r} \int_{\mathcal{N}(a_i*\mathbb{V}, r/\kappa)\cap K} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2} \\ &\leq \int_{\mathrm{Sl}(\hat{a},r)\cap K} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}. \end{aligned}$$

The assumption that $\mathcal{H}_{\mathbb{R}}^{\gamma}(\mathcal{C} \cap K) > 0$ places a quantitative restriction on the size of a separated set in $\mathcal{C}$. Namely, it holds that

$$\liminf_{r\to 0} N_r r^{2\gamma} > 0.$$

The exponent 2 appears here because $\{a_i\}_{i=1}^{N}$ is a $8(r/\kappa)^2$-separated set. Combining these estimates with the definition of γ shows that

$$k = k \liminf_{r\to 0} r^{-2\gamma+(w+1)-p(1-\frac{m}{\alpha})-(w-1)} \lesssim \liminf_{r\to 0} \frac{\int_{\mathrm{Sl}(\hat{a},r)\cap K} g^p \, d\mathcal{H}_{\mathbb{H}}^{2n+2}}{r^{w-1}}.$$

Letting k tend to infinity contradicts (3.14), and yields the desired result. □

This line of reasoning establishes (3.10) and consequently completes the proof of Theorem 1.5.

4. A mapping that increases the dimension of many lines

We now prove Theorem 1.9. The construction is similar in spirit to those given in [8, Theorem 1.3] and [5, Section 4].

Proof of Theorem 1.9. We consider the foliation of $\mathbb{H}$ by left translates of the horizontal subgroup $\mathbb{V}$ defined by the x-axis; the same construction works for any horizontal subgroup of $\mathbb{H}$. Again, we set $W = (\mathbb{V}^\perp, d_{\mathbb{R}^2})$, and we define for $a \in \mathbb{V}^\perp$ and $s \in \mathbb{R}$

$$a(s) = a * (s, 0, 0).$$

Let $p > 4$ and let $\alpha \in [1, \frac{p}{p-2}]$. By the Dimension Comparison Theorem, it suffices to show that there is a compact set $E \subseteq \mathbb{V}^\perp$ and a continuous mapping $f : \mathbb{H} \to \mathbb{R}^2$ with an upper gradient in $\mathrm{L}^p(\mathbb{H})$ such that

$$\dim_{\mathbb{R}} E = 2 - p\left(1 - \frac{1}{\alpha}\right),$$

and $\dim f(a * \mathbb{V}) = \alpha$ for every $a \in E$.

Let

$$\beta = 2 - p\left(1 - \frac{1}{\alpha}\right)$$

and choose $0 < \sigma < 1$ such that

$$4\sigma^{\beta} = 1.$$

We consider the iterated function system defined by the (Euclidean) similarities $f_i\colon W \to W$, $i = 1, \dots, 4$, where

$$\begin{aligned}
f_1((0, y, t)) &= (0, \sigma y, \sigma t),\\
f_2((0, y, t)) &= (0, \sigma y, \sigma t) + (0, 1 - \sigma, 0),\\
f_3((0, y, t)) &= (0, \sigma y, \sigma t) + (0, 0, 1 - \sigma),\\
f_4((0, y, t)) &= (0, \sigma y, \sigma t) + (0, 1 - \sigma, 1 - \sigma).
\end{aligned}$$

The unique compact invariant set F^{α} of this system is also known as a *four-corner set* or *Garnett set*. The set F^{α} can be expressed explicitly in the following way. Let $I = \{0\} \times [0, 1] \times [0, 1]$ be the (Euclidean) unit square in W. For $m \in \mathbb{N}$, let $\mathcal{S}_m$ denote the sequences $\omega = (\omega_1, \dots, \omega_n)$ of length m with entries in the set $\{1, \dots, 4\}$. We employ the convention that $\mathcal{S}_0$ contains the empty sequence. For $\omega \in \mathcal{S}_m$, define

$$f_{\omega} = f_{\omega_1} \circ \dots \circ f_{\omega_m}.$$

Set $F^{\alpha}_{\omega} = f_{\omega}(I)$. Then

$$F^{\alpha} = \bigcap_{m\in\mathbb{N}} \bigcup_{\omega\in\mathcal{S}_m} F^{\alpha}_{\omega}.$$

The iterated function system satisfies the open set condition, and so

$$0 < \mathcal{H}^{\beta}_{\mathbb{R}}(F^{\alpha}) < \infty$$

by Moran's Theorem [25].

Consider a diffeomorphism $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with the property that $\pi_W = P_W \circ \phi$, where $P_W(x, y, t) = (0, y, t)$ is the standard Euclidean orthogonal projection onto W. Fix $n \in \mathbb{N}$ and $\omega \in \mathcal{S}_n$. We define a "column" $\mathcal{C}_{\omega}$ over F^{α}_{ω} by

$$\mathcal{C}_{\omega} = \phi^{-1}(\{(x, y, t) : (0, y, t) \in F^{\alpha}_{\omega} \text{ and } x \in [0, 1]\}).$$

Thus, if $a \in F^{\alpha}_{\omega}$, then the point $a(s)$ of the leaf $a * \mathbb{V}$ is in $\mathcal{C}_{\omega}$ for any $s \in [0, 1]$.

Let X_{ω} be a maximal σ^m-separated set (in the Heisenberg distance) in $\mathcal{C}_{\omega}$. By volume considerations, we see that

$$\operatorname{card} X_{\omega} \lesssim \sigma^{-2m}.$$

Denote

$$\mathcal{Q}_m = \left\{ B_{\mathbb{H}}(z, \sigma^m) : z \in \bigcup_{\omega\in\mathcal{S}_m} X_{\omega} \right\}$$

and

$$\mathcal{Q} = \bigcup_{m\in\mathbb{N}} \mathcal{Q}_m.$$

There exists a quantity $C \geq 1$, depending only on α, such that if ω and ω' are distinct sequences in $\mathcal{S}_n$, then

$$d_{\mathbb{H}}(\mathcal{C}_\omega, \mathcal{C}_{\omega'}) \geq d_{\mathbb{R}}(\mathcal{C}_\omega, \mathcal{C}_{\omega'}) \geq \frac{\sigma^m}{C}.$$

Hence, for some possibly larger quantity $C \geq 1$, also depending only on α, the collection $\{(1/C)B : B \in \mathcal{Q}_m\}$ is disjoint. Since $\mathbb{H}$ is Ahlfors regular, we conclude that

$$\sup_{z\in\mathbb{H}} \sum_{B\in\mathcal{Q}_m} \chi_{100B}(z) < \infty. \tag{4.1}$$

For each $B \in \mathcal{Q}$, we may find a Lipschitz function $\psi_B \colon \mathbb{H} \to [0,1]$ such that $\psi_B|_{\overline{B}} = 1$, the support of ψ_B is contained in $2B$, and

$$\operatorname{Lip} \psi_B \lesssim (\operatorname{diam} B)^{-1}.$$

Let $\xi : \{B \in \mathcal{Q}\} \to \overline{B}_{\mathbb{R}^N}(0,1)$ be any function. For each $m \in \mathbb{N}$, define $f_{\xi,m} : \mathbb{H} \to \mathbb{R}^N$ by

$$f_{\xi,m}(x) = \sum_{B\in\mathcal{Q}_m} \sigma^{\frac{m}{\alpha}} \psi_B(x)\xi(B).$$

Finally, define $f_\xi : \mathbb{H} \to \mathbb{R}^N$ by

$$f_\xi(x) = \sum_{m\in\mathbb{N}} (1+m)^{-2} f_{\xi,m}(x).$$

Then f_ξ is continuous and bounded. We claim that $\operatorname{Lip} f_\xi$ is an upper gradient of f_ξ, and that $\operatorname{Lip} f_\xi \in \mathrm{L}^{\mathrm{p}}(\mathbb{H})$. It suffices to show, for each $n \in \mathbb{N}$, that $\operatorname{Lip} f_{\xi,m}$ is an upper gradient of $f_{\xi,m}$, and that the norms $\{||\operatorname{Lip} f_{\xi,m}||_{\mathrm{L}^{\mathrm{p}}}\}_{m\in\mathbb{N}}$ are uniformly bounded. The first statement follows from the fact that $f_{\xi,m}$ is locally Lipschitz [16]. For the second fact, we calculate, using the bounded overlap condition (4.1), that

$$\begin{aligned} ||\operatorname{Lip} f_{\xi,m}||^p_{\mathrm{L}^p} &\lesssim \sum_{\omega\in\mathcal{S}_m} \sum_{B\in X_\omega} \int_{2B} (\operatorname{diam} B)^{-p} \sigma^{\frac{mp}{\alpha}} \, d\mathcal{H}^4_{\mathbb{H}} \\ &\lesssim 4^m \sigma^{-2m} \sigma^{-mp} \sigma^{\frac{mp}{\alpha}} \sigma^{4m} = (4\sigma^\beta)^m = 1. \end{aligned}$$

We now consider the function ξ as a random variable; intuitively this means that the vectors $\xi(B)$ are chosen randomly. More precisely, we assume that vectors $\{\xi(B)\}_{B\in\mathcal{Q}}$ are independent random variables distributed according to the uniform probability distribution on the closed unit ball $\overline{B}_{\mathbb{R}^N}(0,1)$.

We will prove that for every $\alpha' < \alpha$,

$$\mathbb{E}_\xi \left(\int_{F_\alpha} I_{\alpha'}((f_\xi)_\#(\mathcal{H}^1 \llcorner a * \mathbb{V}) \, d\mathcal{H}^\beta(a) \right) < \infty,$$

which implies that almost surely in ξ, it holds that $\dim f_\xi(a * V) \geq \alpha'$ for $\mathcal{H}^\beta$-almost every $a \in F_\alpha$. Thus, almost surely in ξ, the full-measure set of points $a \in F_\alpha$ where this occurs satisfies the requirements on the set E in the statement of the theorem.

By the Fubini–Tonelli theorem, the definition of the energy functional, and [24, Theorem 1.19], it suffices to show that

$$\int_{[0,1]} \int_{[0,1]} \int_{F_\alpha} \mathbb{E}_\xi \left(|f_\xi(a(s)) - f_\xi(a(s'))|^{-\alpha'} \right) d\mathcal{H}^\beta(a) \, d\mathcal{H}^1(s) \, d\mathcal{H}^1(s') < \infty.$$

For $a \in F$ and $s, s' \in [0,1]$, we write

$$f_\xi(a(s)) - f_\xi(a(s')) = \sum_{B \in \mathcal{Q}} c_B(a, s, s') \, \xi_B,$$

where for $B \in \mathcal{Q}_m$

$$c_B(a, s, s') = (1+m)^{-2} \sigma^{\frac{m}{\alpha}} \big(\psi_B(a(s)) - \psi_B(a(s')) \big).$$

We denote by $c(a, s, s')$ the supremum of the set of numbers $\{c_B(a, s, s')\}_{B \in \mathcal{Q}}$. Note that $c(a, s, s') = |c_B(a, s, s')|$ for some $B \in \mathcal{Q}$, as

$$\sum_{B \in \mathcal{Q}} |c_B(s, s')| < \infty.$$

By [5, Lemma 4.4], since $\alpha' < \alpha < N$ it holds that

$$\mathbb{E}_\xi \left(|f_\xi(a(s)) - f_\xi(a(s'))|^{-\alpha'} \right) \lesssim c(a, s, s')^{-\alpha'}.$$

This is the key probabilistic point of the proof. In view of this, it remains to show that

$$\int_{[0,1]} \int_{[0,1]} \int_F c(a, s, s')^{-\alpha'} \, d\mathcal{H}^\beta(a) d\mathcal{H}^1(s) \, d\mathcal{H}^1(s') < \infty.$$

We will in fact show the stronger statement

$$\sup_{a \in F} \sup_{s \in [0,1]} \int_{[0,1]} c(a, s, s')^{-\alpha'} \, d\mathcal{H}^1(s') < \infty.$$

Fix $a \in F$ and $s \in [0,1]$. For each $s' \in [0,1]$, define $m(s') \in \mathbb{N}$ by

$$2^{-m(s')+2} \leq d_{\mathbb{H}}(a(s), a(s')) < 2^{-m(s')+3}.$$

Find $B \in \mathcal{Q}_{m(s')}$ that contains $a(s)$. Then $a(s') \in 100B \setminus 2B$, and so

$$c(a, s, s')|| \geq |c_B(a, s, s')| = (1 + n(s'))^{-2} \sigma^{\frac{n(s')}{\alpha}}.$$

For each $m \in \mathbb{N}$, denote by E_m the set of points $s' \in [0,1]$ for which $m(s') = m$, and let $B_m \in \mathcal{Q}_m$ contain $a(s)$. By the above argument, $\mathcal{H}^1(E_m) \le \mathcal{H}^1(100B_m \cap a(s)) \lesssim \sigma^m$. Hence

$$\begin{aligned}\int_{[0,1]} c(a,s,s')^{-\alpha'}\, d\mathcal{H}^1(s) &= \sum_{m\in\mathbb{N}} \int_{E_m} c(a,s,s')^{-\alpha'}\, d\mathcal{H}^1(s)\\ &\le \sum_{m\in\mathbb{N}} m^{2\alpha'} \sigma^{\frac{-m\alpha'}{\alpha}} \mathcal{H}^1(E_m)\\ &\lesssim \sum_{m\in\mathbb{N}} m^{2\alpha'} \sigma^{m\left(1-\frac{\alpha'}{\alpha}\right)}.\end{aligned}$$

Since $\alpha' < \alpha$, the final sum above converges to a value independent of $a \in F$ and $s \in [0,1]$. This completes the proof. □

Remark 4.1. It seems likely that a mapping as in Theorem 1.9 can be found for *many* sets $E \subseteq W$ of dimension $2 - p\left(1 - \frac{1}{\alpha}\right)$; the key property is that the set E should be *evenly coverable*, *i.e.*, there exist constants $\sigma, C \ge 1$ such that for all sufficiently small $\epsilon > 0$, there is a cover $\{B(x_k, r_k)\}_{k\in\mathbb{N}}$ of E by balls centered in E such that

i) $\sup_{k\in\mathbb{N}} r_k < \epsilon$;
ii) $\sum_{k\in\mathbb{N}} r_k^{\dim E} < C$;
iii) $\sup_{x\in X} \sum_{k\in\mathbb{N}} \chi_{B(x_k,\sigma r_k)}(x) < C$.

For further discussion of the notion of even coverability, see [8, Section 7].

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