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Cohomology and coquasi-bialgebras in the category of Yetter-Drinfeld modules

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Abstract. We prove that a finite-dimensional Hopf algebra with the dual Chevalley Property over a field of characteristic zero is quasi-isomorphic to a Radford-Majid bosonization whenever the third Hochschild cohomology group in the category of Yetter-Drinfeld modules of its diagram with coefficients in the base field vanishes. Moreover we show that this vanishing occurs in meaningful examples where the diagram is a Nichols algebra.

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Introduction

Let A be a finite-dimensional Hopf algebra over a field $\Bbbk$ of characteristic zero such that the coradical H of A is a sub-Hopf algebra (*i.e.*, A has the dual Chevalley Property). Denote by $\mathcal{D}(A)$ the diagram of A. The main aim of this paper (see Theorem 5.6) is to prove that, if the third Hochschild cohomology group in ${}_{H}^{H}\mathcal{YD}$ of the algebra $\mathcal{D}(A)$ with coefficients in $\Bbbk$ vanishes, in symbols $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{D}(A), \Bbbk) = 0$, then A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}_{H}^{H}\mathcal{YD}$ with $\mathrm{gr}E \cong \mathcal{D}(A)$ as bialgebras in ${}_{H}^{H}\mathcal{YD}$.

The paper is organized as follows. Let H be a Hopf algebra over a field $\Bbbk$. In Section 1 we investigate the properties of coalgebras with multiplication and unit in the category ${}_{H}^{H}\mathcal{YD}$ (in particular of coquasi-bialgebras) and their associated graded coalgebra. The main result of this section, Theorem 1.6, establishes that the associated graded coalgebra $\mathrm{gr}Q$ of a connected coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$ is a connected bialgebra in ${}_{H}^{H}\mathcal{YD}$.

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In Section 2 we study the deformation of coquasi-bialgebras in ${}_{H}^{H}\mathcal{YD}$ by means of gauge transformations. In Proposition 2.5 we investigate its behaviour with respect to bosonization while in Proposition 2.6 with respect to the associated graded coalgebra.

In Section 3 we consider the associated graded coalgebra in case the Hopf algebra H is semisimple and cosemisimple (*e.g.* H is finite-dimensional cosemisimple over a field of characteristic zero). In particular, in Theorem 3.2, we prove that a finite-dimensional connected coquasi-bialgebra Q in ${}_{H}^{H}\mathcal{YD}$ is gauge equivalent to a connected bialgebra in ${}_{H}^{H}\mathcal{YD}$ whenever $\mathrm{H}^{3}_{\mathcal{YD}}(\mathrm{gr}Q, \Bbbk) = 0$. This result is inspired by [18, Proposition 2.3].

In Section 4 we focus on the link between $\mathrm{H}^{n}_{\mathcal{YD}}(B, \Bbbk)$ and the invariants of $\mathrm{H}^{n}(B, \Bbbk)$, where B is a bialgebra in $\mathrm{H}^{n}_{\mathcal{YD}}(B, \Bbbk)$. In particular, in Proposition 4.7 we show that $\mathrm{H}^{n}_{\mathcal{YD}}(B, \Bbbk)$ is isomorphic to $\mathrm{H}^{n}(B, \Bbbk)^{D(H)}$, which is a subspace of $\mathrm{H}^{n}(B, \Bbbk)^{H} \cong \mathrm{H}^{n}(B\#H, \Bbbk)$, see Corollary 4.3.

Section 5 is devoted to the proof of the main result of the paper, the aforementioned Theorem 5.6.

In Section 6 we provide examples where $\mathrm{H}^{n}_{\mathcal{YD}}(B, \Bbbk) = 0$ in case B is the Nichols algebra $\mathcal{B}(V)$ of a Yetter-Drinfeld module V. In particular we show that that $\mathrm{H}^{3}_{\mathcal{YD}}(\mathcal{B}(V), \Bbbk)$ can be zero although $\mathrm{H}^{3}(\mathcal{B}(V)\#H, \Bbbk)$ is non-trivial.

Notation Given a category $\mathcal{C}$ and objects $M, N \in \mathcal{C}$, the notation $\mathcal{C}(M, N)$ stands for the set of morphisms in $\mathcal{C}$ from M to N. This notation will be mainly applied to the case where $\mathcal{C}$ is the category of vector spaces $\mathbf{Vec}_{\Bbbk}$ over a field $\Bbbk$ or $\mathcal{C}$ is the category of Yetter-Drinfeld modules ${}_{H}^{H}\mathcal{YD}$ over a Hopf algebra H. The set of natural numbers including 0 is denoted by $\mathbb{N}_0$ while $\mathbb{N}$ denotes the same set without 0. Given C a coalgebra, we use the Sweedler notation for the coproduct, $\Delta(c) = c_1 \otimes c_2, c \in C$; similarly, for V a left C-comodule, we use the following notation for the coation: $\lambda(v) = v_{-1} \otimes v_0 \in V \otimes C, v \in V$.

1. Connected bialgebras in Yetter-Drinfeld categories

Definition 1.1. Let C be a coalgebra. Denote by C_n the n-th term of the coradical filtration of C and set $C_{-1} := 0$. For every $x \in C$, we set

$$|x| := \min\{i \in \mathbb{N}_0 : x \in C_i\} \qquad \text{and} \qquad \overline{x} := x + C_{|x|-1}.$$

Note that, for $x = 0$, we have $|x| = 0$. One can define the associated graded coalgebra

$$\mathrm{gr}C := \oplus_{i\in\mathbb{N}_0} \frac{C_i}{C_{i-1}}$$

with structure given, for every $x \in C$, by

$$\Delta_{\mathrm{gr}C}(\overline{x}) = \sum_{0 \le i \le |x|} (x_1 + C_{i-1}) \otimes \left(x_2 + C_{|x|-i-1}\right), \tag{1.1}$$

$$\varepsilon_{\mathrm{gr}C}(\overline{x}) = \delta_{|x|,0}\varepsilon_C(x). \tag{1.2}$$

Claim 1.2. For every $i \in \mathbb{N}_0$, take a basis $\left\{\overline{x^{i,j}} \mid j \in B_i\right\}$ of the $\Bbbk$-module C_i/C_{i-1} with $\overline{x^{i,j}} \neq \overline{x^{i,l}}$ for $j \neq l$ and

$$\left|x^{i,j}\right| = i.$$

Then $\left\{x^{i,j} \mid 0 \le i \le n,\ j \in B_i\right\}$ is a basis of C_n and $\left\{x^{i,j} \mid i \in \mathbb{N}_0,\ j \in B_i\right\}$ is a basis of C. Assume that C has a distinguished grouplike element $1 = 1_C \neq 0$ and take $i > 0$. If $\varepsilon\left(x^{i,j}\right) \neq 0$ then we have that

$$\overline{x^{i,j} - \varepsilon\left(x^{i,j}\right)1} = \overline{x^{i,j}}$$

so that we can take $x^{i,j} - \varepsilon\left(x^{i,j}\right)1$ in place of $x^{i,j}$. In other words we can assume

$$\varepsilon\left(x^{i,j}\right) = 0, \text{ for every } i > 0,\ j \in B_i. \tag{1.3}$$

It is well-known that there is a $\Bbbk$-linear isomorphism $\varphi : C \to \mathrm{gr}C$ defined on the basis by $\varphi\left(x^{i,j}\right) := \overline{x^{i,j}}$.

We compute

$$\varepsilon_{\mathrm{gr}C}\varphi\left(x^{i,j}\right) = \varepsilon_{\mathrm{gr}C}\left(\overline{x^{i,j}}\right) \overset{(1.2)}{=} \delta_{i,0}\varepsilon\left(x^{0,j}\right) \overset{(1.3)}{=} \varepsilon\left(x^{i,j}\right).$$

Hence we obtain

$$\varepsilon_{\mathrm{gr}C} \circ \varphi = \varepsilon. \tag{1.4}$$

Let H be a Hopf algebra. A *coalgebra with multiplication and unit* in ${}^H_H\mathcal{YD}$ is a datum $(Q, m, u, \Delta, \varepsilon)$ where (Q, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$, $m : Q \otimes Q \to Q$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$ called multiplication (which may fail to be associative) and $u : \Bbbk \to Q$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$ called unit. In this case we set $1_Q := u(1_\Bbbk)$.

Note that, for every $h \in H$, $k \in \Bbbk$, we have

$$\begin{aligned} h1_Q &= hu(1_\Bbbk) = u(h1_\Bbbk) = u(\varepsilon_H(h)1_\Bbbk) \\ &= \varepsilon_H(h)u(1_\Bbbk) = \varepsilon_H(h)1_Q, \end{aligned} \tag{1.5}$$

$$\begin{aligned} \left(1_Q\right)_{-1} \otimes \left(1_Q\right)_0 &= (u(1_\Bbbk))_{-1} \otimes (u(1_\Bbbk))_0 \\ &= (1_\Bbbk)_{-1} \otimes u((1_\Bbbk)_0) = 1_H \otimes u(1_\Bbbk) = 1_H \otimes 1_Q. \end{aligned} \tag{1.6}$$

Proposition 1.3. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon)$ be a coalgebra with multiplication and unit in ${}_{H}^{H}\mathcal{YD}$. If Q_0 is a subcoalgebra of Q in ${}_{H}^{H}\mathcal{YD}$ such that $Q_0 \cdot Q_0 \subseteq Q_0$, then Q_n is a subcoalgebra of Q in ${}_{H}^{H}\mathcal{YD}$ for every $n \in \mathbb{N}_0$. Moreover $Q_a \cdot Q_b \subseteq Q_{a+b}$ for every $a, b \in \mathbb{N}_0$ and the graded coalgebra $\mathrm{gr}Q$, associated with the coradical filtration of Q, is a coalgebra with multiplication and unit in ${}_{H}^{H}\mathcal{YD}$ with respect to the usual coalgebra structure and with multiplication and unit defined by*

$$\begin{aligned} m_{\mathrm{gr}Q}\left((x + Q_{a-1}) \otimes (y + Q_{b-1})\right) &:= xy + Q_{a+b-1}, \\ u_{\mathrm{gr}Q}(k) &:= k1_Q + Q_{-1} \end{aligned} \tag{1.7}$$

Proof. The coalgebra structure of Q induces a coalgebra structure on $\mathrm{gr}Q$. Since Q_0 is a subcoalgebra of Q in ${}_{H}^{H}\mathcal{YD}$ and, for $n \geq 1$, one has $Q_n = Q_{n-1} \wedge_Q Q_0$, then inductively one proves that Q_n is a subcoalgebra of Q in ${}_{H}^{H}\mathcal{YD}$. As a consequence one gets that $\mathrm{gr}Q$ is a coalgebra in ${}_{H}^{H}\mathcal{YD}$ (this construction can be performed in the setting of monoidal categories under suitable assumptions, see *e.g.* [5, Theorem 2.10]). Let us prove that $\mathrm{gr}Q$ inherits also a multiplication and unit. Let us check that $Q_a \cdot Q_b \subseteq Q_{a+b}$ for every $a, b \in \mathbb{N}_0$. We proceed by induction on $n = a + b$. If $n = 0$ there is nothing to prove. Let $n \geq 1$ and assume that $Q_i \cdot Q_j \subseteq Q_{i+j}$ for every $i, j \in \mathbb{N}_0$ such that $0 \leq i + j \leq n - 1$. Let $a, b \in \mathbb{N}_0$ be such that $n = a + b$. Since $\Delta(Q_a) \subseteq \sum_{i=0}^{a} Q_i \otimes Q_{a-i}$ and $c_{Q,Q}(Q_u \otimes Q_v) \subseteq Q_v \otimes Q_u$, where $c_{Q,Q}$ denotes the braiding in ${}_{H}^{H}\mathcal{YD}$, using the compatibility condition between Δ and m, one easily gets that $\Delta(Q_a \cdot Q_b) \subseteq Q_{a+b-1} \otimes Q + Q \otimes Q_0$.

Therefore $Q_a \cdot Q_b \subseteq Q_{a+b}$. This property implies we have a well-defined map in ${}_{H}^{H}\mathcal{YD}$

$$m_{\mathrm{gr}Q}^{a,b} : \frac{Q_a}{Q_{a-1}} \otimes \frac{Q_b}{Q_{b-1}} \to \frac{Q_{a+b}}{Q_{a+b-1}}$$

defined, for $x \in Q_a$ and $y \in Q_b$, by (1.7). This can be seen as the graded component of a morphism in ${}_{H}^{H}\mathcal{YD}$ that we denote by $m_{\mathrm{gr}Q} : \mathrm{gr}Q \otimes \mathrm{gr}Q \to \mathrm{gr}Q$. Let us check that $m_{\mathrm{gr}Q}$ is a coalgebra morphism in ${}_{H}^{H}\mathcal{YD}$. Consider a basis of Q with terms of the form $x^{i,j}$ as in 1.2. Hence we can write the comultiplication in the form

$$\Delta\left(x^{a,u}\right) = \sum_{s+t \leq a} \sum_{l,m} \eta_{s,t,l,m}^{a,u} x^{s,l} \otimes x^{t,m}.$$

Now, using (1.1), one gets that

$$\Delta_{\mathrm{gr}Q}\left(\overline{x^{a,u}}\right) = \sum_{0 \leq i \leq a} \sum_{l,m} \eta_{i,a-i,l,m}^{a,u} \overline{x^{i,l}} \otimes \overline{x^{a-i,m}}. \tag{1.8}$$

Using that $\Delta_{\mathrm{gr}Q\otimes\mathrm{gr}\,Q} = \left(\mathrm{gr}\,Q \otimes c_{\mathrm{gr}\,Q,\mathrm{gr}Q} \otimes \mathrm{gr}\,Q\right)\left(\Delta_{\mathrm{gr}\,Q} \otimes \Delta_{\mathrm{gr}Q}\right)$ and (1.8), it is straightforward to check that $\left(m_{\mathrm{gr}Q} \otimes m_{\mathrm{gr}Q}\right)\Delta_{\mathrm{gr}Q\otimes\mathrm{gr}Q}\left(\overline{x^{a,u}} \otimes \overline{x^{b,v}}\right) = \Delta_{\mathrm{gr}Q} m_{\mathrm{gr}Q}\left(\overline{x^{a,u}} \otimes \overline{x^{b,v}}\right)$.

Moreover, since $\varepsilon_{\mathrm{gr}Q\otimes\mathrm{gr}Q} = \varepsilon_{\mathrm{gr}Q} \otimes \varepsilon_{\mathrm{gr}Q}$, we get that $\varepsilon_{\mathrm{gr}Q} m_{\mathrm{gr}Q}\left(\overline{x^{a,u}} \otimes \overline{x^{b,v}}\right) = \varepsilon_{\mathrm{gr}Q\otimes\mathrm{gr}Q}\left(\overline{x^{a,u}} \otimes \overline{x^{b,v}}\right)$.

This proves that $m_{\mathrm{gr}Q}$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$.

The fact that $u_{\mathrm{gr}Q} : \Bbbk \to \mathrm{gr}Q$, defined by $u_{\mathrm{gr}Q}(k) := k1_Q + Q_{-1}$ is a coalgebra morphism in ${}_H^H\mathcal{YD}$ easily follows by means of (1.6) and (1.7). □

Definition 1.4 ([2, Definition 5.2]). Let H be a Hopf algebra. We say that $(Q, m, u, \Delta, \varepsilon, \alpha)$ is a *coquasi-bialgebra* in the pre-braided monoidal category ${}_H^H\mathcal{YD}$ if (Q, Δ, ε) is a coalgebra in ${}_H^H\mathcal{YD}$, $m : Q \otimes Q \to Q$ and $u : \Bbbk \to Q$ are coalgebra homomorphisms in ${}_H^H\mathcal{YD}$ and $\alpha \in {}_H^H\mathcal{YD}\left(Q^{\otimes 3}, \Bbbk\right)$ (*braided reassociator*) is a convolution invertible element such that

$$\alpha\,(Q \otimes Q \otimes m) * \alpha\,(m \otimes Q \otimes Q) = (\varepsilon \otimes \alpha) * \alpha\,(Q \otimes m \otimes Q) * (\alpha \otimes \varepsilon)\,, \quad (1.9)$$

$$\alpha\,(Q \otimes u \otimes Q) = \alpha\,(u \otimes Q \otimes Q) = \alpha\,(Q \otimes Q \otimes u) = \varepsilon_{Q\otimes Q}, \quad (1.10)$$

$$m\,(Q \otimes m) * \alpha = \alpha * m\,(m \otimes Q)\,, \quad (1.11)$$

$$m\,(u \otimes Q) = \mathrm{Id}_Q = m\,(Q \otimes u)\,. \quad (1.12)$$

Here $*$ denotes the convolution product, where $Q^{\otimes 3}$ is the tensor product of coalgebras in ${}_H^H\mathcal{YD}$ whence it depends on the braiding of this category. Note that in (1.10) any of the three equalities such as $\alpha\,(u \otimes Q \otimes Q) = \varepsilon_{Q\otimes Q}$ implies that α is unital.

Remark 1.5. When $H = \Bbbk$ we recover the usual definition of coquasi-bialgebra that will be also named an **ordinary coquasi-bialgebra**.

Theorem 1.6. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in ${}_H^H\mathcal{YD}$. Then* $\mathrm{gr}Q$ *is a connected bialgebra in* ${}_H^H\mathcal{YD}$.

Proof. By Proposition 1.3, we know that $\mathrm{gr}Q$ is a coalgebra with multiplication and unit in ${}_H^H\mathcal{YD}$. We have to check that the multiplication is associative and unitary.

Given two coalgebras D, E in ${}_H^H\mathcal{YD}$ endowed with coalgebras filtration $\left(D_{(n)}\right)_{n\in\mathbb{N}_0}$ and $\left(E_{(n)}\right)_{n\in\mathbb{N}_0}$ in ${}_H^H\mathcal{YD}$ such that $D_{(0)}$ and $E_{(0)}$ are one-dimensional, let us check that $C_{(n)} := \sum_{0\le i\le n} D_{(i)} \otimes E_{(n-i)}$ gives a coalgebra filtration on $C := D \otimes E$ in ${}_H^H\mathcal{YD}$. First note that the coalgebra structure of C depends on the braiding. Thus, we have

$$\begin{aligned}\Delta_C\left(C_{(n)}\right) &= \left(D\otimes c_{D,E}\otimes E\right)\left(\Delta_D\otimes\Delta_E\right)\left(\sum_{i=0}^{n} D_{(i)}\otimes E_{(n-i)}\right)\\ &\subseteq \left(D\otimes c_{D,E}\otimes E\right)\left(\sum_{i=0}^{n}\sum_{a=0}^{i}\sum_{b=0}^{n-i} D_{(a)}\otimes D_{(i-a)}\otimes E_{(b)}\otimes E_{(n-i-b)}\right)\\ &\subseteq \sum_{i=0}^{n}\sum_{a=0}^{i}\sum_{b=0}^{n-i} D_{(a)}\otimes c_{D,E}\left(D_{(i-a)}\otimes E_{(b)}\right)\otimes E_{(n-i-b)}\\ &\subseteq \sum_{i=0}^{n}\sum_{a=0}^{i}\sum_{b=0}^{n-i} D_{(a)}\otimes c_{D_{(i-a)},E_{(b)}}\left(D_{(i-a)}\otimes E_{(b)}\right)\otimes E_{(n-i-b)}\\ &\subseteq \sum_{i=0}^{n}\sum_{a=0}^{i}\sum_{b=0}^{n-i} D_{(a)}\otimes E_{(b)}\otimes D_{(i-a)}\otimes E_{(n-i-b)}\\ &\subseteq \sum_{i=0}^{n}\sum_{w=0}^{n}\sum_{\substack{0\leq a\leq i,\\ 0\leq b\leq n-i\\ a+b=w}} D_{(a)}\otimes E_{(b)}\otimes D_{(i-a)}\otimes E_{(n-i-b)}\\ &\subseteq \sum_{w=0}^{n} C_{(w)}\otimes C_{(n-w)}.\end{aligned}$$

Moreover, by [37, Proposition 11.1.1], we have that the coradical of C is contained in $D_{(0)}\otimes E_{(0)}$ and hence it is one-dimensional.

This argument can be used to produce a coalgebra filtration on $C := Q\otimes Q\otimes Q$ using as a filtration on Q the coradical filtration. Let $n>0$ and let $w\in C_{(n)} = \sum_{i+j+k\leq n} Q_i\otimes Q_j\otimes Q_k$. By [6, Lemma 3.69], we have that

$$\Delta_C(w) - w\otimes\left(1_Q\right)^{\otimes 3} - \left(1_Q\right)^{\otimes 3}\otimes w\in C_{(n-1)}\otimes C_{(n-1)}.$$

Thus we get

$$w_1\otimes w_2\otimes w_3 - \Delta_C(w)\otimes\left(1_Q\right)^{\otimes 3} - \Delta_C\left(\left(1_Q\right)^{\otimes 3}\right)\otimes w\in\Delta_C\left(C_{(n-1)}\right)\otimes C_{(n-1)}$$

and hence, tensoring the first relation by $\left(1_Q\right)^{\otimes 3}$ on the right and adding it to the second one, we get

$$\begin{aligned}w_1\otimes w_2\otimes w_3 - w\otimes\left(1_Q\right)^{\otimes 3}\otimes\left(1_Q\right)^{\otimes 3} - \left(1_Q\right)^{\otimes 3}\otimes w\otimes\left(1_Q\right)^{\otimes 3}\\ - \left(1_Q\right)^{\otimes 6}\otimes w\in C_{(n-1)}\otimes C_{(n-1)}\otimes C_{(n-1)}.\end{aligned}$$

For shortness, we set $\nu_n(z) := m(Q\otimes m)(z) + Q_{n-1}$ for every $z\in C$. Thus, by applying to the last displayed relation $C_{(n-1)}\otimes m(Q\otimes m)\otimes C_{(n-1)}$ and factoring out the middle term by Q_{n-1}, we get

$$\left[\begin{array}{c} w_1 \otimes \nu_n(w_2) \otimes w_3 - w \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes (1_Q)^{\otimes 3} + \\ -(1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 3} \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes w \end{array}\right]$$
$$\in C_{(n-1)} \otimes \left(\frac{\nu_n\left(C_{(n-1)}\right)}{Q_{n-1}}\right) \otimes C_{(n-1)} \subseteq C_{(n-1)} \otimes \frac{Q_{n-1}}{Q_{n-1}} \otimes C_{(n-1)} = 0.$$

Thus we can express the first term with respect to the remaining ones as follows

$$\begin{aligned} & w_1 \otimes \nu_n(w_2) \otimes w_3 \\ = \; & w \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} \\ & + (1_Q)^{\otimes 3} \otimes \nu_n\left((1_Q)^{\otimes 3}\right) \otimes w \\ = \; & w \otimes (1_Q + Q_{n-1}) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} \\ & + (1_Q)^{\otimes 3} \otimes (1_Q + Q_{n-1}) \otimes w \\ \overset{n>0}{=} \; & (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3}. \end{aligned}$$

We have so proved that for $n > 0$ and $w \in C_{(n)}$

$$w_1 \otimes \nu_n(w_2) \otimes w_3 = (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3}. \tag{1.13}$$

The same equation trivially holds also in the case $n = 0$ as $C_{(n)}$ is one-dimensional.

Let $x, y, z \in Q$. Then $x \otimes y \otimes z \in C_{(|x|+|y|+|z|)}$ so that

$$\begin{aligned} (\overline{x} \cdot \overline{y}) \cdot \overline{z} &= \left((x + Q_{|x|-1}) \cdot (y + Q_{|y|-1})\right) \cdot (z + Q_{|z|-1}) \\ &= \left((xy) + Q_{|x|+|y|-1}\right) \cdot (z + Q_{|z|-1}) \\ &= (xy)z + Q_{|x|+|y|+|z|-1} \\ &= \omega^{-1}\left((x \otimes y \otimes z)_1\right) \nu_{|x|+|y|+|z|}\left((x \otimes y \otimes z)_2\right) \omega\left((x \otimes y \otimes z)_3\right) \\ &\overset{(1.13)}{=} \omega^{-1}\left(1_Q \otimes 1_Q \otimes 1_Q\right) \nu_{|x|+|y|+|z|}(x \otimes y \otimes z)\, \omega\left(1_Q \otimes 1_Q \otimes 1_Q\right) \\ &= \nu_{|x|+|y|+|z|}(x \otimes y \otimes z) \\ &= x(yz) + Q_{|x|+|y|+|z|-1} = \overline{x} \cdot (\overline{y} \cdot \overline{z}). \end{aligned}$$

Therefore the multiplication is associative. It is also unitary as

$$\overline{x} \cdot \overline{1_Q} = (x + Q_{|x|-1}) \cdot (1_Q + Q_{-1}) = x \cdot 1_Q + Q_{|x|-1} = x + Q_{|x|-1} = \overline{x}$$

and similarly $\overline{1_Q} \cdot \overline{x} = \overline{x}$ for every $x \in Q$. □

2. Gauge transformations

Definition 2.1. Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}_H^H\mathcal{YD}$. A *gauge transformation for* Q is a morphism $\gamma : Q \otimes Q \to \Bbbk$ in ${}_H^H\mathcal{YD}$ which is convolution invertible in ${}_H^H\mathcal{YD}$ and which is also unitary on both entries.

Remark 2.2. For γ as above, let us check that γ^{-1} is unitary whence a gauge transformation too.

First note that for all $x \in Q$, by means of (1.7) and (1.6), one gets

$$\left(1_Q \otimes x\right)_1 \otimes \left(1_Q \otimes x\right)_2 = 1_Q \otimes x_1 \otimes 1_Q \otimes x_2, \tag{2.1}$$

$$\left(x \otimes 1_Q\right)_1 \otimes \left(x \otimes 1_Q\right)_2 = x_1 \otimes 1_Q \otimes x_2 \otimes 1_Q. \tag{2.2}$$

Thus

$$\begin{aligned}\gamma^{-1}\left(1_Q \otimes x\right) &= \gamma^{-1}\left(1_Q \otimes x_1\right)\varepsilon\left(x_2\right) = \gamma^{-1}\left(1_Q \otimes x_1\right)\gamma\left(1_Q \otimes x_2\right)\\ &= \left(\gamma^{-1} * \gamma\right)\left(1_Q \otimes x\right) = \varepsilon\left(x\right)\end{aligned}$$

and similarly $\gamma^{-1}\left(x \otimes 1_Q\right) = \varepsilon\left(x\right)$.

Lemma 2.3. *Let H be a Hopf algebra and let C be a coalgebra in ${}_H^H\mathcal{YD}$. Given a map $\gamma \in {}_H^H\mathcal{YD}\left(C, \Bbbk\right)$, we have that γ is convolution invertible in ${}_H^H\mathcal{YD}\left(C, \Bbbk\right)$ if and only if it is convolution invertible in* $\mathbf{Vec}_{\Bbbk}\left(C, \Bbbk\right)$. *Moreover the inverse is the same.*

Proof. Assume there is a $\Bbbk$-linear map $\gamma^{-1} : C \to \Bbbk$ which is a convolution inverse of γ in $\mathbf{Vec}_{\Bbbk}\left(C, \Bbbk\right)$. By [1, Remark 2.4(ii)], γ^{-1} is left H-linear. Let us check that γ^{-1} is left H-colinear:

$$\begin{aligned}c_{-1} \otimes \gamma^{-1}\left(c_0\right) &= \left(c_1\right)_{-1} 1_H \otimes \gamma^{-1}\left(\left(c_1\right)_0\right)\gamma\left(c_2\right)\gamma^{-1}\left(c_3\right)\\ &= \left(c_1\right)_{-1}\left(c_2\right)_{-1} \otimes \gamma^{-1}\left(\left(c_1\right)_0\right)\gamma\left(\left(c_2\right)_0\right)\gamma^{-1}\left(c_3\right)\\ &\overset{(*)}{=} \left(c_1\right)_{-1} \otimes \gamma^{-1}\left(\left(\left(c_1\right)_0\right)_1\right)\gamma\left(\left(\left(c_1\right)_0\right)_2\right)\gamma^{-1}\left(c_2\right)\\ &= \left(c_1\right)_{-1} \otimes \left(\gamma^{-1} * \gamma\right)\left(\left(c_1\right)_0\right)\gamma^{-1}\left(c_2\right)\\ &= \left(c_1\right)_{-1} \otimes \varepsilon_C\left(\left(c_1\right)_0\right)\gamma^{-1}\left(c_2\right)\\ &\overset{(*)}{=} 1_H \otimes \varepsilon_C\left(c_1\right)\gamma^{-1}\left(c_2\right) = 1_H \otimes \gamma^{-1}\left(c\right)\end{aligned}$$

where in (*) we used that the comultiplication or the counit of C is left H-colinear. Thus γ is convolution invertible in ${}_H^H\mathcal{YD}\left(C, \Bbbk\right)$. The other implication is obvious. □

Proposition 2.4. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$. Let $\gamma : Q \otimes Q \to \Bbbk$ be a gauge transformation for Q. Then*

$$Q^{\gamma} := \left(Q, m^{\gamma}, u, \Delta, \varepsilon, \omega^{\gamma}\right)$$

is a coquasi-bialgebra in ${}_{H}^{H}\mathcal{YD}$, where

$$\begin{aligned} m^{\gamma} &:= \gamma * m * \gamma^{-1} \\ \omega^{\gamma} &:= (\varepsilon \otimes \gamma) * \gamma\,(Q \otimes m) * \omega * \gamma^{-1}\,(m \otimes Q) * \left(\gamma^{-1} \otimes \varepsilon\right). \end{aligned}$$

Proof. The proof is analogue to [24, Proposition XV.3.2] in its dual version. We include some details for the reader's sake. Note that Q^{γ} has the same underlying coalgebra of Q which is a coalgebra in ${}_{H}^{H}\mathcal{YD}$. The unit is also the same and hence it is a coalgebra map in ${}_{H}^{H}\mathcal{YD}$. Since m^{γ} is the convolution product of morphisms in ${}_{H}^{H}\mathcal{YD}$, it results that m^{γ} is in ${}_{H}^{H}\mathcal{YD}$ as well.

Since m is a coalgebra map in ${}_{H}^{H}\mathcal{YD}$ and γ is convolution invertible with convolution inverse γ^{-1}, it follows that m^{γ} is a coalgebra map in ${}_{H}^{H}\mathcal{YD}$.

By means of (2.1) and (2.2), one gets that $m^{\gamma}\left(1_{Q} \otimes x\right) = x = m^{\gamma}\left(x \otimes 1_{Q}\right)$.

Let us consider now ω^{γ}. Since it is the convolution product of morphisms in ${}_{H}^{H}\mathcal{YD}$, it results that ω^{γ} is in ${}_{H}^{H}\mathcal{YD}$ as well.

Let us check that ω^{γ} is unitary. Consider the map $\alpha_2 : Q \otimes Q \to Q \otimes Q \otimes Q$ defined by $\alpha_2 (x \otimes y) = x \otimes 1_{Q} \otimes y$. The equalities (2.2) and (1.7) yield

$$\begin{aligned} (\alpha_2 (x \otimes y))_1 \otimes (\alpha_2 (x \otimes y))_2 &= \alpha_2 \left(x_1 \otimes (x_2)_{-1}\, y_1\right) \otimes \alpha_2 \left((x_2)_0 \otimes y_2\right) \\ &= \alpha_2 \left((x \otimes y)_1\right) \otimes \alpha_2 \left((x \otimes y)_2\right) \end{aligned}$$

so that α_2 is comultiplicative.

Thus

$$\omega^{\gamma}\alpha_2 := (\varepsilon \otimes \gamma)\,\alpha_2 * \gamma\,(Q \otimes m)\,\alpha_2 * \omega\alpha_2 * \gamma^{-1}\,(m \otimes Q)\,\alpha_2 * \left(\gamma^{-1} \otimes \varepsilon\right)\alpha_2$$

and computing the factors of this convolution products one gets

$$\begin{gathered} (\varepsilon \otimes \gamma)\,\alpha_2 = \varepsilon \otimes \varepsilon, \quad \gamma\,(Q \otimes m)\,\alpha_2 = \gamma, \quad \omega\alpha_2 = \varepsilon \otimes \varepsilon, \\ \gamma^{-1}\,(m \otimes Q)\,\alpha_2 = \gamma^{-1}, \quad \left(\gamma^{-1} \otimes \varepsilon\right)\alpha_2 = \varepsilon \otimes \varepsilon \end{gathered}$$

and hence $\omega^{\gamma}\alpha_2 = \gamma * \gamma^{-1} = \varepsilon \otimes \varepsilon$, which means that $\omega^{\gamma}\left(x \otimes 1_{Q} \otimes y\right) = \varepsilon(x)\,\varepsilon(y)$ for every $x, y \in Q$.

Similarly, considering $\alpha_1 : Q \otimes Q \to Q \otimes Q \otimes Q$ defined by $\alpha_1 (x \otimes y) = 1_{Q} \otimes x \otimes y$, one proves that $\omega^{\gamma}\left(1_{Q} \otimes x \otimes y\right) = \varepsilon(x)\,\varepsilon(y)$. A symmetric argument shows that $\omega^{\gamma}\left(x \otimes y \otimes 1_{Q}\right) = \varepsilon(x)\,\varepsilon(y)$ were $D = Q \otimes Q \otimes Q$.

Note that, by Lemma 2.3, ω^{γ} is convolution invertible in ${}_{H}^{H}\mathcal{YD}(D, \Bbbk)$ as it is convolution invertible in $\mathbf{Vec}_{\Bbbk}(D, \Bbbk)$.

Let us check that the multiplication is quasi-associative. By [2, Lemma 2.10 formula (2.7)], we have

$$\begin{aligned}
m^{\gamma}\left(Q\otimes\gamma*m*\gamma^{-1}\right) &= (\varepsilon\otimes\gamma)*m^{\gamma}\,(Q\otimes m)*\left(\varepsilon\otimes\gamma^{-1}\right),\\
\left(\varepsilon\otimes\gamma^{-1}\right)*(\varepsilon\otimes\gamma) &= \varepsilon\otimes\left(\gamma^{-1}*\gamma\right)=\varepsilon\otimes\varepsilon\otimes\varepsilon,\\
m^{\gamma}\left(m^{\gamma}\otimes Q\right) &= m^{\gamma}\left(\gamma*m*\gamma^{-1}\otimes Q\right)\\
&= (\gamma\otimes\varepsilon)*m^{\gamma}\left(m*\gamma^{-1}\otimes Q\right)\\
&= (\gamma\otimes\varepsilon)*m^{\gamma}\,(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right),\\
\left(\gamma^{-1}\otimes\varepsilon\right)*(\gamma\otimes\varepsilon) &= \left(\left(\gamma^{-1}*\gamma\right)\otimes\varepsilon\right)=\varepsilon\otimes\varepsilon\otimes\varepsilon.
\end{aligned}$$

By using these equalities one obtains

$$\begin{aligned}
&m^{\gamma}\left(Q\otimes m^{\gamma}\right)*\omega^{\gamma}\\
&\quad=(\varepsilon\otimes\gamma)*\gamma\,(Q\otimes m)*m\,(Q\otimes m)*\omega*\gamma^{-1}\,(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right),\\
&\omega^{\gamma}*m^{\gamma}\left(m^{\gamma}\otimes Q\right)\\
&\quad=(\varepsilon\otimes\gamma)*\gamma\,(Q\otimes m)*\omega*m\,(m\otimes Q)*\gamma^{-1}\,(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)
\end{aligned}$$

so that $\omega^{\gamma}*m^{\gamma}\,(m^{\gamma}\otimes Q)=m^{\gamma}\,(Q\otimes m^{\gamma})*\omega^{\gamma}$.

It remains to check that ω^{γ} is a reassociator. By [2, Lemma 2.10 formula (2.7)], we have

$$\begin{aligned}
\omega^{\gamma}\left(Q\otimes Q\otimes\gamma*m*\gamma^{-1}\right)&=(\varepsilon\otimes\varepsilon\otimes\gamma)*\omega^{\gamma}\,(Q\otimes Q\otimes m)*\left(\varepsilon\otimes\varepsilon\otimes\gamma^{-1}\right),\\
\omega^{\gamma}\left(\gamma*m*\gamma^{-1}\otimes Q\otimes Q\right)&=(\gamma\otimes\varepsilon\otimes\varepsilon)*\omega^{\gamma}\,(m\otimes Q\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\otimes\varepsilon\right),\\
(\gamma\otimes\varepsilon\otimes\varepsilon)*(\varepsilon\otimes\varepsilon\otimes\gamma)&=\gamma\otimes\gamma=(\varepsilon\otimes\varepsilon\otimes\gamma)*(\gamma\otimes\varepsilon\otimes\varepsilon)\,.
\end{aligned}$$

By using these equalities one obtains

$$\begin{aligned}
&\omega^{\gamma}\left(Q\otimes Q\otimes m^{\gamma}\right)*\omega^{\gamma}\left(m^{\gamma}\otimes Q\otimes Q\right)\\
&=\left[\begin{array}{c}(\varepsilon\otimes\varepsilon\otimes\gamma)*(\varepsilon\otimes\gamma\,(Q\otimes m))*\gamma\,(Q\otimes m\,(Q\otimes m))\\ *\omega\,(Q\otimes Q\otimes m)*\omega\,(m\otimes Q\otimes Q)\\ *\gamma^{-1}\,(m\,(m\otimes Q)\otimes Q)*\left(\gamma^{-1}\,(m\otimes Q)\otimes\varepsilon\right)*\left(\gamma^{-1}\otimes\varepsilon\otimes\varepsilon\right)\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left(\varepsilon\otimes\omega^{\gamma}\right)*\omega^{\gamma}\left(Q\otimes m^{\gamma}\otimes Q\right)*\left(\omega^{\gamma}\otimes\varepsilon\right)\\
&=\left[\begin{array}{c}(\varepsilon\otimes\varepsilon\otimes\gamma)*(\varepsilon\otimes\gamma\,(Q\otimes m))*\gamma\,(Q\otimes m\,(Q\otimes m))\\ *(\varepsilon\otimes\omega)*\omega\,(Q\otimes m\otimes Q)*(\omega\otimes\varepsilon)\\ *\gamma^{-1}\,(m\,(m\otimes Q)\otimes Q)*\left(\gamma^{-1}\,(m\otimes Q)\otimes\varepsilon\right)*\left(\gamma^{-1}\otimes\varepsilon\otimes\varepsilon\right)\end{array}\right].
\end{aligned}$$

Therefore

$$\omega^{\gamma}\left(Q\otimes Q\otimes m^{\gamma}\right)*\omega^{\gamma}\left(m^{\gamma}\otimes Q\otimes Q\right)=\left(\varepsilon\otimes\omega^{\gamma}\right)*\omega^{\gamma}\left(Q\otimes m^{\gamma}\otimes Q\right)*\left(\omega^{\gamma}\otimes\varepsilon\right). \square$$

In analogy to the case of Hopf algebras, one can define the bosonization $E\#H$ of a coquasi-bialgebra in ${}_H^H\mathcal{YD}$ by a Hopf algebra H, see [2, Definition 5.4] for further details on the structure. The following result was originally stated for E a Hopf algebra. Yorck Sommerhäuser suggested the present more general form that deals with the behaviour of the bosonization under a suitable gauge transformation.

Proposition 2.5. *Let H be a Hopf algebra and let $(E, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}_H^H\mathcal{YD}$. Let $\gamma : E\otimes E \to \Bbbk$ be a gauge transformation for E. Set*

$$\Gamma : (E\#H)\otimes(E\#H) \to \Bbbk : (x\#h)\otimes\left(x'\#h'\right) \mapsto \gamma\left(x\otimes hx'\right)\varepsilon_H\left(h'\right).$$

Then Γ is a gauge transformation and $(E\#H)^{\Gamma} = E^{\gamma}\#H$ as ordinary coquasi-bialgebras.

Proof. By [2, Lemma 2.15 and what follows], we have that Γ is convolution invertible H-bilinear and H-balanced. Moreover $\Gamma^{-1}\left((x\#h)\otimes\left(x'\#h'\right)\right) = \gamma^{-1}\left(x\otimes hx'\right)\varepsilon_H\left(h'\right)$. If $\alpha : (E\#H)\otimes(E\#H) \to E\#H$ is H-bilinear and H-balanced, it is easy to check that $\Gamma*\alpha*\Gamma^{-1}$ is H-bilinear and H-balanced too.

In particular, since

$$m_{E\#H}\left((x\#h)\otimes\left(x'\#h'\right)\right) = m\left(x\otimes h_1x'\right)\otimes h_2h'$$

we have that $m_{E\#H}$ is H-bilinear and H-balanced where $E\#H$ carries the left H-diagonal action and the right regular action over H.

Thus $m_{(E\#H)^{\Gamma}} = \Gamma * m_{E\#H} * \Gamma^{-1}$ is H-bilinear and H-balanced. Moreover, since E^{γ} is also a coquasi-bialgebra in ${}_H^H\mathcal{YD}$ we have that $m_{E^{\gamma}\#H} : (E\#H)\otimes(E\#H) \to E\#H$ is H-bilinear and H-balanced too.

Therefore, in order to check that $m_{(E\#H)^{\Gamma}} = m_{E^{\gamma}\#H}$, it suffices to prove that they coincide on elements of the form $(x\#1_H)\otimes\left(x'\#1_H\right)$.

Let us consider the multiplication

$$\begin{aligned}
&m_{(E\#H)^{\Gamma}}\left((x\#1_H)\otimes\left(x'\#1_H\right)\right)\\
&=\left(\Gamma * m_{E\#H} * \Gamma^{-1}\right)\left((x\#1_H)\otimes\left(x'\#1_H\right)\right)\\
&=\Gamma\left((x\#1_H)_1\otimes\left(x'\#1_H\right)_1\right)\cdot m_{E\#H}\left((x\#1_H)_2\otimes\left(x'\#1_H\right)_2\right)\\
&\quad\cdot\Gamma^{-1}\left((x\#1_H)_3\otimes\left(x'\#1_H\right)_3\right).
\end{aligned}$$

Now, from

$$\Delta_{E\#H}(x\#h) = \sum\left(x^{(1)}\#x^{(2)}{}_{-1}h_1\right)\otimes\left(x^{(2)}{}_0\#h_2\right)$$

we get

$$\begin{aligned}&(x\#1_H)_1 \otimes (x\#1_H)_2 \otimes (x\#1_H)_3\\ &= \sum \left(x^{(1)}\#x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}\right) \otimes \left(x^{(2)}{}_0\#x^{(3)}{}_{-1}\right) \otimes \left(x^{(3)}{}_0\#1_H\right)\end{aligned}$$

so that

$$\begin{aligned}
& m_{(E\#H)^\Gamma}\left((x\#1_H)\otimes\left(x'\#1_H\right)\right)\\
=\; & \Gamma\left((x\#1_H)_1\otimes\left(x'\#1_H\right)_1\right)\cdot m_{E\#H}\left((x\#1_H)_2\otimes\left(x'\#1_H\right)_2\right)\\
& \cdot\Gamma^{-1}\left((x\#1_H)_3\otimes\left(x'\#1_H\right)_3\right)\\
=\; & \left[\begin{array}{c}\sum\Gamma\left(x^{(1)}\#x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}\otimes x'^{(1)}\#x'^{(2)}{}_{-1}x'^{(3)}{}_{\langle -2\rangle}\right)\\ \cdot m_{E\#H}\left(x^{(2)}{}_0\#x^{(3)}{}_{-1}\otimes x'^{(2)}{}_0\#x'^{(3)}{}_{-1}\right)\\ \cdot\Gamma^{-1}\left(x^{(3)}{}_0\#1_H\otimes x'^{(3)}{}_0\#1_H\right)\end{array}\right]\\
=\; & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}x'^{(1)}\right)\\ \cdot m_{E\#H}\left(x^{(2)}{}_0\#x^{(3)}{}_{-1}\otimes x'^{(2)}\#x'^{(3)}{}_{-1}\right)\\ \cdot\gamma^{-1}\left(x^{(3)}{}_0\otimes x'^{(3)}{}_0\right)\end{array}\right]\\
=\; & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}x'^{(1)}\right)\\ \cdot m\left(x^{(2)}{}_0\otimes x^{(3)}{}_{\langle -2\rangle}x'^{(2)}\right)\otimes x^{(3)}{}_{-1}x'^{(3)}{}_{-1}\\ \cdot\gamma^{-1}\left(x^{(3)}{}_0\otimes x'^{(3)}{}_0\right)\end{array}\right]\\
=\; & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}x'^{(1)}\right)\\ \cdot m\left(x^{(2)}{}_0\otimes x^{(3)}{}_{-1}x'^{(2)}\right)\otimes\left(x^{(3)}{}_0\otimes x'^{(3)}\right)_{-1}\\ \cdot\gamma^{-1}\left(x^{(3)}{}_0\otimes x'^{(3)}{}_0\right)\end{array}\right]\\
\overset{\gamma^{-1}\text{ colin.}}{=} & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}x'^{(1)}\right)\cdot m\left(x^{(2)}{}_0\otimes x^{(3)}{}_{-1}x'^{(2)}\right)\otimes 1_H\\ \cdot\gamma^{-1}\left(x^{(3)}{}_0\otimes x'^{(3)}\right)\end{array}\right]\\
=\; & \left[\begin{array}{c}\sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}x'^{(1)}\right)m\left(x^{(2)}{}_0\otimes x^{(3)}{}_{-1}x'^{(2)}\right)\\ \gamma^{-1}\left(x^{(3)}{}_0\otimes x'^{(3)}\right)\end{array}\right]\otimes 1_H.
\end{aligned}$$

Now we have

$$\sum (x\otimes y)^{(1)}\otimes(x\otimes y)^{(2)} = \sum x^{(1)}\otimes x^{(2)}{}_{-1}y^{(1)}\otimes x^{(2)}{}_0\otimes y^{(2)}$$

so that

$$\begin{aligned}&\sum (x\otimes y)^{(1)}\otimes(x\otimes y)^{(2)}\otimes(x\otimes y)^{(3)}\\ &= \sum\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}y^{(1)}\right)\otimes\left(x^{(2)}{}_0\otimes x^{(3)}{}_{-1}y^{(2)}\right)\otimes\left(x^{(3)}{}_0\otimes y^{(3)}\right).\end{aligned}$$

Using this equality we can proceed in our computation:

$$
\begin{aligned}
& m_{(E\#H)^{\Gamma}}\left((x\#1_H)\otimes\left(x'\#1_H\right)\right) \\
&=\begin{bmatrix} \sum\gamma\left(x^{(1)}\otimes x^{(2)}{}_{-1}x^{(3)}{}_{\langle -2\rangle}x'^{(1)}\right) \\ \cdot m\left(x^{(2)}{}_{0}\otimes x^{(3)}{}_{-1}x'^{(2)}\right)\gamma^{-1}\left(x^{(3)}{}_{0}\otimes x'^{(3)}\right)\end{bmatrix}\otimes 1_H \\
&=\left[\sum\gamma\left(\left(x\otimes x'\right)^{(1)}\right)\cdot m\left(\left(x\otimes x'\right)^{(2)}\right)\cdot\gamma^{-1}\left(\left(x\otimes x'\right)^{(3)}\right)\right]\#1_H \\
&=\left(\gamma * m * \gamma^{-1}\right)\left(x\otimes x'\right)\#1_H \\
&= m_{E^{\gamma}}\left(x\otimes x'\right)\#1_H \\
&= m_{E^{\gamma}\#H}\left((x\#1_H)\otimes\left(x'\#1_H\right)\right).
\end{aligned}
$$

Finally $u_{(E\#H)^{\Gamma}} = u_{E\#H} = 1_E\#1_H = 1_{E^{\gamma}}\#1_H = u_{E^{\gamma}\#H}$.

As a coalgebra $(E\#H)^{\Gamma}$ coincides with $E\#H$ and hence with $E^{\gamma}\#H$.

Finally let us check that $\omega_{E^{\gamma}\#H}$ and $\omega_{(E\#H)^{\Gamma}}$ coincide. To this aim, let us use the maps $\mho^{*}_{H,-}$ of [2, Lemma 2.15]. First note that $\omega_{E^{\gamma}\#H} = \mho^{3}_{H,E^{\gamma}}\left(\omega_{E^{\gamma}}\right)$ by [2, Proposition 5.3]. Now

$$
\begin{aligned}
\omega_{(E\#H)^{\Gamma}} &= \left(\varepsilon_{E\#H}\otimes\Gamma\right) * \Gamma\left(E\#H\otimes m_{E\#H}\right) * \omega_{E\#H} \\
&\qquad * \Gamma^{-1}\left(m_{E\#H}\otimes E\#H\right) * \left(\Gamma^{-1}\otimes\varepsilon_{E\#H}\right) \\
&= \left(\mho^{1}_{H,E}(\varepsilon)\otimes\mho^{2}_{H,E}(\gamma)\right) * \mho^{2}_{H,E}(\gamma)\left(E\#H\otimes m_{E\#H}\right) * \mho^{3}_{H,E}(\omega) \\
&\qquad * \mho^{2}_{H,E}\left(\gamma^{-1}\right)\left(m_{E\#H}\otimes E\#H\right) \\
&\qquad * \left(\mho^{2}_{H,E}\left(\gamma^{-1}\right)\otimes\mho^{1}_{H,E}(\varepsilon)\right).
\end{aligned}
$$

One easily checks that

$$
\begin{aligned}
\mho^{1}_{H,E}(\varepsilon)\otimes\mho^{2}_{H,E}(\gamma) &= \mho^{3}_{H,E^{\gamma}}(\varepsilon\otimes\gamma), \\
\mho^{2}_{H,E}(\gamma)\left(E\#H\otimes m_{E\#H}\right) &= \mho^{3}_{H,E^{\gamma}}\left(\gamma\left(E\otimes m\right)\right), \\
\mho^{2}_{H,E}\left(\gamma^{-1}\right)\left(m_{E\#H}\otimes E\#H\right) &= \mho^{3}_{H,E^{\gamma}}\left(\gamma^{-1}\left(m\otimes E\right)\right), \\
\mho^{2}_{H,E}\left(\gamma^{-1}\right)\otimes\mho^{1}_{H,E}\left(\varepsilon_E\right) &= \mho^{3}_{H,E^{\gamma}}\left(\gamma^{-1}\otimes\varepsilon\right).
\end{aligned}
$$

Thus we obtain

$$\begin{aligned}\omega_{(E\#H)^\Gamma} &= \mho^3_{H,E^\gamma}(\varepsilon\otimes\gamma) * \mho^3_{H,E^\gamma}(\gamma(E\otimes m)) \\ &\qquad * \mho^3_{H,E}(\omega) * \mho^3_{H,E^\gamma}\left(\gamma^{-1}(m\otimes E)\right) \\ &\qquad * \mho^3_{H,E^\gamma}\left(\gamma^{-1}\otimes\varepsilon\right) \\ &= \mho^3_{H,E^\gamma}\left[(\varepsilon\otimes\gamma) * \gamma(E\otimes m) * \omega * \gamma^{-1}(m\otimes E) * \left(\gamma^{-1}\otimes\varepsilon\right)\right] \\ &= \mho^3_{H,E^\gamma}(\omega_{E^\gamma}) = \omega_{E^\gamma\#H}. \end{aligned}$$

□

Proposition 2.6. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Let $\gamma : Q\otimes Q \to \Bbbk$ be a gauge transformation for Q. Then $\mathrm{gr}\,Q^\gamma$ and $\mathrm{gr}\,Q$ coincide as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. By Proposition 2.4, Q^γ is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. It is obviously connected as it coincides with Q as a coalgebra. By Theorem 1.6, both $\mathrm{gr}\,Q$ and $\mathrm{gr}\,Q^\gamma$ are connected bialgebras in ${}^H_H\mathcal{YD}$. Let us check they coincide.

Note that, by Remark 2.2, we have that γ^{-1} is a gauge transformation, hence it is trivial on $\Bbbk 1_Q\otimes 1_Q$. Let $C := Q\otimes Q$. Let $n > 0$ and let $w\in C_{(n)} = \sum_{i+j\le n} Q_i\otimes Q_j$. By [6, Lemma 3.69], we have that $\Delta_C(w) - w\otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2}\otimes w \in C_{(n-1)}\otimes C_{(n-1)}$. Thus we get

$$w_1\otimes w_2\otimes w_3 - \Delta_C(w)\otimes (1_Q)^{\otimes 2} - \Delta_C\left((1_Q)^{\otimes 2}\right)\otimes w \in \Delta_C\left(C_{(n-1)}\right)\otimes C_{(n-1)}$$

and hence

$$\begin{aligned} w_1\otimes w_2\otimes w_3 - w\otimes (1_Q)^{\otimes 2}\otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2}\otimes w\otimes (1_Q)^{\otimes 2} \\ - (1_Q)^{\otimes 4}\otimes w \in C_{(n-1)}\otimes C_{(n-1)}\otimes C_{(n-1)}. \end{aligned}$$

Since $m\left(C_{(n-1)}\right)\subseteq Q_{n-1}$ we get

$$\begin{aligned} w_1\otimes m(w_2)\otimes w_3 - w\otimes 1_Q\otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2}\otimes m(w)\otimes (1_Q)^{\otimes 2} \\ - (1_Q)^{\otimes 3}\otimes w \in C_{(n-1)}\otimes Q_{n-1}\otimes C_{(n-1)} \end{aligned}$$

and hence

$$w_1\otimes (m(w_2) + Q_{n-1})\otimes w_3 = (1_Q)^{\otimes 2}\otimes (m(w) + Q_{n-1})\otimes (1_Q)^{\otimes 2}. \quad (2.3)$$

Let $x, y \in Q$. We compute

$$\begin{aligned}
\overline{x} \cdot_\gamma \overline{y} &= \left(x + Q_{|x|-1}\right) \cdot_\gamma \left(y + Q_{|y|-1}\right) \\
&= \left(x \cdot_\gamma y\right) + Q_{|x|+|y|-1} \\
&= \gamma\left((x \otimes y)_1\right) m\left((x \otimes y)_2\right) \gamma^{-1}\left((x \otimes y)_3\right) + Q_{|x|+|y|-1} \\
&= \gamma\left((x \otimes y)_1\right)\left(m\left((x \otimes y)_2\right) + Q_{|x|+|y|-1}\right) \gamma^{-1}\left((x \otimes y)_3\right) \\
&\overset{(2.3)}{=} \gamma\left(\left(1_Q\right)^{\otimes 2}\right)\left(m\left(x \otimes y\right) + Q_{|x|+|y|-1}\right) \gamma^{-1}\left(\left(1_Q\right)^{\otimes 2}\right) \\
&= m\left(x \otimes y\right) + Q_{|x|+|y|-1} = \left(x \cdot y\right) + Q_{|x|+|y|-1} = \overline{x} \cdot \overline{y}.
\end{aligned}$$

Note that Q^γ and Q have the same unit so that $\mathrm{gr}Q$ and $\mathrm{gr}Q^\gamma$ have the same unit as well. □

3. (Co)semisimple case

Assume H is a semisimple and cosemisimple Hopf algebra (*e.g.* H is finite-dimensional cosemisimple over a field of characteristic zero). Note that H is then separable (see *e.g.* [34, Corollary 3.7] or [6, Theorem 2.34]) whence finite-dimensional. Let $(Q, m, u, \Delta, \varepsilon)$ be a finite-dimensional coalgebra with multiplication and unit in ${}^H_H\mathcal{YD}$. Assume that the coradical Q_0 is a subcoalgebra of Q in ${}^H_H\mathcal{YD}$ such that $Q_0 \cdot Q_0 \subseteq Q_0$. Let $y^{n,i}$ with $1 \leq i \leq \dim\left(Q_n/Q_{n-1}\right)$ be a basis for Q_n/Q_{n-1}. Consider, for every $n > 0$, the exact sequence in ${}^H_H\mathcal{YD}$ given by

$$0 \longrightarrow Q_{n-1} \xrightarrow{s_n} Q_n \xrightarrow{\pi_n} \frac{Q_n}{Q_{n-1}} \longrightarrow 0\,.$$

Now, since H is semisimple and cosemisimple, by [30, Proposition 7] the Drinfeld double $D(H)$ is semisimple. By a result essentially due to Majid (see [28, Proposition 10.6.16]) and by [32, Proposition 6], we get that the category ${}^H_H\mathcal{YD} \cong {}_{D(H)}\mathfrak{M}$ is a semisimple category. Therefore π_n cosplits, *i.e.*, there is a morphism $\sigma_n : (Q_n/Q_{n-1}) \to Q_n$ in ${}^H_H\mathcal{YD}$ such that $\pi_n\sigma_n = \mathrm{Id}$. Let $u_n : \Bbbk \to Q_n$ be the corestriction of the unit $u : \Bbbk \to Q$ and let $\varepsilon_n = \varepsilon_{|Q_n} : Q_n \to \Bbbk$ be the counit of the subcoalgebra Q_n. Set $\sigma_n' := \sigma_n - u_n \circ \varepsilon_n \circ \sigma_n$. This is a morphism in ${}^H_H\mathcal{YD}$. Moreover

$$\begin{aligned}
\pi_n \circ \sigma_n' &= \pi_n \circ \sigma_n - \pi_n \circ u_n \circ \varepsilon_n \circ \sigma_n \overset{n>0}{=} \mathrm{Id}_{Q_n/Q_{n-1}} - 0 = \mathrm{Id}_{Q_n/Q_{n-1}}, \\
\varepsilon_n \circ \sigma_n' &= \varepsilon_n \circ \sigma_n - \varepsilon_n \circ u_n \circ \varepsilon_n \circ \sigma_n = \varepsilon_n \circ \sigma_n - \varepsilon_n \circ \sigma_n = 0.
\end{aligned}$$

Therefore, without loss of generality we can assume that $\varepsilon_n \circ \sigma_n = 0$. A standard argument on split short exact sequences shows that there exists a morphism $p_n : Q_n \to Q_{n-1}$ in ${}^H_H\mathcal{YD}$ such that $s_n p_n + \sigma_n \pi_n = \mathrm{Id}_{Q_n}$, $p_n s_n = \mathrm{Id}_{Q_{n-1}}$ and $p_n \sigma_n = 0$. We set $x^{n,i} := \sigma_n\left(y^{n,i}\right)$. Therefore

$$y^{n,i} = \pi_n\sigma_n\left(y^{n,i}\right) = \pi_n\left(x^{n,i}\right) = x^{n,i} + Q_{n-1} = \overline{x^{n,i}}.$$

These terms $x^{n,i}$ define a $\Bbbk$-basis for Q. As Q is finite-dimensional, there exists $d \in \mathbb{N}_0$ such that $Q = Q_d$; we fix d minimal. For all $0 \leq a < b$, define the maps

$$\begin{aligned} p_{a,b} &: Q_b \to Q_a, & p_{a,b} &:= p_{a+1} \circ p_{a+2} \circ \cdots \circ p_{b-1} \circ p_b, \\ s_{b,a} &: Q_a \to Q_b, & s_{b,a} &:= s_b \circ s_{b-1} \circ \cdots \circ s_{a+2} \circ s_{a+1}. \end{aligned}$$

Clearly one has $p_{a,b} \circ s_{b,a} = \mathrm{Id}_{Q_a}$. Thus, for $0 \leq i, a < b$ we have

$$p_{i,b} \circ s_{b,a} = \begin{cases} p_{i,b} \circ s_{b,i} \circ s_{i,a} & i > a \\ p_{i,a} \circ p_{a,b} \circ s_{b,a} & i \leq a \end{cases} = \begin{cases} s_{i,a} & i > a \\ p_{i,a} & i \leq a \end{cases} \tag{3.1}$$

Thus we get an isomorphism $\varphi : Q \to \mathrm{gr} Q$ of objects in ${}^H_H\mathcal{YD}$ given by

$$\begin{aligned} \varphi(x) &:= p_{0,d}(x) + \pi_1 p_{1,d}(x) + \pi_2 p_{2,d}(x) + \cdots + \pi_{d-2} p_{d-2,d}(x) \\ &\quad + \pi_{d-1} p_{d-1,d}(x) + \pi_d(x) \\ &= \sum_{0 \leq t \leq d} \pi_t p_{t,d}(x), \text{ for every } x \in Q, \end{aligned}$$

where we set

$$\pi_0 = \mathrm{Id}_{Q_0}, \qquad p_{d,d} = \mathrm{Id}_{Q_d}.$$

For $0 \leq n \leq d$, we have

$$\begin{aligned} \varphi\left(x^{n,i}\right) &= \varphi\left(s_{d,n}\left(x^{n,i}\right)\right) = \varphi\left(s_{d,n}\sigma_n\left(y^{n,i}\right)\right) = \sum_{0 \leq t \leq d} \pi_t p_{t,d} s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right) \\ &= \sum_{n < t \leq d} \pi_t p_{t,d} s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \sum_{0 \leq t \leq n} \pi_t p_{t,d} s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right) \\ &\overset{(3.1)}{=} \sum_{n < t \leq d} \pi_t s_{t,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \sum_{0 \leq t < n} \pi_t p_{t,n}\left(\sigma_n\left(y^{n,i}\right)\right) \\ &\quad + \pi_n p_{n,d} s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right) \\ &= \sum_{n < t \leq d} \pi_t s_{t,t-1} s_{t-1,n}\left(\sigma_n\left(y^{n,i}\right)\right) + \sum_{0 \leq t < n} \pi_t p_{t,n-1} p_{n-1,n}\left(\sigma_n\left(y^{n,i}\right)\right) \\ &\quad + \pi_n p_{n,d} s_{d,n}\left(\sigma_n\left(y^{n,i}\right)\right) \\ &= \sum_{n < t \leq d} \pi_t s_t s_{t-1,n} \sigma_n\left(y^{n,i}\right) + \sum_{0 \leq t < n} \pi_t p_{t,n-1} p_n \sigma_n\left(y^{n,i}\right) + \pi_n \sigma_n\left(y^{n,i}\right) \\ &= 0 + 0 + y^{n,i} = y^{n,i}. \end{aligned}$$

Hence $\varphi\left(x^{n,i}\right) = y^{n,i}$. Since $y^{n,i}$ with $1 \leq i \leq \dim\left(Q_n/Q_{n-1}\right) =: d_n$ form a basis for Q_n/Q_{n-1} we have that

$$h y^{n,i} \in \frac{Q_n}{Q_{n-1}}, \qquad \left(y^{n,i}\right)_{-1} \otimes \left(y^{n,i}\right)_0 \in H \otimes \frac{Q_n}{Q_{n-1}}.$$

Therefore there are $\chi_{t,i}^{n} \in H^{*}$ and $h_{t,i}^{n} \in H$ such that

$$hy^{n,i} = \sum_{1\leq t\leq d_n} \chi_{t,i}^{n}(h)\, y^{n,t}, \qquad \left(y^{n,i}\right)_{-1} \otimes \left(y^{n,i}\right)_{0} = \sum_{1\leq t\leq d_n} h_{i,t}^{n} \otimes y^{n,t}. \quad (3.2)$$

We have

$$\begin{aligned} h\left(h'y^{n,i}\right) &= \sum_{1\leq s\leq d_n} \chi_{s,i}^{n}\left(h'\right) hy^{n,s} = \sum_{1\leq s\leq d_n} \chi_{s,i}^{n}\left(h'\right) \sum_{1\leq t\leq d_n} \chi_{t,s}^{n}(h)\, y^{n,t} \\ &= \sum_{1\leq s\leq d_n} \sum_{1\leq t\leq d_n} \chi_{t,s}^{n}(h)\, \chi_{s,i}^{n}\left(h'\right) y^{n,t}, \\ \left(hh'\right) y^{n,i} &= \sum_{1\leq t\leq d_n} \chi_{t,i}^{n}\left(hh'\right) y^{n,t} \end{aligned}$$

and hence

$$\chi_{t,i}^{n}\left(hh'\right) = \sum_{1\leq s\leq d_n} \chi_{t,s}^{n}(h)\, \chi_{s,i}^{n}\left(h'\right).$$

Moreover

$$y^{n,i} = 1_H y^{n,i} = \sum_{1\leq t\leq d_n} \chi_{t,i}^{n}(1_H)\, y^{n,t}$$

and hence

$$\chi_{t,i}^{n}(1_H) = \delta_{t,i}.$$

We also have

$$\begin{aligned} \left(y^{n,i}\right)_{-1} \otimes \left(\left(y^{n,i}\right)_{0}\right)_{-1} \otimes \left(\left(y^{n,i}\right)_{0}\right)_{0} &= \sum_{1\leq s\leq d_n} h_{i,s}^{n} \otimes \left(y^{n,s}\right)_{-1} \otimes \left(y^{n,s}\right)_{0} \\ &= \sum_{1\leq s\leq d_n} h_{i,s}^{n} \otimes \sum_{1\leq t\leq d_n} h_{s,t}^{n} \otimes y^{n,t} \\ &= \sum_{1\leq s\leq d_n} \sum_{1\leq t\leq d_n} h_{i,s}^{n} \otimes h_{s,t}^{n} \otimes y^{n,t}, \\ \left(\left(y^{n,i}\right)_{-1}\right)_{1} \otimes \left(\left(y^{n,i}\right)_{-1}\right)_{2} \otimes \left(y^{n,i}\right)_{0} &= \sum_{1\leq t\leq d_n} \Delta_H\left(h_{t,i}^{n}\right) \otimes y^{n,t} \end{aligned}$$

so that

$$\Delta_H\left(h_{t,i}^{n}\right) = \sum_{1\leq s\leq d_n} h_{i,s}^{n} \otimes h_{s,t}^{n}.$$

Moreover

$$y^{n,i} = \varepsilon_H\left(\left(y^{n,i}\right)_{-1}\right)\left(y^{n,i}\right)_{0} = \sum_{1\leq t\leq d_n} \varepsilon_H\left(h_{t,i}^{n}\right) y^{n,t}$$

and hence

$$\varepsilon_H\left(h^n_{t,i}\right) = \delta_{t,i}.$$

Finally

$$\begin{aligned}
\left(h_1 y^{n,i}\right)_{-1} h_2 \otimes \left(h_1 y^{n,i}\right)_0 &= \sum_{1\le s\le d_n} \chi^n_{s,i}\left(h_1\right)\left(y^{n,s}\right)_{-1} h_2 \otimes \left(y^{n,s}\right)_0 \\
&= \sum_{1\le s\le d_n} \chi^n_{s,i}\left(h_1\right) \sum_{1\le t\le d_n} h^n_{s,t} h_2 \otimes y^{n,t} \\
&= \sum_{1\le s\le d_n} \sum_{1\le t\le d_n} h^n_{s,t}\chi^n_{s,i}\left(h_1\right) h_2 \otimes y^{n,t}, \\
h_1\left(y^{n,i}\right)_{-1} \otimes h_2\left(y^{n,i}\right)_0 &= \sum_{1\le s\le d_n} h_1 h^n_{i,s} \otimes h_2 y^{n,s} \\
&= \sum_{1\le s\le d_n} h_1 h^n_{i,s} \otimes \sum_{1\le t\le d_n} \chi^n_{t,s}\left(h_2\right) y^{n,t} \\
&= \sum_{1\le s\le d_n} \sum_{1\le t\le d_n} h_1 \chi^n_{t,s}\left(h_2\right) h^n_{i,s} \otimes y^{n,t}.
\end{aligned}$$

Therefore, we get

$$\sum_{1\le s\le d_n} h^n_{s,t}\chi^n_{s,i}\left(h_1\right) h_2 = \sum_{1\le s\le d_n} h_1 \chi^n_{t,s}\left(h_2\right) h^n_{i,s}.$$

We have

$$\begin{aligned}
hx^{n,i} &= h\sigma_n\left(y^{n,i}\right) = \sigma_n\left(hy^{n,i}\right) = \sigma_n\left(\sum_{1\le t\le d_n} \chi^n_{t,i}\left(h\right) y^{n,t}\right) \\
&= \sum_{1\le t\le d_n} \chi^n_{t,i}\left(h\right) x^{n,t}, \\
\left(x^{n,i}\right)_{-1} \otimes \left(x^{n,i}\right)_0 &= \left(\sigma_n\left(y^{n,i}\right)\right)_{-1} \otimes \left(\sigma_n\left(y^{n,i}\right)\right)_0 \\
&= \left(y^{n,i}\right)_{-1} \otimes \sigma_n\left(\left(y^{n,i}\right)_0\right) = \sum_{1\le t\le d_n} h^n_{i,t} \otimes x^{n,t}, \\
\varepsilon_Q\left(x^{n,i}\right) &= \varepsilon_n\left(x^{n,i}\right) = \varepsilon_n\sigma_n\left(y^{n,i}\right) = 0 \text{ for } n > 0.
\end{aligned}$$

If Q is connected, then $d_0 = 1$ so we may assume $y^{0,0} := 1_Q + Q_{-1}$. Since $\pi_0 = \mathrm{Id}_{Q_0}$ we get

$$\sigma_0 = \mathrm{Id}_{Q_0} \circ \sigma_0 = \pi_0 \circ \sigma_0 = \mathrm{Id}_{Q_0}$$

and hence

$$x^{0,0} = \sigma_0\left(y^{0,0}\right) = \sigma_0\left(1_Q + Q_{-1}\right) = 1_Q.$$

Since, by Proposition 1.3, $Q_a \cdot Q_{a'} \subseteq Q_{a+a'}$ for every $a, a' \in \mathbb{N}_0$, we can write the product of two elements of the basis in the form

$$x^{a,l}x^{a',l'} = \sum_{u \leq a+a'} \sum_{v} \mu_{u,v}^{a,l,a',l'} x^{u,v}. \tag{3.3}$$

We compute

$$\begin{aligned}
\overline{x^{a,l}} \cdot \overline{x^{a',l'}} &= \left(x^{a,l} + Q_{a-1}\right)\left(x^{a',l'} + Q_{a'-1}\right) \\
&= \left(x^{a,l}x^{a',l'}\right) + Q_{a+a'-1} \\
&\overset{(3.3)}{=} \left(\sum_{u \leq a+a'} \sum_{v} \mu_{u,v}^{a,l,a',l'} x^{u,v}\right) + Q_{a+a'-1} \\
&= \left(\sum_{v} \mu_{a+a',v}^{a,l,a',l'} x^{a+a',v}\right) + Q_{a+a'-1} \\
&= \sum_{v} \mu_{a+a',v}^{a,l,a',l'} \left(x^{a+a',v} + Q_{a+a'-1}\right) \\
&= \sum_{v} \mu_{a+a',v}^{a,l,a',l'} \overline{x^{a+a',v}}
\end{aligned}$$

which gives

$$\overline{x^{a,l}} \cdot \overline{x^{a',l'}} = \sum_{v} \mu_{a+a',v}^{a,l,a',l'} \overline{x^{a+a',v}}. \tag{3.4}$$

Remark 3.1. Let H be a Hopf algebra and let (A, m_A, u_A) be an algebra in ${}_H^H\mathcal{YD}$. Let $\varepsilon_A : A \to \Bbbk$ be an algebra map in ${}_H^H\mathcal{YD}$. The Hochschild cohomology in a monoidal category is known, see *e.g.* [7]. Consider $\Bbbk$ as an A-bimodule in ${}_H^H\mathcal{YD}$ through ε_A. Then, following [7, 1.24], we can consider an analogue of the standard complex

$${}_H^H\mathcal{YD}(\Bbbk,\Bbbk) \xrightarrow{\partial^0} {}_H^H\mathcal{YD}(A,\Bbbk) \xrightarrow{\partial^1} {}_H^H\mathcal{YD}(A^{\otimes 2},\Bbbk) \xrightarrow{\partial^2} {}_H^H\mathcal{YD}(A^{\otimes 3},\Bbbk) \xrightarrow{\partial^3} \cdots$$

Explicitly, given f in the corresponding domain of ∂^n, for $n = 0, 1, 2, 3$, we have

$$\begin{aligned}
\partial^0(f) &= f(1)\varepsilon_A - \varepsilon_A f(1) = 0, \\
\partial^1(f) &= f \otimes \varepsilon_A - f m_A + \varepsilon_A \otimes f, \\
\partial^2(f) &= f \otimes \varepsilon_A - f(A \otimes m_A) + f(m_A \otimes A) - \varepsilon_A \otimes f, \\
\partial^3(f) &= f \otimes \varepsilon_A - f(A \otimes A \otimes m_A) + f(A \otimes m_A \otimes A) \\
&\quad - f(m_A \otimes A \otimes A) + \varepsilon_A \otimes f.
\end{aligned}$$

For every $n \geq 1$ denote by

$$\mathrm{Z}^n_{\mathcal{YD}}(A,\Bbbk){:=}\ker\left(\partial^n\right),\quad \mathrm{B}^n_{\mathcal{YD}}(A,\Bbbk){:=}\mathrm{Im}\left(\partial^{n-1}\right)\quad\text{and}\quad \mathrm{H}^n_{\mathcal{YD}}(A,\Bbbk){:=}\frac{\mathrm{Z}^n_{\mathcal{YD}}(A,\Bbbk)}{\mathrm{B}^n_{\mathcal{YD}}(A,\Bbbk)}$$

the Abelian groups of n-cocycles, of n-coboundaries and the n-th Hochschild cohomology group in ${}^H_H\mathcal{YD}$ of the algebra A with coefficients in $\Bbbk$. We point out that the construction above works for an arbitrary A-bimodule M in ${}^H_H\mathcal{YD}$ instead of $\Bbbk$.

Our next result is inspired by [18, Proposition 2.3]. Two coquasi-bialgebras Q and Q' in ${}^H_H\mathcal{YD}$ will be called *gauge equivalent* whenever there is some gauge transformation $\gamma : Q \otimes Q \to \Bbbk$ in ${}^H_H\mathcal{YD}$ such that $Q^\gamma \cong Q'$ as coquasi-bialgebras in ${}^H_H\mathcal{YD}$, see Proposition 2.4 for the structure of Q^γ.

Theorem 3.2. *Let H be a semisimple and cosemisimple Hopf algebra and let $(Q,m,u,\Delta,\varepsilon,\omega)$ be a finite-dimensional connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. If $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q,\Bbbk) = 0$ then Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.*

Proof. For $t \in \mathbb{N}_0$, and x, y, z in the basis of Q, we set

$$\omega_t(x\otimes y\otimes z) := \delta_{|x|+|y|+|z|,t}\,\omega(x\otimes y\otimes z).$$

Let us check it defines a morphism $\omega_t : Q\otimes Q\otimes Q \to \Bbbk$ in ${}^H_H\mathcal{YD}$. It is left H-linear as, by means of (3.2), the definition of ω_t and the H-linearity of ω, we can prove that $\omega_t\left(h\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)\right) = \varepsilon_H(h)\,\omega_t\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)$.

Moreover it is left H-colinear as, by means of (3.2), the definition of ω_t and the H-colinearity of ω, we can prove that

$$\begin{aligned}&\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)_{-1}\otimes\omega_t\left(\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right)_0\right)\\ &\quad= 1_H\otimes\omega_t\left(x^{n,i}\otimes x^{n',i'}\otimes x^{n'',i''}\right).\end{aligned}$$

Clearly, for $x, y, z \in Q$ in the basis, one has

$$\sum_{t\in\mathbb{N}_0}\omega_t(x\otimes y\otimes z) = \sum_{t\in\mathbb{N}_0}\delta_{|x|+|y|+|z|,t}\,\omega(x\otimes y\otimes z) = \omega(x\otimes y\otimes z)$$

so that we can formally write

$$\omega = \sum_{t\in\mathbb{N}_0}\omega_t. \tag{3.5}$$

Since ε is trivial on elements in the basis of strictly positive degree, one gets

$$\omega_0 = \varepsilon\otimes\varepsilon\otimes\varepsilon. \tag{3.6}$$

If $\omega = \omega_0$ then Q is a (connected) bialgebra in ${}_H^H\mathcal{YD}$ and the proof is finished. Thus we can assume $\omega \neq \omega_0$ and set

$$\begin{aligned} s &:= \min\{i \in \mathbb{N} : \omega_i \neq 0\}, \\ \overline{\omega}_s &:= \omega_s\left(\varphi^{-1} \otimes \varphi^{-1} \otimes \varphi^{-1}\right), \\ \overline{Q} &:= \mathrm{gr}Q. \end{aligned}$$

Note that $\overline{\omega}_s$ is a morphism in ${}_H^H\mathcal{YD}$ as a composition of morphisms in ${}_H^H\mathcal{YD}$.

Let $n \in \mathbb{N}_0$, let $C^4 = Q \otimes Q \otimes Q \otimes Q$ and let $u \in C^4_{(n)} = \sum_{i+j+k+l \leq n} Q_i \otimes Q_j \otimes Q_k \otimes Q_l$.

A direct computation rewriting the cocycle condition using (3.5) proves that, for every $n \in \mathbb{N}_0$, and $u \in C^4_{(n)}$

$$\begin{aligned} &\sum_{0 \leq i+j \leq n} \left[\omega_i (Q \otimes Q \otimes m) * \omega_j (m \otimes Q \otimes Q)\right](u) \\ &= \sum_{0 \leq a+b+c \leq n} \left[(\varepsilon \otimes \omega_a) * \omega_b (Q \otimes m \otimes Q) * (\omega_c \otimes \varepsilon)\right](u). \end{aligned} \tag{3.7}$$

Next aim is to check that $[\overline{\omega}_s] \in \mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q, \Bbbk)$ *i.e.*, that

$$\begin{aligned} \overline{\omega}_s\left(m_{\overline{Q}} \otimes \overline{Q} \otimes \overline{Q}\right) + \overline{\omega}_s\left(\overline{Q} \otimes \overline{Q} \otimes m_{\overline{Q}}\right) = \left(\varepsilon_{\overline{Q}} \otimes \overline{\omega}_s\right) + \overline{\omega}_s\left(\overline{Q} \otimes m_{\overline{Q}} \otimes \overline{Q}\right) \\ + \left(\overline{\omega}_s \otimes \varepsilon_{\overline{Q}}\right). \end{aligned}$$

This is achieved by evaluating the two sides of the equality above on $\overline{u} := \overline{x} \otimes \overline{y} \otimes \overline{z} \otimes \overline{t}$ where x, y, z, t are elements in the basis and using (3.4). If $\overline{u}$ has homogeneous degree greater than s, then both terms are zero. Otherwise, *i.e.*, if $\overline{u}$ has homogeneous degree at most s, one has $\overline{\omega}_s\left(m_{\overline{Q}} \otimes \overline{Q} \otimes \overline{Q}\right)(\overline{u}) = \omega_s\left(m_Q \otimes Q \otimes Q\right)(u)$ and similarly for the other pieces so that one has to check that

$$\begin{aligned} \omega_s (m \otimes Q \otimes Q)(u) + \omega_s (Q \otimes Q \otimes m)(u) = (\varepsilon \otimes \omega_s)(u) \\ + \omega_s (Q \otimes m \otimes Q)(u) \\ + (\omega_s \otimes \varepsilon)(u). \end{aligned}$$

This equality follows by using (3.7) and the definition of s.

By assumption $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q, \Bbbk) = 0$ so that there exists a morphism $\overline{v} : \overline{Q} \otimes \overline{Q} \to \Bbbk$ in ${}_H^H\mathcal{YD}$ such that

$$\overline{\omega}_s = \partial^2 \overline{v} = \overline{v} \otimes \varepsilon_{\overline{Q}} - \overline{v}\left(\overline{Q} \otimes m_{\overline{Q}}\right) + \overline{v}\left(m_{\overline{Q}} \otimes \overline{Q}\right) - \varepsilon_{\overline{Q}} \otimes \overline{v}.$$

Explicitly, on elements of the basis we get

$$\overline{\omega}_s (\overline{x} \otimes \overline{y} \otimes \overline{z}) = \overline{v}(\overline{x} \otimes \overline{y})\, \varepsilon_{\overline{Q}}(\overline{z}) - \overline{v}(\overline{x} \otimes \overline{y} \cdot \overline{z}) + \overline{v}(\overline{x} \cdot \overline{y} \otimes \overline{z}) - \varepsilon_{\overline{Q}}(\overline{x})\, \overline{v}(\overline{y} \otimes \overline{z}).$$

Define $\overline{\zeta}:\overline{Q}\otimes\overline{Q}\to\Bbbk$ on the basis by setting

$$\overline{\zeta}\left(\overline{x}\otimes\overline{y}\right):=\delta_{|x|+|y|,s}\overline{v}\left(\overline{x}\otimes\overline{y}\right).$$

As we have done for ω_t, one can check that $\overline{\zeta}$ is a morphism in ${}_H^H\mathcal{YD}$.

Moreover on elements in the basis we get

$$\begin{aligned}&\left(\partial^2\overline{\zeta}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)\\ =&\left(\overline{\zeta}\otimes\varepsilon_{\overline{Q}}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)-\overline{\zeta}\left(\overline{Q}\otimes m_{\overline{Q}}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)\\ &+\overline{\zeta}\left(m_{\overline{Q}}\otimes\overline{Q}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)-\left(\varepsilon_{\overline{Q}}\otimes\overline{\zeta}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)\\ =&\,\overline{\zeta}\left(\overline{x}\otimes\overline{y}\right)\varepsilon_{\overline{Q}}\left(\overline{z}\right)-\overline{\zeta}\left(\overline{x}\otimes\overline{y}\cdot\overline{z}\right)+\overline{\zeta}\left(\overline{x}\cdot\overline{y}\otimes\overline{z}\right)-\varepsilon_{\overline{Q}}\left(\overline{x}\right)\overline{\zeta}\left(\overline{y}\otimes\overline{z}\right).\end{aligned}$$

By using (3.4), one gets

$$\overline{\zeta}\left(\overline{x}\otimes\overline{y}\cdot\overline{z}\right)=\delta_{|x|+|y|+|z|,s}\overline{v}\left(\overline{x}\otimes\overline{y}\cdot\overline{z}\right)\quad\text{and}\quad\overline{\zeta}\left(\overline{x}\cdot\overline{y}\otimes\overline{z}\right)=\delta_{|x|+|y|+|z|,s}\overline{v}\left(\overline{x}\cdot\overline{y}\otimes\overline{z}\right).$$

By means of these equalities one gets

$$\begin{aligned}\left(\partial^2\overline{\zeta}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)&=\delta_{|x|+|y|+|z|,s}\left(\partial^2\overline{v}\right)\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)\\ &=\delta_{|x|+|y|+|z|,s}\overline{\omega}_s\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right)\\ &=\delta_{|x|+|y|+|z|,s}\omega_s\left(x\otimes y\otimes z\right)\\ &=\delta_{|x|+|y|+|z|,s}\delta_{|x|+|y|+|z|,s}\omega\left(x\otimes y\otimes z\right)\\ &=\delta_{|x|+|y|+|z|,s}\omega\left(x\otimes y\otimes z\right)\\ &=\omega_s\left(x\otimes y\otimes z\right)=\overline{\omega}_s\left(\overline{x}\otimes\overline{y}\otimes\overline{z}\right).\end{aligned}$$

Therefore $\partial^2\overline{\zeta}=\overline{\omega}_s$. This means that we can assume that $\overline{v}\left(\overline{x}\otimes\overline{y}\right)=0$ for $|x|+|y|\neq s$. Equivalently

$$\overline{v}\left(\overline{x}\otimes\overline{y}\right)=\delta_{|x|+|y|,s}\overline{v}\left(\overline{x}\otimes\overline{y}\right)\text{ for }x,y\text{ in the basis.}\tag{3.8}$$

Set

$$v:=\overline{v}\circ(\varphi\otimes\varphi)\qquad\text{and}\qquad\gamma:=(\varepsilon\otimes\varepsilon)+v.$$

In particular, one gets

$$v\left(x\otimes y\right)=\delta_{|x|+|y|,s}v\left(x\otimes y\right)\text{ for }x,y\text{ in the basis.}\tag{3.9}$$

Note also that both v and γ are morphisms in ${}_H^H\mathcal{YD}$ as they are obtained as composition or sum of morphisms in this category. Let us check that γ is a gauge transformation on Q in ${}_H^H\mathcal{YD}$.

Recall that $x^{0,0} = 1_Q$ is in the basis. For x in the basis, we have $\gamma\left(x \otimes 1_Q\right) = \varepsilon(x) + v\left(x \otimes 1_Q\right)$. Note that

$$\begin{aligned} 0 &= \delta_{|x|,s}\varepsilon(x) = \delta_{|x|+|1_Q|+|1_Q|,s}\omega\left(x\otimes 1_Q\otimes 1_Q\right) \\ &= \omega_s\left(x\otimes 1_Q\otimes 1_Q\right) = \overline{\omega}_s\left(\overline{x}\otimes\overline{1_Q}\otimes\overline{1_Q}\right) \\ &= \overline{v}\left(\overline{x}\otimes\overline{1_Q}\right)\varepsilon_{\overline{Q}}\left(\overline{1_Q}\right) - \overline{v}\left(\overline{x}\otimes\overline{1_Q}\cdot\overline{1_Q}\right) + \overline{v}\left(\overline{x}\cdot\overline{1_Q}\otimes\overline{1_Q}\right) - \varepsilon_{\overline{Q}}(\overline{x})\overline{v}\left(\overline{1_Q}\otimes\overline{1_Q}\right) \\ &\overset{(3.8)}{=} \overline{v}\left(\overline{x}\otimes\overline{1_Q}\right) - \overline{v}\left(\overline{x}\otimes\overline{1_Q}\right) + \overline{v}\left(\overline{x}\otimes\overline{1_Q}\right) - \varepsilon_{\overline{Q}}(\overline{x})\delta_{|1_Q|+|1_Q|,s}\overline{v}\left(\overline{1_Q}\otimes\overline{1_Q}\right) \\ &= v\left(x\otimes 1_Q\right) \end{aligned}$$

so that $v\left(x \otimes 1_Q\right) = 0$ and hence $\gamma\left(x \otimes 1_Q\right) = \varepsilon(x) + v\left(x \otimes 1_Q\right) = \varepsilon(x)$. Similarly one proves $\gamma\left(1_Q \otimes x\right) = \varepsilon(x)$. Hence γ is unital. Note that the coalgebra $C = Q \otimes Q$ is connected as Q is. Thus, in order to prove that $\gamma : Q \otimes Q \to \Bbbk$ is convolution invertible it suffices to check (see [28, Lemma 5.2.10]) that $\gamma_{|\Bbbk 1_Q \otimes \Bbbk 1_Q}$ is convolution invertible. But for $k, k' \in \Bbbk$ we have

$$\gamma\left(k1_Q \otimes k'1_Q\right) = kk'\gamma\left(1_Q \otimes 1_Q\right) = kk'\varepsilon\left(1_Q\right) = kk' = (\varepsilon \otimes \varepsilon)\left(k1_Q \otimes k'1_Q\right)$$

Hence $\gamma_{|\Bbbk 1_Q \otimes \Bbbk 1_Q} = (\varepsilon \otimes \varepsilon)_{|\Bbbk 1_Q \otimes \Bbbk 1_Q}$ which is convolution invertible. Thus there is a $\Bbbk$-linear map $\gamma^{-1} : Q \otimes Q \to \Bbbk$ and such that

$$\gamma * \gamma^{-1} = \varepsilon \otimes \varepsilon = \gamma^{-1} * \gamma.$$

Note that, by Lemma 2.3, $\gamma \in {}^H_H\mathcal{YD}$ implies $\gamma^{-1} \in {}^H_H\mathcal{YD}$.

Therefore γ is a gauge transformation for Q. By Proposition 2.4, Q^γ is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. By Proposition 2.6, we have that $\mathrm{gr}Q^\gamma$ and $\mathrm{gr}Q$ coincide as bialgebras in ${}^H_H\mathcal{YD}$. Hence $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q^\gamma, \Bbbk) = \mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q, \Bbbk) = 0$. Therefore Q^γ fulfills the same requirement of Q as in the statement. Let us check that $(\omega^\gamma)_t = 0$ for $1 \le t \le s$ (this will complete the proof by an induction process as Q is finite-dimensional).

Note that the definition of γ and (3.9) imply

$$\gamma(x \otimes y) = \delta_{|x|+|y|,0}\gamma(x \otimes y) + \delta_{|x|+|y|,s}\gamma(x \otimes y) \quad \text{for } x, y \text{ in the basis.} \tag{3.10}$$

Let $C^2 = Q \otimes Q$ and let $C^2_{(n)} = \sum_{i+j\le n} Q_i \otimes Q_j$. For $u \in C^2_{(2s-1)}$ we have

$$[\gamma * ((\varepsilon \otimes \varepsilon) - v)](u) = (\varepsilon \otimes \varepsilon)(u) - v(u) + v(u) - v(u_1)v(u_2) \overset{(3.9)}{=} (\varepsilon \otimes \varepsilon)(u).$$

Therefore $[\gamma * ((\varepsilon \otimes \varepsilon) - v)]_{|C^2_{(2s-1)}} = (\varepsilon \otimes \varepsilon)_{|C^2_{(2s-1)}}$. By uniqueness of the convolution inverse, we deduce

$$\gamma^{-1}(u) = (\varepsilon \otimes \varepsilon)(u) - v(u), \quad \text{for } u \in C^2_{(2s-1)}. \tag{3.11}$$

Let x, y, z be in the basis. Set $\overline{u} := \overline{x} \otimes \overline{y} \otimes \overline{z}$ and $u := x \otimes y \otimes z$. We compute

$$\begin{aligned}
&\left(\omega^{\gamma}\right)_s(u) = \delta_{|x|+|y|+|z|,s}\,\omega^{\gamma}(u)\\
&= \delta_{|x|+|y|+|z|,s}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\omega*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&= \delta_{|x|+|y|+|z|,s}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*(\omega_0+\omega_s)*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&\overset{(3.6)}{=} \delta_{|x|+|y|+|z|,s}\left[\begin{array}{c}(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)+\\ (\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\omega_s*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\end{array}\right](u)\\
&= \left[\begin{array}{c}\delta_{|x|+|y|+|z|,s}(\varepsilon\otimes\gamma)(u_1)\cdot\gamma(Q\otimes m)(u_2)\cdot\gamma^{-1}(m\otimes Q)(u_3)\cdot\left(\gamma^{-1}\otimes\varepsilon\right)(u_4)+\\ \delta_{|x|+|y|+|z|,s}(\varepsilon\otimes\gamma)(u_1)\cdot\gamma(Q\otimes m)(u_2)\cdot\omega_s(u_3)\cdot\gamma^{-1}(m\otimes Q)(u_4)\cdot\left(\gamma^{-1}\otimes\varepsilon\right)(u_5)\end{array}\right].
\end{aligned}$$

Now, all terms appearing in the last two lines, excepted ω_s, vanish out of degrees 0 and s and coincide with $\varepsilon \otimes \varepsilon \otimes \varepsilon$ on degree 0. On the other hand ω_s vanishes out of s. Since $\gamma := (\varepsilon \otimes \varepsilon) + v$ and in view of (3.11), the term $\delta_{|x|+|y|+|z|,s}$ forces the following simplification

$$\left(\omega^{\gamma}\right)_s(u) = \left[\begin{array}{c}\delta_{|x|+|y|+|z|,s}[(\varepsilon\otimes v)(u)+v(Q\otimes m)(u)-v(m\otimes Q)(u)-(v\otimes\varepsilon)(u)]+\\ +\delta_{|x|+|y|+|z|,s}\,\omega_s(u)\end{array}\right].$$

Now $\omega_s(u) = \overline{\omega}_s(\overline{u})$ while one proves that

$$(\varepsilon\otimes v)(u) = \left(\varepsilon_{\overline{Q}}\otimes\overline{v}\right)(\overline{u}),\ \delta_{|x|+|y|+|z|,s}\,v(m\otimes Q)(u) = \delta_{|x|+|y|+|z|,s}\,\overline{v}\left(m_{\overline{Q}}\otimes\overline{Q}\right)(\overline{u})$$

and similarly for the other pieces of the equality.

Thus one gets

$$\begin{aligned}
\left(\omega^{\gamma}\right)_s(u) &= \left[\begin{array}{c}\delta_{|x|+|y|+|z|,s}\left[\left(\varepsilon_{\overline{Q}}\otimes\overline{v}\right)(\overline{u})+\overline{v}\left(\overline{Q}\otimes m_{\overline{Q}}\right)(\overline{u})-\overline{v}\left(m_{\overline{Q}}\otimes\overline{Q}\right)(\overline{u})\right.\\ \left.-\left(\overline{v}\otimes\varepsilon_{\overline{Q}}\right)(\overline{u})\right]+\delta_{|x|+|y|+|z|,s}\,\overline{\omega}_s(\overline{u})\end{array}\right]\\
&= -\delta_{|x|+|y|+|z|,s}\,\partial^2\overline{v}+\delta_{|x|+|y|+|z|,s}\,\overline{\omega}_s(\overline{u}) = 0.
\end{aligned}$$

For $0 \le t \le s-1$, analogously to the above, we compute

$$\begin{aligned}
\left(\omega^{\gamma}\right)_t(u) &= \delta_{|x|+|y|+|z|,t}\,\omega^{\gamma}(u)\\
&= \delta_{|x|+|y|+|z|,t}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\omega*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&= \delta_{|x|+|y|+|z|,t}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\omega_0*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&\overset{(3.6)}{=} \delta_{|x|+|y|+|z|,t}\left[(\varepsilon\otimes\gamma)*\gamma(Q\otimes m)*\gamma^{-1}(m\otimes Q)*\left(\gamma^{-1}\otimes\varepsilon\right)\right](u)\\
&= \delta_{|x|+|y|+|z|,t}(\varepsilon\otimes\varepsilon\otimes\varepsilon)(u) = \delta_{0,t}(\varepsilon\otimes\varepsilon\otimes\varepsilon)(u).
\end{aligned}$$

Therefore we can now repeat the argument on ω^γ instead of ω. Deforming several times we will get a reassociator, say ω', whose first non trivial component ω'_t, with $t \neq 0$, exceeds the dimension of Q. In other words $\omega' = \omega'_0$ which is trivial. Hence Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$. $\square$

4. Invariants

Given a $\Bbbk$-algebra A, we denote by $\mathrm{H}^n(A, -)$ the n-th right derived functor of $\mathrm{Hom}_{A,A}(A, -)$ in the category of A-bimodules. In other words, for every A-bimodule M, $\mathrm{H}^n(A, M)$ is the Hochschild cohomology group of A with coefficients in M. Denote by $\mathrm{Z}^n(A, M)$ and $\mathrm{B}^n(A, M)$ the Abelian groups of n-cocycles and of n-coboundaries respectively.

Let H be a Hopf algebra, let B be a left H-module algebra and let M be a $B\#H$-bimodule, where $B\#H$ denotes the smash product algebra, see *e.g.* [28, Definition 4.1.3]. Then $\mathrm{H}^n(B, M)$ becomes an H-bimodule as follows. Its structure of left H-module is given via ε_H and its structure of right H-module is defined, for every $f \in \mathrm{Z}^n(B, M)$ and $h \in H$, by setting

$$[f]h := \left[\chi_n^h(M)(f)\right]$$

where, for every $k \in \Bbbk$, $b_1, \ldots, b_n \in B$, we set

$$\begin{aligned}\chi_0^h(M)(f)(k) &:= (1_B\#S(h_1))\, f(k)\,(1_B\#h_2)\\ &\quad\text{for } n = 0 \text{ while and for } n \geq 1\\ \chi_n^h(M)(f)(b_1 \otimes b_2 \otimes \cdots \otimes b_n) &:= (1_B\#S(h_1))\, f(h_2 b_1 \otimes h_3 b_2 \otimes \ldots\\ &\quad \ldots \otimes h_{n+1} b_n)\,(1_B\#h_{n+2}).\end{aligned}$$

Moreover

$$\partial^n \circ \chi_n^h(M) = \chi_{n+1}^h(M) \circ \partial^n, \text{ for every } n \geq -1, \tag{4.1}$$

where $\partial^n : \mathrm{Hom}_\Bbbk\left(B^{\otimes n}, M\right) \to \mathrm{Hom}_\Bbbk\left(B^{\otimes(n+1)}, M\right)$ denotes the differential of the usual Hochschild cohomology.

Denote by $\mathrm{H}^n(B, M)^H$ the space of H-invariant elements of $\mathrm{H}^n(B, M)$.

Proposition 4.1. *Let H be a semisimple Hopf algebra and let B be a left H-module algebra. Denote by $A := B\#H$. Then, for each $n \in \mathbb{N}_0$ and for every A-bimodule M*

$$\mathrm{H}^n(B\#H, M) \cong \mathrm{H}^n(B, M)^H.$$

Proof. We will apply [34, Equation (3.6.1)]. To this aim we have to prove first that A/B is an H-Galois extension such that A is flat as left and right B-module. Now, $A = B\#_\xi H$ for $\xi : H \otimes H \to B$ defined by $\xi(x, y) = \varepsilon_H(x)\varepsilon_H(y)1_A$, *cf.* [28, Definition 7.1.1]. Moreover a direct computation shows that $\iota : B \to A : b \mapsto b\#1_H$ is a right H-extension where A is regarded as a right H-comodule via

$\rho : A \to A \otimes H : b\#h \mapsto (b\#h_1) \otimes h_2$. Thus, by [28, Proposition 7.2.7], we know that $\iota : B \to A$ is H-cleft and hence, by [28, Theorem 8.2.4], it is H-Galois. The B-bimodule structure of A is induced by ι so that, explicitly, we have

$$\begin{aligned} b'(b\#h) &= \left(b'\#1_H\right)(b\#h) = b'b\#h, \\ (b\#h)\,b' &= (b\#h)\left(b'\#1_H\right) = b\left(h_1 b'\right)\#h_2. \end{aligned}$$

Note that $A = B\#H$ is flat as a left B-module as H is a free $\Bbbk$-module ($\Bbbk$ is a field). Now consider the map $\alpha : H \otimes B \to A$ defined by setting $\alpha(h \otimes b) := h_1 b \otimes h_2$ (note that it is defined as the braiding in ${}^H_H\mathcal{YD}$). We have

$$\alpha\left(h \otimes bb'\right) = h_1\left(bb'\right) \otimes h_2 = (h_1 b)\left(h_2 b'\right) \otimes h_3 = (h_1 b\#h_2)\,b' = \alpha(h \otimes b)\,b'$$

so that α is right B-linear where $H \otimes B$ is regarded as a right module via $(h\#b)\,b' := h\#bb'$. Now H is semisimple and hence separable (see [34, Corollary 3.7]). Thus H is finite-dimensional and hence it has bijective antipode S_H. Thus α is invertible with inverse given by $\alpha^{-1}(b\#h) := h_2 \otimes S_H^{-1}(h_1)\,b$. Therefore α is an isomorphism of right B-modules and hence A is flat as a right B-module as $H \otimes B$ is.

We have now the hypotheses necessary to apply [34, Equation (3.6.1)] and obtain

$$\mathrm{H}^n(A, M) \cong \mathrm{Hom}_{-,H}\left(\Bbbk, \mathrm{H}^n(B, M)\right) = \mathrm{Hom}_{\Bbbk}\left(\Bbbk, \mathrm{H}^n(B, M)\right)^H \cong \mathrm{H}^n(B, M)^H.$$

□

Remark 4.2. Proposition 4.1 in the particular case when $M = \Bbbk$ and B is finite-dimensional is [36, Theorem 2.17]. Note that in the notation therein, one has $E(B) = \oplus_{n\in\mathbb{N}_0} E_n(B, \Bbbk)$ where $E_n(B, \Bbbk) = \mathrm{Ext}^n_B(\Bbbk, \Bbbk) \cong \mathrm{H}^n(B, \Bbbk)$. The latter isomorphism is [15, Corollary 4.4, page 170].

Let H be a Hopf algebra and let B be a bialgebra in the braided category ${}^H_H\mathcal{YD}$. Denote by $A := B\#H$ the Radford-Majid bosonization of B by H, see *e.g.* [31, Theorem 1]. Note that A is endowed with an algebra map $\varepsilon_A : A \to \Bbbk$ defined by $\varepsilon_A(b\#h) = \varepsilon_B(b)\,\varepsilon_H(h)$ so that we can regard $\Bbbk$ as an A-bimodule via ε_A. Then we can consider $\mathrm{H}^n(B, \Bbbk)$ as an H-bimodule as follows. Its structure of left H-module is given via ε_H and its structure of right H-module is defined, for every $f \in \mathrm{Z}^n(B, \Bbbk)$ and $h \in H$, by setting

$$[f]\,h := [fh],$$

where $(fh)(z) = f(hz)$, for every $z \in B^{\otimes n}$. The latter is the usual right H-module structure of $\mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)$. Indeed, for every $n \geq -1$, the vector space $\mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)$ is an H-bimodule with respect to this right H-module structure and the left one induced by ε_H.

Corollary 4.3. *Let H be a semisimple Hopf algebra and let B be a bialgebra in the braided category ${}^H_H\mathcal{YD}$. Set $A := B\#H$. Then, for each $n \in \mathbb{N}_0$*

$$\mathrm{H}^n(B\#H, \Bbbk) \cong \mathrm{H}^n(B, \Bbbk)^H$$

and the differential $\partial^n : \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right) \to \mathrm{Hom}_{\Bbbk}\left(B^{\otimes(n+1)}, \Bbbk\right)$ *of the usual Hochschild cohomology is* H*-bilinear.*

Proof. In the particular case $M = \Bbbk$, the right module H-structure used in Proposition 4.1 simplifies as follows. It is defined, for every $f \in \mathrm{Z}^n(B, \Bbbk)$ and $h \in H$, by setting

$$[f]\, h := \left[\chi_n^h(\Bbbk)(f)\right]$$

where, for every $k \in \Bbbk, b_1, \ldots, b_n \in B$, we set

$$\chi_0^h(\Bbbk)(f)(k) := \varepsilon_H(h)\, f(k) \text{ for } n = 0 \text{ while and for } n \geq 1$$
$$\chi_n^h(\Bbbk)(f)(b_1 \otimes b_2 \otimes \cdots \otimes b_n) := f(h_1 b_1 \otimes h_2 b_2 \otimes \cdots \otimes h_n b_n).$$

More concisely $\chi_n^h(\Bbbk)(f)(z) = f(hz)$ for every $n \in \mathbb{N}_0$ and $z \in B^{\otimes n}$ *i.e.* $[f]\, h := [fh]$ where $fh := \chi_n^h(\Bbbk)(f)$.

Now consider the differential $\partial^n : \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right) \to \mathrm{Hom}_{\Bbbk}\left(B^{\otimes(n+1)}, \Bbbk\right)$ of the usual Hochschild cohomology. Note that for each $n \in \mathbb{N}_0$, $\mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)$ is regarded as a bimodule over H using the left H-module structures of its arguments. By (4.1), we have

$$\partial^n \chi_n^h(\Bbbk)(f) = \chi_{n+1}^h(\Bbbk)\, \partial^n(f)$$

Since $\chi_n^h(\Bbbk)(f) = fh$, the last displayed equality becomes $\partial^n(fh) = \partial^n(f)\, h$ for every $n \in \mathbb{N}_0$. Thus ∂^n is right H-linear. Since $hf = \varepsilon_H(h)\, f$ for every $f \in \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)$, $h \in H$, we get that ∂^n is also left H-linear whence H-bilinear. □

Remark 4.4. Note that, in the context of the proof of [18, Proposition 5.1], one has

$$\mathrm{H}^3\left(\mathcal{B}(V)\#\mathbb{C}\left[\mathbb{Z}_p\right], \mathbb{C}\right) \cong \mathrm{H}^3\left(\mathcal{B}(V), \mathbb{C}\right)^{\mathbb{Z}_p}.$$

This is a particular case of Corollary 4.3 where $H = \mathbb{C}\left[\mathbb{Z}_p\right]$, $V \in {}_H^H\mathcal{YD}$ and $B = \mathcal{B}(V)$.

Proposition 4.5. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be Abelian categories. Let* $r, \omega : \mathcal{C} \to \mathcal{D}$ *be exact functors such that* r *is a subfunctor of* ω i.e.*, there is a natural transformation* $\eta : r \to \omega$ *which is a monomorphism when evaluated on objects. If* X *is a subobject of* Y *then* $r(X) = \omega(X) \cap r(Y)$. *Moreover, for every morphism* $f : X \to Y$ *in* $\mathcal{C}$ *one has*

$$\ker(r(f)) = r(\ker(f)) = \omega(\ker(f)) \cap r(X) = \ker(\omega(f)) \cap r(X),$$
$$\mathrm{Im}(r(f)) = \mathrm{Im}(\omega(f)) \cap r(Y) = r(\mathrm{Im}(f)).$$

Proof. The proof is similar to [35, Proposition 1.7, page 138]. □

Remark 4.6. From Corollary 4.3, we have

$$\begin{aligned} \mathrm{H}^n(B,\Bbbk)^H &= \left\{[f] \mid f \in \mathrm{Z}^n(B,\Bbbk), \varepsilon_H(h)[f] = [f]h, \text{ for every } h \in H\right\} \\ &= \left\{[f] \mid f \in \mathrm{Z}^n(B,\Bbbk), [\varepsilon_H(h) f] = [fh], \text{ for every } h \in H\right\} \end{aligned}$$

where, for every $z \in B^{\otimes n}$, we have

$$(fh)(z) = f(hz).$$

Note that, for any H-bimodule M one has

$$\mathrm{Hom}_{H,H}(H, M) \cong M^H = \{m \in M \mid hm = mh, \text{ for every } h \in H\}.$$

Note also that H is a separable $\Bbbk$-algebra whence it is projective in the category of H-bimodules. As a consequence $\mathrm{Hom}_{H,H}(H,-) \cong (-)^H : {}_H\mathfrak{M}_H \to \mathfrak{M}$ is an exact functor (here ${}_H\mathfrak{M}_H$ is the category of H-bimodules and $\mathfrak{M}$ the category of $\Bbbk$-vector spaces). By Proposition 4.5 applied to the case when $r := (-)^H : {}_H\mathfrak{M}_H \to \mathfrak{M}$ and ω is the forgetful functor, for every morphism $f : X \to Y$ of H-bimodules one has

$$\ker\left(f^H\right) = \ker(f) \cap X^H = (\ker(f))^H \quad \text{and} \quad \mathrm{Im}\left(f^H\right) = \mathrm{Im}(f) \cap Y^H = (\mathrm{Im}(f))^H.$$

Still by Corollary 4.3, we know that the differential $\partial^n : \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right) \longrightarrow \mathrm{Hom}_{\Bbbk}\left(B^{\otimes(n+1)}, \Bbbk\right)$ of the usual Hochschild cohomology is H-bilinear. Thus we can apply the argument above to get

$$\begin{aligned} \ker\left(\left(\partial^n\right)^H\right) &= \ker\left(\partial^n\right) \cap \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)^H = \left(\ker\left(\partial^n\right)\right)^H \qquad \text{and} \\ \mathrm{Im}\left(\left(\partial^{n-1}\right)^H\right) &= \mathrm{Im}\left(\partial^{n-1}\right) \cap \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)^H = \left(\mathrm{Im}\left(\partial^{n-1}\right)\right)^H. \end{aligned}$$

Now $\mathrm{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)^H = \mathrm{Hom}_{H,-}\left(B^{\otimes n}, \Bbbk\right)$ so that we get

$$\begin{aligned} \mathrm{Z}^n_{H\text{-Mod}}(B,\Bbbk) &= \mathrm{Z}^n(B,\Bbbk) \cap \mathrm{Hom}_{H,-}\left(B^{\otimes n}, \Bbbk\right) = \mathrm{Z}^n(B,\Bbbk)^H \qquad \text{and} \\ \mathrm{B}^n_{H\text{-Mod}}(B,\Bbbk) &= \mathrm{B}^n(B,\Bbbk) \cap \mathrm{Hom}_{H,-}\left(B^{\otimes n}, \Bbbk\right) = \mathrm{B}^n(B,\Bbbk)^H, \end{aligned}$$

where $\mathrm{Z}^n_{H\text{-Mod}}(B,\Bbbk)$ and $\mathrm{B}^n_{H\text{-Mod}}(B,\Bbbk)$ denotes the the Abelian groups of n-cocycles, of n-coboundaries for the cohomology of the algebra B with coefficients in $\Bbbk$ computed in the monoidal category H-Mod of left H-modules. The corresponding n-th Hochschild cohomology group is

$$\mathrm{H}^n_{H\text{-Mod}}(B,\Bbbk) := \frac{\mathrm{Z}^n_{H\text{-Mod}}(B,\Bbbk)}{\mathrm{B}^n_{H\text{-Mod}}(B,\Bbbk)} = \frac{\mathrm{Z}^n(B,\Bbbk)^H}{\mathrm{B}^n(B,\Bbbk)^H} \cong \left(\frac{\mathrm{Z}^n(B,\Bbbk)}{\mathrm{B}^n(B,\Bbbk)}\right)^H = \mathrm{H}^n(B,\Bbbk)^H.$$

Denote by $D(H)$ the Drinfeld double, see *e.g.* the first structure of [25, Theorem 7.1.1].

Proposition 4.7. *In the setting of Corollary* 4.3 *assume that* H *is also cosemisimple. Then, for* $n \in \mathbb{N}_0$

$$\begin{aligned} &\mathrm{Z}^n_{\mathcal{YD}}(B,\Bbbk) = \mathrm{Z}^n(B,\Bbbk)^{D(H)}, \ \mathrm{B}^n_{\mathcal{YD}}(B,\Bbbk) = \mathrm{B}^n(B,\Bbbk)^{D(H)} \\ \text{and} \quad &\mathrm{H}^n_{\mathcal{YD}}(B,\Bbbk) \cong \mathrm{H}^n(B,\Bbbk)^{D(H)}. \end{aligned}$$

where $\mathrm{Z}^n(B,\Bbbk)$ *and* $\mathrm{B}^n(B,\Bbbk)$ *are regarded as* $D(H)$*-subbimodules of* $\mathrm{Hom}_{\Bbbk}(B^{\otimes n},\Bbbk)$ *whose structure is induced by the left* $D(H)$*-module structures of its arguments.*

Moreover $\mathrm{H}^n(B,\Bbbk)^{D(H)}$ *is a subspace of* $\mathrm{H}^n(B,\Bbbk)^H$.

Proof. For shortness, in this proof, we denote $D(H)$ by D. Consider the analogue of the standard complex as in Remark 3.1

$${}^H_H\mathcal{YD}(\Bbbk,\Bbbk) \xrightarrow{\partial^0} {}^H_H\mathcal{YD}(B,\Bbbk) \xrightarrow{\partial^1} {}^H_H\mathcal{YD}(B^{\otimes 2},\Bbbk) \xrightarrow{\partial^2} \cdots$$

where ∂^n is induced by the differential $\partial^n : \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n},\Bbbk\right) \longrightarrow \mathrm{Hom}_{\Bbbk}\left(B^{\otimes (n+1)},\Bbbk\right)$ of the ordinary Hochschild cohomology. Now, since H is semisimple, it is finite-dimensional (whence it has bijective antipode) so that, by a result essentially due to Majid (see [28, Proposition 10.6.16]) and by [32, Proposition 6], we get a category isomorphism ${}^H_H\mathcal{YD} \cong {}_D\mathfrak{M}$. Thus the complex above can be rewritten as follows

$$\mathrm{Hom}_{D,-}(\Bbbk,\Bbbk) \xrightarrow{\partial^0} \mathrm{Hom}_{D,-}(B,\Bbbk) \xrightarrow{\partial^1} \mathrm{Hom}_{D,-}(B^{\otimes 2},\Bbbk) \xrightarrow{\partial^2} \cdots .$$

Now, since, for each $n \in \mathbb{N}_0$, we have $\mathrm{Hom}_{D,-}\left(B^{\otimes n},\Bbbk\right) = \mathrm{Hom}_{\Bbbk}\left(B^{\otimes n},\Bbbk\right)^D$, we obtain the complex

$$\mathrm{Hom}_{\Bbbk}(\Bbbk,\Bbbk)^D \xrightarrow{\partial^0} \mathrm{Hom}_{\Bbbk}(B,\Bbbk)^D \xrightarrow{\partial^1} \mathrm{Hom}_{\Bbbk}(B^{\otimes 2},\Bbbk)^D \xrightarrow{\partial^2} \cdots .$$

We will write $(\partial^n)^D$ instead of ∂^n when we would like to stress that the map considered is the one induced on invariants. Thus we will write equivalently

$$\mathrm{Hom}_{\Bbbk}(\Bbbk,\Bbbk)^D \xrightarrow{(\partial^0)^D} \mathrm{Hom}_{\Bbbk}(B,\Bbbk)^D \xrightarrow{(\partial^1)^D} \mathrm{Hom}_{\Bbbk}(B^{\otimes 2},\Bbbk)^D \xrightarrow{(\partial^2)^D} \cdots .$$

Now, assume H is also cosemisimple. Since H is both semisimple and cosemisimple, by [30, Proposition 7] the Hopf algebra D is semisimple as an algebra. Thus, as in Remark 4.6 in case of H, the functor $(-)^D : {}_D\mathfrak{M}_D \to \mathfrak{M}$ is exact (here ${}_D\mathfrak{M}_D$ is the category of D-bimodules and $\mathfrak{M}$ the category of $\Bbbk$-vector spaces). By

Proposition 4.5 applied to the case when $r := (-)^D : {}_D\mathfrak{M}_D \to \mathfrak{M}$ and ω is the forgetful functor, for every morphism $f : X \to Y$ of D-bimodules one has

$$\ker\left(f^D\right) = \ker(f) \cap X^D = (\ker(f))^D$$
$$\text{and} \qquad \operatorname{Im}\left(f^D\right) = \operatorname{Im}(f) \cap Y^D = (\operatorname{Im}(f))^D.$$

In particular we get

$$\ker\left((\partial^n)^D\right) = \ker(\partial^n) \cap \operatorname{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)^D = \ker(\partial^n)^D$$
$$\text{and} \qquad \operatorname{Im}\left(\left(\partial^{n-1}\right)^D\right) = \operatorname{Im}\left(\partial^{n-1}\right) \cap \operatorname{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)^D = \operatorname{Im}\left(\partial^{n-1}\right)^D$$

and hence

$$\mathrm{Z}^n_{\mathcal{YD}}(B, \Bbbk) = \mathrm{Z}^n(B, \Bbbk) \cap \operatorname{Hom}_{D,-}\left(B^{\otimes n}, \Bbbk\right) = \mathrm{Z}^n(B, \Bbbk)^D \qquad \text{and}$$
$$\mathrm{B}^n_{\mathcal{YD}}(B, \Bbbk) = \mathrm{B}^n(B, \Bbbk) \cap \operatorname{Hom}_{D,-}\left(B^{\otimes n}, \Bbbk\right) = \mathrm{B}^n(B, \Bbbk)^D.$$

Then we obtain

$$\mathrm{H}^n_{\mathcal{YD}}(B, \Bbbk) = \frac{\mathrm{Z}^n_{\mathcal{YD}}(B, \Bbbk)}{\mathrm{B}^n_{\mathcal{YD}}(B, \Bbbk)} = \frac{\mathrm{Z}^n(B, \Bbbk)^D}{\mathrm{B}^n(B, \Bbbk)^D} \cong \mathrm{H}^n(B, \Bbbk)^D.$$

Let us prove the last part of the statement. The correspondence between the left D-module structure and the structure of Yetter-Drinfeld module over H is written explicitly in [25, Proposition 7.1.6]. In particular $D = H^* \otimes H$ and given $V \in {}^H_H\mathcal{YD}$, the two structures are related by the following equality $(f \otimes h) \rhd v = f\left((h \rhd v)_{-1}\right)(h \rhd v)_0$ for every $f \in H^*, h \in H, v \in V$. Thus $(\varepsilon_H \otimes h) \rhd v = h \rhd v$. Moreover H is a Hopf subalgebra of D via $h \mapsto \varepsilon_H \otimes h$, where D is considered with the first structure of [25, Theorem 7.1.1]. Since the D-bimodule structure of $\mathrm{H}^n(B, \Bbbk)$ is induced by the one of $\operatorname{Hom}_{\Bbbk}\left(B^{\otimes n}, \Bbbk\right)$ which comes from the left D-module structures of its arguments and similarly for the H-bimodule structure of $\mathrm{H}^n(B, \Bbbk)$, we deduce that $\mathrm{H}^n(B, \Bbbk)^D$ is a subspace of $\mathrm{H}^n(B, \Bbbk)^H$. □

Example 4.8. In the setting of the proof of [9, Theorem 4.1.3], a Nichols algebra $\mathcal{B}(V)$ such that $\mathrm{H}^3(\mathcal{B}(V), \Bbbk)^{\mathbb{Z}_m} = 0$ is considered where $\Bbbk$ is a field of characteristic zero. By Proposition 4.7 applied in the case $H = \Bbbk\mathbb{Z}_m$ and $B = \mathcal{B}(V)$, we have that $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}(V), \Bbbk) \cong \mathrm{H}^3(\mathcal{B}(V), \Bbbk)^{D(H)}$ is a subspace of $\mathrm{H}^3(\mathcal{B}(V), \Bbbk)^H = \mathrm{H}^3(\mathcal{B}(V), \Bbbk)^{\mathbb{Z}_m} = 0$. Thus we get $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{B}(V), \Bbbk) = 0$. Therefore, in view of Theorem 3.2, if $(Q, m, u, \Delta, \varepsilon, \omega)$ is a finite-dimensional connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$ such that $\mathrm{gr}Q \cong \mathcal{B}(V)$ (as above) as augmented algebras in ${}^H_H\mathcal{YD}$ (the counit must be the same in order to have the same Yetter-Drinfeld module structure on $\Bbbk$), then we can conclude that Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.

Remark 4.9. Let A be a finite-dimensional coquasi-bialgebra with the dual Chevalley property *i.e.*, the coradical H of A is a coquasi-subbialgebra of A (in particular H is cosemisimple). Assume the coquasi-bialgebra structure of H has trivial reassociator (*i.e.*, it is an ordinary bialgebra) and also assume it has an antipode (*i.e.*, it is a Hopf algebra). Then, by [10, Corollary 6.4], $\mathrm{gr}A$ is isomorphic to $R\#H$ as a coquasi-bialgebra, where R is a suitable connected bialgebra in ${}_{H}^{H}\mathcal{YD}$. Note that $R\#H$ is the usual Radford-Majid bosonization as H has trivial reassociator, see [10, Definition 5.4]. Hence we can compute

$$\mathrm{H}^3\left(\mathrm{gr}A, \Bbbk\right) = \mathrm{H}^3\left(R\#H, \Bbbk\right).$$

Assume further that H is semisimple. Then, by Corollary 4.3, we have

$$\mathrm{H}^n\left(R\#H, \Bbbk\right) \cong \mathrm{H}^n\left(R, \Bbbk\right)^H$$

so that $\mathrm{H}^3(\mathrm{gr}A, \Bbbk) \cong \mathrm{H}^3(R, \Bbbk)^H$. Thus, if $\mathrm{H}^3\left(R, \Bbbk\right)^H = 0$, one gets $\mathrm{H}^3\left(\mathrm{gr}A, \Bbbk\right) = 0$ which is the analogue of the condition [18, Proposition 2.3] (note that our A is the dual of the one considered therein) which guarantees that A is gauge equivalent to an ordinary Hopf algebra, if A has a quasi-antipode and $\Bbbk = \mathbb{C}$. Next we will give another approach to arrive at the same conclusion but just requiring $\mathrm{H}^3_{\mathcal{YD}}\left(R, \Bbbk\right) = 0$. Note that a priori $\mathrm{H}^3_{\mathcal{YD}}\left(R, \Bbbk\right) \cong \mathrm{H}^3\left(R, \Bbbk\right)^{D(H)}$ is smaller than $\mathrm{H}^3\left(R, \Bbbk\right)^H$. We point out that requiring, as above, that H has trivial reassociator is equivalent to asking that $\mathrm{gr}A$ has trivial reassociator (see *e.g.* [10, Proposition 6.2]) which is the standing hypothesis of [18, Proposition 2.3].

5. The dual Chevalley property

The main aim of this section is to prove Theorem 5.6. Let A be a Hopf algebra over a field $\Bbbk$ of characteristic zero such that the coradical H of A is a sub-Hopf algebra (*i.e.*, A has the dual Chevalley Property). Assume H is finite-dimensional so that H is semisimple. By [2, Theorem I], there is a gauge transformation $\zeta : A \otimes A \to \Bbbk$ such that A^{ζ} is isomorphic, as a coquasi-bialgebra, to the bosonization $Q\#H$ of a connected coquasi-bialgebra Q in ${}_{H}^{H}\mathcal{YD}$ by H. By construction ζ is H-bilinear and H-balanced: this follows from [2, Proposition 5.7] (note that gauge transformation $v_B : B \otimes B \to \Bbbk$, used therein for $B := R\#_{\xi}H$, is H-bilinear and H-balanced, as observed in the proof) and the fact that there is an H-bilinear Hopf algebra isomorphism $\psi : B \to A$ (see [2, Proof of Theorem I, page 36 and Theorem 6.1] which is a consequence of [6, Theorem 3.64]) where (R, ξ) is a suitable connected pre-bialgebra with cocycle in ${}_{H}^{H}\mathcal{YD}$ (note that $\zeta = v_B \circ \left(\psi^{-1} \otimes \psi^{-1}\right)$): here by connected pre-bialgebra we mean that the coradical R_0 of R is $\Bbbk 1_R$ (by the properties of 1_R this implies that R_0 is a subcoalgebra in ${}_{H}^{H}\mathcal{YD}$ of R). Assume that A is finite-dimensional. Then $Q\#H$ and hence Q is finite dimensional.

Thus, by Theorem 3.2, if $\mathrm{H}^3_{\mathcal{YD}}\left(\mathrm{gr}Q, \Bbbk\right) = 0$, then Q is gauge equivalent to a connected bialgebra in ${}_{H}^{H}\mathcal{YD}$.

First let us check which condition on A guarantee that $\mathrm{H}^3_{\mathcal{YD}}(\mathrm{gr}Q, \Bbbk) = 0$. Note that by construction $Q = R^v$ (see [2, Proposition 5.7]) where $v := (\lambda\xi)^{-1}$, the convolution inverse of $\lambda\xi$ and $\lambda : H \to \Bbbk$ denotes the total integral on H. Thus we can rewrite gr Q as gr R^v.

Moreover v_B is given by $v_B\left((r\#h)\otimes\left(r'\#h'\right)\right) = v\left(r\otimes hr'\right)\varepsilon_H\left(h'\right)$ for every $r, r' \in R, h, h' \in H$. By [8, Proposition 2.5], gr$R$ inherits the pre-bialgebra structure in ${}^H_H\mathcal{YD}$ of R. This is proved by checking that $R_i \cdot R_j \subseteq R_{i+j}$ for every $i, j \in \mathbb{N}_0$, where R_i denotes the i-th term of the coradical filtration of R. Moreover R_i is a subcoalgebra of R in ${}^H_H\mathcal{YD}$.

Lemma 5.1. *Keep the above hypotheses and notation. Then* gr R^v *and* gr R *coincide as bialgebras in* ${}^H_H\mathcal{YD}$ *where the structures of* gr R *are induced by the ones of* (R, ξ).

Proof. By Theorem 1.6, gr R^v = gr Q is a connected bialgebras in ${}^H_H\mathcal{YD}$.

Note that R^v and R coincide as coalgebras in ${}^H_H\mathcal{YD}$ so that gr R^v and gr R coincide as coalgebras in ${}^H_H\mathcal{YD}$. They also have the same unit. It remains to check that their two multiplications coincide too.

Since ξ is unital, by [6, Proposition 4.8], we have that v is unital and this is equivalent to v^{-1} unital (see the proof therein).

Let $C := R \otimes R$. Let $n > 0$ and let $w \in C_{(n)} = \sum_{i+j\leq n} R_i \otimes R_j$. By [6, Lemma 3.69], we have that

$$\Delta_C(w) - w \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes w \in C_{(n-1)} \otimes C_{(n-1)}.$$

Thus we get

$$w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1_R)^{\otimes 2} - \Delta_C\left((1_R)^{\otimes 2}\right) \otimes w \in \Delta_C\left(C_{(n-1)}\right) \otimes C_{(n-1)}$$

and hence

$$\begin{aligned} w_1 \otimes w_2 \otimes w_3 - w \otimes (1_R)^{\otimes 2} \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes w \otimes (1_R)^{\otimes 2} \\ - (1_R)^{\otimes 4} \otimes w \in C_{(n-1)} \otimes C_{(n-1)} \otimes C_{(n-1)}. \end{aligned}$$

Since $m\left(C_{(n-1)}\right) \subseteq \sum_{i+j\leq n} m\left(R_i \otimes R_j\right) \subseteq R_{n-1}$ we get

$$\begin{aligned} w_1 \otimes m(w_2) \otimes w_3 - w \otimes 1_R \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes m(w) \otimes (1_R)^{\otimes 2} \\ - (1_R)^{\otimes 3} \otimes w \in C_{(n-1)} \otimes R_{n-1} \otimes C_{(n-1)} \end{aligned}$$

and hence

$$w_1 \otimes (m(w_2) + R_{n-1}) \otimes w_3 = (1_R)^{\otimes 2} \otimes (m(w) + R_{n-1}) \otimes (1_R)^{\otimes 2}. \quad (5.1)$$

Let $x, y \in R$. We compute

$$\begin{aligned}
\overline{x} \cdot_v \overline{y} &= \left(x + R_{|x|-1}\right) \cdot_v \left(y + R_{|y|-1}\right) \\
&= (x \cdot_v y) + R_{|x|+|y|-1} = m^v (x \otimes y) + R_{|x|+|y|-1} \\
&= v\left((x \otimes y)_1\right) m\left((x \otimes y)_2\right) v^{-1}\left((x \otimes y)_3\right) + R_{|x|+|y|-1} \\
&= v\left((x \otimes y)_1\right) \left(m\left((x \otimes y)_2\right) + R_{|x|+|y|-1}\right) v^{-1}\left((x \otimes y)_3\right) \\
&\overset{(5.1)}{=} v\left((1_R)^{\otimes 2}\right) \left(m (x \otimes y) + R_{|x|+|y|-1}\right) v^{-1}\left((1_R)^{\otimes 2}\right) \\
&= m (x \otimes y) + R_{|x|+|y|-1} = (x \cdot y) + R_{|x|+|y|-1} = \overline{x} \cdot \overline{y}. \qquad \square
\end{aligned}$$

The following result is inspired by [6, Theorem 3.71].

Lemma 5.2. *Let H be a cosemisimple Hopf algebra. Let C be a left H-comodule coalgebra such that C_0 is a one-dimensional left H-comodule subcoalgebra of C. Let $B = C\#H$ be the smash coproduct of C by H* i.e., *the coalgebra defined by*

$$\begin{aligned}
\Delta_B (c\#h) &= \sum \left(c_1\# (c_2)_{-1} h_1\right) \otimes ((c_2)_0 \#h_2), \\
\varepsilon_B (c\#h) &= \varepsilon_C (c) \varepsilon_H (h).
\end{aligned} \tag{5.2}$$

Then, for every $n \in \mathbb{N}_0$ we have $B_n = C_n\#H$.

Proof. Since C_0 is a subcoalgebra of C in ${}^H\mathfrak{M}$ and, for $n \geq 1$, one has $C_n = C_{n-1} \wedge_C C_0$, then inductively one proves that C_n is a subcoalgebra of C in ${}^H\mathfrak{M}$. Set $B_{(n)} := C_n\#H$ for every $n \in \mathbb{N}_0$. Let us check that $B_{(n)} = B_n$ by induction on $n \in \mathbb{N}_0$.

Let $n = 0$. First note $B = \cup_{m\in\mathbb{N}_0} B_{(m)}$ and, since $\Delta_C (C_m) \subseteq \sum_{0\leq i\leq m} C_i \otimes C_{m-i}$, we also have

$$\begin{aligned}
\Delta_B \left(B_{(m)}\right) = \Delta_B (C_m\#H) &\subseteq \sum_{0\leq i\leq m} \sum \left(C_i\# (C_{m-i})_{-1} (H)_1\right) \otimes \left((C_{m-i})_0 \# (H)_2\right) \\
&\subseteq \sum_{0\leq i\leq m} (C_i\#H) \otimes (C_{m-i}\# (H)) = \sum_{0\leq i\leq m} B_{(i)} \otimes B_{(m-i)}.
\end{aligned}$$

Therefore $\left(B_{(m)}\right)_{m\in\mathbb{N}_0}$ is a coalgebra filtration for B and hence, by [37, Proposition 11.1.1], we get that $B_{(0)} \supseteq B_0$. Since C_0 is one-dimensional, there is a grouplike element $1_C \in C_0$ such that $C_0 = \Bbbk 1_C$. Moreover one checks that C_0 is a subcoalgebra of C in ${}^H\mathfrak{M}$ implies $\sum (1_C)_{-1} \otimes (1_C)_0 = 1_H \otimes 1_C$.

Let $\sigma : H \to C \otimes H : h \mapsto 1_C \otimes h$ be the canonical injection. We have

$$\begin{aligned}
\Delta_B\sigma (h) &= \Delta_B (1_C \otimes h) = \sum \left(1_C\# (1_C)_{-1} h_1\right) \otimes ((1_C)_0 \#h_2) \\
&= \sum (1_C\#1_H h_1) \otimes (1_C\#h_2) = \sum \sigma (h_1) \otimes \sigma (h_2) = (\sigma \otimes \sigma) \Delta_H (h), \\
\varepsilon_B\sigma (h) &= \varepsilon_B (1_C \otimes h) = \varepsilon_C (1_C) \varepsilon_H (h) = \varepsilon_H (h)
\end{aligned}$$

so that σ is a coalgebra map. Since H is cosemisimple and σ an injective coalgebra map we deduce that also $\sigma(H) = C_0 \otimes H = B_{(0)}$ is a cosemisimple subcoalgebra of B whence $B_{(0)} \subseteq B_0$.

Let $n > 0$ and assume that $B_i = B_{(i)}$ for $0 \leq i \leq n-1$. Let $\sum_{i\in I} c_i \# h_i \in B_n$. Then

$$\Delta_B\left(\sum_{i\in I} c_i \# h_i\right) \in B_{n-1} \otimes B + B \otimes B_0 = C_{n-1} \otimes H \otimes C \otimes H + C \otimes H \otimes C_0 \otimes H.$$

Let $p_n : C \to \frac{C}{C_n}$ be the canonical projection. If we apply $(p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H)$ we get

$$\begin{aligned}
0 &= (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H)\, \Delta_B\left(\sum_{i\in I} c_i \# h_i\right) \\
&= (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H)\left(\sum_{i\in I} \big((c_i)_1 \# ((c_i)_2)_{-1}(h_i)_1\big) \otimes \big(((c_i)_2)_0 \# (h_i)_2\big)\right) \\
&= (p_{n-1} \otimes p_0 \otimes H)\left(\sum_{i\in I} (c_i)_1 \otimes (c_i)_2 \otimes h_i\right) \\
&= ((p_{n-1} \otimes p_0)\, \Delta_C \otimes H)\left(\sum_{i\in I} c_i \# h_i\right).
\end{aligned}$$

Thus $\sum_{i\in I} c_i \# h_i \in \ker((p_{n-1} \otimes p_0)\, \Delta_C \otimes H) = \left[\ker((p_{n-1} \otimes p_0)\, \Delta_C)\right] \otimes H = C_n \otimes H = B_{(n)}$. Thus $B_n \subseteq B_{(n)}$. On the other hand, form $\Delta_C(C_n) \subseteq C_{n-1} \otimes C + C \otimes C_0$ we deduce

$$\begin{aligned}
\Delta_B\left(B_{(n)}\right) &= \Delta_B(C_n \otimes H) \\
&\subseteq \sum \big((C_n)_1 \# ((C_n)_2)_{-1}(H)_1\big) \otimes \big(((C_n)_2)_0 \# (H)_2\big) \\
&\subseteq \sum \big(C_{n-1} \# (C)_{-1} H\big) \otimes ((C)_0 \# H) \\
&\quad + \sum \big(C \# (C_0)_{-1} H\big) \otimes ((C_0)_0 \# H) \\
&\subseteq (C_{n-1} \# H) \otimes (C \# H) + (C \# H) \otimes (C_0 \# H) \\
&= B_{(n-1)} \otimes B + B \otimes B_{(0)} = B_{n-1} \otimes B + B \otimes B_0
\end{aligned}$$

and hence $B_{(n)} \subseteq B_n$. □

Definition 5.3. Let A be a Hopf algebra over a field $\Bbbk$ such that the coradical H of A is a sub-Hopf algebra (*i.e.*, A has the dual Chevalley Property). Set $G := \operatorname{gr} A$. There are two canonical Hopf algebra maps

$$\begin{aligned}
\sigma_G &: H \to \operatorname{gr} A : h \mapsto h + A_{-1}, \\
\pi_G &: \operatorname{gr} A \to H : a + A_{n-1} \mapsto a\delta_{n,0}, \qquad n \in \mathbb{N}_0.
\end{aligned}$$

The diagram of A (see [11, page 659]) is the vector space

$$\mathcal{D}(A) := \left\{ d \in \operatorname{gr} A \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}.$$

It is a bialgebra in ${}_H^H\mathcal{YD}$ as follows. $\mathcal{D}(A)$ is a subalgebra of G. The left H-action, the left H-coaction of $\mathcal{D}(A)$, the comultiplication and counit are given respectively by

$$h \triangleright d := \sum \sigma_G(h_1)\, d\sigma_G S(h_2), \qquad \rho(d) = \sum \pi_G(d_1) \otimes d_2,$$
$$\Delta_{D(A)}(d) := \sum d_1 \sigma_G S_H \pi_G(d_2) \otimes d_3, \qquad \varepsilon_{D(A)}(d) = \varepsilon_G(d).$$

Although the following result seems to be folklore, we include here its statement for future reference.

Proposition 5.4. *Let A be a Hopf algebra over a field $\Bbbk$ such that the coradical H of A is a sub-Hopf algebra. Let A' be a Hopf algebra over a field $\Bbbk$. Let $f : A' \to A$ be an isomorphism of Hopf algebras. Then $H' := f^{-1}(H) \cong H$ is the coradical of A' and it is a sub-Hopf algebra of A'. Thus we can identify H' with H. Moreover f induces an isomorphism $\mathcal{D}(f) : \mathcal{D}(A') \to \mathcal{D}(A)$ of bialgebras in ${}_H^H\mathcal{YD}$.*

Proposition 5.5. *Keep the hypotheses and notation of the beginning of the section. Then $\mathcal{D}(A) \cong \mathcal{D}(R\#_\xi H) \cong \operatorname{gr} R$ as bialgebras in ${}_H^H\mathcal{YD}$.*

Proof. Apply Proposition 5.4 to the canonical isomorphism $\psi : B := R\#_\xi H \to A$ that we recalled at the beginning of the section to get that $\mathcal{D}(R\#_\xi H) \cong \mathcal{D}(A)$. Note that, by H-linearity we have

$$\psi(1_R\#h) = \psi((1_R\#1_H)(1_R\#h)) = \psi((1_R\#1_H)h) = \psi(1_R\#1_H)h = h$$

so that $\psi(\Bbbk 1_R \otimes H) = H$ and hence $H' = \psi^{-1}(H) = \Bbbk 1_R \otimes H$ with the notation of Proposition 5.4. Thus $B_0 = \Bbbk 1_R \otimes H = R_0 \otimes H$ so that we can identify B_0 with H via the canonical isomorphism $H \to R_0 \otimes H : h \mapsto 1_R \otimes h$. Its inverse is $R_0 \otimes H \to H : r \otimes h \mapsto \varepsilon_R(r)\, h$. With this identification and by setting $G := \operatorname{gr} B$, we can consider the canonical bialgebra maps

$$\sigma_G\colon H \to \operatorname{gr} B : h \mapsto 1_R\#h + (R\#_\xi H)_{-1},$$
$$\pi_G\colon \operatorname{gr} B \to H : r\#h + (R\#_\xi H)_{n-1} \mapsto \varepsilon_R(r)\, h\delta_{n,0}, \text{ where } r\#h \in (R\#_\xi H)_n,\ n \in \mathbb{N}_0.$$

Since the underlying coalgebra of B is exactly the smash coproduct of R by H and (R, ξ) is a connected pre-bialgebra with cocycle in ${}_H^H\mathcal{YD}$, by Lemma 5.2, we have that $B_n = R_n \otimes H$. Let us compute $\mathcal{D} := \mathcal{D}(B)$. As a vector space it is

$$\mathcal{D} := \left\{ d \in G \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}.$$

By [11, Lemma 2.1], we have that $\mathcal{D} = \oplus_{n\in\mathbb{N}_0}\mathcal{D}^n$ where $\mathcal{D}^n = \mathcal{D}\cap G^n = \mathcal{D}\cap\frac{B_n}{B_{n-1}}$. Let $d := \overline{\sum_{i\in I} r_i\#h_i} \in \mathcal{D}^n$ where we can assume $\sum_{i\in I} r_i\#h_i \in B_n\backslash B_{n-1}$ and, for every $i \in I$, $r_i\#h_i \in B_n\backslash B_{n-1}$. Then $\overline{\sum_{i\in I} r_i\#h_i} = \sum_{i\in I}\overline{r_i\#h_i}$ and hence the fact that d is coinvariant rewrites as

$$\sum_{i\in I}\left(\overline{r_i\#h_i}\right)_1 \otimes \pi_G\left(\left(\overline{r_i\#h_i}\right)_2\right) = \sum_{i\in I}\overline{r_i\#h_i} \otimes 1_H. \tag{5.3}$$

By definition of π_G and (1.1), the left-hand side becomes

$$\sum_{i\in I}\left(\overline{r_i\#h_i}\right)_1 \otimes \pi_G\left(\left(\overline{r_i\#h_i}\right)_2\right) = \sum_{i\in I}\left(\left(r_i\#\left(h_i\right)_1\right) + B_{n-1}\right) \otimes \left(h_i\right)_2$$

so that (5.3) becomes

$$\sum_{i\in I}\left(\left(r_i\#\left(h_i\right)_1\right) + B_{n-1}\right) \otimes \left(h_i\right)_2 = \sum_{i\in I}\overline{r_i\#h_i} \otimes 1_H = \sum_{i\in I}\left(r_i\#h_i + B_{n-1}\right) \otimes 1_H$$

i.e.

$$\sum_{i\in I}\left(r_i\#\left(h_i\right)_1\right) \otimes \left(h_i\right)_2 - \sum_{i\in I} r_i\#h_i \otimes 1_H \in B_{n-1} \otimes H = R_{n-1} \otimes H \otimes H.$$

If we apply $R \otimes \varepsilon_H \otimes H$, we get

$$\sum_{i\in I} r_i \otimes h_i - \sum_{i\in I} r_i\varepsilon_H\left(h_i\right) \otimes 1_H \in R_{n-1} \otimes H = B_{n-1}.$$

Thus $\overline{\sum_{i\in I} r_i\#h_i} = \sum_{i\in I}\overline{r_i\#h_i} = \sum_{i\in I}\left(r_i\#h_i + B_{n-1}\right) = \sum_{i\in I}\left(r_i\varepsilon_H\left(h_i\right) \otimes 1_H\right) + B_{n-1}$.

Since $\sum_{i\in I} r_i\#h_i \in B_n\backslash B_{n-1}$ we get that $\left(\sum_{i\in I} r_i\varepsilon_H\left(h_i\right)\right) \otimes 1_H \notin B_{n-1}$ and hence $\sum_{i\in I} r_i\varepsilon_H\left(h_i\right) \notin R_{n-1}$ and we can write

$$\overline{\sum_{i\in I} r_i\#h_i} = \overline{\left(\sum_{i\in I} r_i\varepsilon_H\left(h_i\right)\right) \otimes 1_H}.$$

Therefore we have proved that the map

$$\varphi_n : \frac{R_n}{R_{n-1}} \to \mathcal{D}^n : \overline{r} \mapsto \overline{r \otimes 1_H},$$

which is well-defined as $\mathcal{D}^n = \mathcal{D} \cap G^n = \mathcal{D} \cap \frac{B_n}{B_{n-1}} = \mathcal{D} \cap \frac{R_n\otimes H}{R_{n-1}\otimes H}$, is also surjective.

It is also injective as $\varphi_n(\overline{r}) = \varphi_n(\overline{s})$ implies $r \otimes 1_H - s \otimes 1_H \in B_{n-1} = R_{n-1} \otimes H$ and hence, by applying $R \otimes \varepsilon_H$, we get $r - s \in R_{n-1}$, *i.e.*, $\overline{r} = \overline{s}$. Therefore φ_n is an isomorphism such that $\overline{\sum_{i\in I} r_i \# h_i} = \varphi_n\left(\overline{\sum_{i\in I} r_i \varepsilon_H(h_i)}\right)$ and hence

$$\varphi_n^{-1}\left(\overline{\sum_{i\in I} r_i \# h_i}\right) = \overline{\sum_{i\in I} r_i \varepsilon_H(h_i)}.$$

Clearly this extends to a graded $\Bbbk$-linear isomorphism

$$\varphi : \mathrm{gr}\, R \to \mathcal{D}.$$

Let us check that φ is a morphism in ${}_H^H\mathcal{YD}$. First note that, for every $r \in R_n$, we have

$$\begin{aligned}\varphi(r + R_{n-1}) &= \delta_{|r|,n}\varphi(r + R_{n-1}) = \delta_{|r|,n}\varphi_n(r + R_{n-1}) = \delta_{|r|,n}\varphi_n(\overline{r}) \\ &= \delta_{|r|,n}\overline{r \otimes 1_H} = \delta_{|r|,n}\left(r \otimes 1_H + \left(R\#_\xi H\right)_{n-1}\right) = r \otimes 1_H \\ &\quad + \left(R\#_\xi H\right)_{n-1}.\end{aligned}$$

Thus

$$\varphi(r + R_{n-1}) = r \otimes 1_H + \left(R\#_\xi H\right)_{n-1}, \text{ for every } r \in R_n. \tag{5.4}$$

For every $r \in R_n \backslash R_{n-1}$, by using (5.4), it is straighforward to prove that $h \triangleright \varphi(\overline{r}) = \varphi(h\overline{r})$. Moreover, by applying (1.1), (5.2), the definition of π_G and (5.4), we get that $\rho\varphi(\overline{r}) = (H \otimes \varphi)\rho(\overline{r})$.

Let us check that φ is a morphism of bialgebras in ${}_H^H\mathcal{YD}$. Fix $r \in R_n \backslash R_{n-1}$. Using the definition of $\Delta_{\mathcal{D}}$, (1.1), (5.2), the definition of π_G, the definition of σ_G, (5.4) and (1.1) again, we obtain $\Delta_{\mathcal{D}}\varphi(\overline{r}) = (\varphi \otimes \varphi)\Delta_{\mathrm{gr}\, R}(\overline{r})$.

Let us check φ is counitary:

$$\begin{aligned}\varepsilon_{\mathcal{D}}\varphi(\overline{r}) &= \varepsilon_G\varphi(\overline{r}) = \varepsilon_G\left(\overline{r \otimes 1_H}\right) \overset{(1.2)}{=} \delta_{n,0}\varepsilon_B(r \otimes 1_H) \\ &= \delta_{n,0}\varepsilon_R(r) \overset{(1.2)}{=} \varepsilon_{\mathrm{gr}\, R}(\overline{r}).\end{aligned}$$

Let us check φ is multiplicative. Let $s \in R_m \backslash R_{m-1}$. Then, by definition of φ, of $m_{\mathcal{D}}$ and of the multiplication of $R\#_\xi H$, we have that

$$\begin{aligned}m_{\mathcal{D}}(\varphi \otimes \varphi)(\overline{s} \otimes \overline{r}) = \sum &\left(s^{(1)}\left(\left(s^{(2)}\right)_{-1} r^{(1)}\right) \# \xi\left(\left(s^{(2)}\right)_0 \otimes r^{(2)}\right)\right) \\ &+ \left(R\#_\xi H\right)_{m+n-1}.\end{aligned}$$

Now write $\sum s^{(1)} \otimes s^{(2)} = \sum_{0\leq i\leq m} s_i \otimes s'_{m-i}$ for some $s_i, s'_i \in R_i$ and similarly $\sum r^{(1)} \otimes r^{(2)} = \sum_{0\leq j\leq n} r_j \otimes r'_{n-j}$ for some $r_j, r'_j \in R_j$. Then

$$\begin{aligned}
m_{\mathcal{D}}(\varphi\otimes\varphi)(\overline{s}\otimes\overline{r}) &= \sum_{\substack{0\leq i\leq m\\0\leq j\leq n}} \left(s_i\left(\left(s'_{m-i}\right)_{-1} r_j\right)\#\xi\left(\left(s'_{m-i}\right)_0\otimes r'_{n-j}\right)\right) + \left(R\#_\xi H\right)_{m+n-1} \\
&= \sum_{\substack{0\leq i\leq m\\0\leq j\leq n}} \delta_{i,m}\delta_{j,n}\left(s_i\left(\left(s'_{m-i}\right)_{-1} r_j\right)\#\xi\left(\left(s'_{m-i}\right)_0\otimes r'_{n-j}\right)\right) \\
&\quad + \left(R\#_\xi H\right)_{m+n-1} \\
&= \sum\left(s_m\left(\left(s'_0\right)_{-1} r_n\right)\#\xi\left(\left(s'_0\right)_0\otimes r'_0\right)\right) + \left(R\#_\xi H\right)_{m+n-1} \\
&\overset{R_0=\Bbbk 1_R}{=} \sum s_m\left(\left(s'_0\right)_{-1} r_n\right)\#\varepsilon_R\left(\left(s'_0\right)_0\right)\varepsilon_R\left(r'_0\right)1_H + \left(R\#_\xi H\right)_{m+n-1} \\
&= \sum s_m\varepsilon_R\left(s'_0\right) r_n\varepsilon_R\left(r'_0\right)\#1_H + \left(R\#_\xi H\right)_{m+n-1} \\
&= \sum_{\substack{0\leq i\leq m\\0\leq j\leq n}} \delta_{i,m}\delta_{j,n}\left(s_i\varepsilon_R\left(s'_{m-i}\right) r_j\varepsilon_R\left(r'_{m-j}\right)\#1_H\right) \\
&\quad + \left(R\#_\xi H\right)_{m+n-1} \\
&= \sum_{\substack{0\leq i\leq m\\0\leq j\leq n}} \left(s_i\varepsilon_R\left(s'_{m-i}\right) r_j\varepsilon_R\left(r'_{m-j}\right)\#1_H\right) + \left(R\#_\xi H\right)_{m+n-1} \\
&= \sum\left(s^{(1)}\varepsilon_R\left(s^{(2)}\right) r^{(1)}\varepsilon_R\left(r^{(2)}\right)\#1_H\right) + \left(R\#_\xi H\right)_{m+n-1} \\
&= \left(sr\#1_H\right) + \left(R\#_\xi H\right)_{m+n-1} \overset{(5.4)}{=} \varphi\left(sr + R_{m+n-1}\right) \\
&= \varphi\left(\left(s+R_{m-1}\right)\left(r+R_{n-1}\right)\right) = \varphi m_{\operatorname{gr} R}\left(\overline{s}\otimes\overline{r}\right).
\end{aligned}$$

Let us check φ is unitary. We have

$$\begin{aligned}
\varphi\left(1_{\operatorname{gr} R}\right) &= \varphi\left(1_R + R_{-1}\right) = \varphi\left(\overline{1_R}\right) \\
&= \overline{1_R\otimes 1_H} = \left(1_R\otimes 1_H\right) + \left(R\#_\xi H\right)_{-1} = 1_B + B_{-1} = 1_G. \qquad \square
\end{aligned}$$

Summing up we have proved that

$$\operatorname{gr} Q \overset{Q=R^v}{=} \operatorname{gr} R^v \overset{\text{Lem. 5.1}}{\cong} \operatorname{gr} R \overset{\text{Pro. 5.5}}{\cong} \mathcal{D}\left(R\#_\xi H\right) \overset{\text{Pro. 5.4}}{\cong} \mathcal{D}(A)$$

as bialgebras in ${}^H_H\mathcal{YD}$. Therefore $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{D}(A), \Bbbk) = 0$ (the Hochschild cohomology in ${}^H_H\mathcal{YD}$ of the algebra $\mathcal{D}(A)$ with values in $\Bbbk$) if, and only if, $\mathrm{H}^3_{\mathcal{YD}}(\operatorname{gr}Q, \Bbbk) = 0$. In this case, by the foregoing, we get that Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.

Now let E be a connected bialgebra in ${}_{H}^{H}\mathcal{YD}$ and let $\gamma : E\otimes E \to \Bbbk$ be a gauge transformation for E such that $Q = E^{\gamma}$. We proved that $A^{\zeta} \cong Q\#H \cong E^{\gamma}\#H$ as coquasi-bialgebras. By Proposition 2.5, we have that $(E\#H)^{\Gamma} = E^{\gamma}\#H$ as an ordinary coquasi-bialgebras. Recall that two coquasi-bialgebras A and A' are called *gauge equivalent* or *quasi-isomorphic* whenever there is some gauge transformation $\gamma : Q \otimes Q \to \Bbbk$ in $\mathbf{Vec}_{\Bbbk}$ such that $A^{\gamma} \cong A'$ as coquasi-bialgebras. We point out that, if A and A' are ordinary bialgebras and $A^{\gamma} \cong A'$, then γ comes out to be a unitary cocycle. This is encoded in the triviality of the reassociators of A and A'.

Theorem 5.6. *Let A be a finite-dimensional Hopf algebra over a field $\Bbbk$ of characteristic zero such that the coradical H of A is a sub-Hopf algebra (*i.e.*, A has the dual Chevalley Property). If $\mathrm{H}^{3}_{\mathcal{YD}}(\mathcal{D}(A), \Bbbk) = 0$, then A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}_{H}^{H}\mathcal{YD}$ by H. Moreover* $\operatorname{gr} E \cong \mathcal{D}(A)$ *as bialgebras in ${}_{H}^{H}\mathcal{YD}$.*

Proof. By the foregoing $A^{\zeta} \cong Q\#H \cong E^{\gamma}\#H = (E\#H)^{\Gamma}$ as coquasi-bialgebras. Now A is quasi-isomorphic to A^{ζ} which is quasi-isomorphic to $E\#H$ so that A is quasi-isomorphic to $E\#H$. Moreover

$$\operatorname{gr} E = \operatorname{gr} E^{\gamma} = \operatorname{gr} Q \cong \mathcal{D}(A).$$

where the first equality holds by Proposition 2.6. □

More generally, given a (finite-dimensional) Hopf algebra A whose coradical H is a sub-Hopf algebra, then if H is also semisimple, we expect that A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}_{H}^{H}\mathcal{YD}$ by H. See *e.g.* [21, Corollary 3.4] and the proof therein and [3,4] for a further clue in this direction.

6. Examples

We notice that the Hochschild cohomology of a finite-dimensional Nichols algebras has been computed in a few examples. We consider here those Nichols algebras to compute $\mathrm{H}^{3}_{\mathcal{YD}}(\mathcal{B}(V), \Bbbk)$.

6.1. Braidings of Cartan type

Let $A = (a_{ij})_{1\leq i,j\leq\theta}$ be a finite Cartan matrix, Δ the corresponding root system, $(\alpha_i)_{1\leq i\leq\theta}$ a set of simple roots and W its Weyl group. Let $w_0 = s_{i_1}\cdots s_{i_M}$ be a reduced expression of the element $w_0 \in W$ of maximal length as a product of simple reflections, $\beta_j = s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})$, $1 \leq j \leq M$. Then $\beta_j \neq \beta_k$ if $j \neq k$ and $\Delta^{+} = \{\beta_j | 1 \leq j \leq M\}$, see [22, page 25 and Proposition 3.6].

Let Γ be a finite Abelian group, $\widehat{\Gamma}$ its group of characters. $\mathcal{D} = (\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, A)$ is a *datum of finite Cartan type* [12] associated to Γ and A if $g_i \in \Gamma$,

$\chi_j \in \widehat{\Gamma}$, $1 \leq i, j \leq \theta$, satisfy $\chi_i(g_i) \neq 1$, $\chi_i(g_j)\chi_j(g_i) = \chi_i(g_i)^{a_{ij}}$ for all i, j. Set $\mathfrak{q} = (q_{ij})_{1\leq i,j\leq\theta}$, where $q_{ij} = \chi_j(g_i)$.

In what follows V denotes the Yetter-Drinfeld module over $\Bbbk\Gamma$, $\dim V = \theta$, with a fixed basis $x_1, \dots, x_\theta$, where the action and the coaction over each x_i is given by χ_i and g_i, respectively. Then the associated braiding is $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all i, j. Let $\mathcal{B}_{\mathfrak{q}} = \mathcal{B}(V)$. The tensor algebra $T(V)$ is $\mathbb{N}_0^\theta$-graded with grading α_i for each x_i. For $\beta = \sum_{i=1}^{\theta} a_i\alpha_i \in \Delta^+$, set

$$g_\beta = g_1^{a_1} \cdots g_\theta^{a_\theta}, \qquad \chi_\beta = \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \qquad q_\beta = \chi_\beta(g_\beta).$$

Given $\alpha, \beta \in \Delta^+$, we denote $q_{\alpha\beta} = \chi_\beta(g_\alpha)$.

We assume as in [12,26] that *the order of q_{ii} is odd for all i, and not divisible by 3 for each connected component of the Dynkin diagram of A of type G_2*. Therefore the order of q_{ii} is the same for all the i in the same connected component J. Given $\beta \in J$, we denote by N_β the order of the corresponding q_{ii} in J, which is also the order of q_β.

By [23] there exist homogeneous elements x_β of degree β, $\beta \in \Delta^+$, such that the Nichols algebra $\mathcal{B}_{\mathfrak{q}}$ of V is presented by generators $x_1, \dots, x_\theta$ and relations

$$\begin{aligned} (\operatorname{ad}_c x_i)^{1-a_{ij}} x_j &= 0, && 1 \leq i \neq j \leq \theta; \\ x_\beta^{N_\beta} &= 0, && \beta \in \Delta_+. \end{aligned}$$

Moreover $\{x_{\beta_1}^{n_1} \dots x_{\beta_M}^{n_M} | 0 \leq n_i < N_{\beta_i}\}$ is a basis of $\mathcal{B}_{\mathfrak{q}}$.

We shall prove that $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{B}_{\mathfrak{q}}, \Bbbk\right) = 0$. We need first some technical results.

Lemma 6.1. *Let $\alpha, \beta \in \Delta^+$. Then either $g_\alpha g_\beta^{N_\beta} \neq e$, or else $\chi_\alpha \chi_\beta^{N_\beta} \neq \epsilon$.*

Proof. Suppose on the contrary that $g_\alpha g_\beta^{N_\beta} = e$, $\chi_\alpha \chi_\beta^{N_\beta} = \epsilon$. Then

$$q_\alpha = \chi_\alpha^{-1}\left(g_\alpha^{-1}\right) = \chi_\beta^{N_\beta}\left(g_\beta^{N_\beta}\right) = q_\beta^{N_\beta^2} = 1,$$

since q_β is a root of unity of order N_β. But this is a contradiction, since $q_\alpha \neq 1$. □

Lemma 6.2. *Let $\alpha, \beta, \gamma \in \Delta^+$ be pairwise different. Then either $g_\alpha g_\beta g_\gamma \neq e$, or else $\chi_\alpha \chi_\beta \chi_\gamma \neq \epsilon$.*

Proof. Suppose on the contrary that $g_\alpha g_\beta g_\gamma = e$ and $\chi_\alpha \chi_\beta \chi_\gamma = \epsilon$. Then

$$\begin{aligned} q_\alpha &= \chi_\alpha^{-1}\left(g_\alpha^{-1}\right) = \chi_\beta \chi_\gamma \left(g_\beta g_\gamma\right) = q_\beta q_\gamma q_{\beta\gamma} q_{\gamma\beta}, \\ q_\beta &= q_\alpha q_\gamma q_{\alpha\gamma} q_{\gamma\alpha}, \\ q_\gamma &= q_\alpha q_\beta q_{\alpha\beta} q_{\beta\alpha}. \end{aligned} \tag{6.1}$$

Notice that α, β, γ belong to the same connected component. Indeed, if γ belongs to a different connected component, then $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = 1$. Thus $q_\beta = q_\alpha q_\gamma = q_\beta q_\gamma^2$, so $q_\gamma^2 = 1$, which is a contradiction. Therefore we may assume that the Dynkin diagram is connected.

One can prove that $q_{s_i(\alpha)} = q_\alpha$ for every $\alpha \in \Delta$. As we observed that $\Delta^+ = \{\beta_j | 1 \leq j \leq M\}$, we deduce that for every $\beta \in \Delta^+$ there is some j such that $q_\beta = q_j$. One can prove that there is some $q \in \Bbbk$ such that $q_\alpha = q^{(\alpha,\alpha)/2}$ and $q_{\alpha\gamma}q_{\gamma\alpha} = q^{(\alpha,\gamma)}$, where $(\cdot,\cdot)$ is the invariant bilinear form on the simple Lie algebra $\mathfrak{g}$ associated with the finite Cartan matrix [13, Chapter VI, Section 1, Proposition 3 and Definition 3] and the basis of the root systems given in [13, Chapter VI, Section 4] should be normalized in such a way that $q = q_\delta$, $(\delta,\delta) = 2$ for each short root $\delta \in \Delta$. Note that $q_\alpha = q^{(\alpha,\alpha)/2} \neq 1$ for all α as $(\alpha,\alpha) \neq 0$. Thus

- $q_\alpha = q_\beta = q_\gamma = q$ if the Dynkin diagram is simply laced,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^2\}$ if the Dynkin diagram has a double arrow,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^3\}$ if the Dynkin diagram is of type G_2.

If the Dynkin diagram is simply laced, then, by (6.1), we have $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = q_{\alpha\beta}q_{\beta\alpha} = q^{-1}$. Then $q^{(\alpha,\gamma)} = q^{-1}$. Now set $n(\alpha,\beta) := 2(\alpha,\beta)/(\beta,\beta) = (\alpha,\beta)$. Then $n(\alpha,\beta)$ is symmetric whence, by [13, Chapter VI, Section 1, page 148] we have $(\alpha,\gamma) = -1$ as the order of q is odd, so $\alpha + \gamma \in \Delta^+$, by [13, VI, Section 1, Corollary, page 149]. Now the same argument we used above shows that also $(\alpha,\beta) = -1 = (\gamma,\beta)$ and hence $(\alpha+\gamma,\beta) = -2$, so $\alpha+\beta+\gamma \in \Delta^+$, since $\alpha+\gamma \neq -\beta$ (as $\alpha+\gamma$ and β are both in Δ^+). But $q_{\alpha+\beta+\gamma} = q_\alpha q_\beta q_\gamma q_{\beta\gamma}q_{\gamma\beta}q_{\alpha\gamma}q_{\gamma\alpha}q_{\alpha\beta}q_{\beta\alpha} = 1$, which is a contradiction.

If the Dynkin diagram has a double arrow, then $q_\alpha, q_\beta, q_\gamma \in \{q, q^2\}$. If $q_\alpha = q_\beta = q_\gamma$, then the proof follows as for the simply-laced case because $n(u,v) = n(v,u)$ for $u, v \in \{\alpha,\beta,\gamma\}$. If $q_\alpha = q_\beta = q$ and $q_\gamma = q^2$, then $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = q^{-2}$, and $q_{\alpha\beta}q_{\beta\alpha} = 1$, by (6.1). Then a simple calculation yields $(\beta,\gamma) = -2$ so that $\beta+\gamma \in \Delta^+$. One also gets $(\alpha,\beta) = 0$ and $(\alpha,\gamma) = -2$ so that $(\alpha,\beta+\gamma) = (\alpha,\beta) + (\alpha,\gamma) = -2 < 0$ by the conditions on the order of q, so again $\alpha+\beta+\gamma \in \Delta^+$; but again we obtain $q_{\alpha+\beta+\gamma} = 1$, which is a contradiction. The proof for $q_\alpha = q_\beta = q^2$ and $q_\gamma = q$ follows analogously.

Finally, if the Dynkin diagram is of type G_2, then a similar analysis gives a contradiction. $\square$

For each $1 \leq k \leq M$, set $\mathcal{B}_\mathfrak{q}(k)$ as the subspace of $\mathcal{B}_\mathfrak{q}$ spanned by $\{x_{\beta_1}^{n_1}, \dots, x_{\beta_k}^{n_k} | 0 \leq n_i < N_{\beta_i}\}$. By [17] this gives an algebra filtration, and the graded algebra $\operatorname{Gr}\mathcal{B}_\mathfrak{q}$ associated to this filtration is presented by generators $\mathbf{x}_\beta$, $\beta \in \Delta^+$, and relations

$$\mathbf{x}_\beta \mathbf{x}_\gamma = q_{\beta\gamma}\mathbf{x}_\gamma \mathbf{x}_\beta, \qquad \mathbf{x}_\beta^{N_\beta} = 0, \qquad \beta < \gamma \in \Delta^+.$$

In [26] $\operatorname{Gr}\mathcal{B}_\mathfrak{q}$ is viewed as an algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, which (as an algebra) is the Nichols algebra of Cartan type $A_1 \times \cdots \times A_1$, M copies, with action and coaction on $\mathbf{x}_\beta$

given by χ_β, g_β, respectively. By [26, Theorem 4.1], $\mathrm{H}^\bullet(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk)$ is the algebra generated by $\xi_\beta, \eta_\beta, \beta \in \Delta^+$, where $\deg \xi_\beta = 2, \deg \eta_\beta = 1$, and relations

$$\xi_\beta\xi_\gamma = q_{\beta\gamma}^{N_\beta N_\gamma}\xi_\gamma\xi_\beta, \quad \eta_\beta\xi_\gamma = q_{\beta\gamma}^{N_\gamma}\xi_\gamma\eta_\beta, \quad \eta_\beta\eta_\gamma = -q_{\beta\gamma}\eta_\gamma\eta_\beta, \quad \beta,\gamma \in \Delta^+.$$

As we assume that all the q_{ii} have odd order, we deduce in particular from the last equality that $\eta_\beta^2 = 0$ for all $\beta \in \Delta^+$. As an algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, the action and coaction on ξ_β is given by $\chi_\beta^{-N_\beta}$, $g_\beta^{-N_\beta}$, while the action and coaction on η_β is given by $\chi_\beta^{-1}, g_\beta^{-1}$.

Theorem 6.3. $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{B}_{\mathrm{q}}, \Bbbk\right) = 0.$

Proof. First we will prove that $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)^D = 0$ for $D := D(\Bbbk\Gamma)$. Now, the invariants are with respect to the D-bimodule structure that $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)$ inherits from $\mathrm{Hom}\left((\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}})^{\otimes 3}, \Bbbk\right)$ (this is a D-bimodule as its arguments are left D-modules). Since the left D-module structure is induced by the one of $\Bbbk$, it is trivial. Thus the invariants of $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)$ as a D-bimodule reduce to its invariants as a right D-module. Since right D-modules are equivalent to left D-modules, via the antipode of D which is invertible as D is finite-dimensional, the right D-module structure of $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)$ becomes the structure of object in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ described above. Thus, in order to prove that $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)^D = 0$ we just have to check that the invariants of $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)$ as a left-left Yetter-Drinfeld modules are zero.

Now, by the defining relations of $\mathrm{H}^\bullet(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk)$, a basis B of $\mathrm{H}^3(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk)$ is given by $\{\xi_\alpha\eta_\beta\} \cup \{\eta_\alpha\eta_\beta\eta_\gamma | \alpha < \beta < \gamma\}$. If $v \in \mathrm{H}^3(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk)$ is invariant, then v is written as a linear combination of elements in the trivial component. Indeed, write $v = \sum_{b\in B} c_b\, b$ for some $c_b \in \Bbbk$, and let g_b, χ_b be the elements describing the component of $b \in B$. Then

$$v = g\cdot v = \sum_{b\in B} c_b\, g\cdot b = \sum_{b\in B} c_b \chi_b(g)\, b, \qquad \text{for all } g \in \Gamma,$$

$$1\otimes v = \rho(v) = \sum_{b\in B} c_b\, \rho\cdot b = \sum_{b\in B} c_b g_b \otimes b.$$

If $c_b \neq 0$, then $\chi_b(g) = 1$ for all $g \in \Gamma$ so $\chi_b = \epsilon$, and $g_b = 1$. Thus b is invariant. We have so proved that the existence of $v \neq 0$ invariant implies the existence of $b \in B$ invariant. Hence, if B has no invariant element then there is no invariant element at all. Note that, for all $h \in H$, we have $h\cdot(\xi_\alpha\eta_\beta) = (\chi_\alpha^{-N_\alpha}\chi_\beta^{-1})(h)\xi_\alpha\eta_\beta$ and $\rho(\xi_\alpha\eta_\beta) = g_\alpha^{-N_\alpha}g_\beta^{-1}\otimes\xi_\alpha\eta_\beta$ so that, by Lemma 6.1, the element $\xi_\alpha\eta_\beta$ is not D-invariant. A similar argument, using Lemma 6.2, shows that also $\eta_\alpha\eta_\beta\eta_\gamma$ is not D-invariant. Thus the elements in B are not D-invariant, so $\mathrm{H}^3\left(\mathrm{Gr}\,\mathcal{B}_{\mathrm{q}}, \Bbbk\right)^D = 0$. Since the elements in $\{x_{\beta_1}^{n_1}\dots x_{\beta_k}^{n_k} | 0 \le n_i < N_{\beta_i}\}$ are eigenvectors for D, we can mimic the argument in [26, Section 5] by taking into account the spectral sequence associated to the filtration of algebras therein; see for example [26, Corollary 5.5] for a similar argument. Thus $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{B}_{\mathrm{q}}, \Bbbk\right) \cong \mathrm{H}^3\left(\mathcal{B}_{\mathrm{q}}, \Bbbk\right)^D = 0$. □

Remark 6.4. Notice that $\mathrm{H}^3_{\mathcal{YD}}\left(\mathcal{B}_{\mathrm{q}}, \Bbbk\right) \cong \mathrm{H}^3\left(\mathcal{B}_{\mathrm{q}}, \Bbbk\right)^{D(\Bbbk\Gamma)} = 0$ although $\mathrm{H}^3\left(\mathcal{B}_{\mathrm{q}}\#\Bbbk\Gamma, \Bbbk\right) \cong \mathrm{H}^3\left(\mathcal{B}_{\mathrm{q}}, \Bbbk\right)^{\Gamma}$ can be non-trivial, see for example [26, Example 5.8].

6.2. Braidings of non-diagonal type

For $n \geq 3$, denotes $\mathcal{FK}_n$ the quadratic algebra [19] with a presentation by generators $x_{(ij)}$, $1 \leq i < j \leq n$, and relations

$$\begin{aligned}
x_{(ij)}^2 &= 0, && 1 \leq i < j \leq n,\\
x_{(ij)}x_{(jk)} &= x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, && 1 \leq i < j < k \leq n,\\
x_{(jk)}x_{(ij)} &= x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, && 1 \leq i < j < k \leq n,\\
x_{(ij)}x_{(kl)} &= x_{(kl)}x_{(ij)}, && \#\{i, j, k, l\} = 4.
\end{aligned}$$

According to [27] each $\mathcal{FK}_n$ is a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group S_n, generated as an algebra by the vector space V_n with basis $\{x_{(ij)} \mid 1 \leq i < j \leq n\}$. The action is described by identifying (ij) with the corresponding transposition in S_n and then consider the conjugation twisted by the sign, while the coaction is given by declaring x_σ a homogeneous element of degree σ. Then the braiding on V_n becomes

$$c(x_\sigma \otimes x_\tau) = \chi(\sigma, \tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma, \qquad \chi(\sigma, \tau) = \begin{cases} 1 & \sigma(i) < \sigma(j),\ \tau = (ij),\ i < j, \\ -1 & \text{otherwise}, \end{cases}$$

where σ and τ are transpositions. Moreover $\mathcal{FK}_n$ projects onto the Nichols algebra $\mathcal{B}(V_n)$. For $n = 3, 4, 5$, it is known that $\mathcal{FK}_n = \mathcal{B}(V_n)$ and has dimension, respectively, 12, 576 and 8294400.

The Hochschild cohomology of $\mathcal{FK}_3$ is a consequence of the results in [36] as follows.

Theorem 6.5. $\mathrm{H}^\bullet_{\Bbbk S_3\text{-Mod}}(\mathcal{FK}_3, \Bbbk)$ *is isomorphic to the graded algebra*

$$\Bbbk[X, U, V]/(U^2V - VU^2), \qquad \textit{where } \deg U = \deg V = 2,\ \deg X = 4.$$

Proof. By [36, Theorem 4.19], we have that $E(B\#\Bbbk S_3)$ is isomorphic to the algebra in the claim, where $B = \mathcal{FK}_3$. By [36, Theorem 2.17], we know that $E(B\#\Bbbk S_3) \cong E(B)^{\Bbbk S_3}$ as graded algebras. As observed in Remark 4.2, we have that $E(B) \cong \mathrm{H}^\bullet(B, \Bbbk)$. By Remark 4.6, we have $\mathrm{H}^\bullet(B, \Bbbk)^{\Bbbk S_3} \cong \mathrm{H}^\bullet_{\Bbbk S_3\text{-Mod}}(\mathcal{FK}_3, \Bbbk)$. □

From this result we get $\mathrm{H}^3_{\Bbbk S_3\text{-Mod}}(\mathcal{FK}_3, \Bbbk) = 0$ so that, by Proposition 4.7 we conclude that the following holds:

Corollary 6.6. $\mathrm{H}^3_{\mathcal{YD}}(\mathcal{FK}_3, \Bbbk) = 0$.

References

[1] A. ARDIZZONI, M. BEATTIE and C. MENINI, *Cocycle deformations for hopf algebras with a coalgebra projection*, J. Algebra **324** (2010), 673–705.

[2] A. ARDIZZONI, M. BEATTIE and C. MENINI, *Gauge deformations for Hopf algebras with the dual Chevalley property*, J. Algebra Appl. **11** (2012), 1250051.

[3] N. ANDRUSKIEWITSCH, I. E. ANGIONO and A. GARCÍA IGLESIAS, *Liftings of Nichols algebras of diagonal type I. Cartan type A*, Int. Math. Res. Not., to appear.

[4] N. ANDRUSKIEWITSCH, I. E. ANGIONO, A. GARCÍA IGLESIAS, A. MASUOKA and C. VAY, *Liftings via cocycle deformations*, J. Pure Appl. Algebra **218** (2014), 684–703.

[5] A. ARDIZZONI and C. MENINI, *Associated Graded Algebras and Coalgebras*, Comm. Algebra **30** (2012), 862–896.

[6] A. ARDIZZONI, C. MENINI and D. ŞTEFAN, *A Monoidal Approach to Splitting Morphisms of Bialgebras*, Trans. Amer. Math. Soc. **359** (2007), 991–1044.

[7] A. ARDIZZONI, C. MENINI and D. ŞTEFAN, *Hochschild cohomology and 'smoothness' in monoidal categories*, J. Pure Appl. Algebra **208** (2007), 297–330.

[8] A. ARDIZZONI, C. MENINI and F. STUMBO, *Small bialgebras with a projection*, J. Algebra **314** (2007), 613–663.

[9] I. E. ANGIONO, *Basic quasi-Hopf algebras over cyclic groups*, Adv. Math. **225** (2010), 3545–3575.

[10] A. ARDIZZONI and A. PAVARIN, *Bosonization for Dual Quasi-Bialgebras and Preantipode*, J. Algebra **390** (2013), 126–159.

[11] N. ANDRUSKIEWITSCH and H.-J. SCHNEIDER, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), 658–691.

[12] N. ANDRUSKIEWITSCH and H-J. SCHNEIDER, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. **171** (2010), 375–417.

[13] N. BOURBAKI, "Groupes et algèbres de Lie", Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[14] K. S. BROWN, "Cohomology of Groups", Graduate Texts in Mathematics, Vol. 87, Springer-Verlag, New York-Berlin, 1982.

[15] H. CARTAN and S. EILENBERG,"Homological Algebra", Princeton University Press, Princeton, N.J., 1956.

[16] S. DĂSCĂLESCU, C. NĂSTĂSESCU and S. RAIANU, "Hopf Algebras. An Introduction", Monographs and Textbooks in Pure and Applied Mathematics, Vol. 235, Marcel Dekker, Inc., New York, 2001.

[17] C. DE CONCINI and C. PROCESI, *Quantum groups*, In: "D-modules, Representation Theory, and Quantum Groups", Lecture Notes in Math., Vol. 1565, Springer, 1993, 31–140.

[18] P. ETINGOF and S. GELAKI, *Liftings of graded quasi-Hopf algebras with radical of prime codimension*, J. Pure Appl. Algebra **205** (2006), 310–322.

[19] S. FOMIN and A. N. KIRILLOV, *Quadratic algebras, Dunkl elements and Schubert calculus*, Progr. Math. **172** (1999), 146–182.

[20] V. GINZBURG and S. KUMAR, *Cohomology of quantum groups at roots of unity*. Duke Math. J. **69** (1993), 179–198.

[21] L. GRUNENFELDER and M. MASTNAK, *Pointed Hopf algebras as cocycle deformations*, preprint, arXiv:1010.4976v1

[22] H. HILLER, "Geometry of Coxeter Groups", Research Notes in Mathematics, Vol. 54, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.
[23] G. LUSZTIG, *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), 89–113.
[24] C. KASSEL, "Quantum Groups", Graduate Texts in Mathematics, Vol. 155, Springer-Verlag, New York, 1995.
[25] S. MAJID, "Foundations of Quantum Group Theory", Cambridge University Press, Cambridge, 1995.
[26] M. MASTNAK, J. PEVTSOVA, P. SCHAUENBURG and S. WITHERSPOON, *Cohomology of finite-dimensional pointed Hopf algebras*, Proc. Lond. Math. Soc. **100** (2010), 377–404.
[27] A. MILINSKI and H. J. SCHNEIDER, *Pointed indecomposable Hopf algebras over Coxeter groups*, Contemp. Math. **267** (2000), 215–236.
[28] S. MONTGOMERY, "Hopf Algebras and their Actions on Rings", CMBS Regional Conference Series in Mathematics, Vol. 82, 1993.
[29] D. E. RADFORD, "Hopf Algebras", Series on Knots and Everything, Vol. 49, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.
[30] D. E. RADFORD, *Minimal quasitriangular Hopf algebras*, J. Algebra **157** (1993), 285–315.
[31] D. E. RADFORD, *The structure of Hopf algebras with a projection*, J. Algebra **92** (1985), 322–347.
[32] D. E. RADFORD and J. TOWBER, *Yetter-Drinfel'd categories associated to an arbitrary bialgebra*, J. Pure Appl. Algebra **87** (1993), 259–279.
[33] P. SCHAUENBURG, *Faithful flatness over Hopf subalgebras: counterexamples*, In: "Interactions between Ring Theory and Representations of Algebras (Murcia)", Lecture Notes in Pure and Appl. Math., Vol. 210, Dekker, New York, 2000, 331–344.
[34] D. ŞTEFAN, *Hochschild cohomology on Hopf Galois extensions*, J. Pure Appl. Algebra **103** (1995), 221–233.
[35] B. STENSTRÖM, "Rings of Quotients. An Introduction to Methods of Ring Theory", Die Grundlehren der Mathematischen Wissenschaften, Band 217, Springer-Verlag, New York-Heidelberg, 1975.
[36] D. ŞTEFAN and C. VAY, *The cohomology ring of the* 12*-dimensional Fomin-Kirillov algebra*, Adv. Math. **291** (2016), 584–620.
[37] M. E. SWEEDLER, "Hopf Algebras", Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

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