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Renormalized volume on the Teichmüller space of punctured surfaces

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Abstract. We define and study the renormalized volume for geometrically finite hyperbolic 3-manifolds, including those with rank-1 cusps. We prove a variation formula, and show that for certain families of convex co-compact hyperbolic metrics g_ε degenerating to a geometrically finite hyperbolic metric g_0 with rank-1 cusps, the renormalized volume converges to the renormalized volume of the limiting metric.

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1. Introduction

The renormalized volume is a geometric quantity for certain infinite-volume hyperbolic 3-dimensional manifolds, namely those which are convex co-compact. Such a manifold X can be compactified into a smooth compact manifold with boundary $\overline{X}$ in such a way that its metric g has the following property: for any smooth function $\rho \in \mathcal{C}^\infty(\overline{X})$ which is a boundary defining function (*i.e.*, $\rho \geq 0$, $\rho^{-1}(0) = \partial\overline{X}$ and $d\rho|_{\partial\overline{X}}$ does not vanish), $\rho^2 g$ extends to a smooth metric on $\overline{X}$. This induces a natural conformal class on the boundary $M := \partial\overline{X}$ by picking the conformal class $[h]$ of $h = (\rho^2 g)|_{TM}$. We call $(M, [h])$ the *conformal boundary* of X. We say that a boundary defining function ρ is a *geodesic boundary defining function* in $\overline{X}$ if $|d\log(\rho)|_g = 1$ near the boundary M. Notice that such a function induces an *equidistant foliation* near M, given by the level sets of ρ. It turns out that there is a one-to-one correspondence $\hat{h} = e^{2\varphi}h \in [h] \mapsto \hat{\rho}$ between geodesic boundary defining functions (or equivalently, equidistant foliations) near M and the elements of the conformal class $[h]$ on M, where $\hat{\rho}$ solves the Hamilton-Jacobi equation

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near M

$$\left|\frac{d\hat{\rho}}{\hat{\rho}}\right|_g = 1, \quad (\hat{\rho}^2 g)|_{TM} = \hat{h}. \tag{1.1}$$

The renormalized volume of (X, g) is the function on $[h]$ defined by

$$\mathrm{Vol}_R(X, g; \hat{h}) := \mathrm{FP}_{z=0} \int_X \hat{\rho}^z \mathrm{dvol}_g$$

where $\hat{\rho}$ is any smooth positive extension to X of the function solving (1.1) and $\mathrm{FP}_{z=0}$ denotes the finite part (or regular value) at $z = 0$ of a meromorphic function in the variable $z \in \mathbb{C}$. In a way, this definition has similarities with the renormalization used to define the determinant of the Laplacian on a compact manifold. In fact, the functional $\varphi \mapsto \mathrm{Vol}_R(X, g; e^{2\varphi} h)$ varies in the same exact way as do the Liouville functional and the logarithm of the determinant of the Laplacian viewed as functionals on $[h]$. Among metrics in the conformal class $[h]$ of constant volume $2\pi|\chi(M)|$, it is maximized at the hyperbolic metric $h^{\mathrm{hyp}} \in [h]$, and we define the *renormalized volume of* (X, g) by

$$\mathrm{Vol}_R(X, g) := \mathrm{Vol}_R(X, g; h^{\mathrm{hyp}}).$$

We remark that the renormalized volume could equivalently be defined by $\mathrm{Vol}_R(X, g; \hat{h}) = a_0$, where a_0 is defined by the asymptotic expansion (for some $a_j \in \mathbb{R}$) as $\epsilon \to 0$

$$\int_{\hat{\rho} \geq \epsilon} \mathrm{dvol}_g = a_2 \epsilon^{-2} + a_1 \log(\epsilon) + a_0 + \mathcal{O}(\epsilon).$$

In this setting, the first general study was done by Krasnov-Schlenker [18], although earlier works of Takhtajan-Teo [28] considered this quantity, and for more general Poincaré-Einstein manifolds the renormalized volume appeared even earlier in works of Henningson-Skenderis [15] and Graham [11] in AdS/CFT correspondence.

When defined in this way, the renormalized volume has many interesting properties:

- the renormalized volume is a Kähler potential for the Weil-Peterson metric on the Teichmüller space of the conformal boundary, when viewed as a function on the deformation space of convex co-compact hyperbolic 3-manifolds. This was proved by Takhtajan-Teo [28] for a class of Kleinian convex co-compact groups, by Krasnov-Schlenker [18] for quasi-Fuchsian manifolds and by Guillarmou-Moroianu [13] for all geometrically finite hyperbolic 3-manifolds without cusps of rank 1;
- $\mathrm{Vol}_R(X, g)$ can be compared to the volume of the convex core $\mathrm{Vol}(C(X))$ by

$$\mathrm{Vol}(C(X)) - 10\chi(M) \leq \mathrm{Vol}_R(X, g) \leq \mathrm{Vol}(C(X)).$$

This inequality is proved by Schlenker [27] for quasi-Fuchsian manifolds, and extended by Bridgeman-Canary [4] to convex co-compact 3-manifolds with incompressible boundary;

- Schlenker [27] proves that for quasi-Fuchsian manifolds, $\mathrm{Vol}_R(X, g)$ is comparable to the Weil-Petersson distance between the two connected components $(M, h_\pm)$ of the conformal boundary. Namely he shows that

$$\mathrm{Vol}_R(X, g) \leq \tfrac{3}{2}\sqrt{2\pi\chi(X)}d_{\mathrm{WP}}(h_+, h_-), \tag{1.2}$$

improving a weaker inequality due to Brock [5]. Moreover, using [5], Schlenker obtains that there exists some $k_1, k_2 > 0$ such that

$$k_1 d_{\mathrm{WP}}(h_+, h_-) - k_2 \leq \mathrm{Vol}_R(X, g).$$

These inequalities have interesting implications about the geometry of hyperbolic 3-manifolds fibering over the circle, *cf.* [17], [6];

- Ciobotaru-Moroianu [8] prove that for almost-Fuchsian manifolds, the renormalized volume is positive except at the Fuchsian locus where it vanishes[1];
- Moroianu [25] proves that the renormalized volume has a critical point on the deformation space of convex co-compact 3-manifolds if the convex core has smooth totally geodesic boundary, and the Hessian of Vol_R is positive definite there. Another proof appeared recently in [30].

Like in the estimate (1.2), it is of interest to understand the properties of Vol_R on the deformation space of convex co-compact hyperbolic 3-manifolds with a given topology. For example, (1.2) shows that Vol_R does not explode as one approaches the boundary of the Teichmüller space viewed as a Bers slice in the quasi-Fuchsian space.

The first goal of this work is to define the renormalized volume for geometrically finite hyperbolic 3-manifolds, focusing on the rank-1 cusps. Contrary to the convex co-compact setting, the existence of equidistant foliations via geodesic boundary defining functions turns out to be quite tricky in the case of rank-1 cusps. A geometrically finite hyperbolic manifold $(X, g) = \Gamma\backslash\mathbb{H}^3$ with rank-1 cusps is the interior of a smooth non-compact manifold $\overline{X} = \Gamma\backslash(\mathbb{H}^3 \cup \Omega_\Gamma)$ with boundary, where $\Omega_\Gamma \subset \mathbb{S}^2$ is the discontinuity set of the Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$. The smooth manifold with boundary $\overline{X}$ has a non-compact boundary $M = \Gamma\backslash\Omega_\Gamma$ equipped with a conformal class $[h]$ induced from the hyperbolic metric g. On this conformal boundary $(M, [h])$, we show in Proposition 2.3 that there exists a unique complete hyperbolic metric $h^{\mathrm{hyp}} \in [h]$ with finite volume and cusps.

Theorem 1.1. *Let (X, g) be a geometrically finite hyperbolic 3-manifold with rank-1 cusps, let $(M, [h])$ be its conformal boundary and let h^{hyp} be the complete hyperbolic metric with finite volume in the conformal class $[h]$. Then there exists a*

[1] The normalization to make it 0 at the Fuchsian locus is actually to choose the metric in the conformal boundary to have Gaussian curvature -4. The same normalization is used in [18]

non-negative smooth boundary defining function ρ on $\overline{X}$ such that $\rho^2 g|_{TM} = h^{\mathrm{hyp}}$ and, outside a finite volume region $\mathcal{V} \subset X$, $|d\log(\rho)|_g = 1$. The function $z \to \int_{X\setminus\mathcal{V}} \rho^z \mathrm{dvol}_g$ admits a meromorphic extension from $\mathrm{Re}(z) > 2$ to a neighborhood of $z = 0$.

We define the renormalized volume by

$$\mathrm{Vol}_R(X, g) := \mathrm{Vol}_g(\mathcal{V}) + \mathrm{FP}_{z=0} \int_{X\setminus\mathcal{V}} \rho^z \mathrm{dvol}_g.$$

In fact, in Proposition 2.4, we show a stronger statement: we prove that for each conformal representative in $[h]$ with certain asymptotic properties near the cusp, there is an associated geodesic boundary defining function and an equidistant foliation, allowing to view Vol_R as a function on $[h]$ like in the convex co-compact case. In Proposition 7.1, we show a variation formula similar to that of the determinant of the Laplacian [2, Theorem 2.9] or the Liouville functional:

$$\mathrm{Vol}_R\left(X, g; e^{2\varphi}h^{\mathrm{hyp}}\right) = \mathrm{Vol}_R\left(X, g; h^{\mathrm{hyp}}\right) - \frac{1}{4}\int_M \left(|\nabla\varphi|^2_{h^{\mathrm{hyp}}} - 2\varphi\right) \mathrm{dvol}_{h^{\mathrm{hyp}}}.$$

There is a diffeomorphism $\psi : [0, \varepsilon)_x \times M \to \overline{X} \setminus \mathcal{V}$ such that $\psi^*\rho = x$, and the metric has a finite expansion in powers of x:

$$\psi^* g = \frac{dx^2 + h_0 + x^2 h_2 + x^4 h_4}{x^2} \tag{1.3}$$

where the coefficients h_0, h_2 and h_4 are symmetric tensors on M and such that $h_0 = h^{\mathrm{hyp}}$, $h_2^0 := h_2 - \frac{1}{2}h^{\mathrm{hyp}}$ is trace-free and divergence-free with respect to h^{hyp}, and $h_4 = \frac{1}{4}h_0(A^2\cdot, \cdot)$ if A is the endomorphism defined by $h_2 = h_0(A\cdot, \cdot)$. The tensor h_2^0 can thus be identified to a cotangent vector to the Teichmüller space $\mathcal{T}(M)$ of M at the metric $h_0 = h^{\mathrm{hyp}}$, and the pair $(h_0, h_2^0) \in T^*\mathcal{T}(M)$ characterizes uniquely g. We call h_2^0 the *second fundamental form of g at M*.

Theorem 1.2. *For $t \in (-1, 1)$, let (X, g^t) be a smooth family of geometrically finite hyperbolic metrics with cusps of rank 1 and let h^t be the unique finite volume hyperbolic representative of the conformal boundary of (X, g^t). Then,*

$$\partial_t \mathrm{Vol}_R\left(X, g^t\right)\big|_{t=0} = -\frac{1}{4}\int_M \left\langle \dot{h}, h_2^0 \right\rangle_h \mathrm{dvol}_h,$$

with $\dot{h} = \partial_t h^t|_{t=0}$, $h = h^t|_{t=0}$, and h_2^0 is the second fundamental form of $g = g^t|_{t=0}$ at M.

The equivalent result was shown by Krasnov-Schlenker [18] (see also [13] for another proof) in the convex co-compact setting. This implies, using a theorem

of Marden [20], that the deformation space of a geometrically finite hyperbolic 3-manifold (X, g) with rank-1 cusps can be viewed as a Lagrangian submanifold $\mathcal{H} \subset T^*\mathcal{T}(M)$, the graph of the exact 1-form on $\mathcal{T}(M)$ given by the exterior differential of the renormalized volume functional $\mathrm{Vol}_R(X, \cdot) : \mathcal{T}(M) \to \mathbb{R}$. Equivalently, the restriction to $\mathcal{H}$ of the Liouville 1-form on $T^*\mathcal{T}(M)$ is exact, and a primitive is given by $\mathrm{Vol}_R(X, \cdot)$ if we identify $\mathcal{H}$ with $\mathcal{T}(M)$ by the canonical projection; we refer to [13] for details in the convex co-compact setting.

Ahlfors-Bers simultaneous uniformisation theorem shows that if (M, h_-) and (M, h_+) are two hyperbolic surfaces of finite volume with n cusps, there exists a unique (up to diffeomorphism) complete hyperbolic metric $g_{(h_-,h_+)}$ on the cylinder $X := \mathbb{R}_t \times M$, which is realized as a quotient $\Gamma\backslash\mathbb{H}^3$ for some quasi-Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ and the obtained manifold is geometrically finite with cusps of rank-1. The quasi-Fuchsian space is the deformation space of such quasi-Fuchsian groups and it identifies to $\mathcal{T}(M) \times \mathcal{T}(M)$ where $\mathcal{T}(M)$ is the Teichmüller space of M. Fixing h_-, the map $h_+ \mapsto g_{(h_-,h_+)}$ provides an embedding of $\mathcal{T}(M)$ into the quasi-Fuchsian deformation space and we view the renormalized volume as a function on $\mathcal{T}(M)$: $h_+ \mapsto \mathrm{Vol}_R(X, g_{(h_-,h_+)})$. We extend to the case with punctures the result proved in the convex co-compact case by Takhtajan-Teo [28], and later by Krasnov-Schlenker [18], Guillarmou-Moroianu [13].

Theorem 1.3. *Set $h_- = h_0 \in \mathcal{T}(M)$, the map $V_{h_0} : h_+ \mapsto \mathrm{Vol}_R\big(X, g_{(h_0,h_+)}\big)$ is a Kähler potential for Weil-Petersson metric on $\mathcal{T}(M)$; more precisely $\overline{\partial}\partial V_{h_0} = \frac{i}{16}\omega_{\mathrm{WP}}$ where ω_{WP} is the Weil-Petersson symplectic form.*

Notice that the same result was proved independently by Park-Takhtajan-Teo [29].

Our last result consists in analyzing the renormalized volume of families of convex co-compact hyperbolic 3-manifolds degenerating to a geometrically finite manifold with rank-1 cusps. We define precisely an *admissible degeneration of convex co-compact metrics* in Definition 6.1, but essentially such a family of metrics $(g_\varepsilon)_{\varepsilon>0}$ on X corresponds to having a disjoint union $H = \cup_{j=1}^{j_1} H_j$ of j_1 simple curves in $M = \partial\overline{X}$ such that:

(1) outside a uniform neighborhood $\mathcal{U}$ of H, $\rho^2 g_\varepsilon$ converges smoothly to a metric on $\overline{X} \setminus \mathcal{U}$ if ρ is a fixed boundary defining function of $\partial\overline{X}$;
(2) in $\mathcal{U}$ near H_j, the metric g_ε is isometric to a certain region of $\langle\gamma_j^\varepsilon\rangle\backslash\mathbb{H}^3$ where $\gamma_j^\varepsilon \in \mathrm{PSL}_2(\mathbb{C})$ is a loxodromic element converging as $\varepsilon \to 0$ to a parabolic element γ_j in such a way that $\alpha_j(\varepsilon)/\ell_j(\varepsilon)$ converges , where $\ell_j(\varepsilon)$ and $\alpha_j(\varepsilon)$ are respectively the translation length and the holonomy angle of γ_j^ε (i.e., γ_j^ε is conjugated to $z \mapsto e^{\ell(\varepsilon)+i\alpha_j(\varepsilon)}z$).

Our last theorem is:

Theorem 1.4. *Assume g_ε is an admissible degeneration of convex co-compact hyperbolic metrics on X, in the sense of Definition* 6.1*, to a geometrically finite hyperbolic metric g_0 with rank-*1 *cusps on X. Then*

$$\lim_{\varepsilon\to 0} \mathrm{Vol}(X, g_\varepsilon) = \mathrm{Vol}_R(X, g_0).$$

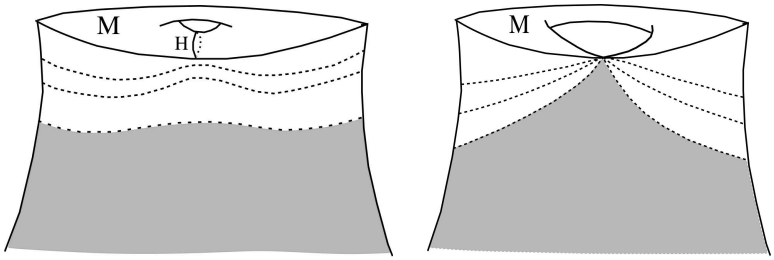

Figure 1.1. We consider a case when the curve H is being pinched in the boundary M. The equidistant foliation is represented by the dotted lines. The first picture corresponds to the convex co-compact case and the second picture is the hyperbolic 3-manifold with a rank-1 cusp. The dark regions are the convex cores.

We show in Proposition 6.2 that such admissible degenerations happen for instance on the boundary of the classical Schottky space. In [4, Theorem 1.3], Bridgeman and Canary show that in other asymptotic regime (when the radius of injectivity of the hyperbolic metric in the domain of discontinuity is going to 0), the renormalized volume tends to $-\infty$.

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2. Renormalized volume for geometrically finite hyperbolic 3-manifolds

2.1. Geometrically finite hyperbolic 3-manifolds

In this section we recall the geometry of geometrically finite hyperbolic manifolds of dimension 3. For more details, we refer to the paper of Bowditch [3] (see also [22, Mazzeo-Phillips] or [12, Guillarmou-Mazzeo]). A manifold X of dimension 3 is said to be geometrically finite hyperbolic if it can be realized as a quotient $X = \Gamma\backslash\mathbb{H}^3$ by a Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{PSO}(3,1)$, so that its action on $\mathbb{H}^3$ has a fundamental domain with finitely many side. In higher dimension, this definition is not very natural and the correct one is given by Bowditc; however we shall restrict here to the 3-dimensional case. If we view $\mathbb{H}^3$ as the open unit ball in $\mathbb{R}^3$, it can be naturally compactified into the closed unit ball $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \mathbb{S}^2$, and elements of $\mathrm{PSL}_2(\mathbb{C})$ acts on $\overline{\mathbb{H}^3}$. We say that X has cusps if Γ contains parabolic elements in $\mathrm{PSL}_2(\mathbb{C})$, *i.e.*, elements which fix only one point in the closed unit ball $\overline{\mathbb{H}^3}$. If for each point $p \in \mathbb{S}^2$ fixed by a parabolic transformation $\gamma_p \in \Gamma$, the subgroup $\Gamma_p \subset \Gamma$ fixing p is the cyclic group generated by the element γ_p, then we say that X has only *cusps of rank* 1, and we will make this assumption for what follows.[2]

[2] Cusps of rank 2 are trivial to deal with for what concerns renormalized volume questions, since they generate ends with finite volume in X.

We view $\mathbb{H}^3$ as the unit ball in $\mathbb{R}^3$. We can add to X a conformal boundary by defining

$$\overline{X} := \Gamma\backslash(\mathbb{H}^3 \cup \Omega)$$

where $\Omega \subset S^2$ is the domain of discontinuity of the group Γ, *i.e.*, the complement in S^2 of the limit set Λ_Γ consisting of accumulation points in the closed unit ball of the orbit of any given point $m \in \mathbb{H}^3$. The manifold $\overline{X}$ is a smooth manifold with boundary and its boundary

$$M := \Gamma\backslash\Omega = \partial\overline{X}$$

is a union of smooth Riemann surfaces, which has cusps if and only if Γ has rank-1 cusps. It inherits a conformal class which is defined to be the conformal class of $\rho^2 g|_{TM}$ where g is the hyperbolic metric on X and ρ is any smooth boundary defining function in $\overline{X}$ (*i.e.*, $\rho \geq 0$, $M = \{\rho = 0\}$ and $d\rho|_M$ never vanishes on M). Note that $\overline{X}$ is not compact if Γ has cusps.

The important geometric fact that we shall use is the following: there exists a compact set $\mathcal{K} \subset \overline{X}$ such that $\overline{X} \setminus \mathcal{K} = \cup_{j=1}^{j_1} \mathcal{U}_j^c$ where $\mathcal{U}_j^c$ are disjoint open sets of $\overline{X}$, called *cusp neighbourhoods*, so that g on $\mathcal{U}_j^c \cap X$ is isometric through a map ι_j to

$$\begin{gathered}\left\{(z = y + ix, w) \in \mathbb{H}^2 \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right);\ |z| > R_j\right\}, \\ \text{with metric } g = \frac{dx^2 + dy^2 + dw^2}{x^2}\end{gathered} \tag{2.1}$$

for some $R_j > 0$; here $\mathbb{H}^2 = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ is viewed as the upper half plane. We shall therefore identify $\mathcal{U}_j^c$ with the region in (2.1). Here j_1 is the number of rank-1 cusps. The compact $\mathcal{K}$ in $\overline{X}$ decomposes further into $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{U}^r$ where $\mathcal{K}_0$ is compact in X and $\mathcal{U}^r$ is a compact set of $\overline{X}$ such that the hyperbolic metric g in the interior of $\mathcal{U}^r$ near M is of the form $g = \overline{g}/\rho^2$ where ρ is a smooth boundary defining function of M and $\overline{g}$ is a smooth metric on $\mathcal{K}$. The boundary M is a non compact Riemann surface with $2j_1$ cusps, and M equipped with the conformal class $[\rho^2 g|_{TM}]$ is called the *conformal boundary* of X. Notice that, using an inversion $(v + iu) = -1/(y + ix)$ in the $\mathbb{H}^2$ factor of (2.1), the neighborhood $\mathcal{U}_j^c \cap X$ with metric g is also isometric to

$$\begin{gathered}\left\{(z = v + iu, w) \in \mathbb{H}^2 \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right);\ |z| < R_j^{-1}\right\}, \\ \text{with metric } g = \frac{du^2 + dv^2 + (u^2 + v^2)^2 dw^2}{u^2}.\end{gathered} \tag{2.2}$$

Using this model for $\mathcal{U}_j^c$, we see that we can compactify $\overline{X}$ into a compact smooth manifold with boundary, denoted $\overline{\mathbf{X}}$, by compactifying the open set (2.2) to

$$\left\{(z = v + iu, w) \in \overline{\mathbb{H}^2} \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right);\ |z| < R_j^{-1}\right\} \tag{2.3}$$

if $\overline{\mathbb{H}^2}$ is the closed upper half-plane of $\mathbb{C}$, and with the smooth structure given by the smooth structure on $\overline{\mathbb{H}^2} \times (\mathbb{R}/\frac{1}{2}\mathbb{Z})$. This compactification amounts to adding a circle at each cusp of $\overline{X}$, and clearly the interior of $\overline{\mathbf{X}}$ is X and $\overline{X}$ is an open set in $\overline{\mathbf{X}}$. We denote by H_j each of these circles defined by $\{u = v = 0\}$ in (2.3), and let $H := \cup_{j=1}^{j_1} H_j$.

There is another natural compactification of $\overline{X}$ (and X) that arises, which corresponds to the real blow-up of H in $\partial\overline{\mathbf{X}}$ in $\overline{\mathbf{X}}$, which we denote by $\overline{X}_c$. This is a smooth manifold with corners of codimension 2 defined as follows: by taking the representation (2.2) of $\mathcal{U}_j^c$, we see that this has closure in $\overline{X}$ diffeomorphic to

$$\left\{(u, v, w) \in \mathbb{R}_+ \times \mathbb{R} \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right);\, u^2 + v^2 < R_j^{-2}\right\}$$

and to define $\overline{X}_c$, we replace this chart by the chart

$$\left\{(r, \theta, w) \in \left[0,\, R_j^{-1}\right) \times \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right)\right\}$$

where $r := \sqrt{u^2 + v^2}$ and $\theta := \arctan \frac{v}{u}$. This corresponds to a real blow-up (denoted $\overline{X}_c = [\overline{\mathbf{X}}; H]$ in [23, Chapter 5]) of the submanifold $\{(u, v, w) \in \overline{\mathcal{U}_j^c};\, u = v = 0\}$, which is a circle. In this way, the manifold with corners $\overline{X}_c$ has two boundary hypersurfaces. One, given by $\theta = \pm\frac{\pi}{2}$, is denoted $\overline{M}$ and is a compactification of M to a smooth surface with boundary, while the other, the *cusp face*, denoted cf, is given by $r = 0$ and is diffeomorphic to a cylinder $[-\frac{\pi}{2}, \frac{\pi}{2}]_\theta \times (\mathbb{R}/\frac{1}{2}\mathbb{Z})_w$ if there is only one cusp of rank 1. More generally, the connected components of cf are in one-to-one correspondence with the cusps of rank 1 of X with each connected component diffeomorphic to $[-\frac{\pi}{2}, \frac{\pi}{2}]_\theta \times (\mathbb{R}/\frac{1}{2}\mathbb{Z})_w$. We will use this extended space $\overline{X}_c$ to describe the analytic structure of the geodesic boundary defining function of M in $\overline{X}$ near the cusps, which allows us to define the renormalized volume in that setting. To summarize, we have the following manifolds and inclusions

$$\partial\overline{X} = M \subset \overline{M}, \quad X \subset \overline{X} \subset \overline{\mathbf{X}}, \quad X \subset \overline{X} \subset \overline{X}_c = \left[\overline{\mathbf{X}}; H\right].$$

2.2. Renormalized volume in the convex co-compact case

A geometrically finite hyperbolic 3-manifold $X = \Gamma\backslash\mathbb{H}^3$ with no cusps is called *convex co-compact*. Such a manifold X can be decomposed as $X = \mathcal{K} \cup \mathcal{U}$ where $\mathcal{K} \subset X$ is a compact region with smooth boundary and $\mathcal{U}$ is isometric to

$$M \times (0, \delta)_\rho, \text{ with metric } g = \frac{d\rho^2 + h((\mathrm{Id} + \frac{1}{2}\rho^2 A)^2 \cdot, \cdot)}{\rho^2} \tag{2.4}$$

where $M = \Gamma\backslash\Omega$ is a compact surface (not necessarily connected), h is a metric on M, A is a symmetric endomorphism of TM satisfying the trace and divergence properties

$$\mathrm{Tr}_h(A) = -\tfrac{1}{2}\mathrm{Scal}_h, \quad \delta_h(A) = \tfrac{1}{2} d\,\mathrm{Scal}_h, \tag{2.5}$$

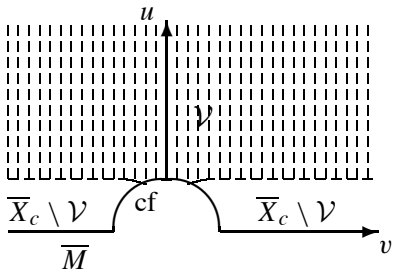

Figure 2.1. The manifold with corners $\overline{X}_c$ (the circle variable w is not represented). The region $\mathcal{V}$ has finite volume and appears in the statement of Proposition 2.4. It corresponds to the region where the geodesic boundary defining function is well-defined.

see [10, Theorem 7.4] or [18] for details. The product form (2.4) will also be written

$$g = \frac{d\rho^2 + h_0 + \rho^2 h_2 + \rho^4 h_4}{\rho^2}$$

with $h_0 = h$, $h_2(\cdot,\cdot) = h(A\cdot,\cdot)$ and $h_4(\cdot,\cdot) := \frac{1}{4}h(A^2\cdot,\cdot)$. The manifold M is compact and, when equipped with the conformal class $[\rho^2 g|_{TM}] = [h]$, is the conformal boundary of X. As above, X can be compactified smoothly into $\overline{X}$ with boundary $\partial\overline{X} = M$ and ρ, viewed as a function on $X \setminus \mathcal{K}$, is a smooth boundary defining function. The function ρ in $\mathcal{U}$ so that the metric has the form (2.4) is not unique and is characterized by the property

$$\left|\frac{d\rho}{\rho}\right|_g = 1 \text{ in } \mathcal{U}, \quad \text{and } (\rho^2 g)|_{TM} = h.$$

In fact, for each metric $\hat{h}$ conformal to h, there is a unique function $\hat{\rho}$ near $\partial\overline{X}$ satisfying $|d\hat{\rho}/\hat{\rho}|_g = 1$ and $\hat{\rho}^2 g|_{\hat{\rho}=0} = \hat{h}$, and we call $\hat{\rho}$ the *geodesic boundary defining function associated to the conformal representative* $\hat{h}$. We just recall briefly the argument of Graham [11], as it will be useful later for the cusp case: take ρ a boundary defining function of $\overline{X}$, then the structure of the hyperbolic metric on $\mathbb{H}^3$ near its boundary implies that $\bar{g} = \rho^2 g$ is smooth up to $\partial\overline{X}$ and $|d\rho/\rho|_g$ is smooth on $\overline{X}$ and equal to 1 at $\partial\overline{X}$ (that follows from the fact that g has curvature -1 in $\mathcal{U}$, see [21]). Then writing $h := (\rho^2 g)|_{TM}$ and $\hat{\rho} = \rho e^{\omega}$ the equation $|d\hat{\rho}/\hat{\rho}|_g = 1$ with the condition $\hat{\rho}^2 g|_{\partial\overline{X}} = \hat{h} = e^{2\varphi}h$ for some $\varphi \in \mathcal{C}^\infty(M)$ is equivalent to the equation

$$2\langle d\omega, d\rho\rangle_{\bar{g}} + \rho|d\omega|^2_{\bar{g}} = \frac{1 - |d\rho^2|_{\bar{g}}}{\rho}, \text{ with boundary condition } \omega|_{\partial\overline{X}} = \varphi.$$

This is a non-characteristic Hamilton-Jacobi equation with smooth coefficients which can be solved near the boundary by the method of characteristics and the

solution is unique. We then extend $\hat{\rho}$ smoothly outside this neighborhood as a positive function in any fashion. The form of the metric g in the collar neighborhood of $M = \partial\overline{X}$ induced by the gradient flow of $\hat{\rho}$ with respect to the metric $\hat{\rho}^2 g$ is of the form $g = (d\hat{\rho}^2 + \hat{h}(\hat{\rho}))/\hat{\rho}^2$ for some one-parameter smooth family $\hat{h}(\hat{\rho})$ of metrics on M parametrized by $\hat{\rho}$, and the constant sectional curvature -1 implies the form (2.4) with (2.5) (using Gauss and Codazzi constraint equations).

If ρ is a geodesic boundary defining function near $\partial\overline{X}$ associated to a conformal representative $h \in [h]$, extended smoothly as a positive function on X, then the form (2.4) of the metric in $\mathcal{U}$ implies that the Riemannian volume measure in $\mathcal{U}$ has the form $\rho^{-3}\mathrm{dvol}_g = G(\rho)d\rho\, d\mathrm{vol}_h$ for some smooth function $G \in \mathcal{C}^\infty([0,\delta))$. It is direct to see (see [1,14] for details) that

$$H(z) := \int_X \rho^z \mathrm{dvol}_g$$

has a meromorphic extension from $\{z \in \mathbb{C}; \mathrm{Re}(z) > 2\}$ to $\mathbb{C}$, with a simple pole at $z = 0$ and the value of the finite part of $H(z)$ at $z = 0$ is independent of the value of ρ in any fixed compact set $\mathcal{K} \subset X$: in fact

$$\mathrm{FP}_{z=0} H(z) = \left(\mathrm{FP}_{z=0} \int_{X\setminus\mathcal{K}} \rho^z \mathrm{dvol}_g\right) + \mathrm{Vol}_g(\mathcal{K}).$$

We define the renormalized volume of X with respect to the conformal representative $h \in [h]$ as

$$\mathrm{Vol}_R(X,h) := \mathrm{FP}_{z=0} \int_X \rho^z \mathrm{dvol}_g.$$

As a function on the set of metrics in the conformal class $[h]$ with volume $2\pi\chi(\partial\overline{X})$, the functional $\mathrm{Vol}_R(X,h)$ has a unique maximum at $h = h^{\mathrm{hyp}}$, the unique hyperbolic metric in the conformal class, see for instance [14, Proposition 3.1].

Definition 2.1. Let X be a convex co-compact hyperbolic 3-manifold with conformal boundary $(M,[h])$ a Riemann surface admitting a hyperbolic metric, *i.e.*, M does not contain genus-1 connected components. Let $h^{\mathrm{hyp}} \in [h]$ be the unique hyperbolic representative in the conformal class $[h]$, and let ρ be the geodesic boundary defining function associated to h^{hyp}, defined uniquely near M and extended smoothly as a positive function in X. The renormalized volume of X is defined to be

$$\mathrm{Vol}_R(X) := \mathrm{FP}_{z=0} \int_X \rho^z \mathrm{dvol}_g = \mathrm{Vol}_R\left(X, h^{\mathrm{hyp}}\right)$$

where g is the hyperbolic metric on X.

The choice of the conformal representative $h^{\mathrm{hyp}} \in [h]$ to be hyperbolic is important and yields particularly interesting properties of the renormalized volume related to quasi-Fuchsian reciprocity and construction of Kähler potential for the Weil-Peterson metric; see [14,18].

2.3. Uniformisation of Riemann surfaces with cusps

Definition 2.2. A *hyperbolic cusp* is a region $\{y > R\}$ of the quotient $\langle z \to z + \frac{1}{2}\rangle \backslash \mathbb{H}^2$ for some $R > 0$, where $z = y + iw$ are coordinates on the hyperbolic half-plane $\mathbb{H}^2$.

This set is also isometric to

$$\left((R,\infty)_y \times (\mathbb{R}/\tfrac{1}{2}\mathbb{Z})_w, h = \frac{dy^2+dw^2}{y^2}\right) \simeq \left(\left(0,\tfrac{1}{R}\right)_v \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right)_w, h = \frac{dv^2}{v^2} + v^2 dw^2\right).$$

A *surface with hyperbolic cusps* (M,h) is a surface isometric outside a compact set to a finite disjoint union of hyperbolic cusps.

We can compactify M into a smooth compact surface $\overline{M}$ with boundary by replacing each cusp end $\left(0, \frac{1}{R}\right)_v \times \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)_w$ by $\left[0, \frac{1}{R}\right)_v \times \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)_w$, that is, by adding circles at infinity of the cusp end.

We can also compactify M to a compact surface Σ by adding a finite number of points, one for each cusp. Define a conformal coordinate near such a point by $\zeta = \exp(4\pi(-y + iw))$. (The factor 4π is needed in order for $e^{4\pi i w}$ to be well-defined for $w \in \mathbb{R}/\frac{1}{2}\mathbb{Z}$.) We compute

$$|d\zeta|^2 = (4\pi)^2|\zeta|^2(dy^2 + dw^2) = (4\pi)^2|\zeta|^2 y^2 h. \tag{2.6}$$

Since h is conformal to $|d\zeta|^2$, we get in this way a conformal structure on Σ. If M is oriented, Σ becomes a compact Riemann surface.

If we take a boundary defining function ρ in a geometrically finite hyperbolic 3-manifold with a certain behaviour near the cusps, we see that the conformal infinity $M = \partial\overline{X}$ will have a metric with a hyperbolic cusp in the conformal class: indeed, set ρ to be a smooth boundary defining function for M in $\overline{X}$ such that

$$\rho = \frac{x}{\sqrt{x^2+y^2}} \text{ in } \mathcal{U}_j^c.$$

Then the metric $h := \rho^2 g|_{TM}$ is a smooth metric on M which is given near the cusps, that is in $\mathcal{U}_j^c \cap M \simeq \{y \in \mathbb{R};\, |y| > R_j\} \times (\mathbb{R}/\frac{1}{2}\mathbb{Z})$, by

$$h = \frac{dy^2+dw^2}{y^2}, \quad |y| > R_j,\ w \in \mathbb{R}/\tfrac{1}{2}\mathbb{Z}$$

or equivalently, using the coordinates (u, v, w) of (2.2), $\rho = u/\sqrt{u^2+v^2}$ and

$$h = \frac{dv^2}{v^2} + v^2 dw^2, \quad 0 < |v| < 1/R_j,\ w \in \mathbb{R}/\tfrac{1}{2}\mathbb{Z}.$$

We define on $\overline{M}$ the space $\dot{\mathcal{C}}^\infty(\overline{M})$ to be the Fréchet subspace of $\mathcal{C}^\infty(\overline{M})$ consisting of functions vanishing to infinite order at $\partial\overline{M}$. We also define $\mathcal{C}_r^\infty(\overline{M})$ to be the

subspace of $\mathcal{C}^\infty(\overline{M})$ consisting of functions f such that $\partial_w f \in \dot{\mathcal{C}}^\infty(\overline{M})$. This corresponds to smooth functions whose Taylor series at the boundary is of the form

$$f(x,\theta) \sim \sum_{k=0}^{\infty} a_k v^k \tag{2.7}$$

where a_k are constants, rather than functions of w. On a surface (M,h) with hyperbolic cusps, we say that a symmetric tensor $h' \in C^\infty(M; S^2(T^*M))$ is a *cusp symmetric tensor* if there exist $A \in C^\infty(\overline{M}; \mathrm{End}(T\overline{M}))$ self-adjoint with respect to h such that $h'(\cdot,\cdot) = h(A\cdot,\cdot)$.

We first claim the following uniformization theorem, see [26, Theorem 3] for a related result for Kähler-Einstein metrics.

Proposition 2.3. *Let h be a metric on a surface M with hyperbolic cusps and let $\overline{M}$ be the compactification as above. There exists a unique conformal factor $\varphi \in \mathcal{C}^\infty(M) \cap L^\infty(M)$ such that $h^{\mathrm{hyp}} = e^{2\varphi}h$ has constant curvature -1 on M. Moreover, $\varphi \in \mathcal{C}^\infty_r(\overline{M})$ and $\varphi|_{\partial\overline{M}} = 0$. More precisely, in every cusp of M,*

$$\varphi(v,w) + \log(1+av) \in \dot{\mathcal{C}}^\infty(\overline{M})$$

for some $a \in \mathbb{R}$ depending on the cusp.

Proof. The surface (M,h) is conformal to the compact Riemann surface Σ with a finite number of points removed, hence its fundamental group is non-commutative and free. The Poincaré–Koebe uniformization theorem implies that M with its induced conformal structure is conformal to a complete hyperbolic quotient. In other words, there exists a unique conformal factor $\varphi \in \mathcal{C}^\infty(M)$ such that the Riemannian metric $h^{\mathrm{hyp}} = e^{2\varphi}h$ is hyperbolic and complete. The complete hyperbolic metric on a punctured Riemann surface is known to have hyperbolic cusps near the punctures, hence there exist isometries Φ between the hyperbolic cusps of h and h^{hyp} near the punctures. Such a Φ is a holomorphic self-map of Σ defined only near the punctures, and $\Phi^* h = e^{2\varphi}h$.

Note that Φ is an isometry, hence it is proper. It follows that it extends continuously as the identity on the punctures. The possible singularities of Φ at the punctures are thus removable since the target surface Σ is compact, so in terms of the complex variable $\zeta = \exp(4\pi(-y+iw))$, we have $\Phi(\zeta) = \zeta f(\zeta)$ with $f(0) = \Phi'(0) \neq 0$. Then using (2.6)

$$h = \frac{|d\zeta|^2}{16\pi^2|\zeta|^2 y^2} = \frac{1}{|\zeta|^2 \log^2|\zeta|}|d\zeta|^2.$$

This implies

$$\Phi^* h = \frac{|\Phi'(\zeta)d\zeta|^2}{|\zeta f(\zeta)|^2 \log^2|\zeta f(\zeta)|} = \frac{|1+\zeta\frac{f'(\zeta)}{f(\zeta)}|^2}{\left(1+\frac{\log|f(\zeta)|}{\log|\zeta|}\right)^2}h.$$

In terms of the boundary-defining function $v = 1/y$,

$$\log|\zeta| = -\frac{4\pi}{v}, \quad \zeta\frac{f'(\zeta)}{f(\zeta)} \in \dot{\mathcal{C}}^\infty(\overline{M}), \quad \log|f(\zeta)| - \log|f(0)| \in \dot{\mathcal{C}}^\infty(\overline{M}).$$

Thus the conformal factor $e^{2\varphi} = \frac{\Phi^* h}{h}$ satifies $\varphi + \log(1 - \frac{\log|f(0)|}{4\pi}v) \in \dot{\mathcal{C}}^\infty(\overline{M})$ near the cusp. □

2.4. The renormalized volume of geometrically finite hyperbolic 3-manifolds

We now wish to define a renormalized volume for a geometrically finite hyperbolic 3-manifold X with rank-1 cusps. We proceed like in the convex co-compact case, by first uniformizing the conformal infinity $(M, [h])$ with the choice of the finite volume hyperbolic metric h in the conformal class and then constructing a geodesic boundary defining function ρ in $\overline{X}$ associated to h. The difficulty here is that the conformal boundary is non-compact and it is not clear what is the behavior of the function ρ near the cusp in $\overline{X}$. We proceed as in Section 2.3: we start by choosing ρ as a smooth boundary defining function near $\partial\overline{X} = M$ which is equal to $\rho = u/\sqrt{u^2+v^2}$ in the model (2.2) of each cusp neighborhood $\mathcal{U}_j^c$; the metric $h \in [h]$ obtained by $h = \rho^2 g|_{TM}$ in the conformal infinity is then hyperbolic outside a compact region of M. Then by Proposition 2.3 we know that there exists a hyperbolic metric $h^{\mathrm{hyp}} = e^{2\varphi}h$, with $\varphi \in \mathcal{C}^\infty(\overline{M})$ and $\varphi|_{\partial\overline{M}} = 0$. We obtain the following Proposition, whose proof is done in Section 6.3.

Proposition 2.4. *Let X be a geometrically finite hyperbolic 3-manifold with rank-1 cusps. Let $(M, [h])$ be the conformal infinity and h^{hyp} be the complete hyperbolic metric with cusps in the conformal class obtained from Proposition 2.3. For each $\psi \in \mathcal{C}_r^\infty(\overline{M})$, consider the conformal representative $\hat{h} := e^{2\psi}h^{\mathrm{hyp}}$. There exists a smooth boundary defining function $\hat{\rho} \in \mathcal{C}^\infty(\overline{X}_c)$ of the boundary hypersurface $\overline{M}$ in $\overline{X}_c$ and a closed set $\mathcal{V} \subset \overline{X}_c$ with finite volume, intersecting $\partial\overline{X}_c$ only at* cf, *such that*

$$\left|\frac{d\hat{\rho}}{\hat{\rho}}\right|_g = 1 \text{ in } \overline{X}_c \setminus \mathcal{V}, \quad \hat{\rho}^2 g|_{TM} = \hat{h}. \tag{2.8}$$

The function $\hat{\rho}$ is defined uniquely near $\overline{M}$. There is a smooth diffeomorphism $\phi : \overline{M} \times [0,\epsilon)_x \to \overline{X}_c \setminus \mathcal{V}$ such that $\phi^*\hat{\rho} = x$ and $\phi^* g$ admits a finite expansion in powers of x,

$$\phi^* g = \frac{dx^2 + \hat{h}_0 + x^2\hat{h}_2 + \frac{1}{4}x^4\hat{h}_4}{x^2} \tag{2.9}$$

where the coefficients $\hat{h}_0 = \hat{h}$ and $\hat{h}_2, \hat{h}_4$ are cusp symmetric tensors such that

$$\mathrm{Tr}_{\hat{h}_0}(\hat{h}_2) = -\frac{1}{2}\mathrm{Scal}_{\hat{h}_0} \quad \textit{and} \quad \delta_{\hat{h}_0}(\hat{h}_2) = \frac{1}{2}d\,\mathrm{Scal}_{\hat{h}_0}$$

and $\hat{h}_4(\cdot,\cdot) = \frac{1}{4}\hat{h}_0(A^2\cdot,\cdot)$ *if* $\hat{h}_2(\cdot,\cdot) = \hat{h}_0(A\cdot,\cdot)$. *Finally, the function* $H(z) := \int_X \hat{\rho}^z \mathrm{dvol}_g$ *admits a meromorphic extension from* $\mathrm{Re}(z) > 2$ *to a neighborhood of* $z = 0$, *with pole of order* 1 *at* $z = 0$.

A smooth boundary defining function of $\overline{M}$ in $\overline{X}_c$ is called *geodesic boundary defining function* associated to $\hat{h}$ if it satisfies (2.8). Similarly to the convex cocompact case, the value of the finite part at $z = 0$ of the integral in any compact subset $\mathcal{V} \subset \overline{X}_c \setminus \overline{M}$ is independent of the value of ρ in $\mathcal{V}$:

$$\mathrm{FP}_{z=0} H(z) = \left(\mathrm{FP}_{z=0} \int_{X\setminus\mathcal{V}} \rho^z \mathrm{dvol}_g\right) + \mathrm{Vol}_g(\mathcal{V}).$$

This is a consequence of the fact that $\mathcal{V}$ has finite volume. In other words, $\mathrm{FP}_{z=0} H(z)$ depends only on the values of ρ in arbitrarily small neighborhoods of $\overline{M}$ in $\overline{X}_c$, and thus it depends only on the conformal representative $\hat{h} \in [h]$ in the conformal infinity.

Now we can define the renormalized volume in this setting:

Definition 2.5. Let X be a geometrically finite hyperbolic 3-manifold with rank-1 cusps, and with conformal boundary $(M, [h])$ admitting a complete hyperbolic metric. Let $h^{\mathrm{hyp}} \in [h]$ be the unique hyperbolic representative in the conformal class $[h]$. For $\psi \in \mathcal{C}^\infty_r(\overline{M})$, let $\hat{\rho} \in \mathcal{C}^\infty(\overline{X}_c)$ be the geodesic boundary defining function of $\overline{M}$ associated to $\hat{h} := e^{2\psi} h^{\mathrm{hyp}}$ defined uniquely by Proposition 2.4 in a neighborhood of $\overline{M}$ in $\overline{X}_c$ and extended in any fashion as a positive smooth function in $\overline{X}_c \setminus \overline{M}$. The renormalized volume of X associated to the conformal representative $\hat{h} = e^{2\psi} h^{\mathrm{hyp}}$ is defined to be

$$\mathrm{Vol}_R(X, \hat{h}) := \mathrm{FP}_{z=0} \int_X \hat{\rho}^z \, \mathrm{dvol}_g$$

where g is the hyperbolic metric on X. We define the renormalized volume of X by

$$\mathrm{Vol}_R(X) := \mathrm{Vol}_R\left(X, h^{\mathrm{hyp}}\right).$$

3. Formation of a cusp from Schottky groups

3.1. Notation

We shall use mainly the representation of $\mathbb{H}^3$ as a half-space $\mathbb{R}^+_x \times \mathbb{C}_z$ in $\mathbb{R}^3$, the boundary then becomes $\partial\mathbb{H}^3 = \{0\} \times \mathbb{C} \simeq \mathbb{C}$.

The intersection of $\mathbb{H}^3 = \mathbb{R}^+_x \times \mathbb{C}_z$ with the Euclidean ball of radius $r > 0$ centered at a boundary point $z \in \partial\mathbb{H}^3$ is called a *half-ball* of $\mathbb{H}^3$, and we denote it by $B(z, r)$. The boundary of a half-ball $B(z, r)$ in $\mathbb{H}^3$ is called a *half-sphere* of center $z \in \mathbb{C}$ and radius $r > 0$, and is denoted by $H(z, r)$. In terms of hyperbolic

geometry, $B(z,r)$ is an unbounded domain with totally geodesic boundary given by the half-sphere $H(z,r) = \partial B(z,r) \cap \mathbb{H}^3$. We say that the ball is *supported* by the disc $D(z,r) \subset \mathbb{C}$ of center z and radius r (this corresponds to $\partial\mathbb{H}^3 \cap \overline{B(z,r)}$). Similarly we say that $H(z,r)$ is *supported* by the circle $C(z,r) = \partial D(z,r)$ in $\partial\mathbb{H}^3 \simeq \mathbb{C}$ (this corresponds to $\partial\mathbb{H}^3 \cap \overline{H(z,r)}$).

3.2. Schottky groups

We shall analyze the behaviour of $\mathrm{Vol}_R(X^\varepsilon)$ for a family $(X^\varepsilon)_{\varepsilon>0}$ of convex cocompact hyperbolic 3-manifolds such that, as $\varepsilon \to 0$, X^ε converges to a hyperbolic 3-manifold X^0 with rank-1 cusps. Here, we take $\varepsilon \geq 0$ to be a continuous parameter, but one could of course also consider sequences. The case that we consider is a smooth (in $\varepsilon > 0$) family of Schottky groups Γ^ε with certain loxodromic generators of $\mathrm{PSL}_2(\mathbb{C})$ converging to parabolic transformations of $\mathrm{PSL}_2(\mathbb{C})$.

We recall that a *marked Schottky group* $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ of genus g with *standard generators* $\gamma_1, \ldots, \gamma_g \in \mathrm{PSL}_2(\mathbb{C})$ is a group generated by these generators such that there exist $2g$ disjoint Jordan curves $(C_{\pm j})_{j=1\ldots,g}$ in $\mathbb{S}^2 = \partial\mathbb{H}^3$ bounding a connected open domain $D \subset \mathbb{S}^2$ with $\gamma_j(D) \cap D = \emptyset$ and $\gamma_j(C_{-j}) = C_{+j}$. Then Γ is free and contains only loxodromic elements, with fundamental domain $D \cup_j C_{\pm j} \subset \mathbb{S}^2$ for the action of Γ on the discontinuity set $\Omega \subset \mathbb{S}^2$ (which is connected open and dense set in $\mathbb{S}^2$). It is shown by Chuckrow [7] that every set of g free generators of a Schottky group is in fact a set of standard generators. A Schottky group is said to be a *classical Schottky group* if there is some set of free generators for which the curves $C_{\pm j}$ can be taken to be circles. A family of circles associated to the generators satisfying the conditions as above will be called a set of *adapted circles*. Such a set is of course not unique. We can view Γ as a discrete group of isometries acting freely and discontinuously on $\mathbb{H}^3$, and as a group of conformal transformations acting freely and discontinuously on the discontinuity set $\Omega \subset \mathbb{S}^2$. To define the Schottky space $\mathcal{S}_g$, we follow Chuckrow [7]: $\mathrm{PSL}_2(\mathbb{C})$ identifies with $P_3(\mathbb{C}) \setminus Y$ where $P_3(\mathbb{C})$ is the 3-dimensional complex projective space, and Y the algebraic submanifold $\{\gamma \in \mathrm{PSL}_2(\mathbb{C});\ \det\gamma = 0\}$. Consider the subset $\mathcal{U}_g$ of $(\mathrm{PSL}_2(\mathbb{C}))^g$ consisting of groups with g generators $\gamma_1, \ldots, \gamma_g$ such that there are at least 3 distincts fixed points among those of γ_j. Then $\mathcal{U}_g$ is an open and connected subset of $(P_3(\mathbb{C}) \setminus Y)^g$. There is an action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{U}_g$ by conjugation:

$$(B, (\gamma_1, \ldots, \gamma_g)) \mapsto (B\gamma_1 B^{-1}, \ldots, B\gamma_g B^{-1})$$

and $\mathcal{U}_g/\mathrm{PSL}_2(\mathbb{C})$ is a complex manifold of dimension $3g-3$ with the natural topology inherited from $(\mathrm{PSL}_2(\mathbb{C}))^g$. One way of fixing coordinates on this space is to fix 3 distincts fixed points of the generators by conjugating the group with an appropriate element of $\mathrm{PSL}_2(\mathbb{C})$. More precisely near a $\Gamma \in \mathcal{U}_g/\mathrm{PSL}_2(\mathbb{C})$, up to conjugation, we can choose the generators γ_j so that 0, 1 and ∞ are the three distinct fixed points among the generators, then there are local complex coordinates on $\mathcal{U}_g/\mathrm{PSL}_2(\mathbb{C})$ near $[\Gamma]$ given by the coefficients $a_j, b_j, c_j, d_j \in \mathbb{C}$ so

that $\gamma_j(z) = (a_j z + b_j)/(c_j z + d_j)$ with $a_j d_j - b_j c_j = 1$ (notice that 3 complex parameters among the γ_j are fixed). The Schottky space $\mathcal{S}_g$ is the subset of $\mathcal{U}_g/\mathrm{PSL}_2(\mathbb{C})$ corresponding to equivalence classes of marked Schottky groups. For a group $\Gamma \in \mathcal{S}_g$, we can always choose the three distinct fixed points to be the repulsive and attractive fixed point of γ_1 and the repulsive fixed point of γ_2, and one then gets global complex coordinates by conjugating the groups so that 0 and ∞ are the attractive and repulsive fixed point of γ_1 and 1 is the repulsive point of γ_2. This system of coordinates is not well adapted to the description of groups tending to the boundary of $\mathcal{S}_g$ with γ_1 becoming parabolic. Chuckrow [7, Lemma 5] showed that $\mathcal{S}_g$ is a connected open subset of $\mathcal{U}_g/\mathrm{PSL}_2(\mathbb{C})$. The Schottky classical space $\mathcal{S}_g^0$ is the open subset of those groups which are classical. Chuckrow [7, Theorems 4 and 5] also showed that boundary points in $\partial\mathcal{S}_g$ are free groups with g generators, without elliptic transformations, which either have a parabolic element or are not discontinuous, and both cases may happen. Marden [19] proved that groups in $\partial\mathcal{S}_g$ are discrete, that $\mathcal{S}_g \setminus \overline{\mathcal{S}_g^0}$ is a non-empty open set, and groups in $\partial\mathcal{S}_g^0$ are discontinuous. Later Jorgensen-Marden-Maskit [16] proved that all points in $\partial\mathcal{S}_g^0$ are geometrically finite Kleinian group with parabolic elements. Thus $\mathcal{S}_g^0$ is better behaved and we will only focus on this space.

3.3. Admissible Schottky groups

We consider a sequence of classical Schottky groups $\Gamma^\varepsilon \in \mathcal{S}_g^0$ where $(\gamma_1^\varepsilon, \dots, \gamma_g^\varepsilon)$ converge to $(\gamma_1^0, \dots, \gamma_g^0)$ with Γ^0 generated by these elements in $\partial\mathcal{S}_g^0 \cap \partial\mathcal{S}_g$ as $\varepsilon \to 0$, so that Γ^0 is a geometrically finite free group with g generators, with parabolic elements. We assume that γ_j^ε is smooth in $\varepsilon \in [0,1]$ for $j \leq g$. We use the coordinates on $\mathcal{S}_g$ as above, so that the fixed points of γ_1^ε are 0 and ∞, and the repulsive point of γ_2^ε is 1. For notational simplicity, we shall sometime remove the 0 superscript for the limiting objects at $\varepsilon = 0$, for instance we shall use γ_j for γ_j^0. We write these elements of $\mathrm{PSL}_2(\mathbb{C})$ as

$$\gamma_j(z) = \frac{a_j z + b_j}{c_j z + d_j}, \quad \gamma_j^\varepsilon(z) = \frac{a_j(\varepsilon) z + b_j(\varepsilon)}{c_j(\varepsilon) z + d_j(\varepsilon)},$$
$$\text{with } a_j d_j - b_j c_j = 1 \text{ and } a_j(\varepsilon) d_j(\varepsilon) - b_j(\varepsilon) c_j(\varepsilon) = 1.$$

For $j < j_0$, the fixed points of γ_j^ε are denoted $p_{\pm j}(\varepsilon)$ and given by

$$p_{\pm j}(\varepsilon) = \tfrac{a_j(\varepsilon) - d_j(\varepsilon)}{2c_j(\varepsilon)} \pm \tfrac{\sqrt{\mathrm{Tr}(\gamma_j^\varepsilon)^2 - 4}}{2c_j(\varepsilon)}. \tag{3.1}$$

(we use the determination of $\sqrt{\cdot}$ on $\mathbb{C} \setminus \mathbb{R}_-$). Up to possibly exchanging γ_j^ε by its inverse in our choice of generators, we can assume that $p_{+j}(\varepsilon)$ is the attractive, and $p_{-j}(\varepsilon)$ the repulsive fixed point. The geodesic in $\mathbb{H}^3$ relating $p_{-j}(\varepsilon)$ to $p_{+j}(\varepsilon)$ is

called the *axis* of γ_j^ε. The Euclidean distance in $\mathbb{C}$ between the two fixed points of γ_j^ε is

$$|p_{+j}(\varepsilon) - p_{-j}(\varepsilon)| = \frac{|\operatorname{Tr}(\gamma_j^\varepsilon)^2 - 4|^{\frac{1}{2}}}{|c_j(\varepsilon)|}. \tag{3.2}$$

Take a family of adapted circles $C_{\pm j}^\varepsilon$ bounding a fundamental domain D^ε. Notice that D^ε has compact closure contained in the region bounded by C_{-1}^ε and C_{+1}^ε in $\mathbb{C}$. If the circles $C_{\pm 1}^\varepsilon$ are not contained in a compact set of $\mathbb{C}$ independent of ε, then all the fixed points of a subsequence of γ_j^ε for $j > 1$ tend to ∞, and that is not possible since the limiting transformations γ_j and γ_k can not have common fixed points if $j \neq k$, according to [19, Lemma 2.3]. For the same reason, D^ε can not shrink to 0 and more generally to a point of $\mathbb{C}$. Up to extraction of a subsequence $\varepsilon_n \to 0$, the circles $C_{\pm j}^{\varepsilon_n}$ then converge to circles or points, and for $j = 1$ the limits $C_{\pm 1}$ are circles. If they are disjoint then the limiting domain D^0 is non-empty and thus, if some circle $C_{\pm j}^{\varepsilon_n}$ converge to a point p, we obtain a contradiction since γ_j would have to map D to p. This shows in that case that all $C_{\pm j}^{\varepsilon_n}$ converge to circles $C_{\pm j}$. If $C_{+1} = C_{-1}$, then since $\gamma_1^\varepsilon \to \gamma_1$, we necessarily have that γ_1 is elliptic or the Identity, but this can not happen by [7] since Γ^0 can not contain elliptic elements and must be a free group with g generators given by $\gamma_1, \dots \gamma_g$. We thus conclude that $D^{\varepsilon_n} \to D^0$ non-empty, bounded by circles $C_{\pm j}$. Necessarily, at least two of the circles $C_{\pm j}$ must intersect at a point since we assume that Γ^0 is not in $\mathcal{S}_g^0$. We will make the assumption that the limiting circles satisfy

$$C_{\pm j} \cap (C_{+k} \cup C_{-k}) = \emptyset, \text{ if } j \neq k. \tag{3.3}$$

Thus there are $g - j_0$ of the generators γ_j that are parabolic for some $0 < j_0 \leq g-1$. Without loss of generality, we choose them to be γ_j for $j = j_0 + 1, \dots, g$. For $j \leq j_0$ the γ_j are loxodromic. If $j > j_0$, we have $\operatorname{Tr}(\gamma_j) = 2$ at the limit and the unique fixed point of the parabolic transformation is $p_j = \frac{a_j - d_j}{2c_j}$, and we have that $|\operatorname{Tr}(\gamma_j^\varepsilon) - 2|^{\frac{1}{2}}/|c_j(\varepsilon)| \to 0$ as $\varepsilon \to 0$. The fundamental domain D^ε for Γ^ε is uniformly bounded in $\mathbb{C}$, and $c_j(\varepsilon) \neq 0$ for $j > 1$ in order to have adapted circles $C_{\pm j}^\varepsilon$ associated to γ_j^ε. To be adapted, the disk bounded by the circle C_{+j}^ε has to contain the point $z = a_j(\varepsilon)/c_j(\varepsilon)$ (which is the image of ∞ by γ_j^ε) and the disk bounded by the circle C_{-j}^ε has to contain $z = -d_j(\varepsilon)/c_j(\varepsilon)$ (which is mapped to ∞ by γ_j^ε); since $C_{\pm j}^\varepsilon$ also contains $p_{\pm j}(\varepsilon)$, we deduce that when $j > j_0$ and $\varepsilon \to 0$, the fact that $p_{\pm j}(\varepsilon) \to p_j$ implies that the radius of $C_{\pm j}^\varepsilon$ is bounded below by $(|\operatorname{Tr}(\gamma_j^\varepsilon)| - |\operatorname{Tr}(\gamma_j^\varepsilon) - 2|^{\frac{1}{2}})/4|c_j(\varepsilon)|$ for small $\varepsilon > 0$. In particular $c_j(\varepsilon)$ converge to $c_j \neq 0$ as otherwise the radius of the adapted circles would tend to ∞. There is a subsequence ε_n where for each j there are adapted circles $C_{\pm j}^{\varepsilon_n}$ associated to the $\gamma_j^{\varepsilon_n}$ that converge to circles $C_{\pm j}^0$ (also denoted $C_{\pm j}$), which are tangent if and only if $j > j_0$ and with $C_{+j} \cap C_{-j} = p_j$ being the fixed point of γ_j. Then the limiting

(tangent) adapted circles for $j > j_0$ have radius bounded from below by $1/2|c_j|$. Moreover, an easy computation shows that γ_j preserve the line $c_j^{-1}(a_j + \mathbb{R}) \subset \mathbb{C}$ that we call the *axis* of γ_j.

The element γ_1^ε is of the form

$$\gamma_1^\varepsilon(z) = q_1(\varepsilon)z, \quad q_1(\varepsilon) \in \mathbb{C},\ |q_1(\varepsilon)| > 1$$

and each γ_j^ε for $j > 1$ can be written as the transformation of $\mathbb{C} \simeq \mathbb{S}^2 \setminus \{\infty\}$ satisfying

$$\theta_j^\varepsilon \circ \gamma_j^\varepsilon(z) = q_j(\varepsilon)\theta_j^\varepsilon(z), \quad \theta_j^\varepsilon(z) := -\frac{z - p_{-j}(\varepsilon)}{z - p_{+j}(\varepsilon)}$$

where $p_{\pm j}(\varepsilon) \in \mathbb{C}$ are the two fixed points of γ_j^ε and $q_j(\varepsilon) \in \mathbb{C}$ is the complex multiplier with $|q_j(\varepsilon)| > 1$ (we take $p_{+j}(\varepsilon)$ to be the attractive fixed point). The multiplier will also be written as

$$q_j(\varepsilon) = e^{\ell_j(\varepsilon) + i\alpha_j(\varepsilon)} \tag{3.4}$$

for some $\ell_j(\varepsilon) > 0$ and $\alpha_j(\varepsilon) \in [0, 2\pi)$. Since for $j > j_0$, γ_j^ε converge to a parabolic element γ_j with fixed point p_j, then $q_j(\varepsilon) \to 1$ since $\mathrm{Tr}(\gamma_j^\varepsilon)^2 - 4 = (q_j(\varepsilon) - 1)^2/q_j(\varepsilon)$ must converge to 0. The axis of γ_j^ε is mapped to $\mathbb{R}^+ \times \{0\} \subset \mathbb{H}^3$ by $(\theta_j^\varepsilon)^{-1}$ in the half-space model $\mathbb{H}^3 = \mathbb{R}_x^+ \times \mathbb{C}_z$

The set D^ε is a fundamental domain for the action of Γ^ε on the discontinuity set $\Omega^\varepsilon \subset \mathbb{C}$. The group acts properly discontinuously on Ω^ε by conformal transformations and the quotient $M^\varepsilon = \Gamma^\varepsilon \backslash D^\varepsilon = \Gamma^\varepsilon \backslash \Omega^\varepsilon$ is a closed Riemann surface of genus g, with conformal structure induced by that of $\mathbb{C}$. It is the conformal boundary of the hyperbolic 3-manifold $X^\varepsilon := \Gamma^\varepsilon \backslash \mathbb{H}^3$. We denote by $F^\varepsilon \subset \mathbb{H}^3$ the fundamental domain for the group Γ^ε with totally geodesic boundary satisfying $\partial F^\varepsilon \cap \partial \mathbb{H}^3 = D^\varepsilon$; in particular $X^\varepsilon = \Gamma^\varepsilon \backslash F^\varepsilon$. Up to extraction of a subsequence, these fundamental domains converge to D^0 and F^0 (that we also denote D and F) and $X^0 = \Gamma^0 \backslash F^0$ is a geometrically finite hyperbolic manifold (that we also denote X).

We define the parameters

$$\lambda_j(\varepsilon) := \frac{|p_{+j}(\varepsilon) - p_{-j}(\varepsilon)|}{\ell_j(\varepsilon)}, \quad \nu_j(\varepsilon) := \frac{\alpha_j(\varepsilon)}{\ell_j(\varepsilon)}. \tag{3.5}$$

Since $c_j(\varepsilon)$ is smooth in $[0, 1]$ and $\mathrm{Tr}(\gamma_j^\varepsilon) = q_j(\varepsilon)^{\frac{1}{2}} + q_j(\varepsilon)^{-\frac{1}{2}}$, we see from (3.2) that if $\nu_j(\varepsilon)$ is smooth, then $\lambda_j(\varepsilon)$ extends smoothly in $\varepsilon \in [0, 1]$ and

$$\lambda_j := \lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = (1 + \nu_j^2)^{\frac{1}{2}}/|c_j|. \tag{3.6}$$

In fact, if $\nu_j(\varepsilon)$ is smooth, then by (3.1) we have that $\frac{p_{+j}(\varepsilon) - p_{-j}(\varepsilon)}{\ell_j(\varepsilon)}$ extends smoothly to $\varepsilon = 0$ with

$$\frac{p_{+j}(\varepsilon) - p_{-j}(\varepsilon)}{\ell_j(\varepsilon)} \to (1 + i\nu_j)/c_j \ \text{ as } \varepsilon \to 0. \tag{3.7}$$

Definition 3.1. For a smooth family of multipliers $\varepsilon \to q(\varepsilon) \in C^\infty([0,1]; \mathbb{C} \setminus D(0,1))$ with $q(0) = 1$ and $|q(\varepsilon)| > 1$ if $\varepsilon > 0$, we say that $q(\varepsilon)$ is *admissible* if $q(\varepsilon) = e^{\ell(\varepsilon)(1+i\nu(\varepsilon))}$ for some real valued functions $\ell(\varepsilon), \nu(\varepsilon)$ smooth in $\varepsilon \in [0,1]$. We say that a smooth family Γ^ε of classical Schottky groups of genus g is admissible if each multiplier $q_j(\varepsilon)$ of the generator γ_j^ε is either admissible or $\gamma_j = \gamma_j^0$ is loxodromic, and if there is a subsequence $\varepsilon_n \to 0$ for which there are $2g$ adapted circles $C_{\pm j}^{\varepsilon_n}$ converging to $C_{\pm j}$ such that if two of the limiting $2g$ circles $C_{\pm j}$ intersect, this can only be $C_{+j} \cap C_{-j} = \{p_j\}$ for j so that γ_j is parabolic with fixed point p_j.

3.4. Canonical circles

The adapted circles $C_{\pm j}^\varepsilon$ associated to the elements γ_j^ε can actually be taken smoothly in $\varepsilon > 0$, but they are not in general of the best form to get a local model description of the geometry with respect to $\varepsilon \to 0$. In addition it is not clear if they can be taken smoothly down to $\varepsilon = 0$, but we will show below that if the family of Schottky groups Γ^ε is admissible, then we can find a smooth family of fundamental domains down to $\varepsilon = 0$, which are bounded by pieces of circles near the punctures.

For this purpose and to obtain a nice description of the degeneration near the punctures, we define the notion of *canonical circles* for a loxodromic transformation.

If $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ is loxodromic with fixed points p_- and p_+ and multiplier $q = e^{\ell(1+i\nu)}$ with $\ell > 0$ and $\theta \circ \gamma \circ \theta^{-1}(z) = qz$ for some $\theta \in \mathrm{PSL}_2(\mathbb{C})$, the canonical circles for γ will be the circles

$$\begin{aligned} \widetilde{C}_\pm := \theta^{-1}\left(\left\{z; |z| = e^{\pm\frac{\ell}{2}}\right\}\right) = \{z \in \mathbb{C}; |z - z_\pm| = r\}, \quad \text{with} \\ z_\mp := p_\pm + \frac{p_\mp - p_\pm}{1 - e^{-\ell}}, \quad r = \frac{|p_+ - p_-|}{2\sinh(\ell/2)}. \end{aligned} \tag{3.8}$$

Lemma 3.2. *Let $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ be loxodromic with multiplier $q = e^{\ell(1+i\nu)}$ and fixed points $p_-, p_+ \in \mathbb{C}$, and let $\widetilde{C}_\mp$ be its associated canonical circles, defined by* (3.8). *Then the transformation γ maps the exterior of the disk $\widetilde{D}_-$ bounded by $\widetilde{C}_-$ to the interior of the disk $\widetilde{D}_+$ bounded by $\widetilde{C}_+$.*

Proof. Consider the two concentric circles $S_\pm := \{|z| = e^{\pm\frac{1}{2}\ell}\}$ and let $m(q)$ be the complex dilation by q in $\mathbb{C}$. Consider the transformations

$$\phi(z) = \frac{z-1}{z+1}, \quad \psi(z) = z + \frac{p_+ + p_-}{p_+ - p_-}, \quad \eta(z) = \frac{p_+ - p_-(\varepsilon)}{2} z. \tag{3.9}$$

The composition $\eta\psi\phi$ maps $\{0, \infty\}$ to $\{p_-, p_+\}$ and $\gamma = (\eta\psi\phi)m(q)(\eta\psi\phi)^{-1}$. The circles $\widetilde{C}_\pm$ are mapped by $\theta := (\eta\psi\phi)^{-1}$ to $\{|z| = e^{\pm\ell/2}\}$ and the lemma follows directly. □

We shall denote by $\widetilde{C}^{\varepsilon}_{\pm j}$ the canonical circles of γ_j^{ε}; a priori they are not adapted circles for the group. For $j > j_0$, assuming that $\nu_j(\varepsilon)$ is smooth in $\varepsilon \in [0,1]$, we see by (3.8) and (3.7) that $\widetilde{C}^{\varepsilon}_{\pm j}$ extends smoothly to $\varepsilon = 0$ with limiting circles $\widetilde{C}^{0}_{\pm j}$, tangent at p_j, with radius r_j and center $z_{\pm j}$ given by

$$z_{\pm j} = p_j \mp \frac{(1 + i\nu_j)}{c_j}, \quad r_j = \lambda_j = \frac{\sqrt{1+\nu_j^2}}{|c_j|}.$$

Note that the lines passing through the centers z_{+j} and z_{-j} intersect the axis of the parabolic transformation γ_j at an angle $\arctan(\nu_j)$.

3.5. Good fundamental domains

For Γ^{ε} a family of admissible Schottky groups we have a subsequence of fundamental domains F^{ε_n} with totally geodesic boundary and with $D^{\varepsilon_n} = \partial\mathbb{H}^3 \cap \partial F^{\varepsilon_n}$ bounded by the adapted circles $C^{\varepsilon_n}_{\pm j}$, and F^{ε_n} and D^{ε_n} are converging to F^0 and D^0, where D^0 is bounded by the limiting circles $C_{\pm j}$. From the limiting domain F^0, we shall construct new fundamental domains $\widetilde{F}^{\varepsilon}$ for Γ^{ε} for small $\varepsilon \geq 0$, called good fundamental domains and constructed by combining canonical circles with the limiting adapted circles $C_{\pm j}$. The domain $\widetilde{D}^{\varepsilon} = \partial\widetilde{F}^{\varepsilon} \cap \partial\mathbb{H}^3$ will be bounded by Jordan curves instead of circles, but their form near the parabolic points p_j will be a good model for the geometry as $\varepsilon \to 0$ near the punctures.

Notice that we can always choose $\delta_0 > 0$ and $\varepsilon_0 > 0$ small enough so that for each $j > j_0$ and $\delta \in (0, \delta_0)$, for all $0 < \varepsilon \leq \varepsilon_0$ the half-ball $B(p_j, \delta) \subset \mathbb{H}^3$ is at positive Euclidean distance from all connected components of $\partial F^0 \setminus \partial F^0 \cap \partial\mathbb{H}^3$ except those half-spheres supported by $C_{\pm j}$.

Recall that $\widetilde{C}^{\varepsilon}_{\pm j}$ are the canonical circles of γ_j^{ε}, and denote by $\widetilde{D}^{\varepsilon}_{\pm j} \subset \mathbb{C}$ the disk bounded by $\widetilde{C}^{\varepsilon}_{\pm j}$. We then show the existence of good fundamental domains:

Lemma 3.3. *There exists $\delta \in (0, \delta_0)$ such that for all $\varepsilon \in [0, \varepsilon_0]$, there exist fundamental domains $\widetilde{F}^{\varepsilon}$ for Γ^{ε} acting on $\mathbb{H}^3$ with the following properties:*

- *the boundary $\partial\widetilde{F}^{\varepsilon}$ is a smooth in $\varepsilon \in [0, \varepsilon_0]$ collection of $2g$ smooth hypersurfaces $(H^{\varepsilon}_{\pm j})_{j=1,\dots,g}$ homeomorphic to half-spheres: more precisely $H^{\varepsilon}_{\pm j}$ is the image of $H^0_{\pm j}$ by a smooth in $\varepsilon \in [0, \varepsilon_0]$ family of diffeomorphisms of $\overline{\mathbb{H}^3}$ equal to* Id *at $\varepsilon = 0$. The closures of $H^{\varepsilon}_{\pm j}$ in $\overline{\mathbb{H}^3}$ are all disjoint except when $\varepsilon = 0$ where $\overline{H^0_{-j}} \cap \overline{H^0_{+j}} = \{p_j\}$ for $j > j_0$;*
- *each γ_j^{ε} maps the exterior of the compact domain bounded by H^{ε}_{-j} in $\overline{\mathbb{H}^3}$ to the interior of the compact domain bounded by H^{ε}_{+j} in $\overline{\mathbb{H}^3}$;*
- *for each $j > j_0$, $\widetilde{F}^{\varepsilon} \cap B(p_j, \delta) = \widetilde{F}^{\varepsilon}_j \cap B(p_j, \delta)$ if $\widetilde{F}^{\varepsilon}_j$ is the fundamental domain with totally geodesic boundary for the cyclic group $\langle\gamma_j^{\varepsilon}\rangle$ and satisfying $\partial\widetilde{F}^{\varepsilon}_j \cap \partial\mathbb{H}^3 = \mathbb{C} \setminus (\widetilde{D}^{\varepsilon}_{+j} \cup \widetilde{D}^{\varepsilon}_{-j})$.*

Proof. For $j \leq j_0$, we can take $H^\varepsilon_{-j} = H^0_j$ to be the half-spheres supported by the circles C^0_{-j} and $H^\varepsilon_{+j} = \gamma^\varepsilon_j(H^0_j)$. Since $\gamma^\varepsilon_j \to \gamma_j$ and $C_{\pm j}$ are adapted circles for the limiting γ_j, clearly for small enough ε the hypersurfaces $H^\varepsilon_{\pm j}$ satisfy the desired properties. Now we deal with the more delicate part, that is when $j > j_0$ and the limiting γ_j is parabolic. We take δ small, but independent of $\varepsilon \geq 0$, and take a large $C > 0$ so that $B(p_j, C\delta)$ is at positive distance from the half-spheres supported by the limiting $C_{\pm k}$ at $\partial\mathbb{H}^3$ for $k \neq j$. Note that C can be taken large by taking δ small (for instance $C \simeq \delta^{-1/2}$ works). We start with $\varepsilon = 0$, where we will modify F^0 to $\widetilde{F}^0$ near the parabolic points p_j. We may assume that $C^0_{-j} \neq \widetilde{C}^0_{-j}$ as otherwise it suffices to take $H^\varepsilon_{\pm j}$ to be the half-spheres supported by $\widetilde{D}^\varepsilon_{\pm j}$, which satisfy the desired properties. By conjugating by $\phi : z \mapsto 1/(z - p_j)$ the parabolic element γ_j becomes a parabolic transformation fixing ∞, thus of the form $z \mapsto z + c$ for some $c \in \mathbb{C}$, which in $\mathbb{H}^3$ acts by $T_c : (x, z) \to (x, z + c)$. The half-balls $B(p_j, \delta)$ and $B(p_j, C\delta)$ are mapped by the Poincaré extension Φ of ϕ to (the interior of) $\mathbb{H}^3 \setminus B(0, \delta^{-1})$ and $\mathbb{H}^3 \setminus B(0, (C\delta)^{-1})$. The circles C^0_{-j} and $\widetilde{C}^0_{-j}$ are mapped to lines L and $\widetilde{L}$ of $\mathbb{C}$ with respective tangent vector $\tau \in \mathbb{C}$ and $\widetilde{\tau} \in \mathbb{C}$, and $\phi(C^0_{+j})$ and $\phi(\widetilde{C}^0_{-j})$ are images of these lines by $z \mapsto z + c$, that is $L + c$ and $\widetilde{L} + c$. The half-spheres of $\mathbb{H}^3$ supported by C^0_{-j} and $\widetilde{C}^0_{-j}$ are mapped to vertical planes $\mathbb{R}^+ \times L$ and $\mathbb{R}^+ \times \widetilde{L}$ by Φ, and the image of the half-spheres supported by C^0_{+j} and $\widetilde{C}^0_{+j}$ are $\mathbb{R}^+ \times (L+c)$ and $\mathbb{R}^+ \times (\widetilde{L}+c)$. Note that $S := \overline{\Phi(F^0 \cap B(p_j, C\delta))} \cap \partial\mathbb{H}^3$ and $\widetilde{S} := \overline{\Phi(\widetilde{F}^0_j \cap B(p_j, C\delta))} \cap \partial\mathbb{H}^3$ are strips in $\mathbb{C} \setminus D(0, (C\delta)^{-1})$ bounded by L and $L + c$ (resp. $\widetilde{L}$ and $\widetilde{L} + c$). For the following part of the proof, we recommend the reader to see Figure 3.1 while reading the argument.

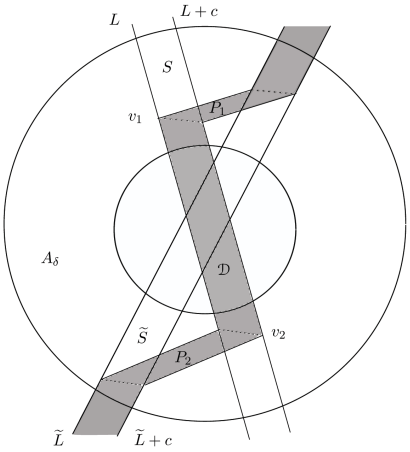

Figure 3.1. The new fundamental domain $\mathcal{D}$ in $\partial\mathbb{H}^3 \simeq \mathbb{C}$ before smoothing is given by the dark region

For $C > 0$ large and $\delta > 0$ small, consider the annulus $A_\delta := \{(C\delta)^{-1} < |z| < \delta^{-1}\}$ in $\mathbb{C}$ viewed as the boundary of the half-space $\mathbb{H}^3$. If $\delta > 0$ is chosen very small, then in A_δ the strips bounded by L and $L+c$ and the strips bounded by $\widetilde{L}$ and $\widetilde{L}+c$ are at a positive distance. We can then take two segments T_1 and T_2 in A_δ with extremities on L and $\widetilde{L}$, which are transverse to the lines with tangent vector $c \in \mathbb{C}$. Then $P_i := \cup_{t=\in[0,1]}(T_i + tc)$ for $i = 1, 2$ are two parallelograms with vertices on L, $L + c$, $\widetilde{L}$, $\widetilde{L} + c$. Then there is a unique fundamental domain $\mathcal{D} \subset \mathbb{C}$ for the translation $z \to z + c$, with boundary made of two piecewise linear curves Z and $Z+c$, with Z containing 5 segments, and such that $\mathcal{D}$ is equal to $\widetilde{S}$ outside $|z| < \delta^{-1}$, to S inside $|z| < (C\delta)^{-1}$, and contains the parallelograms P_1 and P_2. The two points of $\mathcal{D}$ at the largest distance from $\widetilde{S}$ are the vertices v_1 and v_2 of P_1 and P_2 (we choose v_1 to be the one on L), and there is a homotopy h_t (for $t \in [0, 1]$) between $\mathcal{D}$ and $\widetilde{S}$ which can be done in the obvious way by moving v_1 along L toward $v_1' := L \cap \widetilde{L}$ and v_2 along $L + c$ toward $v_2' := (\widetilde{L} + c) \cap (L + c)$ linearly in t. By choosing $C > 0$ large enough, there exists a height $x_0 \in ((C\delta)^{-1}, \delta^{-1})$ so that in the half-space $\mathbb{H}^3$, $\cup_{t\in]0,1]}(\{tx_0\} \times h_t(P_1 \cup P_2))$ is contained in $B(0, \delta^{-1}) \setminus B(0, (C\delta)^{-1})$. We thus take the fundamental domain $\mathcal{F} \subset \mathbb{R}^+_x \times \mathbb{C} = \mathbb{H}^3$ for the quotient $\langle T_c\rangle\backslash\mathbb{H}^3$ given by

$$\mathcal{F} = \Big(\bigcup_{t\in(0,1]} (\{tx_0\} \times h_t(\mathcal{D}))\Big) \cup ([x_0, +\infty) \times \widetilde{S}).$$

This has a piecewise smooth boundary, and can be smoothed out by an arbitrarily small perturbation in $B(0, \delta^{-1}) \setminus B(0, (C\delta)^{-1})$. For convenience we keep the same notation for the smoothed fundamental domain. By construction, $\Phi^{-1}(\mathcal{F}) \cap B(p_j, C\delta)$ gives the desired modification of F^0 inside $B(p_j, C\delta)$ to produce $\widetilde{F}^0$. This construction defines the hypersurfaces $H^0_{\pm j}$, which are the connected components of $\partial\widetilde{F}^0 \setminus \partial\widetilde{F}^0 \cap \partial\mathbb{H}^3$.

Next we want to use a perturbation argument to construct $H^\varepsilon_{\pm j}$ from $H^0_{\pm j}$. For each $j > j_0$, there exists a smooth family in $\varepsilon \in [0, \varepsilon_0]$ of Möbius transformations $A^\varepsilon_j \in \mathrm{PSL}_2(\mathbb{C})$ which map $\widetilde{C}^0_{-j}$ onto $\widetilde{C}^\varepsilon_{-j}$. It is just a composition of a translation and a dilation, and equals Id at $\varepsilon = 0$. Then $H^\varepsilon_{-j} := A^\varepsilon_j H^0_{-j}$ is a smooth hypersurface and define $H^\varepsilon_{+j} := \gamma^\varepsilon_j(H^\varepsilon_{-j})$; both hypersurfaces are disjoint from other $H^\varepsilon_{\pm k}$ for small ε since it is the case for $\varepsilon = 0$. The point $d_j/c_j \in \mathbb{C}$ that is mapped to ∞ by γ_j is in the disk D_{-j} bounded by C_{-j}, and since $A^\varepsilon_j \to \mathrm{Id}$ as $\varepsilon \to 0$, we see that for $\varepsilon > 0$ small enough $d_j(\varepsilon)/c_j(\varepsilon)$ is in the domain bounded by the curve $\partial H^\varepsilon_{-j} \cap \partial\mathbb{H}^3 \subset \mathbb{C}$ and thus property 2) in the lemma is satisfied for this choice of $H^\varepsilon_{\pm j}$. By construction, in $B(p_j, \delta)$ the hypersurfaces $H^\varepsilon_{\pm j}$ are given by pieces of half-spheres supported by $\widetilde{C}^\varepsilon_{\pm j}$, and outside $B(p_j, \delta)$ they are arbitrarily close to $H^0_{\pm j}$ since $A^\varepsilon_j \to \mathrm{Id}$ in $\mathcal{C}^k$-norms, thus we deduce that $H^\varepsilon_{-j} \cap H^\varepsilon_{+j} = \emptyset$ for $\varepsilon > 0$ small enough. These conditions ensure that the domain $\widetilde{F}^\varepsilon$ bounded by the hypersurfaces $H^\varepsilon_{\pm j}$ is a fundamental domain for Γ^ε satisfying all the desired properties of the lemma. □

4. Analysis of the model degeneration

In this section we shall describe more precisely the model geometry for the degeneracy to a rank-1 cusp. Let $\gamma_L \in \mathrm{PSL}_2(\mathbb{C})$ be loxodromic with multiplier $q = e^{\ell(1+i\nu)}$ and fixed points $p_- = 0$ and $p_+ = \lambda\ell$ for some $\lambda > 0$; we write $L = (\ell, \nu, \lambda)$ and we take

$$L \in \mathcal{Q} := (0, 1] \times [-N, N] \times [N^{-1}, N] \tag{4.1}$$

for some $N > 0$ fixed. Using (3.7), the set of those γ_L such that $L \in \mathcal{Q}$ has closure such that the boundary $\{\ell = 0\}$ corresponds to parabolic elements

$$\gamma_L(z) = \frac{z}{cz+1} \text{ with } c = \frac{1+i\nu}{\lambda}, \ L = (0, \nu, \lambda)$$

fixing $p_- = p_+ = 0$. We denote by $\overline{\mathcal{Q}}$ the closure of $\mathcal{Q}$, and we define the *parabolic boundary* of $\mathcal{Q}$ as the set $\{\ell = 0\}$. There is a smooth fibration

$$\begin{gathered} \Pi : \mathcal{X} \to \overline{\mathcal{Q}}, \quad \text{with fibers the manifolds} \\ \Pi^{-1}(L) = \overline{X}_L := \langle \gamma_L \rangle \backslash (\mathbb{H}^3 \cup \Omega_L) \end{gathered} \tag{4.2}$$

where $\Omega_L = \partial\overline{\mathbb{H}^3} \setminus \{0, \lambda\ell\}$ the discontinuity set of the cyclic group $\langle \gamma_L \rangle$. A *cusp region* of $\overline{X}_L$ is the image of a neighborhood $B(0, \delta)$ of $0 \in \partial\overline{\mathbb{H}^3}$ by the covering map $\pi_{\gamma_L} : (\mathbb{H}^3 \cup \Omega_L) \to \overline{X}_L$ and we say that $\cup_{(0,\lambda,\nu)\in\overline{\mathcal{Q}}} \pi_{\gamma_L}(B(0, \delta))$ is the *cusp region* of $\mathcal{X}$.

If $|q| > 1$, consider the isometry of the hyperbolic space $\mathbb{H}^3 = \mathbb{R}^+_x \times \mathbb{C}_z$

$$m(q) : (x, z) \mapsto (|q|x, qz) \tag{4.3}$$

and the quotient of $\mathbb{H}^3$ by the elementary group $\langle m(q) \rangle$ generated by $m(q)$

$$X_{m(q)} := \langle m(q) \rangle \backslash \mathbb{H}^3 \text{ with covering map } \pi_{m(q)} : \mathbb{H}^3 \to \langle m(q) \rangle \backslash \mathbb{H}^3. \tag{4.4}$$

Lemma 4.1. *For $L = (\ell, \nu, \lambda) \in \mathcal{Q}$, let $\gamma_L \in \mathrm{PSL}_2(\mathbb{C})$ be loxodromic with multiplier $q = e^{\ell(1+i\nu)}$ and fixed points $p_- = 0$ and $p_+ = \lambda\ell \in (0, \infty)$, and let $\widetilde{C}^L_\pm$ be its associated canonical circles, defined by* (3.8). *Let $\widetilde{D}^L_\pm \subset \mathbb{C}$ be the disk bounded by $\widetilde{C}^L_\pm$ and $\widetilde{F}_L \subset \mathbb{H}^3$ the fundamental domain for the cyclic group $\langle \gamma_L \rangle$ with totally geodesic boundary satisfying $\partial\widetilde{F}_L \cap \partial\mathbb{H}^3 = \mathbb{C} \setminus (\widetilde{D}^L_+ \cup \widetilde{D}^L_-)$. Let $\pi_{\gamma_L} : \mathbb{H}^3 \to \langle \gamma_L \rangle \backslash \mathbb{H}^3$ denote the covering map, then for $\delta > 0$ small and for $\lambda\ell < \delta$, the set*

$$\mathcal{U}^\delta_L := \pi_{\gamma_L}(B(0, \delta) \cap \widetilde{F}_L) \tag{4.5}$$

is isometric to

$$\pi_{m(q)}\left(\left\{(x, z) \in \mathbb{H}^3 \setminus B(e, \rho); \ e^{-\frac{1}{2}\ell} \le \sqrt{x^2 + |z|^2} \le e^{\frac{1}{2}\ell}\right\}\right) \tag{4.6}$$

where $e(L) \in \mathbb{C} \subset \partial\mathbb{H}^3$ *and* $\rho(L) > 0$ *have asymptotics for small* ℓ

$$e(L) = -1 - \tfrac{\lambda^2\ell^2}{\delta^2} + \mathcal{O}(\tfrac{\lambda^4\ell^4}{\delta^4}), \quad \rho(L) = \tfrac{\lambda\ell}{\delta} + \mathcal{O}(\tfrac{\lambda^2\ell^2}{\delta^2}). \tag{4.7}$$

The isometry from (4.5) *to* (4.6) *is given by*

$$\Theta_L(x,z) := \left(\frac{x\lambda\ell}{|z-\lambda\ell|^2 + x^2}, \frac{-x^2 - |z|^2 + \lambda\ell z}{|z-\lambda\ell|^2 + x^2} \right). \tag{4.8}$$

Proof. We use the notations of the proof of Lemma 3.2. We have a composition $\eta\psi\phi$ which maps $\{0,\infty\}$ to $\{0,\lambda\ell\}$ and $\gamma = \eta\psi\phi\, m(q)(\eta\psi\phi)^{-1}$. We define Θ to be the Poincaré extension of $\theta = (\eta\psi\phi)^{-1}$ to the half-space $\mathbb{H}^3$, thus given by (4.8). We check that the image of $B(0,\delta)$ under $(\eta\psi\phi)^{-1}$ is the complement of the half-ball $B(e,\rho)$ as claimed in the statement of the lemma: $\psi^{-1}\eta^{-1}$ maps $B(0,\delta)$ to the half-ball centered at $(x,z) = (0,-1)$ and radius $2\delta/\lambda\ell$, then ϕ^{-1} maps it to the complement of the half-ball with center and radius

$$(x,z) = \left(0, -1 - \frac{\lambda^2\ell^2}{\delta^2 - \lambda^2\ell^2}\right), \quad \rho = \frac{\delta\lambda\ell}{|\delta^2 - \lambda^2\ell^2|}$$

which proves the claim. □

The following Proposition describes the model manifold $X_{m(q)}$ with more appropriate coordinates; since the proof involves a sequence of tedious (and not very enlightning) computations, we have deferred it in the Appendix.

Proposition 4.2. *Assume that* $L = (\ell, \nu, \lambda) \in \mathcal{Q}$ *with the notation* (4.1)*; then there is an isometry* Φ_L *between the solid torus* (4.4) *and the manifold* $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \mathbb{H}^2_{\zeta=v+iu}$ *equipped with the metric*

$$g_L = \frac{du^2 + dv^2 + ((1+\nu^2)R^4 - 4\nu^2\ell^2u^2)dw^2 + 2\nu(R^2 - 2u^2)dwdv + 4\nu uvdudw}{u^2} \tag{4.9}$$

where $R := \sqrt{u^2 + v^2 + \ell^2}$. *With* $e(L)$, $\rho(L)$ *given by* (4.7)*, the neighborhood*

$$\pi_{m(q)}\left(\left\{ (x,z) \in \mathbb{H}^3 \setminus B(e,\rho);\ e^{-\frac{1}{2}\ell} \le \sqrt{x^2 + |z|^2} \le e^{\frac{1}{2}\ell} \right\} \right) \tag{4.10}$$

is mapped by Φ_L *to the set*

$$\mathcal{W}_L^\delta := \pi_w\left(\left\{ (w,\zeta) \in \left[-\frac{1}{4}, \frac{1}{4} \right) \times \mathbb{H}^2;\ |\zeta - v_L(w)| < \tau_L(w) \right\} \right) \tag{4.11}$$

where $\pi_w : \mathbb{R} \times \mathbb{H}^2 \to (\mathbb{R}/\frac{1}{2}\mathbb{Z}) \times \mathbb{H}^2$ *is the covering map, and* $\tau_L(w)$, $v_L(w)$ *are smooth functions of* $w \in \left[-\frac{1}{4}, \frac{1}{4} \right)$ *which converge uniformly as* $\ell \to 0$ *to some*

$\tau_{\lambda,\nu}(w)$ and $v_{\lambda,\nu}(w)$ satisfying $v_{\lambda,\nu}(w) = \mathcal{O}(\delta^3)$ and $\tau_{\lambda,\nu}(w) = 2\delta/\lambda + \mathcal{O}(\delta^3)$ uniformly in $|w| < 1/4$. Finally, the map

$$(L, x, z) \mapsto (L, \Phi_L \circ \Theta_L(x, z)) \in \left(\mathcal{Q} \times \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right) \times \overline{\mathbb{H}^2}\right)$$

extends smoothly to a neighborhood of the cusp region of $\mathcal{X}$ and is a diffeomorphism with image $V \setminus \{\ell = 0, \zeta = 0\}$ where V is some neighborhood of $\{\ell = 0, \zeta = 0\}$.

Notice that when $\ell \to 0$, the limiting model in Proposition 4.2 is $\left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)_w \times \mathbb{H}^2_{\zeta = v+iu}$ equipped with the metric

$$g_0 = \frac{du^2 + dv^2 + (1+\nu^2)(u^2+v^2)^2 dw^2 + 2\nu(v^2-u^2)dwdv + 4\nu uv dudw}{u^2}. \tag{4.12}$$

Writing $x := \frac{u}{u^2+v^2}$, $y := -\frac{v}{u^2+v^2}$, this becomes

$$g_0 = \frac{dx^2 + dy^2 + (1+\nu^2)dw^2 + 2\nu dwdy}{x^2}$$

and thus taking $(x', y', w') = \left(\frac{x}{\sqrt{1+\nu^2}}, \frac{y}{1+\nu^2}, w + \frac{\nu y}{1+\nu^2}\right)$ and the inverted coordinates $\left(u' = \frac{x'}{x'^2+y'^2}, v' = -\frac{y'}{x'^2+y'^2}, w'\right)$, we obtain

$$g_0 = \frac{dx'^2 + dy'^2 + dw'^2}{x'^2} = \frac{du'^2 + dv'^2 + (u'^2+v'^2)^2 dw'^2}{u'^2} \tag{4.13}$$

which is exactly the model metric of (2.2). We can then write this change of variable

$$u' = (1+\nu^2)^{3/2} u \left(1 - \frac{\nu^2 u^2}{u^2(1+\nu^2)+v^2}\right), \quad v' = \left(1+\nu^2\right) v \left(1 - \frac{\nu^2 u^2}{u^2(1+\nu^2)+v^2}\right)$$
$$w' = w - \frac{\nu}{1+\nu^2}\frac{v}{v^2+u^2} \tag{4.14}$$

and if we take the fundamental domain $\left[-\frac{1}{4}, \frac{1}{4}\right]_w \times \mathbb{H}^2_{iu+v}$ for $\left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)_w \times \mathbb{H}^2$, we see that the corresponding fundamental domain in the coordinates (u', v', w') for the action $w' \mapsto w' + \frac{1}{2}$ becomes

$$\mathcal{D} := \left\{(w', iu' + v') \in \mathbb{R} \times \mathbb{H}^2;\, w' + \frac{\nu}{1+\nu^2}\frac{v}{v^2+u^2} \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right\}. \tag{4.15}$$

This explicit isometry will be used later since it is sometimes more convenient to work in the model (4.13) than in the model (4.12).

The function $U := \frac{u}{R}$ in $\mathcal{W}^\delta_L$ defines the boundary corresponding to $\partial\overline{X}_L$ via $\Phi_L \circ \Theta_L$. We will see later that, near the cusp, this function is a boundary defining function on a space that compactifies $\mathcal{X}$ as a manifold with corners. This function will essentially give the form of the equidistant foliation near the pinched geodesic.

Lemma 4.3. *Let $U := \frac{u}{R}$ be the chosen boundary defining function in $\mathcal{W}_L^\delta$; then the metric $h_L := (1+\nu^2)(U^2 g_L)|_{U=0}$ in the conformal infinity induced by the defining function $U\sqrt{1+\nu^2}$ is given by*

$$h_L := (1+\nu^2)\left(\frac{dv^2}{v^2+\ell^2} + (v^2+\ell^2)(1+\nu^2)dw^2 + 2\nu dv dw\right)$$

and has constant Gaussian curvature -1.

Proof. First we notice that $h_L^\flat := \frac{\ell^2}{(1+\nu^2)(v^2+\ell^2)} h_L$ is flat, since it is given by

$$h_L^\flat = d\theta^2 + \ell^2(1+\nu^2)dw^2 + 2\ell\nu d\theta dw$$

with $\theta := \arctan(v/\ell)$, and thus the Gaussian curvature of h_L is given by

$$\frac{1}{2(1+\nu^2)}\frac{\ell^2}{v^2+\ell^2}\Delta_{h_L^\flat}\left(\log\left(\frac{v^2+\ell^2}{\ell^2}\right)\right) = (\cos\theta)^2\partial_\theta^2(\log\cos(\theta)) = -1$$

which finishes the proof. □

5. Formation of a cusp on surfaces

In this section we discuss the uniformisation on a Riemann surface when there is a degeneration to a surface with cusps.

We start by setting the assumptions. Let N be a compact surface of genus $g \geq 2$ and h_ε a family of smooth metrics on N for $\varepsilon > 0$. Assume that there is a finite family of disjoint smooth embedded circles $(H_j)_{j=1,\dots,j_1}$ on N (for some $j_1 \in \mathbb{N}$) which satisfies the following properties: there exist $A > a > 0$ and some connected open neighborhoods $\mathcal{Z}_j^\varepsilon \subset \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right) \times (-A, A)$ of $\left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right) \times \{0\}$ and some neighborhood $\mathcal{Y}_j^\varepsilon$ of H_j in N such that $\left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right) \times (-a, a) \subset \mathcal{Z}_j^\varepsilon$, and there exist some smooth diffeomorphisms

$$\psi_j^\varepsilon : \mathcal{Z}_j^\varepsilon \to \mathcal{Y}_j^\varepsilon$$

and some parameters $\nu_j(\varepsilon)$, $\ell_j(\varepsilon)$ converging to $\nu_j \in \mathbb{R}$ and 0 as $\varepsilon \to 0$, such that

$$\begin{aligned}&\psi_j^{\varepsilon *} h_\varepsilon \\ =&(1+\nu_j(\varepsilon)^2)\left(\frac{dv^2}{v^2+\ell_j(\varepsilon)^2} + (v^2+\ell_j(\varepsilon)^2)(1+\nu_j(\varepsilon)^2)dw^2 + 2\nu_j(\varepsilon)dvdw\right),\end{aligned} \tag{5.1}$$

where $w \in \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)$ is an angle variable and v is the coordinate obtained by projecting on the second factor. Moreover, we ask that ψ_j^ε is converging in $\mathcal{C}^k$-norms for

all $k \in \mathbb{N}_0$ to some smooth diffeomorphisms $\psi_j^0 : \mathcal{Z}_j^0 \setminus ((\mathbb{R}/\frac{1}{2}\mathbb{Z}) \times \{0\}) \to \mathcal{Y}_j^0 \setminus H_j$ where $\mathcal{Z}_j^0 = \mathrm{Int}(\cap_{\varepsilon>0} \mathcal{Z}_j^\varepsilon)$ and $\mathcal{Y}_j^0 = \mathrm{Int}(\cap_{\varepsilon>0} \mathcal{Y}_j^\varepsilon)$. We finally assume that the metric h_ε converges in $\mathcal{C}^k$-norms on compact sets of $M := N \setminus H$ for all $k \in \mathbb{N}_0$ to a smooth metric h_0 defined on M where $H := \bigcup_{j=1}^{j_0} H_j$. Thus, for $\varepsilon > 0$, the metric h_ε is smooth on N, while for $\varepsilon = 0$, h_0 is a complete metric on M of finite volume with cusp ends.

Notice that near H_j the metric (5.1) can be rewritten under the more standard form

$$h_\varepsilon = \frac{dv^2}{v^2 + \ell_j(\varepsilon)^2} + (v^2 + \ell_j(\varepsilon)^2) dw'^2,$$
$$\text{with } w' := w(1 + \nu_j(\varepsilon)^2) - \frac{\nu_j(\varepsilon)}{\ell_j(\varepsilon)} \arccos\left(\frac{v}{\sqrt{v^2 + \ell_j(\varepsilon)^2}}\right) \tag{5.2}$$

which shows that $H_j = \{v = 0\}$ is a closed geodesic of length $\frac{1}{2}\ell_j(\varepsilon)(1+\nu_j(\varepsilon)^2)$ in this neighborhood. Since $\ell^{-1} \arccos(v/\sqrt{v^2+\ell^2}) = -\int_v^\infty 1/(t^2+\ell^2)dt$, we see that for the limiting case $\ell_j(\varepsilon) = 0$, the change of coordinates above is only well defined (and smooth) in the region $\{v > 0\}$. By changing $\arccos(v/\sqrt{v^2 + \ell_j(\varepsilon)^2})$ to $\arccos(-v/\sqrt{v^2 + \ell_j(\varepsilon)^2})$, we get a smooth change of coordinates at $\varepsilon = 0$ in $\{v < 0\}$. We use the model (5.1) instead of (5.2) since it is more suitable to our 3-dimensional model of Proposition 4.2 for the rank-1 cusp formation.

We can compactify smoothly M into $\overline{M}$ by using ψ_j^0: it suffices to compactify the charts $\mathcal{W}_j^0 \setminus \left(\left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right) \times \{0\}\right)$ made of two disjoint connected components $\{v > 0\}$ and $\{v < 0\}$ by attaching a circle at $v = 0$ on each connected component and defining the smooth structure by saying that v and w are smooth functions. The obtained surface is a smooth surface with $2j_1$ boundary components and with interior given by M. It is important to notice that the isometry between (5.1) and (5.2) at $\ell_j = 0$ (*i.e.*, $\varepsilon = 0$) is not smooth at $v = 0$ since $F_0(v) = -1/v$, thus the smooth compactification we take for M using ψ_j^0 is not the same as the one used in the beginning of Section 2.3, which corresponds rather to compactifying by using the coordinates (w', v) having put metric under the form (5.2).

By the uniformization theorem, we can find for each $\varepsilon > 0$ a unique function $\varphi_\varepsilon \in \mathcal{C}^\infty(N)$ such that the conformal metric

$$h_\varepsilon^{\mathrm{hyp}} = e^{2\varphi_\varepsilon} h_\varepsilon$$

is hyperbolic. Similarly, for $\varepsilon = 0$, Proposition 2.3 insures that we can find $\varphi_0 \in \mathcal{C}^\infty(M)$ such that $h_0^{\mathrm{hyp}} = e^{2\varphi_0} h_0$ is a complete hyperbolic metric of finite volume with cusps on M. In fact, if (w', v) are the coordinates above putting the metric under the form (5.2), Proposition 2.3 shows that $\varphi_0(w', v)$ admits a smooth extension from each connected component of $\{v \neq 0\}$ to both $\{v \geq 0\}$

and $\{v \leq 0\}$ (it is smooth from each side but not globally on an open interval containing 0) with $\varphi_0|_{v=0} = 0$ and $\partial_{w'}\varphi_0 = \mathcal{O}(|v|^\infty)$ near $v = 0$. Viewing now φ_0 as a function of (w, v), we get $\varphi_0(w', v) = \varphi(w(1+\nu^2) - \frac{\nu}{v}, v)$ in $v > 0$ and $\varphi_0(w', v) = \varphi(w(1+\nu^2) + \frac{\nu}{v}, v)$ in $v < 0$, and we easily see that φ_0 admits a smooth extension to $\overline{M}$ such that $\varphi_0|_{\partial\overline{M}} = 0$ and $\partial_w\varphi_0$ vanishes to infinite order at $\partial\overline{M} = \{v = 0\}$.

Proposition 5.1. *Under the assumptions above, we have, as $\varepsilon \to 0$,*

$$\|\varphi_\varepsilon - \varphi_0\|_{\mathcal{C}^0} \to 0.$$

Proof. Let $\widetilde{\varphi}$ be a continuous function on $N \times [0, \varepsilon_0)_\varepsilon$ whose restriction to $\varepsilon = 0$ is given by φ_0 and such that $\widetilde{\varphi}$ is smooth on $(N \times [0, \varepsilon_0)) \setminus (H \times \{0\})$. Moreover, we ask that

$$\partial_w\widetilde{\varphi} = \mathcal{O}((\ell_j(\varepsilon) + |v|)^\infty) \tag{5.3}$$

near $H_j \times \{0\}$; for instance this can be achieved by writing $\varphi_0 = \varphi_{0,1} + \varphi_{0,2}$ with $\mathrm{supp}(\varphi_{0,1}) \cap H = \emptyset$, and $\mathrm{supp}(\varphi_{0,2}) \subset \cup_j \mathcal{Y}_j^\varepsilon$ (where $\mathcal{Y}_j^\varepsilon$ is the collar neighborhood with coordinates v, w as above) and then taking $\widetilde{\varphi} = \widetilde{\varphi}_1 + \widetilde{\varphi}_2$ where $\widetilde{\varphi}_2$ is supported in $\cup_j \mathcal{Y}_j^\varepsilon$ and given in $\mathcal{Y}_j^\varepsilon$ by

$$\widetilde{\varphi}_2(v, w, \varepsilon) = \frac{1}{\ell_j(\varepsilon)} \int \chi\left(\frac{v - v'}{\ell_j(\varepsilon)}\right) \varphi_{0,2}(v', w) dv' \tag{5.4}$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfies $\int \chi = 1$, $\chi \geq 0$ and $\chi(0) = 1$. Using that, near H_j, $\partial_w\varphi_{0,2} = \partial_w\varphi_0 = \mathcal{O}(|v|^\infty)$, we obtain the claim. Consider the new family of metrics

$$\widetilde{h}_\varepsilon = e^{2\widetilde{\varphi}} h_\varepsilon, \quad \varepsilon \in [0, \varepsilon_0),$$

and set $\widetilde{\varphi}_\varepsilon := \varphi_\varepsilon - \widetilde{\varphi}(\cdot, \varepsilon)$ so that $h_\varepsilon^{\mathrm{hyp}} = e^{2\varphi_\varepsilon} h_\varepsilon = e^{2\widetilde{\varphi}_\varepsilon}\widetilde{h}_\varepsilon$. Notice that $\widetilde{\varphi}_0 = 0$ and that $R_{\widetilde{h}_0} = -2$ where R denotes the scalar curvature. Thus, outside any fixed open set containing H, we will have that $R_{\widetilde{h}_\varepsilon} = -2 + o(1)$ as $\varepsilon \to 0$, by the fact that $h_\varepsilon \to h_0$ on M in $\mathcal{C}^k$-norms on compact sets of M. On the other hand, near H, $R_{h_\varepsilon} = -2$ by Lemma 4.3, so by the formula for the scalar curvature under conformal changes of metrics, we have that

$$R_{\widetilde{h}_\varepsilon} = e^{-2\widetilde{\varphi}(\cdot,\varepsilon)}(-2 + 2\Delta_{h_\varepsilon}\widetilde{\varphi}(\cdot, \varepsilon)) \quad \text{near } H.$$

The Laplacian Δ_{h_ε} near H_j is given by

$$\Delta_{h_\varepsilon} = -\partial_v(v^2 + \ell_j(\varepsilon)^2)\partial_v - \frac{(1 + \nu_j(\varepsilon)^2)^{-1}}{v^2 + \ell_j(\varepsilon)^2}\partial_w^2 + 2\frac{\nu_j(\varepsilon)}{1 + \nu_j(\varepsilon)^2}\partial_v\partial_w,$$

therefore using (5.3), (5.4) and the fact that $\varphi_0 \in \mathcal{C}^\infty(\overline{M})$, we deduce that $R_{\widetilde{h}_\varepsilon}$ converges uniformly to $R_{\widetilde{h}_0}$ near H as $\varepsilon \to 0$ and thus $R_{\widetilde{h}_\varepsilon} = -2 + o(1)$ uniformly. In particular, for ε sufficiently small, $R_{\widetilde{h}_\varepsilon}$ will be negative.

Now, again by the formula for the curvature under conformal changes of metrics, we have that

$$-2 = e^{-2\widetilde{\varphi}_\varepsilon}(R_{\widetilde{h}_\varepsilon} + 2\Delta_{\widetilde{h}_\varepsilon}\widetilde{\varphi}_\varepsilon). \tag{5.5}$$

Thus, for $\varepsilon > 0$ sufficiently small so that $R_{\widetilde{h}_\varepsilon}$ is negative, we see that if $\widetilde{\varphi}_\varepsilon$ attains its maximum at p, then

$$\begin{aligned} -2 \geq e^{-2\varphi_\varepsilon(p)} R_{\widetilde{h}_\varepsilon(p)} &\implies e^{2\widetilde{\varphi}_\varepsilon(p)} \leq \frac{R_{\widetilde{h}_\varepsilon}(p)}{-2} \\ &\implies \widetilde{\varphi}_\varepsilon(p) \leq \frac{1}{2}\log\left(\frac{R_{\widetilde{h}_\varepsilon}(p)}{-2}\right) = o(1). \end{aligned}$$

Similarly, if $\widetilde{\varphi}_\varepsilon$ attains its minimum at q, then

$$\widetilde{\varphi}_\varepsilon(q) \geq \frac{1}{2}\log\left(\frac{R_{\widetilde{h}_\varepsilon}(q)}{-2}\right) = o(1).$$

Consequently, $\widetilde{\varphi}_\varepsilon \to 0$ uniformly on M as $\varepsilon \to 0$. Since $\varphi_\varepsilon - \varphi_0 = \widetilde{\varphi}_\varepsilon + \widetilde{\varphi}(\cdot, \varepsilon) - \varphi_0$ and $||\widetilde{\varphi}(\cdot, \varepsilon) - \varphi_0||_{L^\infty} = o(1)$ as $\varepsilon \to 0$, the result follows. □

Remark 5.2. A recent result of Melrose-Zhu [24] shows that in fact φ_ϵ admits a polyhomogeneous expansion on the manifold with corners obtained from $N \times [0, 1)_\varepsilon$ by blowing up $H \times \{0\}$.

The following corollary will be useful to deal with the limit of the renormalized volume under the formation of a rank-1 cusp.

Corollary 5.3. *Let $I_\varepsilon \subset [-1, 1]$ with size $|I_\varepsilon| \to 0$, and let $e^{2\varphi_\varepsilon}$ be the uniformisation factor for h_ε on M so that $h_\varepsilon^{\text{hyp}} = e^{2\varphi_\varepsilon} h_\varepsilon$ is hyperbolic. We have in each collar neighborhood $\mathcal{C}_j$ of H_j,*

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\mathbb{R}/\frac{1}{2}\mathbb{Z}} \int_{I_\varepsilon} |d\varphi_\varepsilon|^2_{h_\varepsilon} dv dw = 0, \\ \lim_{\varepsilon\to 0} \int_M |d\varphi_\varepsilon|^2_{h_\varepsilon} \mathrm{dvol}_{h_\varepsilon} = \int_M |d\varphi_0|^2_{h_0} \mathrm{dvol}_{h_0}. \end{aligned} \tag{5.6}$$

Proof. Since $\varphi_0 \in \mathcal{C}^\infty(\overline{M})$ with $\partial_w \varphi_0 \in \dot{\mathcal{C}}^\infty(\overline{M})$, we see from the form of the metric h_ε in (5.1) that $|d\varphi_0|^2_{h_\varepsilon} \in L^\infty$ with uniform bound with respect to ε, and so

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}/\frac{1}{2}\mathbb{Z}} \int_{I_\varepsilon} |d\varphi_0|^2_{h_\varepsilon} dv dw = 0. \tag{5.7}$$

On the other hand, we know that

$$2\Delta_{h_\varepsilon}\varphi_\varepsilon = -2e^{2\varphi_\varepsilon} - R_{h_\varepsilon}, \quad \text{for } \varepsilon \in [0, \varepsilon_0). \tag{5.8}$$

By the previous proposition, we therefore have that $\|\Delta_{h_\varepsilon}\varphi_\varepsilon - \Delta_{h_0}\varphi_0\|_{\mathcal{C}^0(M)} = o(1)$. Moreover, the form of the metric (5.1) and the fact that $\varphi_0 \in \mathcal{C}^\infty(\overline{M})$ and $\partial_w\varphi_0 \in \dot{\mathcal{C}}^\infty(\overline{M})$ imply that $\|\Delta_{h_\varepsilon}\varphi_0 - \Delta_{h_0}\varphi_0\|_{\mathcal{C}^0(M)} = o(1)$. Now we combine these facts and use integration by parts to show that

$$\int_M |d(\varphi_\varepsilon - \varphi_0)|^2_{h_\varepsilon}\mathrm{dvol}_{h_\varepsilon} = \int_M (\varphi_\varepsilon\Delta_{h_\varepsilon}\varphi_\varepsilon + \varphi_0\Delta_{h_\varepsilon}\varphi_0 - 2\varphi_0\Delta_{h_\varepsilon}\varphi_\varepsilon)\mathrm{dvol}_{h_\varepsilon} = o(1). \tag{5.9}$$

The boundary terms at H are 0 by the properties of h_ε and φ_ε. In particular, as $\varepsilon \to 0$

$$\int_{\mathbb{R}/\frac{1}{2}\mathbb{Z}}\int_{I_\varepsilon} |d(\varphi_\varepsilon - \varphi_0)|^2_{h_\varepsilon} dv dw = o(1). \tag{5.10}$$

The first result in the Corollary then follows by combining (5.7) and (5.10) and using the triangle inequality. The second result follows from (5.9) and the fact that

$$\lim_{\varepsilon\to 0}\int_M |d\varphi_0|^2_{h_\varepsilon}\mathrm{dvol}_{h_\varepsilon} = \int_M |d\varphi_0|^2_{h_0}\mathrm{dvol}_{h_0}. \tag{5.11}$$

This ends the proof. □

6. The boundary defining function used to define the renormalized volume

In this section we analyze the geodesic boundary defining function corresponding to the hyperbolic representative in the conformal infinity when we have a family of convex co-compact hyperbolic 3-manifold converging to a geometrically finite hyperbolic 3-manifold with rank-1 cusps.

6.1. Geometric assumptions on the family of metrics

We fix a compact manifold with boundary $\overline{\mathbf{X}}$ and a family of hyperbolic convex co-compact metrics g_ε, with $\varepsilon > 0$, on the interior X of $\overline{\mathbf{X}}$.

Definition 6.1. We say that *the family g_ε is an admissible degeneration of convex co-compact hyperbolic metrics on X* if g_ε are convex co-compact hyperbolic metrics satisfying the following properties (below, $\mathbb{H}^2$ denotes the open upper half-plane in $\mathbb{C}$ and $\overline{\mathbb{H}^2}$ the closed upper half-plane; we use the topology of $\mathbb{C}$ to define bounded sets in $\overline{\mathbb{H}^2}$):

Assumption 1 (Model near the cusp). There exists a family of j_1 disjoint simples curves $H_1, \dots, H_{j_1}$ in $\partial\overline{\mathbf{X}}$, and disjoint open neighborhoods $\mathcal{U}_j^\varepsilon \subset \overline{\mathbf{X}}$ of H_j, there are diffeomorphisms $\Psi_j^\varepsilon : \mathcal{W}_j^\varepsilon \to \mathcal{U}_j^\varepsilon$ where $\mathcal{W}_j^\varepsilon \subset (\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \overline{\mathbb{H}^2_\zeta}$ are bounded

open sets containing $(\mathbb{R}/\frac{1}{2}\mathbb{Z}) \times \{\zeta \in \overline{\mathbb{H}^2}; |\zeta| < r_j\}$ for some $r_j > 0$, and for $\zeta = v + iu$

$$\Psi_j^{\varepsilon *} g_\varepsilon = \frac{du^2 + dv^2 + ((1+\nu_j(\varepsilon)^2)R^4 - 4\ell_j(\varepsilon)^2\nu_j(\varepsilon)^2u^2)dw^2}{u^2} + \frac{2\nu_j(\varepsilon)(R^2 - 2u^2)dwdv + 4\nu_j(\varepsilon)uvdudw}{u^2} \tag{6.1}$$

for some $\ell_j(\varepsilon) \to 0$ and $\nu_j(\varepsilon) \to \nu_j \in \mathbb{R}$ as $\varepsilon \to 0$, with $R := \sqrt{u^2 + v^2 + \ell_j(\varepsilon)^2}$.

Assumption 2 (Convergence outside the cusp). There exists a hyperbolic metric g_0 on X such that for any fixed boundary defining function $\rho \in \mathcal{C}^\infty(\overline{\mathbf{X}})$, $\rho^2 g_\varepsilon \to \rho^2 g_0$ in all $\mathcal{C}^k$-norms on compact sets of $\overline{\mathbf{X}} \setminus \cup_j H_j$ as $\varepsilon \to 0$. If $\mathcal{W}_j^0 := \mathrm{Int}(\cap_{\varepsilon>0}\mathcal{W}_j^\varepsilon) \subset (\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \overline{\mathbb{H}^2}$ and $\mathcal{U}_j^0 := \mathrm{Int}(\cap_{\varepsilon>0}\mathcal{U}_j^\varepsilon) \subset \overline{\mathbf{X}}$, then Ψ_j^ε converge to a smooth diffeomorphism $\Psi_j^0 : \mathcal{W}_j^0 \setminus (\mathbb{R}/\frac{1}{2}\mathbb{Z}) \times \{0\} \to \mathcal{U}_j^0 \setminus H_j$ in all $\mathcal{C}^k$-norms.

Under these assumptions, the metric g_0 has rank-1 cusps. This follows from the convergence of $\Psi_j^\varepsilon, \mathcal{U}_j^\varepsilon, \mathcal{W}_j^\varepsilon$ and the fact that (6.1) has a limiting metric as $\varepsilon \to 0$ which is isometric to a neighborhood (2.2) of a rank-1 cusp. The *degenerating curve* $H \subset \partial\overline{X}$ is the submanifold given by $H := \cup_{j=1}^{j_1} H_j$.

Proposition 6.2. *Let $\Gamma^\varepsilon \subset \mathrm{PSL}_2(\mathbb{C})$ be an admissible family of classical Schottky groups of genus* $\mathbf{g}$ *in the sense of Definition* 3.1. *Then for each $\varepsilon > 0$, $X^\varepsilon := \Gamma^\varepsilon \backslash \mathbb{H}^3$ is isometric to (X, g_ε) where X is the interior of a solid torus of genus* $\mathbf{g}$ *and g_ε is an admissible degeneration of convex co-compact hyperbolic metrics on X in the sense of Definition 6.1.*

Proof. We can write the hyperbolic manifold as $X^\varepsilon = \Gamma^\varepsilon \backslash \widetilde{F}^\varepsilon$ where $\widetilde{F}^\varepsilon$ are good fundamental domains constructed in Lemma 3.3. The metric on X^ε is the hyperbolic metric $g_{\mathbb{H}^3}$ on $\widetilde{F}^\varepsilon$, which descends smoothly to the quotient by Γ^ε. In fact, we can also consider the closure $\overline{X^\varepsilon}$ obtained from the action of Γ^ε on the closure of $\widetilde{F}^\varepsilon$ in $\mathbb{H}^3 \cup \Omega^\varepsilon$ where $\Omega^\varepsilon \subset \mathbb{S}^2$ is the set of discontinuity of Γ^ε. These can be put together into a smooth fibration

$$\Pi : \mathcal{X} \to [0, 1] \tag{6.2}$$

such that $\Pi^{-1}(\varepsilon) = \overline{X^\varepsilon}$ has interior equipped with the complete hyperbolic metric $\hat{g}_\varepsilon$ induced from $g_{\mathbb{H}^3}$. For $\varepsilon > 0$, X^ε is naturally the interior of a solid torus $\overline{X^\varepsilon}$ of genus $\mathbf{g}$, while when $\varepsilon = 0$, there are cusps of rank 1. So as we have seen in Section 2, the conformal compactification is no longer a solid torus, it is a solid torus with a circle removed for each rank-1 cusp. In fact, by Lemma 4.1 and Proposition 4.2, for each cusp point p_j, we have an isometry $\Psi_j^\varepsilon := (\Phi_j^\varepsilon \circ \Theta_j^\varepsilon)^{-1}$ from a neighborhood of $\zeta = 0$ in $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \mathbb{H}^2_\zeta$ to a neighborhood of p_j in $\widetilde{F}^\varepsilon$, where $\Theta_j^\varepsilon = \Theta_{L(\gamma_j^\varepsilon)}$ is given by (4.8) and $\Phi_j^\varepsilon = \Phi_{L(\gamma_j^\varepsilon)}$ is given by Proposition 4.2 with

$L(\gamma_j^\varepsilon) = (\ell_j(\varepsilon), \nu_j(\varepsilon), \lambda_j(\varepsilon))$ smooth in $\varepsilon \in [0,1]$ (in Section 4 we take the fixed points $p_{-j}^\varepsilon = 0$ and $p_{+j}^\varepsilon = \lambda_j(\varepsilon)\ell_j(\varepsilon)$ but we can always reduce to this case by composing with a smooth family of translations and rotations). Moreover, these combine to give a smooth diffeomorphism $\Psi_j : \mathcal{W}_j \to \mathcal{U}_j$ from a neighborhood $\mathcal{W}_j$ of $\{\zeta = 0, \varepsilon = 0\}$ in $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \overline{\mathbb{H}^2_\zeta} \times [0,1]_\varepsilon \setminus \{\zeta = \varepsilon = 0\}$ into a neighborhood $\mathcal{U}_j \subset \mathcal{X}$ of the cusp point p_j in $\Pi^{-1}(0) \subset \mathcal{X}$. This follows from the last statement of Proposition 4.2.

The diffeomorphisms Ψ_j give us a natural way to compactify uniformly down to $\varepsilon = 0$ by simply replacing $\mathcal{W}_j$ by its closure $\overline{\mathcal{W}}_j$ in $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \overline{\mathbb{H}^2_\zeta} \times [0,1]_\varepsilon$. Indeed, we can consider a compactification

$$\overline{\Pi} : \overline{\mathcal{X}} \to [0,1] \tag{6.3}$$

of (6.2) such that $\overline{\Pi}^{-1}(\varepsilon) = \overline{X^\varepsilon}$ for $\varepsilon > 0$ and $\overline{\Psi}_j : \overline{\mathcal{W}}_j \to \overline{\mathcal{U}}_j$, which restrict to Ψ_j on $\mathcal{W}_j$, is a diffeomorphism from $\overline{\mathcal{W}}_j$ to a neighborhood $\overline{\mathcal{U}}_j$ of the circle $H_j \subset \overline{\Pi}^{-1}(0)$ corresponding to the cusp point p_j. Here, $\overline{\mathcal{X}}$ is now a compact manifold with corners and $\overline{\Pi}$ is a surjective submersion. Moreover, the fibres of $\overline{\Pi}$ are manifolds with boundary, more precisely solid tori of genus $\mathbf{g}$. Choosing a horizontal connection for (6.3), we can then use parallel transport to obtain a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{G} & \overline{X} \times [0,1] \\ & \searrow^{\overline{\Pi}} & \downarrow \mathrm{pr}_2 \\ & & [0,1] \end{array} \tag{6.4}$$

where $\mathrm{pr}_2 : \overline{X} \times [0,1] \to [0,1]$ is the projection on the second factor, $\overline{X}$ is a fixed manifold with boundary and G is a diffeomorphism of manifolds with corners. In the statement of Proposition 6.2, it suffices then to take X to be the interior of $\overline{X}$ with family of metrics $g_\varepsilon = G_* \hat{g}_\varepsilon$ on the slices $\mathrm{pr}_2^{-1}(\varepsilon)$ where $\hat{g}_\varepsilon$ is the induced family of hyperbolic metrics on the fibres of $\Pi : \mathcal{X} \to [0,1]$. The family of diffeomorphisms associated to each cusp point p_j in Definition 6.1 can then be taken to be $G \circ \Psi_j(\cdot, \varepsilon)$ for $\varepsilon \geq 0$. □

6.2. The Hamilton-Jacobi equation outside the cusps

We consider an admissible degenerating family of convex co-compact metrics g_ε on $X = \mathrm{Int}(\overline{\mathbf{X}})$ in the sense of Definition 6.1 and we keep the notations of Section 6.1. The manifold (X, g_0) is geometrically finite hyperbolic with cusps of rank-1 and $\overline{X} = \overline{\mathbf{X}} \setminus H$ where H is the degenerating curve in the boundary of $\overline{\mathbf{X}}$. Recall that $\mathcal{K}$ is a compact subset of $\partial\overline{X}$ where Assumption 2 is satisfied in Definition 6.1. Let h_0^{hyp} be the uniformizing metric on the conformal boundary $M = \partial\overline{\mathbf{X}} \setminus H = \partial\overline{X}$, given by Proposition 2.3; it is a complete hyperbolic metric with finite volume. We

define ρ_0 the geodesic boundary defining function of M in $\mathcal{K}$ near $\partial\overline{X}$ to be the solution of the Hamilton-Jacobi equation

$$\left|\frac{d\rho_0}{\rho_0}\right|^2_{g_0} = 1, \quad (\rho_0^2 g_0)|_{TM} = h_0^{\mathrm{hyp}}.$$

This equation is non-characteristic at $M \cap \mathcal{K}$ and has a unique solution near $M \cap \mathcal{K}$, just as in the convex co-compact case (see the discussion of Section 2.2). We first want to define a geodesic boundary defining function for g_ε by the equation

$$\left|\frac{d\hat{\rho}_\varepsilon}{\hat{\rho}_\varepsilon}\right|^2_{g_\varepsilon} = 1, \quad \hat{\omega}_\varepsilon|_{\rho=0} = 0 \tag{6.5}$$

where $\hat{\rho}_\varepsilon = e^{\hat{\omega}_\varepsilon}\rho_0$; notice that $\hat{\omega}_0 = 0$. We first show

Lemma 6.3. *There exists $\delta > 0$ such that for all $\varepsilon \geq 0$, the Hamilton-Jacobi equation* (6.5) *has a solution $\hat{\omega}_\varepsilon$ in $\mathcal{K} \cap \{\rho_0 < \delta\}$ and $\hat{\omega}_\varepsilon$ converges to* 0 *in $\mathcal{C}^k$-norms there for all k.*

Proof. The equation can be written as

$$2\langle d\hat{\omega}_\varepsilon, d\rho_0\rangle_{\bar{g}_\varepsilon} + \rho_0 |d\hat{\omega}_\varepsilon|^2_{\bar{g}_\varepsilon} = \frac{1 - |d\rho_0^2|_{\bar{g}_\varepsilon}}{\rho_0}, \text{ with boundary condition } \hat{\omega}_\varepsilon|_{\rho_0=0} = 0,$$

where $\bar{g}_\varepsilon := \rho_0^2 g_\varepsilon$ converges in $\mathcal{C}^\infty(\mathcal{K}; S^2T^*\overline{X})$ to $\rho^2 g_0$ as $\varepsilon \to 0$. This is a uniform family (with respect to ε) of non-characteristic Hamiton-Jacobi equations, which converge in $\mathcal{C}^\infty(\mathcal{K})$ to a non-characteristic Hamiton-Jacobi equation as $\varepsilon \to 0$. This is solved by the method of characteristics and thus it admits a solution in a uniform neighborhood of $\rho_0 = 0$, converging smoothly to $\hat{\omega}_0 = 0$ as $\varepsilon \to 0$. □

Notice that $\hat{\rho}_\varepsilon$ is not exactly the geodesic boundary function that we would need to compute the renormalized volume but we will see later that the renormalized volume there can be expressed easily in terms of this boundary defining function. The function we are interested in is

$$\rho_\varepsilon = e^{\omega_\varepsilon}\hat{\rho}_\varepsilon \tag{6.6}$$

where ω_ε is the solution of

$$\left|\frac{d\rho_\varepsilon}{\rho_\varepsilon}\right|^2_{g_\varepsilon} = 1, \quad \omega_\varepsilon|_{\rho_0=0} = \varphi_\varepsilon$$

and φ_ε is the uniformization factor such that $h_\varepsilon^{\mathrm{hyp}} := e^{2\varphi_\varepsilon} h_\varepsilon$ is hyperbolic if $h_\varepsilon := (\rho_0^2 g_\varepsilon)|_{\rho_0=0}$; The Hamilton-Jacobi equation (6.6) has a unique solution in $\mathcal{K}$ near M and in particular one has $\omega_0|_{\mathcal{K}\cap M} = \varphi_0 = 0$.

6.3. The Hamilton-Jacobi equation near the cusps

In the model from Proposition 4.2, which is a neighborhood of $\{\zeta=0\}$ in $(\mathbb{R}/\frac{1}{2}\mathbb{Z})_w \times \mathbb{H}^2_{\zeta=v+iu}$, it will be useful to forget the ε parameter and consider now $(\ell(\varepsilon), \nu(\varepsilon))$ as independent parameters (ℓ, ν); we shall study the geodesic boundary defining function as a function of (ℓ, ν), where $\nu \in \mathbb{R}$ is a bounded parameter and $\ell \in [0, \ell_0]$ for some fixed small $\ell_0 > 0$. We view ν as a parameter moving in a compact set, and since the metric has a uniform behavior in terms of ν in this set, we shall not emphasize the dependence in ν in the notations. The metric g_ε of (4.9) will be rewritten as

$$g_\ell = \frac{du^2+dv^2+((1+\nu^2)R^4-4\ell^2\nu^2u^2)dw^2+2\nu(R^2-2u^2)dwdv+4\nu uvdudw}{u^2} \tag{6.7}$$

with $R = \sqrt{u^2+v^2+\ell^2}$.

We thus consider for the moment just a neighborhood of a cusp, that is we set

$$\overline{\mathcal{U}} := \left\{(w, u, v) \in \left(\mathbb{R}/\tfrac{1}{2}\mathbb{Z}\right) \times [0, 1) \times \mathbb{R};\, u^2+v^2 < 1\right\}.$$

and $\mathcal{U}$ its interior. Consider the submanifold $H := \{u = v = \ell = 0\} \subset \overline{X} \times [0, \ell_0)$ which corresponds to the cusp, and let $\overline{\mathcal{U}}_\ell$ be the blow-up of $\overline{\mathcal{U}} \times [0, \ell_0)$ at H, defined to be

$$\overline{\mathcal{U}}_\ell = \left(\overline{\mathcal{U}} \times [0, \ell_0) \setminus H\right) \sqcup SH$$

where $SH \subset T(\overline{X} \times [0, \ell_0))|_H$ is the normal inward pointing spherical bundle of H. There is a blow-down map $\beta : \overline{\mathcal{U}}_\ell \to \overline{\mathcal{U}} \times [0, \ell_0)$, which is the identity outside SH and the projection $SH \to H$ on the base when restricted to SH. The set $\overline{\mathcal{U}}$ has a natural structure of smooth manifold with corners of codimension 2 in a way that the functions u, v, ℓ, R lift by β to smooth functions on $\overline{\mathcal{U}}_\ell$; we will use the same notations for these functions and for their lifts to $\overline{\mathcal{U}}_\ell$. There are three boundary hypersurfaces in $\overline{\mathcal{U}}_\ell$: the face denoted $\mathcal{F}_\ell$ whose interior is diffeomorphic to $\{\ell = 0, u \neq 0\} \subset \overline{\mathcal{U}} \times [0, \ell_0)$, the face denoted $\mathcal{F}_u$ whose interior is diffeomorphic to $\{u = 0, \ell \neq 0\} \subset \overline{\mathcal{U}} \times [0, \ell_0)$, and the front face $\mathcal{F}_R = SH$ given by the equation $R = 0$. See Figure 6.1. We notice that $\mathcal{F}_\ell$ is naturally diffeomorphic to a neighborhood of cf in the manifold $\overline{X}_c$ defined in Section 2.3, with cf identified with $\mathcal{F}_R \cap \mathcal{F}_\ell$; thus studying what happens on $\mathcal{F}_\ell$ is equivalent to consider a neighborhood of the cusp in $\overline{X}_c$.

Consider the following smooth variables on $\overline{\mathcal{U}}_\ell$:

$$\left(U = \frac{u}{R}, v, w, \ell\right). \tag{6.8}$$

They provide coordinates outside $\mathcal{F}_R$. In fact, when restricted to $\mathcal{F}_\ell$, (U, v, w) provides smooth coordinates on $\mathcal{F}_\ell$ near the corner $\mathcal{F}_R \cap \mathcal{F}_u$, with U being a smooth

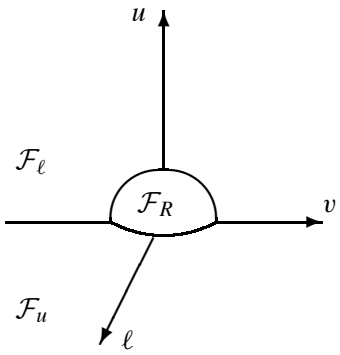

Figure 6.1. The manifold with corners $\overline{\mathcal{U}}_\ell$

defining function for $\mathcal{F}_u$ and v being a smooth defining function for $\mathcal{F}_R$. We will also sometime use the smooth variable

$$V = \frac{v}{R} = \pm\sqrt{1 - U^2 - \frac{\ell^2}{R^2}}$$

on $\overline{\mathcal{U}}_\ell$. Then we have

$$\frac{du}{u} = \frac{dU}{U(1-U^2)} + \frac{V dv}{R(1-U^2)}.$$

Hence, we see that the dual vector fields $u\partial_u$ and ∂_v to du/u and dv become in the coordinates (U, v, w)

$$u\partial_u \to U(1-U^2)\partial_U, \quad \partial_v \to \partial_v - \frac{VU}{R}\partial_U. \tag{6.9}$$

In terms of the variables U, v, w, the metrics g_ℓ is given by

$$\begin{aligned} g_\ell = {} & \frac{dU^2}{U^2(1-U^2)^2} + \frac{2vdUdv}{U(1-U^2)(v^2+\ell^2)} + \frac{v^2dv^2}{(v^2+\ell^2)^2} + \frac{(1-U^2)dv^2}{U^2(v^2+\ell^2)} \\ & + \frac{4vv}{U(1-U^2)}dUdw + \left((1+v^2)\frac{(v^2+\ell^2)}{U^2(1-U^2)} - 4\ell^2v^2\right)dw^2 \\ & + 2v\left(\frac{2v^2}{v^2+\ell^2} + \frac{(1-2U^2)}{U^2}\right)dwdv. \end{aligned} \tag{6.10}$$

In particular, looking at the conformal family of metrics $\overline{g}_\ell = U^2 g_\ell$, we see that when pulling-back to $\{U = 0\} = \mathcal{F}_u$ this metric, one has

$$h_\ell := \overline{g}_\ell\big|_{U=0} = \frac{dv^2}{v^2+\ell^2} + (1+v^2)(v^2+\ell^2)dw^2 + 2vdvdw, \tag{6.11}$$

which corresponds to the model (5.1) for the formation of a cusp obtained by pinching a closed geodesic. In general, as described in the previous section, the global hyperbolic representative of this conformal infinity will be slightly different, of the form

$$h_\ell^{\mathrm{hyp}} = e^{2\varphi_\ell} h_\ell \tag{6.12}$$

for some family of smooth functions φ_ℓ, which is obtained by uniformization and has the properties of Proposition 2.3 for the case $\ell = 0$. By Proposition 5.1 (with ℓ playing the role of ε here), the uniformizing factor φ_ℓ will tend to φ_0 as $\ell \to 0$ on the interior of $\mathcal{F}_u$. Since we want to work in a more general setting than the uniformized metric, we now just fix an arbitrary family of smooth functions φ_ℓ so that $\varphi_\ell \to \varphi_0$ on $\mathcal{F}_u$ as $\ell \to 0$ with the requirement that φ_0 satisfies the properties of Proposition 2.3, *i.e.*, it extends smoothly to the closure of $\mathcal{F}_u$ in $\mathcal{F}_\ell$ and $\partial_w \varphi_0$ vanishes to infinite order at $\mathcal{F}_R = \{v = U = 0\}$ in $\mathcal{F}_\ell \cap \mathcal{F}_u$.

To define the renormalized volume associated to such a choice of representative in the conformal class at the boundary, we need to construct a boundary defining function ρ_ℓ of the face $\mathcal{F}_u = \{U = 0\}$ such that

$$\begin{aligned} &(1) \quad \rho_\ell = e^{\omega_\ell} U \text{ for some function } \omega_\ell \text{ satisfying } \omega_\ell|_{U=0} = \varphi_\ell \\ &(2) \quad \left|\frac{d\rho_\ell}{\rho_\ell}\right|^2_{g_\ell} = 1. \end{aligned} \tag{6.13}$$

We solve this first order differential equation near the face $\mathcal{F}_R$. This is an equation of Hamilton-Jacobi type which we write explicitly in terms of the coordinates U, v, w. First, in the coordinates u, v, w and in matrix form, the family of dual metrics g_ℓ^{-1} on the cotangent space is given by

$$g_\ell^{-1} = \frac{u^2}{R^4} \begin{pmatrix} R^4 + 4v^2u^2v^2 & 2v^2uv(R^2-2u^2) & -2vuv \\ 2v^2uv(R^2-2u^2) & R^4 + v^2(R^2-2u^2)^2 & -v(R^2-2u^2) \\ -2vuv & -v(R^2-2u^2) & 1 \end{pmatrix}. \tag{6.14}$$

Since $\frac{d\rho_\ell}{\rho_\ell} = \frac{(R^2-u^2)du}{uR^2} - \frac{vdv}{R^2} + d\omega_\ell$, the equation $|\frac{d\rho_\ell}{\rho_\ell}|^2_{g_\ell} = 1$ becomes

$$\begin{aligned} &\frac{(1-U^2)^2}{u^2}|du|^2_{g_\ell} + |d\omega_\ell|^2_{g_\ell} + \frac{v^2}{R^4}|dv|^2_{g_\ell} + 2(1-U^2)\left\langle \frac{du}{u}, d\omega_\ell \right\rangle_{g_\ell} \\ &\qquad -\frac{2v(1-U^2)}{R^2}\langle dv, \frac{du}{u}\rangle_{g_\ell} - \frac{2v}{R^2}\langle dv, d\omega_\ell\rangle_{g_\ell} = 1. \end{aligned} \tag{6.15}$$

Now, we compute (recall that $V = v/R$)

$$\begin{aligned} &\frac{(1-U^2)^2}{u^2}|du|^2_{g_\ell} + \frac{v^2}{R^4}|dv|^2_{g_\ell} - \frac{2v(1-U^2)}{R^2}\left\langle dv, \frac{du}{u}\right\rangle_{g_\ell} \\ &= (1-U^2)^2 + V^2U^2\Big(4v^2(1-U^2)^2 + 1 + v^2(1-2U^2)^2 - 4v^2(1-U^2)(1-2U^2)\Big), \end{aligned} \tag{6.16}$$

and

$$\begin{aligned}\left\langle \frac{du}{u}, d\omega_\ell \right\rangle_{g_\ell} &= (1+4\nu^2U^2V^2)(u\partial_u\omega_\ell)+2\nu^2U^2(1-2U^2)VR\partial_v\omega_\ell \\ &\quad -2\nu U^2\frac{V}{R}\partial_w\omega_\ell, \\ \langle dv, d\omega_\ell\rangle_{g_\ell} &= -2\nu^2U^2(1-2U^2)VR(u\partial_u\omega_\ell)+U^2(1+\nu^2(1-2U^2)^2)R^2\partial_v\omega_\ell \\ &\quad -U^2\nu(1-2U^2)\partial_w\omega_\ell,\end{aligned} \tag{6.17}$$

and from (6.14), $|d\omega_\ell|^2_{g_\ell}$ is of the form

$$|d\omega_\ell|^2_{g_\ell} = (u\partial_u\omega_\ell)^2 + \frac{U^2}{R^2}|\partial_w\omega_\ell|^2 + \nu U^2 P_0\left(U^2, V; u\partial_u\omega_\ell, R\partial_v\omega_\ell, \frac{1}{R}\partial_w\omega_\ell\right)$$

for some polynomial $P_0(x, y; X, Y, Z)$, which is quadratic in (X, Y, Z) with coefficients which are polynomial functions in (x, y), independent of ℓ and depending smoothly on ν. Gathering these computations with (6.15), we obtain that $|d\rho_\ell/\rho_\ell|_{g_\ell} = 1$ is equivalent to

$$\begin{aligned}&2\left(1-U^2+2\nu^2V^2U^2(3-4U^2)\right)(u\partial_u\omega_\ell) + U^2VQ_1(U^2)(R\partial_v\omega_\ell) \\ &\quad + \nu U^2VQ_2(U^2)(\frac{1}{R}\partial_w\omega_\ell) + (u\partial_u\omega_\ell)^2 + \frac{U^2}{R^2}|\partial_w\omega_\ell|^2 \\ &\quad + \nu U^2P_0\left(U^2, V; u\partial_u\omega_\ell, R\partial_v\omega_\ell, \frac{1}{R}\partial_w\omega_\ell\right) = U^2Q_3(U^2, V)\end{aligned}$$

where Q_i are polynomials, and thus using (6.9) and dividing by U we get an equation of the form

$$\begin{aligned}&2\left((1-U^2)^2+V^2U^2Q_0(U^2)\right)\partial_U\omega_\ell + UQ_1(U^2)v\partial_v\omega_\ell \\ &\quad + U(1-U^2)^2(\partial_U\omega_\ell)^2 + \frac{U}{R^2}|\partial_w\omega_\ell|^2 + \nu UVQ_2(U^2)\left(\frac{1}{R}\partial_w\omega_\ell\right) \\ &\quad + \nu UP_1\left(U^2, V, R; U\partial_U\omega_\ell, R\partial_v\omega_\ell, \frac{1}{R}\partial_w\omega_\ell\right) = UQ_3(U^2, V)\end{aligned} \tag{6.18}$$

for some polynomials Q_i, and P_1 having the same properties as P_0, and Q_2. This equation is of the form $F(D\omega_\ell, \omega_\ell, z) = 0$, where $x = (U, v, w)$, $D\omega_\ell = (\partial_U\omega_\ell,$

$\partial_v\omega_\ell$, $\partial_w\omega_\ell$) and

$$\begin{aligned} F(p,z,x) &= F(p_U,p_v,p_w,z,U,v,w)\\ &= 2\Big[(1-U^2)^2+V^2U^2Q_0(U^2)\Big]p_U + UvQ_1(U^2)p_v\\ &\quad + \nu\frac{UV}{R}Q_2(U^2)p_w + U(1-U^2)^2p_U^2 + \frac{Up_w^2}{R^2}\\ &\quad + \nu U P_1\Big(U^2,V;Up_U,Rp_v,\frac{p_w}{R}\Big) - UQ_3(U^2,V). \end{aligned} \tag{6.19}$$

In this definition, notice that the dependence in z and w is in fact trivial. Now, since $\partial_{p_U}F|_{U=0}=2\neq 0$, the equation with initial condition $\omega_\ell|_{U=0}=\varphi_\ell$ is non-characteristic. It can therefore be resolved using the method of characteristics for U small outside $R=0$. In general, the equations for the characteristics are given by (denoting $(x_1,x_2,x_3)=(U,v,w)$ and $(p_1,p_2,p_3)=(p_U,p_v,p_w)$)

$$\begin{aligned} \dot p_i(s) &= -\partial_{x_i}F(p(s),z(s),x(s)) - \partial_z F(p(s),z(s),x(s)),\\ \dot z(s) &= \sum_i \partial_{p_i}F(p(s),z(s),x(s))p_i(s),\\ \dot x_i(s) &= \partial_{p_i}F(p(s),z(s),x(s)). \end{aligned} \tag{6.20}$$

where a dot is used to denote a derivative with respect to the parameter s. We notice that, when $\nu=0$, these equations have smooth coefficients except for all terms containing p_w/R. Thus they are smooth outside the face $\mathcal{F}_R=\{R=0\}$, in particular they restrict on the face $\mathcal{F}_\ell\setminus\{R\neq 0\}$ corresponding to the rank-1 cusp limiting case. We will need to solve these equations with the following initial conditions on the face $\mathcal{F}_u=\{U=0\}$ (we restrict for the moment to the region $U=0$, $R\neq 0$)

$$\begin{aligned} &U(0)=0,\quad v(0)=v_0,\quad w(0)=w_0,\quad z(0)=\varphi_\ell(v_0,w_0),\\ &p_v(0)=\partial_v\varphi_\ell(v_0,w_0),\quad p_w(0)=\partial_w\varphi_\ell(v_0,w_0),\quad p_U(0)=0, \end{aligned} \tag{6.21}$$

where the last condition follows from the fact $F(p(0),z(0),x(0))=0$. The behavior of the solution for U small near the face $R=0$ can possibly be singular because of the singularity of the coefficients containing some R^{-1} in F there. The solution ω_ℓ will be given by

$$D\omega_\ell(U(s),v(s),w(s))=(p_U(s),p_v(s),p_w(s)),\quad \omega(U(s),v(s),w(s))=z(s) \tag{6.22}$$

with initial condition $\omega_\ell(0,v_0,w_0)=\varphi_\ell(v_0,w_0)$. We analyze the solution near $\mathcal{F}_R$ when $\ell=0$ ($\mathcal{F}_R\cap\mathcal{F}_\ell$ corresponds to $\{u=v=0\}$ inside $\mathcal{F}_\ell$).

Proposition 6.4. *For each $\varphi_0 \in \mathcal{C}^\infty(\mathcal{F}_\ell \cap \mathcal{F}_u)$ with $\partial_w \varphi_0$ vanishing to infinite order at $\mathcal{F}_R$, there exists a unique smooth function ω_0 on $\mathcal{F}_\ell$ defined in a neighborhood of $\mathcal{F}_u \cap \mathcal{F}_R$ in $\mathcal{F}_\ell$ such that $\partial_w \omega_0$ vanishes to infinite order at $\mathcal{F}_R$ and $\rho_0 = e^{\omega_0} U$ is a boundary defining function of $\mathcal{F}_\ell \cap \mathcal{F}_u$ with the property that*

$$(\rho_0^2 g_0)\Big|_{\mathcal{F}_u} = e^{2\varphi_0} h_0, \quad \left|\frac{d\rho_0}{\rho_0}\right|_{g_0} = 1.$$

Proof. We need to investigate if the equation (6.20) can be solved in a uniform way as the initial condition v_0 in (6.21) approaches zero. We first change coordinates and use the coordinates (u', v', w') of (4.14) in which the metric g_0 has the simpler form (4.13). In fact, since the metric in the new coordinates has the same form as in the original coordinates (u, v, w) but with ν replaced by 0, we are reduced to solve a Hamilton-Jacobi equation which has the same form as (6.18) but with $\nu = 0$, and in the coordinates (U', v', w') where $U' := u'/\sqrt{u'^2 + v'^2}$. Our first goal is to prove that ω_0 viewed in the (U', v', w') coordinates is smooth near $U' = 0$, and then to come back to the original coordinates using (4.14) to deduce the desired result.

We are reduced to analyze the solution of (6.18) when $\nu = 0$, which we now do (for convenience of notations we keep the expression of this equation with the variable (U, v, w) for the moment, having in mind that they really mean (U', v', w')). We also allow w and w_0 to be in $\mathbb{R}$ instead of $\mathbb{R}/\frac{1}{2}\mathbb{Z}$, which is the same as viewing the equation in the universal covering, since we need to work in the domain (4.15) where the coordinates (U', v', w') are valid. We notice that since we assume $\nu = 0$, each of the singular terms in the equations (6.20) comes now with a p_w factor. From the initial conditions and the independence of F with respect to w, we have that $p_w(s) = \partial_w \varphi_0(v_0, w_0)$ for all s. On the other hand, by hypothesis, we know that $\partial_w \varphi_0(v_0, w_0) = \mathcal{O}(|v_0|^\infty)$ when $v_0 \to 0$. To solve the ODE (6.20) uniformly as $v_0 \to 0$, we now check that for $v_0 \neq 0$, $v(s)$ cannot approach zero rapidly.

Lemma 6.5. *There exists a positive constant K depending on φ_0 but not on v_0 and w_0, as well as $C > 0$ such that*

$$|v(s)| \geq |v_0| e^{-Cs} \quad \text{and} \quad U(s) \geq s \quad \text{for} \quad s \leq K.$$

Proof. We will consider the case $v_0 > 0$, since the case $v_0 < 0$ can be dealt with in a similar fashion. First we use that for $\ell = 0$, we have that

$$R = \frac{v}{\sqrt{1 - U^2}}, \quad V = \sqrt{1 - U^2}.$$

Set $y = \log v$, then from (6.19) and (6.20), we can write, as long as $U < 1$,

$$\dot{y} = U Q_1(U^2) + U\left(A_1(U^2) e^y + A_2(U^2) p_U + A_3(U^2) e^y p_V + A_4(U^2) e^{-y} p_w\right)$$

for some polynomials A_i in the variable U^2. Consider the vector $\overrightarrow{X} = (p_U(s), p_v(s) - p_v(0), U(s), y(s) - y(0))$, where $y(0) = \log v_0$. Since p_w is in fact independent of s, we see from (6.19) and (6.20) that there exists a positive constant K_1

depending on φ_0 such that

$$\frac{d}{ds}|\overrightarrow{X}(s)| \leq K_1$$

whenever $|\overrightarrow{X}(s)| \leq \frac{1}{K_1}$. This means that

$$|\overrightarrow{X}(s)| \leq K_1 s$$

for $s \leq \frac{1}{K_1^2}$. In particular, there exists $C > 0$ such that

$$|y(s) - y(0)| \leq K_1 s \quad \Longrightarrow \quad v(s) \geq v_0 e^{-Cs}$$

for $s \leq \frac{1}{K_1^2}$. This gives the first half of the result for some big constant K. Now, for $|\overrightarrow{X}(s)|$ sufficiently small, notice that $\dot{U} \geq 1$. Integrating, we get that $U(s) \geq s$ for s sufficiently small. Taking the constant K smaller if necessary gives the result. □

We have $p_w(s) = p_w(0) = \partial_w \varphi_0(v_0, w_0)$ which decreases rapidly when v_0 tends to zero, and $V = v/R$ is close to 1 when U is small, thus using Lemma 6.5, there is $C > 0$ such that for $s < 1/K$, we have $U(s) \geq s$ and

$$\left|\frac{p_w(s)}{R(s)^2}\right| \leq 2\left|\frac{\partial_w \varphi_0(v_0, w_0)}{v(s)^2}\right| \leq Ce^{CU(s)}\left|\frac{\partial_w \varphi_0(v_0, w_0)}{v_0^2}\right| = \mathcal{O}(|v_0|^\infty).$$

Using this rapid vanishing as $v_0 \to 0$, by looking at (6.20) and (6.19), we deduce that the $(U(s), v(s), w(s))$ extend smoothly as the initial condition $x_i(0)$, $p_i(0)$ tend to $\{v = 0\}$, on a uniform time $s \in [0, s_0]$ with $s_0 > 0$. In fact, with the initial condition $v_0 = 0$, we have by (6.19) that $v(s) = p_w(s) = 0$ for all s and the ODE simply becomes in the region $U < 1$ (using that $V = \sqrt{1 - U^2}$ in that case)

$$\begin{gathered} \dot{p_U} = L_1(U, p_U), \quad \dot{p_v} = L_2(U, p_v, p_U), \quad \dot{z} = L_3(U, P_U), \\ \dot{U} = L_4(U, P_U), \quad \dot{w} = 0 \end{gathered} \tag{6.23}$$

for some polynomials L_j with $L_4(0, p_U) = 2$. In particular, we see that the curves $U(s)$, $v(s)$ are tangent to the face $v = 0$ (as long as $U < 1$) and they are transverse to $U = 0$; moreover $U(s) = 2s + \mathcal{O}(s^2)$ near $U = 0$. We thus obtain that $\psi : (s, v_0, w_0) \to (U(s), v(s), w(s))$ is a smooth local diffeomorphism on $[0, \epsilon) \times [0, \epsilon) \times \mathbb{R}$ for small $\epsilon > 0$ and there is $\epsilon > 0$ such that for each point $w_0 \in \mathbb{R}$ it is a diffeomorphism from $[0, \epsilon) \times [0, \epsilon) \times (w_0 - \epsilon, w_0 + \epsilon)$ on its image. Moreover, it is easily seen that $\psi(s, v_0, w_0 + \frac{1}{2}) = (U(s), v(s), w(s) + \frac{1}{2})$. The same hold in the region $v_0 \leq 0$ and this implies that ω_0 given by (6.22) for $\ell = 0$ extends as a smooth function of (U, v, w) near each $(0, 0, w_0)$ on $\{U \geq 0, v \geq 0\}$ and on $\{U \geq 0, v \leq 0\}$, in some neighborhood which has uniform size with respect to w_0. We also have that $\partial_w \omega_0(\psi(s, v_0, w_0)) = p_w(s) = \partial_w \varphi_0(v_0, w_0) = \mathcal{O}(|v_0|^\infty)$, thus $\partial_w \omega_0 = \mathcal{O}(|v|^\infty)$ uniformly where it is defined.

We have thus proved that in the (U', v', w') coordinates, ω_0 lifted to the universal cover is smooth in $[0, \epsilon) \times [0, \epsilon) \times \mathbb{R}$, and $\partial_{w'}\omega_0 = \mathcal{O}(|v'|^\infty)$. To conclude the proof, we need to come back to the original coordinates (U, v, w) by using (4.14):

$$U' = \frac{U\sqrt{1+v^2}}{\sqrt{1+v^2U^2}}, \quad v' = \frac{v(1+v^2)}{1+v^2U^2}, \quad w' = w - \frac{v}{1+v^2}\frac{1-U^2}{v}.$$

First it is clear that $\omega_0(U', v', w')$ is smooth when viewed as a function of (U, v, w) except possibly at $v = 0$ where w' is a singular function of v. Similarly to the discussion of Section 5 (which corresponds to an analysis in the boundary $U = 0$), the fact that $\partial_{w'}\omega_0 = \mathcal{O}(|v'|^\infty)$ actually implies that ω_0 is smooth in the variable (U, v, w) since $D\omega_0$ admits a smooth extension to $v = 0$. This achieves the proof of Proposition 6.4, as (U, v, w) are smooth coordinates near $\mathcal{F}_R \cap \mathcal{F}_u$ on the face $\mathcal{F}_\ell$, and U' is a smooth function of U. □

Corollary 6.6. *Let ρ_0 be the function of Proposition* 6.4. *There exists a diffeomorphism $\phi : [0, \varepsilon)_s \times \mathcal{O} \to \mathcal{Q} \subset \mathcal{F}_\ell$ with $\mathcal{Q}$ a neighborhood of $\mathcal{F}_u \cap \mathcal{F}_R$ in $\mathcal{F}_\ell$ and $\mathcal{O}$ a neighborhood of $\mathcal{F}_u \cap \mathcal{F}_R$ in $\mathcal{F}_u \cap \mathcal{F}_\ell$ such that $\phi^*\rho_0 = s$ and*

$$\phi^* g_0 = \frac{ds^2 + h_0(A_s \cdot \cdot)}{s^2}$$

where A_s is a one-parameter smooth family of smooth endomorphisms of $T\mathcal{O}$ up to $\mathcal{F}_R$, for $s \in [0, \varepsilon)$, so that $h_s(\cdot, \cdot) := h_0(A_s \cdot \cdot)$ is a smooth family of cusp symmetric tensors.

Proof. The diffeomorphism is given by $\phi(s, v_0, w_0) = \phi_s(v_0, w_0)$ where ϕ_s is the flow at time s of the gradient $\nabla^{\rho_0^2 g_0}\rho_0$ of ρ_0 with respect to $\rho_0^2 g_0$. First, we notice that this flow is exactly the diffeomorphism $\phi(s, v_0, w_0) = x(s/2)$ where $x(s)$ is the integral curve studied in the proof of the previous proposition — satisfying (6.20) with initial condition $x(0) = (0, v_0, w_0)$. Since $(\phi^*U)/s$ is a smooth function on $[0, \varepsilon) \times \mathcal{O}$ for some small neighborhood $\mathcal{O}$ of $\{0\} \times \{0\} \times \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)$ in $(U, v, w) \in [0, \varepsilon) \times [0, \varepsilon) \times \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)$, the metric $s^2\phi^* g_0$ is given by a positive smooth function times $\phi^*(U^2 g_0)$ with g_0 given in (6.10) (for $\ell = 0$). To prove the statement, it suffices to check that for vector fields $Z_1 := v_0\partial_{v_0}$ and $Z_2 := v_0^{-1}\partial_{w_0}$, we have that $\phi^*(U^2 g_0)(Z_i, Z_j)$ is smooth near $s = v_0 = 0$ for $i, j \in \{1, 2\}$. Since $\phi(s, 0, w_0) \subset \{v = 0\}$ by the analysis of the proof in the previous proposition, writing $\phi(s, v_0, w_0) = (U, v, w)$ we get $v = v_0(1 + v_0 f(s, v_0, w_0))$ and $w = w_0 + v_0 k(s, v_0, w_0)$ for some smooth functions f, k, and thus

$$\phi_*(v_0\partial_{v_0}) = vW_1, \quad \phi_*(v_0^{-1}\partial_{w_0}) = v^{-1}\partial_w + W_2$$

for some smooth vector field W_1, W_2 near $v = U = 0$. By inspecting (6.10) for $\ell = 0$, $\phi^*(U^2 g_0)(Z_i, Z_j)$ is smooth near $s = v = 0$. The same argument works in the region $v \leq 0$ covering the other neighborhood of $\mathcal{F}_R \cap \mathcal{F}_u$ in $\mathcal{F}_\ell$. □

6.4. Proof of Proposition 2.4

We decompose the hyperbolic 3-manifold with rank-1 cusps (X, g) as in Section 2.1 into a region $\mathcal{K} \subset \overline{X}$ and some cusp neighborhoods $\mathcal{U}_j^c$ for $j = 1, \dots, j_1$. Recall that $\overline{X}$ can be compactified into $\overline{X}_c$. Then we fix a boundary defining function ρ in a neighborhood of $M = \partial\overline{X}$, which is equal to $\rho = u/\sqrt{u^2+v^2}$ in the coordinates of the model (2.2) of $\mathcal{U}_j^c$. The hyperbolic metric g there, as given by the model (2.2), corresponds to the case $\ell = 0$, $\nu = 0$ in the expression (6.7) and $U = u/R$ is the chosen defining function of $\partial\overline{X}$ in these coordinates. Let h^{hyp} be the unique hyperbolic metric on M in the conformal class of $h := (\rho^2 g)|_{TM}$. Let $\psi \in \mathcal{C}_r^\infty(\overline{M})$ and $\hat{h} = e^{2\psi}h^{\mathrm{hyp}}$. By Proposition 2.3, we have $e^{2\psi}h^{\mathrm{hyp}} = e^{2(\psi+\varphi)}h$ for some $\varphi \in \mathcal{C}_r^\infty(\overline{M})$. Since we still have that $\psi + \varphi \in \mathcal{C}_r^\infty(\overline{M})$, Proposition 6.4 shows that there exists a smooth defining function $\hat{\rho}$ of $\overline{M}$ on a neighborhood of $\mathrm{cf} \cap \overline{M}$ in $\overline{X}_c$ (as explained above, $\overline{M}$ corresponds to $\mathcal{F}_u$ and cf to $\mathcal{F}_R$ in the model $\mathcal{F}_\ell$ of $\overline{X}_c$ near the cusps), such that $|d\hat{\rho}/\hat{\rho}|_g = 1$ with $\rho^2 g|_{TM} = e^{2\psi}h^{\mathrm{hyp}}$; it is unique where it is defined. On the other hand, outside $\mathcal{U}_j^c$, this equation is also a smooth non-characteristic Hamilton-Jacobi equation, thus the solution $\hat{\rho}$ defined near $\overline{M} \cap \mathrm{cf}$ can be extended uniquely as a solution also in a whole neighborhood of $\overline{M}$ in $\overline{X}_c$, giving the desired function $\hat{\rho}$. Considering the maps $\phi : [0, \epsilon)_x \to \overline{M} \to \overline{X}_c$ given by $\phi(x, y) = \phi_x(y)$ where ϕ_s the flow at time s of the gradient $\nabla^{\hat{\rho}^2 g}\hat{\rho}$ of $\hat{\rho}$ with respect to $\hat{\rho}^2 g$, we see by using Corollary 6.6 (recall that ϕ is the gradient flow in the proof of that Corollary) that on $(0, \epsilon)_x \times M$

$$\phi^* g = \frac{dx^2 + \hat{h}_x}{x^2}$$

for some 1-parameter family $\hat{h}_x$ of smooth metrics on M depending smoothly on $x \in [0, \varepsilon)$, and h_x are actually a smooth family of cusp symmetric tensors. Since g is hyperbolic, we know (as it is a local computation) from [10, Theorem 7.4] that the dependence of $\hat{h}_x$ is a polynomial of order 2 in x^2

$$\hat{h}_x = \hat{h}((\mathrm{Id} + x^2 A)\cdot, \cdot)$$

with $\mathrm{Tr}(A) = -\frac{1}{2}\mathrm{Scal}_{\hat{h}}$ and $\delta_{\hat{h}}(A) = \frac{1}{2}d\mathrm{Scal}_{\hat{h}}$. It remains to check that the complement of the region $\phi([0, \epsilon) \times \overline{M})$, called $\mathcal{V}$, has finite volume with respect to g in X. Clearly, $\mathcal{K} \cap (X \setminus \{\hat{\rho} < \epsilon\})$ is compact in X thus has finite volume. Now we analyze the region $\mathcal{U}_j^c \setminus \{\hat{\rho} < \epsilon\}$. To show that it has finite volume, it suffices to use that $\hat{\rho}$ is a defining function of $\mathcal{F}_u \cap \mathcal{F}_\ell$ in the blown-up space $\mathcal{F}_\ell$ of $\overline{\mathcal{U}}_j^c$ around the region $H = \{(u, v) = 0\}$ representing the cusp, and so $\{\hat{\rho} \geq \epsilon\}$ is contained in some region $\{U \geq c\epsilon\}$ for some $c > 0$. Now the volume form of the metric g in coordinates (u, v, w) is $\frac{u^2+v^2}{u^3}dudvdw$ and a simple computation shows that

$$\int_0^1 \int_{-Cu}^{Cu} \left(\frac{u}{\sqrt{u^2+v^2}}\right)^z \frac{u^2+v^2}{u^3} dvdu < \infty \tag{6.24}$$

for all $z \in \mathbb{C}$. By taking $z = 0$ we see that the region has finite volume for any finite constant $C > 0$. It remains to show that if $\hat{\rho}$ is extended smoothly to $\overline{X}_c$ as a boundary defining function of $\overline{M}$ (positive in $\overline{X}_c \setminus \overline{M}$) then $H(z) = \int_X \hat{\rho}^z \mathrm{dvol}_g$ is meromorphic in $\{\mathrm{Re}(z) > -\epsilon\}$ for some $\epsilon > 0$. We can split the integral as an integral near $\partial\overline{X} \cap \mathcal{K}$ and the meromorphy of this part follows directly from the fact that $\hat{\rho}$ is a smooth boundary defining function there, and there remains the integral in each $\mathcal{U}_j^c$. The part of the integral in $\mathcal{V}$ clearly gives holomorphy in z by (6.24). For the integral in $\mathcal{U}_j^c \setminus \mathcal{V}$, we notice that the volume form in the coordinates (U, R, w) near $\mathcal{F}_u = \{U = 0\}$ in the model $\mathcal{F}_\ell$ (isometric to $\mathcal{U}_j^c$ with $v = 0$) of Section 6.3 is given by $dU dR dw/(U^3\sqrt{1-U^2})$ and thus from the fact that $\hat{\rho}/U$ is a smooth positive function in these coordinates near $U = 0$, the meromorphy of the remaining part of the integral $H(z)$ follows by Taylor expanding $\hat{\rho}/U$ at $U = 0$. □

6.5. Taylor expansion of the boundary defining function to second order

For $\ell > 0$ fixed, it is also straightforward to solve the equations (6.15) near the degenerating curve and find ω_ℓ and ρ. The function ω_ℓ will be smooth in s, so smooth in U. In particular, at $U = 0$, it has an expansion of the form

$$\omega_\ell \sim \sum_{j=0}^{\infty} a_j U^k. \tag{6.25}$$

To compute the limit as $\ell \to 0$ of the renormalized volume, we will need to know the terms of order 0 and 2. By assumption, we have that $a_0 = \varphi_\ell$. We now compute a_1 and a_2.

Proposition 6.7. *Near $v = 0$, the coefficients a_1 and a_2 in the expansion* (6.25) *are given by $a_1 = 0$ and*

$$a_2 = -\frac{1}{4}\left(|d\varphi_\ell|^2_{h_\ell} + (1+v^2)\left(1 - \frac{\ell^2}{v^2+\ell^2}\right) - 2 + 2(v^2-1)v\partial_v\varphi_\ell - \frac{2vv}{v^2+\ell^2}\partial_w\varphi_\ell\right).$$

Proof. We see directly from (6.18) that $a_1 = 0$. Then notice that by (6.14), the metric dual to $\overline{g}_\ell = U^2 g_\ell$ is smooth near $\mathcal{F}_u \setminus (\mathcal{F}_u \cap \mathcal{F}_R)$ and as $U \to 0$

$$\begin{aligned} |d\omega_\ell|^2_{g_\ell} &= U^2\left((v^2+\ell^2)((1+v^2)(\partial_v\varphi_\ell)^2) + \frac{(\partial_w\varphi_\ell)^2}{v^2+\ell^2} - 2v\partial_v\varphi_\ell\partial_w\varphi_\ell\right) + \mathcal{O}(U^3) \\ &= U^2|d\varphi_\ell|^2_{h_\ell} + \mathcal{O}(U^3) \end{aligned}$$

where h_ℓ is given by (6.11). Combining this with (6.15), (6.16) and (6.17), we have

$$-2 + \frac{v^2}{v^2+\ell^2}(1+v^2) + 4a_2 + 2(v^2-1)v\partial_v\varphi_\ell - 2v\frac{v}{v^2+\ell^2}\partial_w\varphi_\ell + |d\varphi_\ell|^2_{h_\ell} = 0$$

which achieves the proof. □

7. Variation formulas for the renormalized volume and Kähler potential for Weil-Petersson metric

In this section we describe the properties of the renormalized volume as a function on the conformal class of the conformal boundary: we first compute the variation of the renormalized volume for families of hyperbolic metrics with rank 1-cusps, and we show that Vol_R is a Kähler potential for Weil-Petersson metric in a Bers slice of the quasi-Fuchsian deformation space.

7.1. Variation formula

Arguing as in [14, Proposition 3.11], we have the following variation formula for the renormalized volume under a change of conformal representative in the conformal boundary.

Proposition 7.1. *Let X be a geometrically finite hyperbolic 3-manifold. Let h^{hyp} be the unique hyperbolic representative in the conformal boundary of g and let $\hat{h} := e^{2\psi}h^{\mathrm{hyp}}$ with $\psi \in \mathcal{C}^\infty_r(\overline{M})$. If ρ and $\hat{\rho}$ are geodesic boundary defining functions associated to h^{hyp} and $\hat{h}$ given by Proposition* 2.4*, we have*

$$\mathrm{Vol}_R(X,\hat{h}) = \mathrm{Vol}_R(X,h^{\mathrm{hyp}}) - \frac{1}{4}\int_M (|\nabla\psi|^2_{h^{\mathrm{hyp}}} - 2\psi)\mathrm{dvol}_{h^{\mathrm{hyp}}}.$$

For any $\chi \in \mathcal{C}^\infty_c(\overline{X})$ satisfying $\chi = \sum_{k=0}^2 \chi_k\rho^2 + \mathcal{O}(\rho^3)$ at $\partial\overline{X}$, with $\chi_k \in \mathcal{C}^\infty_c(M)$

$$\begin{aligned}\mathrm{FP}_{z=0}\int_X \hat{\rho}^z\chi\,\mathrm{dvol}_g = \mathrm{FP}_{z=0}\int_X \rho^z\chi\,\mathrm{dvol}_g \\ - \frac{1}{4}\int_M (\chi_0(|\nabla\psi|^2_{h^{\mathrm{hyp}}} - 2\psi) - 4\chi_2\psi)\mathrm{dvol}_{h^{\mathrm{hyp}}}.\end{aligned} \tag{7.1}$$

Proof. First, by Proposition 2.4, associated to both h^{hyp} (resp. to $\hat{h}$), there are product coordinates $[0,\epsilon)_x \times \overline{M}$ near $\overline{M}$ in the compactification $\overline{X}_c$ of X in which g is of the form

$$g = \frac{dx^2 + h_0 + x^2h_2 + x^4h_4}{x^2}$$

with $h_0 = h^{\mathrm{hyp}}$ (resp. $h_0 = \hat{h}$) , h_2, h_4 some smooth cusp symmetric tensors such that $\mathrm{Tr}_{h_0}(h_2) = -\frac{1}{2}\mathrm{Scal}_{h_0}$ and $\delta_{h_0}(h_2) = \frac{1}{2}d\mathrm{Scal}_{h_0}$. The complement of the regions covered by these coordinates has finite volume, thus the part of the integrals above over the region $x > \epsilon/2$ are trivial to deal with. On the other hand, by the proof of Proposition 2.4, we can also solve the Hamilton-Jacobi equation $|\frac{dx}{x} + d\omega|^2_g = 1$ near $\overline{M}$ with initial condition $\omega|_{\overline{M}} = \psi$. From the symmetries of this equation, we see that ω must have an even expansion in x at $\overline{M}$, $\omega \sim \sum_{j=0}^\infty \omega_{2j}x^{2j}$. As

in [14, Lemma 3.6], putting this expansion back in the Hamilton-Jacobi equation, we compute that (the computation is local)

$$\omega_2 = -\frac{1}{4}|\nabla\omega_0|^2_{h_0}, \quad \text{with } \omega_0 = \psi.$$

On the other hand, the volume form of g is given by $\mathrm{dvol}_g = v(x)\mathrm{dvol}_{h_0}\frac{dx}{x^3}$ with $v(x) = v_0 + x^2 v_2 + \mathcal{O}(x^3)$ for $v_0 = 1$ and $v_2 = -\frac{1}{4}\mathrm{Scal}_{h_0}$. Hence, we compute just as in the proof of [14, Lemma 3.5] that

$$\begin{aligned}\mathrm{Vol}_R(X, \hat{h}) &= \mathrm{Vol}_R(X) + \int_M (v_0\omega_2 + v_2\omega_0)\ \mathrm{dvol}_{h^{\mathrm{hyp}}} \\ &= \mathrm{Vol}_R(X) - \frac{1}{4}\int_M (|\nabla\omega_0|^2_{h^{\mathrm{hyp}}} - 2\omega_0)\mathrm{dvol}_{\mathrm{h}^{\mathrm{hyp}}}.\end{aligned} \tag{7.2}$$

For (7.1), the calculation is similar but one has to replace $v(x)$ by $v(x)\chi(x)$ in the reasoning, thus $v_0\omega_2$ and $v_2\omega_0$ become $v_0\chi_0\omega_2$ and $(v_2\chi_0 + v_0\chi_2)\omega_0$. □

First we say that (X, g^t) for $t \in (-t_0, t_0)$ is a smooth family of geometrically finite hyperbolic manifolds: if $g := g^0$ is a geometrically finite metric on X with j_1 cusps of rank-1, represented by some disjoint curves $H = \cup_{j=1}^{j_1} H_j$ in the boundary $\partial\overline{\mathbf{X}}$ of the compactification $\overline{\mathbf{X}}$ as in Section 2.1, g^t is hyperbolic for all t and there is a neighborhood $\mathcal{U}_j$ of H_j in $\overline{\mathbf{X}}$ such that $\rho^2 g^t$ extends to a smooth metric on $\overline{\mathbf{X}} \setminus \cup_j \mathcal{U}_j$ if ρ is a boundary defining function of $\partial\overline{\mathbf{X}}$, and there exists a smooth family of diffeomorphisms $\psi_j^t : \mathcal{U}_j \to \psi^t(\mathcal{U}_j) \subset \overline{\mathbb{H}^2_\zeta} \times \left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)_w$ such that for $\zeta = v + iu \in \mathbb{H}^2$

$$(\psi_j^t)_* g^t = \frac{du^2 + dv^2 + (u^2 + v^2)dw^2}{u^2}.$$

For such a family of metrics, it is easy by extending $(\psi_j^t)^{-1} \circ \psi_j^0$ to $\overline{\mathbf{X}}$ to construct a diffeomorphism θ^t of $\mathbf{X}$ such that $\rho^2(\theta^t)^* g^t$ extend smoothly as a metric on $\overline{X} = \overline{\mathbf{X}} \setminus H$ and near H_j

$$(\psi_j^0)_*(\theta^t)^* g^t = \frac{du^2 + dv^2 + (u^2 + v^2)dw^2}{u^2}.$$

We can thus reduce the analysis to the family of metrics $(\theta^t)^* g^t$ with a cusp singularity at H, which we do now, and to avoid heavy notation we write g^t instead of $(\theta^t)^* g^t$. Denote by h^t the hyperbolic metric in the conformal boundary of (X, g^t): it is a smooth family in t of hyperbolic metric with finite volume and cusps. Proceeding as in the proof of Proposition 6.4 and using h^t as the representative of the conformal infinity of g^t, we can then solve the Hamilton-Jacobi equation

$$\left|\frac{d\rho^t}{\rho^t}\right|_{g^t} = 1, \quad (\rho^t)^2 g^t\Big|_{TM} = h^t$$

smoothly in t to get a smooth family of boundary defining functions ρ^t of $\overline{M}$ in $\overline{X}$. As we have seen in the proof of Proposition 2.4, the gradient vector field $\nabla^{\overline{g}^t}\rho^t$, where $\overline{g}^t = (\rho^t)^2 g^t$, is defined and smooth in a neighborhood of $\overline{M}$ in $\overline{X}_c$ and is tangent to the cusp face cf. Integrating this vector field for each t then gives a smooth family of collar neighborhood $\phi^t : \overline{M} \times [0, \epsilon)_x \to \overline{X}$ such that

$$(\phi^t)^* g^t = \frac{dx^2 + h_0^t + x^2 h_2^t + x^4 h_4^t}{x^2} \tag{7.3}$$

with h_{2j}^t some smooth families (in t) of cusps symmetric tensors such that $h_0^t = h^t$.

Theorem 7.2. *Let (X, g^t) be a smooth family of geometrically finite hyperbolic metrics. Let h^t be the unique hyperbolic representative of the conformal infinity of g^t and h_2^t the second fundamental form at $\partial\overline{X}$ given by* (7.3). *If $\mathrm{Vol}_R^t(X)$ denotes the renomalized volume of (X, g^t), then*

$$\partial_t \mathrm{Vol}_R^t(X)\big|_{t=0} = -\frac{1}{4}\int_M \langle \dot{h}_0, h_2 - h_0\rangle_{h_0} \mathrm{dvol}_{h_0},$$

where $h_2 = h_2^t\big|_{t=0}$, $h_0 = h^t|_{t=0}$ and the dot denotes a derivative in the t variable evaluated at $t = 0$.

Proof. The proof is very similar to the proof of [14, Theorem 5.3] and is based on Schläfli formula, but here one has to be careful about the degeneracy near the cusps to perform the argument. Like in [14, Theorem 5.3], we can pull-back g^t (using an extension of $(\phi^t)^{-1} \circ \phi^0$) by a family of diffeomorphisms of $\overline{X}_c$ which is the Identity outside a neighborhood of $\overline{M}$, so that the new metric is isometric to the right hand side of (7.3) near $\overline{M}$ via the diffeomorphism $\phi := \phi^0$ that is independent of t. For $\delta \in (0, \delta_0)$, consider the region $V_\delta := \phi(\overline{M} \times [0, \delta)) \subset \overline{X}$. Then, as in the proof of Proposition 2.4, $\overline{X} \setminus V_\delta$ is of finite volume with respect to g^t, and we claim that

$$\partial_t \mathrm{Vol}(\overline{X} \setminus V_\delta, g^t)\big|_{t=0} = \frac{1}{2}\int_{\rho=\delta} \left(\dot{H} + \frac{1}{2}\langle \dot{g}, \mathrm{II}\rangle_g\right) \mathrm{dvol}_g, \tag{7.4}$$

where H^t is the mean curvature of $\phi(\overline{M} \times \{\delta\})$ with respect to the metric g^t, II^t is its second fundamental form and $g := g^t|_{t=0}$. The proof of (7.4) is then the same as the one of [14, Lemma 5.1]: using the variation formula for the scalar curvature, we find

$$-4\partial_t \mathrm{Vol}(\overline{X} \setminus V_\delta, g^t) = \int_X (\Delta_g \mathrm{Tr}_g(\dot{g}) + d^*\delta^g(\dot{g}))\mathrm{dvol}_g$$

and the integration by parts of $\Delta_g \mathrm{Tr}_g(\dot{g})$ and $d^*\delta^g\dot{g}$ can be done but there could possibly be a new contribution coming from the cusp face cf in the compactification $\overline{X}_c$ of $\overline{X}$ ($\mathrm{cf} \cap \{\rho = \delta\}$ corresponds to the cusp point at infinity of the Riemann surface $\{\rho = \delta\}$). In order to analyze this, we can apply Green's formula on $\{R \geq \lambda, \rho \geq \delta\}$ where R is a boundary defining function of cf. If $\phi' : \mathrm{cf} \times [0, 1) \to \overline{X}$ is

a collar neighborhood of the cusp face in $\overline{X}$, we know from the local form (2.2) of g^t that

$$\mathrm{Area}(\phi'(\mathrm{cf}\times\{\lambda\}))\cap(\overline{X}\setminus V_\delta)=\mathcal{O}(\lambda^2).$$

It is direct to check (using (6.14) with $\nu=0$) that $\partial_R \mathrm{Tr}_g(\dot g)$ and $\delta^g(\dot g)(\partial_R)$ are uniformly bounded in λ on $\phi'(\mathrm{cf}\times\{\lambda\})\cap(\overline{X}\setminus V_\delta)$, where ∂_R is the unit normal vector to $\phi'(\mathrm{cf}\times\{\lambda\})$ with respect to g. This means that there is in fact no contribution coming from the cusp face when we take the limit $\lambda\searrow 0$. Thus, when we integrate by parts, we obtain the same formula as in [14] and (7.4) follows. □

7.2. A Kähler potential for the Weil-Petersson metric

Consider the quasi-Fuchsian space associated to Riemann surfaces with n-cusps. For each pair (M,h_-) and (M,h_+) of hyperbolic surfaces of finite volume with n cusps, denoting $h:=(h_-,h_+)$, there exists a unique (up to diffeomorphism) complete hyperbolic metric $g=g_h$ on the cylinder $X:=\mathbb{R}_t\times M$, which is realized as a quotient $\Gamma\backslash\mathbb{H}^3$ for $\Gamma\subset \mathrm{PSL}_2(\mathbb{C})$ a quasi-Fuchsian group. The quasi-Fuchsian space is identified with $\mathcal{T}(M)\times\mathcal{T}(M)$ where $\mathcal{T}(M)$ is the Teichmüller space of M. Fixing $h_-=h_0$, the map $h_+\mapsto g_h$ provides an embedding of $\mathcal{T}(M)$ into the quasi-Fuchsian deformation space, called Bers embedding, and we view the renormalized volume as a function on $\mathcal{T}(M)$: $V_{h_0}: h_+\mapsto \mathrm{Vol}_R(X,g_h)$.

Proof of Theorem 1.3. First, we notice that applying the proof of Proposition 7.1 in [14], mutatis mutandis, we can compute the Hessian of the renormalized volume at the Fuchsian locus $h_+=h_-$:

$$\mathrm{Hess}_{h_0}(V_{h_0})(k)=\frac{1}{8}\int_M |k|^2_{h_0}\mathrm{dvol}_{h_0}=\frac{1}{8}|k|^2_{\mathrm{WP}}. \tag{7.5}$$

By Theorem 7.2 and the form $x^{-2}(dx^2+(1+\frac{x^2}{2})^2h_0)$ of the quasi-Fuchsian metric for $h_-=h_+=h_0$, the metric $h_+=h_0$ is a critical point of V_{h_0} and a direct computation (as in [18, Section 8]) shows that for $k_1,k_2\in T\mathcal{T}_{h_0}(M)$

$$\overline{\partial}\partial V_{h_0}(h_0).(k_1,k_2)=\frac{i}{4}\Big(\mathrm{Hess}_{h_0}(V_{h_0})(Jk_1,k_2)-\mathrm{Hess}_{h_0}(V_{h_0})(k_1,Jk_2)\Big)$$

if J is the complex structure on $T\mathcal{T}(M)$. Combining with (7.5), we obtain that $\overline{\partial}\partial V_{h_0}(h_0)=\frac{i}{16}\omega_{\mathrm{WP}}(h_0)$.

To obtain the final result we need to show that $\overline{\partial}\partial V_{h_-}(h_0)$ does not depend on h_-. This follows from quasi-Fuchsian reciprocity like in [18, Proposition 8.9]. For the convenience of the reader, we briefly repeat the argument. Thus, consider the Lie derivative $\mathcal{L}_{Y_-}(d\partial V_{h_-}(h_+))$ with respect to the variable h_-, where Y_- is a vector field on $\mathcal{T}(M)$ and h_+ is fixed. To prove it vanishes, it suffices to show that $\mathcal{L}_{Y_-}(\partial V_{h_-}(h_+))$ is the exterior derivative of a function of h_+. We claim that

$\mathcal{L}_{Y_-}(\partial V_{h_-}(h_+)) = dF_{(h_-,Y_-)}(h_+)$ where $F_{(h_-,Y_-)}$ is the function on $\mathcal{T}(M)$ defined by

$$F_{(h_-,Y_-)}(h_+) = \partial \mathrm{Vol}_R(h_-, h_+).(Y_-, 0).$$

To prove this, we will need to use the Bers embedding of Teichmüller space and especially its relation to Vol_R. The universal cover of M is the upper half plane $U = \mathbb{H}^2 \subset \mathbb{C}$ and (after composing by an isometry) the metric h_+ lifts to the hyperbolic metric $g_{\mathbb{H}^2} = \frac{|dz|^2}{\mathrm{Im}(z)^2}$. The covering map is denoted $\pi : \mathbb{H}^2 \to M$, the fundamental group $\pi_1(M)$ is represented by a Fuchsian co-compact group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$. The metric h_- on M lifts by π to a metric $\widetilde{h}_-$ on $\mathbb{H}^2$ which is Γ-invariant and of curvature -1. Using the map $z \mapsto \bar{z}$, we can equip the lower half-plane $L = \{\mathrm{Im}(z) < 0\} \subset \mathbb{C}$ with the metric $\widetilde{h}_-$ (the orientation of h_- is then reversed), and this metric can be written as $\widetilde{h}_- = a(z)|dz + \mu d\bar{z}|^2$ for some smooth $a > 0$ and some complex valued Beltrami coefficient μ with $|\mu| < 1$. Extend μ by 0 on U, then by Ahlfors-Bers result, there is a unique quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ which satisfies

$$\partial_{\bar{z}} f = \mu \partial_z f$$

and f fixes the points $0, 1, \infty$. Notice that f is a conformal map from $(\mathbb{C}, |dz + \mu d\bar{z}|^2)$ to $(\mathbb{C}, |dz|^2)$, and thus f is holomorphic in U. The Bers embedding is the map

$$\Theta_{h_+} : h_- \mapsto S(f|_U), \quad S(f) = \left(\partial_z \left(\frac{\partial_z^2 f}{\partial_z f} \right) - \frac{1}{2} \left(\frac{\partial_z^2 f}{\partial_z f} \right)^2 \right) dz^2.$$

where S is the Schwarzian derivative. The element $S(f)$ is a holomorphic quadratic differential with respect to the complex structure of h_+ on U, which is Γ-equivariant, thus descends to an element in $(T^*_{h_+}\mathcal{T})^{1,0}$ if $\mathcal{T}$ denotes the Teichmüller space of M. The Bers map Θ_{h_+} is holomorphic as a map $\mathcal{T} \to (T^*_{h_+}\mathcal{T})^{1,0}$. The group $\Gamma' := f\Gamma f^{-1}$ is a quasi-Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Let $J = f^{-1}$, where now we consider $f : U \to \Omega_+$. Here Ω_+ is the upper component of the domain of discontinuity of Γ' on $\mathbb{C}$ and $\Gamma'\backslash\Omega_+$, equipped with the complex structure induced by $\mathbb{C}$, is conformal to (M, h_+) by f. One has moreover $J^* g_{\mathbb{H}^2} = e^{\phi}|dz|^2$ for some smooth Liouville field ϕ on Ω_+, Γ'-equivariant, and $e^{\phi}|dz|^2$ is a hyperbolic metric on Ω_+. Thus

$$\partial_z \partial_{\bar{z}} \phi = \frac{1}{2} e^{\phi}.$$

We have also $J^* S(J) = -S(f)$ and we would like to express $S(f)$ in terms of the Liouville field ϕ. Remark that

$$|\partial_z J|^2 = e^{\phi}(\mathrm{Im}(J(z)))^2,$$

thus

$$\partial_z \phi = \frac{\partial_z^2 J}{\partial_z J} + i \frac{\partial_z J}{\mathrm{Im} J}.$$

Now we compute

$$\partial_z^2\phi - \frac{1}{2}(\partial_z\phi)^2 = S(J) + i\frac{\partial_z^2 J}{\mathrm{Im}J} - \frac{1}{2}\frac{(\partial_z J)^2}{(\mathrm{Im}J)^2} - \frac{1}{2}\left(i\frac{\partial_z J}{\mathrm{Im}J}\right)^2 - i\frac{\partial_z^2 J}{\partial_z J}.\frac{\partial_z J}{\mathrm{Im}J} \quad (7.6)$$
$$= S(J)$$

and thus $\Theta_{h_+}(h_-) = -J^*((\partial_z^2\phi - \frac{1}{2}(\partial_z\phi)^2)dz^2)$. Next we can use Epstein's description of the equidistant foliation in [9], combined with Theorem 7.2, which show that $\partial V_{h_-}(h_+) = \Theta_{h_+}$: if we lift the quasi-Fuchsian hyperbolic metric to $\mathbb{H}^3$ (in the half-space model $\mathbb{H}^3 = (0,\infty)_x \times \mathbb{C}_z$), the geodesic boundary defining function ρ associated to h_+ and the equidistant foliation given by the level sets $\{\rho = \mathrm{const}\}$ also lifts to $\mathbb{H}^3$, the lift of the boundary metric h_+ is given by $e^\phi dz^2$ on the domain of discontinuity $\Omega_+ \subset \mathbb{C} = \partial\mathbb{H}^3$, and [9, formula (5.5)] gives near Ω_+ as $\rho \to 0$

$$g_{\mathbb{H}^3} = \frac{d\rho^2 + e^\phi|dz|^2 + (\mathrm{Re}((\partial_z^2\phi - \frac{1}{2}(\partial_z\phi)^2)dz^2) + \partial_z\partial_{\bar z}\phi|dz|^2)\rho^2 + \mathcal{O}(\rho^4)}{\rho^2}.$$

This implies that $dV_{h_-}(h_+) = \frac{1}{4}\,\mathrm{Re}(\Theta_{h_+}(h_-))$ and thus we obtain

$$\partial V_{h_-}(h_+) = \frac{1}{4}\Theta_{h_+}(h_-)$$

by using (7.6). The same holds by reversing the role of h_- and h_+. On the other hand, if $\Theta(h_-, h_+) := \Theta_{h_+}(h_-)$, one has for any section $Y_\pm$ of $T\mathcal{T}(M)$ (here $Y_\pm$ depends only on the $h_\pm$ variable),

$$\begin{aligned}
\mathrm{Re}\langle\mathcal{L}_{Y_-}\Theta_{h_+}, Y_+\rangle &= \mathrm{Re}(\mathcal{L}_{(Y_-,0)}\Theta)_{(h_-,h_+)}(0, Y_+)\\
&= 4\mathcal{L}_{(Y_-,0)}d\mathrm{Vol}_R(h_-, h_+).(0, Y_+)\\
&= 4\nabla^2\mathrm{Vol}_R(h_-, h_+).((Y_-, 0), (0, Y_+)) \qquad (7.7)\\
&= 4\mathcal{L}_{Y_+}(d\mathrm{Vol}_R(h_-, h_+).(Y_-, 0))\\
&= \mathrm{Re}\langle\mathcal{L}_{Y_+}\Theta_{h_-}, Y_-\rangle.
\end{aligned}$$

Since $\Theta_{h_\pm}$ is a family of holomorphic differentials on $\mathcal{T}(M)$ that depends holomorphically on $h_\pm$, we see that (7.7) in fact implies the quasi-Fuchsian reciprocity

$$\langle\mathcal{L}_{Y_-}\Theta_{h_+}, Y_+\rangle = \langle\mathcal{L}_{Y_+}\Theta_{h_-}, Y_-\rangle. \quad (7.8)$$

Coming back to the renormalized volume, this finally yields

$$\begin{aligned}
4\langle\mathcal{L}_{Y_-}(\partial V_{h_-}(h_+)), Y_+\rangle &= \langle\mathcal{L}_{Y_-}\Theta_+, Y_+\rangle = \langle\mathcal{L}_{Y_+}\Theta_-, Y_-\rangle\\
&= 4\langle\mathcal{L}_{Y_+}(\partial V_{h_+}(h_-)), Y_-\rangle \qquad (7.9)\\
&= 4\langle dF_{(h_-,Y_-)}(h_+), Y_+\rangle
\end{aligned}$$

as claimed. □

8. Limit of the renormalized volume under the formation of a rank-1 cusp

We consider an admissible degeneration of convex co-compact hyperbolic metrics g_ε on a manifold X in the sense of Definition 6.1; X is thus the interior of a smooth compact manifold $\overline{\mathbf{X}}$ with boundary $N := \partial\overline{\mathbf{X}}$, with degenerating curve $H = \cup_{j=1}^{j_1} H_j \subset N$ and $\overline{X} = \overline{\mathbf{X}} \setminus H$. Recall that $\overline{X}_c$ is the smooth manifold with corners obtained by blowing-up H in $\overline{\mathbf{X}}$, with boundary faces $\overline{M}$ and cf, see Section 2.3. The goal of this Section is to show

Theorem 8.1. *Let g_ε be an admissible degeneration of convex co-compact hyperbolic metrics on X in the sense of Definition 6.1, with limiting geometrically finite hyperbolic metric g_0. Then*

$$\lim_{\varepsilon\to 0} \mathrm{Vol}_R(X, g_\varepsilon) = \mathrm{Vol}_R(X, g_0).$$

8.1. Limit far from the cusp

First we describe the limit of the renormalized volume of the part far from the cusp, that is in a fixed compact region $\mathcal{K} \subset \overline{X}$.

Proposition 8.2. *Let $\rho_\varepsilon \in \mathcal{C}^\infty(\overline{\mathbf{X}})$ be a geodesic boundary defining function such that $h_\varepsilon := (\rho_\varepsilon^2 g_\varepsilon)|_N$ is the unique hyperbolic metric in the conformal boundary (ρ_ε is uniquely defined near N). Let $\rho_0 \in \mathcal{C}^\infty(\overline{X}_c)$ be a geodesic boundary defining function of $\overline{M}$ of Proposition 2.4 with $h_0 := (\rho_0^2 g_0)|_{TM}$ being the unique finite volume hyperbolic metric in the conformal boundary (ρ_0 is uniquely defined near $\overline{M}$). Let θ_ε be a family of smooth functions on $\overline{\mathbf{X}}$ vanishing in a uniform neighborhood of the degenerating curve H and converging in all $\mathcal{C}^k$-norms to θ. The following limit holds*

$$\lim_{\varepsilon\to 0}\left(\mathrm{FP}_{z=0}\int_X \theta_\varepsilon \rho_\varepsilon^z \,\mathrm{dvol}_{g_\varepsilon}\right) = \mathrm{FP}_{z=0}\int_X \theta\rho_0^z \,\mathrm{dvol}_{g_0}.$$

Proof. Let $\mathcal{K}$ be a compact neighborhood of $\mathrm{supp}\,\theta$. First, we can write $\mathrm{dvol}_{g_\varepsilon} = e^{G_\varepsilon}\mathrm{dvol}_{g_0}$ for some smooth function G_ε converging to 0 in $\mathcal{C}^\infty(\mathcal{K})$. We use the notation of Section 6.2: the geodesic boundary defining function $\hat\rho_\varepsilon$ in $\mathcal{K}$ is defined by (6.5). Then we get

$$\int_X \theta_\varepsilon \hat\rho_\varepsilon^z \,\mathrm{dvol}_{g_\varepsilon} - \int_X \theta\rho_0^z \,\mathrm{dvol}_{g_0} = \int_X \rho_0^z \left(\theta_\varepsilon e^{G_\varepsilon + z\hat\omega_\varepsilon} - \theta\right)\mathrm{dvol}_{g_0} \tag{8.1}$$

where $\hat\rho_\varepsilon = e^{\hat\omega_\varepsilon}\rho_0$, and $\hat\omega_\varepsilon$ and $\theta_\varepsilon - \theta$ converge to 0 in $\mathcal{C}^\infty(\mathcal{K})$ by Lemma 6.3. Now the volume form of g_0 near $\rho_0 = 0$ is of the form $\rho_0^{-3} e^H d\rho_0 d\mu$ where $d\mu$ is a smooth measure on $\mathcal{K}\cap M$ and H a smooth function of $\mathcal{K}$, thus writing

$$e^{z\hat\omega_\varepsilon} = 1 + z\hat\omega_\varepsilon + z^2 F_\varepsilon$$

for some smooth function F_ε on $\mathbb{C}_z \times \mathcal{K}$ and using that for small $\delta > 0$, $\int_0^\delta \rho^{z-1} d\rho_0$ has a pole of order 1 at $z = 0$ with residue 1, we directly obtain that

$$\mathrm{FP}_{z=0} \int_X \rho_0^z \left(\theta_\varepsilon e^{G_\varepsilon + z\hat{\omega}_\varepsilon} - \theta\right) \mathrm{dvol}_{g_0} = \mathrm{FP}_{z=0} \int_X \rho_0^z (\theta_\varepsilon e^{G_\varepsilon} - \theta) \,\mathrm{dvol}_{g_0} + \frac{1}{2} \int_{\mathcal{K}\cap M} \partial_{\rho_0}^2 \left(\theta_\varepsilon e^{G_\varepsilon + H} \hat{\omega}_\varepsilon\right) |_{\rho_0=0} \, d\mu$$

where ∂_{ρ_0} is the vector field given by the gradient of ρ_0 with respect to $\rho^2 g_0$. Using that $G_\varepsilon \to 0$, $\hat{\omega}_\varepsilon \to 0$ and $\theta_\varepsilon \to \theta$ in $\mathcal{C}^\infty(\mathcal{K})$, as $\varepsilon \to 0$, we obtain that the finite part of (8.1) at $z = 0$ converges to 0 as $\varepsilon \to 0$. We write $h_\varepsilon = e^{2\varphi_\varepsilon} \hat{h}_\varepsilon$. To conclude, we may use Proposition 7.1, which of course also works in the convex co-compact case: that is for each $\varepsilon > 0$, we get with $\theta_\varepsilon = \sum_{k=0}^2 \theta_{\varepsilon,k} \rho_\varepsilon^k + \mathcal{O}(\rho_\varepsilon^3)$ for some $\theta_{\varepsilon,k} \in \mathcal{C}_0^\infty(M)$

$$\mathrm{FP}_{z=0} \int_X \hat{\rho}_\varepsilon^z \theta_\varepsilon \,\mathrm{dvol}_{g_\varepsilon} = \mathrm{FP}_{z=0} \int_X \rho_\varepsilon^z \theta_\varepsilon \,\mathrm{dvol}_{g_\varepsilon} - \frac{1}{4} \int_{\mathcal{K}\cap M} \left(\theta_{\varepsilon,k} \left(|d\varphi_\varepsilon|_{h_\varepsilon}^2 + \mathrm{Scal}_{h_\varepsilon} \varphi_\varepsilon\right) - 4\theta_{\varepsilon,k}\varphi_\varepsilon\right) \mathrm{dvol}_{h_\varepsilon}.$$

By assumption we have $\theta_{\varepsilon,k} \to \theta_k$ with $\theta = \sum_{k=0}^2 \theta_k \rho_0^k + \mathcal{O}(\rho_0^3)$. Using Proposition 5.1 and Corollary 5.3 we directly obtain that (recall that $\varphi_0 = 0$)

$$\lim_{\varepsilon\to 0} \int_{\mathcal{K}\cap M} \left(\theta_{\varepsilon,0}(|d\varphi_\varepsilon|_{h_\varepsilon}^2 + \mathrm{Scal}_{h_\varepsilon}\varphi_\varepsilon) - 4\theta_{\varepsilon,2}\varphi_\varepsilon\right) \mathrm{dvol}_{h_\varepsilon} = 0,$$

which achieves the proof since (8.1) has finite part at $z = 0$ tending to 0. □

8.2. Limit near the cusp

We next study the behaviour of the renormalized volume in the regions $\mathcal{U}_j^\varepsilon$ containing the degeneration. We notice that Theorem 8.1 follows from Propositions 8.2 and the following

Proposition 8.3. *With the notations and assumptions of Proposition 8.2 and Theorem 8.1, we have*

$$\lim_{\varepsilon\to 0} \mathrm{FP}_{z=0} \int_X (1-\theta^\varepsilon)\rho_\varepsilon^z \mathrm{dvol}_{g_\varepsilon} = \mathrm{FP}_{z=0} \int_X (1-\theta)\rho_0^z \mathrm{dvol}_{g_0}.$$

Proof. We can assume that $(1-\theta^\varepsilon)$ is supported in $\cup_j \mathcal{U}_j^\varepsilon$, we are reduced to a local analysis and we can use the model $\overline{\mathcal{U}}_\ell$ with metric g_ℓ of Section 6.3, where we have forgot the ε parameter and use rather ℓ with $\ell \to 0$, and $\nu = \nu(\ell)$ is converging to some limit ν_0 as $\ell \to 0$. First, an easy computation gives that the volume form of g_ℓ is given by

$$\mathrm{dvol}_{g_\ell} = \frac{R^2 du dv dw}{u^3}$$

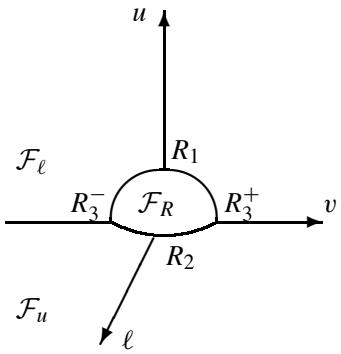

Figure 8.1. The manifold with corners $\overline{\mathcal{U}}_\ell$

where $R^2 = u^2 + v^2 + \ell^2$. We need to prove that

$$\lim_{\ell\to 0} \mathrm{FP}_{z=0} \int_{(u,v,w,\ell)\in\overline{\mathcal{U}}_\ell} \rho_\ell^z \chi \frac{R^2 du dv dw}{u^3} = \mathrm{FP}_{z=0} \int \rho_0^z \chi \frac{R^2 du dv dw}{u^3} \tag{8.2}$$

where $\rho_\ell = \rho_\varepsilon$ is the function solving (6.13) with $e^{2\varphi_\ell} h_\ell$ being hyperbolic if h_ℓ is given by (6.11), and $\chi \in \mathcal{C}_c^\infty(\overline{\mathcal{U}}_\ell)$ is independent of ℓ and equal to 1 near $u = v = 0$. To study the renormalized integral (8.2) we decompose $\overline{\mathcal{U}}_\ell$ in several regions, see Figure 8.1.

We start with a region of finite volume (with the notations of Section 6.3)

$$R_1(\ell) = \{(u, v, w) \mid u \le \delta,\ -1 \le V \le 1,\ 0 < L \le 1\},$$

where we use the following coordinates,

$$u, \quad V = \frac{v}{u}, \quad L = \frac{\ell}{u}, \quad w. \tag{8.3}$$

In fact, for $\ell > 0$ fixed, we have that

$$0 \le L \le 1,\ 0 \le u \le \delta \implies \ell \le u \le \delta.$$

Take δ so that χ is supported in $\sqrt{u^2 + v^2} \le \delta$. In these coordinates, the volume form of g_ℓ is for ℓ fixed given by

$$\mathrm{dvol}_{g_\ell} = \frac{(\ell^2 + u^2 + v^2) du dv dw}{u^3} = (1 + V^2 + L^2) du dV dw.$$

Restricted to this region, the volume is thus clearly finite and there is no need to renormalize. Thus,

$$\mathrm{FP}_{z=0} \int_{R_1(\ell)} \rho_\ell^z \chi \frac{R^2 du dv dw}{u^3} = \int_{-\frac{1}{4}}^{-\frac{1}{4}} \int_0^1 \int_\ell^\delta \chi(u, Vu, w) \left(1 + V^2 + \frac{\ell^2}{u^2}\right) du dV dw.$$

We can use dominated convergence (using $L^2\ \mathbb{1}_{[\ell,\delta]}(u) \leq \mathbb{1}_{[0,\delta]}$) to deduce that

$$\begin{aligned}
&\lim_{\ell\to 0} \mathrm{FP}_{z=0} \int_{(u,V,L,w)\in R_1} \rho_\ell^z \chi \frac{R^2 du dv dw}{u^3} \\
&= \int_{-\frac{1}{4}}^{-\frac{1}{4}} \int_0^1 \int_0^\delta \chi(u, Vu, w)\left(1+V^2\right) du dV dw \qquad (8.4)\\
&= \mathrm{FP}_{z=0} \int_{R_1(0)} \rho_0^z \chi \frac{(u^2+v^2) du dv dw}{u^3}.
\end{aligned}$$

Next we analyze the region $R_2(\ell)$ near the intersection $\mathcal{F}_u \cap \mathcal{F}_R$ but away from the corners $\mathcal{F}_R \cap \mathcal{F}_u \cap \mathcal{F}_\ell$. In this region, we can use the coordinates

$$\ell, \quad \widetilde{U} = \frac{u}{\ell}, \quad \widetilde{V} = \frac{v}{\ell}, \quad w.$$

In these coordinates, we can define more precisely the region $R_2(\ell)$ by

$$R_2(\ell) = \{(u, v, w) \mid\ 0 \leq \widetilde{U} \leq 1,\ -1 \leq \widetilde{V} \leq 1\}$$

In these coordinates, the volume form of g_ℓ is given (for ℓ fixed) by

$$\mathrm{dvol}_{g_\ell} = \frac{\ell(1+\widetilde{U}^2+\widetilde{V}^2) d\widetilde{U} d\widetilde{V} dw}{\widetilde{U}^3}.$$

Since $U := \frac{u}{R} = \frac{\widetilde{U}}{\sqrt{1+\widetilde{U}^2+\widetilde{V}^2}}$ and $\rho_\ell = e^{\omega_\ell} U$ with the notation of (6.13), we have

$$\begin{aligned}
&\mathrm{FP}_{z=0} \int_{R_2(\ell)} \chi \rho_\ell^z \frac{R^2 du dv dw}{u^3} \\
&= \mathrm{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^1 \int_0^1 \rho_\ell^z \chi \frac{\ell(1+\widetilde{U}^2+\widetilde{V}^2) d\widetilde{U} d\widetilde{V} dw}{\widetilde{U}^3} \\
&= \mathrm{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^1 \int_0^1 \chi \frac{\ell \widetilde{U}^z e^{z\omega_\ell}(1+\widetilde{U}^2+\widetilde{V}^2) d\widetilde{U} d\widetilde{V} dw}{(1+\widetilde{U}^2+\widetilde{V}^2)^{\frac{z}{2}} \widetilde{U}^3} \\
&= A_1(\ell) + A_2(\ell) + A_3(\ell)
\end{aligned}$$

with

$$\begin{aligned}
A_1(\ell) &:= \mathrm{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^1 \int_0^1 \chi \frac{\ell \widetilde{U}^z (1+\widetilde{U}^2+\widetilde{V}^2) d\widetilde{U} d\widetilde{V} dw}{\widetilde{U}^3}, \\
A_2(\ell) &:= \mathrm{res}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^1 \int_0^1 \chi \frac{\ell \widetilde{U}^z \omega_\ell (1+\widetilde{U}^2+\widetilde{V}^2) d\widetilde{U} d\widetilde{V} dw}{\widetilde{U}^3}, \\
A_3(\ell) &:= -\frac{1}{2} \mathrm{res}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^1 \int_0^1 \chi \frac{\ell \widetilde{U}^z \log(1+\widetilde{U}^2+\widetilde{V}^2)(1+\widetilde{U}^2+\widetilde{V}^2) d\widetilde{U} d\widetilde{V} dw}{\widetilde{U}^3}.
\end{aligned}$$

For $j = 0, 1$, the function $\ell\chi(\ell\widetilde{U}, \ell\widetilde{V}, w)(1 + \widetilde{U}^2 + \widetilde{V}^2)(\log(1 + \widetilde{U}^2 + \widetilde{V}^2))^j$ converges to 0 in $\mathcal{C}^k$-norms for all k, and thus it is direct to see $\lim_{\ell\to 0} A_1(\ell) = \lim_{\ell\to 0} A_3(\ell) = 0$. For the second term, we use the Taylor expansion of ω_ℓ in terms of $\widetilde{U}$ using (6.25):

$$\omega_\ell = a_0 + a_2 U^2 + \mathcal{O}(U^3) = a_0 + \frac{a_2\widetilde{U}^2}{1+\widetilde{V}^2} + \mathcal{O}(\widetilde{U}^3).$$

Thus, we compute that

$$\begin{aligned} A_2(\ell) &= \ell \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^{1} \Big((a_0 + a_2)(\chi(0, \ell\widetilde{V}, w) + a_0\ell^2(1+\widetilde{V}^2)\partial_u^2\chi(0, \ell\widetilde{V}, w) \Big) dV dw \\ &= \ell \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-1}^{1} \chi(0, \ell\widetilde{V}, w)(a_0 + a_2) dV dw + \mathcal{O}(\ell^3). \\ &= \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\ell}^{\ell} \chi(0, v, w) \Big(\varphi_\ell - \frac{1}{4}|d\varphi_\ell|^2_{h_\ell} \\ &\qquad + \frac{C_1\ell^2 + C_2 v\partial_w\varphi_\ell}{(\ell^2+v^2)} + C_3 v\partial_v\varphi_\ell + \frac{1}{2} \Big) dv dw, \end{aligned} \tag{8.5}$$

where C_j are constant depending smoothly on ν, and we used that $a_0 = \varphi_\ell$ is uniformly bounded in ℓ in the second line. From Proposition 5.1 and Corollary 5.3, we see that

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\ell}^{\ell} \chi(0, v, w) \left(\varphi_\ell - \frac{1}{4}|d\varphi_\ell|^2_{h_\ell} + \frac{C_1\ell^2}{(\ell^2+v^2)} \right) dv dw \to 0.$$

Using Cauchy-Schwartz and $|d\varphi_\ell|_{h_\ell} \geq C(|(v^2+\ell^2)^{-\frac{1}{2}}\partial_w\varphi_\ell| + |v\partial_v\varphi_\ell|)$ we also get that

$$\begin{aligned} &\int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\ell}^{\ell} |\chi(0, v, w)| \left(\frac{C_2|v\partial_w\varphi_\ell|}{(\ell^2+v^2)} + C_3|v\partial_v\varphi_\ell| \right) dv dw \\ &\leq C' \Big(\sqrt{\ell}||d\varphi_\ell||_{L^2} + \left(\int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\ell}^{\ell} |d\varphi_\ell|^2_{h_\ell} dv dw \right)^{\frac{1}{2}} \Big) \end{aligned}$$

for some C' independent of ℓ, thus this converges to 0 by Corollary 5.3, and we conclude that $\lim_{\ell\to 0} A_2(\ell) = 0$ and

$$\lim_{\ell\to 0} \mathrm{FP}_{z=0} \int_{R_2(\ell)} \chi\rho_{\tilde{\ell}}^z \frac{R^2 du dv dw}{u^3} = 0.$$

Next we consider the coordinates, smooth near the corners $\mathcal{F}_R \cap \mathcal{F}_u \cap \mathcal{F}_\ell$,

$$v, \widehat{U} = \frac{u}{|v|}, \hat{L} = \frac{\ell}{|v|}, w$$

and take the region $R_3(\ell) \cup R_4(\ell)$ given by

$$R_3(\ell) = \left\{(u, v, w) \mid |v| \leq \delta,\ \hat{L} \leq 1, \widehat{U} \leq 1\right\}.$$

We see that χ can written as $\sum_{j=1}^4 \chi\ \mathbb{1}_{R_j(\ell)}$. In these coordinates, the volume form of g_ℓ is given for fixed ℓ by

$$\mathrm{dvol}_{g_\ell} = \frac{\left(1 + \hat{L}^2 + \widehat{U}^2\right) d\widehat{U} dv dw}{\widehat{U}^3}.$$

Thus, since $U = \frac{\widehat{U}}{\sqrt{1+\hat{L}^2+\widehat{U}^2}}$, we have

$$\begin{aligned}
&\mathrm{FP}_{z=0} \int_{R_3(\ell)} \chi \rho_\ell^z \frac{R^2 du dv dw}{u^3} \\
&= \mathrm{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \int_0^1 \chi \frac{\hat{U}^z e^{z\omega_\ell} \left(1 + \frac{\ell^2}{v^2} + \widehat{U}^2\right) d\widehat{U} dv dw}{\left(1 + \frac{\ell^2}{v^2} + \hat{U}^2\right)^{z/2} \widehat{U}^3} \qquad (8.6) \\
&= I_1(\ell) + I_2(\ell) + I_3(\ell).
\end{aligned}$$

with

$$\begin{aligned}
I_1(\ell) &:= \mathrm{FP}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \int_0^1 \chi \frac{\widehat{U}^z \left(1 + \frac{\ell^2}{v^2} + \widehat{U}^2\right) d\widehat{U} dv dw}{\widehat{U}^3}, \\
I_2(\ell) &:= \mathrm{res}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \int_0^1 \chi \frac{\widehat{U}^z \omega_\ell \left(1 + \frac{\ell^2}{v^2} + \widehat{U}^2\right) d\widehat{U} dv dw}{\widehat{U}^3}, \\
I_3(\ell) &:= -\frac{1}{2} \mathrm{res}_{z=0} \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \int_0^1 \chi \log\left(1 + \widehat{U}^2 + \frac{\ell^2}{v^2}\right) \\
&\qquad \cdot \frac{\widehat{U}^z \left(1 + \frac{\ell^2}{v^2} + \widehat{U}^2\right) d\widehat{U} dv dw}{\widehat{U}^3}.
\end{aligned}$$

We notice that, in view of the smoothness of ω_ℓ as a function of U, v, w, these three terms also make sense for $\ell = 0$, and (8.6) for $\ell = 0$ is given by $\sum_{j=1}^3 I_j(0)$. To

conclude the proof, we want to prove that $I_j(\ell) \to I_j(0)$ as $\ell \to 0$ for $j = 1, 2, 3$. For the first term, we compute that

$$I_1(\ell) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \left(\left(1 + \frac{\ell^2}{v^2} \right) q_1(v, w) + q_2(v, w) \right) dv dw, \tag{8.7}$$

where q_1 and q_2 are smooth and independent of ℓ, and it is then clear that

$$\lim_{\ell \to 0} I_1(\ell) = I_1(0)$$

To deal with $I_3(\ell)$, we can proceed similarly: we remark that for $\ell \geq 0$, the integrand in $I_3(\ell)$ is of the form $\hat{U}^{z-3} Q(\hat{U}, \frac{\ell^2}{v^2}, v, w)$ where Q is some smooth function of its parameters, thus it is straightforward to see that

$$I_3(\ell) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} q_3 \left(v, \frac{\ell^2}{v^2}, w \right) dv dw$$

for some smooth function q_3 of its parameters. We conclude as in the case of I_1 that

$$\lim_{\ell \to 0} I_3(\ell) = I_3(0).$$

Finally we study $I_2(\ell)$. From the expansion (6.25), we have for $\ell \geq 0$ that

$$\omega_\ell = a_0 + a_2 U^2 + \mathcal{O}(U^3) = a_0 + \frac{a_2 \widehat{U}^2}{1 + \frac{\ell^2}{v^2}} + \mathcal{O}(\widehat{U}^3).$$

Hence, we compute that for $\ell \geq 0$

$$\begin{aligned} I_2(\ell) &= \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \Big((a_0 + a_2)(\chi(0, v, w) + a_0 (v^2 + \ell^2) \partial_u^2 \chi(0, v, w) \Big) dv dw \\ &= \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \chi(0, v, w) \Big(\varphi_\ell - \frac{1}{4} |d\varphi_\ell|^2_{h_\ell} \\ &\qquad + \frac{C_1 \ell^2 + C_2 v \partial_w \varphi_\ell}{(\ell^2 + v^2)} + C_3 v \partial_v \varphi_\ell + \frac{1}{2} \Big) dv dw \\ &\quad + \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\ell \leq |v| \leq \delta} \varphi_\ell (v^2 + \ell^2) \partial_u^2 \chi(0, v, w) dv dw. \end{aligned}$$

for some constant C_j depending smoothly on ν. By Proposition 5.1, the last line is continuous at $\ell = 0$, and using Corollary 5.6 with the stronger estimate (5.9), it is direct to check (like we did for the term $A_2(\ell)$) that $I_2(\ell)$ is continuous at $\ell = 0$, *i.e.*, $\lim_{\ell \to 0} I_2(\ell) = I_2(0)$. □

9. Appendix

Proof of Proposition 4.2.

We will construct Φ_L in two steps, as a composition $\Phi_L = \Xi_L \circ \Upsilon_L$. Let us first construct the diffeomorphism Υ_L, which is done by changing coordinates on $X_{m(q)}$.

If $r = \sqrt{x^2 + |z|^2}$ is the Euclidean radial coordinate in $\mathbb{H}^3 = \mathbb{R}^+_x \times \mathbb{H}^2_z$, then the hyperbolic metric takes the form in the Euclidean radial coordinates (r, ω) with $\omega \in \mathbb{S}^2$

$$g_{\mathbb{H}^3} = \frac{dr^2 + r^2 g_{\mathbb{S}^2}}{r^2 \omega_x^2}$$

where $r\omega_x = x$ and $\omega_x = x(\omega)$ is the vertical coordinate on the sphere. We denote by $\omega_1 = \mathrm{Re}(z(\omega))$ and $\omega_2 = \mathrm{Im}(z(\omega))$ the coordinates of ω in the horizontal direction z. Consider the stereographic projection $\mathbb{S}^2 \to \mathbb{R}^2$ from the point $(x, z) = (0, -1) \in \mathbb{S}^2 \subset \mathbb{R}^3$, providing coordinates $\hat{u}, \hat{v} \in \mathbb{R}^2$ so that

$$\hat{u} = \frac{\omega_x}{\omega_1 + 1}, \quad \hat{v} = \frac{\omega_2}{\omega_1 + 1}, \text{ and the metric } g_{\mathbb{S}^2} = \frac{4(d\hat{u}^2 + d\hat{v}^2)}{(1 + \hat{u}^2 + \hat{v}^2)^2}.$$

In the coordinates $(r, \hat{u}, \hat{v}) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, the hyperbolic metric takes the form

$$g_{\mathbb{H}^3} = \frac{(1 + \hat{u}^2 + \hat{v}^2)^2 dr^2}{4\hat{u}^2 r^2} + \frac{d\hat{u}^2 + d\hat{v}^2}{\hat{u}^2}.$$

Notice that $\hat{v} + i\hat{u}$ define coordinates on the hyperbolic plane $\mathbb{H}^2$ (viewed as the upper half-space in $\mathbb{C}$), and the stereographic projection is an isometry from the half-sphere $H(0, 1)$ equipped with the metric induced from $\mathbb{H}^3$ to this hyperbolic plane. The action $z \mapsto qz = e^{\ell(1+i\nu)}z$ in $\mathbb{C}$ corresponds in $\mathbb{H}^3$ to a dilation by e^ℓ centered at $(x, z) = (0, 0)$ followed by a hyperbolic rotation $R_{\mathbb{H}^3}(\nu\ell, x)$ of angle $\nu\ell$ around the x axis in $\mathbb{H}^3 = \mathbb{R}^+_x \times \mathbb{C}_z$. The latter is an elliptic isometry for $g_{\mathbb{H}^3}$ and so its restriction to $H(0, 1)$ becomes an elliptic isometry of the hyperbolic half-plane $\mathbb{H}^2$ with coordinate $z = \hat{v} + i\hat{u}$, fixing the point $z = i$, and considering the derivative at this point shows that $R_{\mathbb{H}^3}(\nu\ell, x)|_{H(0,1)}$, viewed in the variable $z = \hat{v} + i\hat{u} \in \mathbb{H}^2$ via the stereographic projection, acts as the hyperbolic rotation of angle $\nu\ell$ and center $z = i \in \mathbb{H}^2$. We denote by

$$R_{\nu\ell} = \begin{pmatrix} \cos\frac{\nu\ell}{2} & \sin\frac{\nu\ell}{2} \\ -\sin\frac{\nu\ell}{2} & \cos\frac{\nu\ell}{2} \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

this hyperbolic rotation.

In the quotient (4.6), the fundamental domain is $e^{-\frac{1}{2}\ell} \leq r \leq e^{\frac{1}{2}\ell}$ so to have coordinates with uniform behavior with respect to the deformation parameters ℓ, we introduce the rescaled coordinates

$$u' = \ell\hat{u}, \quad v' = \ell\hat{v}, \quad w = \frac{\log r}{2\ell}.$$

We denote by $\Upsilon_L : (x, z) \mapsto (w, v' + iu')$ the diffeomorphism corresponding to the change of coordinates. In these coordinates, the hyperbolic metric on $e^{-\frac{1}{2}\ell} \leq r \leq e^{\frac{1}{2}\ell}$ takes the form:

$$(\Upsilon_L)_* g_{\mathbb{H}^3} = \frac{du'^2 + dv'^2 + (\ell^2 + u'^2 + v'^2)^2 dw^2}{u'^2},$$

where $w \in [-\frac{1}{4}, \frac{1}{4}]$. Moreover the transformation γ_L becomes in these coordinates

$$(w, v' + iu') \mapsto \left(w + \tfrac{1}{2}, \ell R_{-v\ell}(\ell^{-1}(v' + iu'))\right).$$

The intersection of the half-sphere $\partial B(e(L), \rho(L))$ of (4.6) with the half-sphere $H(0, e^{2\ell w})$ (with $|w| < 1/4$) is the half-circle obtained by intersecting the plane

$$\mathrm{Re}(z) = \kappa(w, \ell, \delta) = \frac{e(L)^2 + e^{4\ell w} - \rho(L)^2}{2e(L)}$$

with $H(0, e^{2\ell w})$. Under the stereographic projection $H(0, e^{2\ell w}) \to \{(x, z); \mathrm{Re}(z) = 0\} = \mathbb{R}^2$ from the point $(x, z) = (0, -e^{2\ell w})$, a small computation shows that it is thus sent to the half circle centered at 0 of radius

$$e^{2\ell w}\sqrt{\frac{e^{2\ell w} + \kappa(w, \ell(L), \delta)}{e^{2\ell w} - \kappa(w, \ell, \delta)}} = \frac{r_\lambda(w)}{\ell} + \mathcal{O}_\delta(1), \quad r_\lambda(w) := \left(\frac{\lambda^2}{2\delta^2} - 4w^2\right)^{-\frac{1}{2}}$$

where we have used (4.7) in the last equality. Consequently, the intersection of the half-ball $B(e(L), \rho(L))$ of (4.6) with the half-sphere $H(0, e^{2\ell w})$ (with $|w| < 1/4$) becomes, in the coordinates $\zeta' = v' + iu' \in \mathbb{H}^2$, a half-disc of the form

$$\mathrm{Im}(\zeta') > 0, \quad |\zeta'| \leq \ell\sqrt{\frac{(e^{2\ell w} - \kappa(w, \ell, \delta))e^{2\ell w}}{e^{2\ell w} + \kappa(w, \ell, \delta)}} = r_\lambda(w) + \mathcal{O}_\delta(\ell). \tag{9.1}$$

and thus, taking δ small enough (independent of ℓ) so that $\lambda/\delta - 4 > 1$ this set is asymptotic to the half-disk

$$\{\zeta' \in \mathbb{C}; \mathrm{Im}(\zeta') > 0, |\zeta'| \leq r_\lambda(w)\}. \tag{9.2}$$

We have thus showed the following:

Lemma 9.1. *There is an isometry Υ_L between $\langle\gamma_L\rangle\backslash\mathbb{H}^3$ and*

$$X_{\gamma_q} := \langle\gamma_q\rangle\backslash\left(\mathbb{R}_w \times \mathbb{H}^2_{\zeta'=v'+iu'}, \ \frac{du'^2 + dv'^2 + (\ell^2 + u'^2 + v'^2)^2 dw^2}{u'^2}\right), \tag{9.3}$$

where γ_q is the map

$$\gamma_q : (w, \zeta') \mapsto \left(w + \tfrac{1}{2}, \frac{\cos(\nu\ell/2)\zeta' + \ell \sin(\nu\ell/2)}{-\ell^{-1}\sin(\nu\ell/2)\zeta' + \cos(\nu\ell/2)} \right).$$

Moreover, if $\delta > 0$ is small enough, the model neighborhood (4.6) *is mapped via Υ_L to*

$$\pi_{\gamma_q}\left(\left\{ (w, \zeta') \in [-\tfrac{1}{4}, \tfrac{1}{4}) \times \mathbb{H}^2; |\zeta'| < r_q(w) \right\} \right) \tag{9.4}$$

where $\pi_{\gamma_q} : \mathbb{R} \times \mathbb{H}^2 \to X_{\gamma_q}$ is the covering map, and $r_q(w)$ is the radius of the half-circle given by equation (9.1) *and converging to $r_\lambda(w) > 0$ with $r_\lambda(w) = \mathcal{O}(\delta)$ uniformly in $|w| < 1/4$.*

Notice that, if $\ell \to 0$, then γ_q converges to some transformation $\gamma_\nu : (w, \zeta') \mapsto \left(w + \frac{1}{2}, P_\nu(\zeta') \right)$ with $P_\nu \in \mathrm{PSL}_2(\mathbb{R})$ the parabolic transformation $\zeta' \mapsto \frac{2\zeta'}{\nu\zeta'+2}$, and X_{γ_q} converges to

$$X_{\gamma_\nu} := \langle \gamma_\nu \rangle \backslash \left(\mathbb{R}_w \times \mathbb{H}^2_{\zeta'=v'+iu'},\ g_0 = \frac{du'^2 + dv'^2 + (u'^2 + v'^2)^2 dw^2}{u'^2} \right).$$

Conjugating by an inversion $\zeta' \mapsto -1/\zeta'$ on $\mathbb{H}^2$, P_ν becomes the transformation $\zeta' \mapsto \zeta' - \nu/2$ and the transformation γ_ν viewed in the coordinates $(w, y + ix)$ defined by $y + ix = -1/(v + iu)$, is the parabolic isometry of $\mathbb{H}^3 = \mathbb{R}_w \times \mathbb{H}^2_{y+ix}$ fixing ∞ and given by $T_\nu : (w, y + ix) \mapsto (w + \frac{1}{2}, y - \frac{\nu}{2} + ix)$. Then X_{γ_ν} is isometric to $\langle T_\nu \rangle \backslash \mathbb{H}^3$, which is the model of a hyperbolic cusp of rank 1. Clearly, the model of Lemma 9.1 extends smoothly to the parabolic boundary $\{\ell = 0\}$ of $\overline{\mathcal{Q}}$.

We also need to control the change of coordinates from the neighborhood $\mathcal{U}_L^\delta$ of (4.5) to this new model when $\ell \to 0$, that is we want to know $\Upsilon_L \circ \Theta_L$. A direct computation gives

$$\begin{aligned} r^2(\Theta_L(x,z)) &= \frac{x^2\lambda^2\ell^2 + |x^2 + |z|^2 - z\lambda\ell|^2}{(x^2 + |z - \lambda\ell|^2)^2}, \\ \omega_x(\Theta_L(x,z)) = \frac{x\lambda\ell}{\eta_L(x,z)}, \quad \omega_1(\Theta_L(x,z)) &= \frac{-x^2 - |z|^2 + \mathrm{Re}(z)\lambda\ell}{\eta_L(x,z)} \\ \omega_2(\Theta_L(x,z)) &= \frac{\mathrm{Im}(z)\lambda\ell}{\eta_L(x,z)} \end{aligned} \tag{9.5}$$

with $\eta_L(x,z) := \sqrt{(x^2 + \mathrm{Im}(z)^2)\lambda^2\ell^2 + (x^2 + |z|^2 - \mathrm{Re}(z)\lambda\ell)^2}$, thus

$$\begin{aligned} u'(\Theta_L(x,z)) &= \frac{x}{\lambda(x^2 + \mathrm{Im}(z)^2)}(\eta_L(x,z) + x^2 + |z|^2 - \mathrm{Re}(z)\lambda\ell), \\ v'(\Theta_L(x,z)) &= \frac{\mathrm{Im}(z)}{\lambda(x^2 + \mathrm{Im}(z)^2)}(\eta_L(x,z) + x^2 + |z|^2 - \mathrm{Re}(z)\lambda\ell). \end{aligned} \tag{9.6}$$

Notice that $\Upsilon_L \circ \Theta_L$ extends smoothly in a neighborhood of the cusp region of $\mathcal{X}$ of the form

$$\mathcal{V}^\delta := \{(L, x, z) \in \overline{\mathcal{Q}} \times \pi_{\gamma_L}(\overline{B(0,\delta)}); (x, z) \in \overline{\widetilde{F}_L} \setminus \{0\}\}.$$

Indeed one has $w(\Theta_L(x,z)) = \frac{\log(r\circ\Theta_L(x,z))}{2\ell}$, and by (9.5) we can write it under the form $w(\Theta_L(x,z)) = \frac{\log(1+\ell F(L,x,z))}{2\ell}$ for some $F(L,x,z)$ smooth in $\mathcal{V}^\delta$ and thus w extends smoothly in $\mathcal{V}^\delta$. It is also easily checked that (u', v') extend smoothly to $\mathcal{V}^\delta$ by (9.6). The inverse also admits a smooth extension to $\{\ell = 0, (u', v') \neq (0,0)\}$ by a similar computation.

To finish the proof of the Proposition, we shall construct a diffeomorphism Ξ_L corresponding to a new change of coordinates. In the $\mathbb{H}^3 = \mathbb{R}_w \times \mathbb{H}^2_{\zeta'=v'+iu'}$ hyperbolic space, we define the function

$$\mu(w, \zeta') := d_{\mathbb{H}^2}(\zeta'; i\ell)$$

which is invariant under the transformation γ_q. One has in particular

$$\cosh(\mu) = \frac{u'^2 + v'^2 + \ell^2}{2u'\ell}.$$

Let us make the following change of coordinates on $[-1/4, 1/4] \times \mathbb{H}^2$, which defines Ξ_L,

$$\Xi_L : (w, \zeta') \mapsto (w, \zeta := \ell R_{-2\nu\ell w}(\ell^{-1}\zeta'))$$

where $R_\theta \in \mathrm{PSL}_2(\mathbb{R})$ is the hyperbolic rotation of angle θ and center i. The transformation γ_q becomes in the (w, ζ) coordinates (*i.e.*, after conjugation with Ξ_L) the transformation

$$\Xi_L \circ \gamma_q \circ (\Xi_L)^{-1} : (w, \zeta) \mapsto (w + \frac{1}{2}, \zeta).$$

We see that Ξ_L extends smoothly to $\{\ell = 0; |\zeta'| < \delta\}$ if δ is small enough, with value

$$\Xi_{(0,\nu,\lambda)}(w, \zeta') = \frac{\zeta'}{\nu w \zeta' + 1}$$

and the same holds for its inverse. Thus we deduce that $\Phi_L := \Xi_L \circ \Upsilon_L$ is such that $(L, x, z) \mapsto \Phi_L \circ \Theta_L$ extends smoothly to $\mathcal{V}^\delta$ if $\delta > 0$ is chosen small enough. If we write $\zeta = v + iu \in \mathbb{H}^2$, the function $\cosh(\mu)$ is clearly invariant by rotation, so

$$\frac{u'^2 + v'^2 + \ell^2}{u'} = \frac{u^2 + v^2 + \ell^2}{u} \tag{9.7}$$

and we compute

$$u' = \frac{u}{|-\ell^{-1}\sin(\nu\ell w)\zeta + \cos(\nu\ell w)|^2}, \quad d\zeta' = \frac{d\zeta + dw(\nu\zeta^2 + \nu\ell^2)}{(-\ell^{-1}\sin(\nu\ell w)\zeta + \cos(\nu\ell w))^2}. \tag{9.8}$$

Therefore the metric g_L becomes in the new coordinates.

$$g_L := (\Xi_L \circ \Upsilon_L)_* g_{\mathbb{H}^3} = \frac{du^2 + dv^2 + ((1+v^2)R^4 - 4v^2\ell^2u^2)dw^2}{u^2} + \frac{2v(R^2 - 2u^2)dwdv + 4vuvdudw}{u^2}$$

where $R := \sqrt{u^2 + v^2 + \ell^2}$. Here we notice that the change of coordinates $v' + iu' \mapsto v + iu$ for a fixed w is a hyperbolic rotation of angle $-2v\ell w$ and center $i\ell$ in $\mathbb{H}^2$. In particular it maps the half-circle (9.1) (which is a geodesic of $\mathbb{H}^2$) to the half-circle in $\mathbb{H}^2$ which intersects the real axis at the two points

$$v_\pm(q) = \frac{\pm r_q(w)\cos(v\ell w) + \ell \sin(v\ell w)}{\mp r_q(w)\ell^{-1}\sin(v\ell w) + \cos(v\ell w)} = \frac{\pm r_q(w)}{1 \mp v w r_q(w)} + \mathcal{O}(\ell).$$

This shows that the region (9.4) in the coordinates (w, ζ) becomes the set

$$\left\{(w,\zeta) \in \left[-\frac{1}{4}, \frac{1}{4}\right) \times \mathbb{H}^2;\ |\zeta - v_q(w)| \le \tau_q(w)\right\} / \left\{w \sim w + \frac{1}{2}\right\}$$

for $v_q(w) = \frac{1}{2}(v_+(q) + v_-(q))$ and $\tau_q(w) = \frac{1}{2}(v_+(q) - v_-(q))$ which clearly converge as $\ell \to 0$, and satisfy the desired properties (recall that $r_q = r_\lambda(w) + o(1)$ as $\ell \to 0$ with the notation of (9.2)).

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