Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 93-111

Hartogs-type extension for unbounded sets in $\mathbb{C}^2$ via construction of Riemann domains

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Abstract. This paper generalizes the results in [2] about holomorphic extension of CR functions from the boundary of a domain, and gives a class of counterexamples to the conjecture formulated in [2] and [4] that for tube-like domains Ω in $\mathbb{C}^2$ which do not contain complex lines in the closure $\overline{\Omega}$, any CR function defined on the boundary $b\Omega$ can be holomorphically extended to Ω. This investigation has led to interesting new results about Hartogs-type extension for unbounded domains in $\mathbb{C}^2$ via construction of Riemann domains. It should be noted that the Hartogs-type theorems not only play an important role in Complex Analysis of Several Variables, but are also significant in Algebraic Geometry and Partial Differential Equations.

Mathematics Subject Classification (2010): 32V25 (primary); 32V10, 32D15, 32D26 (secondary).

1. Introduction

The classic Hartogs theorem roughly states that if Ω is a bounded domain in $\mathbb{C}^n$, for $n \geq 2$, then any holomorphic function defined in a neighborhood of the boundary $b\Omega$ can be holomorphically extended to the inside of Ω. Later on, this theorem was generalized by Severi [29], Kneser [14], Fichera [9], Martinelli [17] to the assertion that CR functions defined only on the boundary $b\Omega$ extend holomorphically to the inside of Ω, see [25]. In the case of extensions of CR functions, there are some differentiability assumptions on the boundary $b\Omega$. If the holomorphic extension concerns any CR function, we will call Ω a *Hartogs-type domain*.

In recent years there has been a renewed interest in Hartogs type theorems; just to mention few: Merker and Porten [18–20], Porten [24], Coltoiu and Ruppenthal [6], Ohsawa [21], Øvrelid and Vassiliadou [22], Harz-Shcherbina-Tomassini [12], and the authors [2–4]. The Hartogs-type theorems not only play an impor-

The second named author has partially supported by the Missouri University of Science and Technology "visiting faculty" program.

Received February 9, 2015; accepted in revised form October 11, 2015.

Published online March 2017.

tant role in Complex Analysis of Several Variables, but are also significant in Algebraic Geometry and Partial Differential Equations. Except for the standard version of the Hartogs theorem, there are many variations, like Ivashkovich [13] for meromorphic maps into compact Kähler manifolds, Laurent-Thiebaut and Leiterer [15] for differential forms, Palamodov [23] or Damiano-Struppa-A.Vajiac-M.Vajiac [7] for systems of partial differential equations, or Sarkis [26] and Dwilewicz and Merker [8] for complex projective spaces. The latter can be considered for compact complex manifolds. In the case of complex projective spaces, the holomorphic extension is proved for CR functions defined on compact real hypersurfaces, which are globally minimal (any two points of the hypersurface can be connected by a piece wise smooth curve which is tangent to the complex space). In view of some deep work of Siu [27,28], there is evidence to suggest that all smooth compact hypersurfaces in $\mathbb{CP}^n$ are globally minimal. From these short comments it is clear that Hartogs-type extension theorems are related to many deep and important problems in various areas of mathematics. For historical remarks on the classical Hartogs result [11], classical Bochner tube theorem [1] and its generalization for CR functions [5], see [2] and Range [25].

Most of the previous work on the Hartogs theorem considered bounded domains in $\mathbb{C}^n$ or domains inside compact manifolds, such as $\mathbb{CP}^n$. In a series of papers [2–4] carried out by the authors, the Hartogs theorem in unbounded domains has been considered. For an unbounded domain in $\mathbb{C}^n$, there is an obvious necessary condition for a domain to be of Hartogs type, namely that Ω cannot contain any global analytic varieties. In this paper, we focus on $\mathbb{C}^2$ with coordinates (z, w) and define $\pi(z, w) = z$. In the case of tube-like domains Ω in $\mathbb{C}^2$, *i.e.*, where $\pi(\Omega)$ is bounded, then the necessary condition is equivalent to the requirement that Ω cannot contain any complex line. In [2] and [4] we gave examples that show that this condition is not sufficient for a domain to be of Hartogs-type. Also in these papers we stated the following:

Conjecture. If a tube-like domain in $\mathbb{C}^2$ does not contain complex lines in its closure, then the domain is of Hartogs-type.

It should be mentioned that, in 1987, G. Lupacciolu [16] proved a theorem related to this conjecture: *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an unbounded domain with connected boundary $\partial\Omega$ of class C^1. If we can find an algebraic hypersurface W of $\mathbb{C}^n$ such that $\widetilde{W} \cap \widetilde{\partial\Omega} = \emptyset$, where tilde means the closure of these sets in $\mathbb{CP}^n \supset \mathbb{C}^n = \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}_\infty$, then any continuous CR function on $\partial\Omega$ extends holomorphically and uniquely to Ω.*

In the already mentioned paper by Harz-Shcherbina-Tomassini [12], also related results to the conjecture were obtained. Since a formulation of the main results is pretty technical, we refer the reader to that paper. Here we only mention that for each $n \geq 2$ the authors constructed (1) an unbounded closed pseudoconcave complete pluripolar set $\mathcal{E}$ in $\mathbb{C}^n$ which contains no analytic variety of positive dimension, (2) an unbounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$, and (3) a smooth CR function f on $\partial\Omega$ which has a single-valued holomorphic extension exactly to the set $\overline{\Omega} \setminus \mathcal{E}$.

It appears that the above formulated conjecture does not hold and in this paper we prove:

Theorem I (see Corollary 3.3). *There exists a connected tube-like domain Ω in $\mathbb{C}^2$ with smooth C^∞ connected boundary $b\Omega$ and with the following properties:*

(a) *Ω is a Reinhardt domain with respect to w,* i.e., *if $(z, w) \in \Omega$, then $(z, e^{i\theta} w) \in \Omega$ for all $\theta \in \mathbb{R}$;*
(b) *the closure $\overline{\Omega}$ of the domain does not contain complex lines;*
(c) *there is a smooth CR function on $b\Omega$ that cannot be holomorphically extended to Ω,* i.e., *Ω is not of Hartogs type.*

In [2], we stated a result that the above formulated conjecture is true for Reinhardt domains with respect to w. However this result is not correct in view of Theorem I above. The process of fixing this result uncovered some interesting ideas on analytic extensions. Thus the purpose of this paper is two-fold:

(i) provide a class of counter examples to the conjecture formulated above;
(ii) provide an extra condition on the fibers $\pi^{-1}(z)\cap(\mathbb{C}^2\setminus\overline{\Omega})$ which when satisfied does ensure that Ω is of Hartogs type.

The extra condition in 2. is stated in terms of an equivalence relation defined on the points in the fiber $\pi^{-1}(z) \cap (\mathbb{C}^2 \setminus \overline{\Omega})$. Although the equivalence relation is a bit technical to state in full generality (see sec. 4), this relation, in particular, implies that any two points (z, w_1) and (z, w_2) which lie in the same connected component of $\mathbb{C}^2 \setminus \overline{\Omega}$ are equivalent. Using this equivalence relation, we show the following results.

Theorem II (see Corollary 4.6). *Let Ω be a domain in $\mathbb{C}^2$ which satisfies the following conditions:*

(a) *there such is z^* that $\pi^{-1}(z^*)\cap(\mathbb{C}^2\setminus\overline{\Omega})$ has at least two non-equivalent points;*
(b) *for every $z_0 \in \pi(\mathbb{C}^2 \setminus \overline{\Omega})$ there is an open neighborhood V of z_0 such that for each component H of $\pi^{-1}(V) \cap (\mathbb{C}^2 \setminus \overline{\Omega})$, and for every $z \in V$, all points in $\pi^{-1}(z) \cap H$ are equivalent.*

Then there exists a holomorphic function defined on $\mathbb{C}^2 \setminus \overline{\Omega}$ which cannot be holomorphically extended onto $\mathbb{C}^2$.

Theorem III (see Theorem 5.1). *Let Ω be a tube-like domain in $\mathbb{C}^2$ which is Reinhardt with respect to w. Moreover, assume $\overline{\Omega}$ does not contain complex lines. If for any $z \in \pi(\mathbb{C}^2 \setminus \overline{\Omega})$ all points of $\pi^{-1}(z) \cap (\mathbb{C}^2 \setminus \overline{\Omega})$ are equivalent, then Ω is a Hartogs-type domain.*

The organization of the paper is as follows: after notation and definitions in Section 2 we give a counterexample to the conjecture, in Section 3, based on the construction of a Riemann domain in terms of (connected) components. The main results of the paper are proved in Sections 4 and 5.

2. Notation and definitions

In this paper we consider subsets and domains of $\mathbb{C}^2$ and also we will use the projection $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}, \pi(z, w) = z$, where (z, w) are coordinates in $\mathbb{C}^2$.

If S is a subset of $\mathbb{C}^2$, the topology and differential structure (if applicable) on S is induced by $\mathbb{C}^2$. A *subset* $S_0 \subset S$ *is connected* if any two points $p_0, p_1 \in S_0$ can be joined by a C^1-piecewise curve $\varphi(t)$, for $0 \leq t \leq 1$, with $\varphi(t) \in S_0$, subject to $\varphi(0) = p_0$ and $\varphi(1) = p_1$. By a *component*, $S_0 \subset S$, we mean a connected subset of S such that S_0 is both closed and open in S. By a *domain* in $\mathbb{C}^2$ we mean an open and connected subset, sometimes we will also assume that the boundary is connected and of some class C^k - the assumptions will always be specified.

We will use the notion of Riemann domain following Robert C. Gunning [10, Chapter H, page 72]:

> *A* ***Riemann domain*** *of dimension n is a complex manifold M together with a nonsingular holomorphic mapping* $P : M \longrightarrow \mathbb{C}^n$, *called the* ***projection*** *of M to* $\mathbb{C}^n$.

And we will employ the following statement [10, Chapter H, page 72]:

> *If M is any Hausdorff topological space that admits a local homeomorphism* $P : M \longrightarrow \mathbb{C}^n$, *then P induces on M the structure of a complex manifold, the mapping P itself providing local coordinates on M. With this complex structure, M is evidently a Riemann domain.*

Finally, we will also use the following crucial property of one-dimensional Riemann domains, see [10, Chapter P, page 171, Theorem 6]:

> *Any one-dimensional Riemann domain is holomorphically convex and has the property that holomorphic functions separate points.*

3. A counterexample to the conjecture

In this section we give a counterexample to the statement in [2], which has been mentioned in the Introduction. The counterexample is based on the construction of a Riemann domain of dimension $n = 1$. The proofs are sometimes only outlined since more precise arguments are given in Section 4.

3.1. Construction of a Riemann domain based on (connected) components

We consider a domain $W \subset \mathbb{C}^2$ which satisfies the following condition:

> *For every* $z_0 \in \pi(W)$ *there is an open neighborhood* $V \subset \mathbb{C}$ *of* z_0 *such that for each component H of* $\pi^{-1}(V) \cap W$, *and for every* $z \in V$, *the fiber* $\pi^{-1}(z) \cap H$ *is connected or empty* (*see Figure* 3.1). (3.1)

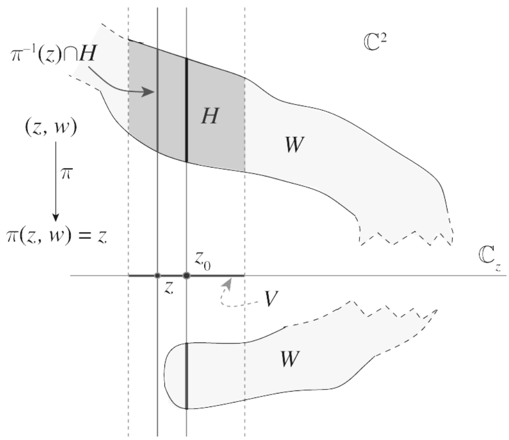

Figure 3.1. The domain W and the projection.

For example (see Figure 3.2), the domain

$$W = \left\{(z, w) \in \mathbb{C}^2;\ \operatorname{Re} z < \left(|w|^2 - 1\right)^2\right\}$$

does not satisfy (3.1) for $z_0 = 0$. As another example, consider the spiral

$$S = \left\{\left(e^{i|w|}, w\right) \in \mathbb{C} \times \mathbb{C}\right\} = \left\{(z, w);\ z = e^{i|w|}\right\}$$

and let W be a tubular neighborhood of S of uniform (but small) thickness. Then W satisfies (3.1).

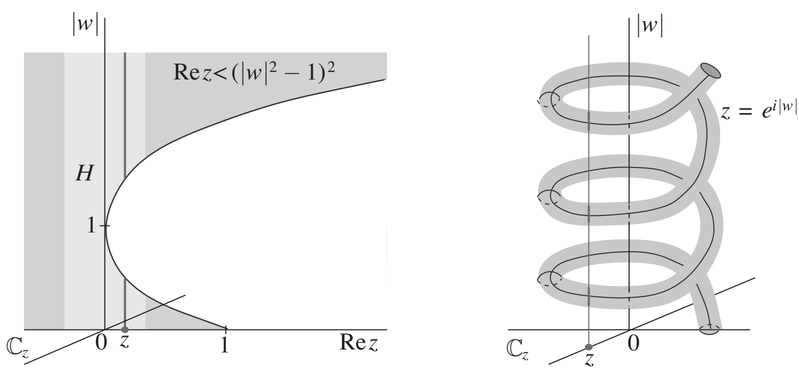

Figure 3.2. Examples.

Define the following equivalence relation on W:

$$(z_1, w_1) \sim (z_2, w_2)$$
$$\text{if and only if } \begin{cases} z_1 = z_2 & \text{and} \\ (z_1, w_1),\ (z_2, w_2) & \text{belong to the same} \\ & \text{component of } \pi^{-1}(z_1) \cap W. \end{cases} \tag{3.2}$$

Let $\widetilde{W}$ be the set of equivalence classes of $(z, w) \in W$, denoted $[z, w]$. Let $\widetilde{\pi} : W \longrightarrow \widetilde{W} = W/\sim$ be defined as $\widetilde{\pi}(z, w) = [z, w]$. Define $P : \widetilde{W} \longrightarrow \mathbb{C}$ by $P([z, w]) = z$. Note that $P \circ \widetilde{\pi} = \pi$ as maps from $\mathbb{C}^2$ to $\mathbb{C}$.

We endow $\widetilde{W}$ with the quotient topology whereby a set $U \subset \widetilde{W}$ is open if and only if $\widetilde{\pi}^{-1}(U)$ is open in $\mathbb{C}^2$. We now describe the basis of neighborhoods for each point $Z_0 = [z_0, w_0] \in \widetilde{W}$ in this topology. Choose a neighborhood $V \subset \mathbb{C}$ of z_0 satisfying condition (3.1), and let H be the unique component of $\pi^{-1}(V) \cap W$ containing (z_0, w_0). Then H is an open set in $\mathbb{C}^2$ and $\pi(H)$ is an open neighborhood of z_0 in $\mathbb{C}$. Let $\{V_n\}$ be a basis sequence of connected open neighborhoods in $\mathbb{C}$ of z_0 such that $V \supset \pi(H) \supset \overline{V}_1 \supset V_1 \supset \overline{V}_2 \supset \dots$,

$$V_n \supset \overline{V}_{n+1}, \qquad \bigcap_n V_n = \{z_0\}.$$

Denote $H_n := \pi^{-1}(V_n) \cap H$, for $n = 1, 2, \dots$. In view of (3.1), $\pi^{-1}(z) \cap H$ is connected and nonempty for each $z \in V_n \subset \pi(H)$ and so $[\pi^{-1}(z) \cap H]$ is an element of $\widetilde{W}$. Since $\bigcap_n V_n = \{z_0\}$, we see that $\widetilde{H}_n = \widetilde{\pi}(H_n)$ is a local neighborhood basis for Z_0. It is also easy to show that $P : \widetilde{W} \longrightarrow \mathbb{C}$ is a local homeomorphism and hence that $\widetilde{W}$ is Hausdorff (details are in Section 4). Consequently, $\widetilde{W}$ together with the projection P, is a Riemann domain.

3.2. Non-extendable holomorphic functions

In order to get a non-extendable holomorphic function, we need another assumption, namely:

$$\text{\textit{There exists a point } } z^* \in \pi(W) \text{ \textit{such that} } \pi^{-1}(z^*) \cap W \text{ \textit{has at least two components. Equivalently, this means that} } P^{-1}(z^*) \text{ \textit{has at least two distinct points, say} } [z^*, w_1] \text{ \textit{and} } [z^*, w_2] \text{ \textit{in} } \widetilde{W}. \tag{3.3}$$

Proposition 3.1. *If W satisfies* (3.1) *and* (3.3) *then there is a holomorphic function $F : W \longrightarrow \mathbb{C}$ which cannot be extended to an entire function on $\mathbb{C}^2$.*

Proof. We use the fact alluded to earlier in Gunning (see Section 2) namely the fact that there is a holomorphic function $f : \widetilde{W} \longrightarrow \mathbb{C}$ which separates points in $\widetilde{W}$. Define $F : W \longrightarrow \mathbb{C}$ by $F(z, w) = f([z, w])$. Since all points (z, w) in a connected component of $\pi^{-1}(z) \cap W$ lie in the same equivalence class, $[z, w]$, it is clear that F is well defined. It is also clear that F is independent of w on any connected component of $\pi^{-1}(z) \cap W$ since $F(z, w)$ is the constant given by $f([z, w])$. If F had an extension $\mathfrak{F}$ to all of $\mathbb{C}^2$, then $\mathfrak{F}$ would have to be globally independent of w. On the other hand, since W satisfies (3.3), $\widetilde{W}$ has two distinct points, $[z^*, w_1]$ and $[z^*, w_2]$ and so

$$\mathfrak{F}(z^*, w_1) = f([z^*, w_1]) \neq f([z^*, w_2]) = \mathfrak{F}(z^*, w_2)$$

and this contradicts the fact that $\mathfrak{F}$ is independent of w. Thus, F has no extension to $\mathbb{C}^2$. □

3.3. Example of a Reinhardt domain which is not of Hartogs type

We consider domains $W \subset \mathbb{C}^2$ which are Reinhardt in w, *i.e.*, if $(z, w) \in W$, then $(z, e^{i\theta} w) \in W$ for all $\theta \in \mathbb{R}$.

Proposition 3.2. *There is a Reinhardt domain in $\mathbb{C}^2$ satisfying the assumptions* (3.1) *and* (3.3) *and such that:*

(i) $W \supset (\mathbb{C} \setminus \overline{D}(0, R)) \times \mathbb{C}$ *for some* $0 < R < +\infty$*;*
(ii) *the boundary of W is connected;*
(iii) $\mathbb{C}^2 \setminus W$ *is connected.*

Proof. We take the space $\mathbb{C} \times \mathbb{R}_+$ with coordinates $(z = x + iy, s)$. In this space we define a surface S, which will be the union of two surfaces S_1 and S_2.
Definition of S_1. The surface S_1 is defined as:

$$S_1 = \{(z, 2) \in \mathbb{C} \times \mathbb{R}_+;\ z \in \mathbb{C}\} \setminus \Big\{(z = re^{i\theta},\ 2) \in \mathbb{C} \times \mathbb{R}_+; \\ \text{for } 1 \le r \le 4, \text{ and } 0 \le \theta \le \frac{2\pi}{3}\Big\},$$

S_1 (in shaded gray) is illustrated in Figure 3.3.

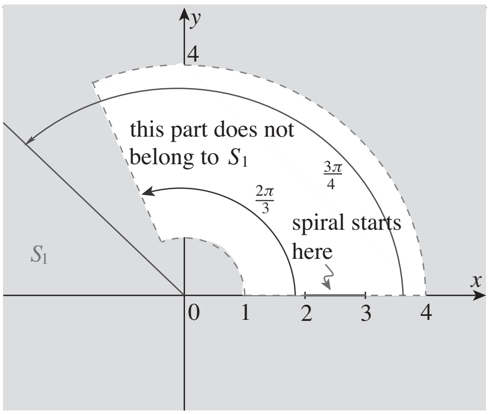

Figure 3.3. The domain S_1.

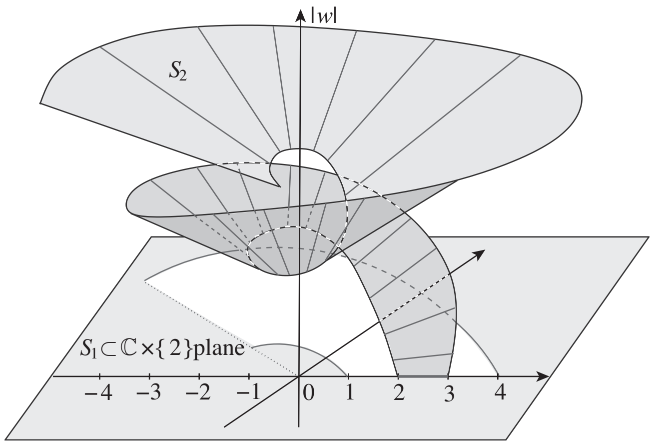

Figure 3.4. A spiral inside the cylinder $|z| < 7$.

Definition of S_2. The surface S_2 (see Figure 3.4) is defined in polar coordinates as

$$S_2 = \left\{ (\rho \cos\theta,\ \rho \sin\theta,\ 2+\theta);\ \text{for } \rho(\theta) < \rho < R(\theta),\ \text{and } 0 \le \theta < \frac{11}{4}\pi \right\},$$

where the functions $r(\theta)$ and $R(\theta)$ are

$$r(\theta) = 2 - \frac{2\theta}{3\pi} \quad \textit{for} \quad 0 \le \theta \le \frac{11}{4}\pi$$
$$R(\theta) = 3 + \frac{2\theta}{3\pi} \quad \textit{for} \quad 0 \le \theta \le \frac{11}{4}\pi.$$

We note that the surface $S = S_1 \cup S_2$ in Figure 3.4 is smooth (C^∞) except on the line segment

$$\{(z = r,\ w = 2);\ \ \text{for } 2 < r < 3\}$$

that connects S_1 and S_2. Moreover S_1 and S_2 can be easily deformed near the interval so that $S_1 \cup S_2$ is smooth.

We define the domain W as the union of three pieces:

$$W = W_1 \cup W_2 \cup [(\mathbb{C} \setminus \overline{D(0,7)}) \times \mathbb{C}],$$

where W_1 and W_2 are neighborhoods of S_1 and S_2, defined as follows:

$$W_1 = \left\{(z,w) \in \mathbb{C}^2;\ (z,2) \in S_1,\ 1 < |w| < 3\right\}$$
$$W_2 = \left\{(z,w) \in \mathbb{C}^2;\ \left(z = \rho e^{i\theta},\ 2+\theta\right) \in S_2,\ \left||w| - (2+\theta)\right| < 1,\ 0 \le \theta < \frac{11}{4}\pi\right\}.$$

By construction, the domain W satisfies the conditions (i)-(iii) of Proposition 3.2. It remains to show that W satisfies the assumptions (3.1) and (3.3). Condition (3.3) is satisfied by any $z^* \in \mathbb{C}$ with $2 < |z^*| < 3$.

To check condition (3.1), take any point $z_0 \in \mathbb{C}$. For z_0 we want to show that there is an open neighborhood V of z_0 such that for each connected component H of $\pi^{-1}(V) \cap W$, and for every $z \in V$, the fiber $\pi^{-1}(z) \cap H$ is either connected or empty. If $z_0 \in \mathbb{C}$, then we can let V be any disc $|z - z_0| < \delta$, where $\delta > 0$ is sufficiently small. The proposition is proved. □

Corollary 3.3. *There exists a domain Ω in $\mathbb{C}^2$ with the following properties:*

1. *Ω is a connected tube-like domain with smooth C^∞ connected boundary $b\Omega$;*
2. *Ω is a Reinhardt domain with respect to w;*
3. *the complement $\mathbb{C}^2 \setminus \overline{\Omega}$ is connected;*
4. *the closure $\overline{\Omega}$ of the domain does not contain complex lines;*
5. *there is a smooth CR function on $b\Omega$ that cannot be holomorphically extended to Ω,* i.e., *Ω is not of Hartogs type.*

Proof. It is *"almost"* enough to take $\Omega = \mathbb{C}^2 \setminus \overline{W}$, where W was constructed in the proof of Proposition 3.2. *"Almost"* means that we have to change a few things:

1° Ω is "slightly" bigger than $\mathbb{C}^2 \setminus W$, *i.e.*, $\mathbb{C}^2 \setminus W \subset \Omega$, in particular, $b\Omega \subset W$;
2° the boundary $b\Omega$ is *"close"* enough to bW in order to guarantee that no vertical complex lines are contained in $\overline{\Omega}$;
3° the boundary $b\Omega$ is C^∞.

It is easy to achieve these properties, for instance we "shrink" the domain W by a distance ε, where $\varepsilon > 0$ is sufficiently small. After shrinking the domain W, we can smooth-out near the crease where W_1 joins with W_2 as well as the juncture of W_1 with the cylinder, and also the boundaries bW_1 and bW_2 which lie inside the cylinder $|z| < 7 + \varepsilon$.

To show that there exists a smooth CR function on $b\Omega$ that cannot be holomorphically extended to Ω, we take the holomorphic function constructed in Proposition 3.1 for the domain W constructed in Proposition 3.2. Obviously the restriction of the function to the boundary $b\Omega$ is CR and it cannot be holomorphically extended to Ω. The corollary is proved. □

Remark 3.4. It should be noted that although the domain Ω from the corollary looks complicated, topologically it is pretty simple. Namely it is easy to see that

- Ω is homeomorphic to $D(0,1) \times \mathbb{C}$,
- $\mathbb{C}^2 \setminus \overline{\Omega}$ is homeomorphic to $[\mathbb{C} \setminus \overline{D(0,1)}] \times \mathbb{C}$, so in particular, the complement of $\overline{\Omega}$ is connected.

Finally we make some comments about the above corollary. In papers [2] and [4] by the authors, we formulated by following:

Conjecture. A tube-like domain in $\mathbb{C}^2$ which does not contain complex varieties in its closure is a Hartogs-type domain. (By complex varieties we mean closed complex varieties in $\mathbb{C}^2$ of dimension at least one.)

In view of Corollary 3.3, this conjecture is false and further conditions are needed to ensure that a tube-like domain is Hartogs. The conjecture does not even hold for Reinhardt domains in $\mathbb{C}^2$ with respect to one coordinate.

4. Construction of Riemann domains based on equivalent components

In the previous section we described an equivalence relation on the set W by stating that $(z, w_1) \sim (z, w_2)$ if both points lie in the same connected component of $\pi^{-1}(z) \cap W$ (and points with different z-coordinates are not equivalent). We also constructed a domain whose boundary is not of Hartogs type by ensuring that there were at least two non-equivalent points $(z, w_1) \not\sim (z, w_2)$ in W. One might guess that if W has the property that if $(z, w_1) \sim (z, w_2)$ for all such points in W, then its boundary is of Hartogs type. This result is true for Reinhardt domains with respect to the w-variable. In fact, this result holds for domains with a weaker form of equivalence which we now describe.

4.1. Definition of conjugate and equivalent components

Let W be a domain in $\mathbb{C}^2$ and $\pi : W \longrightarrow \mathbb{C}$ the projection $\pi(z, w) = z$.

We say that two points (z, w_1) and (z, w_2) are *conjugate* if there are two C^1 paths

$$t \to \left(z^t, w_\alpha^t\right) = (z(t), w_\alpha(t)) \in W, \quad \text{for } \alpha = 1, 2 \text{ and } t \in [0, 1]$$

such that

$$z_\alpha(0) = z, \quad w_\alpha(0) = w_\alpha \text{ for } \alpha = 1, 2$$
$$w_1(t) = w_2(t) \qquad \text{for } t \text{ near } 1,$$

see Figure 4.1. Note that it is required that the z-coordinate of both path functions be the same. In particular, if $z_1 \neq z_2$, then (z_1, w_1) and (z_2, w_2) are *not conjugate*. Let Z_α^t be the component of $\pi^{-1}(z(t))$ containing (z^t, w_α^t). Note that from

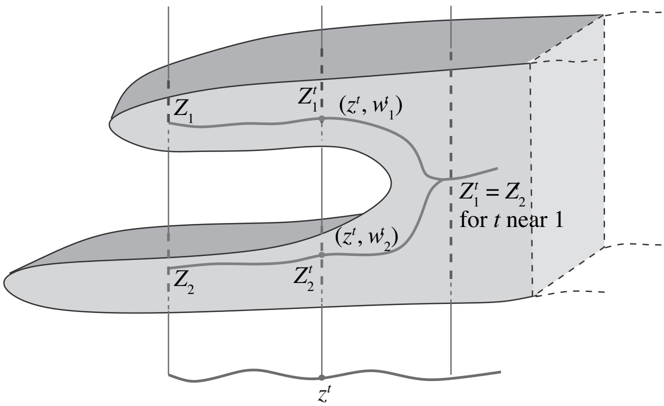

Figure 4.1. Conjugate components.

the definition of the paths, it may be the case that $Z_1^t \cap Z_2^t$ is empty for small t but that one must have $Z_1^t = Z_2^t$ for t near 1. It is also clear that any two points which are equivalent as defined in Section 3 are conjugate via this definition (indeed just connect (z, w_1) and (z, w_2) to each other along a path that only varies in w).

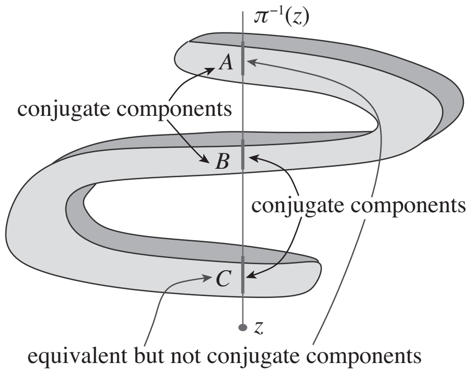

Figure 4.2. Illustration of conjugate and equivalent components.

The above definition of conjugation is reflexive and symmetric, but it is not transitive. For example in Figure 4.2, A is conjugate to B and B is conjugate to C, but A is *not* conjugate to C because any two paths starting at A and C and connecting to a common point cannot have the same z-component. To obtain an equivalence relation, define $(z, w_1) \sim (z, w_2)$ if there is a finite chain of points

$$(z, w_1) = (z, \zeta_1),\ (z, \zeta_2), \ldots, (z, \zeta_N) = (z, w_2) \in W$$

such that (z, ζ_j) is conjugate to (z, ζ_{j+1}) for $j = 1, \ldots, N-1$. It is easy to show that "$\sim$" is transitive and hence is an equivalence relation.

Let $\widehat{W}$ be the set of all equivalence classes of W under the above defined equivalence relation. Denote by $\widehat{\pi} : W \longrightarrow \widehat{W}$ the natural projection $\widehat{\pi}(z, w) = [z, w]$, which the equivalence class of (z, w), and define $P : \widehat{W} \longrightarrow \mathbb{C}$ as $P([z, w]) = z$. We endow $\widehat{W}$ with the quotient topology, *i.e.*, $\widehat{U} \subset \widehat{W}$ is open if $\widehat{\pi}^{-1}(\widehat{U})$ is open in W.

4.2. Examples

Example 4.1. In Figure 4.2 and Figure 4.3 there are components which are conjugate, components which are equivalent but not conjugate, and components which are not equivalent.

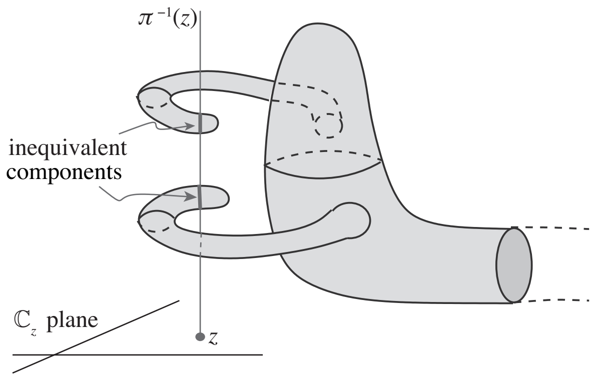

Figure 4.3. Inequivalent components.

Example 4.2. In Figure 4.4 we construct a Reinhardt domain W in $\mathbb{C}^2$ such that there exists $z_0 \in \pi(W)$ and there exists a component Y_0 of $\pi^{-1}(z_0)$ with the following property:

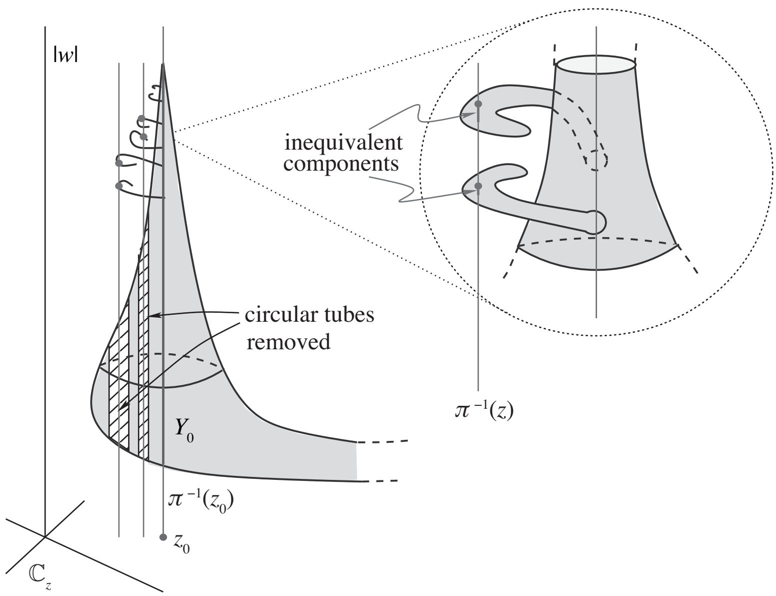

Figure 4.4. Inequivalent components in a neighborhood.

For any neighborhood V of z_0, let $H = H_V$ be a component of $\pi^{-1}(V)$ containing Y_0, then there exists $z \in V$, for $z \neq z_0$, such that $\pi^{-1}(z) \cap H$ has inequivalent points.

4.3. Properties of the set of equivalence classes of components

In this subsection we make the following additional assumption about the set $W \subset \mathbb{C}^2$. Let W be an open connected subset of $\mathbb{C}^2$ with connected boundary bW and connected complement $\mathbb{C}^2 \setminus \overline{W}$. Moreover let W satisfy the following condition:

For every $z_0 \in \pi(W)$, there is an open set V of z_0 in $\mathbb{C}$ such that for each component H of $\pi^{-1}(V) \cap W$, and for every $z \in V$, all points in $\pi^{-1}(z) \cap H$ are equivalent. (4.1)

Remark 4.3. We would like to justify the assumption (4.1) imposed on the domain W. Later, we need to ensure that the quotient topology on $\widehat{W}$ is locally homeomorphic to an open domain in $\mathbb{C}$, and also that $\widehat{W}$ is Hausdorff. Without the assumption (4.1), as Example 4.2 shows, there will be a problem to prove these properties.

Lemma 4.4. *Under the assumptions on the domain W, including* (4.1)*, we have:*
(i) *there is a natural projection $P : \widehat{W} \longrightarrow \mathbb{C}$ onto $\pi(W)$, where $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$, is given by $\pi(z, w) = z$;*
(ii) *P is a local homeomorphism;*
(iii) *$\widehat{W}$ is a Hausdorff space;*
(iv) *$\widehat{W}$ is a Riemann domain.*

Proof. If $z \in \pi(W)$, then $\pi^{-1}(z) \cap W$ is not empty, say $(z, w) \in \pi^{-1}(z) \cap W$, and the point (z, w) determines the class $[z, w] \in \widehat{W}$. Clearly, $P([z, w]) = z$, so (i) is proved.

To prove the second part of the lemma, let $\widehat{Z_0} \in \widehat{W}$ with $P(\widehat{Z_0}) = z_0$. By assumption, there is a neighborhood V of z_0 such that, for the component H containing Z_0 and for each $z \in V$, all points of $H \cap \pi^{-1}(z)$ are equivalent. This implies that the projection π is one-to-one from $(H \cap \pi^{-1}(V))/\!\sim$ onto $V \cap \pi(H)$ and actually a homeomorphism, taking into account the quotient topology on $\widehat{W}$.

We will prove that $\widehat{W}$ is Hausdorff. Take two distinct points $\widehat{Z_1} = [z_1, w_1]$ and $\widehat{Z_2} = [z_2, w_2]$. There are two cases to consider: (1) $z_1 \neq z_2$ and (2) $z_1 = z_2 = z_0$ but $[z_0, w_1] \neq [z_0, w_2]$.

If $P(\widehat{Z_1}) = z_1 \neq z_2 = P(\widehat{Z_2})$, then clearly we can find open disjoint neighborhoods V_1 of z_1 and V_2 of z_2 in $\mathbb{C}$. The sets $P^{-1}(V_1)$ and $P^{-1}(V_2)$ are open and disjoint neighborhoods of $\widehat{Z_1}$ and $\widehat{Z_2}$.

So now we consider the case where $\widehat{Z_1} = [z_0, w_1]$ and $\widehat{Z_2} = [z_0, w_2]$, with $\widehat{Z_1} \neq \widehat{Z_2}$, as in Figure 4.5. We can find an open neighborhood V of z_0 and components H_1 and H_2 in $\pi^{-1}(V) \cap W$ containing (z_0, w_1) and (z_0, w_2), respectively. The components H_1 and H_2 are disjoint, because otherwise, if $H_1 = H_2 =: H$ and in each fiber $\pi^{-1}(z) \cap H$ all points are equivalent, we get contradiction with the assumption $\widehat{Z_1} \neq \widehat{Z_2}$. We claim that there is a neighborhood V_0 of z_0 in $\mathbb{C}$ such that $[H_1 \cap \pi^{-1}(V_0)]$ and $[H_2 \cap \pi^{-1}(V_0)]$ are disjoint. If not, then for any neighborhood U of z_0 in $\mathbb{C}$ there is $z = z^U$, $z^U \in U$, and there are two points $(z^U, \zeta_1) \in \pi^{-1}(z) \cap H_1$ and $(z^U, \zeta_2) \in \pi^{-1}(z) \cap H_2$ that are equivalent, *i.e.*,

$$\left(z^U, \zeta_1\right) \sim \left(z^U, \zeta_2\right).$$

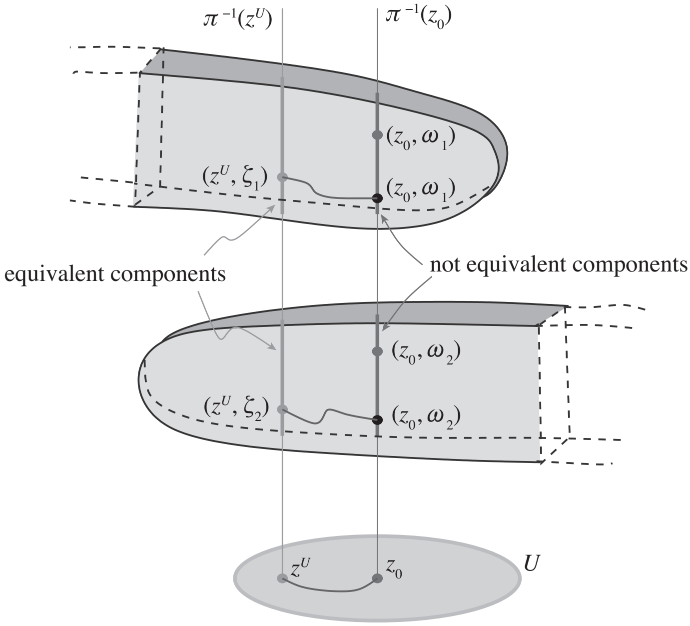

Figure 4.5. Illustration of the proof that $\widehat{W}$ is Hausdorff; actually the situation in the figure cannot take place under the assumptions of the theorem and when U is sufficiently small.

On the other hand, if the neighborhood U is sufficiently small, using the assumption (4.1), we can find two curves

$$(z(t), w_1(t)) \quad \textit{and} \quad (z(t), w_2(t)) \quad \textit{for} \quad 0 \le t \le 1$$

that join the points

$$\left(z^U, \zeta_1\right) \textit{ with } \left(z_0, \widetilde{w}_1\right) \qquad \textit{and} \qquad \left(z^U, \zeta_2\right) \textit{ with } \left(z_0, \widetilde{w}_2\right),$$

for some choice of $\widetilde{w}_1,\ \widetilde{w}_2$ with

$$\left(z_0, \widetilde{w}_1\right) \sim (z_0, w_1) \quad \textit{and} \quad \left(z_0, \widetilde{w}_2\right) \sim (z_0, w_2).$$

So we get $(z_0, w_1) \sim (z_0, w_2)$ by transitivity and this implies $\widehat{Z}_1 = \widehat{Z}_2$, contrary to the assumption. Therefore we conclude that the open sets corresponding to $[H_1 \cap \pi^{-1}(V_0)]$ and $[H_2 \cap \pi^{-1}(V_0)]$ are disjoint neighborhoods of $\widehat{Z}_1$, $\widehat{Z}_2$ respectively and hence $\widehat{W}$ is Hausdorff.

Using points (ii) and (iii) just proved, we can introduce a complex structure on $\widehat{W}$ using the local homeomorphism P. Consequently, $\widehat{W}$ is a one-dimensional complex manifold and the mapping P is holomorphic. Also the natural projection $\widehat{\pi} : W \longrightarrow \widehat{W}$ is holomorphic. □

4.4. Construction of a non-extendable holomorphic function and domains which are not of Hartogs type

In this section we prove the following result analogous to Proposition 3.1.

Theorem 4.5. *Suppose that W is a connected open subset of $\mathbb{C}^2$ with connected boundary bW and connected complement. Also assume that W satisfies condition (4.1) from Subsection 4.3, and additionally there is z^* such that $\pi^{-1}(z^*) \cap W$ has at least two inequivalent points. Then there is a holomorphic function $f : W \longrightarrow \mathbb{C}$ that cannot be extended to a holomorphic function $F : \pi^{-1}(\pi(W)) \longrightarrow \mathbb{C}$, in particular, F cannot be extended to an entire function on $\mathbb{C}^2$.*

Proof. The proof here is very similar to the one we gave in Subsection 3.2. By assumption, there is a $z^* \in \mathbb{C}$ and $w_1, w_2 \in \mathbb{C}$ such that $[z^*, w_1], [z^*, w_2] \in \widehat{W}$ with $[z^*, w_1] \neq [z^*, w_2]$. By Gunning [10, Theorem 6 in Chapter P, page 171], there is a holomorphic function f on $\widehat{W}$ with $f([z^*, w_1]) \neq f([z^*, w_2])$.

Define the function $F : W \longrightarrow \mathbb{C}$ as

$$F(z, w) = f([z, w]), \quad \textit{that is} \quad F = f \circ \widehat{\pi} \quad \textit{on } \widehat{W}.$$

The function F is well defined on W and is holomorphic since f and $\widehat{\pi}$ are holomorphic. Moreover, from the definition of the function F and from the condition (4.1), the function F is locally independent of w, therefore

$$\frac{\partial F}{\partial w}(z, w) = 0.$$

We conclude now that F cannot have a holomorphic extension $G : \pi^{-1}(\pi(W)) \longrightarrow \mathbb{C}$. This extension would satisfy $\frac{\partial G}{\partial w} \equiv 0$, and so G is independent of w,

$$G(z, w) = G(z) \quad \textit{for} \quad z \in \pi(W).$$

This implies

$$f\Big([z^*, w_1]\Big) = G\big(z^*\big) = f\Big([z^*, w_2]\Big)$$

which contradicts $f([z^*, w_1]) \neq f([z^*, w_2])$. □

Corollary 4.6. *Let Ω be a domain in $\mathbb{C}^2$ with connected smooth boundary and which satisfies the conditions:*

(a) *for every $z_0 \in \pi(\mathbb{C}^2 \setminus \overline{\Omega})$ there is an open neighborhood V of z_0 such that for each component H of $\pi^{-1}(V) \cap (\mathbb{C}^2 \setminus \overline{\Omega})$, and for every $z \in V$, all points in $\pi^{-1}(z) \cap H$ are equivalent;*
(b) *there is z^* such that $\pi^{-1}(z^*) \cap (\mathbb{C}^2 \setminus \overline{\Omega})$ has at least two non-equivalent points.*

Then there exists a holomorphic function defined on $\mathbb{C}^2 \setminus \overline{\Omega}$ which cannot be holomorphically extended onto $\mathbb{C}^2$.

Proof. Since Ω is connected and the smooth boundary $b\Omega$ is connected, it is an easy exercise to prove that the complement $\mathbb{C}^2 \setminus \overline{\Omega}$ is connected. Since $W = \mathbb{C}^2 \setminus \overline{\Omega}$ satisfies the conditions (a) and (b), we can use Theorem 4.5 to find a holomorphic function on W which cannot be holomorphically extended to $\pi^{-1}(\pi(W))$, in particular to $\mathbb{C}^2$, which is what we wanted to prove. □

5. Reinhardt domains which are of Hartogs type

In this section we give a sufficient condition for a Reinhardt tube-like domain to be of Hartogs type.

Theorem 5.1. *Let $\Omega \subset \mathbb{C}^2$ be a tube-like domain along the w-axis,* i.e., $\pi(\Omega)$ *is bounded where $\pi(z, w,) = z$, with the following properties:*

1. *Ω is open, connected and with smooth C^∞ connected boundary $M = b\Omega$;*
2. *$\pi(\overline{\Omega}) \subset \pi\left(\mathbb{C}^2 \setminus \overline{\Omega}\right)$;*
3. *Ω is a Reinhardt domain with respect to w.*

If for any $z \in \pi(\mathbb{C}^2 \setminus \overline{\Omega})$ all points of $\pi^{-1}(z) \cap (\mathbb{C}^2 \setminus \overline{\Omega})$ are equivalent, then Ω is a Hartogs-type domain, i.e., *any smooth (C^∞) CR function f defined on $b\Omega$ can be holomorphically extended to Ω.*

Proof. Let us start with some elementary observations. We have

$$\pi(\overline{\Omega}) \subset \overline{\pi(\Omega)},$$

and also we note that $\pi(\Omega)$ and $\pi(\mathbb{C}^2 \setminus \overline{\Omega})$ are open sets. Assumption 1. implies that the complement of $\overline{\Omega}$ is connected, and assumption 2. implies that $\pi(\mathbb{C}^2 \setminus \overline{\Omega}) = \mathbb{C}$. The projection $\pi(\overline{\Omega})$ is bounded but not necessarily compact. If $R > 0$ then $\pi\left(\overline{\Omega} \cap \{|w| \leq R\}\right)$ is compact. Since Ω is a Reinhardt tube-like domain with respect to w, the complement $\mathbb{C}^2 \setminus \overline{\Omega}$ is also a Reinhardt domain with respect to w. □

Let f be a C^∞ CR function defined on the boundary $M = b\Omega$. Since Ω is a tube-like domain, we know that it can be represented as the difference of two holomorphic functions

$$f = \left(f^+ - f^-\right)\big|_{b\Omega}, \quad f^+ \in \mathscr{H}(\Omega) \cap C^\infty(\overline{\Omega}), \quad f^- \in \mathscr{H}\left(\mathbb{C}^2 \setminus \overline{\Omega}\right) \cap C^\infty\left(\mathbb{C}^2 \setminus \Omega\right). \tag{5.1}$$

Take the function f^- defined on $\mathbb{C}^2 \setminus \overline{\Omega}$ and take a point $P = (z_0, w_1) \in \mathbb{C}^2 \setminus \overline{\Omega}$ and expand it into the Laurent series with respect to w for small $|z - z_0|$ and small $\big||w| - |w_1|\big|$,

$$f^-(z, w) = \sum_{j=-\infty}^{\infty} a_j(z)\, w^j. \tag{5.2}$$

We note that the coefficients $a_j(z)$ are holomorphic functions for z near z_0.

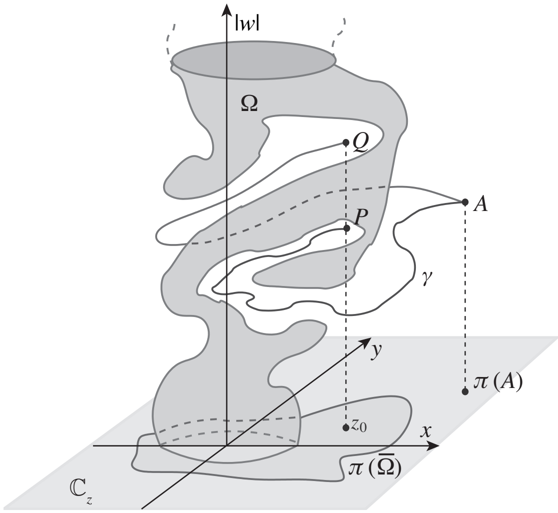

Figure 5.1. Tube-like Reinhardt domain and curves joining the points A, P and Q.

Suppose $Q = (z_0, w_2)$ is another point in the same fiber, $\pi^{-1}(z_0)$, as P (see Figure 5.1). A Laurent series expansion of f^- for z near z_0 and small $\big|\,|w| - |w_2|\big|$ gives

$$f^-(z, w) = \sum_{j=0}^{\infty} b_j(z)\, w^j. \tag{5.3}$$

We claim $a_j(z) = b_j(z)$ for z near z_0. We use the property that (z_0, w_1) and (z_0, w_2) are equivalent. Since equivalent points are endpoints of a finite chain of conjugate points, we can assume that (z_0, w_1) and (z_0, w_2) are conjugate. Therefore, there are two paths, C_1 and C_2, of the form

$$C_1 : [0, 1] \ni t \longrightarrow (z(t), w_1(t)), \quad P = (z(0), w_1(0)) = (z_0, w_1)$$
$$C_2 : [0, 1] \ni t \longrightarrow (z(t), w_2(t)), \quad Q = (z(0), w_2(0)) = (z_0, w_2)$$

such that $w_1(t) = w_2(t)$ for t near $t = 1$. For $j = 1, 2$, let $S_j(t)$ be the component of $\pi^{-1}(z(t))$ in $\mathbb{C}^2 \setminus \overline{\Omega}$ containing $(z(t), w_j(t))$. Clearly $S_1(t) = S_2(t)$ for t near $t = 1$. By the uniqueness of the Laurent coefficients, we obtain that $a_j(z) = b_j(z)$ for z near $z(1)$. By unique continuation, we see that $a_j \equiv b_j$ in a neighborhood of the entire path $\{z(t);\ 0 \le t \le 1\}$. In particular $a_j \equiv b_j$ near z_0.

Having shown that the Laurent coefficients a_j and b_j are the same, we will now show that $f^- \in \mathscr{H}(\mathbb{C}^2 \setminus \overline{\Omega})$ extends to an entire function.

Let A be a point in $\mathbb{C}^2 \setminus \overline{\Omega}$ with $\pi(A) \notin \overline{\pi(\Omega)}$ (see Figure 5.1). Let z_0 be a point in $\pi(\overline{\Omega})$. Since $\pi(\overline{\Omega}) \subset \pi(\mathbb{C}^2 \setminus \overline{\Omega})$ (by assumption), we can find a point $P \in \mathbb{C}^2 \setminus \overline{\Omega}$ such that $\pi(P) = z_0$. Since $\mathbb{C}^2 \setminus \overline{\Omega}$ is connected, there is a smooth curve γ joining point A with point P.

For $z \notin \pi(\overline{\Omega})$ the function $w \mapsto f^-(z,w)$ is entire and thus the Laurent coefficient functions $a_j(z) = 0$ for $j < 0$. By unique continuation and the uniqueness of Laurent coefficients (which we have just proved), we conclude that $a_j(z) = 0$ for $j < 0$ for (z,w) near the entire path γ. In particular, $a_j(z) = 0$ for $j < 0$ and z near z_0. Since $z_0 \in \pi(\overline{\Omega})$ was arbitrarily chosen, we conclude that $a_j \equiv 0$ on $\pi(\overline{\Omega})$, for $j < 0$, and hence

$$f^-(z,w) = \sum_{j=0}^{\infty} a_j(z) w^j, \tag{5.4}$$

is entire on $\mathbb{C}^2$.

In view of (5.1), f can be extended to Ω as $f^+ - f^- \in \mathscr{H}(\Omega)$, thus establishing that Ω is a Hartogs domain.

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