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An inscribed radius estimate for mean curvature flow in Riemannian manifolds

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Abstract. We consider a family of embedded, mean convex hypersurfaces in a Riemannian manifold which evolve by the mean curvature flow. We show that, given any number $T > 0$ and any $\delta > 0$, we can find a constant C with the following property: if $t \in [0, T)$ and p is a point on M_t where the curvature is greater than C, then the inscribed radius is at least $\frac{1}{(1+\delta)H}$ at the point p. The constant C depends only on δ, T, and the initial data.

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1. Introduction

In a recent paper [3], we established a sharp bound for the inscribed radius for mean convex hypersurfaces in Euclidean space which evolve by mean curvature flow. In this paper we generalize this result to the case of a flow in an ambient Riemannian manifold. Let X be a Riemannian manifold of dimension $n+1$, and let $F : M \times [0, T) \to X$ be a family of closed, embedded, mean convex hypersurfaces in X which evolve by mean curvature flow. As in [3], we define a function μ by

$$\mu(x,t) = \sup_{y \in M,\, 0 < d(F(x,t),F(y,t)) \leq \frac{1}{2}\,\mathrm{inj}(X)} \left(-\frac{2\,\langle \exp^{-1}_{F(x,t)}(F(y,t)),\, \nu(x,t) \rangle}{d(F(x,t),\, F(y,t))^2} \right).$$

Note that $\lambda_1 \leq \ldots \leq \lambda_n \leq \mu$, where the λ_i are the principal curvatures. For hypersurfaces in Euclidean space, the reciprocal of $\mu(x,t)$ is equal to the inscribed radius of M_t at the point x.

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Theorem 1.1. *Let $\delta > 0$ and $T > 0$ be given positive numbers. Then the function μ satisfies an estimate of the form*

$$\mu \leq (1 + \delta)\, H + C(X, M_0, \delta, T)$$

for all $t \in [0, T)$ and all points on M_t.

In the special case where X is the Euclidean space $\mathbb{R}^{n+1}$, it follows from general results of Brian White that the ratio $\frac{\mu}{H}$ is uniformly bounded from above (*cf.* [13–15]). An alternative proof was given by Sheng and Wang [12]. Andrews [1] recently gave another proof of that fact; his argument is based on a maximum principle for a two-point function. This technique was developed in earlier work of Huisken [7] on the curve shortening flow in the plane.

In a recent paper [3], we showed that, for any mean convex solution to the mean curvature flow in Euclidean space, we have an estimate of the form $\mu \leq (1 + \delta)\, H + C$, where C is a positive constant that depends only on δ and the initial hypersurface M_0.

One of the key insights in [3] is that the evolution equation for the function μ contains a gradient term which has a favorable sign (see also [2]). This makes it possible to use integral estimates and Stampacchia iteration to prove the desired bound for μ. The proof of Theorem 1.1 follows the same strategy, but requires some adaptations due to the background geometry.

We next define

$$\rho(x, t) = \max \left\{ \sup_{y \in M,\, 0 < d(F(x,t), F(y,t)) \leq \frac{1}{2} \operatorname{inj}(X)} \frac{2 \left\langle \exp_{F(x,t)}^{-1}(F(y, t)), \nu(x, t) \right\rangle}{d(F(x, t), F(y, t))^2}, 0 \right\}.$$

Note that $-\rho \leq \lambda_1 \leq \ldots \leq \lambda_n$. For hypersurfaces in Euclidean space, the reciprocal of $\rho(x, t)$ has a geometric interpretation as the outer radius of M_t at the point x.

Theorem 1.2. *Let $\delta > 0$ and $T > 0$ be given positive numbers. Then the function ρ satisfies an estimate of the form*

$$\rho \leq \delta\, H + C(X, M_0, \delta, T)$$

for all $t \in [0, T)$ and all points on M_t.

We note that Theorem 1.2 is a refinement of the convexity estimate of Huisken and Sinestrari [8,9]; see also [13–15].

Having extended the noncollapsing estimate to Riemannian manifolds, we can conclude that the curvature derivative estimates of Haslhofer and Kleiner (*cf.* [6, Theorem 1.8']) also hold for mean curvature flow of mean convex hypersurfaces in Riemannian manifolds. The constant in the interior gradient estimate will depend on the noncollapsing constant and also on the length of the time interval $[0, T)$.

2. Evolution of the inscribed radius under mean curvature flow

Given any point $q \in X$, we define a function $\psi_q : X \to \mathbb{R}$ by $\psi_q(p) = \frac{1}{2}\, d(p,q)^2$, where $d(p,q)$ denotes the Riemannian distance in X. Moreover, let us put $\Xi_{q,p} := (\operatorname{Hess} \psi_q)_p - g$. Clearly, $\Xi_{q,p}$ is a symmetric bilinear form on T_pX, and we have $|\Xi_{q,p}| \leq O(d(p,q)^2)$.

Proposition 2.1. *Consider a point $(\bar{x},\bar{t}) \in M \times [0,T)$ such that $\lambda_n(\bar{x},\bar{t}) < \mu(\bar{x},\bar{t})$ and $\mu(\bar{x},\bar{t}) \geq 8\, \operatorname{inj}(X)^{-1}$. We further assume that U is an open neighborhood of $\bar{x}$ and $\Phi : U \times (\bar{t}-\alpha, \bar{t}] \to \mathbb{R}$ is a smooth function such that $\Phi(\bar{x},\bar{t}) = \mu(\bar{x},\bar{t})$ and $\Phi(x,t) \geq \mu(x,t)$ for all points $(x,t) \in U \times (\bar{t}-\alpha,\bar{t}]$. Then*

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi - |A|^2\, \Phi + \sum_{i=1}^n \frac{1}{\Phi - \lambda_i}\, (D_i \Phi)^2 \leq C\, H + C\, \Phi + C\, \sum_{i=1}^n \frac{1}{\Phi - \lambda_i}$$

at the point $(\bar{x},\bar{t})$. Here, C is a positive constant that depends only on the ambient manifold X and the initial hypersurface M_0.

Proof. Let us define a function $Z : M \times M \times [0,T) \to \mathbb{R}$ by

$$\begin{aligned} Z(x,y,t) &= \Phi(x,t)\, \psi_{F(y,t)}(F(x,t)) - \Big\langle \nabla \psi_{F(y,t)}\big|_{F(x,t)}, \nu(x,t) \Big\rangle \\ &= \frac{1}{2}\, \Phi(x,t)\, d(F(x,t),F(y,t))^2 + \Big\langle \exp_{F(x,t)}^{-1}(F(y,t)), \nu(x,t) \Big\rangle. \end{aligned}$$

By assumption, we have $Z(x,y,t) \geq 0$ whenever $x \in U$, $t \in (\bar{t}-\alpha,\bar{t}]$, and $d(F(x,t),F(y,t)) \leq \frac{1}{2}\operatorname{inj}(X)$. Moreover, we can find a point $\bar{y} \in M$ such that $0 < d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t})) \leq \frac{1}{2}\operatorname{inj}(X)$ and $Z(\bar{x},\bar{y},\bar{t}) = 0$. It is clear that $\Phi(\bar{x},\bar{t})\, d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t})) \leq 2$, so $d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t})) \leq \frac{1}{4}\operatorname{inj}(X)$. This implies

$$\begin{aligned} 0 = \frac{\partial Z}{\partial x_i}(\bar{x},\bar{y},\bar{t}) &= \frac{1}{2}\, \frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})\, d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2 \\ &\quad - \Phi(\bar{x},\bar{t})\, \Big\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y},\bar{t})), \frac{\partial F}{\partial x_i}(\bar{x},\bar{t}) \Big\rangle \\ &\quad + h_i^j(\bar{x},\bar{t})\, \Big\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y},\bar{t})), \frac{\partial F}{\partial x_j}(\bar{x},\bar{t}) \Big\rangle \\ &\quad - \Xi_{F(\bar{y},\bar{t}),F(\bar{x},\bar{t})}\Big(\frac{\partial F}{\partial x_i}(\bar{x},\bar{t}), \nu(\bar{x},\bar{t})\Big). \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} &\Big\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y},\bar{t})), \frac{\partial F}{\partial x_i}(\bar{x},\bar{t}) \Big\rangle \\ &= \frac{1}{2}\, \frac{1}{\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t})}\, \Big(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t}) + O(1)\Big)\, d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2. \end{aligned}$$

We now differentiate one more time. Using the Codazzi equations, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}, \bar{t}) \\
&= \frac{1}{2}\, \Delta\Phi(\bar{x}, \bar{t})\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2 \\
&\quad - 2\, \frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t}) \left\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t}) \right\rangle \\
&\quad + \frac{\partial H}{\partial x_i}(\bar{x}, \bar{t}) \left\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t}) \right\rangle \\
&\quad + H(\bar{x}, \bar{t})\, \Phi(\bar{x}, \bar{t}) \left\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) \right\rangle \\
&\quad - |A(\bar{x}, \bar{t})|^2 \left\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) \right\rangle \\
&\quad + n\, \Phi(\bar{x}, \bar{t}) - H(\bar{x}, \bar{t}) \\
&\quad + O\Big(d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t})) + H(\bar{x}, \bar{t})\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2\Big),
\end{aligned}$$

hence

$$\begin{aligned}
&\sum_{i=1}^n \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}, \bar{t}) \\
&\leq \frac{1}{2} \bigg(\Delta\Phi(\bar{x}, \bar{t}) + |A(\bar{x}, \bar{t})|^2\, \Phi(\bar{x}, \bar{t}) \\
&\qquad - \sum_{i=1}^n \frac{2}{\Phi(\bar{x}, \bar{t}) - \lambda_i(\bar{x}, \bar{t})} \left(\frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t}) \right)^2 \bigg)\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2 \\
&\quad + \frac{\partial H}{\partial x_i}(\bar{x}, \bar{t}) \left\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t}) \right\rangle \\
&\quad + H(\bar{x}, \bar{t})\, \Phi(\bar{x}, \bar{t}) \left\langle \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) \right\rangle \\
&\quad + n\, \Phi(\bar{x}, \bar{t}) - H(\bar{x}, \bar{t}) \\
&\quad + O\big(d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t})) + H(\bar{x}, \bar{t})\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2\big) \\
&\quad + O\left(\sum_{i=1}^n \frac{1}{\Phi(\bar{x}, \bar{t}) - \lambda_i(\bar{x}, \bar{t})} \left| \frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t}) \right| d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2 \right).
\end{aligned}$$

We next compute

$$\begin{aligned}
0 &= \frac{\partial Z}{\partial y_i}(\bar{x}, \bar{y}, \bar{t}) \\
&= \left\langle \Big(D \exp_{F(\bar{x},\bar{t})}^{-1}\Big)_{F(\bar{y},\bar{t})} \left(\frac{\partial F}{\partial y_i}(\bar{y}, \bar{t}) \right), \nu(\bar{x}, \bar{t}) + \Phi(\bar{x}, \bar{t})\, \exp_{F(\bar{x},\bar{t})}^{-1}(F(\bar{y}, \bar{t})) \right\rangle.
\end{aligned}$$

Note that the vector $\nu(\bar{x},\bar{t}) + \Phi(\bar{x},\bar{t})\,\exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t}))$ has unit length. From this, we deduce that

$$\Big(D\exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}(\nu(\bar{y},\bar{t})) = \nu(\bar{x},\bar{t}) + \Phi(\bar{x},\bar{t})\,\exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})) + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\big).$$

Moreover, we can arrange that

$$\begin{aligned}&\Big(D\exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}\left(\frac{\partial F}{\partial y_i}(\bar{y},\bar{t})\right)\\ &= \frac{\partial F}{\partial x_i}(\bar{x},\bar{t}) - 2\,\frac{\big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})),\, \frac{\partial F}{\partial x_i}(\bar{x},\bar{t})\big\rangle}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2}\,\frac{\exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t}))}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2}\\ &\quad + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\big).\end{aligned}$$

This gives

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y},\bar{t}) &= \frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})\Big\langle\Big(D\exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}\left(\frac{\partial F}{\partial y_i}(\bar{y},\bar{t})\right), \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t}))\Big\rangle\\ &\quad - \big(\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t})\big)\Big\langle\Big(D\exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}\left(\frac{\partial F}{\partial y_i}(\bar{y},\bar{t})\right), \frac{\partial F}{\partial x_i}(\bar{x},\bar{t})\Big\rangle\\ &\quad + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big)\\ &= -\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})\,\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})), \frac{\partial F}{\partial x_i}(\bar{x},\bar{t})\Big\rangle\\ &\quad - (\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t}))\left(1 - 2\,\frac{\big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})),\, \frac{\partial F}{\partial x_i}(\bar{x},\bar{t})\big\rangle^2}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2}\right)\\ &\quad + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t})) + H(\bar{x},\bar{t})\,d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\big)\\ &= -(\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t}))\\ &\quad + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t})) + H(\bar{x},\bar{t})\,d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\big)\end{aligned}$$

for each i. Summation over i gives

$$\begin{aligned}\sum_{i=1}^n \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y},\bar{t}) &= -n\,\Phi(\bar{x},\bar{t}) + H(\bar{x},\bar{t})\\ &\quad + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t})) + H(\bar{x},\bar{t})\,d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\big).\end{aligned}$$

Finally, we have

$$\sum_{i=1}^n \frac{\partial^2 Z}{\partial y_i^2}(\bar{x}, \bar{y}, \bar{t})$$
$$= n\Phi(\bar{x},\bar{t}) - H(\bar{y},\bar{t}) \Big\langle \Big(D \exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}(\nu(\bar{y},\bar{t})), \nu(\bar{x},\bar{t}) + \Phi(\bar{x},\bar{t}) \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t}))\Big\rangle$$
$$+ O\big(d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))\big).$$

Thus, we conclude that

$$\sum_{i=1}^n \left(\frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}, \bar{t}) + 2\,\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x}, \bar{y}, \bar{t}) + \frac{\partial^2 Z}{\partial y_i^2}(\bar{x}, \bar{y}, \bar{t})\right)$$
$$\leq \frac{1}{2}\Bigg(\Delta\Phi(\bar{x}, \bar{t}) + |A(\bar{x}, \bar{t})|^2\,\Phi(\bar{x}, \bar{t})$$
$$- \sum_{i=1}^n \frac{2}{\Phi(\bar{x}, \bar{t}) - \lambda_i(\bar{x}, \bar{t})}\,\left(\frac{\partial\Phi}{\partial x_i}(\bar{x}, \bar{t})\right)^2\Bigg)\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2$$
$$+ \frac{\partial H}{\partial x_i}(\bar{x}, \bar{t})\,\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t})\Big\rangle$$
$$+ H(\bar{x}, \bar{t}) + H(\bar{x}, \bar{t})\,\Phi(\bar{x}, \bar{t})\,\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t})\Big\rangle$$
$$- H(\bar{y}, \bar{t})\Big\langle\Big(D \exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}(\nu(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) + \Phi(\bar{x}, \bar{t})\,\exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t}))\Big\rangle$$
$$+ O\big(d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t})) + H(\bar{x}, \bar{t})\,d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2\big)$$
$$+ O\left(\sum_{i=1}^n \frac{1}{\Phi(\bar{x}, \bar{t}) - \lambda_i(\bar{x}, \bar{t})}\,\Big|\frac{\partial\Phi}{\partial x_i}(\bar{x}, \bar{t})\Big|\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2\right).$$

On the other hand, we have

$$\frac{\partial Z}{\partial t}(\bar{x}, \bar{y}, \bar{t}) = \frac{1}{2}\,\frac{\partial\Phi}{\partial t}(\bar{x}, \bar{t})\,d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2$$
$$+ H(\bar{x}, \bar{t}) + H(\bar{x}, \bar{t})\,\Phi(\bar{x}, \bar{t})\,\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t})\Big\rangle$$
$$- H(\bar{y}, \bar{t})\Big\langle\Big(D \exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})}(\nu(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) + \Phi(\bar{x},\bar{t})\,\exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t}))\Big\rangle$$
$$+ \sum_{i=1}^n \frac{\partial H}{\partial x_i}(\bar{x}, \bar{t})\,\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t})\Big\rangle$$
$$+ H(\bar{x}, \bar{t})\,\Xi_{F(\bar{y},\bar{t}),F(\bar{x},\bar{t})}(\nu(\bar{x}, \bar{t}), \nu(\bar{x}, \bar{t})).$$

Consequently,

$$\begin{aligned}
0 &\geq \frac{\partial Z}{\partial t}(\bar{x}, \bar{y}, \bar{t}) - \sum_{i=1}^{n} \left(\frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}, \bar{t}) + 2\, \frac{\partial^2 Z}{\partial x_i\, \partial y_i}(\bar{x}, \bar{y}, \bar{t}) + \frac{\partial^2 Z}{\partial y_i^2}(\bar{x}, \bar{y}, \bar{t}) \right) \\
&\geq \frac{1}{2} \Bigg(\frac{\partial \Phi}{\partial t}(\bar{x}, \bar{t}) - \Delta\Phi(\bar{x}, \bar{t}) - |A(\bar{x}, \bar{t})|^2\, \Phi(\bar{x}, \bar{t}) \\
&\qquad + \sum_{i=1}^{n} \frac{2}{\Phi(\bar{x}, \bar{t}) - \lambda_i(\bar{x}, \bar{t})} \left(\frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t}) \right)^2 \Bigg)\, d(F(\bar{x}, \bar{t}),\, F(\bar{y}, \bar{t}))^2 \\
&\quad - O\big(d(F(\bar{x}, \bar{t}),\, F(\bar{y}, \bar{t})) + H(\bar{x}, \bar{t})\, d(F(\bar{x}, \bar{t}),\, F(\bar{y}, \bar{t}))^2 \big) \\
&\quad - O\left(\sum_{i=1}^{n} \frac{1}{\Phi(\bar{x}, \bar{t}) - \lambda_i(\bar{x}, \bar{t})} \left| \frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t}) \right| d(F(\bar{x}, \bar{t}),\, F(\bar{y}, \bar{t}))^2 \right).
\end{aligned}$$

We now multiply both sides by $\frac{2}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2}$. Using the estimate

$$\begin{aligned}
\frac{1}{d(F(\bar{x},\bar{t}), F(\bar{y},\bar{t}))} &\leq \frac{\left|\left\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})), \nu(\bar{x},\bar{t}) \right\rangle\right|}{d(F(\bar{x},\bar{t}), F(\bar{y},\bar{t}))^2} \\
&\qquad + \sum_{i=1}^{n} \frac{\left|\left\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})), \frac{\partial F}{\partial x_i}(\bar{x},\bar{t}) \right\rangle\right|}{d(F(\bar{x},\bar{t}), F(\bar{y},\bar{t}))^2} \\
&\qquad \leq \frac{1}{2}\, \Phi(\bar{x},\bar{t}) + \sum_{i=1}^{n} \frac{1}{2}\, \frac{1}{\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t})} \left(\left| \frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t}) \right| + O(1) \right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{\partial \Phi}{\partial t}(\bar{x},\bar{t}) - \Delta\Phi(\bar{x},\bar{t}) - |A(\bar{x},\bar{t})|^2\, \Phi(\bar{x},\bar{t}) + \sum_{i=1}^{n} \frac{2}{\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t})} \left(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t}) \right)^2 \\
&\leq O\left(H(\bar{x},\bar{t}) + \Phi(\bar{x},\bar{t}) + \sum_{i=1}^{n} \frac{1}{\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t})} + \sum_{i=1}^{n} \frac{1}{\Phi(\bar{x},\bar{t}) - \lambda_i(\bar{x},\bar{t})} \left| \frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t}) \right| \right).
\end{aligned}$$

From this, the assertion follows. □

Corollary 2.2. *The function μ satisfies the evolution equation*

$$\frac{\partial \mu}{\partial t} - \Delta\mu - |A|^2\,\mu + \sum_{i=1}^n \frac{1}{\mu - \lambda_i}\,(D_i\mu)^2 \leq C\,H + C\,\mu + C\,\sum_{i=1}^n \frac{1}{\mu - \lambda_i}$$

on the set $\{\lambda_n < \mu\} \cap \{\mu \geq 8\,\mathrm{inj}(X)^{-1}\}$. Here, $\Delta\mu$ is interpreted in the sense of distributions. Moreover, C is a positive constant that depends only on the ambient manifold X and the initial hypersurface M_0.

Proof. It is easy to see that the function μ is locally Lipschitz continuous and semi-convex (see [3, Proposition 2.1]). Hence, by a theorem of Alexandrov, the function μ admits a second order Taylor expansion around almost every point (*cf.* [5, Section 6.4]). Let χ_{ij} denote the spatial Hessian of μ in the sense of Alexandrov. Note that χ is defined outside a set of measure zero. It follows from Proposition 2.1 that

$$\frac{\partial \mu}{\partial t} - \mathrm{tr}(\chi) - |A|^2\,\mu + \sum_{i=1}^n \frac{1}{\mu - \lambda_i}\,(D_i\mu)^2 \leq C\,H + C\,\mu + C\,\sum_{i=1}^n \frac{1}{\mu - \lambda_i}$$

at almost every point in the set $\{\lambda_n < \mu\} \cap \{\mu \geq 8\,\mathrm{inj}(X)^{-1}\}$. On the other hand, it is well known that the Hessian in the sense of Alexandrov is dominated by the distributional second derivative (see [5, Section 6.4]). Consequently, $\mathrm{tr}(\chi) \leq \Delta\mu$, where $\Delta\mu$ is interpreted in the sense of distributions. From this, the assertion follows. □

Corollary 2.3. *We have*

$$\sup_{t \in [0,T)} \sup_{M_t} \frac{\mu}{H} \leq C,$$

where C is a constant that depends only on the ambient manifold X, the initial hypersurface M_0, and on T.

Proof. The ratio $\frac{\mu}{H}$ saisfies an evolution equation of the form

$$\frac{\partial}{\partial t}\Big(\frac{\mu}{H}\Big) - \Delta\Big(\frac{\mu}{H}\Big) - 2\,\Big\langle \frac{\nabla H}{H}, \nabla\Big(\frac{\mu}{H}\Big)\Big\rangle \leq C + C\,\frac{\mu}{H} + C\,\sum_{i=1}^n \frac{1}{H\,(\mu - \lambda_i)}.$$

It follows from results in [9] that

$$\sup_{t \in [0,T)} \sup_{M_t} \frac{|\lambda_i| + 1}{H} \leq K,$$

where K is a constant that depends only on the ambient manifold X, the initial hypersurface M_0, and on T. Hence, if $\frac{\mu}{H} \geq 2K$, then $\frac{\mu-\lambda_i}{H} \geq K$, and therefore $\frac{1}{H\,(\mu-\lambda_i)} \leq \frac{1}{K\,H^2} \leq K$. Thus, we conclude that

$$\frac{\partial}{\partial t}\Big(\frac{\mu}{H}\Big) - \Delta\Big(\frac{\mu}{H}\Big) - 2\,\Big\langle \frac{\nabla H}{H}, \nabla\Big(\frac{\mu}{H}\Big)\Big\rangle \leq C + C\,\frac{\mu}{H}$$

whenever $\frac{\mu}{H} \geq 2K$. Hence, the assertion follows from the maximum principle. □

3. An auxiliary inequality

In this section, we will consider a single hypersurface $M_{\bar{t}}$ for some fixed time $\bar{t}$. We will suppress $\bar{t}$ in the notation, as we will only work with a fixed hypersurface. By the convexity estimate of Huisken and Sinestrari (*cf.* [9, Remark 3.9]), we have a pointwise estimate of the form $\lambda_1 \geq -\varepsilon\,H - K_1(\varepsilon)$, where ε is an arbitrary positive real number.

Proposition 3.1. *Consider a point $\bar{x} \in M$ such that $\lambda_n(\bar{x}) < \mu(\bar{x})$ and $\mu(\bar{x}) \geq 8\,\mathrm{inj}(X)^{-1}$. Furthermore, we assume that U is an open neighborhood of $\bar{x}$ and $\Phi : U \to \mathbb{R}$ is a smooth function such that $\Phi(\bar{x}) = \mu(\bar{x})$ and $\Phi(x) \geq \mu(x)$ for all $x \in U$. Then*

$$\begin{aligned}
0 \leq \Delta\Phi &+ \frac{1}{2}\,|A|^2\,\Phi - \frac{1}{2}\,H\,\Phi^2 + \frac{1}{2}\,n^3\,\big(n\varepsilon\,\Phi + K_1(\varepsilon)\big)\,\Phi^2 \\
&+ \sum_{i=1}^n \frac{1}{\Phi - \lambda_i}\,(|D_i\Phi| + C)\,|D_i H| \\
&+ \big(H + n^3\,(n\varepsilon\,\Phi + K_1(\varepsilon))\big)\sum_{i=1}^n \frac{1}{(\Phi - \lambda_i)^2}\,\big((D_i\Phi)^2 + C\big) \\
&+ C\,\Phi + C\,\sum_{i=1}^n \frac{1}{\Phi - \lambda_i}
\end{aligned}$$

at the point $\bar{x}$. Here, C is a positive constant that depends only on X, M_0, and T.

Proof. As above, we define

$$Z(x, y) = \frac{1}{2}\,\Phi(x)\,d(F(x), F(y))^2 + \big\langle \exp_{F(x)}^{-1}(F(y)), \nu(x)\big\rangle.$$

By assumption, we have $Z(x, y) \geq 0$ whenever $x \in U$ and $d(F(x), F(y)) \leq \frac{1}{2}\,\mathrm{inj}(X)$. Moreover, there exists a point $\bar{y} \in M$ such that $0 < d(F(\bar{x}), F(\bar{y})) \leq$

$\frac{1}{2}\,\mathrm{inj}(X)$ and $Z(\bar{x},\bar{y})=0$. As above, it is easy to see that $\Phi(\bar{x})\,d(F(\bar{x}),F(\bar{y}))\leq 2$, so $d(F(\bar{x}),F(\bar{y}))\leq\frac{1}{4}\,\mathrm{inj}(X)$. Moreover, we have $H(\bar{x})\leq C\,\Phi(\bar{x})$ and $H(\bar{y})\leq C\,\Phi(\bar{x})$ for some constant C that depends only on the ambient manifold X.

It follows from results in Section 2 that

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y})\\
&\leq\frac{1}{2}\Bigg(\Delta\Phi(\bar{x})+|A(\bar{x})|^2\,\Phi(\bar{x})-H(\bar{x})\,\Phi(\bar{x})^2-\sum_{i=1}^{n}\frac{2}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})\right)^2\\
&\qquad+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})+O(1)\right)\frac{\partial H}{\partial x_i}(\bar{x})\Bigg)\,d(F(\bar{x}),F(\bar{y}))^2\\
&\quad+n\,\Phi(\bar{x})-H(\bar{x})\\
&\quad+O\big(d(F(\bar{x}),F(\bar{y}))\big)+O\left(\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left|\frac{\partial\Phi}{\partial x_i}(\bar{x})\right|d(F(\bar{x}),F(\bar{y}))^2\right).
\end{aligned}$$

Moreover, we have

$$\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y})=-(\Phi(\bar{x})-\lambda_i(\bar{x}))+O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big)$$

and

$$\frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y})=\Phi(\bar{x})-h_{ii}(\bar{y})+O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big).$$

In particular, we have $h_{ii}(\bar{y})\leq\Phi(\bar{x})+O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big)$, hence $H(\bar{y})\leq n\,\Phi(\bar{x})+O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big)$. Consequently, the convexity estimate of Huisken and Sinestrari (see Remark 3.9 in [9]) implies that $h_{ii}(\bar{y})\geq-\varepsilon\,H(\bar{y})-K_1(\varepsilon)\geq-n\varepsilon\,\Phi(\bar{x})-K_1(\varepsilon)-O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big)$. From this, we deduce that

$$\frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y})\leq\Phi(\bar{x})+n\varepsilon\,\Phi(\bar{x})+K_1(\varepsilon)+O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big).$$

Note that $\Phi(\bar{x})-\lambda_i(\bar{x})\geq 0$ for $i=1,\dots,n$. From this, we deduce that $\max_{1\leq i\leq n}(\Phi(\bar{x})-\lambda_i(\bar{x}))\leq\sum_{i=1}^{n}(\Phi(\bar{x})-\lambda_i(\bar{x}))\leq n\Phi(\bar{x})$, hence $\sum_{i=1}^{n}\frac{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}{\Phi(\bar{x})^2}\leq$

n^3. Thus, we conclude that

$$\begin{aligned}
&\sum_{i=1}^n \left(\frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2\,\frac{\Phi(\bar{x})-\lambda_i(\bar{x})}{\Phi(\bar{x})}\,\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) + \frac{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}{\Phi(\bar{x})^2}\,\frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y})\right) \\
&\leq \frac{1}{2}\left(\Delta\Phi(\bar{x}) + |A(\bar{x})|^2\,\Phi(\bar{x}) - H(\bar{x})\,\Phi(\bar{x})^2 - \sum_{i=1}^n \frac{2}{\Phi(\bar{x})-\lambda_i(\bar{x})}\,\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})\right)^2\right. \\
&\qquad\left. + \sum_{i=1}^n \frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\,\left(\frac{\partial\Phi}{\partial x_i}(\bar{x}) + O(1)\right)\,\frac{\partial H}{\partial x_i}(\bar{x})\right) d(F(\bar{x}),F(\bar{y}))^2 \\
&\quad + n\,\Phi(\bar{x}) - H(\bar{x}) - \sum_{i=1}^n \frac{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}{\Phi(\bar{x})} \\
&\quad + \sum_{i=1}^n \frac{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}{\Phi(\bar{x})^2}\,(n\varepsilon\,\Phi(\bar{x}) + K_1(\varepsilon)) \\
&\quad + O\big(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\big) \\
&\quad + O\left(\sum_{i=1}^n \frac{1}{\Phi(\bar{x},\bar{t})-\lambda_i(\bar{x},\bar{t})}\,\left|\frac{\partial\Phi}{\partial x_i}(\bar{x},\bar{t})\right| d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\right) \\
&\leq \frac{1}{2}\left(\Delta\Phi(\bar{x}) + |A(\bar{x})|^2\,\Phi(\bar{x}) - H(\bar{x})\,\Phi(\bar{x})^2\right. \\
&\qquad - \sum_{i=1}^n \frac{2}{\Phi(\bar{x})-\lambda_i(\bar{x})}\,\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})\right)^2 \\
&\qquad\left. + \sum_{i=1}^n \frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\,\left(\frac{\partial\Phi}{\partial x_i}(\bar{x}) + O(1)\right)\,\frac{\partial H}{\partial x_i}(\bar{x})\right) d(F(\bar{x}),F(\bar{y}))^2 \\
&\quad + H(\bar{x}) - \frac{|A(\bar{x})|^2}{\Phi(\bar{x})} + n^3\,(n\varepsilon\,\Phi(\bar{x}) + K_1(\varepsilon)) \\
&\quad + O\left(d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))\right) \\
&\quad + O\left(\sum_{i=1}^n \frac{1}{\Phi(\bar{x},\bar{t})-\lambda_i(\bar{x},\bar{t})}\,\left|\frac{\partial\Phi}{\partial x_i}(\bar{x},\bar{t})\right| d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\right).
\end{aligned}$$

We now multiply both sides by $\frac{2}{d(F(\bar{x}),F(\bar{y}))^2}$. Using the identity

$$\begin{aligned}
\frac{1}{d(F(\bar{x}),F(\bar{y}))^2} &= \frac{\left\langle \exp^{-1}_{F(\bar{x})}(F(\bar{y})),\,\nu(\bar{x})\right\rangle^2}{d(F(\bar{x}),F(\bar{y}))^4} + \sum_{i=1}^n \frac{\left\langle \exp^{-1}_{F(\bar{x})}(F(\bar{y})),\,\frac{\partial F}{\partial x_i}(\bar{x})\right\rangle^2}{d(F(\bar{x}),F(\bar{y}))^4} \\
&= \frac{1}{4}\left(\Phi(\bar{x})^2 + \sum_{i=1}^n \frac{1}{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}\,\left(\frac{\partial\Phi}{\partial x_i}(\bar{x}) + O(1)\right)^2\right),
\end{aligned}$$

we derive the estimate

$$
\begin{aligned}
&\frac{2}{d(F(\bar{x}),F(\bar{y}))^2}\sum_{i=1}^{n}\biggl(\frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y})+2\,\frac{\Phi(\bar{x})-\lambda_i(\bar{x})}{\Phi(\bar{x})}\,\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y})\\
&\qquad\qquad+\frac{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}{\Phi(\bar{x})^2}\,\frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y})\biggr)\\
&\leq \Delta\Phi(\bar{x})+|A(\bar{x})|^2\,\Phi(\bar{x})-H(\bar{x})\,\Phi(\bar{x})^2\\
&\quad-\sum_{i=1}^{n}\frac{2}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})\right)^2+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})+O(1)\right)\frac{\partial H}{\partial x_i}(\bar{x})\\
&\quad+\frac{1}{2}\left(H(\bar{x})-\frac{|A(\bar{x})|^2}{\Phi(\bar{x})}+n^3\,(n\varepsilon\,\Phi(\bar{x})+K_1(\varepsilon))\right)\\
&\quad\times\left(\Phi(\bar{x})^2+\sum_{i=1}^{n}\frac{1}{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})+O(1)\right)^2\right)\\
&\quad+O\left(\Phi(\bar{x})+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left|\frac{\partial\Phi}{\partial x_i}(\bar{x})\right|\right)\\
&=\Delta\Phi(\bar{x})+\frac{1}{2}\,|A(\bar{x})|^2\,\Phi(\bar{x})-\frac{1}{2}\,H(\bar{x})\,\Phi(\bar{x})^2+\frac{1}{2}\,n^3\,(n\varepsilon\,\Phi(\bar{x})+K_1(\varepsilon))\,\Phi(\bar{x})^2\\
&\quad-\sum_{i=1}^{n}\frac{2}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})\right)^2+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})+O(1)\right)\frac{\partial H}{\partial x_i}(\bar{x})\\
&\quad+\frac{1}{2}\left(H(\bar{x})-\frac{|A(\bar{x})|^2}{\Phi(\bar{x})}+n^3(n\varepsilon\Phi(\bar{x})+K_1(\varepsilon))\right)\sum_{i=1}^{n}\frac{1}{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}\left(\frac{\partial\Phi}{\partial x_i}(\bar{x})+O(1)\right)^2\\
&\quad+O\left(\Phi(\bar{x})+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}+\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\left|\frac{\partial\Phi}{\partial x_i}(\bar{x})\right|\right).
\end{aligned}
$$

Since the function Z attains a local minimum at the point $(\bar{x},\bar{y})$, we have

$$
\sum_{i=1}^{n}\left(\frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y})+2\,\frac{\Phi(\bar{x})-\lambda_i(\bar{x})}{\Phi(\bar{x})}\,\frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y})+\frac{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}{\Phi(\bar{x})^2}\,\frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y})\right)\geq 0.
$$

Putting these facts together, we obtain

$$\begin{aligned}0 &\leq \Delta\Phi(\bar{x}) + \frac{1}{2}\,|A(\bar{x})|^2\,\Phi(\bar{x}) - \frac{1}{2}\,H(\bar{x})\,\Phi(\bar{x})^2 + \frac{1}{2}\,n^3(n\varepsilon\,\Phi(\bar{x}) + K_1(\varepsilon))\,\Phi(\bar{x})^2 \\ &- \sum_{i=1}^n \frac{2}{\Phi(\bar{x})-\lambda_i(\bar{x})}\Big(\frac{\partial\Phi}{\partial x_i}(\bar{x})\Big)^2 + \sum_{i=1}^n \frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\Big(\frac{\partial\Phi}{\partial x_i}(\bar{x}) + O(1)\Big)\,\frac{\partial H}{\partial x_i}(\bar{x}) \\ &+ \frac{1}{2}\Big(H(\bar{x}) - \frac{|A(\bar{x})|^2}{\Phi(\bar{x})} + n^3(n\varepsilon\Phi(\bar{x}) + K_1(\varepsilon))\Big)\sum_{i=1}^n \frac{1}{(\Phi(\bar{x})-\lambda_i(\bar{x}))^2}\Big(\frac{\partial\Phi}{\partial x_i}(\bar{x}) + O(1)\Big)^2 \\ &+ O\Big(\Phi(\bar{x}) + \sum_{i=1}^n \frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})} + \sum_{i=1}^n \frac{1}{\Phi(\bar{x})-\lambda_i(\bar{x})}\,\Big|\frac{\partial\Phi}{\partial x_i}(\bar{x})\Big|\Big).\end{aligned}$$

From this, the assertion follows. ☐

Corollary 3.2. *We have*

$$\begin{aligned}0 \leq \Delta\mu &+ \frac{1}{2}\,|A|^2\,\mu - \frac{1}{2}\,H\,\mu^2 + \frac{1}{2}\,n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\,\mu^2 \\ &+ \sum_{i=1}^n \frac{1}{\mu-\lambda_i}\,(|D_i\mu| + C)\,|D_i H| \\ &+ \big(H + n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\big)\sum_{i=1}^n \frac{1}{(\mu-\lambda_i)^2}\,((D_i\mu)^2 + C) \\ &+ C\,\mu + C\,\sum_{i=1}^n \frac{1}{\mu-\lambda_i}\end{aligned}$$

on the set $\{\lambda_n < \mu\} \cap \{\mu \geq 8\,\mathrm{inj}(X)^{-1}\}$. *Here,* $\Delta\mu$ *is interpreted in the sense of distributions*.

Corollary 3.3. *We have*

$$\begin{aligned}0 \leq &-\int_{M_t} \langle\nabla\eta, \nabla\mu\rangle + \frac{1}{2}\int_{M_t} \eta\,\big(|A|^2\,\mu - H\,\mu^2 + n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\,\mu^2\big) \\ &+ \int_{M_t} \eta\,\sum_{i=1}^n \frac{1}{\mu-\lambda_i}\,(|D_i\mu| + C)\,|D_i H| \\ &+ \int_{M_t} \eta\,\big(H + n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\big)\sum_{i=1}^n \frac{1}{(\mu-\lambda_i)^2}\,((D_i\mu)^2 + C) \\ &+ C\int_{M_t} \eta\,\mu + C\int_{M_t} \eta\,\sum_{i=1}^n \frac{1}{\mu-\lambda_i}\end{aligned}$$

for every nonnegative test function η *which is supported in the set* $\{\lambda_n < \mu\} \cap \{\mu \geq 8\,\mathrm{inj}(X)^{-1}\}$.

4. Proof of Theorem 1.1

Let us fix positive real numbers $\delta > 0$ and $T > 0$. By the convexity estimate of Huisken and Sinestrari (*cf.* [9, Remark 3.9]), we can find a constant $K_0 \geq 8\,\mathrm{inj}(X)^{-1}\left(\inf_{t\in[0,T)} \inf_{M_t} \min\{H, 1\}\right)^{-1}$ such that

$$(n-1)\,\lambda_1 \geq -\frac{\delta}{2}\,H - K_0\,\min\{H, 1\}$$

for $t \in [0, T)$. Here, K_0 is a constant that depends only on X, M_0, δ, and T.

For each $\sigma \in (0, \frac{1}{2})$, we define

$$f_\sigma = H^{\sigma-1}\,(\mu - (1+\delta)\,H) - K_0$$

and

$$f_{\sigma,+} = \max\{f_\sigma, 0\}.$$

On the set $\{f_\sigma \geq 0\}$, we have

$$\mu \geq (1+\delta)\,H + K_0\,H^{1-\sigma} \geq (1+\delta)\,H + K_0\,\min\{H, 1\},$$

hence

$$\mu - \lambda_n \geq \sum_{i=1}^{n-1} \lambda_i + \delta\,H + K_0\,\min\{H, 1\} \geq \frac{\delta}{2}\,H.$$

In particular, we have $\{f_\sigma \geq 0\} \subset \{\lambda_n < \mu\} \cap \{\mu \geq 8\,\mathrm{inj}(X)^{-1}\}$. By Corollary 2.3, we can find a constant $\Lambda \geq 1$, depending only on X, M_0, and T, such that $\mu \leq \Lambda\,H$ and $|A|^2 \leq \Lambda\,H^2$ for $t \in [0, T)$.

Proposition 4.1. *Given any $\delta > 0$, we can find a positive constant c_0, depending only on δ and the initial hypersurface M_0, with the following property: if $p \geq \frac{1}{c_0}$ and $\sigma \leq c_0\,p^{-\frac{1}{2}}$, then we have*

$$\frac{d}{dt}\left(\int_{M_t} f_{\sigma,+}^p\right) \leq C\,\sigma\,p \int_{M_t} f_{\sigma,+}^p + \sigma\,p\,K_0^p \int_{M_t} |A|^2 + (Cp)^p\,|M_t|$$

for almost all $t \in [0, T)$. Here, C is a positive constant that depends only on X, M_0, δ, and T, but not on σ and p.

Proof. By Corollary 2.2, we have

$$\frac{\partial \mu}{\partial t} - \Delta\mu - |A|^2\,\mu + \sum_{i=1}^{n} \frac{1}{\mu - \lambda_i}\,(D_i\mu)^2 \leq C\,H$$

on the set $\{f_\sigma \geq 0\}$, where $\Delta\mu$ is interpreted in the sense of distributions, and C is a positive constant that depends only on X, M_0, δ, and T. A straightforward calculation gives

$$\begin{aligned}
&\frac{\partial}{\partial t} f_\sigma - \Delta f_\sigma - 2\,(1-\sigma)\left\langle \frac{\nabla H}{H}, \nabla f_\sigma \right\rangle + \sum_{i=1}^n \frac{H^{\sigma-1}}{\mu - \lambda_i}\,(D_i\mu)^2 - \sigma\,|A|^2\,(f_\sigma + K_0) \\
&\leq -\sigma\,(1-\sigma)\,H^{\sigma-3}\,(\mu - (1+\delta)\,H)\,|\nabla H|^2 + C\,H^\sigma \\
&\leq C\,H^\sigma
\end{aligned}$$

on the set $\{f_\sigma \geq 0\}$, where Δf_σ is again interpreted in the sense of distributions. This implies

$$\begin{aligned}
\frac{d}{dt}\left(\int_{M_t} f_{\sigma,+}^p\right) \leq &-p(p-1)\int_{M_t} f_{\sigma,+}^{p-2}\,|\nabla f_\sigma|^2 + 2\,(1-\sigma)\,p\int_{M_t} f_{\sigma,+}^{p-1}\left\langle \frac{\nabla H}{H}, \nabla f_\sigma \right\rangle \\
&- p\int_{M_t} \sum_{i=1}^n \frac{f_{\sigma,+}^{p-1}\,H^{\sigma-1}}{\mu - \lambda_i}\,(D_i\mu)^2 + \sigma\,p\int_{M_t} |A|^2\,f_{\sigma,+}^{p-1}\,(f_\sigma + K_0) \\
&+ \int_{M_t}\left(C\,p\,H^\sigma\,f_{\sigma,+}^{p-1} - H^2\,f_{\sigma,+}^p\right).
\end{aligned}$$

The integral of $|A|^2\,f_{\sigma,+}^{p-1}\,(f_\sigma + K_0)$ has an unfavorable sign. To estimate this term, we put $\varepsilon = \frac{\delta}{4n^4\Lambda^2}$. Applying Corollary 3.3 to the test function $\eta = \frac{f_{\sigma,+}^p}{H}$ gives

$$\begin{aligned}
&\frac{1}{2}\int_{M_t} \left(H\,\mu^2 - |A|^2\,\mu - n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\,\mu^2\right)\frac{f_{\sigma,+}^p}{H} \\
&\leq -\int_M \left\langle \nabla\left(\frac{f_{\sigma,+}^p}{H}\right), \nabla\mu \right\rangle + \int_M \frac{f_{\sigma,+}^p}{H}\sum_{i=1}^n \frac{1}{\mu - \lambda_i}\,(|D_i\mu| + C)\,|D_i H| \\
&\quad + \int_M \frac{f_{\sigma,+}^p}{H}\left(H + n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\right)\sum_{i=1}^n \frac{1}{(\mu - \lambda_i)^2}\,((D_i\mu)^2 + C) \\
&\quad + C\int_{M_t} \frac{f_{\sigma,+}^p}{H}\,\mu + C\int_{M_t} \frac{f_{\sigma,+}^p}{H}\sum_{i=1}^n \frac{1}{\mu - \lambda_i} \\
&\leq p\int_{M_t} \frac{f_{\sigma,+}^{p-1}}{H}\,|\nabla\mu|\,|\nabla f_\sigma| + C\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,(|\nabla\mu| + 1)\,|\nabla H| \\
&\quad + C\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,|\nabla\mu|^2 + C\int_{M_t} f_{\sigma,+}^p.
\end{aligned}$$

Here, C is a positive constant which depends on X, M_0, δ, and T, but not on σ and p. On the set $\{f_\sigma \geq 0\}$, we have $\mu \geq (1+\delta)\,H$. Moreover, the convexity estimate of Huisken and Sinestrari implies that $|A|^2 \leq (1+\varepsilon)\, H^2 + K_2(\varepsilon)$. Consequently, we have

$$\begin{aligned}
&H\,\mu^2 - |A|^2\,\mu - n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\,\mu^2 \\
&\geq (1+\delta)\,H^2\,\mu - |A|^2\,\mu - n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\,\Lambda^2\,H^2 \\
&\geq (\delta-\varepsilon)\,H^2\,\mu - n^3\,(n\varepsilon\,\mu + K_1(\varepsilon))\,\Lambda^2\,H^2 - K_2(\varepsilon)\,\mu \\
&\geq \frac{\delta}{2}\,H^2\,\mu - C\,H
\end{aligned}$$

on the set $\{f_\sigma \geq 0\}$. Therefore, we obtain

$$\begin{aligned}
\int_{M_t} |A|^2\, f_{\sigma,+}^p \leq C\,p \int_{M_t} \frac{f_{\sigma,+}^{p-1}}{H}\,|\nabla\mu|\,|\nabla f_\sigma| + C\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,(|\nabla\mu|+1)\,|\nabla H| \\
+ C\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,|\nabla\mu|^2 + C\int_{M_t} f_{\sigma,+}^p,
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T. Using the pointwise inequality

$$f_{\sigma,+}^{p-1}\,(f_\sigma + K_0) \leq 2\, f_{\sigma,+}^p + K_0^p,$$

we obtain

$$\begin{aligned}
&\int_{M_t} |A|^2\, f_{\sigma,+}^{p-1}\,(f_\sigma + K_0) \\
&\leq C\,p \int_{M_t} \frac{f_{\sigma,+}^{p-1}}{H}\,|\nabla\mu|\,|\nabla f_\sigma| + C\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,(|\nabla\mu|+1)\,|\nabla H| \\
&\quad + C\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,|\nabla\mu|^2 + C\int_{M_t} f_{\sigma,+}^p + K_0^p \int_{M_t} |A|^2,
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T. Putting these facts together, we conclude that

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{M_t} f_{\sigma,+}^p\Big) \\
&\leq -p(p-1)\int_{M_t} f_{\sigma,+}^{p-2}\,|\nabla f_\sigma|^2 + 2\,(1-\sigma)\,p\int_{M_t} f_{\sigma,+}^{p-1}\,\Big\langle \frac{\nabla H}{H}, \nabla f_\sigma \Big\rangle \\
&\quad - p\int_{M_t} \sum_{i=1}^n \frac{f_{\sigma,+}^{p-1}\,H^{\sigma-1}}{\mu-\lambda_i}\,(D_i\mu)^2 + C\,\sigma\,p^2 \int_{M_t} \frac{f_{\sigma,+}^{p-1}}{H}\,|\nabla\mu|\,|\nabla f_\sigma| \\
&\quad + C\,\sigma\,p\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,(|\nabla\mu|+1)\,|\nabla H| + C\,\sigma\,p\int_{M_t} \frac{f_{\sigma,+}^p}{H^2}\,|\nabla\mu|^2 \\
&\quad + C\,\sigma\,p\int_{M_t} f_{\sigma,+}^p + \sigma\,p\,K_0^p \int_{M_t} |A|^2 + \int_{M_t} \big(C\,p\,H^\sigma\, f_{\sigma,+}^{p-1} - H^2\, f_{\sigma,+}^p\big),
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T. Using the identity

$$\frac{\nabla H}{H} = \frac{\nabla \mu - H^{1-\sigma}\, \nabla f_\sigma}{(1-\sigma)\,\mu + \sigma\,(1+\delta)\,H},$$

we obtain

$$\left\langle \frac{\nabla H}{H}, \nabla f_\sigma \right\rangle \leq \frac{\langle \nabla \mu, \nabla f_\sigma \rangle}{(1-\sigma)\,\mu + \sigma\,(1+\delta)\,H} \leq C\,H^{-1}\,|\nabla \mu|\,|\nabla f_\sigma|$$

and

$$\frac{|\nabla H|}{H} \leq C\,\frac{|\nabla \mu|}{H} + C\,\frac{|\nabla f_\sigma|}{f_{\sigma,+}}.$$

This implies

$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_{M_t} f_{\sigma,+}^p\bigg) \\
&\leq -p(p-1)\int_{M_t} f_{\sigma,+}^{p-2}\,|\nabla f_\sigma|^2 - p\int_{M_t}\sum_{i=1}^n \frac{f_{\sigma,+}^{p-1}\,H^{\sigma-1}}{\mu-\lambda_i}\,(D_i\mu)^2 \\
&\quad + C\left(p+\sigma\,p^2\right)\int_{M_t}\frac{f_{\sigma,+}^{p-1}}{H}\,|\nabla\mu|\,|\nabla f_\sigma| + C\,\sigma\,p\int_{M_t}\frac{f_{\sigma,+}^p}{H^2}\,|\nabla\mu|^2 \\
&\quad + C\,\sigma\,p\int_{M_t}\frac{f_{\sigma,+}^p}{H^2}\,|\nabla\mu| + C\,\sigma\,p\int_{M_t}\frac{f_{\sigma,+}^{p-1}}{H}\,|\nabla f_\sigma| \\
&\quad + C\,\sigma\,p\int_{M_t} f_{\sigma,+}^p + \sigma\,p\,K_0^p\int_{M_t}|A|^2 + \int_{M_t}\Big(C\,p\,H^\sigma\,f_{\sigma,+}^{p-1} - H^2\,f_{\sigma,+}^p\Big),
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T.

Therefore, we can find a positive constant c_0, depending only on δ, M_0, X, and T, with the following property: if $p \geq \frac{1}{c_0}$ and $\sigma \leq c_0\,p^{-\frac{1}{2}}$, then we have

$$\frac{d}{dt}\bigg(\int_{M_t} f_{\sigma,+}^p\bigg) \leq C\,\sigma\,p\int_{M_t} f_{\sigma,+}^p + \sigma\,p\,K_0^p\int_{M_t}|A|^2 + \int_{M_t}\Big(C\,p\,H^\sigma\,f_{\sigma,+}^{p-1} - H^2\,f_{\sigma,+}^p\Big)$$

for almost all $t \in [0,T)$. Finally, since H is uniformly bounded from below on bounded time intervals, we have the pointwise estimate

$$C\,p\,H^\sigma\,f_{\sigma,+}^{p-1} - \frac{1}{C}\,H^2\,f_{\sigma,+}^p \leq (C'p)^p\,H^{2-(2-\sigma)\,p} \leq (C''p)^p.$$

This completes the proof of Proposition 4.1. □

As usual, we can now use the Michael-Simon Sobolev inequality (*cf.* [11]) and Stampacchia iteration to show that

$$\mu \leq (1+\delta)\,H + C(X, M_0, \delta, T)$$

for all $t \in [0,T)$ and all points on M_t. This is the desired noncollapsing estimate.

5. Proof of Theorem 1.2

Finally, we give the proof of Theorem 1.2.

Proposition 5.1. *The function ρ satisfies*

$$\frac{\partial \rho}{\partial t} - \Delta\rho - |A|^2\,\rho + \sum_{i=1}^n \frac{1}{\rho + \lambda_i}\,(D_i\rho)^2 \leq C\,H + C\,\rho + C\,\sum_{i=1}^n \frac{1}{\rho + \lambda_i}$$

on the set $\{\rho + \lambda_1 > 0\} \cap \{\rho \geq 8\,\mathrm{inj}(X)^{-1}\}$. Here, $\Delta\rho$ is interpreted in the sense of distributions. Moreover, C is a positive constant that depends only on the ambient manifold X and the initial hypersurface M_0.

The proof of Proposition 5.1 is analogous to the proof of Corollary 2.2 above. In fact, it suffices to reverse the orientation of M_t everywhere in the argument.

Corollary 5.2. *We have*

$$\sup_{t\in[0,T)}\,\sup_{M_t} \frac{\rho}{H} \leq C,$$

where C is a constant that depends only on the ambient manifold X, the initial hypersurface M_0, and on T.

Proof. The ratio $\frac{\rho}{H}$ saisfies an evolution equation of the form

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{H}\Big) - \Delta\Big(\frac{\rho}{H}\Big) - 2\Big\langle \frac{\nabla H}{H}, \nabla\Big(\frac{\rho}{H}\Big)\Big\rangle \leq C + C\,\frac{\rho}{H} + C\sum_{i=1}^n \frac{1}{H(\rho+\lambda_i)}.$$

It follows from results in [9] that

$$\sup_{t\in[0,T)}\,\sup_{M_t} \frac{|\lambda_i| + 1}{H} \leq K,$$

where K is a constant that depends only on the ambient manifold X, the initial hypersurface M_0, and on T. Hence, if $\frac{\rho}{H} \geq 2K$, then $\frac{\rho+\lambda_i}{H} \geq K$, and therefore $\frac{1}{H\,(\rho+\lambda_i)} \leq \frac{1}{K\,H^2} \leq K$. Thus, we conclude that

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{H}\Big) - \Delta\Big(\frac{\rho}{H}\Big) - 2\Big\langle \frac{\nabla H}{H}, \nabla\Big(\frac{\rho}{H}\Big)\Big\rangle \leq C + C\,\frac{\rho}{H}$$

whenever $\frac{\rho}{H} \geq 2K$. Hence, the assertion follows from the maximum principle. □

We next state an auxiliary result:

Proposition 5.3. *Consider a point $(\bar{x}, \bar{t}) \in M \times [0, T)$ such that $\rho(\bar{x}, \bar{t}) + \lambda_1(\bar{x}, \bar{t}) > 0$ and $\rho(\bar{x}, \bar{t}) \geq 8\,\mathrm{inj}(X)^{-1}$. We further assume that $U \subset M \times [0, T)$ is an open neighborhood of $\bar{x}$ and $\Phi : U \times (\bar{t} - \alpha, \bar{t}] \to \mathbb{R}$ is a smooth function such that $\Phi(\bar{x}, \bar{t}) = \rho(\bar{x}, \bar{t})$ and $\Phi(x, t) \geq \rho(x, t)$ for all points $(x, t) \in U \times (\bar{t} - \alpha, \bar{t}]$. Then*

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} H\, \Phi^2 - \sum_{i=1}^{n} \frac{1}{\Phi + \lambda_i} \left(|D_i \Phi| + L\right)\left(|D_i H| + L\right) - \sum_{i=1}^{n} \frac{H}{(\Phi + \lambda_i)^2} \Big((D_i \Phi)^2 + L\Big) \leq L\, H$$

at the point $(\bar{x}, \bar{t})$. Here, L is a positive constant that depends only on the ambient manifold X, the initial hypersurface M_0, and on T.

Proof. We define

$$\begin{aligned} W(x, y, t) &= \Phi(x, t)\, \psi_{F(y,t)}(F(x, t)) + \big\langle \nabla \psi_{F(y,t)}\big|_{F(x,t)}, \nu(x, t)\big\rangle \\ &= \frac{1}{2}\, \Phi(x, t)\, d(F(x, t), F(y, t))^2 - \Big\langle \exp^{-1}_{F(x,t)}(F(y, t)), \nu(x, t)\Big\rangle. \end{aligned}$$

By assumption, we have $W(x, y, t) \geq 0$ whenever $x \in U$, $t \in (\bar{t} - \alpha, \bar{t}]$, and $d(F(x, t), F(y, t)) \leq \frac{1}{2}\,\mathrm{inj}(X)$. Moreover, we can find a point $\bar{y}$ such that $0 < d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t})) \leq \frac{1}{2}\,\mathrm{inj}(X)$ and $W(\bar{x}, \bar{y}, \bar{t}) = 0$. From this, we deduce that $\Phi(\bar{x}, \bar{t})\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t})) \leq 2$, hence $d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t})) \leq \frac{1}{4}\,\mathrm{inj}(X)$. As in Section 2, we compute

$$\begin{aligned} 0 = \frac{\partial W}{\partial x_i}(\bar{x}, \bar{y}, \bar{t}) &= \frac{1}{2}\, \frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t})\, d(F(\bar{y}, \bar{t}), F(\bar{x}, \bar{t}))^2 \\ &\quad - \Phi(\bar{x}, \bar{t}) \left\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t}) \right\rangle \\ &\quad - h_i^j(\bar{x}, \bar{t}) \left\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_j}(\bar{x}, \bar{t}) \right\rangle \\ &\quad + \Xi_{F(\bar{y},\bar{t}), F(\bar{x},\bar{t})}\left(\frac{\partial F}{\partial x_i}(\bar{x}, \bar{t}), \nu(\bar{x}, \bar{t}) \right). \end{aligned}$$

Let us pick geodesic normal coordinates around $\bar{x}$ such that $h_{ij}(\bar{x}, \bar{t})$ is a diagonal matrix. The relation $\frac{\partial W}{\partial x_i}(\bar{x}, \bar{y}, \bar{t}) = 0$ implies

$$\begin{aligned} &\left\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t}) \right\rangle \\ &= \frac{1}{2}\, \frac{1}{\Phi(\bar{x}, \bar{t}) + \lambda_i(\bar{x}, \bar{t})} \left(\frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{t}) + O(1) \right) d(F(\bar{x}, \bar{t}), F(\bar{x}, \bar{t}))^2. \end{aligned}$$

In the next step, we use the identity

$$\begin{aligned}
\frac{\partial W}{\partial t}(\bar{x}, \bar{y}, \bar{t}) &= \frac{1}{2}\,\frac{\partial \Phi}{\partial t}(\bar{x}, \bar{t})\, d(F(\bar{x}, \bar{t}), F(\bar{y}, \bar{t}))^2 \\
&- H(\bar{x}, \bar{t}) + H(\bar{x}, \bar{t})\, \Phi(\bar{x}, \bar{t})\, \Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}\big(F(\bar{y}, \bar{t})\big), \nu(\bar{x}, \bar{t})\Big\rangle \\
&+ H(\bar{y}, \bar{t}) \Big\langle \Big(D \exp^{-1}_{F(\bar{x},\bar{t})}\Big)_{F(\bar{y},\bar{t})} (\nu(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) - \Phi(\bar{x}, \bar{t})\, \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t}))\Big\rangle \\
&- \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}(\bar{x}, \bar{t}) \Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y}, \bar{t})), \frac{\partial F}{\partial x_i}(\bar{x}, \bar{t})\Big\rangle \\
&- H(\bar{x}, \bar{t})\, \Xi_{F(\bar{y},\bar{t}),F(\bar{x},\bar{t})}(\nu(\bar{x}, \bar{t}), \nu(\bar{x}, \bar{t})).
\end{aligned}$$

The terms $H(\bar{y}, \bar{t})$ and $\big\langle (D \exp^{-1}_{F(\bar{x},\bar{t})})_{F(\bar{y},\bar{t})}(\nu(\bar{y}, \bar{t})), \nu(\bar{x}, \bar{t}) - \Phi(\bar{x}, \bar{t}) \exp^{-1}_{F(\bar{x},\bar{t})} (F(\bar{y}, \bar{t}))\big\rangle$ are nonnegative. This gives

$$\begin{aligned}
&\frac{\partial W}{\partial t}(\bar{x},\bar{y},\bar{t}) \\
&\geq \frac{1}{2}\bigg(\frac{\partial \Phi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t})\Phi(\bar{x},\bar{t})^2 \\
&\qquad - \sum_{i=1}^{n} \frac{1}{\Phi(\bar{x},\bar{t})+\lambda_i(\bar{x},\bar{t})}\Big(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t}) + O(1)\Big)\frac{\partial H}{\partial x_i}(\bar{x},\bar{t})\bigg)|F(\bar{x},\bar{t}) - F(\bar{y},\bar{t})|^2 \\
&\quad - H(\bar{x},\bar{t}) + O\big(H(\bar{x},\bar{t})d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2\big).
\end{aligned}$$

We now multiply both sides by $\frac{2}{|F(\bar{x},\bar{t})-F(\bar{y},\bar{t})|^2}$. Using the relation

$$\begin{aligned}
\frac{1}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2} &= \frac{\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})), \nu(\bar{x},\bar{t})\Big\rangle^2}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^4} \\
&\quad + \sum_{i=1}^{n} \frac{\Big\langle \exp^{-1}_{F(\bar{x},\bar{t})}(F(\bar{y},\bar{t})), \frac{\partial F}{\partial x_i}(\bar{x},\bar{t})\Big\rangle^2}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^4} \\
&= \frac{1}{4}\bigg(\Phi(\bar{x},\bar{t})^2 + \sum_{i=1}^{n} \frac{1}{(\Phi(\bar{x},\bar{t})+\lambda_i(\bar{x}))^2}\Big(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t}) + O(1)\Big)^2\bigg),
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\frac{2}{d(F(\bar{x},\bar{t}),F(\bar{y},\bar{t}))^2}\frac{\partial W}{\partial t}(\bar{x},\bar{y},\bar{t}) \\
&\geq \frac{\partial \Phi}{\partial t}(\bar{x},\bar{t})+H(\bar{x},\bar{t})\Phi(\bar{x},\bar{t})^2-\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x},\bar{t})+\lambda_i(\bar{x},\bar{t})}\left(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})+O(1)\right)\frac{\partial H}{\partial x_i}(\bar{x},\bar{t}) \\
&\quad -\frac{1}{2}H(\bar{x},\bar{t})\left(\Phi(\bar{x},\bar{t})^2+\sum_{i=1}^{n}\frac{1}{(\Phi(\bar{x},\bar{t})+\lambda_i(\bar{x},\bar{t}))^2}\left(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})+O(1)\right)^2\right) \\
&\quad -O(H(\bar{x},\bar{t})) \\
&=\frac{\partial \Phi}{\partial t}(\bar{x},\bar{t})+\frac{1}{2}H(\bar{x},\bar{t})\Phi(\bar{x},\bar{t})^2-\sum_{i=1}^{n}\frac{1}{\Phi(\bar{x},\bar{t})+\lambda_i(\bar{x},\bar{t})}\left(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})+O(1)\right)\frac{\partial H}{\partial x_i}(\bar{x},\bar{t}) \\
&\quad -\frac{1}{2}H(\bar{x},\bar{t})\sum_{i=1}^{n}\frac{1}{(\Phi(\bar{x},\bar{t})+\lambda_i(\bar{x},\bar{t}))^2}\left(\frac{\partial \Phi}{\partial x_i}(\bar{x},\bar{t})+O(1)\right)^2-O(H(\bar{x},\bar{t})).
\end{aligned}$$

Since $\frac{\partial W}{\partial t}(\bar{x}, \bar{y}, \bar{t}) \leq 0$, the assertion follows. □

Corollary 5.4. *We have*

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} - \sum_{i=1}^{n} \frac{1}{\rho+\lambda_i}\,(|D_i\rho| + L)\,(|D_i H| + L) - \sum_{i=1}^{n} \frac{H}{(\rho+\lambda_i)^2}\big((D_i\rho)^2 + L\big) \\
&\leq L\, H
\end{aligned}$$

almost everywhere on the set $\{\rho + \lambda_1 > 0\} \cap \{\rho \geq 8\,\mathrm{inj}(X)^{-1}\}$.

Let $\delta > 0$ be given. The convexity estimate of Huisken and Sinestrari (see [9, Remark 3.9]) implies that we can find a constant

$$K_0 \geq 8\,\mathrm{inj}(X)^{-1}\left(\inf_{t\in[0,T)}\inf_{M_t}\min\{H, 1\}\right)^{-1},$$

depending only on X, M_0, δ, and T, such that

$$\lambda_1 \geq -\frac{\delta}{2}\,H - K_0\,\min\{H, 1\}.$$

For each $\sigma \in (0, \frac{1}{2})$, we put

$$g_\sigma = H^{\sigma-1}\,(\rho - \delta\,H) - K_0$$

and

$$g_{\sigma,+} = \max\{g_\sigma, 0\}.$$

On the set $\{g_\sigma \geq 0\}$, we have

$$\rho \geq \delta\, H + K_0\, H^{1-\sigma} \geq \delta\, H + K_0\, \min\{H, 1\},$$

hence

$$\rho + \lambda_1 \geq \frac{\delta}{2}\, H.$$

In particular, we have $\{g_\sigma \geq 0\} \subset \{\rho + \lambda_1 > 0\} \cap \{\rho \geq 8\, \mathrm{inj}(X)^{-1}\}$. Furthermore, by Corollary 5.2, there exists a constant $\Lambda \geq 1$, depending only on X, M_0, and T, such that $\rho \leq \Lambda\, H$ and $|A|^2 \leq \Lambda\, H^2$ for $t \in [0, T)$.

Proposition 5.5. *Given any $\delta > 0$, there exists a positive constant c_0, depending only on X, M_0, δ, and T, with the following property: if $p \geq \frac{1}{c_0}$ and $\sigma \leq c_0\, p^{-\frac{1}{2}}$, then we have*

$$\frac{d}{dt}\Big(\int_{M_t} g_{\sigma,+}^p\Big) \leq C\, \sigma\, p \int_{M_t} g_{\sigma,+}^p + \sigma\, p\, K_0^p \int_{M_t} |A|^2 + C\, p^p \int_{M_t} H^{2-(2-\sigma)p}$$

for almost all t.

Proof. For abbreviation, we define a function ω by

$$\begin{aligned}\omega = \Delta\rho - \sum_{i=1}^n \frac{1}{\rho + \lambda_i}\, (D_i\rho)^2 \\ - \sum_i \frac{1}{\rho + \lambda_i}\, (|D_i\rho| + L)\, (|D_i H| + L) - \sum_{i=1}^n \frac{H}{(\rho + \lambda_i)^2}\, \Big((D_i\rho)^2 + L\Big),\end{aligned}$$

where L is the constant in Corollary 5.4. Combining Proposition 5.1 and Corollary 5.4, we obtain

$$\frac{\partial\rho}{\partial t} - \Delta\rho - |A|^2\, \rho + \sum_{i=1}^n \frac{1}{\rho + \lambda_i}\, (D_i\rho)^2 \leq -\max\{\omega + |A|^2\, \rho, 0\} + C\, H$$

on the set $\{g_\sigma \geq 0\}$. From this, we deduce that

$$\begin{aligned}&\frac{\partial}{\partial t} g_\sigma - \Delta g_\sigma - 2\,(1-\sigma)\, \Big\langle \frac{\nabla H}{H}, \nabla g_\sigma \Big\rangle + 2 \sum_{i=1}^n \frac{H^{\sigma-1}}{\rho + \lambda_i}\, (D_i\rho)^2 - \sigma\, |A|^2\, (g_\sigma + K_0) \\ &\leq -H^{\sigma-1} \max\Big\{\omega + |A|^2\, \rho, 0\Big\} - \sigma\,(1-\sigma)\, H^{\sigma-3}\, (\rho - \delta\, H)\, |\nabla H|^2 + C\, H^\sigma\end{aligned}$$

on the set $\{g_\sigma \geq 0\}$. Note that $g_\sigma \leq H^{\sigma-1}\,\rho$ by definition of g_σ. Since $\sigma \in (0, \frac{1}{2})$, we have $2\sigma\,\frac{g_\sigma}{\rho} \leq H^{\sigma-1}$ at each point on the hypersurface. This implies

$$\begin{aligned}
&\frac{\partial}{\partial t} g_\sigma - \Delta g_\sigma - 2\,(1-\sigma)\left\langle \frac{\nabla H}{H}, \nabla g_\sigma \right\rangle + \sum_{i=1}^n \frac{H^{\sigma-1}}{\rho + \lambda_i}\,(D_i\rho)^2 \\
&\leq -H^{\sigma-1}\,\max\left\{\omega + |A|^2\,\rho, 0\right\} + \sigma\,|A|^2\,(g_\sigma + K_0) + C\,H^\sigma \\
&\leq -2\sigma\,\frac{g_\sigma}{\rho}\,\left(\omega + |A|^2\,\rho\right) + \sigma\,|A|^2\,(g_\sigma + K_0) + C\,H^\sigma \\
&= -2\sigma\,\frac{g_\sigma}{\rho}\,\omega + \sigma\,|A|^2\,(K_0 - g_\sigma) + C\,H^\sigma
\end{aligned}$$

on the set $\{g_\sigma \geq 0\}$. Therefore, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{M_t} g_{\sigma,+}^p\right) \\
&\leq -p(p-1)\int_{M_t} g_{\sigma,+}^{p-2}\,|\nabla g_\sigma|^2 + 2\,(1-\sigma)\,p\int_{M_t} g_{\sigma,+}^{p-1}\left\langle \frac{\nabla H}{H}, \nabla g_\sigma \right\rangle \\
&\quad - p\int_{M_t}\sum_{i=1}^n \frac{g_{\sigma,+}^{p-1}\,H^{\sigma-1}}{\rho+\lambda_i}\,(D_i\rho)^2 - 2\sigma\,p\int_{M_t}\frac{g_{\sigma,+}^p}{\rho}\,\omega \\
&\quad + \sigma\,p\int_{M_t} g_{\sigma,+}^{p-1}\,(K_0 - g_\sigma)\,|A|^2 + \int_{M_t}\left(C\,H^\sigma\,g_{\sigma,+}^{p-1} - H^2\,g_{\sigma,+}^p\right).
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
-\int_{M_t}\frac{g_{\sigma,+}^p}{\rho}\,\omega &\leq C\,p\int_{M_t}\frac{g_{\sigma,+}^{p-1}}{H}\,|\nabla\rho|\,|\nabla g_\sigma| \\
&\quad + C\int_{M_t}\frac{g_{\sigma,+}^p}{H^2}\,(|\nabla\rho|+1)\,(|\nabla H|+1) + C\int_{M_t}\frac{g_{\sigma,+}^p}{H^2}\,(|\nabla\rho|^2+1),
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T. Putting these facts together, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{M_t} g_{\sigma,+}^p\Big) \\
&\leq -p(p-1)\int_{M_t} g_{\sigma,+}^{p-2}\,|\nabla g_\sigma|^2 + 2\,(1-\sigma)\,p\int_{M_t} g_{\sigma,+}^{p-1}\,\Big\langle \frac{\nabla H}{H}, \nabla g_\sigma\Big\rangle \\
&\quad - p\int_{M_t}\sum_{i=1}^n \frac{g_{\sigma,+}^{p-1}\,H^{\sigma-1}}{\rho+\lambda_i}\,(D_i\rho)^2 + C\,\sigma\,p^2\int_{M_t}\frac{g_{\sigma,+}^{p-1}}{H}\,|\nabla\rho|\,|\nabla g_\sigma| \\
&\quad + C\,\sigma\,p\int_{M_t}\frac{g_{\sigma,+}^p}{H^2}\,(|\nabla\rho|+1)\,(|\nabla H|+1) + C\,\sigma\,p\int_{M_t}\frac{g_{\sigma,+}^p}{H^2}\,(|\nabla\rho|^2+1) \\
&\quad + \sigma\,p\,K_0^p\int_{M_t}|A|^2 + \int_{M_t}\Big(C\,H^\sigma\,g_{\sigma,+}^{p-1} - H^2\,g_{\sigma,+}^p\Big),
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T. Using the identity

$$\frac{\nabla H}{H} = \frac{\nabla\rho - H^{1-\sigma}\,\nabla g_\sigma}{(1-\sigma)\,\rho + \sigma\,\delta\,H},$$

we obtain

$$\Big\langle \frac{\nabla H}{H}, \nabla g_\sigma\Big\rangle \leq \frac{\langle\nabla\rho, \nabla g_\sigma\rangle}{(1-\sigma)\,\rho+\sigma\,\delta\,H} \leq C\,H^{-1}\,|\nabla\rho|\,|\nabla g_\sigma|$$

and

$$\frac{|\nabla H|}{H} \leq C\,\frac{|\nabla\rho|}{H} + C\,\frac{|\nabla g_\sigma|}{g_{\sigma,+}}.$$

This gives

$$\begin{aligned}
\frac{d}{dt}\Big(\int_{M_t} g_{\sigma,+}^p\Big) &\leq -p(p-1)\int_{M_t} g_{\sigma,+}^{p-2}\,|\nabla g_\sigma|^2 - p\int_{M_t}\sum_{i=1}^n\frac{g_{\sigma,+}^{p-1}\,H^{\sigma-1}}{\rho+\lambda_i}\,(D_i\rho)^2 \\
&\quad + C\big(p+\sigma p^2\big)\int_{M_t}\frac{g_{\sigma,+}^{p-1}}{H}\,|\nabla\rho|\,|\nabla g_\sigma| + C\,\sigma\,p\int_{M_t}\frac{g_{\sigma,+}^p}{H^2}\,|\nabla\rho|^2 \\
&\quad + C\,\sigma\,p\int_{M_t}\frac{g_{\sigma,+}^p}{H^2}\,(|\nabla\rho|+1) + C\,\sigma\,p\int_{M_t}\frac{g_{\sigma,+}^{p-1}}{H^2}\,|\nabla g_\sigma| \\
&\quad + \sigma\,p\,K_0^p\int_{M_t}|A|^2 + \int_{M_t}\Big(C\,H^\sigma\,g_{\sigma,+}^{p-1} - H^2\,g_{\sigma,+}^p\Big),
\end{aligned}$$

where C is a positive constant that depends only on X, M_0, δ, and T.

Consequently, there exists a positive constant c_0, depending only X, M_0, δ, and T, with the following property: if $p \geq \frac{1}{c_0}$ and $\sigma \leq c_0\, p^{-\frac{1}{2}}$, then we have

$$\frac{d}{dt}\left(\int_{M_t} g_{\sigma,+}^p\right) \leq C\sigma\, p\int_{M_t} g_{\sigma,+}^p + \sigma\, p K_0^p \int_{M_t} |A|^2 + \int_{M_t}\left(C\, H^\sigma\, g_{\sigma,+}^{p-1} - H^2 g_{\sigma,+}^p\right).$$

Finally, since we have a lower bound for the function H for $t \in [0, T)$, we obtain a pointwise upper bound for the function $C\, H^\sigma\, g_{\sigma,+}^{p-1} - H^2\, g_{\sigma,+}^p$ for all $t \in [0, T)$. This yields

$$\frac{d}{dt}\left(\int_{M_t} g_{\sigma,+}^p\right) \leq C\,\sigma\, p\int_{M_t} g_{\sigma,+}^p + \sigma\, p\, K_0^p \int_{M_t} |A|^2 + C\, p^p \int_{M_t} H^{2-(2-\sigma)p}.$$

This completes the proof of Proposition 5.5. □

As above, we can use now the Michael-Simon Sobolev inequality (*cf.* [11]) and Stampacchia iteration to show that

$$\rho \leq \delta\, H + C(X, M_0, \delta, T)$$

for all $t \in [0, T)$ and all points on M_t.

References

[1] B. ANDREWS, *Non-collapsing in mean-convex mean curvature flow*, Geom. Topol. **16** (2012), 1413–1418.

[2] S. BRENDLE, *Embedded minimal tori in S^3 and the Lawson conjecture*, Acta Math. **211** (2013), 177–190.

[3] S. BRENDLE, *A sharp bound for the inscribed radius under mean curvature flow*, Invent. Math. **202** (2015), 217–237.

[4] S. BRENDLE and G. HUISKEN, *Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$*, Invent. Math. **203** (2016), 615–654.

[5] L. C. EVANS and R. F. GARIEPY, "Measure Theory and Fine Properties of Functions", Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[6] R. HASLHOFER and B. KLEINER, *Mean curvature flow of mean convex hypersurfaces*, Comm. Pure Appl. Math., to appear.

[7] G. HUISKEN, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), 127–133.

[8] G. HUISKEN and C. SINESTRARI, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. **8** (1999), 1–14.

[9] G. HUISKEN and C. SINESTRARI, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), 45–70.

[10] G. HUISKEN and C. SINESTRARI, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. **175** (2009), 137–221.

[11] J. H. MICHAEL and L. M. SIMON, *Sobolev and mean value inequalities on generalized submanifolds of $\mathbb{R}^n$*, Comm. Pure Appl. Math. **26** (1973), 316–379.

[12] W. SHENG and X. J. WANG, *Singularity profile in the mean curvature flow*, Methods Appl. Anal. **16** (2009), 139–155.

[13] B. WHITE, *The size of the singular set in mean curvature flow of mean convex sets*, J. Amer. Math. Soc. **13** (2000), 665–695.
[14] B. WHITE, *The nature of singularities in mean curvature flow of mean convex sets*, J. Amer. Math. Soc. **16** (2003), 123–138.
[15] B. WHITE, *Subsequent singularities in mean-convex mean curvature flow*, Calc. Var. **54** (2015), 1457–1468.

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