Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVI (2016), 1409-1445

The monodromy representation of Lauricella's hypergeometric function F_C

YOSHIAKI GOTO

Abstract. We study the monodromy representation of the system E_C of differential equations satisfied by Lauricella's hypergeometric function F_C of m variables. Our representation space is the twisted homology group associated with an integral representation of F_C. We find generators of the fundamental group of the complement of the singular locus of E_C, and we give relations for these generators. We express the circuit transformations along these generators, using the intersection forms defined on the twisted homology group and its dual.

Mathematics Subject Classification (2010): 33C65 (primary); 32S40, 14F35 (secondary).

1. Introduction

Lauricella's hypergeometric series F_C of m variables $x_1, \ldots, x_m$ with complex parameters $a, b, c_1, \ldots, c_m$ is defined by

$$F_C(a, b, c; x) = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{(a, n_1 + \cdots + n_m)(b, n_1 + \cdots + n_m)}{(c_1, n_1)\cdots(c_m, n_m) n_1! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m},$$

where $x = (x_1, \ldots, x_m)$, $c = (c_1, \ldots, c_m)$, $c_1, \ldots, c_m \notin \{0, -1, -2, \ldots\}$, and $(c_1, n_1) = \Gamma(c_1 + n_1)/\Gamma(c_1)$. This series converges in the domain

$$D_C := \left\{ (x_1, \ldots, x_m) \in \mathbb{C}^m \;\middle|\; \sum_{k=1}^{m} \sqrt{|x_k|} < 1 \right\},$$

and admits an Euler-type integral representation (2.3). The system $E_C(a, b, c)$ of differential equations satisfied by $F_C(a, b, c; x)$ is a holonomic system of rank 2^m with the singular locus S given in (2.1). There is a fundamental system of solutions to $E_C(a, b, c)$ in a simply connected domain in $D_C - S$, which is given in terms of

Received September 21, 2014; accepted September 17, 2015.
Published online December 2016.

Lauricella's hypergeometric series F_C with different parameters; see (2.2) for their expressions.

In the case $m = 2$, the series $F_C(a, b, c; x)$ and the system $E_C(a, b, c)$ are called Appell's hypergeometric series $F_4(a, b, c; x)$ and system $E_4(a, b, c)$ of differential equations. The monodromy representation of $E_4(a, b, c)$ has been studied from several different points of view, see [5,6,8,12]. On the other hand, there were few results of the monodromy representation for general m. In [2] Beukers studies the monodromy representation of A-hypergeometric system and gives representation matrices for many kinds of hypergeometric systems as examples of his main theorem. However, it seems that his method is not applicable for Lauricella's F_C.

In this paper we study the monodromy representation of $E_C(a, b, c)$ for general m, by using twisted homology groups associated with the integral representation (2.3) of $F_C(a, b, c; x)$ and the intersection form defined on the twisted homology groups. Our consideration is based on the method for Appell's $E_4(a, b, c)$ in [5].

Let X be the complement of the singular locus S. The fundamental group of X is generated by $m + 1$ loops $\rho_0, \ \rho_1, \ldots, \ \rho_m$ which satisfy

$$\rho_i \rho_j = \rho_j \rho_i \quad (1 \le i, j \le m), \quad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \le k \le m).$$

Here, ρ_k $(1 \le k \le m)$ turns the divisor $(x_k = 0)$, and ρ_0 turns the divisor

$$\prod_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \left(1 + \sum_{k=1}^{m} \varepsilon_k \sqrt{x_k} \right) = 0$$

around the point $\left(\frac{1}{m^2}, \ldots, \frac{1}{m^2} \right)$. In the appendix, we show this claim by applying the Zariski theorem of Lefschetz type. Note that, for $m = 2$, an explicit expression of the fundamental group of X is given in [8].

We thus investigate the circuit transformations $\mathcal{M}_i$ along ρ_i, for $0 \le i \le m$. We use the 2^m twisted cycles $\{\Delta_I\}_{I \subset \{1, \ldots, m\}}$ constructed in [4], which represent elements in the m-th twisted homology group and correspond to the solutions (2.2) to $E_C(a, b, c)$. We obtain the representation matrix of $\mathcal{M}_k$ $(1 \le k \le m)$ with respect to the basis $\{\Delta_I\}_I$ easily. The eigenvalues of $\mathcal{M}_k$ are $\exp(-2\pi\sqrt{-1}c_k)$ and 1. Both eigenspaces are 2^{m-1}-dimensional and spanned by half subsets of $\{\Delta_I\}_I$. On the other hand, it is difficult to represent $\mathcal{M}_0$ directly with respect to the basis $\{\Delta_I\}_I$. Thus we study the structure of the eigenspaces of $\mathcal{M}_0$. We find out that it is quite simple; our main theorem (Theorem 5.6) is stated as follows. The eigenvalues of $\mathcal{M}_0$ are $(-1)^{m-1} \exp(2\pi\sqrt{-1}(c_1 + \cdots + c_m - a - b))$ and 1. The eigenspace W_0 of eigenvalue $(-1)^{m-1} \exp(2\pi\sqrt{-1}(c_1 + \cdots + c_m - a - b))$ is one-dimensional and spanned by the twisted cycle $D_{1 \cdots m}$ defined by some bounded chamber. Further, the eigenspace W_1 of eigenvalue 1 is characterized as the orthogonal complement of $W_0 = \mathbb{C} D_{1 \cdots m}$ with respect to the intersection form.

As a corollary, we express the linear map $\mathcal{M}_i$ $(0 \le i \le m)$ by using the intersection form. Our expressions are independent of the choice of a basis of the twisted homology group. To represent $\mathcal{M}_i$ by a matrix with respect to a given basis, it is sufficient to evaluate some intersection numbers. In particular, the images of any twisted cycles by $\mathcal{M}_0$ are determined only from the intersection number with the eigenvector $D_{1\cdots m}$; see Corollary 5.7. In Section 6, we give the simple representation matrix of $\mathcal{M}_i$ with respect to a suitable basis, and write down the examples for $m = 2$ and $m = 3$.

The irreducibility condition of the system $E_C(a, b, c)$ is known to be

$$a - \sum_{i \in I} c_i, \quad b - \sum_{i \in I} c_i \notin \mathbb{Z}$$

for any subset I of $\{1, \ldots, m\}$, as in [7]. Throughout this paper, we assume that the parameters $a,\ b$, and $c = (c_1, \ldots, c_m)$ are generic, which means that we add other conditions to the irreducibility condition; for details, refer to Remark 7.6.

ACKNOWLEDGEMENTS. The author thanks Professor Keiji Matsumoto for his useful advice and constant encouragement. He is also grateful to Professor Jyoichi Kaneko for helpful discussions. He thanks the referee for suggesting some improvement in the previous version of the article.

2. Differential equations and integral representations

In this section we collect some facts about Lauricella's F_C and the system E_C of differential equations that it satisfies.

Notation 2.1. (i) Throughout this paper, the letter k always stands for an index running from 1 to m. If no confusion is possible, $\sum_{k=1}^{m}$ and $\prod_{k=1}^{m}$ are often simply denoted by $\sum$ (or $\sum_k$) and $\prod$ (or $\prod_k$), respectively. For example, under this convention $F_C(a, b, c; x)$ is expressed as

$$F_C(a, b, c; x) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{\left(a, \sum n_k\right)\left(b, \sum n_k\right)}{\prod (c_k, n_k) \cdot \prod n_k!} \prod x_k^{n_k}.$$

(ii) For a subset I of $\{1, \ldots, m\}$, we denote the cardinality of I by $|I|$.

Let ∂_k $(1 \le k \le m)$ be the partial differential operator with respect to x_k. We set $\theta_k := x_k \partial_k, \theta := \sum_k \theta_k$. Lauricella's $F_C(a, b, c; x)$ satisfies differential equations

$$[\theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b)]\, f(x) = 0, \quad 1 \le k \le m.$$

The system generated by them is called Lauricella's hypergeometric system $E_C(a, b, c)$ of differential equations.

Fact 2.2 ([7,11]). The system $E_C(a,b,c)$ is a holonomic system of rank 2^m with the singular locus

$$
\begin{aligned}
S &:= \left(\prod_k x_k \cdot R(x) = 0\right) \subset \mathbb{C}^m \\
R(x_1,\dots,x_m) &:= \prod_{\varepsilon_1,\dots,\varepsilon_m=\pm 1} \left(1 + \sum_k \varepsilon_k \sqrt{x_k}\right).
\end{aligned}
\tag{2.1}
$$

If $c_1,\dots,c_m \notin \mathbb{Z}$, then the vector space of solutions to $E_C(a,b,c)$ in a simply connected domain in $D_C - S$ is spanned by the following 2^m functions:

$$
f_I := \prod_{i\in I} x_i^{1-c_i} \cdot F_C\left(a + |I| - \sum_{i\in I} c_i,\, b + |I| - \sum_{i\in I} c_i,\, c^I; x\right), \tag{2.2}
$$

where I is a subset of $\{1,\dots,m\}$, and the row vector $c^I = (c^I_1,\dots,c^I_m)$ of $\mathbb{C}^m$ is defined by

$$
c^I_k = \begin{cases} 2 - c_k & (k \in I) \\ c_k & (k \notin I). \end{cases}
$$

Note that the solution (2.2) for $I = \emptyset$ is $f(= f_\emptyset) = F_C(a,b,c;x)$, and $R(x)$ is an irreducible polynomial of degree 2^{m-1} in $x_1,\dots,x_m$.

Fact 2.3 (Euler-type integral representation [1, Example 3.1]). For sufficiently small positive real numbers $x_1,\dots,x_m$, if $c_1,\dots,c_m, a - \sum c_k \notin \mathbb{Z}$, then $F_C(a,b,c;x)$ admits the following integral representation:

$$
\begin{aligned}
F_C(a,b,c;x) = & \frac{\Gamma(1-a)}{\prod \Gamma(1-c_k) \cdot \Gamma\left(\sum c_k - a - m - 1\right)} \\
& \cdot \int_\Delta \prod t_k^{-c_k} \cdot \left(1 - \sum t_k\right)^{\sum c_k - a - m} \cdot \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} dt_1 \wedge \cdots \wedge dt_m,
\end{aligned}
\tag{2.3}
$$

where Δ is the twisted cycle made by an m-simplex [1, Sections 3.2-3].

This twisted cycle coincides with $\Delta_\emptyset = \Delta$ introduced in Section 4. In the case of $m = 2$, we show a figure of Δ in Example 4.1.

3. Twisted homology groups and local systems

For twisted homology groups and the intersection form between twisted homology groups, refer to [1,13], or [4, Section 3].

Put $X := \mathbb{C}^m - S$ and

$$v(t) := 1 - \sum_k t_k, \quad w(t,x) := \prod_k t_k \cdot \left(1 - \sum_k \frac{x_k}{t_k}\right),$$

$$\mathfrak{X} := \left\{ (t,x) \in \mathbb{C}^m \times X \;\middle|\; \prod_k t_k \cdot v(t) \cdot w(t,x) \neq 0 \right\}.$$

There is a natural projection

$$pr : \mathfrak{X} \to X; \ (t,x) \mapsto x,$$

and we define $T_x := pr^{-1}(x)$ for any $x \in X$. We regard T_x as an open submanifold of $\mathbb{C}^m$ by the coordinates $t = (t_1, \ldots, t_m)$. We consider the twisted homology groups on T_x with respect to the multivalued function

$$\begin{aligned} u_x(t) :=& \prod t_k^{1-c_k+b} \cdot v(t)^{\sum c_k - a - m + 1} w(t,x)^{-b} \\ =& \prod t_k^{1-c_k} \cdot \left(1 - \sum t_k\right)^{\sum c_k - a - m + 1} \cdot \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} \end{aligned}$$

(the second equality holds under the coordination of branches). We denote the k-th twisted homology group by $H_k(T_x, u_x)$, and the locally finite one by $H_k^{lf}(T_x, u_x)$.

Facts 3.1 ([1,4]).

(i) $H_k(T_x, u_x) = 0, \ H_k^{lf}(T_x, u_x) = 0$, for $k \neq m$.
(ii) $\dim H_m(T_x, u_x) = 2^m$.
(iii) The natural map $H_m(T_x, u_x) \to H_m^{lf}(T_x, u_x)$ is an isomorphism (the inverse map is called the regularization).

Hereafter, we identify $H_m^{lf}(T_x, u_x)$ with $H_m(T_x, u_x)$, and call an m-dimensional twisted cycle by a twisted cycle simply. Note that the intersection form I_h is defined between $H_m(T_x, u_x)$ and $H_m(T_x, u_x^{-1})$.

For $x, x' \in X$ and a path τ in X from x to x', there is the canonical isomorphism

$$\tau_* : H_m(T_x, u_x) \to H_m(T_{x'}, u_{x'}).$$

Hence the family

$$\mathcal{H} := \bigcup_{x \in X} H_m(T_x, u_x)$$

forms a local system on X.

Let δ be a twisted cycle in T_x for a fixed x. If x' is a sufficiently close point to x, there is a unique twisted cycle δ' such that $\int_{\delta'} u_{x'}\varphi$ is obtained by the analytic continuation of $\int_\delta u_x \varphi$, where

$$\varphi := \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot \left(1 - \sum t_k\right)}.$$

Thus we can regard the integration $\int_\delta u_x\varphi$ as a holomorphic function in x. Fact 2.3 means that the integral $\int_\Delta u_x\varphi$ represents $F_C(a,b,c;x)$ modulo Gamma factors. Let Sol be the sheaf on X whose sections are holomorphic solutions to $E_C(a,b,c)$. The stalk Sol_x at $x \in X$ is the space of local holomorphic solutions near x.

Fact 3.2 ([4]). For any $x \in X$,

$$\Phi_x : H_m(T_x, u_x) \to Sol_x;\ \delta \mapsto \int_\delta u_x\varphi$$

is an isomorphism.

4. Twisted cycles corresponding to the solutions f_I

Fact 2.2 implies that Sol_x is a $\mathbb{C}$-vector space of dimension 2^m and spanned by f_I's, for $x \in D_C - S$. In [4], we construct twisted cycles Δ_I that correspond to f_I, for all subsets I of $\{1,\dots,m\}$. In this section, we review the construction of Δ_I briefly.

We construct the twisted cycles $\Delta_I \in H_m(T_x, u_x)$, for fixed sufficiently small positive real numbers $x_1,\dots,x_m$. We set $J := I^c = \{1,\dots,m\} - I$. We consider

$$M_I := \mathbb{C}^m - \left(\bigcup_k (s_k = 0) \cup (v_I = 0) \cup (w_I = 0)\right),$$

where v_I and w_I are polynomials in $s_1,\dots,s_m$ defined by

$$v_I := \prod_{i\in I} s_i \cdot \left(1 - \sum_{i\in I}\frac{x_i}{s_i} - \sum_{j\in J} s_j\right),\ w_I := \prod_{j\in J} s_j \cdot \left(1 - \sum_{i\in I} s_i - \sum_{j\in J}\frac{x_j}{s_j}\right).$$

Let u_I be a multivalued function on M_I defined as

$$u_I := \prod_k s_k^{C_k} \cdot v_I^A \cdot w_I^B,$$

where

$$A := \sum c_k - a - m + 1, \quad B := -b,$$
$$C_i := c_i - 1 - A\ (i \in I), \quad C_j := 1 - c_j - B\ (j \in J).$$

Note that if $I = \emptyset$, then $u_\emptyset$ and $M_\emptyset$ coincide with u_x and T_x in Section 3, respectively. We construct the twisted cycle $\tilde{\Delta}_I$ in M_I with respect to u_I. Let ε be a positive real number satisfying $\varepsilon < \frac{1}{m+1}$ and $x_k < \frac{\varepsilon^2}{m}$ (we use the assumption $\varepsilon_1 = \cdots = \varepsilon_m = \varepsilon$ in [4, Section 4]). We consider the closed subset

$$\sigma_I := \left\{ (s_1,\dots,s_m) \in \mathbb{R}^m \;\middle|\; s_k \geq \varepsilon,\ \begin{array}{c} 1 - \sum_{i\in I} s_i \geq \varepsilon, \\ 1 - \sum_{j\in J} s_j \geq \varepsilon \end{array} \right\}$$

which is a direct product of an $|I|$-simplex and an $(m-|I|)$-simplex, and is contained in the bounded domain

$$\left\{ (s_1,\dots,s_m)\in\mathbb{R}^m \;\middle|\; s_k>0,\; \begin{array}{c} 1-\sum_{i\in I}\frac{x_i}{s_i}-\sum_{j\in J}s_j>0, \\ 1-\sum_{i\in I}s_i-\sum_{j\in J}\frac{x_j}{s_j}>0 \end{array} \right\}.$$

The orientation of σ_I is induced from the natural embedding $\mathbb{R}^m\subset\mathbb{C}^m$. We construct a twisted cycle from $\sigma_I\otimes u_I$. Set $L_1:=(s_1=0),\dots,L_m:=(s_m=0)$, $L_{m+1}:=(1-\sum_{i\in I}s_i=0)$, $L_{m+2}:=(1-\sum_{j\in J}s_j=0)$, and let $U(\subset\mathbb{R}^m)$ be the bounded chamber surrounded by $L_1,\dots,L_m,L_{m+1},L_{m+2}$, then σ_I is contained in U. Note that we do not consider the hyperplane L_{m+1} (respectively L_{m+2}), when $I=\emptyset$ (respectively $I=\{1,\dots,m\}$). For $K\subset\{1,\dots,m+2\}$, we consider $L_K:=\cap_{p\in K}L_p$, $U_K:=\overline{U}\cap L_K$ and $T_K:=\varepsilon$-neighborhood of U_K. Then we have

$$\sigma_I=U-\bigcup_K T_K.$$

Using these neighborhoods T_K, we can construct a twisted cycle $\tilde{\Delta}_I$ in the same manner as [1, Section 3.2.4].

We briefly explain the expression of $\tilde{\Delta}_I$. For $p=1,\dots,m+2$, let l_p be the $(m-1)$-face of σ_I given by $\sigma_I\cap\overline{T_p}$, and let S_p be a positively oriented circle with radius ε in the orthogonal complement of L_p starting from the projection of l_p to this space and surrounding L_p. Then $\tilde{\Delta}_I$ is written as

$$\sigma_I\otimes u_I+\sum_{\emptyset\neq K\subset\{1,\dots,m+2\}}\prod_{p\in K}\frac{1}{d_p}\cdot\left(\left(\bigcap_{p\in K}l_p\right)\times\prod_{p\in K}S_p\right)\otimes u_I,$$

where

$$d_i:=\gamma_i-1(i\in I) \qquad d_j:=\gamma_j^{-1}-1(j\in J)$$
$$d_{m+1}:=\beta^{-1}-1 \qquad d_{m+2}:=\alpha^{-1}\prod\gamma_k-1$$

and $\alpha:=e^{2\pi\sqrt{-1}a}$, $\beta:=e^{2\pi\sqrt{-1}b}$, $\gamma_k:=e^{2\pi\sqrt{-1}c_k}$. We often omit "$\otimes u_I$".

Example 4.1. In the case of $m=2$ and $I=\emptyset$, we have

$$\begin{aligned}\tilde{\Delta}=&\sigma+\frac{S_1\times l_1}{1-\gamma_1^{-1}}+\frac{S_2\times l_2}{1-\gamma_2^{-1}}+\frac{S_4\times l_4}{1-\alpha^{-1}\gamma_1\gamma_2}\\ &+\frac{S_1\times S_2}{(1-\gamma_1^{-1})(1-\gamma_2^{-1})}+\frac{S_2\times S_4}{(1-\gamma_2^{-1})(1-\alpha^{-1}\gamma_1\gamma_2)}+\frac{S_4\times S_1}{(1-\alpha^{-1}\gamma_1\gamma_2)(1-\gamma_1^{-1})},\end{aligned}$$

where the 1-chains l_j satisfy $\partial\sigma=l_1+l_2+l_4$ (see Figure 4.1), and the orientation of each direct product is induced from those of its components. Note that the face l_3 does not appear in this case.

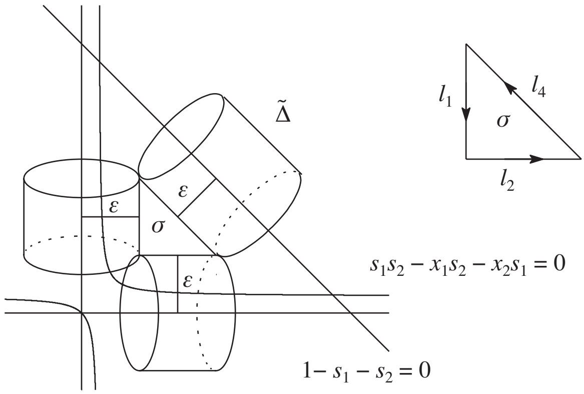

Figure 4.1. $\tilde{\Delta}(=\Delta)$ for $m=2$.

Using the bijection

$$\iota_I : M_I \to T_x; \quad \iota_I(s_1,\dots,s_m) := (t_1,\dots,t_m),$$
$$t_i = \frac{x_i}{s_i} \ (i \in I),\ t_j = s_j \ (j \in J),$$

we define the twisted cycle Δ_I in $T_x(=M_\emptyset)$ as $\Delta_I := (-1)^{|I|}(\iota_I)_*(\tilde{\Delta}_I)$. Note that $\iota_I(\sigma_I)$ is contained in the bounded domain $\{(t_1,\dots,t_m) \in \mathbb{R}^m \mid t_1,\dots,t_m, v(t), w(t,x) > 0\}$ which is denoted by $D_{1\cdots m}$ in Section 5.

We regard $\{\Delta_I\}_I$ as the 2^m twisted cycles Δ_I's arranged as $(\Delta, \Delta_1, \Delta_2, \dots, \Delta_m, \Delta_{12}, \Delta_{13}, \dots, \Delta_{1\cdots m})$. For a twisted cycle δ with respect to u_x, we denote by $\delta^\vee$ the twisted cycle with respect to u_x^{-1}, which is defined by the same construction as used for δ.

Fact 4.2 ([4]). We have

$$\Phi_x(\Delta_I) = \frac{\prod_{i\in I}\Gamma(c_i-1)\cdot\prod_{j\notin I}\Gamma(1-c_j)\cdot\Gamma\left(\sum_k c_k - a - m + 1\right)\Gamma(1-b)}{\Gamma\left(\sum_{i\in I}c_i - a - |I| + 1\right)\Gamma\left(\sum_{i\in I}c_i - b - |I| + 1\right)}\cdot f_I.$$

The intersection matrix $H := \left(I_h(\Delta_I, \Delta_{I'}^\vee)\right)_{I,I'}$ is diagonal. Further, the (I,I)-entry $H_{I,I}$ of H is

$$H_{I,I} = (-1)^{|I|}\cdot\frac{\prod_{j\notin I}\gamma_j\cdot\left(\alpha - \prod_{i\in I}\gamma_i\right)\left(\beta - \prod_{i\in I}\gamma_i\right)}{\prod_k(\gamma_k - 1)\cdot\left(\alpha - \prod_k \gamma_k\right)(\beta - 1)}.$$

Therefore, the Δ_I's form a basis of $H_m(T_x, u_x)$.

5. Monodromy representation

Put $\dot{x} := \left(\frac{1}{2m^2}, \ldots, \frac{1}{2m^2}\right) \in X$. For $\rho \in \pi_1(X, \dot{x})$ and $g \in Sol_{\dot{x}}$, let $\rho_* g$ be the analytic continuation of g along ρ. Since $\rho_* g$ is also a solution to $E_C(a, b, c)$, the map $\rho_* : Sol_{\dot{x}} \to Sol_{\dot{x}};\ g \mapsto \rho_* g$ is a $\mathbb{C}$-linear automorphism which satisfies $(\rho \cdot \rho')_* = \rho'_* \circ \rho_*$ for $\rho, \rho' \in \pi_1(X, \dot{x})$. Here, the composition $\rho \cdot \rho'$ of loops ρ and ρ' is defined as the loop going first along ρ, and then along ρ'. We thus obtain a representation

$$\mathcal{M}' : \pi_1(X, \dot{x}) \to GL(Sol_{\dot{x}})$$

of $\pi_1(X, \dot{x})$, where $GL(V)$ is the general linear group on a $\mathbb{C}$-vector space V. Since we can identify $Sol_{\dot{x}}$ with $H_m(T_{\dot{x}}, u_{\dot{x}})$ by Fact 3.2, the representation $\mathcal{M}'$ is equivalent to

$$\mathcal{M} : \pi_1(X, \dot{x}) \to GL(H_m(T_{\dot{x}}, u_{\dot{x}})).$$

Note that, for $\rho \in \pi_1(X, \dot{x})$, the map $\mathcal{M}(\rho) : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ coincides with the canonical isomorphism $\rho_* : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ in the local system $\mathcal{H}$. The representation $\mathcal{M}$ (and $\mathcal{M}'$) is called the monodromy representation, which is the main object in this paper.

For $1 \le k \le m$, let ρ_k be the loop in X defined by

$$\rho_k : [0, 1] \ni \theta \mapsto \left(\frac{1}{2m^2}, \ldots, \frac{e^{2\pi\sqrt{-1}\theta}}{2m^2}, \ldots, \frac{1}{2m^2}\right) \in X,$$

where $\frac{e^{2\pi\sqrt{-1}\theta}}{2m^2}$ is the k-th entry of $\rho_k(\theta)$. We take a positive real number ε_0 so that $\varepsilon_0 < \min\left\{\frac{1}{2m^2}, \frac{1}{(m-2)^2} - \frac{1}{m^2}\right\}$, and we define the loop ρ_0 in X as $\rho_0 := \tau_0 \rho'_0 \overline{\tau_0}$, where

$$\tau_0 : [0, 1] \ni \theta \mapsto \left((1-\theta) \cdot \frac{1}{2m^2} + \theta \cdot \left(\frac{1}{m^2} - \varepsilon_0\right)\right)(1, \ldots, 1) \in X,$$

$$\rho'_0 : [0, 1] \ni \theta \mapsto \left(\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}\right)(1, \ldots, 1) \in X,$$

and $\overline{\tau_0}$ is the reverse path of τ_0.

Remark 5.1. The loop ρ_k ($1 \le k \le m$) turns the hyperplane $(x_k = 0)$, and ρ_0 turns the hypersurface $(R(x) = 0)$ around the point $\left(\frac{1}{m^2}, \ldots, \frac{1}{m^2}\right)$, positively. Note that $\left(\frac{1}{m^2}, \ldots, \frac{1}{m^2}\right)$ is the nearest to the origin in $(R(x) = 0) \cap (x_1 = x_2 = \cdots = x_m) = \left\{\frac{1}{m^2}(1, \ldots, 1), \frac{1}{(m-2)^2}(1, \ldots, 1), \ldots\right\}$.

Theorem 5.2. *The loops $\rho_0, \rho_1, \ldots, \rho_m$ generate the fundamental group $\pi_1(X, \dot{x})$. Moreover, if $m \ge 2$, then they satisfy the following relations:*

$$\rho_i \rho_j = \rho_j \rho_i \quad (1 \le i, j \le m), \qquad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \le k \le m).$$

Remark 5.3. It is shown in [8] that if $m = 2$, then $\pi_1(X, \dot{x})$ is the group generated by ρ_0, ρ_1, ρ_2 with the relations in Theorem 5.2.

We show this theorem in Appendix A. By this theorem, for the study of the monodromy representation $\mathcal{M}$, it is sufficient to investigate $m + 1$ linear maps

$$\mathcal{M}_i := \mathcal{M}(\rho_i) \quad (0 \le i \le m).$$

Proposition 5.4. *For $1 \le k \le m$, the eigenvalues of $\mathcal{M}_k$ are γ_k^{-1} and 1. The eigenspace of $\mathcal{M}_k$ of eigenvalue γ_k^{-1} is spanned by the twisted cycles*

$$\Delta_I, \quad k \in I \subset \{1, \dots, m\}.$$

That of eigenvalue 1 is spanned by

$$\Delta_I, \quad k \notin I \subset \{1, \dots, m\}.$$

In particular, both eigenspaces are of dimension 2^{m-1}.

Proof. By Fact 4.2, the twisted cycle Δ_I corresponds to the solution

$$f_I = \prod_{i \in I} x_i^{1-c_i} \cdot F_C\left(a + |I| - \sum_{i \in I} c_i, b + |I| - \sum_{i \in I} c_i, c^I; x\right)$$

to $E_C(a, b, c)$. Since the series F_C defines a single-valued function around the origin, we have

$$\mathcal{M}'(\rho_k)(f_I) = \begin{cases} \gamma_k^{-1} f_I & k \in I \\ f_I & k \notin I. \end{cases}$$

Therefore, we obtain this proposition. □

Corollary 5.5. *For $1 \le k \le m$, the linear map $\mathcal{M}_k : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ is expressed as*

$$\mathcal{M}_k : \delta \mapsto \delta - \left(1 - \gamma_k^{-1}\right) \sum_{I \ni k} \frac{I_h\left(\delta, \Delta_I^{\vee}\right)}{I_h\left(\Delta_I, \Delta_I^{\vee}\right)} \Delta_I.$$

Further, the representation matrix M_k of $\mathcal{M}_k$ with respect to the basis $\{\Delta_I\}_I$ is the diagonal matrix whose (I, I)-entry is

$$\begin{cases} \gamma_k^{-1} & I \ni k \\ 1 & I \not\ni k. \end{cases}$$

Proof. We prove the first claim. By Proposition 5.4, $H_m(T_{\dot{x}}, u_{\dot{x}})$ is decomposed into the direct sum of the eigenspaces: $H_m(T_{\dot{x}}, u_{\dot{x}}) = (\bigoplus_{I \ni k} \mathbb{C}\Delta_I) \oplus (\bigoplus_{I \not\ni k} \mathbb{C}\Delta_I)$. Then it is sufficient to show that the claim holds for $\delta = \Delta_I$. This is clear by Fact 4.2 and Proposition 5.4. The second claim is obvious. □

For each subset $I \subset \{1, \dots, m\}$, we define a chamber D_I which gives an element in $H_m(T_{\dot{x}}, u_{\dot{x}})$. For $I = \{1, \dots, m\}$, we put

$$D_{1\cdots m} := \{(t_1, \dots, t_m) \in \mathbb{R}^m \mid t_k > 0 \ (1 \le k \le m),\ v(t) > 0,\ w(t, \dot{x}) > 0\}.$$

For $I = \emptyset$, we put

$$D_\emptyset = D := \{(t_1, \dots, t_m) \in \mathbb{R}^m \mid t_k < 0 \ (1 \le k \le m)\}.$$

For $I \neq \emptyset, \{1, \dots, m\}$, we put

$$D_I := \left\{ (t_1, \dots, t_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} t_i > 0 \ (i \in I),\ t_j < 0 \ (j \notin I), \\ v(t) > 0,\ (-1)^{m-|I|+1} w(t, \dot{x}) > 0 \end{array} \right\}.$$

The arguments of the factors of $u_{\dot{x}}(t)$ are defined as follows:

	$t_i (i \in I)$	$t_j (j \notin I)$	$v(t)$	$w(t, \dot{x})$
$D_{1\cdots m}$	0	$-$	0	0
D	$-$	$-\pi$	0	$-m\pi$
otherwise	0	$-\pi$	0	$-(m - \lvert I\rvert + 1)\pi$

By the identification of $H_m^{lf}(T_x, u_x)$ and $H_m(T_x, u_x)$ (see below Fact 3.1), we can consider that the (open) chamber D_I defines an element in $H_m(T_x, u_x)$. Note that if $m = 2$, then $D,\ D_1,\ D_2$, and D_{12} are equal to $\Delta_6,\ \Delta_7,\ \Delta_8$, and Δ_5 in [5], respectively. We state our main results:

Theorem 5.6. *The eigenvalues of* $\mathcal{M}_0$ *are* $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$ *and* 1. *The eigenspace* W_0 *of* $\mathcal{M}_0$ *of eigenvalue* $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$ *is spanned by* $D_{1\cdots m}$, *and hence is one-dimensional. The eigenspace* W_1 *of* $\mathcal{M}_0$ *of eigenvalue* 1 *is spanned by*

$$D_I, \quad I \subsetneq \{1, \dots, m\},$$

and expressed as

$$W_1 = \{\delta \in H_m(T_{\dot{x}}, u_{\dot{x}}) \mid I_h(\delta, D_{1\cdots m}^\vee) = 0\}.$$

In particular, this space is $(2^m - 1)$*-dimensional.*

The proof of this theorem is given in Section 7.

Corollary 5.7. *The linear map* $\mathcal{M}_0 : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ *is expressed as*

$$\mathcal{M}_0 : \delta \mapsto \delta - \left(1 + (-1)^m \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}\right) \frac{I_h\left(\delta, D_{1\cdots m}^\vee\right)}{I_h\left(D_{1\cdots m}, D_{1\cdots m}^\vee\right)} D_{1\cdots m}.$$

Proof. By Theorem 5.6, we have $H_m(T_{\dot{x}}, u_{\dot{x}}) = W_0 \oplus W_1 = \mathbb{C}D_{1\cdots m} \oplus W_1$. Then it is sufficient to show that the claim holds for $\delta = D_{1\cdots m}$ and $\delta \in W_1$. This is clear by Theorem 5.6. □

Proposition 5.8. *We have*

$$I_h\left(D_{1\cdots m}, \Delta_I^\vee\right) = I_h\left(\Delta_I, \Delta_I^\vee\right) = I_h\left(\Delta_I, D_{1\cdots m}^\vee\right). \tag{5.1}$$

Thus we obtain

$$D_{1\cdots m} = \sum_{I\subset\{1,\dots,m\}} \Delta_I, \tag{5.2}$$

$$I_h\left(D_{1\cdots m}, D_{1\cdots m}^\vee\right) = \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(\beta-1)\left(\alpha - \prod_k \gamma_k\right)}. \tag{5.3}$$

This proposition is also proved in Section 7. By this proposition, we obtain the following corollary.

Corollary 5.9. *The linear map $\mathcal{M}_0$ is expressed as*

$$\mathcal{M}_0 : \delta \mapsto \delta - \frac{(\beta-1)\left(\alpha - \prod_k \gamma_k\right)}{\alpha\beta} I_h\left(\delta, D_{1\cdots m}^\vee\right) D_{1\cdots m}.$$

Let M_0 be the representation matrix of $\mathcal{M}_0$ with respect to the basis $\{\Delta_I\}_I$. Then we have

$$M_0 = E_{2^m} - \frac{(\beta-1)\left(\alpha - \prod_k \gamma_k\right)}{\alpha\beta} NH,$$

where E_{2^m} is the unit matrix of size 2^m, N is the $2^m \times 2^m$ matrix with all entries 1, and $H = \left(I_h(\Delta_I, \Delta_{I'}^\vee)\right)_{I,I'}$ is the intersection matrix given in Fact 4.2.

Proof. The expression of $\mathcal{M}_0$ follows immediately from Corollary 5.7 and (5.3). To obtain the representation matrix, we have to show that the representation matrix of the linear map $\delta \mapsto I_h(\delta, D_{1\cdots m}^\vee)D_{1\cdots m}$ is given by NH. By Proposition 5.8, we have

$$\begin{aligned} I_h\left(\Delta_I, D_{1\cdots m}^\vee\right) D_{1\cdots m} &= I_h\left(\Delta_I, \Delta_I^\vee\right) D_{1\cdots m} = \sum_{I'} I_h\left(\Delta_I, \Delta_I^\vee\right) \Delta_{I'} \\ &= (\Delta, \Delta_1, \Delta_2, \dots, \Delta_m, \Delta_{12}, \Delta_{13}, \dots, \Delta_{1\cdots m}) \begin{pmatrix} I_h\left(\Delta_I, \Delta_I^\vee\right) \\ I_h\left(\Delta_I, \Delta_I^\vee\right) \\ \vdots \\ I_h\left(\Delta_I, \Delta_I^\vee\right) \end{pmatrix}, \end{aligned}$$

and hence the claim is proved. □

Remark 5.10. Let ρ_∞ be a loop in X turning the hyperplane $L_\infty \subset \mathbb{P}^m$ at infinity. Because of

$$\rho_\infty = \eta_\varepsilon(\ell_1 \cdots \ell_m \ell_{1\cdots 1}\ell_{1\cdots 10} \cdots \ell_{0\cdots 0})^{-1},$$

we can express $\mathcal{M}(\rho_\infty)$ by Corollaries 5.5, 5.9, equalities (A.1) and (A.2); see Appendix A, for the notations η_ε and ℓ_*. However, it is too complicated to be written down. Here we give the eigenvalues of $\mathcal{M}(\rho_\infty)$. Similarly to [9, Section 2.3], it turns out that $x_m^{-a} f(\frac{x_1}{x_m}, \dots, \frac{x_{m-1}}{x_m}, \frac{1}{x_m})$ is a solution to $E_C(a,b,c)$ if and only if $f(\xi_1,\dots,\xi_m)$ is a solution to $E_C(a, a-c_m+1, (c_1,\dots,c_{m-1}, a-b+1))$ with variables $\xi_1,\dots,\xi_m$. Then an argument similar to that used for Proposition 5.4 shows that the eigenvalues of $\mathcal{M}(\rho_\infty)$ are α and β. Moreover, both eigenspaces are of dimension 2^{m-1}.

6. Representation matrices

For $0 \le i \le m$, the matrix representation of $\mathcal{M}_i$ with respect to the basis $\{\Delta_I\}_I$ is given by M_i in Corollaries 5.5 and 5.9. However, M_0 is too complicated to be written down. In this section we give another basis $\{\Delta'_I\}_I$ of $H_m(T_{\dot{x}}, u_{\dot{x}})$ and write down the representation matrix of $\mathcal{M}_i$ with respect to this basis.

In this and the next sections, we use the following formulas.

Lemma 6.1. *For a positive integer n and complex numbers $\lambda_1,\dots,\lambda_n$, we have*

$$\sum_{N\subset\{1,\dots,n\}} \prod_{l\in N} \frac{\lambda_l}{1-\lambda_l} = \prod_{l=1}^{n} \frac{1}{1-\lambda_l}, \qquad \sum_{N\subset\{1,\dots,n\}} \prod_{l\in N} \frac{1}{\lambda_l - 1} = \prod_{l=1}^{n} \frac{\lambda_l}{\lambda_l-1}, \tag{6.1}$$

$$\sum_{N\subset\{1,\dots,n\}} \prod_{l\in N}(1-\lambda_l)\prod_{l\notin N}\lambda_l = \sum_{N\subset\{1,\dots,n\}} (-1)^{|N|}\prod_{l\in N}(\lambda_l-1)\prod_{l\notin N}\lambda_l = 1, \tag{6.2}$$

$$\sum_{N\subset\{1,\dots,n\}} \prod_{l\in N}(\lambda_l - 1) = \prod_{l=1}^{n}\lambda_l. \tag{6.3}$$

Proof. Because of

$$1 + \frac{\lambda_l}{1-\lambda_l} = \frac{1}{1-\lambda_l}, \quad 1 + \frac{1}{\lambda_l - 1} = \frac{\lambda_l}{\lambda_l - 1},$$

we obtain (6.1) by induction on n. The equalities (6.2) and (6.3) follow from the first and the second ones of (6.1), respectively. □

Let P be the $2^m \times 2^m$ matrix whose (N, I)-entry is

$$\begin{cases} \alpha\beta \prod_{j\notin I} \dfrac{\gamma_j - 1}{\gamma_j} \cdot \dfrac{\prod_{n\in N}\gamma_n}{\left(\alpha - \prod_{n\in N}\gamma_n\right)\left(\beta - \prod_{n\in N}\gamma_n\right)} & (N \subset I) \\ 0 & (N \not\subset I) \end{cases}$$

and $\{\Delta'_I\}_I$ be the basis of $H_m(T_{\dot{x}}, u_{\dot{x}})$ defined as

$$\begin{aligned}&\left(\Delta', \Delta'_1, \Delta'_2, \dots, \Delta'_m, \Delta'_{12}, \Delta'_{13}, \dots, \Delta'_{1\cdots m}\right)\\ &= (\Delta, \Delta_1, \Delta_2, \dots, \Delta_m, \Delta_{12}, \Delta_{13}, \dots, \Delta_{1\cdots m})\, P.\end{aligned}$$

Namely, Δ'_I is defined by

$$\Delta'_I = \alpha\beta \prod_{j\notin I} \frac{\gamma_j - 1}{\gamma_j} \cdot \sum_{N\subset I} \frac{\prod_{n\in N}\gamma_n}{\left(\alpha - \prod_{n\in N}\gamma_n\right)\left(\beta - \prod_{n\in N}\gamma_n\right)} \Delta_N.$$

Note that P is an upper triangular matrix.

Lemma 6.2. *We have*

$$\frac{\left(\alpha - \prod_k \gamma_k\right)\left(\beta - \prod_k \gamma_k\right)}{\alpha\beta\prod_k \gamma_k}\Delta'_{1\cdots m} + \sum_{I\subsetneq\{1,\dots,m\}} \left(\frac{1}{\prod_{i\in I}\gamma_i} + (-1)^{m-|I|}\frac{\prod_k \gamma_k}{\alpha\beta}\right)\Delta'_I = D_{1\cdots m}.$$

Proof. By the definition, the left-hand side is equal to

$$\begin{aligned}&\frac{\left(\alpha - \prod_k \gamma_k\right)\left(\beta - \prod_k \gamma_k\right)}{\alpha\beta\prod_k \gamma_k}\cdot\alpha\beta \sum_{N\subset\{1,\dots,m\}} \frac{\prod_{n\in N}\gamma_n}{\left(\alpha - \prod_{n\in N}\gamma_n\right)\left(\beta - \prod_{n\in N}\gamma_n\right)}\Delta_N\\ &+ \sum_{I\subsetneq\{1,\dots,m\}}\Bigg[\prod_{j\notin I}(\gamma_j - 1)\left(\frac{\alpha\beta}{\prod_k\gamma_k} + (-1)^{m-|I|}\prod_{i\in I}\gamma_i\right)\\ &\qquad\times\sum_{N\subset I}\frac{\prod_{n\in N}\gamma_n}{\left(\alpha - \prod_{n\in N}\gamma_n\right)\left(\beta - \prod_{n\in N}\gamma_n\right)}\Delta_N\Bigg].\end{aligned}\tag{6.4}$$

Clearly the coefficient of $\Delta_{1\cdots m}$ in (6.4) is 1. The coefficient of Δ_N ($N \neq \{1, \dots, m\}$) is

$$\begin{aligned}&\frac{\prod_{n\in N}\gamma_n}{\left(\alpha - \prod_{n\in N}\gamma_n\right)\left(\beta - \prod_{n\in N}\gamma_n\right)}\\ &\times\left(\frac{\left(\alpha - \prod_k\gamma_k\right)\left(\beta - \prod_k\gamma_k\right)}{\prod_k\gamma_k} + \sum_{\substack{I\supset N\\ I\neq\{1,\dots,m\}}}\prod_{j\notin I}(\gamma_j - 1)\left(\frac{\alpha\beta}{\prod_k\gamma_k} + (-1)^{m-|I|}\prod_{i\in I}\gamma_i\right)\right)\end{aligned}$$

which equals to 1 by the equalities (6.2) and (6.3). Therefore, by using (5.2), we conclude that (6.4) is equal to

$$\sum_{I\subset\{1,\dots,m\}} \Delta_I = D_{1\cdots m}. \qquad \square$$

Corollary 6.3. *For $0 \le i \le m$, let M_i' be the representation matrix of $\mathcal{M}_i$ with respect to the basis $\{\Delta_I'\}_I$. Then we have*

$$M_0' = E_{2^m} - N_0, \quad M_k' = M_k + N_k \ (1 \le k \le m),$$

where N_i is defined as follows. The (I, I')-entry of N_0 (respectively N_k) is zero, except in the case of $I' = \emptyset$ (respectively $k \in I'$ and $I = I' - \{k\}$). The $(I, \emptyset)$-entry of N_0 is

$$\begin{cases} \dfrac{\left(\alpha - \prod_k \gamma_k\right)\left(\beta - \prod_k \gamma_k\right)}{\alpha\beta \prod_k \gamma_k} & I = \{1, \dots, m\} \\ \dfrac{1}{\prod_{i\in I} \gamma_i} + (-1)^{m-|I|} \dfrac{\prod_k \gamma_k}{\alpha\beta} & \text{otherwise.} \end{cases}$$

The $(I' - \{k\}, I')$-entry of N_k is 1.

In particular, M_k' $(1 \le k \le m)$ is upper triangular, M_0' is lower triangular, and the $(\emptyset, \emptyset)$-entry of M_0' is

$$1 - \left(1 + (-1)^m \frac{\prod \gamma_k}{\alpha\beta}\right) = (-1)^{m-1} \prod \gamma_k \cdot \alpha^{-1}\beta^{-1}.$$

Proof. First, we evaluate M_0'. By Corollary 5.9, it is sufficient to show that the matrix representation of the linear map

$$\delta \mapsto \frac{(\beta - 1)\left(\alpha - \prod_k \gamma_k\right)}{\alpha\beta} I_h(\delta, D_{1\cdots m}^\vee) D_{1\cdots m}$$

is given by N_0. By Fact 4.2 and Proposition 5.8, we have

$$\frac{(\beta - 1)\left(\alpha - \prod_k \gamma_k\right)}{\alpha\beta} I_h(\Delta_{I'}', D_{1\cdots m}^\vee) D_{1\cdots m} = \left(\sum_{N\subset I'} (-1)^{|N|}\right) \prod_{i\in I'} \frac{\gamma_i}{\gamma_i - 1} \cdot D_{1\cdots m},$$

and hence we obtain

$$\frac{(\beta-1)\left(\alpha-\prod_k \gamma_k\right)}{\alpha\beta} I_h(\Delta'_{I'}, D^\vee_{1\cdots m}) D_{1\cdots m} = \begin{cases} D_{1\cdots m} & I' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Thus Lemma 6.2 shows the claim.

Next, we evaluate M'_k $(1 \le k \le m)$. We have to show that

$$\mathcal{M}_k(\Delta'_I) = \begin{cases} \Delta'_I & k \notin I \\ \gamma_k^{-1}\Delta'_I + \Delta'_{I-\{k\}} & k \in I. \end{cases}$$

If $k \notin I$, then the subsets N of I also satisfy $k \notin N$, and hence we have $\mathcal{M}_k(\Delta_N) = \Delta_N$ by Proposition 5.4. This implies that $\mathcal{M}_k(\Delta'_I) = \Delta'_I$, for $k \notin I$. We assume $k \in I$. For a subset N of $I - \{k\}$, we have

$$\mathcal{M}_k(\Delta_N) = \Delta_N = \left(\gamma_k^{-1} + \frac{\gamma_k - 1}{\gamma_k}\right)\Delta_N, \quad \mathcal{M}_k(\Delta_{N\cup\{k\}}) = \gamma_k^{-1}\Delta_{N\cup\{k\}}.$$

Then we obtain

$$\begin{aligned} \mathcal{M}_k(\Delta'_I) &= \gamma_k^{-1}\Delta'_I + \frac{\gamma_k - 1}{\gamma_k}\cdot\alpha\beta\prod_{j\notin I}\frac{\gamma_j - 1}{\gamma_j}\cdot\sum_{N\subset I-\{k\}}\frac{\prod_{n\in N}\gamma_n}{\left(\alpha-\prod_{n\in N}\gamma_n\right)\left(\beta-\prod_{n\in N}\gamma_n\right)}\Delta_N \\ &= \gamma_k^{-1}\Delta'_I + \alpha\beta\prod_{j\notin I-\{k\}}\frac{\gamma_j - 1}{\gamma_j}\cdot\sum_{N\subset I-\{k\}}\frac{\prod_{n\in N}\gamma_n}{\left(\alpha-\prod_{n\in N}\gamma_n\right)\left(\beta-\prod_{n\in N}\gamma_n\right)}\Delta_N \\ &= \gamma_k^{-1}\Delta'_I + \Delta'_{I-\{k\}}. \end{aligned}$$

□

Example 6.4. We write down M'_i $(0 \le i \le m)$ for $m = 2, 3$.

(i) In the case of $m = 2$, the representation matrices M'_0, M'_1, M'_2 are as follows:

$$M'_0 = \begin{pmatrix} -\frac{\gamma_1\gamma_2}{\alpha\beta} & 0 & 0 & 0 \\ -\frac{1}{\gamma_1} + \frac{\gamma_1\gamma_2}{\alpha\beta} & 1 & 0 & 0 \\ -\frac{1}{\gamma_2} + \frac{\gamma_1\gamma_2}{\alpha\beta} & 0 & 1 & 0 \\ -\frac{(\alpha-\gamma_1\gamma_2)(\beta-\gamma_1\gamma_2)}{\alpha\beta\gamma_1\gamma_2} & 0 & 0 & 1 \end{pmatrix},$$

$$M'_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{\gamma_1} \end{pmatrix}, \quad M'_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{\gamma_2} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_2} \end{pmatrix}.$$

These are equal to the transposed matrices of those in [5, Remark 4.4].

(ii) In the case of $m = 3$, the representation matrices M'_0, M'_1, M'_2, M'_3 are as follows:

$$M'_0 = \begin{pmatrix} \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0&0&0&0&0&0&0 \\ -\frac{1}{\gamma_1} - \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 1&0&0&0&0&0&0 \\ -\frac{1}{\gamma_2} - \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0&1&0&0&0&0&0 \\ -\frac{1}{\gamma_3} - \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0&0&1&0&0&0&0 \\ -\frac{1}{\gamma_1\gamma_2} + \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0&0&0&1&0&0&0 \\ -\frac{1}{\gamma_1\gamma_3} + \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0&0&0&0&1&0&0 \\ -\frac{1}{\gamma_2\gamma_3} + \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0&0&0&0&0&1&0 \\ -\frac{(\alpha-\gamma_1\gamma_2\gamma_3)(\beta-\gamma_1\gamma_2\gamma_3)}{\alpha\beta\gamma_1\gamma_2\gamma_3} & 0&0&0&0&0&0&1 \end{pmatrix},$$

$$M'_1 = \begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 0&\frac{1}{\gamma_1}&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&0&\frac{1}{\gamma_1}&0&0&0 \\ 0&0&0&0&0&\frac{1}{\gamma_1}&0&0 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&\frac{1}{\gamma_1} \end{pmatrix}, \quad M'_2 = \begin{pmatrix} 1&0&1&0&0&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&0&\frac{1}{\gamma_2}&0&0&0&0&0 \\ 0&0&0&1&0&0&1&0 \\ 0&0&0&0&\frac{1}{\gamma_2}&0&0&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&\frac{1}{\gamma_2}&0 \\ 0&0&0&0&0&0&0&\frac{1}{\gamma_2} \end{pmatrix},$$

$$M'_3 = \begin{pmatrix} 1&0&0&1&0&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&\frac{1}{\gamma_3}&0&0&0&0 \\ 0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&\frac{1}{\gamma_3}&0&0 \\ 0&0&0&0&0&0&\frac{1}{\gamma_3}&0 \\ 0&0&0&0&0&0&0&\frac{1}{\gamma_3} \end{pmatrix}.$$

7. Proof of the main theorem

In this section we prove Theorem 5.6. Since $\dim H_m(T_{\dot{x}}, u_{\dot{x}}) = 2^m$, it is sufficient to show that D_I's are eigenvectors and linearly independent. First, we evaluate the intersection numbers $I_h(\Delta_I, D^{\vee}_{I'})$. Second, we show the linear independence of $\{D_I\}_I$ by evaluating the determinant of the matrix $\big(I_h(\Delta_I, D^{\vee}_{I'})\big)_{I,I'}$. Third, we prove the properties of the eigenspace of $\mathcal{M}_0$ of eigenvalue 1. Finally, we show that $D_{1\cdots m}$ is an eigenvector of $\mathcal{M}_0$ of eigenvalue $(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$.

7.1. An expression of $D_{1\cdots m}$

We prove Proposition 5.8 using imaginary cycles and the Δ_I's introduced in Section 4.

Fix any $s_0 \in \sigma_I$, and set

$$\sqrt{-1}\mathbb{R}_I^m := \left\{ s_0 + \sqrt{-1}(\eta_1, \ldots, \eta_m) \mid (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \right\} \subset M_I,$$

which is called an imaginary cycle. By arguments similar to those in the proof of [4, Proposition 4.3 and Theorem 4.4], we can prove that the integration of $u\varphi$ on $(\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)$ also gives the solution f_I to $E_C(a,b,c)$, under some conditions for the parameters a, b, c. Therefore, $(\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee$ is orthogonal to the cycles $\Delta_{I'}$ ($I' \neq I$) with respect to I_h (*cf.* [5, Proof of Lemma 4.1]), and hence $(\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee$ is a constant multiple of $\Delta_I^\vee$. Note that both $D_{1\cdots m}$ and $\iota_I(\sigma_I)$ intersect $\iota_I(\sqrt{-1}\mathbb{R}_I^m)$ at $\iota_I(s_0)$ transversally. Since $D_{1\cdots m}$ and $\iota_I(\sigma_I)$ have a same orientation (*cf.* [4, Remark 4.5 (i)]), we have

$$I_h\left(D_{1\cdots m}, (\iota_I)_*\left(\sqrt{-1}\mathbb{R}_I^m\right)^\vee\right) = I_h\left(\Delta_I, (\iota_I)_*\left(\sqrt{-1}\mathbb{R}_I^m\right)^\vee\right).$$

Thus we obtain

$$\Delta_I^\vee = \frac{I_h\left(\Delta_I, \Delta_I^\vee\right)}{I_h(D_{1\cdots m}, (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee)} \cdot (\iota_I)_*\left(\sqrt{-1}\mathbb{R}_I^m\right)^\vee,$$

which implies the first equality of (5.1) because of

$$\begin{aligned} I_h\left(D_{1\cdots m}, \Delta_I^\vee\right) &= \frac{I_h\left(\Delta_I, \Delta_I^\vee\right)}{I_h\left(D_{1\cdots m}, (\iota_I)_*\left(\sqrt{-1}\mathbb{R}_I^m\right)^\vee\right)} \cdot I_h\left(D_{1\cdots m}, (\iota_I)_*\left(\sqrt{-1}\mathbb{R}_I^m\right)^\vee\right) \\ &= I_h\left(\Delta_I, \Delta_I^\vee\right). \end{aligned}$$

The second equality of (5.1) is shown as

$$I_h\left(\Delta_I, D_{1\cdots m}^\vee\right) = (-1)^m I_h\left(D_{1\cdots m}, \Delta_I^\vee\right)^\vee = (-1)^m I_h\left(\Delta_I, \Delta_I^\vee\right)^\vee = I_h\left(\Delta_I, \Delta_I^\vee\right),$$

where $g(\alpha, \beta, \gamma_1, \ldots, \gamma_m)^\vee := g(\alpha^{-1}, \beta^{-1}, \gamma_1^{-1}, \ldots, \gamma_m^{-1})$ for $g(\alpha, \beta, \gamma_1, \ldots, \gamma_m) \in \mathbb{C}(\alpha, \beta, \gamma_1, \ldots, \gamma_m)$. The orthogonality of the Δ_I's implies

$$D_{1\cdots m} = \sum_I \frac{I_h\left(D_{1\cdots m}, \Delta_I^\vee\right)}{I_h\left(\Delta_I, \Delta_I^\vee\right)} \Delta_I = \sum_I \Delta_I,$$

which is equality (5.2). Hence the self-intersection number of $D_{1\cdots m}$ is

$$I_h\left(D_{1\cdots m}, D_{1\cdots m}^{\vee}\right) = \sum_I I_h\left(\Delta_I, \Delta_I^{\vee}\right)$$

$$= \sum_I (-1)^{|I|} \frac{\prod_{j\notin I}\gamma_j \cdot \left(\alpha - \prod_{i\in I}\gamma_i\right)\left(\beta - \prod_{i\in I}\gamma_i\right)}{\prod_k(\gamma_k - 1)\cdot\left(\alpha - \prod_k \gamma_k\right)(\beta - 1)} = \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(\beta - 1)\left(\alpha - \prod_k \gamma_k\right)}.$$

At the last equality, we use (6.3). Therefore, Proposition 5.8 is proved.

7.2. Intersection numbers

For $I, I' \subset \{1,\dots,m\}$, we evaluate the intersection number $I_h(\Delta_I, D_{I'}^{\vee})$. By Proposition 5.8, we may assume $I' \neq \{1,\dots,m\}$. We set

$$J := \{1,\dots,m\} - I, \quad J' := \{1,\dots,m\} - I',$$
$$I_0 := I\cap I', \quad I_1 := I \cap J', \quad J_0 := J \cap I', \quad J_1 := J\cap J'.$$

Using ι_I, we have $I_h(\Delta_I, D_{I'}^{\vee}) = I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^{\vee})$, where $\tilde{D}_{I'} := (-1)^{|I|}\cdot(\iota_I)_*^{-1}(D_{I'})$. Note that the orientation of $\tilde{D}_{I'}$ is also induced from the natural embedding $\mathbb{R}^m \subset \mathbb{C}^m$. Thus σ_I and $\tilde{D}_{I'}$ have the same orientation. For $I' \neq \emptyset$, $\tilde{D}_{I'}$ is a chamber

$$\left\{(s_1,\dots,s_m)\in\mathbb{R}^m \,\middle|\, \begin{array}{l} s_i > 0\ (i\in I'),\ s_j < 0\ (j\notin I'), \\ (-1)^{|I_1|}v_I(s) > 0,\ (-1)^{|I_1|+|J'|+1}w_I(s) > 0 \end{array}\right\}$$

loaded the branch of u_I by the assignment of arguments as follows:

	$s_i(i\in I')$	$s_i(i\in I_1)$	$s_i(i\in J_1)$	$v_I(s)$	$w_I(s)$
argument	0	π	$-\pi$	$\lvert I_1\rvert\pi$	$(\lvert I_1\rvert - (\lvert J'\rvert+1))\pi$

In fact, the conditions for v_I and w_I are simply given by

$$1 - \sum_{i\in I}\frac{x_i}{s_i} - \sum_{j\in J} s_j > 0, \quad 1 - \sum_{i\in I} s_i - \sum_{j\in J}\frac{x_j}{s_j} < 0,$$

respectively, because $|J'| = |I_1| + |J_1|$. In the case $I' = \emptyset$ (then $I_0 = J_0 = \emptyset$), $\tilde{D}_{\emptyset} = \tilde{D}$ is a chamber

$$\{(s_1,\dots,s_m)\in\mathbb{R}^m \mid s_k < 0\ (1\le k\le m)\}$$

loaded the branch of u_I by the assignment of arguments as follows:

	$s_i(i\in I_1)$	$s_i(i\in J_1)$	$v_I(s)$	$w_I(s)$
argument	π	$-\pi$	$\lvert I_1\rvert\pi$	$(\lvert I_1\rvert - m)\pi$

Lemma 7.1. *If $I' \neq \emptyset$ and $I \subset J'$, we have $I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^{\vee}) = 0$.*

Proof. By the assumption, we have $J_0 = J \cap I' = I' \neq \emptyset$. For $(s_1, \ldots, s_m) \in \tilde{D}_{I'}$, we show that at least one of the s_j's ($j \in J_0$) satisfies $0 < s_j < mx_j$. Because of $mx_j < m \cdot \frac{\varepsilon^2}{m} < \varepsilon$, it implies that the chamber $\tilde{D}_{I'}$ is included in the ε-neighborhood of $(s_j = 0)$, and hence $\tilde{D}_{I'}$ does not intersect $\tilde{\Delta}_I$. Thus, the lemma is proved. We assume that all of the s_j's ($j \in J_0$) satisfy $s_j \geq mx_j$. By

$$0 > 1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j} = 1 - \sum_{i \in I_1} s_i - \sum_{j \in J_0} \frac{x_j}{s_j} - \sum_{j \in J_1} \frac{x_j}{s_j},$$

$s_i < 0$ ($i \in I_1$) and $s_j < 0$ ($j \in J_1$), we have

$$1 < 1 - \sum_{i \in I_1} s_i - \sum_{j \in J_1} \frac{x_j}{s_j} < \sum_{j \in J_0} \frac{x_j}{s_j}.$$

However, the inequalities

$$\sum_{j \in J_0} \frac{x_j}{s_j} \leq \sum_{j \in J_0} \frac{x_j}{mx_j} = \sum_{j \in J_0} \frac{1}{m} \leq 1$$

lead to a contradiction to $1 < \sum_{j \in J_0} \frac{x_j}{s_j}$. □

We consider in the case of $I' \neq \emptyset$. By Lemma 7.1, we may assume that $I \not\subset J'$. If we consider $x_1, \ldots, x_m \to 0$, the condition $(-1)^{|I_1|} v_I(s) > 0$ may be replaced with $1 - \sum_{j \in J} s_j > 0$, and $(-1)^{|I_1|+|J'|+1} w(s) > 0$ may be replaced with $1 - \sum_{i \in I} s_i < 0$ to judge if s belongs to a central area of $\tilde{D}_{I'}$. This observation means that we can evaluate the intersection number $I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^{\vee})$ like that of the regularization of V_I and $V_{I'}'^{\vee}$ by omitting the difference of the branches of u_I, where

$$V_I := \left\{ (s_1, \ldots, s_m) \in \mathbb{R}^m \;\middle|\; s_k > 0,\ 1 - \sum_{i \in I} s_i > 0,\ 1 - \sum_{j \in J} s_j > 0 \right\},$$

$$V_{I'}' := \left\{ (s_1, \ldots, s_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{c} s_k > 0\ (k \in I'),\ s_k < 0\ (k \in J'), \\ 1 - \sum_{i \in I} s_i < 0,\ 1 - \sum_{j \in J} s_j > 0 \end{array} \right\}. \tag{7.1}$$

Note that the chamber $V'_{I'}$ is not empty, because of $I \not\subset J'$. In the case of $I' = \emptyset$, we can see that the above claim is valid, by replacing (7.1) with

$$V' := \{(s_1, \dots, s_m) \in \mathbb{R}^m \mid s_k < 0 \ (1 \leq k \leq m)\}$$

(note that $1 - \sum_{i\in I} s_i > 0$ and $1 - \sum_{j\in J} s_j > 0$ hold clearly). Recall that when we construct the twisted cycle $\tilde{\Delta}_I$, the exponents of $(s_i = 0)$, $(s_j = 0)$, $(1 - \sum_{i\in I} s_i = 0)$ and $(1 - \sum_{j\in J} s_j = 0)$ are

$$c_i - 1, \quad 1 - c_j, \quad -b, \quad \sum_{k=1}^{m} c_k - a - m + 1,$$

respectively, where $i \in I$ and $j \in J$; see [4, Section 4].

Theorem 7.2. *For $I' \neq \emptyset$, we have*

$$\begin{aligned} I_h\left(\tilde{\Delta}_I, \tilde{D}^\vee_{I'}\right) = (-1)^{m-|J_1|-1} \cdot \prod_{k\in J'} \frac{1}{1-\gamma_k} \cdot \frac{1}{1-\beta} \\ \cdot \left[1 + \sum_{\substack{K_I \subsetneq I_0 \\ K_J \subset J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) \right. \\ \left. + \frac{\alpha}{\prod_k \gamma_k - \alpha} \sum_{\substack{K_I \subseteq I_0 \\ K_J \subsetneq J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) \right]. \end{aligned} \tag{7.2}$$

For $I' = \emptyset$, we have

$$I_h(\tilde{\Delta}_I, \tilde{D}^\vee) = (-1)^{|I|} \cdot \prod_{k=1}^{m} \frac{1}{1-\gamma_k}. \tag{7.3}$$

Proof. Let s_0 be an intersection point of $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$. We denote the difference of the branches of u_I at s_0 by $\chi_{I,I'}$, namely,

$$\chi_{I,I'} := \frac{\text{the value } u_I(s_0) \text{ with respect to the branch defined on } \tilde{\Delta}_I}{\text{the value } u_I(s_0) \text{ with respect to the branch defined on } \tilde{D}_{I'}}.$$

Note that $\chi_{I,I'}$ is independent of the choice of the intersection point s_0. We prove the theorem by two steps.

Step 1: We show that

$$I_h\left(\tilde{\Delta}_I, \tilde{D}_{I'}^{\vee}\right) = \chi_{I,I'} \cdot (-1)^{m-(|J'|+1)} \cdot \prod_{i\in I_1} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in J_1} \frac{1}{\gamma_j^{-1} - 1} \cdot \frac{1}{\beta^{-1} - 1}$$
$$\cdot \left[1 + \sum_{\substack{K_I \subsetneq I_0 \\ K_J \subset J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{1}{\gamma_j^{-1} - 1} \right) \right. \tag{7.4}$$
$$\left. + \frac{1}{\alpha^{-1}\prod_k \gamma_k - 1} \sum_{\substack{K_I \subseteq I_0 \\ K_J \subsetneq J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{1}{\gamma_j^{-1} - 1} \right) \right] \quad (I' \neq \emptyset),$$

$$I_h(\tilde{\Delta}_I, \tilde{D}^{\vee}) = \chi_{I,\emptyset} \cdot (-1)^{m-m} \cdot \prod_{i\in I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in J} \frac{1}{\gamma_j^{-1} - 1}. \tag{7.5}$$

We prove (7.4), by using results in [10]. Obviously, we have

$$\overline{V_I} \cap \overline{V'_{I'}} = \left\{ (s_1, \dots, s_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} s_j = 0 \ (j \in J'),\ 1 - \sum_{i\in I} s_i = 0, \\ s_i \geq 0 \ (i \in I'),\ 1 - \sum_{j\in J} s_j \geq 0 \end{array} \right\},$$

which implies that the intersection number $I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^{\vee})$ is equal to the product of

$$\chi_{I,I'} \cdot \prod_{i\in I\cap J'} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in J\cap J'} \frac{1}{\gamma_j^{-1} - 1} \cdot \frac{1}{\beta^{-1} - 1}$$

and the self-intersection number of the twisted cycle determined by the chamber

$$\left\{ (s_1, \dots, s_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} s_j = 0 \ (j \in J'),\ 1 - \sum_{i\in I} s_i = 0, \\ s_i > 0 \ (i \in I'),\ 1 - \sum_{j\in J} s_j > 0 \end{array} \right\}$$

in the $(m-(|J'|+1))$-dimensional space $L := \bigcap_{j\in J'}(s_j = 0) \cap (1 - \sum_{i\in I} s_i = 0)$. To evaluate this self-intersection number, we investigate the non-empty intersections of $(s_i = 0)$ $(i \in I')$, $(1 - \sum_{j\in J} s_j = 0)$ with L.

(i) Without $(1 - \sum_{j\in J} s_j = 0)$: we choose subsets K of I' such that $\bigcap_{k\in K}(s_k = 0) \cap L \neq \emptyset$. By the condition $1 - \sum_{i\in I} s_i = 0$, we have

$$\bigcap_{k\in K}(s_k = 0) \cap L \neq \emptyset \Leftrightarrow K \cap I \subsetneq I \Leftrightarrow K = K_I \cup K_J \ (K_I \subsetneq I,\ K_J \subset J).$$

(ii) With $(1-\sum_{j\in J} s_j = 0)$: we choose subsets K of I' such that $\bigcap_{k\in K}(s_k = 0)\cap(1-\sum_{j\in J} s_j = 0)\cap L \neq \emptyset$. By the conditions $1-\sum_{i\in I} s_i = 0$ and $1-\sum_{j\in J} s_j = 0$, we have

$$\begin{aligned}&\bigcap_{k\in K}(s_k=0)\cap\Big(1-\sum_{j\in J}s_j=0\Big)\cap L\neq\emptyset\\ &\Leftrightarrow K\cap I\subsetneq I,\ K\cap J\subsetneq J \Leftrightarrow K=K_I\cup K_J\ (K_I\subsetneq I,\ K_J\subsetneq J).\end{aligned}$$

Therefore, the self-intersection number is equal to

$$\begin{aligned}(-1)^{m-(|J'|+1)}\cdot\Bigg[1+\sum_{\substack{K_I\subsetneq I_0\\ K_J\subset J_0}}\Bigg(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\Bigg)\\ +\frac{1}{\alpha^{-1}\prod_k\gamma_k-1}\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\Bigg(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\Bigg)\Bigg],\end{aligned}$$

and hence (7.4) is proved. We can obtain the equality (7.5) in a similar way.

Step 2: We evaluate $\chi_{I,I'}$. We consider the differences of the branches of the factors of u_I at an intersection point of $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$.

(i) The argument of s_k on $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$ are given follows:

	$k\in I'=I_0\cup J_0$	$k\in I_1$	$k\in J_1$
$\tilde{\Delta}_I$	0	π	π
$\tilde{D}_{I'}$	0	π	$-\pi$

Since the exponent of s_j $(j\in J)$ is $C_j = 1-c_j+b$, the contribution by the branch of $\prod_k s_k^{C_k}$ is $\prod_{j\in J_1}(\gamma_j^{-1}\beta)$.

(ii) We have

$$v_I=\prod_{i\in I}s_i\cdot\Bigg(1-\sum_{j\in J}s_j-\sum_{i\in I}\frac{x_i}{s_i}\Bigg),$$

and the term $\sum_{i\in I}\frac{x_i}{s_i}$ does not concern the difference of the branches. By (i) and the fact that $s\in V'_{I'}$ satisfies $1-\sum_{j\in J}s_j>0$, both the argument of v_I on $\tilde{\Delta}_I$ and that on $\tilde{D}_{I'}$ are $|I_1|\pi$, and hence the contribution by the branch of v_I^A is 1.

(iii) We have

$$w_I=\prod_{j\in J}s_j\cdot\Bigg(1-\sum_{i\in I}s_i-\sum_{j\in J}\frac{x_j}{s_j}\Bigg),$$

and the term $\sum_{j\in J}\frac{x_j}{s_j}$ does not concern the difference of the branches. By (i) and the fact that $s\in V'_{I'}$ satisfies

$$\begin{cases}1-\sum\limits_{i\in I}s_i<0 & I'\neq\emptyset\\ 1-\sum\limits_{i\in I}s_i>0 & I'=\emptyset,\end{cases}$$

the arguments of w_I on $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$ at the intersection points are as follows:

$$(\text{argument on }\tilde{\Delta}_I)=\begin{cases}(|J_1|+1)\pi & I'\neq\emptyset\\ |J_1|\pi & I'=\emptyset,\end{cases}$$

$$(\text{argument on }\tilde{D}_{I'})=\begin{cases}(|I_1|-|J'|-1)\pi & I'\neq\emptyset\\ (|I_1|-m)\pi=-|J_1|\pi & I'=\emptyset.\end{cases}$$

Here, note that $m=|J'|=|I_1|+|J_1|$, if $I'=\emptyset$. Because of $|J'|=|I_1|+|J_1|$, we obtain

$$\begin{aligned}&(\text{difference of the arguments of }w_I)\\ &=\begin{cases}(|J_1|+1)\pi-(|I_1|-|J'|-1)\pi=2(|J_1|+1)\pi & I'\neq\emptyset\\ |J_1|\pi-(-|J_1|)\pi=2|J_1|\pi & I'=\emptyset.\end{cases}\end{aligned}$$

Since the exponent of w_I is $B=-b$, the contribution by the branch of w_I^B is

$$\begin{cases}\beta^{-(|J_1|+1)} & I'\neq\emptyset\\ \beta^{-|J_1|} & I'=\emptyset.\end{cases}$$

We thus have

$$\chi_{I,I'}=\prod_{j\in J_1}(\gamma_j^{-1}\beta)\cdot\beta^{-(|J_1|+1)}\ \ (I'\neq\emptyset),\qquad \chi_{I,\emptyset}=\prod_{j\in J_1}(\gamma_j^{-1}\beta)\cdot\beta^{-|J_1|}.$$

By Step 1, we obtain (7.2) and (7.3). □

To simplify the equality (7.2), we use Lemma 6.1. We summarize the results in this subsection.

Corollary 7.3. *If $I'\neq\emptyset,\{1,\dots,m\}$ then we have*

$$I_h\left(\Delta_I,D_{I'}^{\vee}\right)=(-1)^{|I|+|I'|-1}\cdot\prod_{k=1}^{m}\frac{1}{1-\gamma_k}\cdot\frac{\prod\limits_{i\in I_0}\gamma_i-1}{1-\beta}\cdot\frac{\prod\limits_{k}\gamma_k-\alpha\prod\limits_{j\in J_0}\gamma_j}{\prod_k\gamma_k-\alpha}. \tag{7.6}$$

This equality holds even if $I\subset J'$. For $I'=\emptyset$, we have

$$I_h\left(\Delta_I,D^{\vee}\right)=(-1)^{|I|}\cdot\prod_{k=1}^{m}\frac{1}{1-\gamma_k}. \tag{7.7}$$

Proof. Recall that $I_h(\Delta_I, D^\vee_{I'}) = I_h(\tilde{\Delta}_I, \tilde{D}^\vee_{I'})$. The equality (7.7) coincides with that in Theorem 7.2. If $I \subset J'$, then we have $I_0 = I \cap I' = \emptyset$, and hence $\prod_{i\in I_0}\gamma_i - 1 = 0$. Thus the right-hand side of (7.6) is 0, which is compatible with Lemma 7.1. Then we have to show that the right-hand side of (7.2) is equal to that of (7.6). By (6.1), we have

$$1 + \sum_{\substack{K_I \subsetneq I_0 \\ K_J \subset J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) = (-1)^{|I_0|} \cdot \left(\prod_{i\in I_0} \gamma_i - 1 \right) \cdot \prod_{k\in I'} \frac{1}{1-\gamma_k},$$

$$\sum_{\substack{K_I \subsetneq I_0 \\ K_J \subsetneq J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right)$$
$$= (-1)^{|I_0|} \cdot \left(\prod_{i\in I_0} \gamma_i - 1 \right) \cdot \left(1 - \prod_{j\in J_0} \gamma_j \right) \cdot \prod_{k\in I'} \frac{1}{1-\gamma_k}.$$

Therefore, we obtain

$$\begin{aligned}
I_h\left(\Delta_I, D^\vee_{I'}\right) &= I_h\left(\tilde{\Delta}_I, \tilde{D}^\vee_{I'}\right) \\
&= (-1)^{m-|J_1|-1} \cdot \prod_{k\in J'} \frac{1}{1-\gamma_k} \cdot \frac{1}{1-\beta} \cdot (-1)^{|I_0|} \cdot \left(\prod_{i\in I_0} \gamma_i - 1 \right) \\
&\quad \times \prod_{k\in I'} \frac{1}{1-\gamma_k} \cdot \left(1 + \frac{\alpha}{\prod_k \gamma_k - \alpha} \cdot \left(1 - \prod_{j\in J_0} \gamma_j \right) \right) \\
&= (-1)^{|I_1|+|J_0|-1} \cdot \prod_{k=1}^{m} \frac{1}{1-\gamma_k} \cdot \frac{\prod_{i\in I_0} \gamma_i - 1}{1-\beta} \cdot \frac{\prod_k \gamma_k - \alpha \prod_{j\in J_0} \gamma_j}{\prod_k \gamma_k - \alpha}.
\end{aligned}$$

Here we use $m = |I_0| + |I_1| + |J_0| + |J_1|$. Further, since

$$|I_1| + |J_0| = |I \cap I'^c| + |I^c \cap I'| = |I \cup I'| - |I \cap I'| = |I| + |I'| - 2|I \cap I'|,$$

we have $(-1)^{|I_1|+|J_0|-1} = (-1)^{|I|+|I'|-1}$. □

Lemma 7.4. *If $I' \neq \{1, \dots, m\}$ then $I_h(D_{1\cdots m}, D^\vee_{I'}) = 0$.*

Proof. This is obvious, since

$$\overline{D_{1\cdots m}} \subset \{(s_1, \dots, s_m) \in \mathbb{R}^m \mid s_k > x_k \ (1 \le k \le m)\},$$
$$\overline{D_{I'}} \cap \{(s_1, \dots, s_m) \in \mathbb{R}^m \mid s_k \ge x_k \ (1 \le k \le m)\} = \emptyset.$$ □

7.3. Linear independence

Let Λ_0 be the matrix $(I_h(\Delta_I, D_{I'}))_{I,I'}$ with $I,\ I'$ arranged in the same way as in the basis $\{\Delta_I\}_I$ (see Section 3). In this subsection, we evaluate the determinant of Λ_0.

Theorem 7.5. *We have*

$$\det\Lambda_0 = \begin{cases} -\left(\alpha\beta-\prod_{k=1}^{m}\gamma_k\right)\dfrac{\left(\prod_k\gamma_k+\alpha\right)^{2^{m-1}-1}}{(1-\beta)^{2^m-1}\left(\prod_k\gamma_k-\alpha\right)^{2^{m-1}}}\cdot\prod_{k=1}^{m}\dfrac{1}{(1-\gamma_k)^{2^{m-1}}} & m\text{: odd},\\[2ex] \left(\alpha\beta+\prod_{k=1}^{m}\gamma_k\right)\dfrac{\left(\prod_k\gamma_k+\alpha\right)^{2^{m-1}-2}}{(1-\beta)^{2^m-1}\left(\prod_k\gamma_k-\alpha\right)^{2^{m-1}-1}}\cdot\prod_{k=1}^{m}\dfrac{1}{(1-\gamma_k)^{2^{m-1}}} & m\text{: even}.\end{cases}$$

In particular, we obtain $\det\Lambda_0 \neq 0$*, hence* $\{D_I\}_I$ *is linearly independent.*

Remark 7.6. In this paper we assume that the parameters $a,\ b$, and $c=(c_1,\dots,c_m)$ are generic. In fact, it is sufficient for our proof of Theorem 5.6 to assume the irreducibility condition of the system $E_C(a,b,c)$

$$a-\sum_{i\in I}c_i,\quad b-\sum_{i\in I}c_i\notin\mathbb{Z}\quad (I\subset\{1,\dots,m\}),$$

and the conditions

$$c_1,\dots,c_m\notin\mathbb{Z},\quad a-\sum_{k=1}^{m}c_k\notin\frac{1}{2}\mathbb{Z},\quad a+b-\sum_{k=1}^{m}c_k+\frac{m+1}{2}\notin\mathbb{Z}.$$

To compute $\det\Lambda_0$, we change Λ_0 by elementary transformations, while keeping the determinant unchanged, as follows. Add the first, second, ..., (2^m-1)-th row of Λ_0 to the 2^m-th row of Λ_0; then 2^m-th row becomes

$$\begin{aligned}&\left(I_h\left(\sum_I\Delta_I, D^\vee\right),\dots,I_h\left(\sum_I\Delta_I, D^\vee_{2\cdots m}\right),I_h\left(\sum_I\Delta_I, D^\vee_{1\cdots m}\right)\right)\\ &=\left(I_h\left(D_{1\cdots m}, D^\vee\right),\dots,I_h\left(D_{1\cdots m}, D^\vee_{2\cdots m}\right),I_h\left(D_{1\cdots m}, D^\vee_{1\cdots m}\right)\right)\\ &=\left(0,\dots,0,I_h\left(D_{1\cdots m}, D^\vee_{1\cdots m}\right)\right)\end{aligned}$$

by Lemma 7.4. It means that

$$\det \Lambda_0 = I_h\left(D_{1\cdots m}, D^{\vee}_{1\cdots m}\right) \cdot \det \Lambda',$$

where Λ' is the leading principal minor of Λ_0 of size $2^m - 1$. By Proposition 5.8 and Corollary 7.3, we have

$$\det \Lambda_0 = \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(1-\beta)^{2^m-1}\left(\prod_k \gamma_k - \alpha\right)^{2^m-1}} \cdot \prod_{k=1}^{m} \frac{1}{(1-\gamma_k)^{2^m-1}} \cdot \det \Lambda,$$

where Λ is a $(2^m-1)\times(2^m-1)$ matrix whose (I, I')-entry is

$$\Lambda_{I,I'} := (-1)^{|I|+|I'|-1} \cdot \left(\prod_{i\in I\cap I'} \gamma_i - 1\right) \cdot \left(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j\in I^c\cap I'} \gamma_j\right) \quad I' \neq \emptyset,$$

$$\Lambda_{I,\emptyset} := (-1)^{|I|}.$$

We write

$$\Lambda = \begin{pmatrix} \Lambda(0,0) & \Lambda(0,1) & \cdots & \Lambda(0,m-1) \\ \Lambda(1,0) & \Lambda(1,1) & \cdots & \Lambda(1,m-1) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda(m-1,0) & \Lambda(m-1,1) & \cdots & \Lambda(m-1,m-1) \end{pmatrix},$$

where $\Lambda(k,k')$ is the $\binom{m}{k} \times \binom{m}{k'}$ matrix. Note that the entries of $\Lambda(k,k')$ are the (I, I')-entries of Λ with $|I| = k,\ |I'| = k'$.

We compute $\det \Lambda$. Put $\Lambda^{(0)} := \Lambda$. We take $\Lambda^{(n)}$ by induction on n as follows: for $n \geq 1$, we define $\Lambda^{(n)}$ by replacing the columns of I' ($|I'| \geq n+1$) of $\Lambda^{(n-1)}$ with

$$\Lambda^{(n-1)}_{*,I'} + \sum_{\substack{K'\subset I' \\ |K'|=n}} (-1)^{|I'|+n+1} \frac{\prod_k \gamma_k + (-1)^n \alpha \prod_{j\in K'^c\cap I'} \gamma_j}{\prod_k \gamma_k + (-1)^n \alpha} \cdot \Lambda^{(n-1)}_{*,K'},$$

where $\Lambda^{(n-1)}_{*,I'}$ is the column of I' of $\Lambda^{(n-1)}$. Straightforward calculations show the following result:

Lemma 7.7.

(i) $\det \Lambda^{(n)} = \det \Lambda$, $\Lambda^{(n)}_{\emptyset,\emptyset} = 1$;

(ii) *If* $|I'| \geq n+1$, *then*

$$\Lambda^{(n)}_{I,I'} = (-1)^{|I|+|I'|-1} \cdot \Bigg[\Big(\prod_{i \in I \cap I'} \gamma_i - 1 \Big) \cdot \Big(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \Big) - \sum_{\substack{K \subset I \cap I' \\ 0<|K| \leq n}} \Big(\prod_{i \in K} (\gamma_i - 1) \cdot \Big(\prod_{k=1}^{m} \gamma_k + (-1)^{|K|} \alpha \prod_{j \in K^c \cap I'} \gamma_j \Big) \Big) \Bigg];$$

(iii) $k \leq n \Longrightarrow \Lambda^{(n)}(k,k') = O \ (k' > k)$;
(iv) $\Lambda^{(n)}(1,1), \ldots, \Lambda^{(n)}(n+1,n+1)$ *are diagonal;*
(v) $1 \leq |I| \leq n+1 \Longrightarrow \Lambda^{(n)}_{I,I} = -\prod_{i \in I}(\gamma_i - 1) \cdot \big(\prod_k \gamma_k + (-1)^{|I|}\alpha \big)$.

Note that the columns of I' for $|I'| \leq n$ and the rows of I for $|I| \leq n-1$ are equal to those of $\Lambda^{(n-1)}$. Using this lemma, we prove Theorem 7.5.

Proof of Theorem 7.5. By Lemma 7.7, $\Lambda^{(m-2)}$ is the lower triangular matrix whose diagonal entries are given by (i) and (v). Hence we obtain

$$\begin{aligned} \det \Lambda_0 &= \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(1-\beta)^{2^m-1} \big(\prod_k \gamma_k - \alpha \big)^{2^m-1}} \cdot \prod_{k=1}^{m} \frac{1}{(1-\gamma_k)^{2^m-1}} \cdot \det \Lambda^{(m-2)} \\ &= (-1)^m \cdot \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(1-\beta)^{2^m-1} \big(\prod_k \gamma_k - \alpha \big)^{2^m-1}} \cdot \prod_{k=1}^{m} \frac{1}{(1-\gamma_k)^{2^{m-1}}} \\ &\quad \times \prod_{\emptyset \neq I \subsetneq \{1,\ldots,m\}} \Big(\prod_{k=1}^{m} \gamma_k + (-1)^{|I|} \alpha \Big). \end{aligned}$$

If m is odd we have

$$\prod_{\emptyset \neq I \subsetneq \{1,\ldots,m\}} \Big(\prod_{k=1}^{m} \gamma_k + (-1)^{|I|} \alpha \Big) = \Big(\prod_{k=1}^{m} \gamma_k - \alpha \Big)^{2^{m-1}-1} \cdot \Big(\prod_{k=1}^{m} \gamma_k + \alpha \Big)^{2^{m-1}-1}.$$

If m is even we have

$$\prod_{\emptyset \neq I \subsetneq \{1,\ldots,m\}} \Big(\prod_{k=1}^{m} \gamma_k + (-1)^{|I|} \alpha \Big) = \Big(\prod_{k=1}^{m} \gamma_k - \alpha \Big)^{2^{m-1}} \cdot \Big(\prod_{k=1}^{m} \gamma_k + \alpha \Big)^{2^{m-1}-2}.$$

Therefore, the proof of Theorem 7.5 is completed. □

7.4. The eigenspace of $\mathcal{M}_0$ associated to 1

By Lemma 7.4 and Theorem 7.5, to prove Theorem 5.6 we have to show that

- $\mathcal{M}_0(D_I) = D_I$ for $I \subsetneq \{1, \dots, m\}$,
- $\mathcal{M}_0(D_{1\cdots m}) = \left[(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}\right] \cdot D_{1\cdots m}$.

In this subsection we show the first claim. The second one is proved in the next subsection.

Hereafter, we use the coordinates $(s_1, \dots s_m) = \left(\dfrac{t_1}{x_1}, \dots, \dfrac{t_m}{x_m}\right)$. The functions $v(t)$ and $w(t, x)$ are expressed as

$$1 - \sum_{k=1}^{m} x_k s_k, \quad \prod_{k=1}^{m} (x_k s_k) \cdot \left(1 - \sum_{k=1}^{m} \frac{1}{s_k}\right),$$

respectively. Let

$$v'(s, x) := 1 - \sum_{k=1}^{m} x_k s_k, \quad w'(s) := \prod_{k=1}^{m} s_k \cdot \left(1 - \sum_{k=1}^{m} \frac{1}{s_k}\right).$$

If $x_1, \dots, x_m$ are positive real numbers then we have

$$t_k \gtreqless 0 \Leftrightarrow s_k \gtreqless 0, \quad v(t) \gtreqless 0 \Leftrightarrow v'(s, x) \gtreqless 0, \quad w(t, x) \gtreqless 0 \Leftrightarrow w'(s) \gtreqless 0,$$

and hence the expressions of the D_I's are as follows:

$$\begin{aligned} D_{1\cdots m}&: s_k > 0\ (1 \le k \le m),\ v'(s,x) > 0,\ w'(s) > 0, \\ D&: s_k < 0\ (1 \le k \le m), \\ D_I\ (\text{otherwise})&: s_i > 0\ (i \in I),\ s_j < 0\ (j \notin I),\ v'(s,x) > 0,\ (-1)^{m-|I|+1} w'(s) > 0. \end{aligned}$$

Note that, if $x = (x_1, \dots, x_m)$ moves, then only the divisor $(v'(s, x) = 0)$ varies.

Recall that the loop ρ_0 is homotopic to the composition $\tau_0 \rho'_0 \overline{\tau_0}$, where

$$\tau_0 : [0, 1] \ni \theta \mapsto \left((1 - \theta) \cdot \frac{1}{2m^2} + \theta \cdot \left(\frac{1}{m^2} - \varepsilon_0\right)\right)(1, \dots, 1) \in X,$$

$$\rho'_0 : [0, 1] \ni \theta \mapsto \left(\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}\right)(1, \dots, 1) \in X,$$

for a sufficiently small positive real number ε_0. Since variations along the paths τ_0 and $\overline{\tau_0}$ give trivial transformations of the cycles D_I's, we have to consider the variation along ρ'_0 for a sufficiently small ε_0. Let $x \to \left(\frac{1}{m^2}, \dots, \frac{1}{m^2}\right)$, then $(v'(s, x) = 0)$ and $(w'(s) = 0)$ are tangent at $(s_1, \dots, s_m) = (m, \dots, m)$. Thus $D_{1\cdots m}$ is a vanishing cycle. Each D_I ($I \subsetneq \{1, \dots, m\}$) survives as $x \to \left(\frac{1}{m^2}, \dots, \frac{1}{m^2}\right)$, and its variation along ρ'_0 is too slight to change the branch of u_x on it. This implies that $\mathcal{M}_0(D_I) = D_I$ for $I \subsetneq \{1, \dots, m\}$.

7.5. An eigenvector of $\mathcal{M}_0$ associated to the eigenvalue $(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$

In this subsection, we show $\mathcal{M}_0(D_{1\cdots m}) = [(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}] \cdot D_{1\cdots m}$. As mentioned in the previous subsection, it is sufficient to consider the variation of $D_{1\cdots m}$ along ρ_0' for a sufficiently small ε_0. Thus we may consider that $D_{1\cdots m}$ is contained in a small neighborhood of $s = (m, \dots, m)$ in $\mathbb{R}^m$.

Putting $x_1 = \cdots = x_m = \frac{1}{m^2} - \varepsilon_0$, we have

$$v'(s, \rho_0'(0)) = 1 - \left(\frac{1}{m^2} - \varepsilon_0\right)\sum_{k=1}^{m} s_k.$$

We use the coordinates system

$$(s_1', \dots, s_{m-1}', s_m') := \left(s_1 - m, \dots, s_{m-1} - m, \sum_{k=1}^{m} s_k - m^2\right).$$

Note that $s_l = s_l' + m$ $(1 \le l \le m-1)$ and $s_m = s_m' - \sum_{l=1}^{m-1} s_l' + m$. Then the origin $(s_1', \dots, s_m') = (0, \dots, 0)$ corresponds to $(s_1, \dots, s_m) = (m \dots, m)$. Let U be a small neighborhood of $(s_1', \dots, s_m') = (0, \dots, 0)$ so that $s_k > 0$ $(1 \le k \le m)$. In U, we have

$$v'(s, \rho_0'(0)) > 0 \Leftrightarrow 1 - \left(\frac{1}{m^2} - \varepsilon_0\right)(s_m' + m^2) > 0 \Leftrightarrow s_m' < \frac{m^2}{\frac{1}{m^2} - \varepsilon_0} \cdot \varepsilon_0,$$

$$w'(s) > 0 \Leftrightarrow 1 - \sum_{k=1}^{m} \frac{1}{s_k} > 0 \Leftrightarrow s_m' > \sum_{l=1}^{m-1} s_l' - m + \frac{1}{1 - \sum\limits_{l=1}^{m-1} \frac{1}{s_l' + m}}.$$

Hence $D_{1\cdots m}$ is expressed as

$$\left\{ (s_1', \dots, s_m') \in U \;\middle|\; \sum_{l=1}^{m-1} s_l' - m + \frac{1}{1 - \sum\limits_{l=1}^{m-1} \frac{1}{s_l'+m}} < s_m' < \frac{m^2}{\frac{1}{m^2} - \varepsilon_0} \cdot \varepsilon_0 \right\}.$$

Let θ move from 0 to 1, then the arguments of $\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}$ at the start point and the end point are equal. Thus the argument of $\frac{m^2}{\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}} \cdot \varepsilon_0 e^{2\pi\sqrt{-1}\theta}$ increases by 2π, when θ moves from 0 to 1. Put

$$f(s_1', \dots, s_{m-1}') := \sum_{l=1}^{m-1} s_l' - m + \frac{1}{1 - \sum\limits_{l=1}^{m-1} \frac{1}{s_l'+m}}.$$

Then $(s'_1, \dots, s'_{m-1}) = (0, \dots, 0)$ is a critical point of f, and the Hessian matrix $H_f(0, \dots, 0)$ at this point is positive definite. The Morse lemma implies that f is expressed as

$$\sum_{l=1}^{m-1} z_l^2,$$

with appropriate coordinates $(z_1, \dots, z_{m-1})$ around the origin. Therefore, the claim $\mathcal{M}_0(D_{1\cdots m}) = [(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}] \cdot D_{1\cdots m}$ is obtained from the following result:

Lemma 7.8. *For $y, \lambda, \mu \in \mathbb{C}$, we put*

$$Z_y := \mathbb{C}^m - \left(\left(z_m - \sum_{l=1}^{m-1} z_l^2 = 0 \right) \cup (y - z_m = 0) \right) \subset \mathbb{C}^m,$$

$$\nu_y(z) := \left(z_m - \sum_{l=1}^{m-1} z_l^2 \right)^{\lambda} \cdot (y - z_m)^{\mu},$$

where $z_1, \dots, z_m$ are coordinates of $\mathbb{C}^m$. We consider the twisted homology groups $H_m(Z_y, \nu_y)$ $(y \in \mathbb{C})$. Let $\delta_y \in H_m(Z_y, \nu_y)$ $(y > 0)$ be expressed by the twisted cycle defined by the domain

$$D(y) := \left\{ (z_1, \dots, z_m) \in \mathbb{R}^m \;\middle|\; \sum_{l=1}^{m-1} z_l^2 < z_m < y \right\},$$

and let δ' be the element in $H_m(Z_1, \nu_1)$, which is obtained by the deformation of δ_1 along $y = e^{2\pi\sqrt{-1}\theta}$ as $\theta : 0 \to 1$. Then we have

$$\delta' = (-1)^{m-1} e^{2\pi\sqrt{-1}(\lambda+\mu)} \cdot \delta_1.$$

Proof. It is easy to see that the domain $D(y)$ is expressed by $(\xi_1, \dots, \xi_m) \in [0, 1]^m$ as

$$z_l = (2\xi_l - 1) \sqrt{y\xi_m \prod_{j=l+1}^{m-1} (1 - (2\xi_j - 1)^2)} \quad (1 \le l \le m-1),$$
$$z_m = y\xi_m.$$

The functions $z_m - \sum_{l=1}^{m-1} z_l^2$ and $y - z_m$ are expressed as

$$y\xi_m \left(1 - \sum_{l=1}^{m-1} (2\xi_l - 1)^2 \prod_{j=l+1}^{m-1} \left(1 - \left(2\xi_j - 1\right)^2 \right) \right), \quad y(1 - \xi_m), \tag{7.8}$$

respectively. We consider the variation along $y = e^{2\pi\sqrt{-1}\theta}$ as $\theta : 0 \to 1$. The expression of the domain $D(1)$ by $(\xi_1, \dots, \xi_m) \in [0,1]^m$ is changed. However, by a bijection

$$r : \xi_l \mapsto 1 - \xi_l \ (1 \le l \le m-1), \quad \xi_m \mapsto \xi_m,$$

the expression coincides with the original one with contributions to orientation. Further, both arguments of $z_m - \sum_{l=1}^{m-1} z_l^2$ and $y - z_m$ increase by 2π, and the expressions (7.8) are invariant under the bijection r. Therefore, we obtain

$$\delta' = (-1)^{m-1} e^{2\pi\sqrt{-1}(\lambda+\mu)} \cdot \delta_1. \qquad \square$$

Appendix

A. The fundamental group

In this appendix we prove Theorem 5.2. We assume $m \ge 2$.

We regard $\mathbb{C}^m$ as a subset of $\mathbb{P}^m$ and put $L_\infty := \mathbb{P}^m - \mathbb{C}^m$. Then we can consider that $S \cup L_\infty$ is a hypersurface in $\mathbb{P}^m$, and

$$X = \mathbb{C}^m - S = \mathbb{P}^m - (S \cup L_\infty).$$

By a special case of the Zariski theorem of Lefschetz type (refer to [3, Proposition 4.3.1]), the inclusion $L - (L \cap (S \cup L_\infty)) \hookrightarrow X$ induces a surjection

$$\eta : \pi_1 (L - (L \cap (S \cup L_\infty))) \to \pi_1(X),$$

for a line L in $\mathbb{P}^m$, which intersects $S \cup L_\infty$ transversally and avoids its singular parts. Note that generators of $\pi_1(L-(L\cap(S\cup L_\infty)))$ are given by $m+2^{m-1}$ loops going once around each of the intersection points in $L \cap S \subset \mathbb{C}^m$. To define loops in X explicitly, we specify such a line L in the following way. Let $r_1, \dots, r_{m-1}$ be positive real numbers satisfying

$$r_1 < \frac{1}{4}, \quad r_k < \frac{r_{k-1}}{4} \ \text{ for } \ 2 \le k \le m-1,$$

and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{m-1})$ be sufficiently small positive real numbers such that $\varepsilon_1 < \dots < \varepsilon_{m-1}$. We consider lines

$$L_0 : (x_1, \dots, x_{m-1}, x_m) = (r_1, \dots, r_{m-1}, 0) + t(0, \dots, 0, 1) \quad t \in \mathbb{C},$$
$$L_\varepsilon : (x_1, \dots, x_{m-1}, x_m) = (r_1, \dots, r_{m-1}, 0) + t(\varepsilon_1, \dots, \varepsilon_{m-1}, 1) \quad t \in \mathbb{C}$$

in $\mathbb{C}^m$. We identify L_ε with $\mathbb{C}$ by the coordinate t. The intersection point $L_\varepsilon \cap (x_k = 0)$ is coordinated by $t = -\frac{r_k}{\varepsilon_k} < 0$, for $1 \le k \le m-1$. The intersection point $L_\varepsilon \cap (x_m = 0)$ is coordinated by $t = 0$. L_ε and $(R(x) = 0)$ intersect at 2^{m-1} points. We coordinate the intersection points $L_\varepsilon \cap (R(x) = 0)$ by $t =$

$t_{a_1\cdots a_{m-1}}$, $(a_1, \ldots, a_{m-1}) \in \{0, 1\}^{m-1}$. The correspondence is as follows. We denote the coordinates of the intersection points $L_0 \cap (R(x) = 0)$ by

$$t^{(0)}_{a_1\cdots a_{m-1}} := \left(1 + \sum_{k=1}^{m-1} (-1)^{a_k} \sqrt{r_k}\right)^2 .$$

By this definition, we have

$$\begin{aligned} & t^{(0)}_{a_1\cdots a_{m-1}} < t^{(0)}_{a'_1\cdots a'_{m-1}} \\ & \iff a_1 - a'_1 = \cdots = a_{r-1} - a'_{r-1} = 0,\ a_r = 1,\ a'_r = 0 \\ & \iff a_1 \cdots a_{m-1} > a'_1 \cdots a'_{m-1}, \end{aligned}$$

where $a_1 \cdots a_{m-1}$ is regarded as a binary number. For example, if $m = 4$ then

$$t^{(0)}_{111} < t^{(0)}_{110} < t^{(0)}_{101} < t^{(0)}_{100} < t^{(0)}_{011} < t^{(0)}_{010} < t^{(0)}_{001} < t^{(0)}_{000}.$$

Since L_ε is sufficiently close to L_0, $t_{a_1\cdots a_{m-1}}$ is supposed to be arranged near to $t^{(0)}_{a_1\cdots a_{m-1}}$.

We can show that L_0 does not pass the singular part of $(R(x) = 0)$. This implies that for sufficiently small ε_k's, L_ε also avoids the singular parts of $S \cup L_\infty$. Thus, $\eta_\varepsilon : \pi_1 (L_\varepsilon - (L_\varepsilon \cap (S \cup L_\infty))) \to \pi_1(X)$ is a surjection.

Let ℓ_k be the loop in $L_\varepsilon - (L_\varepsilon \cap S)$ going once around the intersection point $L_\varepsilon \cap (x_k = 0)$, and let $\ell_{a_1\cdots a_{m-1}}$ be the loop in $L_\varepsilon - (L_\varepsilon \cap S)$ going once around the intersection point $t_{a_1\cdots a_{m-1}}$. Each loop approaches the intersection point through the upper half-plane of the t-space; see Figure A.1.

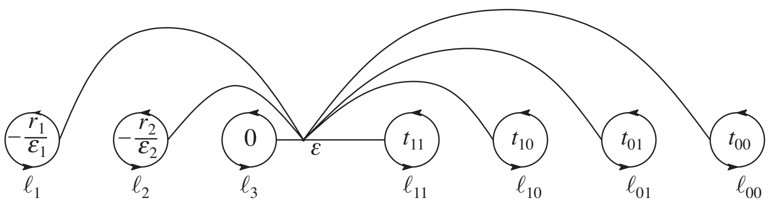

Figure A.1. ℓ_* for $m = 3$.

It is easy to see that

$$\eta_\varepsilon(\ell_k) = \rho_k \ (1 \le k \le m), \quad \eta_\varepsilon(\ell_{1\cdots1}) = \rho_0. \tag{A.1}$$

Further, we have

$$\rho_i \rho_j = \rho_j \rho_i \quad \text{for} \quad 1 \le i, j \le m,$$

since the fundamental group of $(\mathbb{C}^\times)^m$ is Abelian. To investigate relations among the $\eta_\varepsilon(\ell_{a_1\cdots a_{m-1}})$'s, we consider these loops in $L_0 - (L_0 \cap S)$. By the above definition, we can define the $\ell_{a_1\cdots a_{m-1}}$'s as loops in $L_0 - (L_0 \cap S)$. Since L_0 is sufficiently close to L_ε, the image of $\ell_{a_1\cdots a_{m-1}}$ under

$$\eta : \pi_1 (L_0 - (L_0 \cap (S \cup L_\infty))) \to \pi_1(X)$$

coincides with $\eta_\varepsilon(\ell_{a_1\cdots a_{m-1}})$ as elements in $\pi_1(X)$. Though η is not a surjection, relations among the $\eta(\ell_{a_1\cdots a_{m-1}})$'s in $\pi_1(X)$ can be regarded as those among the $\eta_\varepsilon(\ell_{a_1\cdots a_{m-1}})$'s.

Lemma A.1.

(i) $\eta(\ell_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}) = \rho_k \eta(\ell_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}})\rho_k^{-1}$.
(ii) $\eta(\ell_{1\dots 1}) = \rho_{m-1}\eta(\ell_{1\dots 1}\ell_{1\dots 10}\ell_{1\dots 1}^{-1})\rho_{m-1}^{-1}$.

Temporarily, we admit this lemma. By (i), we have

$$\begin{aligned} \eta_\varepsilon(\ell_{a_1\cdots a_{m-1}}) = \eta(\ell_{a_1\cdots a_{m-1}}) &= \left(\rho_1^{b_1}\cdots\rho_{m-1}^{b_{m-1}}\right)\cdot\eta(\ell_{1\dots 1})\cdot\left(\rho_1^{b_1}\cdots\rho_{m-1}^{b_{m-1}}\right)^{-1} \\ &= \left(\rho_1^{b_1}\cdots\rho_{m-1}^{b_{m-1}}\right)\cdot\rho_0\cdot\left(\rho_1^{b_1}\cdots\rho_{m-1}^{b_{m-1}}\right)^{-1} \end{aligned} \tag{A.2}$$

as elements in $\pi_1(X)$, where $(b_1,\dots,b_{m-1}) := (1-a_1,\dots,1-a_{m-1})$. This implies that the loops $\rho_0,\dots,\rho_m$ generate $\pi_1(X)$, since the images of the ℓ_k's and $\ell_{a_1\cdots a_{m-1}}$'s by η_ε generate $\pi_1(X)$. By (ii) and the above argument, we obtain

$$\begin{aligned} \rho_0 = \eta(\ell_{1\dots 1}) &= \rho_{m-1}\eta\left(\ell_{1\dots 1}\ell_{1\dots 10}\ell_{1\dots 1}^{-1}\right)\rho_{m-1}^{-1} \\ &= \rho_{m-1}\cdot\rho_0\cdot\rho_{m-1}\rho_0\rho_{m-1}^{-1}\cdot\rho_0^{-1}\cdot\rho_{m-1}^{-1}, \end{aligned}$$

that is, $(\rho_0\rho_{m-1})^2 = (\rho_{m-1}\rho_0)^2$. Changing the definitions of L_0 and L_ε, we obtain the relations

$$(\rho_0\rho_k)^2 = (\rho_k\rho_0)^2 \quad (1 \le k \le m).$$

For example, if we put

$$L_\varepsilon : (x_1, x_2, \dots, x_m) = (0, r_1, \dots, r_{m-1}) + t(1, \varepsilon_1, \dots, \varepsilon_{m-1}) \quad t \in \mathbb{C},$$

then a similar argument shows $(\rho_0\rho_m)^2 = (\rho_m\rho_0)^2$. Therefore, the proof of Theorem 5.2 is complete.

Proof of Lemma A.1. For $\theta \in [0,1]$, let $L(\theta)$ be the line defined by

$$\begin{aligned} L(\theta) :&(x_1,\dots,x_k,\dots,x_{m-1},x_m) \\ &= (r_1,\dots,e^{2\pi\sqrt{-1}\theta}r_k,\dots,r_{m-1},0) + t(0,\dots,0,1) \quad (t \in \mathbb{C}). \end{aligned}$$

Note that $L(0) = L(1) = L_0$. We identify $L(\theta)$ with $\mathbb{C}$ by the coordinate t. It is easy to see that the intersection points of $L(\theta)$ and $(R(x) = 0)$ are given by the following 2^{m-1} elements:

$$t^{(\theta)}_{a_1\cdots a_{m-1}} := \left(1 + \sum_{\substack{j=1 \\ j\neq k}}^{m-1}(-1)^{a_j}\sqrt{r_j} + (-1)^{a_k}\sqrt{r_k}e^{\pi\sqrt{-1}\theta}\right)^2.$$

The points $1+\sum_{j\neq k}(-1)^{a_j}\sqrt{r_j}+(-1)^{a_k}\sqrt{r_k}e^{\pi\sqrt{-1}\theta}$ are in the right half-plane for any $\theta \in [0, 1]$, since $\sum_{j=1}^{m-1}\sqrt{r_j} < \sum_{j=1}^{m-1} 2^{-j} < 1$. Let θ move from 0 to 1, then

(a) $t^{(1)}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}} = t^{(0)}_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}}$,
$\quad t^{(1)}_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}} = t^{(0)}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$,

(b) $t^{(\theta)}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$ moves in the upper half-plane,

(c) $t^{(\theta)}_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}}$ moves in the lower half-plane.

For example, the $t_{a_1a_2a_3}$'s move as Figure A.2, for $m = 4$ and $k = 2$.

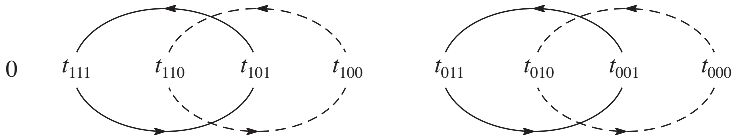

Figure A.2. $t_{a_1a_2a_3}$ for $m = 4,\ k = 2$.

We put $P(\theta) := \mathbb{C} - \{t^{(\theta)}_{a_1\cdots a_{m-1}} \mid a_j \in \{0, 1\}\}$ that is regarded as a subset of $L(\theta)$. Let ε' be a sufficiently small positive real number, and we consider the fundamental group $\pi_1(P(\theta), \varepsilon')$. As mentioned above, the $\ell_{a_1\cdots a_{m-1}}$'s are defined as elements in $\pi_1(P(0), \varepsilon') = \pi_1(P(1), \varepsilon')$. Let θ move from 0 to 1, then the $\ell_{a_1\cdots a_{m-1}}$'s define the elements in each $\pi_1(P(\theta), \varepsilon')$ naturally. The properties (a), (b), (c) imply the following.

Lemma A.2. *$\ell_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$ in $\pi_1(P(0), \varepsilon')$ changes to $\ell_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}}$ in $\pi_1(P(1), \varepsilon')$.*

We give the proof of this lemma below. By this variation, the base point moves around the divisor $(x_k = 0)$, since the base point $\varepsilon' \in P(\theta)$ corresponds to the point $(r_1, \dots, e^{2\pi\sqrt{-1}\theta}r_k, \dots, r_{m-1}, \varepsilon') \in L(\theta)$. It implies the conjugation by ρ_k in $\pi_1(X)$. Hence we obtain the relation (i).

To prove (ii), we use a similar argument for $k = m-1$ and $\ell_{1\cdots 1} \in \pi_1(P(0), \varepsilon')$. Let θ move from 0 to 1, then $\ell_{1\cdots 1}$ changes into a loop in $P(1)$, which goes once around $t^{(1)}_{1\cdots 1} = t^{(0)}_{1\cdots 10}$ and approaches this point through the lower half-plane (see Figure A.3). Since such a loop is homotopic to $\ell_{1\cdots 1}\ell_{1\cdots 10}\ell^{-1}_{1\cdots 1}$, we obtain (ii). □

Proof of Lemma A.2. We show that the variations of the $t_{a'_1\cdots a'_{m-1}}$'s do not interfere with the moving of the loop $\ell_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$. We put $\tilde{t}^{(\theta)}_{a_1\cdots a_{m-1}} := 1 + \sum_{j\neq k}(-1)^{a_j}\sqrt{r_j} + (-1)^{a_k}\sqrt{r_k}e^{\pi\sqrt{-1}\theta}$. This satisfies $(\tilde{t}^{(\theta)}_{a_1\cdots a_{m-1}})^2 = t^{(\theta)}_{a_1\cdots a_{m-1}}$. Since each $\tilde{t}^{(\theta)}_{a_1\cdots a_{m-1}}$ is in the right half-plane, $t^{(\theta)}_{a_1\cdots a_{m-1}}$ does not meet the half-line $(-\infty, 0] \subset \mathbb{R}$. For each θ, $\tilde{P}(\theta) :=$ (the right half-plane) $-\{\tilde{t}^{(\theta)}_{a_1\cdots a_{m-1}} \mid a_j \in \{0, 1\}\}$ is homeomorphic to $P(\theta) - (-\infty, 0]$ by the map

$$h : \tilde{P}(\theta) \longrightarrow P(\theta) - (-\infty, 0]; \quad z \longmapsto z^2.$$

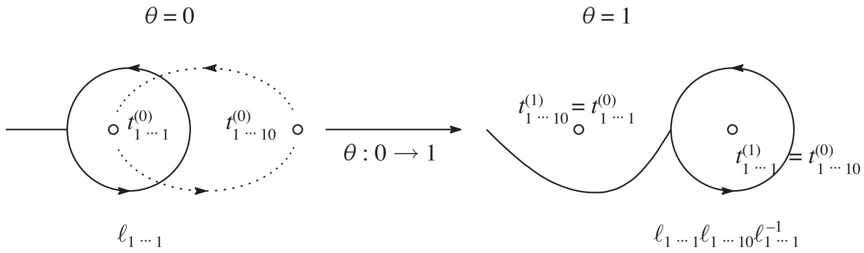

Figure A.3. The variation of $\ell_{1\dots 1}$.

It is sufficient to show that the points $\tilde{t}^{(\theta)}_{a_1\cdots a_{m-1}}$'s do not interfere with the moving of the loop $\tilde{\ell}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$ in $\tilde{P}(\theta)$, which satisfies $h_*(\tilde{\ell}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}})=\ell_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$. Since each $\tilde{t}^{(\theta)}_{a'_1\cdots a'_{k-1}1a'_{k+1}\cdots a'_{m-1}}$ moves in lower half-plane, it does not interfere with the moving of $\tilde{\ell}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$. We consider the variation of $\tilde{t}^{(\theta)}_{a'_1\cdots a'_{k-1}0a'_{k+1}\cdots a'_{m-1}}$ for $(a'_1,\dots,a'_{k-1},a'_{k+1},\dots,a'_{m-1})\neq(a_1,\dots,a_{k-1},a_{k+1},\dots,a_{m-1})$. By definition, $\tilde{t}^{(\theta)}_{a'_1\cdots a'_{k-1}0a'_{k+1}\cdots a'_{m-1}}-\tilde{t}^{(\theta)}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$ does not depend on θ. Thus, $\tilde{t}^{(\theta)}_{a'_1\cdots a'_{k-1}0a'_{k+1}\cdots a'_{m-1}}$ moves parallel to $\tilde{t}^{(\theta)}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$. This implies that $\tilde{t}^{(\theta)}_{a'_1\cdots a'_{k-1}0a'_{k+1}\cdots a'_{m-1}}$ does not interfere with the moving of $\tilde{\ell}_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}$. Therefore, the proof is complete. □

References

[1] K. AOMOTO and M. KITA, "Theory of Hypergeometric Functions", translated by K. Iohara, Springer Monographs in Mathematics, Springer-Verlag, Tokyo, 2011.

[2] F. BEUKERS, *Monodromy of A-hypergeometric functions*, preprint, arXiv:1101.0493.

[3] A. DIMCA, "Singularities and Topology of Hypersurfaces", Universitext, Springer-Verlag, New York, 1992.

[4] Y. GOTO, *Twisted cycles and twisted period relations for Lauricella's hypergeometric function F_C*, Internat. J. Math. **24** (2013), 1350094.

[5] Y. GOTO and K. MATSUMOTO, *The monodromy representation and twisted period relations for Appell's hypergeometric function F_4*, Nagoya Math. J. **217** (2015), 61–94.

[6] Y. HARAOKA and Y. UENO, *Rigidity for Appell's hypergeometric series F_4*, Funkcial. Ekvac. **51** (2008), 149–164.

[7] R. HATTORI and N. TAKAYAMA, *The singular locus of Lauricella's F_C*, J. Math. Soc. Japan **66** (2014), 981–995.

[8] J. KANEKO, *Monodromy group of Appell's system (F_4)*, Tokyo J. Math. **4** (1981), 35–54.

[9] M. KATO, *The Riemann problem for Appell's F_4*, Mem. Fac. Sci. Kyushu Univ. Ser. A **47** (1993), 227–243.

[10] M. KITA and M. YOSHIDA, *Intersection theory for twisted cycles II —Degenerate arrangements*, Math. Nachr. **168** (1994), 171–190.

[11] G. LAURICELLA, *Sulle funzioni ipergeometriche a più variabili*, Rend. Circ. Mat. Palermo **7** (1893), 111–158.

[12] K. TAKANO, *Monodromy group of the system for Appell's F_4*, Funkcial. Ekvac. **23** (1980), 97–122.
[13] M. YOSHIDA, "Hypergeometric Functions, my Love. Modular Interpretations of Configuration Spaces", Aspects of Mathematics, Vol. E 32, Vieweg & Sohn, Braunschweig, 1997.

General Education
Otaru University of Commerce
Midori 3-5-21, Otaru, Hokkaido
047-8501, Japan
goto@res.otaru-uc.ac.jp