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Boundary regularity of Dirichlet minimizing Q-valued functions

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Abstract. We prove Hölder continuity at the boundary for Dirichlet mimizing Q-valued functions. Almgren introduced multivalued/Q-valued functions to study regularity of minimal surfaces in higher codimension. The Hölder continuity in the interior for Dirichlet minimizers is an outcome of Almgren's original theory [2], to which the work of C. De Lellis and E. N. Spadaro has given a simpler alternative approach [7]. We extend the Hölder regularity for Dirichlet minimizing Q-valued functions up to the boundary assuming C^1 regularity of the domain and $C^{0,\alpha}$ regularity of the boundary data with $\alpha > \frac{1}{2}$.

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Introduction

Multivalued maps with focus on Dirichlet integral minimizing maps have been introduced by F. Almgren in his pioneering work [2]. Namely, he considered Q-valued functions, where Q denotes the number of values the function takes, counting multiplicity. His purpose was the development of a proof of a regularity result on area minimizing rectifiable currents. The author recommends [10] for a motivation of their definition and for an overview of Almgren's program. This article also compares different modern approachs to Q-valued functions inspired for instance by a metric analysis and surveys some recent contributions. A complete modern revision of Almgren's original theory and results can be found in [7]. We follow their notation, compare Section 1.

Having introduced a Dirichlet energy for Q-valued functions, a Dirichlet minimizer is characterised by the fact that it has least energy with respect to compact variations. Examples of such minimizers are generated by complex varieties as nicely proven in [16]. Concerning their regularity, one knows that they are Hölder continuous in the interior. This is already contained in Almgren's original theory and nicely presented in [7].

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Almgren's theory has been extended in several directions. The papers [3, 5, 12, 20] consider Q-valued functions mapping into non-euclidean ambient spaces, while [11, 13, 19, 21, 22] focus on other objects in the Q-valued setting like differential inclusions, geometric flows and quasi minima, and [6, 15] extend some theorems to more general energy functionals. Nonetheless many regularity questions concerning these functions remain open. Some of them have been already proposed by Almgren himself and can be found in [1] and [10].

We address the following regularity question concerning Almgren's multivalued functions, posed for example by C. De Lellis in [10, Section 8, (7)]:

Are Dirichlet minimizers continuous, or even Hölder, up to the boundary if the boundary data are sufficient regular?

The following result gives a rather general first answer:

Theorem 0.1. *Let $\frac{1}{2} < s \leq 1$ be given. There is a constant $\alpha = \alpha(N, Q, n, s) > 0$ with the property that, if*

(a1) $\Omega \subset \mathbb{R}^N$ *is a bounded* C^1 *regular domain,*
(a2) $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ *is Dirichlet minimizing,*
(a3) $u\big|_{\partial\Omega} \in C^{0,s}(\partial\Omega)$

then $u \in C^{0,\alpha}(\overline{\Omega})$.

To my knowledge, the only boundary regularity theorem proved in this context prior to Theorem 0.1 is contained in [18] where, assuming the domain of the Dirichlet minimizer is a 2-dimensional disk, the author proved that continuity holds up to the boundary if the boundary data is continuous. We will give a proof on different lines that continuity extends up the boundary for Lipschitz regular domains (*cf.* Section 4.2).

The equivalent "classical" statement of Theorem 0.1 for single-valued harmonic functions states:

$$f : \Omega \to \mathbb{R}^n \text{ harmonic, } f\big|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega) \text{ for some } 0 < \beta < 1 \text{ then } f \in C^{0,\beta}(\overline{\Omega}).$$

Note that harmonic functions f with finite energy belong to $W^{1,2}(\Omega, \mathbb{R}^n)$ and $f \in W^{1,2}(\Omega)$ if and only if $f\big|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega)$. Now, $H^{\frac{1}{2}}(\partial\Omega)$ can be characterised using the Gagliardo semi-norm $\int_{\partial\Omega\times\partial\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^N}\, dxdy$ that is controlled by the $C^{0,\beta}(\partial\Omega)$-norm for $\beta > \frac{1}{2}$. Nonetheless our result is suboptimal in the sense that for classical harmonic functions the modulus of continuity does not depend on finiteness of energy. So that $f\big|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega) \cap C^{0,\beta}(\partial\Omega)$ for any $0 < \beta < 1$ implies $u \in C^{0,\beta}(\overline{\Omega})$. In contrast, the Hölder exponent we claim in Theorem 0.1 is not explicit. For dimension three and higher that is not really surprising since the optimal (or even an explicit) exponent is not known in the interior so far.

The result of Theorem 0.1 is unsatisfactory for planar domains, because in this case Dirichlet minimizers are Hölder continuous with exponent at least $\frac{1}{Q}$, which is optimal. It would be desirable that the Hölder continuity with the same exponent extends to the boundary. We obtain the two-dimensional case of Theorem 0.1 by "lifting it" to three dimensions. So we get a "bad", non-explicit exponent. On the other hand we can prove, as mentioned, that continuity extends up the boundary on 2-dimensional Lipschitz regular domains if the boundary data is continuous. Concerning the optimal exponent we can give a partial first answer. At least on conical subsets of Ω the interior regularity extends up to the boundary for boundary data $u\big|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$, with $\beta > \frac{1}{2}$.

Outline of this article: Section 1 recalls the basic definition and results on Q-valued functions that are of interest in our context, Section 2 fixes notation and general assumptions, Section 3 contains the proof of Theorem 0.1 for dimension three and higher, Section 4 considers the two-dimensional setting. Finally the appendix with Sections A, B and C provides tools needed in the proof. So in Appendix A we prove certain properties of functions in a fractional Sobolev space $W^{s,2}$ with $\frac{1}{2} < s < 1$. It contains for instance an interpolation lemma in the spirit of Luckhaus with boundaries functions in a fractional Sobolev space $W^{s,2}$. These results are extended to Q-valued functions in Section B. Furthermore we present a concentration compactness result for Q-valued functions. It is along the same lines and indeed inspired by C. De Lellis and E. Spadaro's version [9, Lemma 3.2] and a $W^{s,p}$ selection criterion, needed in the two-dimensional setting.

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1. Q-valued functions

As announced this section recalls the basic definitions and results on Q-valued functions needed in here. The theory is presented omitting the actual proofs. They can be found for instance in C. De Lellis and E. Spadaro's work [7]. More refined results are presented in the appendix. A concentration compactness result is presented therein. It is along the same lines and indeed inspired by C. De Lellis and E. Spadaro's version [9, Lemma 3.2]. Furthermore an interpolation lemma in the spirit of Luckhaus with boundary functions in a fractional Sobolev space and a $W^{s,p}$, $s > \frac{1}{2}$ selection criterion is proved.

From now on Q, Q_1, $Q_2, \ldots$ will always denote natural numbers.

Definition 1.1. $\big(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G}\big)$ denotes the space of unordered Q-tuples given by

$$\mathcal{A}_Q(\mathbb{R}^n) = \left\{ T = \sum_{i=1}^{Q} [\![t_i]\!] \colon t_i \in \mathbb{R}^n, i = 1, \dots, Q \right\}.$$

$\mathcal{A}_Q(\mathbb{R}^{\mathcal{N}})$ can be made a complete metric space by defining the distance between two points as

$$\mathcal{G}(S,T)^2 = \min_{\sigma \in \mathcal{P}_Q} \sum_{i=1}^{Q} |s_i - t_{\sigma(i)}|^2$$

with $\mathcal{P}_Q$ denoting the permutation group of $\{1, \dots, Q\}$.

We use the convention $[\![t]\!] = \delta_t$ for a Dirac measure at a point $t \in \mathbb{R}^n$. Considering $T = \sum_{i=1}^{Q} [\![t_i]\!]$ as a sum of Q Dirac measures one notice that $\mathcal{A}_Q(\mathbb{R}^n)$ corresponds to the set of 0-dimensional integral currents of mass Q and positive orientation. Hence we will write

$$\operatorname{spt}(T) = \left\{ t_1, \dots, t_Q \colon T = \sum_{i=1}^{Q} [\![t_i]\!] \right\} \subset \mathbb{R}^n.$$

Furthermore $\mathcal{A}_Q(\mathbb{R}^n)$ is endowed with an intrinsic addition:

$$+ \colon \mathcal{A}_{Q_1}(\mathbb{R}^n) \times \mathcal{A}_{Q_2}(\mathbb{R}^n) \to \mathcal{A}_{Q_1+Q_2}(\mathbb{R}^n) \quad S + T = \sum_{i=1}^{Q_1} [\![s_i]\!] + \sum_{i=1}^{Q_2} [\![t_i]\!].$$

We define a translation operator

$$\oplus \colon \mathcal{A}_Q(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{A}_Q(\mathbb{R}^n) \quad T \oplus s = \sum_{i=1}^{Q} [\![t_i + s]\!].$$

$\mathcal{A}_Q(\mathbb{R}^N)$ is a complete metric space, so the notion of measurability, continuity and more generally the notion of modulus of continuity, Hölder and Lipschitz continuity is defined for functions taking values in $\mathcal{A}_Q(\mathbb{R}^N)$, *i.e.*, $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n), \Omega \subset \mathbb{R}^N$.

As it has been shown in [7, Proposition 0.4] for any measurable function $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ we can find a measurable selection, *i.e.*,

$$v = (v_1, \dots, v_Q) : \Omega \to (\mathbb{R}^n)^Q \text{ measurable so that } u(x) = [v](x) = \sum_{i=1}^{Q} [\![v_i(x)]\!].$$

$[v]$ denotes the natural embedding of $(\mathbb{R}^n)^Q \hookrightarrow \mathcal{A}_Q(\mathbb{R}^n)$ as introduced in [5]. In particular selections of higher regularity are considered in [5, 7, Proposition 1.2] and in the Appendix B.3.

We will write $|u(x)| = \sqrt{\sum_{i=1}^{Q} |v_i(x)|^2} = \mathcal{G}(u(x), Q[\![0]\!])$.

Definition 1.2. The Sobolev space $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is defined as the set of measurable functions $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ that satisfy

(w1) $x \mapsto \mathcal{G}(u(x), T) \in W^{1,2}(\Omega, \mathbb{R}_+)$ for every $T \in \mathcal{A}_Q(\mathbb{R}^n)$;
(w2) $\exists \varphi_j \in L^2(\Omega, \mathbb{R}_+)$ for $j = 1, \dots, N$ so that $|D_j \mathcal{G}(u(x), T)| \le \varphi_j(x)$ for any $T \in \mathcal{A}_Q(\mathbb{R}^n)$ and a.e. $x \in \Omega$.

It is not difficult to show the existence of a set of minimal functions $\tilde{\varphi}_j$, in the sense that $\tilde{\varphi}_j(x) \le \varphi_j(x)$ for a.e. x and any φ_j satisfying property (w2), [7, Proposition 4.2]. Such a minimal bound is denoted by $|D_j u|$ and is explicitly characterised by

$$|D_j u|(x) = \sup \left\{ |D_j \mathcal{G}(u(x), T_i)| : \{T_i\}_{i \in \mathbb{N}} \text{ dense in } \mathcal{A}_Q(\mathbb{R}^n) \right\}.$$

The Sobolev "semi-norm", or Dirichlet energy, is defined by integrating the measurable function $|Du|^2 = \sum_{j=1}^N |D_j u|^2$:

$$\int_\Omega |Du|^2 = \int_\Omega \sum_{j=1}^J |D_j u|^2. \tag{1.1}$$

Note that, strictly speaking, it is not a "semi-norm". $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is not a linear space since $\mathcal{A}_Q(\mathbb{R}^n)$ lacks this property.

A function $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ is said to be Dirichlet minimizing if

$$\begin{aligned} &\int_\Omega |Du|^2 \\ &= \inf \left\{ \int_\Omega |Dv|^2 : v \in W^{1,2}\big(\Omega, \mathcal{A}_Q(\mathbb{R}^n)\big), \mathcal{G}\,(u(x), v(x)) \in W_0^{1,2}(\Omega, \mathbb{R}_+) \right\}. \end{aligned} \tag{1.2}$$

On Lipschitz regular domains $\Omega \subset \mathbb{R}^N$ one has a continuous trace operator as for classical single valued Sobolev functions, *e.g.* [7, Proposition 4.5]

$$\circ\big|_{\partial\Omega} : W^{1,2}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^n)\right) \to L^2(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n)).$$

The definition of $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$, Definition 1.2, implies that on a Lipschitz regular domain $\Omega \subset \mathbb{R}^N$ one has that $\mathcal{G}(u(x), v(x)) \in W_0^{1,2}(\Omega)$ corresponds to $u\big|_{\partial\Omega} = v\big|_{\partial\Omega}$ for any $u, v \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$.

As a consequence of a Rademacher theorem for multivalued Lipschitz functions, [7, Section 1.3 and Theorem 1.13], a Sobolev function $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is a.e. approximately differentiable in the sense that:

(1) $\exists \mathcal{U}_x : \Omega \to \mathcal{A}_Q(\mathbb{R}^n \times \mathrm{Hom}(\mathbb{R}^N, \mathbb{R}^n))$, $x \mapsto \mathcal{U}_x = \sum_{i=1}^Q [\![(u_i(x), U_i(x))]\!]$ measurable with $U_i(x) = U_j(x)$ whenever $u_i(x) = u_j(x)$;
(2) $\mathcal{U}_x$ defines a 1-jet $J\mathcal{U}_x : \Omega \times \mathbb{R}^N \to \mathcal{A}_Q(\mathbb{R}^n)$ by $J\mathcal{U}_x(y) = \sum_{i=1}^Q [\![u_i(x) + U_i(x)(y - x)]\!]$, which has the additional property that $J\mathcal{U}_x(x) = u(x)$ for a.e. $x \in \Omega$;

(3) for a.e. $x \in \Omega$, $\exists E_x \subset \Omega$ having density 1 in x so that $\mathcal{G}(u(y), J\mathcal{U}_x(y)) = o(|y-x|)$ on E_x.

As one may guess the 1-jet corresponds to a first order "Taylor expansion", that becomes apparent in the proof of Rademacher's theorem, [7, Theorem 1.13]. One can show that $|D_j u|(x) = \sum_{i=1}^{Q} |U_i(x)e_j|^2$ for a.e. $x \in \Omega$, [7, Proposition 2.17]. From now on we will write $Du_i(x)$ for $U_i(x)$ and $D_j u_i(x)$ for $U_i(x)e_j$.

A useful tool is Almgren's bi-Lipschitz embedding of $\mathcal{A}_Q(\mathbb{R}^n)$ into some $\mathbb{R}^N$. A remark of Brian White improved it, compare [7, Theorem 2.1 and Corollary 2.2]:

Theorem 1.3 (bi-Lipschitz embedding). *There exists $m = m(Q,n)$ and an injective map $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^m$ with the properties*

(i) $\mathrm{Lip}(\boldsymbol{\xi}) \le 1$ *and* $\mathrm{Lip}(\boldsymbol{\xi}^{-1}|_{\boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))}) \le C(Q,n)$;
(ii) $\forall T \in \mathcal{A}_Q(\mathbb{R}^n)$ $\exists \delta = \delta(T) > 0$ *such that* $|\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)| = \mathcal{G}(T,S)$ *for all* $S \in B_\delta(T) \subset \mathcal{A}_Q(\mathbb{R}^n)$.

There is a retraction $\boldsymbol{\rho} : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ because of (i) *and the Lipschitz extension theorem,* e.g. [7, *Theorem* 1.7].

As a consequence $|Du|(x) = |D\boldsymbol{\xi} \circ u|(x)$ for a.e. $x \in \Omega$ for any $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$.

We want to remark that the image of $\mathcal{A}_Q(\mathbb{R}^n)$ under $\boldsymbol{\xi}$ in $\mathbb{R}^m$ is not convex neither a C^2 manifold. Moreover there is no "nearest point" projection not even in a tubular neighborhood.

Two cornerstones in the context of Dirichlet minimizers that are of interest for us in the following are (*cf.* [7, Theorem 0.8 and Theorem 0.9]):

Theorem 1.4 (Existence of Dirichlet minimizers). *Let $v \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. Then there exists a* (*not necessarily unique*) *Dirichlet minimizing $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ with $\mathcal{G}(u(x), v(x)) \in W_0^{1,2}(\Omega, \mathbb{R}_+)$.*

Theorem 1.5 (Interior Hölder continuity). *There is a constant $\alpha_0 = \alpha_0(N,Q) > 0$ with the property that if $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dirichlet minimizing, then $u \in C^{0,\alpha_0}(K, \mathcal{A}_Q(\mathbb{R}^n))$ for any $K \subset \Omega \subset \mathbb{R}^N$ compact. Indeed, $|Du|$ is an element of the Morrey space $L^{2,N-2-2\alpha_0}$ with the estimate*

$$r^{2-N-2\alpha_0} \int_{B_r(x)} |Du|^2 \le R^{2-N-2\alpha_0} \int_{B_R(x)} |Du|^2 \text{ for } r \le R,\ B_R(x) \subset \Omega. \tag{1.3}$$

For two-dimensional domains $\alpha_0(2,Q) = \frac{1}{Q}$ is explicit and optimal.

Both results had been proven first by Almgren in [2] and nicely reviewed by C. De Lellis and E. Spadaro in [7].

J. Almgren presents in [2, Theorem 2.16] an example of non-uniqueness for the Dirichlet problem: there are two Dirichlet minimizers $f \neq h \in W^{1,2}(B_1, \mathcal{A}_2(\mathbb{R}^2))$, $B_1 \subset \mathbb{R}^2$, with $f = h$ on ∂B_1. Given any other minimzer that agrees with f or h at the boundary must be either f or h.

2. General assumptions and further notation

From now on, if not indicated differently, we will consider the following setting: $\Omega \subset \mathbb{R}^N$ is a bounded C^1-regular domain, *i.e.*, for every $z \in \partial\Omega$ there exists $R = R(z) > 0$, $F = F_z \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ so that (up to a rotation)

$$\Omega \cap B_R(z) = \left\{z + (x', x_N) \colon |x| < R,\ x_N > F(x')\right\}.$$

In particular we will write

$$\Omega_F = \left\{(x', x_N) \colon\ x_N > F(x')\right\} \text{ for } F \in C^1\left(\mathbb{R}^{N-1}, \mathbb{R}\right).$$

Since $\partial\Omega$ is compact, the C^1 regularity implies that

(A1) for any given $\epsilon_F > 0, \exists R = R(\Omega, \epsilon_F) > 0$ with the property that for any $z \in \Omega$ there is $F \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ with $F(0) = 0$, $\operatorname{grad} F(0) = 0$, $\|\operatorname{grad} F\|_\infty < \epsilon_F$ and (up to a rotation):

$$\Omega \cap B_R(z) = \left\{z + (x', x_N) \colon |x| < R,\ x_N > F(x')\right\} = z + \Omega_F \cap B_R.$$

In other words, $\partial\Omega$ is locally the graph of a C^1 function with small gradient over the tangent space $T_z\partial\Omega$.

For the rescaled situation around a point we will write

$$\Omega_{z,r} = \left\{x \in \mathbb{R}^N \colon z + rx \in \Omega\right\} \text{ for } z \in \overline{\Omega}, r > 0.$$

In particular for the "graphical" situation Ω_F at a boundary point $z \in \partial\Omega$ we have

$$\Omega_{z,r} \cap B_1 = \left\{z + (x', x_N) \colon |x| < 1, x_N > F_{0,r}(x')\right\} = z + \Omega_{F_{0,r}} \cap B_1$$

with $F_{0,r}(x') = r^{-1}F(rx')$ (observe that $\left\|\operatorname{grad}(F_{0,r})\right\|_{\infty, B_1} = \|\operatorname{grad} F\|_{\infty, B_r}$).

The boundary portion in the graphical case will be denoted by

$$\Gamma_F = \partial\Omega_F \cap B_1 = \left\{(x', x_N) \colon |x| < 1, x_N = F(x')\right\}.$$

The blow up at a boundary point will always converge to the special case of the upper half space $\mathbb{R}^N_+$. This coincides in our notation with $\Omega_0 = \mathbb{R}^N_+$, *i.e.*, $F = 0$.

Fractional Soblev spaces, named $W^{s,2}$, occur naturally when dealing with boundary regularity for elliptic problems. A short introduction is given in the Appendix A. We define the Gagliardo semi-norms for $0 < s < 1$ and m dimensional submanifolds $\Sigma \subset \mathbb{R}^N$

$$\llbracket f \rrbracket^2_{s,\Sigma} = \int_{\Sigma\times\Sigma} \frac{|f(x) - f(y)|^2}{|x-y|^{m+2s}}\, d(x,y), \quad f \in L^2(\Sigma)$$

$$\llbracket u \rrbracket^2_{s,\Sigma} = \int_{\Sigma\times\Sigma} \frac{\mathcal{G}(u(x), u(y))^2}{|x-y|^{m+2s}}\, d(x,y), \quad u \in L^2\left(\Sigma, \mathcal{A}_Q(\mathbb{R}^n)\right).$$

The notation $[\![\cdot]\!]_{s,\Sigma}$ has been chosen in similarity to the classical notation $[\cdot]_{\alpha,\Sigma}$ for the Hölder semi-norm with exponent α. With a little abuse of notation we will use the same symbol for $s = 1$:

$$[\![f]\!]_{1,\Sigma}^2 = \int_\Sigma |D_\tau f|^2, \quad f \in W^{1,2}(\Sigma)$$
$$[\![u]\!]_{1,\Sigma}^2 = \int_\Sigma |D_\tau u|^2, \quad u \in W^{1,2}\left(\Sigma, \mathcal{A}_Q(\mathbb{R}^n)\right)$$

where D_τ denotes the total tangential derivative on Σ. For single valued functions $f \in W^{1,2}(\Sigma)$ and an orthonormal frame $\tau_1, \dots, \tau_m$ of $T_x\Sigma$ we have $|D_\tau f(x)|^2 = \sum_{j=1}^Q |\frac{\partial f}{\partial \tau_j}|^2$. In the case of a multivalued function u we make use of the approximately differentiability of Sobolev functions: for a.e. $x \in \Sigma$ we have $|D_\tau u|^2(x) = \sum_{j=1}^m \sum_{i=1}^Q |U_i(x)\tau_j|^2$ where $U_i(x)$ are the elements of the 1-jet $J\mathcal{U}_x$, *cf.* the discussion below definition 1.2 for precise statement of the approximate differentiability and the definition of the 1-jet.

3. Hölder continuity for $N \geq 3$

The following is a a more precise version of Theorem 0.1 and the main result of the paper.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ C^1-regular, $N \geq 3$, $Q, n \in \mathbb{N}$ and $\frac{1}{2} < s \leq 1$. If for some ball $B_{3R}(y) \subset \mathbb{R}^N$, $y \in \mathbb{R}^N$*

- (a1) *$u \in W^{1,2}(\Omega \cap B_{3R}(y), \mathcal{A}_Q(\mathbb{R}^n))$ is Dirichlet minimizing;*
- (a2) *$u\big|_{\partial\Omega} \in W^{s,2}(\partial\Omega \cap B_{3R}(y), \mathcal{A}_Q(\mathbb{R}^n))$ and for some $0 < \beta < 1$ there is a constant $M_u > 0$ so that*

$$r^{2(s-\beta)-(N-1)} [\![u]\!]_{s,B_r(z)\cap\partial\Omega}^2 \leq M_u^2 \quad \forall z \in \partial\Omega \cap B_{2R}(y), 0 < r < R$$

holds, there are constants $C, \alpha_1 > 0$ depending on N, n, Q, s and $R_{\partial\Omega} > 0$ depending on $\partial\Omega$ such that

- (i) *$|Du|$ is an element of the Morrey space $L^{2,N-2+2\alpha}(\Omega \cap B_R(y))$ for any $0 < \alpha < \min\{\alpha_1, \beta\}$, and more precisely the following estimate holds*

$$r^{2-N-2\alpha} \int_{B_r(x)\cap\Omega} |Du|^2 \leq 2^N R_0^{2-N-2\alpha} \int_{B_{2R_0}(x)\cap\Omega} |Du|^2 + C \frac{R_0^{2(\beta-\alpha)}}{\beta - \alpha} M_u^2 \quad (3.1)$$

  for any ball $B_r(x)$ with $x \in B_R(y) \cap \Omega$ and $r < \frac{R_0}{2} := \min\{R, R_{\partial\Omega}\}$.
- (ii) *$u \in C^{0,\alpha}(\overline{\Omega \cap B_R(y)})$.*

The following lemma provides a relation between the assumption (a2) above and the Hölder continuity of u at the boundary portion $\partial\Omega \cap B_R(y)$.

Lemma 3.2.

(i) (a2) *is satisfied if* $u\big|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega \cap B_{3R}(y))$ *for* $\frac{1}{2} < \beta < 1$, i.e., *there is a dimensional constant* $C > 0$ *so that for* $0 < s < \beta$

$$r^{2(s-\beta)-(N-1)} [\![u]\!]^2_{s, B_r(z)\cap\partial\Omega} \leq \frac{C}{\beta - s}[u]^2_{\beta,\partial\Omega}$$
$$\forall z \in \partial\Omega \cap B_{2R}(y), r < \min\{R, R(\Omega, 1)\};$$

(ii) *if* (a2) *holds then* $u\big|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega\cap B_{2R}(y))$, i.e., *there is a dimensional constant* $C > 0$ *so that*

$$\mathcal{G}(u(x_1), u(x_2)) \leq CM|x_1 - x_2|^\beta$$
$$\forall x_1, x_2 \in \partial\Omega \cap B_{2R}(y), |x_1 - y_2| \leq \min\left\{R, \frac{R(\Omega, 1)}{2}\right\}.$$

Proof. Claim (i): Given $z \in \partial\Omega \cap B_{2R}(y)$, let $R(\Omega, 1) > 0, F = F_z \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ be the radius and function definded in (A1). For any $0 < r < \min\{R, R(\Omega, 1)\}$ writing $x = (x', F(x')) \in \partial\Omega$ and $B'_r = B_r(0) \subset \mathbb{R}^{N-1}$ we have

$$\begin{aligned}
&\int_{B_r(z)\cap\partial\Omega\times B_r(z)\cap\partial\Omega} \frac{\mathcal{G}(u(x), u(y))^2}{|x - y|^{N-1+2s}}\, dxdy \\
&\leq [u]^2_{\beta,\partial\Omega} \int_{B_r(z)\cap\partial\Omega\times B_r(z)\cap\partial\Omega} |x - y|^{2(\beta-s)-(N-1)}\, d(x, y) \\
&\leq [u]^2_{\beta,\partial\Omega}(1 + \|\mathrm{grad}(F)\|^2_\infty)^2 \int_{B'_r\times B'_r} |x' - y'|^{2(\beta-s)-(N-1)}\, d(x', y') \\
&\leq \frac{4(N-1)\omega^2_{N-1}}{2(\beta - s)}[u]^2_{\beta,\partial\Omega}\, r^{2(\beta-s)+(N-1)}.
\end{aligned}$$

Claim (ii): As we observed in (A1) $\partial\Omega$ is locally a graph, so we can transform it to a local question on $\mathbb{R}^{N-1}$. Furthermore making use of Almgren's bilipschitz embedding, Theorem 1.3, it is sufficient to check it for single valued functions. Hence (ii) is equivalent to check that

There is a dimensional constant $C = C(n) > 0$ *so that if* $f \in W^{s,2}(\mathbb{R}^n, \mathbb{R}^m)$ *and* $M_f > 0$ *are given with the property that*

$$r^{2(s-\beta)-n} [\![f]\!]^2_{s,B_r(z)} \leq M_f^2 \quad \forall B_r(z) \subset \mathbb{R}^n, 0 < r < R_0, \tag{3.2}$$

then $f \in C^{0,\beta}(\mathbb{R}^n, \mathbb{R}^m)$ *with*

$$|f(x) - f(y)| \leq CM_f|x - y|^\beta \quad \forall |x - y| < R_0. \tag{3.3}$$

Let us write $f(z,r) = \fint_{B_r(z)} f$ for any $B_r(z) \subset \mathbb{R}^n$, then using twice Cauchy's inequality we have

$$\fint_{B_r(z)} |f - f(z,r)| \leq |B_r(z)|^{-2} \int_{B_r(z)\times B_r(z)} |f(x) - f(y)|\, d(x,y)$$
$$\leq |B_r(z)|^{-2} \int_{B_r(z)} \left(\int_{B_r(z)} |x-y|^{n+2s}\, dy \right)^{\frac{1}{2}} \left(\int_{B_r(z)} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\, dy \right)^{\frac{1}{2}} dx$$
$$\leq \left(\frac{4^n}{\omega_N^2} r^{2s-n} [\![f]\!]^2_{s,B_r(z)} \right)^{\frac{1}{2}} \leq C r^\beta\, M_f.$$

Hence for any $r < R_0$ and $k \in \mathbb{N}$

$$|f(z, 2^{-k-1}r) - f(z, 2^{-k}r)| \leq 2^n \fint_{B_{2^{-k}r}(z)} |f - f(z, 2^{-k}r)| \leq C M_f\, r^\beta\, 2^{-\beta k},$$

i.e., $k \mapsto f(z, 2^{-k}r)$ is a Cauchy sequence because $\sum_{k=0}^{\infty} |f(z, 2^{-k-1}r) - f(z, 2^{-k}r)| \leq \frac{CM_f}{1-2^{-\beta}} r^\beta$. Furthermore for any two Lebesgue points $z_1, z_2 \in \mathbb{R}^n$ with $|z_1 - z_2| = r < R_0$ we have

$$|f(z_1) - f(z_2)| \leq \sum_{i=1}^{2} |f(z_i) - f(z_i, r)| + \fint_{B_r(z_i)\cap B_r(z_2)} |f(x) - f(z_i)|\, dx$$
$$\leq \sum_{i=1}^{2} \frac{CM_f}{1-2^{-\beta}} r^\beta + \frac{CM_f}{1-2^{-\beta}} r^\beta \leq 4 \frac{CM_f}{1-2^{-\beta}} r^\beta;$$

this shows that f has a representative in $C^{0,\beta}$. □

The core of the proof of Theorem 3.1 is the estimate stated in Proposition 3.3 below. To make its proof more accessible it is presented in the next subsection and split into several lemmas.

Proposition 3.3. *For any $\frac{1}{2} < s \leq 1$ there are constants $\epsilon_0 > 0$, $0 < \delta < \frac{1}{N-2}$ and $C > 0$ depending on N, n, Q, s with the property that, if* (A1) *holds with $\epsilon_F \leq \epsilon_0$, then*

$$\int_{\Omega_F \cap B_1} |Du|^2 \leq \left(\frac{1}{N-2} - \delta \right) \int_{\mathcal{S}^{N-1}\cap \Omega_F} |D_\tau u|^2 + C [\![u]\!]^2_{s,\Gamma_F} \tag{3.4}$$

for any Dirchilet minimizer $u \in W^{1,2}(B_1 \cap \Omega_F, \mathcal{A}_Q(\mathbb{R}^n))$.

Let us take the previous proposition, *i.e.*, the estimate (3.4) for granted and close the argument in the proof of Theorem 3.1.

Proof of Theorem 3.1. Let ϵ_0, δ be the constants of Proposition 3.3. Fix $\alpha_1 \leq \alpha_0$ (α_0 being the Hölder exponent of Theorem 1.4) so that $(N-2+2\alpha_1)\left(\frac{1}{N-2}-\delta\right) \leq 1$. Let $R_0 = \min\{R, R(\Omega, \epsilon_0)\}$, where $R(\Omega, \epsilon_0)$ denotes the radius defined in (A1) corresponding to ϵ_0.

Due to the choice of R_0, for any $0 < r \leq R_0$, $z \in \partial\Omega \cap B_{2R}(y)$ the rescaled map

$$u_{z,r}(x) = u(z + rx) \quad \text{for } x \in B_1 \cap \Omega_{z,r}$$

belongs to $W^{1,2}(\Omega_{z,r} \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and satisfies the assumptions of the Proposition 3.3. One readily checks that for $\frac{1}{2} < s \leq 1$

$$[\![u_{z,r}]\!]^2_{s, B_1\cap\partial\Omega_{z,r}} = r^{2s-(N-1)} [\![u]\!]^2_{s, B_r(z)\cap\partial\Omega}.$$

Applying (3.4) and assumption (a2) we get

$$\begin{aligned}
r^{2-N}\int_{B_r(z)\cap\Omega} |Du|^2 &= \int_{B_1\cap\Omega_{z,r}} |Du_{z,r}|^2 \\
&\leq \left(\frac{1}{N-2}-\delta\right)\int_{\mathcal{S}^{N-1}\cap\Omega_{z,r}} |D_\tau u_{z,r}|^2 + C [\![u_{z,r}]\!]^2_{s, B_1\cap\partial\Omega_{z,r}} \\
&\leq \frac{1}{N-2+2\alpha_1} r^{3-N}\int_{\partial B_r(z)\cap\Omega} |D_\tau u|^2 + C r^{2\beta} M_u^2.
\end{aligned}$$

Hence for a.e. $0 < r < R_0$ and $0 < \alpha < \min\{\alpha_1, \beta\}$

$$\begin{aligned}
&-\frac{\partial}{\partial r}\left(r^{2-N-2\alpha}\int_{B_r(z)\cap\Omega} |Du|^2\right) \\
&= -r^{2-N-2\alpha}\int_{\partial B_r(z)\cap\Omega} |Du|^2 + (N-2+2\alpha) r^{-1-2\alpha} r^{2-N}\int_{B_r(z)\cap\Omega} |Du|^2 \\
&\leq r^{2-N-2\alpha}\int_{\partial B_r(z)\cap\Omega}\left(|D_\tau u|^2 - |Du|^2\right) + (N-2+2\alpha) C r^{2(\beta-\alpha)-1} M_u^2 \\
&\leq (N-2+2\alpha) C r^{2(\beta-\alpha)-1} M_u^2.
\end{aligned}$$

Integrating in r we achieve the following inequality for any $z \in \partial\Omega \cap B_{2R}(y)$ and $0 < r \leq R_0$:

$$r^{2-N-2\alpha}\int_{B_r(z)\cap\Omega} |Du|^2 - R_0^{2-N-2\alpha}\int_{B_{R_0}(z)\cap\Omega} |Du|^2 \leq \frac{C}{\beta-\alpha} R_0^{2(\beta-\alpha)} M_u^2. \quad (3.5)$$

Now we can conclude (3.1). If $x \in \overline{\Omega \cap B_R(y)}$ satisfies $\operatorname{dist}(x, \partial\Omega) > \frac{R_0}{2}$, then $B_r(x) \subset B_{\frac{R_0}{2}}(x) \subset \Omega \cap B_{2R}(y)$ for any $0 < r < \frac{R_0}{2}$ and so, by (1.3) in Theorem 1.5

$$\begin{aligned} r^{2-N-2\alpha} \int_{B_r(x)} |Du|^2 &\leq \left(\frac{R_0}{2}\right)^{2-N-2\alpha} \int_{B_{\frac{R_0}{2}}(x)} |Du|^2 \\ &\leq 2^N R_0^{2-N-2\alpha} \int_{B_{2R_0}(x)\cap\Omega} |Du|^2. \end{aligned} \tag{3.6}$$

Assume therefore $x \in \overline{\Omega \cap B_R(y)}$ has $\operatorname{dist}(x, \partial\Omega) \leq \frac{R_0}{2}$. Fix $z \in \partial\Omega$ so that $\operatorname{dist}(x, \partial\Omega) = |x - z|$, *i.e.*, $z \in \partial\Omega \cap B_{2R}(y)$. Given $0 < r \leq \frac{R_0}{2}$ we set $r_1 = \max\{r, |x - z|\}, r_2 = r_1 + |x - z| \leq 2r_1 \leq R_0$ and so

$$\begin{aligned} r^{2-N-2\alpha}\int_{B_r(x)\cap\Omega} |Du|^2 &\leq r_1{}^{2-N-2\alpha} \int_{B_{r_1}(x)\cap\Omega} |Du|^2 \\ &\leq \left(\frac{r_2}{r_1}\right)^{N-2+2\alpha} r_2^{2-N-2\alpha} \int_{B_{r_2}(z)\cap\Omega} |Du|^2 \\ &\leq 2^N \left(R_0^{2-N-2\alpha}\int_{B_{R_0}(z)\cap\Omega} |Du|^2 + \frac{C}{\beta-\alpha} R_0^{2(\beta-\alpha)} M_u^2 \right) \\ &\leq 2^N \left(R_0^{2-N-2\alpha}\int_{B_{2R_0}(x)\cap\Omega} |Du|^2 + \frac{C}{\beta-\alpha} R_0^{2(\beta-\alpha)} M_u^2 \right), \end{aligned} \tag{3.7}$$

(ii) is a consequence of (i) by the theory of Campanato spaces as follows: (i) implies that $|Du|$ is an element of the Morrey space $L^{2,N-2+2\alpha}(\Omega \cap B_R(y))$. Ω is C^1-regular and therefore by Poincarés inequality this implies that $\boldsymbol{\xi} \circ u$ is an element of the Campanato space $\mathcal{L}^{2,N+2\alpha}(\Omega \cap B_R(y))$, compare for instance [4, Proposition 3.7]. Furthermore one has the equivalence $\mathcal{L}^{2,N+2\alpha}(\Omega \cap B_R(y)) = C^{0,\alpha}(\overline{\Omega \cap B_R(y)})$, [4, Theorem 2.9]. □

3.1. Proof of Proposition 3.3

The proof will be divided into two parts and each part is devoted one subsection.

Subsection 3.1.1: We show that it is necessary and sufficient for a Dirichlet minimizer on the upper half ball $B_1 \cap \{x_N > 0\}$ to be trivial that it has constant boundary data on $B_1 \cap \{x_N = 0\}$.

Subsection 3.1.2: We show that if proposition failed we could construct a non-trivial Dirichlet minimizer on the upper half ball $B_1 \cap \{x_N > 0\}$ with constant boundary data contradicting the previous step.

3.1.1. *Non-existence of certain non-trivial minimizers*

In this subsection we consider Dirichlet minimizers on the upper half ball $B_{1+} = B_1 \cap \{x_N > 0\}$ and we will show that they have to be trivial under certain assumptions. We will use the following notation: $\mathcal{S}_+^{N-1} = \mathcal{S}^{N-1} \cap \{x_N > 0\}$ and $\Gamma_0 = B_1 \cap \{x_N = 0\}$.

Proposition 3.4. *Every 0-homogeneous Dirichlet minimizer in B_{1+} with $u\big|_{\Gamma_0} =$* const. *is trivial,* i.e., *constant.*

Corollary 3.5. *A Dirichlet minimizer on B_{1+} with $u\big|_{\Gamma_0} = const.$ satisfying*

$$\int_{B_{1+}} |Du|^2 = \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau u|^2 \tag{3.8}$$

needs to be constant.

They are both consequence of an appropriately chosen inner variation:

Lemma 3.6 (A special kind of inner variation). *Given a Dirichlet minimizer $u \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$ with $u\big|_{\Gamma_0} =$* const. *and a vector field $X = (X_1, \dots, X_N) \in C_c^1(B_1, \mathbb{R}^N)$ with $e_N \cdot X(x', 0) = X_N(x', 0) \geq 0$ on Γ_0, then*

$$0 \leq \int_{B_{1+}} |Du|^2 \operatorname{div}(X) - 2 \sum_{i=1}^{Q} \langle Du_i : Du_i\, DX \rangle. \tag{3.9}$$

Proof. Let u and X be given and set $T = u\big|_{\Gamma_0}(x)$ for $x \in \Gamma_0$. Observe that for $x_N > 0$ and $0 < t < t_0$ sufficiently small

$$\begin{aligned} x_N + t X_N(x', x_N) &= x_N + t \left(X_N(x', x_N) - X_N(x', 0)\right) + t X_N(x', 0) \\ &\geq (1 - t \,\|DX_N\|_\infty) x_N + t X_N\big(x', 0\big) \geq 0. \end{aligned}$$

Hence for $t_0 > 0$ small

$$\Phi_t(x) = x + t X(x)$$

defines a 1-parameter family of C^1-diffeomorphism that satisfy

$$A_t = \Phi_t(B_{1+}) \subset B_{1+} \text{ for } 0 \leq t \leq t_0.$$

So

$$v_t(x) = \begin{cases} u \circ \Phi_t^{-1}(x) & \text{for } x \in A_t \\ T & \text{for } x \in B_1^+ \setminus A_t \end{cases}$$

defines a C^1 family of competitors to u. Standard calculations, compare for instance [7, Proposition 3.1], give

$$\begin{aligned} D\Phi_t^{-1} \circ \Phi_t &= (D\Phi_t)^{-1} = \sum_{k=0}^{\infty} (-t)^k \,(DX)^k = 1 - t DX + o(t) \\ \det(D\Phi_t) &= 1 + t \operatorname{div}(X) + o(t) \end{aligned}$$

so that

$$|Dv_t|^2 \circ \Phi_t = \sum_{i=1}^{Q} |Du_i D\Phi_t^{-1} \circ \Phi_t|^2 = \sum_{i=1}^{Q} |Du_i\,(1 - tDX + o(t))|^2$$
$$= \sum_{i=1}^{Q} |Du_i|^2 - 2t \sum_{i=1}^{Q} \langle Du_i : Du_i DX\rangle + o(t).$$

In conclusion we found that for all $0 \le t \le t_0$

$$\int_{B_{1+}} |Dv_t|^2 = \int_{A_t} |Dv_t|^2 = \int_{B_{1+}} |Dv_t|^2 \circ \Phi_t\, |\det D\Phi_t|$$
$$= \int_{B_{1+}} |Du|^2 + t \int_{B_{1+}} |Du|^2 \operatorname{div}(X) - 2 \sum_{i=1}^{Q} \langle Du_i : Du_i DX\rangle + o(t).$$

Since $\int_{B_{1+}} |Dv_t|^2 \ge \int_{B_{1+}} |Du|^2$, we necessarily have

$$0 \le \int_{B_{1+}} |Du|^2 \operatorname{div}(X) - 2 \sum_{i=1}^{Q} \langle Du_i : Du_i DX\rangle. \qquad \square$$

Proof of Proposition 3.4. Since u is 0-homogeneous, $u(x) = u\left(\frac{x}{|x|}\right)$ for a.e. x. Thus $\frac{\partial u}{\partial r}(x) = 0$ for a.e. $x \in B_{1+}$, which corresponds to

$$0 = \frac{\partial u}{\partial r}(x) = \sum_{i=1}^{Q} \left[\!\!\left[\sum_{j=1}^{N} D_j u_i(x) \frac{x_j}{|x|} \right]\!\!\right]. \tag{3.10}$$

Fix $0 < R < 1$ and consider the vector field $X(x) = \eta(|x|)e_N = (0, \dots, \eta(|x|))$ with

$$\eta(r) = \begin{cases} 1 - \frac{r}{R} & r \le R \\ 0 & r \ge R. \end{cases}$$

Thus we have $X_N(x) \ge 0$ and $DX(x) = \eta'(|x|)e_N \otimes \frac{x}{|x|}$. This gives $\operatorname{div}(X)(x) = \eta'(|x|)\frac{x_N}{|x|}$ and due to (3.10)

$$\langle Du_i : Du_i DX\rangle = \sum_{j=1}^{N} \left\langle \frac{x_j}{|x|} D_j u_i, D_N u_i \right\rangle \eta'(|x|) = 0 \text{ for a.e. } x.$$

Using $\eta'(|x|) = -\frac{1}{R}\mathbf{1}_{B_R}(x)$ and applying Lemma 3.6 we get

$$0 \le -\frac{1}{R} \int_{B_{R+}} |Du|^2 \frac{x_N}{|x|}.$$

This is only possible for $|Du| = 0$ on B_{R+} and so $|Du| = 0$ on B_{1+}. $\square$

Proof of Corollary 3.5. Let $u \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$ be as assumed. Observe that (3.8) implies that $u \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$. Hence $v(x) = u\left(\frac{x}{|x|}\right)$ defines a 0-homogeneous competitor using $u\big|_{\Gamma_0} = \text{const}$.

$$\int_{B_{1+}} |Dv|^2 = \frac{1}{N-2}\int_{\mathcal{S}_+^{N-1}} |D_\tau v|^2 = \frac{1}{N-2}\int_{\mathcal{S}_+^{N-1}} |D_\tau u|^2 = \int_{B_{1+}} |Du|^2,$$

where we used firstly the 0-homogeneity of v, then $u\big|_{\mathcal{S}_+^{N-1}} = v\big|_{\mathcal{S}_+^{N-1}}$ and finally (3.8). Therefore v has to be minimizing as well, and moreover $Dv = 0$ as a consequence of Proposition 3.4. This proves the corollary since then $Du = 0$ as well. □

3.1.2. *Contradiction argument*

As announced we want to establish now by contradiction the estimate of Proposition 3.3:

$$\int_{\Omega_F\cap B_1} |Du|^2 \le \left(\frac{1}{N-2} - \delta\right)\int_{\mathcal{S}^{N-1}\cap\Omega_F} |D_\tau u|^2 + C\,[\![u]\!]_{s,\Gamma_F}^2.$$

At the end of the proof we add some comments about in which sense one can consider this estimate as optimal.

Proof of Proposition 3.3. If $u \notin W^{1,2}(\mathcal{S}^{N-1}\cap\Omega_F, \mathcal{A}_Q(\mathbb{R}^n))\cap W^{s,2}(\Gamma_F, \mathcal{A}_Q(\mathbb{R}^n))$ the RHS of (3.4) is infinite and so there is nothing to prove. Hence, assuming that the proposition would not hold, we can find sequences $F(k) \in C^1(\mathbb{R}^{N-1}, \mathbb{R})$ defining the sets $\Omega_{F(k)}$ as introduced in (A1) with $\epsilon_{F(k)} < \frac{1}{k}$, *i.e.*, $F_k(0) = 0$, grad $F_k(0) = 0$, $\|\text{grad}\, F_k\|_\infty < \frac{1}{k}$, and an associated $u(k) \in W^{1,2}(B_1 \cap \Omega_{F(k)}, \mathcal{A}_Q(\mathbb{R}^n))$ failing (3.4), *i.e.*,

$$\int_{\Omega_{F(k)}\cap B_1} |Du(k)|^2 > \left(\frac{1}{N-2} - \frac{1}{k}\right)\int_{\mathcal{S}^{N-1}\cap\Omega_{F(k)}} |D_\tau u(k)|^2 + k\,[\![u(k)]\!]_{s,\Gamma_{F(k)}}^2. \quad (3.11)$$

We may assume that the LHS of (3.11) is 1 by dividing each $u(k)$ by its Dirichlet energy $\left(\int_{\Omega_{F(k)}\cap B_1} |Du(k)|^2\right)^{-\frac{1}{2}}$. We also assume, w.l.o.g., $k > k_0 > 4$.

To every k we may fix a C^1-diffeomorphism $G(k) : \overline{B_{1+}} \to \overline{\Omega_{F(k)} \cap B_1}$, arguing for example on the base of Lemma C.2. $F(k) \to F_0 = 0$ in C^1 as $k \to \infty$ and therefore $G(k), G(k)^{-1} \to \mathbf{1}$ in C^1 ($\mathbf{1}$ denotes the indentiy map on $\mathbb{R}^N$).

We consider now instead of the sequence $u(k)$ itself the sequence $v(k) = u(k)\circ G(k) \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$. Up to order $o(1)$ $v(k)$ has the same properties as $u(k)$

since $G(k), G(k)^{-1} \to \mathbf{1}$ in C^1, *i.e.*,

$$\int_{B_1^+} |Dv(k)|^2 = (1+o(1)) \int_{\Omega_{F(k)}} |Du(k)|^2 \leq 1 + o(1);$$

$$\int_{\mathcal{S}_+^{N-1}} |D_\tau v(k)|^2 = (1+o(1)) \int_{\mathcal{S}^{N-1}\cap\Omega_{F(k)}} |D_\tau u(k)|^2 < \frac{1+o(1)}{\frac{1}{N-2} - \frac{1}{k}} < 2N; \quad (3.12)$$

$$[\![v(k)]\!]^2_{s,\Gamma_0} = (1+o(1)) [\![u(k)]\!]^2_{s,\Gamma_{F(k)}} \leq \frac{1+o(1)}{k} \leq \frac{1}{2k},$$

(3.11) with LHS $= 1$ provides the upper bounds. The second and third show that $v(k)\big|_{\partial B_{1+}} \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathcal{A}_Q(\mathbb{R}^n)) \cap W^{s,2}(\Gamma_0, \mathcal{A}_Q(\mathbb{R}^n))$.

We apply the concentration compactness Lemma B.4 to the sequences $v(k)$, $T(k)$. For a subsequence $v(k')$ we can find functions $b_j \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$, a sequence of points $t_j(k') \in \mathbb{R}^n$ so that the "traveling sheets" $b(k') = \sum_{j=1}^J (b_j \oplus t_j(k'))$, satisfies among others $\mathcal{G}(b(k'), v(k')) \to 0$ in $L^2(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$. We will prove now that the b_j satisfy also the following:

(i) $b_j\big|_{\mathcal{S}_+^{N-1}} \in W^{1,2}(\mathcal{S}_+^{N-1}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ and $b_j\big|_{\Gamma_0} = \text{const.}$;

(ii) $b_j \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ is Dirichlet minimizing and

$$\sum_{j=1}^J \int_{B_{1+}} |Db_j|^2 = \lim_{k'\to\infty} \int_{B_{1+}} |Dv(k')|^2 = \lim_{k'\to\infty} \int_{\Omega_{F_{k'}}\cap B_1} |Du(k')|^2 = 1;$$

(iii) $\int_{B_{1+}} |Db_j|^2 \leq \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2$ for all j.

Proof of (i): The concentration compactness lemma states that $\mathcal{G}(v(k'), b(k')) \to 0$ in $L^2(B_{1+})$ and $D\boldsymbol{\xi} \circ v(k') \rightharpoonup D\boldsymbol{\xi} \circ b(k')$ in $L^2(B_{1+}, \mathbb{R}^m)$ weakly. This implies that $\mathcal{G}(v(k'), b(k')) \to 0$ in $L^2(\mathcal{S}_+^{N-1})$ and $D_\tau\boldsymbol{\xi} \circ v(k') \rightharpoonup D_\tau\boldsymbol{\xi} \circ b(k')$ in $L^2(\mathcal{S}_+^{N-1}, \mathbb{R}^m)$, because we had seen in (3.12) that $D\boldsymbol{\xi} \circ v(k')$ is uniformly bounded in $L^2(\mathcal{S}_+^{N-1}, \mathbb{R}^m)$. The lower semicontinuity of energy together with (3.12) then states

$$\begin{aligned} \frac{1}{N-2} \sum_{j=1}^J \int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2 &= \frac{1}{N-2} \int_{\mathcal{S}_+^{N-1}} \sum_{j=1}^J |D_\tau \boldsymbol{\xi} \circ b_j|^2 \\ &\leq \liminf_{k'\to\infty} \left(\left(\frac{1}{N-2} - \frac{1}{k'} \right) \int_{\mathcal{S}_+^{N-1}} |D_\tau \boldsymbol{\xi} \circ v(k')|^2 \right) \\ &\leq 1. \end{aligned} \quad (3.13)$$

$\mathcal{G}(v\big|_{\Gamma_0}(k'), b\big|_{\Gamma_0}(k')) \to 0$ in $L^2(\Gamma_0)$ due to the weak convergence in the interior. Hence, due to dominated convergence, for any $\delta > 0$ and (3.12)

$$\sum_{j=1}^{J} \int_{\substack{\Gamma_0\times\Gamma_0 \\ |x-y|\geq\delta}} \frac{\mathcal{G}(b_j\big|_{\Gamma_0}(x), b_j\big|_{\Gamma_0}(y))^2}{|x-y|^{N-1+2s}}\, d(x,y)$$
$$= \lim_{k'\to\infty} \int_{\substack{\Gamma_0\times\Gamma_0 \\ |x-y|\geq\delta}} \frac{\mathcal{G}(v\big|_{\Gamma_0}(k')(x), v\big|_{\Gamma_0}(k')(y))^2}{|x-y|^{N-1+2s}}\, d(x,y) \leq \lim_{k'\to\infty} \frac{2}{k'} = 0;$$

consequently $b_j\big|_{\Gamma_0} = \text{const.}$ for all j.

Proof of (ii): Let $G : \overline{B_1} \to \overline{B_{1+}}$ be the bilipschitz map constructed in Lemma C.1. $\llbracket v(k') \circ G \rrbracket_{s,\mathcal{S}^{N-1}}$ is uniformly bounded: firstly apply Corollary B.1 to estimate

$$\llbracket v(k')\circ G\rrbracket_{s,\mathcal{S}^{N-1}} \leq C\left(\llbracket v(k') \circ G\rrbracket_{s,\mathcal{S}^{N-1}\cap\left\{x_N>\frac{-1}{\sqrt{5}}\right\}} + \llbracket v(k') \circ G\rrbracket_{s,\mathcal{S}^{N-1}\cap\left\{x_N<\frac{-1}{\sqrt{5}}\right\}}\right);$$

secondly G is bilipschitz and $G(\mathcal{S}^{N-1}\cap\{x_N > \frac{-1}{\sqrt{5}}\}) = \mathcal{S}^{N-1}_+$ and $G(\mathcal{S}^{N-1}\cap\{x_N < \frac{-1}{\sqrt{5}}\}) = \Gamma_0$, so that

$$\llbracket v(k') \circ G\rrbracket_{s,\mathcal{S}^{N-1}\cap\{x_N>\frac{-1}{\sqrt{5}}\}} \leq C\llbracket v(k')\rrbracket_{s,\mathcal{S}^{N-1}_+}$$
$$\llbracket v(k') \circ G\rrbracket_{s,\mathcal{S}^{N-1}\cap\{x_N<\frac{-1}{\sqrt{5}}\}} \leq C\llbracket v(k')\rrbracket_{s,\Gamma_0};$$

thirdly the interpolation property $\llbracket f\rrbracket^2_{s,\mathcal{S}^{N-1}_+} \leq C\int_{\mathcal{S}^{N-1}_+}|Df|^2$ gives

$$\llbracket v(k')\rrbracket_{s,\mathcal{S}^{N-1}_+} \leq \left\||Dv(k')|\right\|_{L^2(\mathcal{S}^{N-1}_+)}.$$

In conclusion we combine all of them and use (3.12) to conclude

$$\llbracket v(k') \circ G\rrbracket_{s,\mathcal{S}^{N-1}} \leq C\left(\left\||Dv(k')|\right\|_{L^2(\mathcal{S}^{N-1}_+)} + \llbracket v(k')\rrbracket_{s,\Gamma_0}\right) \leq C\,(2N).$$

The same bound holds for $b_j \circ G \in W^{s,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$ because of the lower semicontinuity of energy established in (3.13). Furthermore in the proof of (i) we showed that $\mathcal{G}(v(k'), b(k')) \to 0$ in $L^2(\mathcal{S}^{N-1}_+)$ and $L^2(\Gamma_0)$, so that

$$\left\|\mathcal{G}(v(k') \circ G, b(k') \circ G)\right\|_{L^2(\mathcal{S}^{N-1})} = o(1).$$

Let $\delta > 0$ be given and sufficient small so that we can apply the interpolation Lemma B.2. To every k' we fix an interpolation $\varphi(k') \in W^{s,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^n))$

between $v(k') \circ G$ and $b(k') \circ G$, *i.e.*, $\varphi(k')(x) = v(k') \circ G(x)$, $\varphi(k')((1-\lambda)\,x) = b(k') \circ G(x)$ for all $x \in \mathcal{S}^{N-1}$ and

$$\begin{aligned}\int_{B_1 \setminus B_{1-\lambda}} |Dw(k')|^2 &\le C\delta \left(\llbracket v(k') \circ G \rrbracket^2_{s,\mathcal{S}^{N-1}} + \llbracket b(k') \circ G \rrbracket^2_{s,\mathcal{S}^{N-1}} \right) \\ &\quad + \frac{C}{\delta^\alpha} \left\| \mathcal{G}(v(k') \circ G, b(k') \circ G) \right\|^2_{L^2(\mathcal{S}^{N-1})} \\ &\le \delta\, C\, 4N + \frac{C}{\delta^\alpha} o(1).\end{aligned}$$

To check the minimizing property let $c_j \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ be an arbitrary competitor to b_j for $j = 1, \ldots, J$. Set $c(k') = \sum_{j=1}^J \left(c_j \oplus t_j(k') \right)$. For $0 < R \le 1$ we define $\psi_R = G \circ \frac{1}{R} \circ G^{-1}(x) = \frac{e_N}{2} + \frac{1}{R}\left(x - \frac{e_N}{2} \right)$. So we found

$$\int_{C_R} |Dc(k') \circ \psi_R|^2 = R^{N-2} \int_{B_{1+}} |Dc(k')|^2 \le \int_{B_{1+}} |Dc(k')|^2$$

with $C_R = \psi_R^{-1}(B_{1+}) = G(B_R) \subset B_{1+}$. We define $C(k') \in W^{1,2}(B_{1+}, \mathcal{A}_Q(\mathbb{R}^n))$

$$C(k') = \begin{cases} \varphi(k') \circ G^{-1} & \text{if } x \in B_{1+} \setminus C_{1-\lambda} = G(B_1 \setminus B_{1-\lambda}) \\ c(k') \circ \psi_{1-\lambda} & \text{if } x \in C_{1-\lambda}. \end{cases}$$

$C(k') \circ G(k') \in W^{1,2}(\Omega_{F(k)} \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ is now an admissible competitor to $u(k')$ and therefore

$$\begin{aligned}(1 - o(1)) \int_{B_{1+}} |Dv(k')|^2 &\le \int_{\Omega_{F(k)} \cap B_1} |Du(k)|^2 \le (1 + o(1)) \int_{B_{1+}} |DC(k')|^2 \\ &\le (1 + o(1))\, C \int_{B_1 \setminus B_{1-\lambda}} |D\varphi(k')|^2 + (1 + o(1)) \int_{B_{1+}} |Dc(k')|^2 \\ &\le C \left(\delta + \frac{C}{\delta^\alpha} o(1) \right) + (1 + o(1)) \sum_{j=1}^J \int_{B_{1+}} |Dc_j|^2.\end{aligned}$$

Pass to the lim inf and apply the lower semicontinuity ensured by the concentration compactness Lemma B.4 to conclude

$$\sum_{j=1}^J \int_{B_{1+}} |Db_j|^2 \le \liminf_{k' \to \infty} (1 - o(1)) \int_{B_{1+}} |Dv(k')|^2 \le C\delta + \sum_{j=1}^J \int_{B_{1+}} |Dc_j|^2.$$

δ can be chosen arbitrary small and C is a dimensional constant so that b_j has to be Dirichlet minimizing for every $j = 1, \ldots, J$. The strong convergence in energy follows choosing $c_j = b_j$ for every j in the inequality above.

Proof of (iii): Having established (i) and (ii), $a_j(x) = b_j\left(\frac{x}{|x|}\right) \in W^{1,2}(B_{1+}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ is well-defined and an admissible competitor.

$$\int_{B_{1+}} |Db_j|^2 \le \int_{B_{1+}} |Da_j|^2 = \frac{1}{N-2}\int_{\mathcal{S}_+^{N-1}} |D_\tau a_j|^2 = \frac{1}{N-2}\int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2$$

for every j due to the 0-homogeneity of a_j and $a_j\big|_{\mathcal{S}_+^{N-1}} = b_j\big|_{\mathcal{S}_+^{N-1}}$.

The maps b_j constructed above with the properties (i), (ii), (iii) contradict Corollary 3.5. Firstly we found due to (ii), that

$$\begin{aligned}\sum_{j=1}^{J}\int_{B_{1+}} |Db_j|^2 &= \lim_{k'\to\infty}\int_{\Omega_{F(k')}\cap B_1} |Du(k')|^2\\ &\ge \lim_{k'\to\infty}\left(\frac{1}{N-2}-\frac{1}{k'}\right)\int_{\Omega_{F(k)}\cap\mathcal{S}^{N-1}} |D_\tau u(k')|^2\\ &= \lim_{k'\to\infty}\left(\frac{1}{N-2}-\frac{1}{k'}\right)\int_{S_+^{N-1}} |D_\tau v(k')|^2\\ &\ge \frac{1}{N-2}\sum_{j=1}^{J}\int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2.\end{aligned}$$

Combining this with (iii) gives, for $j = 1, \dots, J$

$$\int_{B_{1+}} |Db_j|^2 = \frac{1}{N-2}\int_{\mathcal{S}_+^{N-1}} |D_\tau b_j|^2.$$

Corollary 3.5 states now that $Db_j = 0$ on B_{1+} because $b_j\big|_{\Gamma_0} =$ const. by (i). This contradicts (ii), because $1 = \int_{\Omega_{F(k')}\cap B_1} |Du(k')|^2$ for all k'.

Hence the proposition must hold. □

Having in mind the actual proof of Theorem 3.1 we used from the estimate (3.4) two properties, the scaling property $\llbracket u_{z,r}\rrbracket^2_{s,B_1\cap\partial\Omega_{z,r}} = r^{2s-(N-1)}\llbracket u\rrbracket^2_{s,B_r(z)\cap\partial\Omega}$ and the existence of positive constants $\beta, M_u > 0$ both depending possibly on u so that in combination

$$\llbracket u_{z,r}\rrbracket_{s,B_1\cap\partial\Omega_{z,r}} \le r^\beta M_u.$$

Essentially one would like to replace the $W^{s,2}(\Gamma_F, \mathcal{A}_Q(\mathbb{R}^n))$-norm with a weaker norm with the same scaling property. Actually the $C^{0,\beta}$-Hölder norm, $[u]_{\beta,\Sigma} = \sup_{x,y\in\Sigma}\frac{\mathcal{G}(u(x),u(y))}{|x-y|^\beta}$, for any $0 < \beta < 1$ shares this property since

$$[u_{r,z}]_{\beta,\partial\Omega_{z,r}\cap B_1} = r^\beta [u]_{\beta,\partial\Omega\cap B_1(z)} \le r^\beta [u]_{\beta,\partial\Omega}.$$

So it would be desirable to replace the $W^{s,2}(\partial\Omega)$-norm, ($s > \frac{1}{2}$) with a Hölder-norm with exponent $\beta < \frac{1}{2}$ since it would get us closer to the already mentioned classical result: $u \in W^{1,2}(\Omega)$ harmonic with $u\big|_{\partial\Omega} \in C^{0,\beta}(\partial\Omega)$ for some $\beta > 0$ implies $u \in C^{0,\beta}\overline{\Omega}$.

Nonetheless we cannot hope to prove an estimate like (3.4) by contradiction if the fractional Sobolev norm ($s > \frac{1}{2}$) is replaced by an $C^{0,\beta}$-Hölder norm, $\beta < \frac{1}{2}$ because vanishing of energy through the boundary needs to be excluded. Bounds on $W^{s,2}(\partial\Omega)$-, or $C^{0,s}(\partial\Omega)$-norms with $s < \frac{1}{2}$ are insufficient. This is demonstrated by the following two dimensional example on the disc $B_1 \subset \mathbb{R}^2$. It uses polar coordinates $x = \binom{r\cos(\theta)}{r\sin(\theta)} = re^{i\theta}$.

Example 3.7. For any $\epsilon > 0$ there is a sequence of harmonic functions $f_k \in W^{1,2}(B_1,\mathbb{R})$ and a positive constant $c > 0$ with the following properties: for all k we have $\int_{B_1}|Df_k|^2 > c$, $f_k(e^{i\theta}) = 0$ for $|\theta| > \epsilon$. Furthermore $f_k \to 0$ uniformly on B_1 and $\|f_k\|_{s,\mathcal{S}^1}$, $[f_k]_{s,\mathcal{S}^1} \to 0$ for every $s < \frac{1}{2}$.

Proof of Example 3.7. Given $0 < \epsilon < \frac{\pi}{2}$, fix a smooth, symmetric, non-negative bump function η with $\eta(0) > 0$ and $\eta(\theta) = 0$ for $|\theta| \geq \epsilon$. Let $\sum_{l=0}^{\infty} a_l\cos(l\theta)$ be the Fourier series of $\eta(\theta)$. It is converging uniformly to η in the C^∞ topology since η is smooth and $\sum_{l=0}^{\infty} l^m|a_l| < \infty$ for all $m \in \mathbb{N}$. Fix $k_0 \in \mathbb{N}$ sufficiently large so that $2|a_k| < a_0 = \eta(0)$ for $k \geq k_0$ and set $A = \sum_{l=0}^{\infty}(l+1)|a_l| \geq \left(\sum_{l=0}^{\infty}(l+1)a_l^2\right)^{\frac{1}{2}}$. The addition theorem $2\cos(l\theta)\cos(k\theta) = \cos((l+k)\theta) + \cos((l-k)\theta)$ shows that the harmonic extension of $2\eta(\theta)\cos(k\theta)$ in B_1 is

$$\begin{aligned} g_k(r\,e^{i\theta}) &= \sum_{l=0}^{\infty} a_l\left(r^{l+k}\cos((l+k)\theta) + r^{|l-k|}\cos((l-k)\theta)\right) \\ &= \sum_{m=0}^{\infty}(a_{m-k}+a_{m+k})r^m\cos(m\theta) \quad \text{ with } a_{m-k} = 0 \text{ for } m < k. \end{aligned}$$

For $k \geq k_0$ we obtain a lower bound

$$\frac{1}{\pi}\int_{B_1}|Dg_k|^2 = \sum_{m=1}^{\infty} m(a_{m-k}+a_{m+k})^2 \geq k(a_0+a_{2k})^2 \geq \frac{k}{4}a_0^2$$

and an upper bound

$$\frac{1}{\pi}\int_{B_1}|Dg_k|^2 = \sum_{m=1}^{\infty} m(a_{m-k}+a_{m+k})^2 \leq 2\sum_{l=0}^{\infty}(l+k)a_l^2 + |l-k|a_l^2 \leq 4kA^2.$$

We consider now the sequence of harmonic functions on B_1 given by $f_k(x) = \frac{g_k(x)}{k^{\frac{1}{2}}} \in W^{1,2}(B_1)$. f_k has the desired properties: using the equivalence

(i) $\frac{1}{4}a_0^2 \leq \frac{1}{\pi}\int_{B_1}|Df_k|^2 = \|f_k\|^2_{\frac{1}{2},\mathcal{S}^1} \leq 4A^2$ for all $k \geq k_0$;

(ii) $f_k(e^{i\theta}) = 0$ for $|\theta| > \epsilon$ and all k;

(iii) $\|f_k\|_\infty \leq \frac{2\|\eta\|_\infty}{k^{\frac{1}{2}}} \to 0$ as $k \to \infty$;

(iv) for any $0 < s < \frac{1}{2}$

$$\|f_k\|^2_{s,\mathcal{S}^1} = \sum_{m=0}^{\infty} \frac{m^{2s}}{k}(a_{m-k} + a_{m+k})^2 \leq 8k^{2s-1}A^2$$

$$[f_k]_{s,\mathcal{S}^1} \leq \sum_{m=0}^{\infty} \frac{m^s}{k^{\frac{1}{2}}}|a_{m-k} + a_{m+k}| \leq 2k^{s-\frac{1}{2}}\sum_{l=0}^{\infty}(l+1)|a_l|$$

converging to 0 as $k \to \infty$.

(iii) follows from the maximum principle on harmonic functions. The fact that the $W^{s,2}$-norm on $\mathcal{S}^1$ corresponds to the sum in (iii) is a classical result of interpolation theory with weights. In case of the Hölder norm one checks that $[\varphi]_{\beta,\mathcal{S}^1} \leq \sum_{l=0}^{\infty} l^\beta |c_l|$ in case of a a converging Fourier series $\varphi(\theta) = \sum_{l=0}^{\infty} c_l \cos(l\theta)$. □

4. Boundary regularity in dimension $N = 2$

4.1. Global Hölder regularity

In this section we will show that Theorem 3.1 extends directly to two dimensions. We can consider the two dimensional case as a special case of a certain minimizer on a three dimensional domain.

Lemma 4.1. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be a minimizer on a domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, then $U(x,t) = u(x)$ is an element of $W^{1,2}(\Omega \times I, \mathcal{A}_Q(\mathbb{R}^n))$ for any bounded open interval $I \subset \mathbb{R}$. Moreover U is Dirichlet minimizing.*

Proof. Assuming the contrary there exists $V \in W^{1,2}(\Omega \times I, \mathcal{A}_Q(\mathbb{R}^n))$ with $V = U$ on the boundary of $\Omega \times I$, *i.e.*, $(x,t) \mapsto \mathcal{G}(U(x,t), V(x,t)) \in W^{1,2}_0(\Omega \times I)$ and

$$\int_{\Omega\times I}|DV|^2 < \int_{\Omega\times I}|DU|^2 = |I|\int_\Omega |Du|^2; \tag{4.1}$$

the second equality actually shows that $U \in W^{1,2}(\Omega \times I, \mathcal{A}_Q(\mathbb{R}^n))$.

Consider the subset $J \subset I$

$$J = \left\{t \in I : x \mapsto v_t(x) = V(x,t) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \text{ and } v_t\big|_{\partial\Omega} = u\big|_{\partial\Omega}\right\};$$

then by Fubini's theorem $|I \setminus J| = 0$.

Furthermore by (4.1) there must be a $t \in J$ with

$$\int_\Omega |Dv_t|^2 dx < \int_\Omega |Du|^2. \tag{4.2}$$

v_t for $t \in J$ satisfying (4.2) is an admissible competitor to u, but (4.2) violates the minimality of u. □

Remark 4.2. The converse of this lemma holds as well in the following sense, compare [7, Lemma 3.24]: if $u(x) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $U(x,t) = u(x)$ is Dirichlet minimizing on $\Omega \times \mathbb{R}$ then u itself is minimizing in Ω, in the sense of compact perturbations:

$$\int_{\{U \neq V\}} |DU|^2 \leq \int_{\{U \neq V\}} |DV|^2$$

for all $V \in W^{1,2}(\Omega \times \mathbb{R}, \mathcal{A}_Q(\mathbb{R}^n))$ with $\overline{\{U \neq V\}}$ compact.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^2$, C^1-regular, $Q, n \in \mathbb{N}$ and $\frac{1}{2} < s \leq 1$. If for some ball $B_{3R}(y) \subset \mathbb{R}^N$, $y \in \mathbb{R}^N$*

(a1) *$u \in W^{1,2}(\Omega \cap B_{3R}(y), \mathcal{A}_Q(\mathbb{R}^n))$ is Dirichlet minimizing;*
(a2) *$u\big|_{\partial\Omega} \in W^{s,2}(\partial\Omega \cap B_{3R}(y), \mathcal{A}_Q(\mathbb{R}^n))$*

holds, there are constants $C, \alpha_1 > 0$ depending on n, Q, s and $R_{\partial\Omega} > 0$ depending on $\partial\Omega$ such that

(i) *$|Du|$ is an element of the Morrey space $L^{2,N-2+2\alpha}(\Omega \cap B_R(y))$ for any $0 < \alpha < \min\{\alpha_1, s - \frac{1}{2}\}$, and more precisely the following estimate holds*

$$\begin{aligned} r^{-2\alpha} \int_{B_r(x)\cap\Omega} |Du|^2 \leq 2^7 R_0^{-2\alpha} &\int_{B_{2R_0}(x)\cap\Omega} |Du|^2 \\ &+ C \frac{R_0^{2s-1-2\alpha}}{2s-1-2\alpha} [\![u]\!]^2_{\partial\Omega\cap B_{3R}(y)} \end{aligned} \tag{4.3}$$

for any ball $B_r(x)$ with $x \in B_R(y) \cap \Omega$ and $r < \frac{R_0}{2} := \min\{R, R_{\partial\Omega}\}$.

(ii) *$u \in C^{0,\alpha}(\overline{\Omega \cap B_R(y)})$.*

Proof. Set $\Omega_I = \Omega \times]-4L, 4L[\subset \mathbb{R}^3$ for some large $L > 3R \geq 0$. The boundary portion $\partial\Omega \cap B_{3R}(y) \times]-3L, 3L[$ is C^1-regular by assumption on the regularity of $\partial\Omega$. $U(x,t) = u(x)$ is an element of $W^{1,2}(\Omega_I, \mathcal{A}_Q(\mathbb{R}^n))$ and Dirichlet minimizing as seen in Lemma 4.1. For any $(z, t_0) \in \partial\Omega\times]-2R, 2R[$ and $0 < r < R$ we found

$$\begin{aligned}
&r^{2(s-\beta)-2} \llbracket U \rrbracket^2_{s, B_r(z,t_0)\cap\partial\Omega_I} \\
&\le r^{2(s-\beta)-2} \llbracket U \rrbracket^2_{s,(B_r(z)\cap\partial\Omega)\times]t_0-r,t_0+r[} \\
&= r^{2(s-\beta)-2} \int_{B_r(z)\cap\partial\Omega\times B_r(z)\cap\partial\Omega} \int_{t_0-r}^{t_0+r}\int_{t_0-r}^{t_0+r} \frac{\mathcal{G}(u(x), u(y))^2}{\left(|x-y|^2+(t_1-t_2)^2\right)^{\frac{2+2s}{2}}} dt_1 dt_2 d(x,y) \\
&\le C2r^{2(s-\beta)-1} \int_{B_r(z)\cap\partial\Omega\times B_r(z)\cap\partial\Omega} \frac{\mathcal{G}(u(x), u(y))^2}{|x-y|^{1+2s}} d(x, y) \\
&\le 2C\, r^{2(s-\beta)-1} \llbracket u \rrbracket^2_{s,\partial\Omega\cap B_{3R}(y)}.
\end{aligned}$$

We have applied above the following auxiliary calculation. Let $\alpha > 0$ and $J = [a, a+\delta]$. After the change of variables $t_1 = a + rx$, $t_2 = a + ry$, we have

$$\begin{aligned}
&\int_{J\times J} \frac{1}{\left(r^2+(t_1-t_2)^2\right)^{\frac{\alpha+1}{2}}} d(t_1, t_2) = 2r^{1-\alpha} \int_{\substack{[0,\frac{\delta}{r}]\times[0,\frac{\delta}{r}] \\ x\ge y}} \frac{1}{\left(1+(x-y)^2\right)^{\frac{\alpha+1}{2}}} d(x, y) \\
&= 2r^{1-\alpha} \int_0^{\frac{\delta}{r}} \int_0^{\frac{\delta}{r}-y} \frac{1}{\left(1+z^2\right)^{\frac{\alpha+1}{2}}}\, dz dy \le 2r^{-\alpha}\delta \int_0^{\infty} \frac{1}{\left(1+z^2\right)^{\frac{\alpha+1}{2}}} \\
&= C|J|r^{-\alpha}.
\end{aligned}$$

The dimensional constant $C = 2\int_0^\infty \frac{1}{(1+z^2)^{\frac{\alpha+1}{2}}} \le \frac{\alpha+1}{\alpha}$ is therefore finite.

Combining all obtained estimates we found that U satisfies the assumption of Theorem 3.1 on the ball $B_{3R}(y, 0) \subset \mathbb{R}^3$ with $\beta = s - \frac{1}{2}$ and $M_U = \llbracket u \rrbracket_{s,\partial\Omega\cap B_{3R}(y)}$ in (a2).

Apply Theorem 3.1, in particular (3.1), to U on a point $(x, 0) \in \Omega\times]-L, L[$ with $r < \frac{R_0}{4} < L$. This gives the desired (4.3), because

$$\begin{aligned}
&r^{-2\alpha} \int_{B_r(x)\cap\Omega} |Du|^2 = \frac{r^{-2\alpha}}{2r} \int_{-r}^{r} \int_{B_r(x)\cap\Omega} |DU|^2 \le 2^2(2r)^{-1-2\alpha} \int_{B_{2r}((x,0))\cap\Omega_I} |DU|^2 \\
&\le 2^5 \left(R_0^{-1-2\alpha} \int_{B_{2R_0}((x,0))\cap\Omega_I} |DU|^2 + C\frac{R_0^{2(\beta-\alpha)}}{\beta-\alpha} M_U^2 \right) \\
&\le 2^7 R_0^{-2\alpha} \int_{B_{2R_0}(x)\cap\Omega} |Du|^2 + C\frac{R_0^{2s-1-2\alpha}}{2s-1-2\alpha} \llbracket u \rrbracket^2_{s,\partial\Omega\cap B_{3R}(y)}.
\end{aligned}$$

(ii) $u \in C^{0,\alpha}(\overline{\Omega \cap B_R(y)})$ follows by the same arguments outlined in the proof to Theorem 3.1. □

4.2. Continuity up to boundary

That continuity extends up to the boundary for 2-dimensional ball has been proven by W. Zhu in [18]. His idea is based on the Courant-Lebesgue lemma and can be modified to work on Lipschitz regular domains as well. We will give here a different proof, that on a first glimpse does not seem to be so restricted to the 2-dimensional setting as it is for Zhu's proof due to the Courant-Lebesgue lemma. Our proof uses an interplay of classical trace estimates and energy decay. We shortly recall the classical trace estimates and their proof. The proof here is taken from [17, Lemma 13.5]. As introduced in the general assumptions, Section 2, we use the notation $\Omega_F = \{(x', x_N) \colon x_N > F(x')\}$ for $F : \mathbb{R}^{N-1} \to \mathbb{R}$.

Lemma 4.4. *For F Lipschitz continuous and $1 < p < \infty$, one has*

$$\left\| \frac{f(x', x_N) - f\big|_{\partial\Omega_F}(x')}{x_N - F(x')} \right\|_{L^p(\tilde{\Omega})} \leq \frac{p}{p-1} \left\| \frac{\partial f}{\partial x_N} \right\|_{L^p(\tilde{\Omega})} \quad \forall f \in W^{1,p}(\Omega_F, \mathbb{R}); \tag{4.4}$$

and any subset $\tilde{\Omega} \subset \Omega_F$ of the following type:

$$\tilde{\Omega} = \left\{(x', x_N) \colon x' \in \Omega', F(x') < x_N < G(x')\right\}$$

$\tilde{\Omega} \subset \mathbb{R}^{N-1}$ and $G \geq F$ continuous.

Equivalently one has

$$\left\| \frac{\mathcal{G}(u(x', x_N), u\big|_{\partial\Omega_F}(x'))}{x_N - F(x')} \right\|_{L^p(\tilde{\Omega})} \leq \frac{p}{p-1} \, \| |D_N u| \|_{L^p(\tilde{\Omega})} \qquad \forall u \in W^{1,p}(\Omega_F, \mathcal{A}_Q(\mathbb{R}^n)). \tag{4.5}$$

Proof. For $p > 1$ Hardy's inequality, compare for instance with [17, Lemma 13.4], states that, if $h \in L^p(\mathbb{R}_+)$, $g(t) := \frac{1}{t} \int_0^t h(s) ds \in L^p(\mathbb{R}_+)$ satisfies

$$\|g\|_p \leq \frac{p}{p-1} \, \|f\|_p \, . \tag{4.6}$$

For $f \in C^1_c(\overline{\Omega_F})$ set

$$h(t) := \mathbf{1}_{[0, G(x') - F(x')]}(t) \, \frac{\partial f}{\partial x_N}\big(x', F(x') + t\big).$$

Apply Hardy's inequality to it and observe that for $0 < t < G(x') - F(x')$ and $t = x_N - F(x')$

$$g(t) = \frac{f\big(x', F(x') + t\big) - f\big(x', F(x')\big)}{t} = \frac{f(x', x_N) - f\big|_{\partial\Omega_F}(x')}{x_N - F(x')}.$$

Hence take the power p and integrate in $x' \in \Omega'$ to conclude (4.5). By a density argument the inequality extends to all of $W^{1,p}(\Omega_F)$.

For a Lipschitz continuous $u \in W^{1,p}(\Omega_F, \mathcal{A}_Q(\mathbb{R}^n))$, we have $u\big|_{\partial\Omega_F}(x') = u(x', F(x'))$. $k(t) := \mathcal{G}(u(x', F(x')+t), u(x', F(x')))$ is Lipschitz continuous in t. Furthermore $k'(t) \le |D_N u|(x', F(x')+t)$ for a.e. x'. Apply Hardy's inequality this time to $h(t) = \mathbf{1}_{[0,G(x')-F(x')]}(t)\, k'(t)$, take the power p and integrate in $x' \in \Omega'$. This shows (4.5) under the additional assumption that u is Lipschitz. It extends by density to all of $W^{1,p}(\Omega_F, \mathcal{A}_Q(\mathbb{R}^n))$. □

Proposition 4.5. *Given a Dirichlet minimizer $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ on a Lipschitz regular domain $\Omega \subset \mathbb{R}^N$ and that satisfies*

(a1) *$u\big|_{\partial\Omega}$ is continuous in $z_0 \in \partial\Omega$;*

(a2) *$N = 2$ or*

$$r^{2-N} \int_{B_r(z)\cap\Omega} |Du|^2 \to 0 \text{ as } r \to 0 \tag{4.7}$$

uniformly for all $z \in \partial\Omega \cap B_R(z_0)$ for some $R > 0$;

then u is continuous on $\Omega \cup \{z_0\}$.

Proof. Observe that in case of $N = 2$, $r^{2-N} \int_{B_r(z)\cap\Omega} |Du|^2 = \int_{B_r(z)\cap\Omega} |Du|^2 \to 0$ uniformly due to the absolute continuity of the integral and $|Du|^2 \in L^1(\Omega)$. Hence it is sufficient to prove the proposition under the assumption that (4.7) holds. u is Hölder continuous in the interior (Theorem 1.5) and so it remains to check that continuity extends up to z_0. So we may assume that $\Omega = \Omega_F$ for some Lipschitz continuous F, with Lipschitz norm $\mathrm{Lip}(F) < L$. Furthermore let $z_0 = (z', z_N) = (z', F(z')) \in \partial\Omega_F$.

Consider a generic sequence $x_k = (x_k', x_{N,k})$ converging to z_0 from the interior. Set $r_k = x_{N,k} - F(x_k') > 0$ and $\epsilon = \frac{1}{2\sqrt{1+L^2}}$. Then $B_{2\epsilon r_k}(x_k) \subset \Omega_F$ for all k and

$$r_k^2 \le 2(x_{N,k} - z_N)^2 + 2\big(F(z') - F(x_k')\big)^2 \le \frac{1}{2\epsilon^2}|x_k - z_0|^2. \tag{4.8}$$

To show continuity we have to check that $\mathcal{G}(u(x_k), u\big|_{\partial\Omega_F}(z_0))$ is of order $o(1)$. The triangle inequality and convexity gives

$$\begin{aligned}\frac{1}{3}\mathcal{G}(u(x_k), u\big|_{\partial\Omega_F}(z_0))^2 &\le \mathcal{G}(u(x_k), u(x))^2 \\ &\quad + \mathcal{G}\left(u(x), u\big|_{\partial\Omega_F}(x')\right)^2 + \mathcal{G}\left(u\big|_{\partial\Omega_F}(x'), u\big|_{\partial\Omega_F}(z_0)\right)^2.\end{aligned}$$

Integration in $x \in B_{\epsilon r_k}(x_k)$ gives

$$\begin{aligned}\frac{1}{3}\mathcal{G}\left(u(x_k), u\big|_{\partial\Omega_F}(z_0)\right)^2 &\le \fint_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u(x_k), u(x))^2 \\ &+ \fint_{B_{\epsilon r_k}(x_k)} \mathcal{G}\left(u(x), u\big|_{\partial\Omega_F}(x')\right)^2 + \fint_{B_{\epsilon r_k}(x_k)} \mathcal{G}\left(u\big|_{\partial\Omega_F}(x'), u\big|_{\partial\Omega_F}(z_0)\right)^2.\end{aligned}$$

It is sufficient to check that all integrals are of order $o(1)$.

$$\fint_{B_{\epsilon r_k}(x_k)} \mathcal{G}\left(u\big|_{\partial\Omega_F}(x'), u\big|_{\partial\Omega_F}(z_0)\right)^2 \leq \sup_{x\in B_{|x_k-z_0|}(z_0)} \mathcal{G}\left(u\big|_{\partial\Omega_F}(x'), u\big|_{\partial\Omega_F}(z_0)\right)^2 = o(1)$$

where we used (4.8) and assumption (a1).

For a fixed k set $\tilde{\Omega} = \{(x', x_N) : x' \in \Omega', F(x') < x_N < G(x')\}$ with $\Omega' = B_{\epsilon r_k}(x'_k) \subset \mathbb{R}^{N-1}$, $G(x') = x_{N,k} + \epsilon r_k$. The trace estimate, Lemma 4.4, states

$$\frac{1}{r_k^2}\int_{\tilde{\Omega}} \mathcal{G}\left(u(x), u\big|_{\partial\Omega_F}(x')\right)^2 \leq 4\int_{\tilde{\Omega}} \frac{\mathcal{G}\left(u(x), u\big|_{\partial\Omega_F}(x')\right)^2}{|x_N - F(x')|^2} \leq 16\int_{\tilde{\Omega}} |Du|^2;$$

where we used $x_N - F(x') \leq 2r_k$ on $\tilde{\Omega}$. We set $z_k = (x'_k, F(x'_k))$ then $B_{\epsilon r_k}(x_k) \subset \tilde{\Omega} \subset B_{2r_k}(z_k) \cap \Omega_F$ and assumption (a2) gives

$$\fint_{B_{\epsilon r_k}(x_k)} \mathcal{G}\left(u(x), u\big|_{\partial\Omega_F}(x')\right)^2 \leq \frac{16}{\omega_N \epsilon^N} r_k^{2-N} \int_{B_{2r_k}(z_k)\cap\Omega_F} |Du|^2 = o(1).$$

Finally the first integral is estimated using the internal Hölder continuity result, Theorem 1.5 or [7, Theorem 3.9]: $B_{2\epsilon r_k}(x_k) \subset \Omega_F$, so that for positive C, β

$$\mathcal{G}(u(x), u(x_k))^2 \leq C\left(\frac{|x - x_k|}{\epsilon r_k}\right)^{2\beta} (\epsilon r_k)^{2-N} \int_{B_{2\epsilon r_k}(x_k)} |Du|^2 \text{ for all } x \in B_{\epsilon r_k}(x_k).$$

Integration in x and $B_{2\epsilon r_k}(x_k) \subset B_{2r_k}(z_k)$ gives

$$\fint_{B_{\epsilon r_k}(x_k)} \mathcal{G}(u(x), u(x_k))^2 \leq \frac{C}{(\epsilon r_k)^{N-2}} \int_{B_{2\epsilon r_k}(x_k)} |Du|^2 \leq \frac{C}{\epsilon^{N-2}} r_k^{2-N} \int_{B_{2r_k}(z_k)} |Du|^2;$$

that is of order $o(1)$ by assumption (a2). □

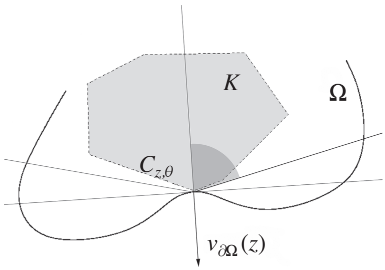
4.3. Partial improvement of the Hölder exponent

In the introduction we mentioned already that it would be desirable to extend the optimal Hölder exponent $\frac{1}{Q}$ in the interior up to the boundary. We want to present in this subsection a partial improvement of Theorem 4.3.

We will say a closed subset $K \subset \overline{\Omega}$ touches $\partial\Omega$ in a point $z \in \partial\Omega$ non-tangential, if there is a radius $R > 0$, a cone

$$C_{z,\theta} = \left\{x \in \mathbb{R}^2 : |x|\cos(\theta) < -\langle \nu_{\partial\Omega}(z), x\rangle\right\}$$

with $\theta < \frac{\pi}{2}$ and $\nu_{\partial\Omega}(z)$ denoting the outward pointing normal to $\partial\Omega$ at z such that $K \cap \overline{B_R(z)} \subset \overline{C_{z,\theta} \cap B_R(z)}$. This is sketched in the figure.



The improvement can now be formulated as:

Theorem 4.6. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ satisfy the assumptions of Theorem* 4.3*, additional $u\big|_{\partial\Omega} \in C^{0,\frac{1}{Q}}(\partial\Omega \cap B_{3R}(y))$, $K \subset \overline{\Omega}$ touching $\partial\Omega$ non-tangential in only one point $z \in \partial\Omega \cap B_R(y)$ then $u \in C^{0,\alpha}(K)$ with $\alpha = \frac{1}{Q}$ for $Q > 2$, $0 < \alpha < \frac{1}{2}$ for $Q = 2$.*

Shrinking $R > 0$ if necessary we may assume that $C_{z,\theta} \cap B_R(z) \subset \Omega$. $K \setminus B_R(z)$ is a compact subset of Ω hence the interior regularity theory holds. It remains to prove the regularity for conical subsets $C_{z,\theta} \cap B_R(z)$. The precise statement of Theorem 4.6 is:

Corollary 4.7. *Let $\frac{1}{2} < s \leq 1$ and $C_\theta = \{x = (x_1, x_2)\colon |x|\cos(\theta) \leq x_2\}$ with $0 < \theta < \frac{\pi}{2}$ (a cone). Under the assumptions*

(a1) *$u \in W^{1,2}(\Omega_F \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ Dirichlet minimizing*
(a2) *$u\big|_{\partial\Omega_F} \in W^{s,2}(\Gamma_F, \mathcal{A}_Q(\mathbb{R}^n))$ and for some $0 < \gamma$ there is a constant $M_u > 0$ so that*

$$r^{2(s-\gamma)-1}[\![u]\!]^2_{s,B_r\cap\Gamma_F} \leq M_u^2,$$

then there exists $0 < R < 1$ depending on $u(0)$ and θ so that, for any $\alpha < \min\{\gamma, \frac{1}{2}\}$ and $\alpha \leq \frac{1}{Q}$ the following holds:

(i) *$|Du|$ is an element of the Morrey space $L^{2,2\alpha}(\Omega_F \cap B_{\frac{R}{2}} \cap C_\theta)$, more precisely*

$$r^{-2\alpha}\int_{B_r(x)\cap\Omega_F}|Du|^2 \leq \frac{4}{\delta^{2\alpha}}\left(\int_{B_R\cap\Omega_F}|Du|^2 + \frac{CR^{2(\gamma-\alpha)}}{\gamma-\alpha}M_u^2\right) \tag{4.9}$$

where $\delta = \cos(\theta) - \cos(\frac{2\theta+\pi}{4})$;
(ii) *$u \in C^{0,\alpha}(\overline{\Omega_F \cap B_{\frac{R}{2}} \cap C_\theta})$.*

Concerning the optimality of the achieved Hölder exponent and assumption (a2) consider the following:

Remark 4.8. (a2) is obviously always satisfied for $\gamma = s - \frac{1}{2}$.

(a2) is satisfied for $\gamma > \frac{1}{2}$ and any $s < \gamma$ if $u\big|_{\Gamma_F} \in C^{0,\gamma}(\Gamma_F)$ as we have seen in Lemma 3.2. Furthermore this implies that

$$u \in C^{0,\alpha}(\overline{\Omega_F \cap B_R \cap C_\theta}) \text{ with } \alpha = \frac{1}{Q} \text{ for } Q > 2 \text{ and any } \alpha < \frac{1}{2} \text{ for } Q = 2,$$

i.e., the optimal exponent extends on cones up to the boundary.

The proof of the corollary follows similar lines as in the higher dimensional case. We will prove an improved estimate in the spirit of Proposition 3.3, that will lead eventually to Corollary 4.7. Before we present this final argument we prove the preliminary lemmas. As in the previous sections: $B_{1+} = B_1 \cap \{x_2 > 0\}$, $\mathcal{S}^1 = \partial B_1$, $\mathcal{S}^1_+ = \mathcal{S}^1 \cap \{x_2 > 0\}$, and $\Gamma_0 = B_1 \cap \{x_2 = 0\}$.

Lemma 4.9. *Let $\frac{1}{2} < s \leq 1$ be given, then there is a constant $C = C(s)$ so that any single valued harmonic function $f \in W^{1,2}(B_{1+})$ satisfies*

$$\int_{B_{1+}} |Df|^2 \leq (1+\epsilon) \int_{\mathcal{S}^1_+} |D_\tau f|^2 + \frac{C}{\epsilon} \int_{\Gamma_0} [\![f]\!]^2_{s,\Gamma_0} \quad \forall \epsilon > 0. \tag{4.10}$$

Proof. In a first step we show the existence of $C = C(s)$ such that any classical single-valued harmonic $h \in W^{1,2}(B_{1+})$ satisfies

$$\int_{B_{1+}} |Dh|^2 \leq C \left(\int_{\mathcal{S}^1_+} |D_\tau h|^2 + [\![h]\!]^2_{s,\Gamma_0} \right). \tag{4.11}$$

If $h \notin W^{s,2}(\Gamma_0)$ the RHS is $+\infty$ so there is nothing to check. $G : \overline{B_1} \to \overline{B_{1+}}$ denotes the bilipschitz map of Lemma C.1. Let $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$ be the Fourier series of $h \circ G\big|_{\mathcal{S}^1} = h\big|_{\partial B_{1+}} \circ G$. Its harmonic extension is then

$$\tilde{h}(r\, e^{i\theta}) = \sum_{k\in\mathbb{Z}} a_k r^k e^{ik\theta}.$$

h is harmonic, hence minimizing the Dirichlet energy, and $\tilde{h} \circ G^{-1}$ is an admissible competitor, so that

$$\int_{B_{1+}} |Dh|^2 \leq \int_{B_{1+}} \left| D\big(\tilde{h} \circ G^{-1}\big) \right|^2 \leq C \int_{B_1} \left| D\tilde{h} \right|^2 = C 2\pi \sum_{k\in\mathbb{Z}} |k| |a_k|^2.$$

It remains to estimate the series on the RHS.

For $s = 1$ we estimate

$$2\pi \sum_{k\in\mathbb{Z}} |k||a_k|^2 \leq 2\pi \sum_{k\in\mathbb{Z}} k^2|a_k|^2 = \int_{S^1_+} \left|D_\tau \tilde{h}\right|^2 + \int_{S^1_-} \left|D_\tau \tilde{h}\right|^2$$
$$\leq C\left(\int_{S^1_+} |D_\tau h|^2 + \int_{\Gamma_0} |D_\tau h|^2\right).$$

The constant C depends only on the Lipschitz norms of G, G^{-1}.

For $\frac{1}{2} < s < 1$ we note that it is classical that for a function $f(\theta) = \sum_{k\in\mathbb{Z}} b_k e^{ik\theta}$ on $\mathcal{S}^1$ the series $|b_0|^2 + \sum_{k\in\mathbb{Z}} |k|^{2s}|b_k|^2$ is an equivalent norm to $\|f\|^2_{L^2(\mathcal{S}^1)} + \llbracket f \rrbracket^2_{s,\mathcal{S}^1}$. So that we get in a first step

$$2\pi \sum_{k\in\mathbb{Z}} |k||a_k|^2 \leq 2\pi \sum_{k\in\mathbb{Z}} |k|^{2s}|a_k|^2 \leq C\llbracket \tilde{h} \rrbracket^2_{s,\mathcal{S}^1};$$

secondly Corollary A.5 gives

$$\llbracket \tilde{h} \rrbracket^2_{s,S^1} \leq C\left(\llbracket \tilde{h} \rrbracket^2_{s,\mathcal{S}^1\cap\{x_2>\frac{1}{5}\}} + \llbracket \tilde{h} \rrbracket^2_{s,\mathcal{S}^1\cap\{x_2<\frac{1}{5}\}}\right);$$

thirdly G is Lipschitz continuous and $G(\mathcal{S}^1 \cap \{x_2 > \frac{1}{5}\}) = \mathcal{S}^1_+$, $G(\mathcal{S}^1 \cap \{x_2 < \frac{1}{5}\}) = \Gamma_0$ so that

$$\llbracket \tilde{h} \rrbracket^2_{s,\mathcal{S}^1\cap\{x_2>\frac{1}{5}\}} + \llbracket \tilde{h} \rrbracket^2_{s,\mathcal{S}^1\cap\{x_2<\frac{1}{5}\}} \leq C\left(\llbracket h \rrbracket^2_{s,\mathcal{S}^1_+} + \llbracket h \rrbracket^2_{s,\Gamma_0}\right);$$

finally combining these with the interpolation property $\llbracket f \rrbracket_{s,\mathcal{S}^1_+} \leq C\llbracket \cdot \rrbracket_{1,\mathcal{S}^1_+}$ we estimate

$$2\pi \sum_{k\in\mathbb{Z}} |k||a_k|^2 \leq C\left(\int_{\mathcal{S}^1_+} |D_\tau h|^2 + \llbracket h \rrbracket^2_{s,\Gamma_0}\right).$$

Hence (4.11) holds.

Now we are able to improve (4.11) to (4.10). Let f be the harmonic function as assumed. We may assume $f \in W^{s,2}(\Gamma_0)$ otherwise the RHS is $+\infty$ and (4.10) holds trivially. Define the linear function

$$l(x_1, x_2) = \frac{f(1,0) - f(-1,0)}{2}x_1 + \frac{f(1,0) + f(-1,0)}{2}.$$

The same calculations as in Lemma 3.2 give a constant $C = C(s)$ with

$$\llbracket l \rrbracket^2_{s,\Gamma_0} \leq C\,\|\text{grad}\, l\|_\infty = C|f(1,0) - f(-1,0)|.$$

We achieved that $f(1,0)-l(1,0)=0=f(-1,0)-l(-1,0)$ and hence Corollary A.4 provides that

$$\tilde{h}(x)=\begin{cases}0 & \text{if } x\in\mathcal{S}^1_+\\ f(x)-l(x) & \text{if } x\in\Gamma_0\end{cases}$$

is an element of $W^{s,2}(\mathcal{S}^1_+\cup\Gamma_0)$. Hence there is a unique harmonic $h\in W^{1,2}(B_{1+})$ with $h\big|_{\mathcal{S}^1_+\cup\Gamma_0}=\tilde{h}$. $g=f-(h+l)$ is harmonic in B_{1+} and satisfies $g(x)=0$ on Γ_0. The antisymmetric reflexion

$$\tilde{g}(x_1,x_2)=\begin{cases}g(x_1,x_2) & \text{if } x_2\geq 0\\ -g(x_1,-x_2) & \text{if } x_2\leq 0\end{cases}$$

is by means of the Schwarz reflexion principle harmonic in B_1 with

$$2\int_{B_{1+}}|Dg|^2=\int_{B_1}|D\tilde{g}|^2\leq\int_{S^1}|D_\tau\tilde{g}|^2=2\int_{S^1_+}|D_\tau g|^2.$$

Young's inequality for $2\langle D_\tau f, D_\tau l\rangle\leq\epsilon|D_\tau f|^2+\frac{1}{\epsilon}\,\|\mathrm{grad}\,l\|^2_\infty$ gives

$$\begin{aligned}\int_{S^1_+}|D_\tau g|^2 &\leq (1+\epsilon)\int_{S^1_+}|D_\tau f|^2+\left(1+\frac{1}{\epsilon}\right)\pi\,\|\mathrm{grad}\,l\|^2_\infty\\ &\leq (1+\epsilon)\int_{S^1_+}|D_\tau f|^2+\frac{C}{\epsilon}[\![f]\!]^2_{s,\Gamma_0}\end{aligned}$$

where we used $\mathrm{grad}\,l=\frac{f(1,0)-f(-1,0)}{2}$ and $W^{s,2}(\Gamma_0)\subset C^{0,s-\frac{1}{2}}(\Gamma_0)$. Young's inequality for $2\langle D_i f, D_i(h+l)\rangle\geq-\epsilon|D_i f|^2-\frac{1}{\epsilon}|D_i(h+l)|^2$ gives

$$\int_{B_{1+}}|Dg|^2\geq(1-\epsilon)\int_{B_{1+}}|Df|^2-\frac{1}{\epsilon}\int_{B_{1+}}|D(h+l)|^2;$$

applying (4.11) we may conclude

$$\begin{aligned}\int_{B_{1+}}|D(h+l)|^2 &\leq C\left(\int_{S^1_+}|D_\tau(h+l)|^2+[\![h+l]\!]^2_{s,\Gamma_0}\right)\\ &\leq C\left(\pi\,\|\mathrm{grad}\,l\|^2_\infty+[\![f]\!]^2_{s,\Gamma_0}\right)\leq C[\![f]\!]^2_{s,\Gamma_0}.\end{aligned}$$

□

Lemma 4.9 behaves well under perturbations of B_{1+}, as made quantitative in the following corollary.

Corollary 4.10. *Let $\frac{1}{2}<s\leq 1$. There is a constant $C>0$ so that to any $\epsilon>0$ there is $\epsilon_F=\epsilon_F(\epsilon)>0$ so that any single valued harmonic function $f\in W^{1,2}(\Omega_F\cap B_1)$ satisfies*

$$\int_{\Omega_F\cap B_1}|Df|^2\leq(1+\epsilon)\int_{\Omega_F\cap S^1}|D_\tau f|^2+\frac{C}{\epsilon}[\![f]\!]^2_{s,\Gamma_F}.$$

Proof. This follows as a perturbation of the previous lemma making use of the bilipschitz equivalence of $\Omega_F \cap B_1$ and B_{1+}, *i.e.*, fix

$$G_F : \overline{B_{1+}} \to \overline{\Omega_F \cap B_1}$$

as given by Lemma C.2. Hence $\|DG_F - \mathbf{1}\|_\infty, \left\|DG_F^{-1} - \mathbf{1}\right\|_\infty < 10 \|\text{grad } F\|_\infty \leq 10\epsilon_F$. Let f as assumed with finite RHS, otherwise there is nothing to prove. $f \circ G_F \in W^{1,2}(B_{1+})$ hence there is an unique harmonic $\tilde{f} \in W^{1,2}(B_{1+})$ with $\tilde{f}\big|_{S^1 \cup \Gamma_0} = f \circ G_F\big|_{S^1 \cup \Gamma_0}$. $f, \tilde{f}$ are Dirichlet minimizer on their domains so that

$$\int_{\Omega_F \cap B_1} |Df|^2 \leq \int_{\Omega_F \cap B_1} \left| D\left(\tilde{f} \circ G_F^{-1}\right)\right|^2 \leq (1 + 10\epsilon_F)^4 \int_{B_{1+}} \left|D\tilde{f}\right|^2.$$

The previous lemma showed that, for some constant $C > 0$,

$$\begin{aligned}\int_{B_{1+}} \left|D\tilde{f}\right|^2 &\leq (1+\epsilon_1) \int_{S^1_+} \left|D_\tau \tilde{f}\right|^2 + \frac{C}{\epsilon_1} \llbracket \tilde{f} \rrbracket^2_{s,\Gamma_0} \\ &\leq (1+\epsilon_1)(1+10\epsilon_F)^3 \int_{S^1 \cap \Omega_F} |D_\tau f|^2 + \frac{C}{\epsilon_1}(1+10\epsilon_F)^5 \llbracket f \rrbracket^2_{s,\Gamma_F}.\end{aligned}$$

We conclude choosing $\epsilon_1 = \frac{\epsilon}{2}$ and then $\epsilon_F > 0$ sufficient small for $(1 + \frac{\epsilon}{2})(1 + 10\epsilon_F)^7 \leq 1 + \epsilon$. □

We can use the obtained results to get an estimate for Dirichlet minimizers in the spirit of Proposition 3.3. As in the proof of the concentration compactness lemma, Lemma B.4, we need the separation $\text{sep}(T)$ of a Q-point $T = \sum_{i=1}^{Q} \llbracket t_i \rrbracket \in \mathcal{A}_Q(\mathbb{R}^n)$, defined as

$$\text{sep}(T) = \begin{cases} 0 & \text{if } T = Q\llbracket t \rrbracket \\ \min_{t_i \neq t_j} |t_i - t_j| & \text{otherwise.} \end{cases}$$

Lemma 4.11. *For $\frac{1}{2} < s \leq 1$ and $\epsilon > 0$, there is a constant $C = C(s) > 0$ with the property that if* (A1) *holds with $\epsilon_F = \epsilon_F(\epsilon) > 0$ then*

$$\int_{B_r \cap \Omega_F} |Du|^2 \leq (1+\epsilon) \int_{\partial B_r \cap \Omega_F} |D_\tau u|^2 + \frac{C}{\epsilon} r^{2s-1} \llbracket u \rrbracket^2_{s, B_r \cap \Omega_F} \forall 0 < r < R_0$$

for any Dirichlet minimizing $u \in W^{1,2}(\Omega_F \cap B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and $R_0 = R_0(u(0)) > 0$.

Proof. As usual we may assume that the RHS is finite. Let $\epsilon_F > 0$ be the constant of the previous Corollary 4.10 and $\|\text{grad } F\|_{\infty, B_1} < \epsilon_F$.

Suppose $\text{sep}(u(0)) = 0$, *i.e.*, $u(0) = Q\llbracket p \rrbracket$ for some $p \in \mathbb{R}^n$. Since we assumed the RHS is finite $u \in W^{1,2}(\partial B_r \cap \Omega_F, \mathcal{A}_Q(\mathbb{R}^n))$. Fix for such a radius $t_- < 0 < t_+$ and $-\frac{\pi}{2} < \theta_+ < \theta_- < \frac{3\pi}{2}$ so that

$$\partial B_r \cap \Omega_F = \left\{ x_+ = (rt_+, F(rt_+)) = re^{i\theta_+}, x_- = (rt_-, F(rt_-)) = re^{i\theta_-} \right\}.$$

There is $b = (b_1, \dots, b_Q) \in W^{1,2}([\theta_+, \theta_-], \mathbb{R}^{nQ})$ so that $[b(\theta)] = u_{0,r}(e^{i\theta}) = u(re^{-\theta})$ for $\theta_+ \leq \theta \leq \theta_-$ due to the 1-dim. $W^{1,2}$-selection criterion [7, Proposition 1.2]. There are $a(t) = (a_1, \dots, a_Q) \in W^{s',2}([0, t_+], \mathbb{R}^{nQ})$ and $b(t) = (b_1, \dots, b_Q) \in W^{s',2}([t_-, 0], \mathbb{R}^{nQ})$ for any $s' < s$ with $[a(t)] = u(rt, F(rt))$, $[b(t)] = u(rt, F(rt))$ respectively due to the $W^{s,2}$-selection, Lemma B.7. Permuting a and c if necessary we may assume that $a(t_+) = b(\theta_+), c(t_-) = b(\theta_-)$. We may define

$$g(x) = \begin{cases} a(x_1) & \text{if } rx \in B_r \cap \Gamma_F, x_1 \geq 0 \\ b(\theta) & \text{if } rx = re^{i\theta} \in \partial B_r \cap \Omega_F \\ c(x_1) & \text{if } rx \in B_r \cap \Gamma_F, x_1 \leq 0. \end{cases}$$

$g = (g_1, \dots, g_Q) \in W^{s',2}(\partial(B_1, (\Omega_F)_{0,r}), \mathbb{R}^{nQ})$ as a consequence of Corollary A.4. $[g(x)] = \sum_{i=1}^{Q} [\![g_i(x)]\!] = u_{0,r}(x)$ for all $x \in \partial(B_1 \cap (\Omega_F)_{0,r})$. Hence there is $h = (h_1, \dots h_Q) \in W^{1,2}(B_1 \cap (\Omega_F)_{0,r}, \mathbb{R}^{nQ})$ harmonic with g as boundary values. $[h] = \sum_{i=1}^{Q} [\![h_i]\!]$ is a competitor to $u_{0,r}$ so that

$$\int_{B_r \cap \Omega_F} |Du|^2 = \int_{B_1 \cap (\Omega_F)_{0,r}} |Du_{0,r}|^2 \leq \int_{B_1 \cap (\Omega_F)_{0,r}} |D[h]|^2 = \int_{B_1 \cap (\Omega_F)_{0,r}} |Dh|^2.$$

The previous Corollary 4.10 applies to h since $\|\text{grad } F_{0,r}\|_{\infty, B_1} = \|\text{grad } F\|_{\infty, B_r} < \epsilon_F$. So, we find for a fixed $\frac{1}{2} < s' < s$, *e.g.* $s' = \frac{1+2s}{4}$,

$$\begin{aligned} \int_{B_1 \cap (\Omega_F)_{0,r}} |Dh|^2 &\leq (1+\epsilon) \int_{S^1 \cap (\Omega_F)_{0,r}} |D_\tau h|^2 + \frac{C}{\epsilon} [\![h]\!]^2_{s', (\Gamma_F)_{0,r}} \\ &\leq (1+\epsilon) r \int_{\partial B_r \cap \Omega_F} |D_\tau u|^2 + \frac{C}{\epsilon} r^{2s-1} [\![u]\!]^2_{s, \Omega_F \cap B_r} \end{aligned}$$

considering in the last line $[h(x)] = [g(x)] = u_{0,r}(x)$ for $x \in \partial(B_1 \cap (\Omega_F)_{0,r})$ and $[\![h]\!]_{s', (\Gamma_F)_{0,r}} \leq C [\![u_{0,r}]\!]_{s, (\Gamma_F)_{0,r}} = C r^{2s-1} [\![u]\!]^2_{s, \Omega_F \cap B_r}$ from the $W^{s,2}$-selection, Lemma B.7.

If $\text{sep}(u(0)) > 0$, *i.e.*, $u(0) = \sum_{j=1}^{J} Q_j [\![p_j]\!], |p_i - p_j| \geq \text{sep}(u(0))$ for $i \neq j$. Fix $R_0 > 0$ so that

$$R_0^{\tilde{\alpha}} [u]_{\tilde{\alpha}, \Omega_F \cap B_{R_0}} < \frac{1}{3} \text{sep}(u(0))$$

where $[\cdot]_{\tilde{\alpha}, \Omega_F \cap B_{R_0}}$ denotes the Hölder semi-norm on $\Omega_F \cap B_{R_0}$ with exponent $\tilde{\alpha} > 0$ provided by Theorem 4.3. Hence there are Dirichlet minimizing $u_j \in W^{1,2}(\Omega_F \cap B_{R_0}, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ with

$$\mathcal{G}(u_j(x), Q_j [\![p_j]\!]) < \frac{1}{3} \text{sep}(u(0)) \text{ for all } x \in \Omega_F \cap B_{R_0}. \tag{4.12}$$

To each u_j the assumption $\mathrm{sep}(u_j(0)) = 0$ is satisfied. So, by the previous considerations, for a.e. $0 < r \le R_0$

$$\begin{aligned}\int_{B_r\cap\Omega_F}|Du|^2 &= \sum_{j=1}^{J}\int_{B_r\cap\Omega_F}|Du_j|^2\\ &\le \sum_{j=1}^{J}(1+\epsilon)r\int_{\partial B_r\cap\Omega_F}|D_\tau u_j|^2 + \frac{C}{\epsilon}r^{2s-1}[\![u_j]\!]^2_{s,\Omega_F\cap B_r}\\ &= (1+\epsilon)r\int_{\partial B_r\cap\Omega_F}|D_\tau u|^2 + \frac{C}{\epsilon}r^{2s-1}[\![u]\!]^2_{s,\Omega_F\cap B_r}\end{aligned}$$

where we used in the last step that $\mathcal{G}(u(x),u(y))^2 = \sum_{j=1}^{J}\mathcal{G}(u_j(x),u_j(y))^2$ to (4.12). $\square$

As Theorem 3.1 follows from Proposition 3.3, we can now use Lemma 4.11 to give the final argument leading to the Hölder estimate of Corollary 4.7.

Proof of Corollary 4.7. Let $\alpha > 0$ be given as stated. Fix $\epsilon > 0$ so that $1+\epsilon \le \frac{1}{2\alpha}$ and $0 < R < 1$ sufficient small so that

(1) $R \le R_0$ when R_0 is the radius of the previous Lemma, 4.11;
(2) $\|\mathrm{grad}\, F\|_{\infty, B_R\cap\Omega_F} < \cos(\frac{2\theta+\pi}{4})$.

(2) ensures that $C_\theta \cap B_R \subset C_{\frac{2\theta+\pi}{4}} \cap B_R \subset \Omega_F \cap B_1$. Following the steps in the proof of Theorem 3.1 for a.e. $0 < r \le R$

$$\begin{aligned}-\frac{\partial}{\partial r}r^{-2\alpha}\int_{B_r\cap\Omega_F}|Du|^2 &= -r^{-2\alpha}\int_{\partial B_r\cap\Omega_F}|Du|^2 + 2\alpha r^{-2\alpha-1}\int_{B_r\cap\Omega_F}|Du|^2\\ &\le \frac{C}{\epsilon}r^{(2s-1-2\alpha)-1}[\![u]\!]^2_{s,B_r\cap\Gamma_F} \le \frac{C}{\epsilon}r^{2(\gamma-\alpha)-1}M_u^2.\end{aligned}$$

Integration in $0 < r \le R$ gives

$$r^{-2\alpha}\int_{B_r\cap\Omega_F}|Du|^2 \le R^{-2\alpha}\int_{B_R\cap\Omega_F}|Du|^2 + \frac{CR^{2(\gamma-\alpha)}}{\gamma-\alpha}M_u^2. \tag{4.13}$$

By definition of $\delta = \cos(\theta) - \cos(\frac{2\theta+\pi}{4})$, for all $x \in B_{\frac{R}{2}} \cap C_\theta$ we have $B_{\delta|x|}(x) \subset C_{\frac{2\theta+\pi}{4}} \cap B_R$. Let $x \in B_{\frac{R}{2}} \cap C_\theta$ and $0 < r < \frac{R}{2}$ be given, set $r_1 = \max\{r, \delta|x|\}$ and $r_2 = r_1 + |x| \le \frac{2}{\delta}r_1$. We found

$$\begin{aligned}r^{-2\alpha}\int_{B_r(x)\cap\Omega_F}|Du|^2 \le r_1^{-2\alpha}\int_{B_{r_1}(x)\cap\Omega_F}|Du|^2 &\le \frac{2^{2\alpha}}{\delta^{2\alpha}}r_2^{-2\alpha}\int_{B_{r_2}(x)\cap\Omega_F}|Du|^2\\ &\le \frac{4}{\delta^{2\alpha}}\left(\int_{B_R\cap\Omega_F}|Du|^2 + \frac{CR^{2(\gamma-\alpha)}}{\gamma-\alpha}M_u^2\right),\end{aligned}$$

where we applied at first the internal estimate since $\alpha \leq \frac{1}{Q}$ and finally the just established (4.13). Having established (i), (ii) follows as indicated in the proof of Theorem 3.1. □

Appendix

A. Fractional Sobolev spaces

Recall that we defined the fractional Sobolev space $W^{s,2}(\Gamma), 0 < s < 1$ on an $(N-1)$-dimensional Lipschitz-manifold $\Gamma \subset \mathbb{R}^N$ to be the subset of $L^2(\Gamma)$ with $\|f\|^2_{W^{s,2}(\Gamma)} = \|f\|^2_{L^2(\Gamma)} + [\![f]\!]^2_{s,\Gamma} < \infty$. In the first subsection we outline how this definition fits into the general framework of of fractional Sobolev spaces. Furthermore we state some general estimates. In particular we give a sufficient condition to patch two fractional Sobolev functions together.

The second subsection is devoted to prove an interpolation lemma in the the spirit of Luckhaus for fractional Sobolev spaces.

A.1. General facts

Essential there are three ways to define $W^{s,2}(\mathbb{R}^N)/\, H^s(\mathbb{R}^N)$ for $0 < s < 1$:

(a) using Fourier transform:

$$H^s\big(\mathbb{R}^N\big) = \{u \in L^2\big(\mathbb{R}^N\big) \ |\xi|^s \mathcal{F}u(\xi) \in L^2\big(\mathbb{R}^N\big)\};$$

(b) using real interpolation:

$$W^{s,2}\big(\mathbb{R}^N\big) = \Big(W^{1,2}\big(\mathbb{R}^N\big), L^2\big(\mathbb{R}^N\big)\Big)_{1-s,2};$$

(c) using the the Gagliardo semi-norm $[\![\cdot]\!]_{s,\mathbb{R}^N}$

$$W^{s,2}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \colon [\![u]\!]^2_{s,\mathbb{R}^N} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, d(x,y); < \infty \right\}.$$

All of these define the same Banach space. Their equivalence can be found for instance in [17]: (a) = (c) corresponds to Lemma 16.3 or Lemma 35.2, (a) = (b) can be found in Lemma 23.1.

For a bounded open domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary one has essential three possible definitions for $W^{s,2}(\Omega)$, compare [17, Section 34 and Section 36]:

(a) as restriction

$$W^{s,2}(\Omega) = \text{ space of restrictions of functions in } W^{s,2}\big(\mathbb{R}^N\big);$$

(b) using interpolation

$$W^{s,2}(\Omega) = \Big(W^{1,2}(\Omega), L^2(\Omega)\Big)_{1-s,2};$$

(c) using the Gagliardo semi-norm

$$W^{s,2}(\Omega) = \left\{u \in L^2(\Omega)\colon \llbracket u \rrbracket^2_{s,\Omega} = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, d(x,y) < \infty\right\}.$$

Once again it turns out that all of them are equivalent, compare [17, Section 34] for (a) = (b) and [17, Lemma 36.1] for (c) = (a).

Furthermore we remark that all these definition behave well under bilipschitz maps $F : \Omega \to \Omega'$ and multiplication with smooth functions *e.g.* partitions of unity. Therefore, to give a definition of $W^{s,2}(\Gamma)$ for a $(N-1)$-dimensional Lipschitz-manifold $\Gamma \subset \mathbb{R}^N$ it is sufficient to give a definition for the graphical case, *i.e.*, let $\Omega_F = \{(x', x_N)\colon x_N > F(x')\}$ with $F \in \mathrm{Lip}(\mathbb{R}^{N-1}, \mathbb{R})$ we define for the manifold $\Gamma_F = \partial\Omega_F$

$$W^{s,2}(\partial\Omega_F) = \left\{u \in L^2(\partial\Omega_F)\colon u(x', F(x')) \in W^{s,2}\big(\mathbb{R}^{N-1}\big)\right\}.$$

Using the Gagliardo semi-norm we can give an equivalent global definition in the case of $\Gamma = \partial\Omega$

$$\llbracket u \rrbracket^2_{s,\partial\Omega} = \int_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N-1+2s}} d(x,y).$$

Having defined the fractional Sobolev spaces we collect now several estimates that might be known but for which we could not find any reference. For $s > \frac{1}{2}$ the trace operator $|_{\partial\mathbb{R}^N_+} : W^{s,2}(\mathbb{R}^N_+) \to W^{s-\frac{1}{2},2}(\mathbb{R}^{N-1})$ is a bounded, linear and surjective map, compare [17, Lemma 16.1, Lemma 16.3]. Additionally it satisfies the following estimate.

Lemma A.1. *For $\frac{1}{2} < s < \frac{3}{2}$ one has*

$$\left\| \frac{u(x', x_N) - u\big|_{\{x_N=0\}}(x')}{|x_N|^s} \right\|_{L^2(\mathbb{R}^N)} \le C(s) \left\| |\xi_N|^s \mathcal{F}u \right\|_{L^2(\mathbb{R}^N)} \le C(s)\|u\|_{H^s(\mathbb{R}^N)}. \quad \text{(A.1)}$$

Proof. We define $v_{x_N}(x') = u(x', x_N)$, then $\mathcal{F}v_{x_N}(\xi') = \int_{\mathbb{R}} e^{2i\pi\xi_N x_N} \mathcal{F}u(\xi', \xi_N)d\xi_N$ and $\mathcal{F}'u\big|_{\partial\mathbb{R}^N_+}(\xi') = \mathcal{F}v_0(\xi') = \int_{\mathbb{R}} \mathcal{F}u(\xi', \xi_N)d\xi_N$; hence by Cauchy inequality

$$\begin{aligned}|\mathcal{F}v_{x_N}(\xi') - \mathcal{F}v_0(\xi')|^2 &= \left|\int_{\mathbb{R}} (e^{2i\pi\xi_N x_N} - 1)\mathcal{F}u(\xi', \xi_N)d\xi_N\right|^2 \\ &\le 4\left(\int_{\mathbb{R}} \frac{|\sin(\pi\xi_N x_N)|}{|\xi_N x_N|^\alpha} x_N d\xi_N\right) x_N^{\alpha-1} \left(\int_{\mathbb{R}} |\sin(\pi\xi_N x_N)||\xi_N|^\alpha |\mathcal{F}u|^2(\xi', \xi_N)d\xi_N\right).\end{aligned}$$

Multiply this by $|x_N|^{-2s}$ and integrate in x_N to conclude

$$\begin{aligned}&\int_{\mathbb{R}} |x_N|^{-2s} |\mathcal{F}v_{x_N}(\xi') - \mathcal{F}v_0(\xi')|^2 dx_N \\ &\leq 4C(\alpha) \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\sin(\pi \xi_N x_N)|}{|\xi_N x_N|^{1+2s-\alpha}} |\xi_N| dx_N \right) |\xi_N|^{2s} |\mathcal{F}u|^2 (\xi', \xi_N) d\xi_N \\ &= 4C(\alpha)^2 \int_{\mathbb{R}} |\xi_N|^{2s} |\mathcal{F}u|^2 (\xi', \xi_N) d\xi_N\end{aligned}$$

where $C(\alpha) = \int_{\mathbb{R}} \frac{\sin(\pi t)}{|t|^\alpha} dt < \infty$ for $\alpha = 1+2s-\alpha$ (note that $1 < \frac{1}{2}+s = \alpha < 2$). This gives the desired result by integrating in ξ', since

$$\int_{\mathbb{R}^N} \frac{|u(x', x_N) - u\big|_{\partial\Omega_F}(x')|^2}{|x_N|^{2s}} dx = \int_{\mathbb{R}} |x_N|^{-2s} \int_{\mathbb{R}^{N-1}} |\mathcal{F}v_{x_N}(\xi') - \mathcal{F}v_0(\xi')|^2 d\xi' dx_N. \qquad \square$$

As a corollary we obtain a tool that will allow us to check if a function $u \in W^{s,2}(\Omega)$ is an extension to a function $v \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, Corollary A.3. In fact one is an extension of the other if their traces coincide.

As introduced before: $\Omega_F = \{x \in \mathbb{R}^N : x_N > F(x')\}$ with F Lipschitz continuous

Corollary A.2. *For $\frac{1}{2} < s < 1$ and $u \in W^{s,2}(\Omega_F)$, one has*

$$\left\| \frac{u(x', x_N) - u\big|_{\partial\Omega_F}(x')}{|x_N - F(x')|^s} \right\|_{L^2(\Omega_F)} \leq C \llbracket u \rrbracket_{s,\Omega_F^N}. \tag{A.2}$$

Proof. Using the bilipschitz mapping $(x', x_N) \mapsto (x', x_N - F(x'))$ and $v(x', x_N) = u(x', F(x') + x_N) \in W^{s,2}(\mathbb{R}^N_+)$ together with

$$\int_{\Omega_F} \frac{|u(x', x_N) - u\big|_{\partial\Omega_F}(x')|^2}{|x_N - F(x')|^{2s}} dx = \int_{R^N_+} \frac{|u(x', x_N + F(x')) - u\big|_{\partial\Omega_F}(x')|^2}{|x_N|^{2s}} dx;$$

one has only to consider the case $F = 0$, *i.e.*, $\Omega_F = \mathbb{R}^N_+$.

Furthermore we can reduce it to the case of Lemma A.1 extending u by $u(x', -x_N)$ for $x_N < 0$ to obtain $u \in W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ and $\| |\xi_N|^s |\mathcal{F}u| \|_{L^2(\mathbb{R}^N)} \leq C \llbracket u \rrbracket_{s,\mathbb{R}^N}$ for $0 < s < 1$, *e.g.* [17, Lemma 16.3]. $\square$

Corollary A.3. *$v \in L^2(\mathbb{R}^{N-1})$ is the trace of $u \in W^{s,2}(\Omega_F)$ if*

$$\left\| \frac{u(x', x_N) - v(x')}{|x_N - F(x')|^s} \right\|_{L^2(\Omega_F)} < \infty, \; s > \frac{1}{2}. \tag{A.3}$$

In particular it implies $v \in W^{s-\frac{1}{2},2}(\mathbb{R}^{N-1})$.

Proof.

$$
\begin{aligned}
&\int_{\mathbb{R}^{N-1}} |v(x') - u\big|_{\partial\Omega_F}(x')|^2 dx' \\
&\leq 2\epsilon^{2s}\frac{1}{\epsilon}\int_0^\epsilon \int_{\mathbb{R}^{N-1}} \frac{|v(x') - u(x', F(x') + x_N)|^2}{|x_N|^{2s}} \\
&\quad + 2\epsilon^{2s}\frac{1}{\epsilon}\int_0^\epsilon \int_{\mathbb{R}^{N-1}} \frac{|u(x', F(x') + x_N) - u\big|_{\partial\Omega_F}(x')|^2}{|x_N|^{2s}} dx' dx_N \\
&\leq 2\epsilon^{2s-1}\left(\left\| \frac{u(x', x_N) - v(x')}{|x_N - F(x')|^s} \right\|^2_{L^2(\Omega_F)} + \left\| \frac{u(x', x_N) - u\big|_{\partial\Omega_F}(x')}{|x_N - F(x')|^s} \right\|^2_{L^2(\Omega_F)} \right)
\end{aligned}
$$

converging to 0 as $\epsilon \to 0$ hence $v = u\big|_{\partial\Omega_F}$. □

Corollary A.4. *Let $u \in W^{s,2}(\Omega_F)$ and $v \in W^{s,2}(\mathbb{R}^N \setminus \Omega_F)$ for $s > \frac{1}{2}$ satisfying $u\big|_{\partial\Omega_F} = v\big|_{\partial\Omega_F}$ then*

$$
U(x) = \begin{cases} u(x) & \text{if } x \in \Omega_F \\ v(x) & \text{if } x \in \mathbb{R}^N \setminus \Omega_F \end{cases} \tag{A.4}
$$

defines an element in $W^{s,2}(\mathbb{R}^N)$ satisfying

$$
[\![U]\!]_{s,\mathbb{R}^N} \leq C\left([\![u]\!]_{s,\Omega_F} + [\![v]\!]_{s,\mathbb{R}^N\setminus\Omega_F}\right). \tag{A.5}
$$

Proof. As before using the bilipschitz mapping $(x', x_N) \mapsto (x', x_N - F(x'))$ one has only to consider the case $F = 0$; then

$$
\begin{aligned}
\|U\|^2_{L^2(\mathbb{R}^N)} &= \|u\|^2_{L^2(\mathbb{R}^N_+)} + \|v\|^2_{L^2(\mathbb{R}^N_-)} \\
\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} d(y, x) &= 2\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-} \frac{|u(x) - v(y)|^2}{|x - y|^{N+2s}} d(y, x) \\
&\quad + \int_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} d(y, x) \\
&\quad + \int_{\mathbb{R}^N_-\times\mathbb{R}^N_-} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} d(y, x).
\end{aligned}
$$

The last two summands are obviously bounded. The first one can be estimated as follows

$$\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-}\frac{|u(x)-v(y)|^2}{|x-y|^{N+2s}}d(y,x)\le 3\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-}\frac{|u\big|_{\partial\Omega_F}(x')-v\big|_{\partial\Omega_F}(y')|^2}{|x-y|^{N+2s}}d(y,x) \quad (A.6)$$

$$+3\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-}\frac{|u(x)-u\big|_{\partial\Omega_F}(x')|^2}{|x-y|^{N+2s}}d(y,x)+3\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-}\frac{|v\big|_{\partial\Omega_F}(y')-v(y)|^2}{|x-y|^{N+2s}}d(y,x). \quad (A.7)$$

For the first integral, (A.6), we have

$$\begin{aligned}&\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-}\frac{\left|u\big|_{\partial\Omega_F}(x')-v\big|_{\partial\Omega_F}(y')\right|^2}{|x-y|^{N+2s}}d(y,x)\\ &\le C_1\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}\frac{\left|u\big|_{\partial\Omega_F}(x')-v\big|_{\partial\Omega_F}(y')\right|^2}{|x'-y'|^{N-2+2s}}d(y',x')\le C\llbracket u\rrbracket^2_{s,\mathbb{R}^N_+},\end{aligned}$$

where we used firstly

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}_-}\frac{1}{|x-y|^{N+2s}}d(x_N,y_N)&=\int_{\mathbb{R}_+\times\mathbb{R}_+}\frac{\left(1+(t+\tau)^2\right)^{-\frac{N}{2}-s}}{|x'-y'|^{N-2+2s}}d(\tau,t)\\ &=\frac{C_1}{|x-y|^{N-2+2s}}\end{aligned}$$

by means of the change of variables $x_N=|x'-y'|t$, $y_N=-|x'-y'|\tau$ and then $u\big|_{\partial\mathbb{R}^N_+}=v\big|_{\partial\mathbb{R}^N_+}$ together with the continuity of the trace operator $\big|_{\partial\mathbb{R}^N_+}:W^{s,2}(\mathbb{R}^N_+)\to W^{s-\frac{1}{2},2}(\mathbb{R}^{N-1})$, compare [17, Lemma 16.1, Lemma 16.3].

For the second and third integral, (A.7), we proceed similarly. For instance for the second

$$\begin{aligned}\int_{\mathbb{R}^N_+\times\mathbb{R}^N_-}\frac{\left|u(x)-u\big|_{\partial\mathbb{R}^N_+}(x')\right|^2}{|x-y|^{N+2s}}d(y,x)&\le C_2\int_{\mathbb{R}^N_+}\frac{\left|u(x',x_N)-u\big|_{\partial\mathbb{R}^N_+}(x')\right|^2}{|x_N|^{2s}}dx\\ &\le C\llbracket u\rrbracket^2_{s,\mathbb{R}^N_+}\end{aligned}$$

where we used

$$\int_{\mathbb{R}^N_-}\frac{1}{|x-y|^{N+2s}}dy=x_N^{-2s}\int_{\mathbb{R}^N_+}\frac{1}{|z+e_N|^{N+2s}}dz=x_N^{-2s}C_2,$$

by means of the change of variables $(y',y_N)=(x'-x_Nz',-x_Nz_N)$, $x_N>0$ and afterwards we apply Lemma A.2.

The constants C_1, C_2 are indeed finite since $(t+\tau)^2 \geq t^2+\tau^2$ and thus

$$C_1 \leq \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{r\,dr\,d\theta}{(1+r^2)^{\frac{N}{2}+s}} = \frac{\pi}{2N-4+4s}$$
$$C_2 \leq \int_{\mathbb{R}^N \setminus B_1(-e_N)} \frac{1}{|z+e_N|^{N+2s}} dz = \frac{N\omega_N}{2s}. \qquad \square$$

For our purpose a particular version of Corollary A.4 is needed:

Corollary A.5. *For any given $-1 < a < 1$ and $\frac{1}{2} < s \leq 1$ there is a constant $C >$ with the property, that if $u \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N > a\})$, $v \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N < a\})$ with $u\big|_{\mathcal{S}^{N-1}\cap\{x_N=a\}} = v\big|_{\mathcal{S}^{N-1}\cap\{x_N=a\}}$ then*

$$U(x) = \begin{cases} u(x) & \text{if } x \in \mathcal{S}^{N-1}, x_N > a \\ v(x) & \text{if } x \in \mathcal{S}^{N-1}, x_N < a \end{cases} \tag{A.8}$$

defines an element in $W^{s,2}(\mathcal{S}^{N-1})$ satisfying

$$[\![U]\!]_{s,\mathcal{S}^{N-1}} \leq C\left([\![u]\!]_{s,\mathcal{S}^{N-1}\cap\{x_N>a\}} + [\![v]\!]_{s,\mathcal{S}^{N-1}\cap\{x_N<a\}}\right). \tag{A.9}$$

Proof. We can apply Corollary A.4 locally using a partition of unity $\{\theta_i\}_{i=1}^L$ subordinate to a coordinated atlas $(U_i, \varphi_i)_{i=1,\dots,L}$. More detailed, we may choose a smooth atlas $(U_i, \varphi_i)_{i=1,\dots,L}$ with the additional property that every chart $\varphi_i : U_i \subset \mathcal{S}^{N-1} \to V_i \subset \mathbb{R}^{N-1}$ satisfies $\varphi_i(U_i \cap \{x_N \geq a\}) = V_i \cap \{y_{N-1} \geq a\}$. We may now apply Corollary A.4 to each pair $u|_{U_i} \circ \varphi_i^{-1}, v|_{U_i} \circ \varphi_i^{-1}$ and obtain functions $U_i \in W^{s,2}(V_i)$. Using a subordinated partition of unity $\{\theta_i\}_{i=1}^L$, the function $U(x) = \sum_{i=1}^L \theta_i(x) U_i \circ \varphi_i(x)$ agrees by construction with u on $S^+ = \mathcal{S}^{N-1} \cap \{x_N > a\}$ and with v on $S^- = \mathcal{S}^{N-1} \cap \{x_N < a\}$. Furthermore it satisfies for a constant $C > 0$

$$[\![U]\!]_{s,\mathcal{S}^{N-1}} \leq \|U\|_{W^{s,2}(\mathcal{S}^{N-1})} \leq C\left(\|u\|_{W^{s,2}(S^+)} + \|v\|_{W^{s,2}(S^-)}\right)$$

because every U_i does. To pass to the desired inequality (A.9) we proceed as follows: given u, v satisfying the assumption, we can apply the above construction to

$$\tilde{u} = u - \fint_{\partial S^+} u\big|_{\partial S^+}, \ \ \tilde{v} = v - \fint_{\partial S^-} v\big|_{\partial S^-},$$

because $\tilde{u}, \tilde{v}$ still satisfy the assumptions as a consequence of $u\big|_{\partial S^+} = v\big|_{\partial S^-}$. We obtain $\tilde{U}$ and U with $\tilde{U} = U - \fint_{\partial S^+} u\big|_{\partial S^+}$. We can now conclude by applying the Poincaré inequality, since

$$[\![\tilde{U}]\!]_{s,\mathcal{S}^{N-1}} = [\![U]\!]_{s,\mathcal{S}^{N-1}}$$
$$\|\tilde{u}\|_{W^{s,2}(S^+)} = \left\| u - \fint_{\partial S^+} u\big|_{\partial S^+} \right\|_{L^2(S^+)} + [\![u]\!]_{s,S^+} \leq C[\![u]\!]_{s,S^+}$$
$$\|\tilde{v}\|_{W^{s,2}(S^-)} = \left\| v - \fint_{\partial S^-} v\big|_{\partial S^+} \right\|_{L^2(S^-)} + [\![v]\!]_{s,S^-} \leq C[\![v]\!]_{s,S^-}. \qquad \square$$

A.2. Interpolation lemma for fractional Sobolev spaces

A classical result due to S. Luckhaus is concerned with the extension of a map that is defined on the boundary of an annulus $\partial(B_1 \setminus B_{1-\lambda})$ into the interior. We want to give an extension to fractional Sobolev spaces. In contrast to Luckhaus original result our version does not provide an L^∞ estimate.

Lemma A.6. *Let $\frac{1}{2} < s < 1$ there is a constant $C > 0$, $\alpha = \frac{N-1}{N+1} + \frac{1}{(s-\frac{1}{2})(1+\frac{1}{N})} > 0$ and a continuous function $\lambda \in C^0([0,1], \mathbb{R}_+)$ with $\lambda(0) = 0$ depending only on the dimension N and s such that the following holds: suppose $\delta^{(s-\frac{1}{2})(1+\frac{1}{N})} < \frac{1}{6}$ be given, let $u, v \in W^{s,2}(\mathcal{S}^{N-1}, \mathbb{R}^m)$ then there exists $\varphi \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathbb{R}^m)$ with the following properties*

$$\varphi(x) = \begin{cases} u(x) & \text{if } |x| = 1 \\ v\left(\frac{x}{1-\lambda}\right) & \text{if } |x| = 1-\lambda \end{cases} \tag{A.10}$$

$$\int_{B_1 \setminus B_{1-\lambda}} |D\varphi|^2 \leq C\delta \left([\![u]\!]^2_{s,\mathcal{S}^{N-1}} + [\![v]\!]^2_{s,\mathcal{S}^{N-1}} \right) + \frac{12}{\delta^\alpha} \int_{\mathcal{S}^{N-1}} |u-v|^2.$$

Remark A.7. One gets easily the version of Lemma A.6 for $s = 1$. In this case $\alpha = 1$, $\lambda(\delta) = \delta$ as follows: given $u, v \in W^{1,2}(\mathcal{S}^{N-1}, \mathbb{R}^m)$ we define the linear interpolation on the cylinder $\mathcal{S}^{N-1} \times [0, \delta]$

$$\phi(y,t) = \left(1 - \frac{t}{\delta}\right) u(y) + \frac{t}{\delta} v(y),$$

(compare claim 2 in the proof). Using polar coordinates we obtain the desired extension $\varphi(x) = \phi(y, 1-r) \in W^{1,2}(B_1 \setminus B_{1-\delta}, \mathbb{R}^m)$, with $r = |x|$, $y = \frac{x}{|x|}$. One checks that φ satisfies the right traces as in (A.10) and

$$\int_{B_1 \setminus B_{1-\lambda}} |D\varphi|^2 \leq 4 \int_{\mathcal{S}^{N-1} \times [0,\lambda]} |D\phi|^2 \leq 2\delta \int_{\mathcal{S}^{N-1}} |D_\tau u|^2 + |D_\tau v|^2 + \frac{4}{\delta} \int_{\mathcal{S}^{N-1}} |u-v|^2.$$

Proof. The proof is split into 2 parts.

(1) finding "good" extensions to functions $f \in W^{s,}(\mathcal{S}^{N-1}, \mathbb{R}^m)$ to $\mathcal{S}^{N-1} \times \mathbb{R}$,
(2) extend/interpolate between the extensions U, V of $u, v \in W^{s,}(\mathcal{S}^{N-1}, \mathbb{R}^m)$

Claim 1. There exists a dimensional constant $C = C(N,s) > 0$ such that for any $f \in W^{s,2}(\mathcal{S}^{N-1}, \mathbb{R}^m)$ there exists $F \in W^{1,2}(\mathcal{S}^{N-1} \times \mathbb{R}, \mathbb{R}^m)$ with $F(x,0) = f(x)$ for a.e. $x \in \mathcal{S}^{N-1}$ and

$$\begin{aligned} \|DF\|_{L^2(\mathcal{S}^{N-1} \times [0,\delta])} &\leq C\delta^{\frac{2s-1}{2N}} [\![f]\!]_{s,\mathcal{S}^{N-1}} \\ \left\| \frac{F(x,t) - f(x)}{t^{s+\frac{1}{2}}} \right\|_{L^2(\mathcal{S}^{N-1} \times \mathbb{R})} &\leq C [\![f]\!]_{s,\mathcal{S}^{N-1}}. \end{aligned} \tag{A.11}$$

Proof of Claim 1. Let us fix some notation. $P_S(x', x_N) = \frac{x'}{1+x_N}$, $P_N(x', x_N) = \frac{x'}{1-x_N}$ denote the stereographic projection from $\mathcal{S}^{N-1}$ to $\mathbb{R}^{N-1}$ with respect to the south and north pole respectively. Furthermore θ_1, θ_2 is a fixed smooth partition of unity on $\mathcal{S}^{N-1}$ with respect to the open sets $U_1 = \mathcal{S}^{N-1} \cap \{x_N > -\frac{1}{2}\}$, $U_2 = \mathcal{S}^{N-1} \cap \{x_N < \frac{1}{2}\}$. Finally $\eta \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ is a fixed non-negative cut-off function with $\eta(t) = 1$ for $t \leq \frac{1}{2}$ and $\eta(t) = 0$ for $t \geq \frac{3}{4}$.

Let $f \in H^s(\mathcal{S}^{N-1}, \mathbb{R}^m)$ be given. We define

$$f_1'(y) = \eta\left(-\langle e_N, P_S^{-1}(y)\right) f\left(P_S^{-1}(y)\right)$$
$$f_2'(y) = \eta\left(-\langle e_N, P_N^{-1}(y)\right) f\left(P_N^{-1}(y)\right).$$

So $f_i' \in H^s(\mathbb{R}^{N-1}, \mathbb{R}^m)$, $i = 1, 2$ with $\operatorname{supp}(f_i) \subset B_7(0)$. The choice of η and the partition of unity ensures that that $f(x) = \theta_1(x) f_1'(P_S(x)) + \theta_2(x) f_2'(P_N(x))$.

The trace operator $\circ|_{\{x_N=0\}}$ is a surjective map from $H^{s+\frac{1}{2}}(\mathbb{R}^N)$ to $H^s(\mathbb{R}^{N-1})$ for all $s > 0$, *e.g.* [17, Lemma 16.1]. Hence there exists $F_i' \in H^{s+\frac{1}{2}}(\mathbb{R}^N)$ with $F_i'|_{\{x_N=0\}} = f_i'$, $\operatorname{supp}(F_I') \subset B_8(0)$ satisfying

$$\left\| F_i' \right\|_{H^{s+\frac{1}{2}}(\mathbb{R}^N)} \leq C \left\| f_i' \right\|_{H^s(\mathbb{R}^{N-1})}$$

and so $DF_i' \in H^{s-\frac{1}{2}}(\mathbb{R}^N)$ and by Lemma A.1

$$\begin{aligned} \left\| DF_i' \right\|_{H^{s-\frac{1}{2}}(\mathbb{R}^N)} &\leq C \left\| f_i' \right\|_{H^s(\mathbb{R}^{N-1})}, \\ \left\| \frac{F_i'(x, x_N) - f_i'(x)}{|x_N|^{s+\frac{1}{2}}} \right\|_{L^2(\mathbb{R}^N)} &\leq C \left\| f_i' \right\|_{H^s(\mathbb{R}^{N-1})}. \end{aligned} \tag{A.12}$$

The assumption that $\operatorname{supp}(F_i') \subset B_8(0)$ is not restrictive since due to $\operatorname{supp}(f_i') \subset B_7(0)$ we may pass to $\theta' F_i'$ for a smooth cut-off function θ', with $\theta' = 0$ for $|x| \geq 8$ and $\theta' = 1$ for $|x| \leq 7$.

The Sobolev embedding theorem for fractional Sobolev spaces, $s - \frac{1}{2} < \frac{N}{2}$, [17, Lemma 32.1], states $H^{s-\frac{1}{2}}(\mathbb{R}^N) \subset L^{p(s)}(\mathbb{R}^N)$ with $\frac{1}{p(s)} = \frac{1}{2} - \frac{2s-1}{2N}$. By Hölders inequality we get for any $\delta > 0$

$$\begin{aligned} \left\| DF_i' \right\|_{L^2(B_8 \times [0,\delta])} &\leq |B_8 \times [0, \delta]|^{\frac{1}{2} - \frac{1}{p(s)}} \left\| DF_i' \right\|_{L^{p(s)}(B_8 \times [0,\delta])} \\ &\leq C\delta^{\frac{2s-1}{2N}} \left\| DF_i' \right\|_{H^{s-\frac{1}{2}}(\mathbb{R}^N)} \leq C\delta^{\frac{2s-1}{2N}} \left\| f_i' \right\|_{H^s(\mathbb{R}^{N-1})}. \end{aligned}$$

We define now an extension of f on $\mathcal{S}^{N-1} \times \mathbb{R}$ by

$$F(x, t) = \theta_1(x) F_1'(P_S(x), t) + \theta_1(x) F_2'(P_N(x), t).$$

The stereographic projections are uniformly Lipschitz, on the open sets U_1 and U_2 respectively so we have $F \in H^{s+\frac{1}{2}}(\mathcal{S}^{N-1} \times \mathbb{R})$, $F|_{t=0} = f$ satisfying

$$\begin{aligned}\|DF\|_{L^2(\mathcal{S}^{N-1}\times[0,\delta])} &\leq C \sum_{i=1,2} \|DF_i'\|_{L^2(B_8\times[0,\delta])} + \|F_i'\|_{L^2(B_8\times[0,\delta])} \\ &\leq C \sum_{i=1,2} \|DF_i'\|_{L^2(B_8\times[0,\delta])} \leq C \sum_{i=1,2} \delta^{\frac{2s-1}{2N}} \|f_i'\|_{H^s(\mathbb{R}^{N-1})} \\ &\leq C\delta^{\frac{2s-1}{2N}} \left(\llbracket f \rrbracket_{s,\mathcal{S}^{N-1}} + \|f\|_{L^2(\mathcal{S}^{N-1})}\right)\end{aligned}$$

where we used Cauchy's inequality to estimate the L^2 norms of F_i, that are compactly supported, and the just established bounds.

Similarly we get

$$\begin{aligned}\left\|\frac{F(x,t)-f(x)}{|t|^{s+\frac{1}{2}}}\right\|_{L^2(\mathcal{S}^{N-1}\times\mathbb{R})} &\leq C \sum_{i=1,2} \left\|\frac{F_i'(x,x_N)-f_i'(x)}{|x_N|^{s+\frac{1}{2}}}\right\|_{L^2(\mathbb{R}^N)} \\ &\leq C\left(\llbracket f \rrbracket_{s,\mathcal{S}^{N-1}} + \|f\|_{L^2(\mathcal{S}^{N-1})}\right).\end{aligned}$$

Finally the L^2 term can be absorbed by the following trick: let f be given as assumed and $m(f) = \fint_{\mathcal{S}^{N-1}} f$ its mean. $f - m(f)$ is still admissible, $\llbracket f \rrbracket_{s,\mathcal{S}^{N-1}} = \llbracket f - m(f) \rrbracket_{s,\mathcal{S}^{N-1}}$, and so by Cauchy's inequality $\|f - m(f)\|_{L^2(\mathcal{S}^{N-1})} \leq C\llbracket f \rrbracket_{s,\mathcal{S}^{N-1}}$. If $\tilde{F}$ is the just constructed extension to $f - m(f)$ then $F = \tilde{F} + m(f)$ is an admissible extension for f and it satisfies the claimed bounds:

$$\begin{aligned}\|DF\|_{L^2(\mathcal{S}^{N-1}\times[0,\delta])} &= \left\|D\tilde{F}\right\|_{L^2(\mathcal{S}^{N-1}\times[0,\delta])} \\ &\leq C\delta^{\frac{2s-1}{2N}}\left(\llbracket f - m(f) \rrbracket_{s,\mathcal{S}^{N-1}} + \|f - m(f)\|_{L^2(\mathcal{S}^{N-1})}\right) \\ &= C\delta^{\frac{2s-1}{2N}} \llbracket f \rrbracket_{s,\mathcal{S}^{N-1}}\end{aligned}$$

$$\left\|\frac{F(x,t)-f(x)}{|t|^{s+\frac{1}{2}}}\right\|_{L^2(\mathcal{S}^{N-1}\times\mathbb{R})} = \left\|\frac{\tilde{F}(x,t)-\tilde{f}(x)}{|t|^{s+\frac{1}{2}}}\right\|_{L^2(\mathcal{S}^{N-1}\times\mathbb{R})} \leq C\llbracket f \rrbracket_{s,\mathcal{S}^{N-1}}.$$

Claim 2. There exists an interpolation as stated.

Proof of Claim 2. Let $u, v \in W^{s,2}(\mathcal{S}^{N-1}, \mathbb{R}^n)$ be given as assumed. We fix two extensions $U, V \in W^{1,2}(\mathcal{S}^{N-1} \times \mathbb{R}, \mathbb{R}^n)$ with the properties stated in claim 1.

Recall that for any nonnegative function $g \in L^1(\Omega)$, $\Omega \subset \mathbb{R}^m$, Chebychef's inequality states $|\{x \in \Omega\colon g(x) > \lambda\}| \leq \frac{1}{\lambda}\int_\Omega g$. Hence the choice $\lambda = \frac{C}{|\Omega|}\int_\Omega g$ gives that $g(x) \leq \frac{C}{|\Omega|}\int_\Omega g$ up to a set of measure $\frac{\Omega}{C}$.

We may apply this argument to the case $C = 5$, $\Omega = [0,\delta]$ and $g_1(t) = \int_{\mathcal{S}^{N-1}\times\{t\}} |DU|^2$, $g_2(t) = \int_{\mathcal{S}^{N-1}\times\{t\}} \frac{|U(x,t)-u(x)|^2}{|t|^{2s+1}}$, g_3, g_4 equivalently V, v replacing U, u. Hence there exists $t_0 \in]0,\delta[$ satisfying

$$g_i(t_0) \leq \frac{5}{\delta}\int_0^\delta g_i(t)dt \text{ for } 1 \leq i \leq 4$$

and $U(x,t_0), V(x,t_0) \in W^{1,2}(\mathcal{S}^{N-1},\mathbb{R}^n)$. Furthermore let us fix $\mu > 0$, determined later. We partition the interval $[0, 2\delta+\mu]$ by $a_0 = 0, a_1 = t_0, a_2 = \delta, a_3 = \delta+\mu, a_4 = 2\delta+\mu-t_0, a_5 = 2\delta+\mu$. We define the extension ϕ separately on the "intervals" $\Omega_i = \mathcal{S}^{N-1}\times[a_{i-1},a_i]$, $1\le i\le 5$ and we will write $\phi_i = \phi|_{\Omega_i}$.

Let $\phi_1(x,t) = U(x,t)$, *i.e.*, $\phi_1(x,0) = u(x)$ for a.e. $x\in\mathcal{S}^{N-1}$, similiar we set $\phi_5(x,t) = V(x,a_5-t)$, *i.e.*, $\phi_5(x,a_5) = v(x)$ for a.e. $x\in\mathcal{S}^{N-1}$. Using (A.11) we have

$$\int_{\Omega_1}|D\phi_1|^2 = \int_{\Omega_1}|DU|^2 \le C t_0^{\frac{2s-1}{N}} \llbracket u \rrbracket^2_{s,\mathcal{S}^{N-1}},$$

similarly $\int_{\Omega_5}|D\phi_5|^2 \le C t_0^{\frac{2s-1}{N}} \llbracket v \rrbracket^2_{s,\mathcal{S}^{N-1}}$. For $\phi_2(x,t) = U(x,t_0)$ and $\phi_4(x,t) = V(x,t_0)$ the particular choice of t_0 gives

$$\int_{\Omega_2}|D\phi_2|^2 \le \frac{\delta-t_0}{\delta}\int_{\mathcal{S}^{N-1}\times[0,\delta]}|DU|^2 \le C\delta^{\frac{2s-1}{N}}\llbracket u \rrbracket^2_{s,\mathcal{S}^{N-1}},$$

equivalently $\int_{\Omega_2}|D\phi_2|^2 \le C\delta^{\frac{2s-1}{N}}\llbracket v \rrbracket^2_{s,\mathcal{S}^{N-1}}$. Finally we can interpolate linearly between $U(x,t_0)$ and $V(x,t_0)$ on Ω_3, *i.e.*, $\phi_3(x,t) = \left(1-\frac{t-a_3}{\mu}\right)U(x,t_0) + \frac{t-a_3}{\mu}V(x,t_0)$ with

$$\int_{\Omega_3}|D\phi_3|^2 \le \frac{\mu}{2}\int_{\mathcal{S}^{N-1}\times\{t_0\}}|DU|^2+|DV|^2 + \frac{1}{\mu}\int_{\mathcal{S}^{N-1}\times\{t_0\}}|U-V|^2.$$

The first integral is estimated as before by

$$I_1 \le \frac{\mu}{2\delta}\int_{\mathcal{S}^{N-1}\times[0,\delta]}|DU|^2+|DV|^2 \le C\frac{\mu}{\delta}\delta^{\frac{2s-1}{N}}\left(\llbracket u \rrbracket^2_{s,\mathcal{S}^{N-1}} + \llbracket v \rrbracket^2_{s,\mathcal{S}^{N-1}}\right).$$

We use the second estimate in (A.11) for the second integral and obtain

$$\begin{aligned}\frac{1}{\mu}\int_{\mathcal{S}^{N-1}\times\{t_0\}}|U-V|^2 &\le \frac{3}{\mu}t_0^{2s+1}\int_{\mathcal{S}^{N-1}\times\{t_0\}}\frac{|U-u|^2}{t_0^{2s+1}}+\frac{|V-v|^2}{t_0^{2s+1}} + \frac{3}{\mu}\int_{\mathcal{S}^{N-1}}|u-v|^2\\ &\le \frac{3}{\mu\delta}t_0^{2s+1}\int_{\mathcal{S}^{N-1}\times[0,\delta]}\frac{|U-u|^2}{t^{2s+1}}+\frac{|V-v|^2}{t^{2s+1}} + \frac{3}{\mu}\int_{\mathcal{S}^{N-1}}|u-v|^2\\ &\le C\frac{t_0^{2s+1}}{\mu\delta}\left(\llbracket u \rrbracket^2_{s,\mathcal{S}^{N-1}} + \llbracket v \rrbracket^2_{s,\mathcal{S}^{N-1}}\right) + \frac{3}{\mu}\int_{\mathcal{S}^{N-1}}|u-v|^2.\end{aligned}$$

$\mu\delta^{\frac{2s-1}{N}}$ is equal to $\frac{\delta^{2s+1}}{\mu}$ for the choice $\mu = \delta^{s+\frac{1}{2}-\frac{2s-1}{2N}}$. Since $\frac{2s-1}{N} \le (s-\frac{1}{2})(1+\frac{1}{N})$ for $s\ge\frac{1}{2}$ we have

$$\delta_0 = \delta^{(s-\frac{1}{2})(1+\frac{1}{N})} = \frac{\delta^{2s+1}}{\mu\delta} \ge \frac{\mu\delta^{\frac{2s-1}{N}}}{\delta}$$

and $\mu = \delta_0^\alpha, \alpha = \frac{N-1}{N+1} + \frac{1}{(s-\frac{1}{2})(1+\frac{1}{N})}$. If we set $\lambda(\delta_0) = 2\delta + \mu = a_5$ and collect all estimates we find

$$\int_{\mathcal{S}^{N-1}\times[0,a_5]} |D\phi|^2 \leq C\delta_0 \left([\![u]\!]_{s,\mathcal{S}^{N-1}}^2 + [\![v]\!]_{s,\mathcal{S}^{N-1}}^2 \right) + \frac{3}{\delta_0^\alpha} \int_{\mathcal{S}^{N-1}} |u-v|^2.$$

Finally define $\varphi(x) = \phi(y, 1-r) \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathbb{R}^m)$, with $r = |x|$, $y = \frac{x}{|x|}$ One checks that φ satisfies (A.10) and

$$\int_{B_1\setminus B_{1-\lambda}} |D\varphi|^2 \leq 4 \int_{\mathcal{S}^{N-1}\times[0,\lambda]} |D\phi|^2. \qquad \square$$

B. Q-valued functions

B.1. Fractional Sobolev spaces for Q-valued functions

As before we restrict ourself to $0 < s \leq 1$. $\mathcal{A}_Q(\mathbb{R}^n)$ fails to be a linear space, so $L^2(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is not a Banach space. Hence we are not in a setting for classical interpolation methods. Nonetheless there are two ways to define $W^{s,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ in a natural way:

(a) using Almgren's bilipschitz embedding $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^m$, Theorem 1.3,

$$W^{s,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) = \left\{ u \in L^2(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \colon \boldsymbol{\xi} \circ u \in W^{s,2}(\Omega, \mathbb{R}^m) \right\};$$

(b) using the Gagliardo norm

$$W^{s,2}(\Omega) = \left\{ u \in L^2(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \colon [\![u]\!]_{s,\Omega}^2 = \int_{\Omega\times\Omega} \frac{\mathcal{G}(u(x), u(y))^2}{|x-y|^{N+2s}} d(x,y) < \infty \right\}.$$

The equivalence of both definitions follows from the bilipschitz property of $\boldsymbol{\xi}$, *i.e.*, $c|\boldsymbol{\xi}\circ u(x) - \boldsymbol{\xi}\circ u(y)| \leq \mathcal{G}(u(x), u(y)) \leq |\boldsymbol{\xi}\circ u(x) - \boldsymbol{\xi}\circ u(y)|$ for some $c = c(n, Q)$. This implies

$$c [\![\boldsymbol{\xi} \circ u]\!]_{s,\Omega}^2 \leq [\![u]\!]_{s,\Omega}^2 \leq [\![\boldsymbol{\xi} \circ u]\!]_{s,\Omega}^2. \tag{B.1}$$

We saw that all definitions of $W^{s,2}(\Omega, \mathbb{R}^m)$ are equivalent in case of a bounded Lipschitz regular domain $\Omega \subset \mathbb{R}^N$.

Combining the definition of $W^{s,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ as suggested in (a) with (B.1) nearly all the statements for single valued functions pass over to the Q-valued setting. For the sake of completeness we state two of them for Q-valued functions:

Corollary B.1. *To any given $-1 < a < 1$ and $\frac{1}{2} < s \leq 1$ there is a constant $C >$ with the property that, if $u \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N > a\}, \mathcal{A}_Q(\mathbb{R}^n))$, $v \in W^{s,2}(\mathcal{S}^{N-1} \cap \{x_N < a\}, \mathcal{A}_Q(\mathbb{R}^n))$ with $u\big|_{\mathcal{S}^{N-1}\cap\{x_N=a\}} = v\big|_{\mathcal{S}^{N-1}\cap\{x_N=a\}}$, then*

$$U(x) = \begin{cases} u(x) & \text{if } x \in \mathcal{S}^{N-1}, x_N > a \\ v(x) & \text{if } x \in \mathcal{S}^{N-1}, x_N < a \end{cases} \tag{B.2}$$

defines an element in $W^{s,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$ *satisfying*

$$\llbracket U \rrbracket_{s,\mathcal{S}^{N-1}} \le C \left(\llbracket u \rrbracket_{s,\mathcal{S}^{N-1}\cap\{x_N>a\}} + \llbracket v \rrbracket_{s,\mathcal{S}^{N-1}\cap\{x_N<a\}} \right). \tag{B.3}$$

Lemma B.2. *Let* $\frac{1}{2} < s < 1$ *there is a constant* $C > 0, \alpha = \frac{N-1}{N+1} + \frac{1}{(s-\frac{1}{2})(1+\frac{1}{N})} > 0$ *and a continuous function* $\lambda \in C^0([0,1], \mathbb{R}_+)$ *with* $\lambda(0) = 0$ *depending only on the dimension* N,*s and* Q *such that the following holds: suppose* $\delta^{(s-\frac{1}{2})(1+\frac{1}{N})} < \frac{1}{6}$ *be given, let* $u, v \in W^{s,2}(\mathcal{S}^{N-1}, \mathcal{A}_Q(\mathbb{R}^n))$ *then there exists* $\varphi \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^n))$ *with the following properties*

$$\varphi(x) = \begin{cases} u(x) & \text{if } |x| = 1 \\ v\left(\frac{x}{1-\lambda}\right) & \text{if } |x| = 1-\lambda \end{cases} \tag{B.4}$$

$$\int_{B_1\setminus B_{1-\lambda}} |D\varphi|^2 \le C\delta \left(\llbracket u \rrbracket^2_{s,\mathcal{S}^{N-1}} + \llbracket v \rrbracket^2_{s,\mathcal{S}^{N-1}} \right) + \frac{C}{\delta^\alpha} \int_{\mathcal{S}^{N-1}} |u-v|^2.$$

Proof. First apply Lemma A.6 to $\boldsymbol{\xi} \circ u, \boldsymbol{\xi} \circ v$. We obtain $\tilde{\varphi} \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathbb{R}^m)$. The retraction $\varphi = \boldsymbol{\rho} \circ \tilde{w} \in W^{1,2}(A_{1,R}, \mathcal{A}_Q(\mathbb{R}^n))$ has the desired properties, since the energy estimate changes only by a constant depending on n, Q. □

Remark B.3. With the same argument Lemma B.2 holds as well for $s = 1$ with $\alpha = 1, \lambda(\delta) = \delta$, using instead of Lemma A.6 the Remark A.7.

B.2. Concentration compactness for Q-valued functions

Let $\Omega \subset \mathbb{R}^N$ be given, then there is a concentration compactness lemma for sequences $u(k) \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ with uniformly bounded energy.

Lemma B.4. *Given a sequence* $u(k) \in W^{1,2}(\Omega, \mathcal{A}_Q(R^n))$ *with*

$$\limsup_{k\to\infty} \int_\Omega |Du(k)|^2 \le \infty$$

for a subsequence, not relabelled, we can find:

(i) *functions* $b_l \in W^{1,2}(\Omega, \mathcal{A}_{Q_l}(\mathbb{R}^n))$ *for* $l = 1, \ldots, J$, $\sum_{l=1}^L Q_l = Q$;
(ii) *a sequence of points* $t_l(k) \in \mathbb{R}^n$, $l = 1, \ldots, J$ *with* $\limsup_{k\to\infty} |t_l(k) - t_m(k)| = +\infty$ *for* $l \neq m$ *and* $\mathcal{G}(u(k), b(k)) \to 0$ *in* L^2 *for the "travelling sheets"* $b(k) = \sum_{l=1}^L (b_l \oplus t_l(k))$.

Moreover, the following two additional properties hold:

(a) *if* $\Omega' \subset \Omega$ *is open and* E_k *is a sequence of measurable sets with* $|E_k| \to 0$, *then*

$$\liminf_{k\to\infty} \int_{\Omega'\setminus E_k} |Du(k)|^2 - \int_{\Omega'} |Db(k)|^2 \geq 0.$$

(b) $$\limsup_{k\to\infty} \int_{\Omega} (|Du(k)| - |Db(k)|)^2 \leq \limsup_{k\to\infty} \int_{\Omega} |Du(k)|^2 - |Db(k)|^2.$$

Before we give the proof we recall the definition of the separation $\operatorname{sep}(T)$ of a Q-point $T = \sum_{i=1}^{Q} [\![t_i]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$.

$$\operatorname{sep}(T) = \begin{cases} 0 & \text{if } T = Q[\![t]\!] \\ \min_{t_i \neq t_j} |t_i - t_j| & \text{otherwise .} \end{cases}$$

The following results are of essential use in the context of the separation and needed for the proof of the concentration compactness lemma. The first gives a kind of relation between $\operatorname{diam}(\operatorname{spt}(T))$ and $\operatorname{sep}(T)$, see [7, Lemma 3.8]; the second gives a retraction $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_T$ based on $\operatorname{sep}(T)$, see [7, Lemma 3.7]

Lemma B.5. *For every* $\epsilon > 0$ *there exists* $\beta = \beta(\epsilon, Q) > 0$ *with the property that to any* $T \in \mathcal{A}_Q(\mathbb{R}^n)$ *there exists* $S = S(T) \in \mathcal{A}_Q(\mathbb{R}^n)$ *with*

$$\operatorname{spt}(S) \subset \operatorname{spt}(T), \quad \mathcal{G}(T, S) < \epsilon \operatorname{sep}(S) \text{ and } \beta \operatorname{diam}(\operatorname{spt}(T)) < \operatorname{sep}(S).$$

Lemma B.6. *For a given* $T \in \mathcal{A}_Q(\mathbb{R}^n)$ *and* $0 < 4s < \operatorname{sep}(T)$ *there exists a* 1-*Lipschitz retraction*

$$\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_T : \mathcal{A}_Q(\mathbb{R}^n) \to \overline{B_s(T)} = \{S \in \mathcal{A}_Q(T) \colon \mathcal{G}(S, T) \leq s\}$$

with the property that

(i) $\boldsymbol{\vartheta}(S) = S$ *if* $\mathcal{G}(S, T) \leq s$;
(ii) $\mathcal{G}(\boldsymbol{\vartheta}(S_1), \boldsymbol{\vartheta}(S_2)) < \mathcal{G}(S_1, S_2)$ *if* $\mathcal{G}(S_1, T) > s$.

Proof of Lemma B.4. By the generalised Poincaré inequality [7, Proposition 2.12] we can pick a sequence of means $T(k) \in \mathcal{A}_Q(\mathbb{R}^n)$ satisfying $\int_\Omega \mathcal{G}(u(k), T(k))^2 \leq C \int_\Omega |Du(k)|^2$. Now we distinguish two cases depending on these $T(k)$. The second will be handled by induction on the first.

Case 1 *and basis of the induction:* $\liminf_{k\to\infty} \operatorname{diam}(\operatorname{spt}(T(k))) < \infty$ ($\operatorname{diam}(\operatorname{spt}(T(k))) = 0$ for $Q = 1$):

Passing to an appropriate subsequence, not relabelled, $\operatorname{diam}(\operatorname{spt}(T(k))) < C$ for all k. Set $L = 1$, and as splitting keep the sequence itself, *i.e.*, $T(k) = T_1(k)$. For every k fix a $t_1(k) \in \operatorname{spt}(T(k))$.

Hence we have

$$\begin{aligned}
&\limsup_k \int_\Omega |u(k) \oplus (-t_1(k))|^2 \\
&= \limsup_k \int_\Omega \mathcal{G}(u(k), Q[\![t_1(k)]\!])^2 \\
&\leq \limsup_k 2 \int_\Omega \mathcal{G}(u(k), T(k))^2 + 2|\Omega|\mathcal{G}(T(k), Q[\![t_1(k)]\!])^2 < \infty.
\end{aligned}$$

Passing to an appropriate subsequence there is $b = b_1 \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ with $u(k) \oplus (-t_1(k)) \to b$ in L^2. This proves (i), (ii) since $\mathcal{G}(u(k) \oplus -t_1(k), b) = \mathcal{G}(u(k), b \oplus t_1(k)) = \mathcal{G}(u(k), b(k))$. Furthermore, the established properties imply that $\boldsymbol{\xi} \circ u(k) \rightharpoonup \boldsymbol{\xi} \circ b(k)$ in $W^{1,2}(\Omega, \mathbb{R}^m)$. The additional property (a) follows, because $\mathbf{1}_{\Omega' \setminus A_k} \to \mathbf{1}_{\Omega'}$ in $L^2(\Omega)$ and so $\mathbf{1}_{\Omega' \setminus A_k} D\boldsymbol{\xi} \circ u(k) \rightharpoonup \mathbf{1}_{\Omega'} D\boldsymbol{\xi} \circ b(k)$. The property (b) is a further consequence of $L^2(\Omega)$ being a Hilbert space. We have seen that $f_k = D\boldsymbol{\xi} \circ u(k) \rightharpoonup f = D\boldsymbol{\xi} \circ b(k)$ weakly in $L^2(\Omega)$ and so passing to the limit in

$$\| |f_k| - |f| \|^2_{L^2(\Omega)} \leq \|f_k - f\|^2_{L^2(\Omega)} = \|f_k\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} - 2\langle f_k - f, f\rangle_{L^2(\Omega)},$$

gives the desired inequality.

Case 2 *and the induction step:* $\liminf_k \operatorname{diam}(\operatorname{spt}(T(k))) = +\infty$

Suppose the lemma holds for $Q' < Q$. For each $T(k)$ we pick $S(k) \in \mathcal{A}_Q(\mathbb{R}^n)$. According to Lemma B.5, $\epsilon = \frac{1}{10}$, *i.e.*, let $\sigma_k = \operatorname{sep}(S(k))$, $S(k) = \sum_{j=1}^{J(k)} Q_j(k)[\![s_j(k)]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$, then $\beta(\frac{1}{10}, Q) \operatorname{diam}(\operatorname{spt}(T(k))) < \sigma_k$ and $\mathcal{G}(T(k), S(k)) < \frac{\sigma_k}{10}$. Passing to an appropriate subsequence, not relabelled, we may further assume that $J(k) > 1$ and $Q_j(k)$ do not depend on k. Fix the associated 1-Lipschitz retractions of B.6 $\boldsymbol{\vartheta}_k : \mathcal{A}_Q(\mathbb{R}^N) \to \overline{B_{\frac{1}{5}s(S(k))}(S(k))}$, *i.e.*, $\mathcal{H}^0\Big(\operatorname{spt}(\boldsymbol{\vartheta}_k(T)) \cap B_{\frac{\sigma_k}{5}(s_j)}\Big) = Q_j$ for all $T \in \mathcal{A}_Q(\mathbb{R}^n)$ and $j = 1, \dots, J$. These retractions $\boldsymbol{\vartheta}_k$ define new sequences $v_j(k)$ in $W^{1,2}(\Omega, \mathcal{A}_{Q_j}(\mathbb{R}^n))$ by $\boldsymbol{\vartheta}_k \circ u(k) = v_1(k) + \cdots v_J(k)$ with $v_j(k) \in B_{\frac{\sigma_k}{5}}(s_j)$

Each sequence $v_j(k)$, $j = 1, \dots, J$ satisfies itself the assumptions of the lemma, because $\sum_{j=1}^J |Dv_j(k)|^2 = |D\boldsymbol{\vartheta}_k \circ u(k)|^2 \leq |Du(k)|^2$ a.e. as a consequence of $\boldsymbol{\vartheta}_k$ being a retraction. Furthermore we record some properties: defining $A_k = \{x : \boldsymbol{\vartheta}_k \circ u(k)(x) \neq u(k)(x)\} = \{x : \mathcal{G}(u(k), S(k)) > \frac{\sigma_k}{5}\} \subset \{x : \mathcal{G}(u(k), T(k)) \geq \frac{\sigma_k}{10}\} = B_k$ (subsets of Ω) we have

(1) $|B_k| \to 0$ as $k \to \infty$, because by the generalised Poincaré inequality

$$\begin{aligned}
|B_k| &\leq \left(\frac{10}{\sigma_k}\right)^{2^*} \int_{B_k} \mathcal{G}(u(k), T(k))^{2^*} \\
&\leq \left(\frac{10}{\sigma_k}\right)^{2^*} C \left(\int_\Omega |Du(k)|^2\right)^{\frac{2^*}{2}} \to 0;
\end{aligned}$$

(2) $\mathcal{G}(u(k), \boldsymbol{\vartheta}_k \circ u(k)) \to 0$ in L^2 as $k \to \infty$, since

$$\begin{aligned}\int_\Omega \mathcal{G}(u(k), \boldsymbol{\vartheta}_k \circ u(k))^2 &= \int_{A_k} \mathcal{G}(u(k), \boldsymbol{\vartheta}_k \circ u(k))^2 \\ &\le 2\int_{B_k} \mathcal{G}(v_k, T(k))^2 + \mathcal{G}(\boldsymbol{\vartheta}_k \circ u(k), \boldsymbol{\vartheta}_k \circ T(k))^2 \\ &\le 4\left(\frac{10}{\sigma_k}\right)^{2^*-2} \int_{B_k} \mathcal{G}(u(k), T(k))^{2^*} \\ &\le \frac{C}{\sigma_k^{2^*-2}} \left(\int_\Omega |Du(k)|^2\right)^{\frac{2^*}{2}} \to 0;\end{aligned}$$

(3) $\limsup_{k\to\infty} \operatorname{dist}(\operatorname{spt}(v_i(x)), \operatorname{spt}(v_j(x))) = +\infty$, since $v_j \in v_j(k) \in B_{\frac{\sigma_k}{5}}(s_j)$ and so $\operatorname{dist}(\operatorname{spt}(v_i(x)), \operatorname{spt}(v_j(x))) \ge |s_i - s_j| - 2\frac{\sigma_k}{5} = \frac{3}{5}\sigma_k$.

Due to the induction hypothesis the lemma holds for each sequence $v_j(k)$, *i.e.*, we can find $b_{j,l} \in W^{1,2}(\Omega, \mathcal{A}_{Q_{j,l}}(\mathbb{R}^n))$, with $\sum_{l=1}^{L_j} Q_{j,l} = Q_j$, sequences of points $t_{j,l}(k) \in B_{\frac{\sigma_k}{5}}(s_j)$ satisfying the conditions (i), (ii). Furthermore the additional properties (a),(b) hold.

We claim that these sequences work as well for $u(k)$. We relabel by setting $L = \sum_{j=1}^J L_j$, $K_j = \sum_{i=1}^{j-1} L_i$ and $b_{K_j+l} = b_{j,l}$, $t_{K_j+l}(k) = t_{j,l}(k)$ and $Q_{K_j+l} = Q_{j,l}$ for $j \in \{1, \dots, J\}$ and $l \in \{1, \dots, L_j\}$. Property (ii) holds because $|t_l(k) - t_m(k)| \ge \frac{3}{5}\sigma_k$ if $l \le K_j < m$ for some j and $|t_l(k) - t_k(m)| \to \infty$ by induction hypothesis if $K_{j-1} < l < m \le K_j$. Furthermore

$$\begin{aligned}\mathcal{G}(u(k), b(k)) &\le \mathcal{G}(u(k), v(k)) + \mathcal{G}(v(k), b(k)) \\ &\le \mathcal{G}(u(k), v(k)) + \sum_{j=1}^{L} \mathcal{G}(v_j(k), b_j(k))\end{aligned}$$

where $v(k) = \sum_{j=1}^L v_j(k)$, $b(k) = \sum_{j=1}^L b_j(k)$ and $b_j(k) = \sum_{l=K_{j-1}+1}^{K_j} (b_l \oplus t_l(k))$ for each j. $\mathcal{G}(u(k), v(k)) \to 0$ in L^2 as seen before and $\mathcal{G}(v_j(k), b_j(k)) \to 0$ in L^2 once again by induction hypothesis for all j. Moreover the additional property (a) holds because if $|E_k| \to 0$ so does $|E_k \cup B_k| \to 0$ and $|Du(k)| = |Dv(k)|$ on $\Omega \setminus B_k$. So

$$\lim_{k\to\infty} \int_{\Omega\setminus E_k} |Du(k)|^2 - \int_\Omega |Db(k)|^2 \ge \lim_{k\to\infty} \int_{\Omega\setminus E_k \cup B_k} |Dv(k)|^2 - \int_\Omega |Db(k)|^2 \ge 0.$$

To check the additional property (b) we may pass firstly to a further subsequence such that all lim sup's are actually limits. We use again the fact that $Du(k) = Dv(k)$

on $\Omega \setminus B_k$ and so

$$\int_\Omega (|Du(k)| - |Db(k)|)^2$$
$$= \int_\Omega (|Dv(k)| - |Db(k)|)^2 + \int_{B_k} |Du(k)|^2 - |Dv(k)|^2 - 2(|Du(k)| - |Dv(k)|)|Db(k)|.$$

The $v(k)'s$ were obtained by retraction so $|Du(k)| \geq |Dv(k)|$ a.e. hence the last term is negative and using induction hypothesis the claim follows

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega (|Du(k)| - |Db(k)|)^2 \\
&\leq \lim_{k\to\infty} \int_\Omega (|Dv(k)| - |Db(k)|)^2 + \int_{B_k} |Du(k)|^2 - |Dv(k)|^2 \\
&\leq \lim_{k\to\infty} \int_\Omega |Dv(k)|^2 - |Db(k)|^2 + \int_{B_k} |Du(k)|^2 - |Dv(k)|^2 \\
&= \lim_{k\to\infty} \int_\Omega |Du(k)|^2 - |Db(k)|^2.
\end{aligned}$$

□

B.3. $W^{s,p}$-selection for $s > \frac{1}{p}$

The proof of this lemma is due to Camillo De Lellis, but has not been published so far.

Lemma B.7. *Let $s > \frac{1}{p}$, $Q \in \mathbb{N}$ be given, then for $u \in W^{s,p}([0,1], \mathcal{A}_Q(\mathbb{R}^n))$ we can find $v = (v_1, \ldots, v_Q) : [0,1] \to (\mathbb{R}^n)^Q$ with the property that*

(i) $$[v(t)] = \sum_{i=1}^{Q} [\![v_i(t)]\!] = u(t) \text{ for all } t \in [0,1];$$

(ii) $v \in W^{s',p}([0,1], (\mathbb{R}^n)^Q)$ *for any $s' < s$,* i.e.*, there is a positive constant C depending on Q and p, s, s' so that*

$$\int_{[0,1]\times[0,1]} \frac{|v(x) - v(y)|^p}{|x-y|^{1+ps'}} d(x,y) \leq C \int_{[0,1]\times[0,1]} \frac{\mathcal{G}(u(x), u(y))^p}{|x-y|^{1+ps}} d(x,y).$$

Proof. The lemma is a consequence of the results on regular selections of multivalued functions, [5, Theorem 1.1], and the following estimate

$$\begin{aligned}
&\int_{0\leq x\leq y\leq 1} \frac{\max_{\sigma,\tau\in[x,y]} |f(\sigma) - f(\tau)|^p}{|x-y|^{1+ps'}} \, d(x,y) \\
&\leq C \int_{0\leq\sigma\leq\tau\leq 1} \frac{|f(\sigma) - f(\tau)|^p}{|\sigma-\tau|^{1+ps}} \, d(\sigma,\tau)
\end{aligned} \tag{B.5}$$

for a constant C depending only on $p, s' < s$.

We start with proving (B.5). $W^{s,p}([0,1]) \subset C^{0,s-\frac{1}{p}}([0,1])$ for $ps > 1$, *i.e.*, for any $\sigma, \tau \in [0,1]$

$$|f(\sigma) - f(\tau)| \leq C \llbracket f \rrbracket_{s,p,[0,1]} \tag{B.6}$$

where we used the abbreviation $\llbracket f \rrbracket^p_{s,p,[a,b]} = \int_{[a,b]\times[a,b]} \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\, d(x,y)$. (B.6) is classical and can for example be deduced from Lemma 3.2. To do so extend f to $\tilde{f} \in W^{s,p}([-1,3],\mathbb{R}^n)$ by

$$\tilde{f} = \begin{cases} f(-t) & \text{if } -1 < t < 0 \\ f(t) & \text{if } 0 < t < 1 \\ f(1-t) & \text{if } 1 < t < 2. \end{cases}$$

The means $\tilde{f}(x,r) = ⨍_{x-r}^{x+r} \tilde{f}$ are well-defined for all $x \in [0,1]$ and $r < 1$. (B.6) for $\tilde{f}$ in the case of $p = 2$ agrees with (3.3) in Lemma 3.2 since (3.2) is satisfied with $\beta = \frac{1}{2}$; for general p the calculations have to be adapted classically. We conclude: for all $\sigma, \tau \in [0,1]$

$$|f(\sigma) - f(\tau)| = |\tilde{f}(\sigma) - \tilde{f}(\tau)| \leq C \llbracket \tilde{f} \rrbracket_{s,p,[-1,2]} \leq C \llbracket f \rrbracket_{s,p,[0,1]}.$$

For any $f \in W^{s,p}([a,b],\mathbb{R}^n)$ we may apply (B.6) to the rescaled function $f_{a,\rho}(t) = f(a+\rho t)$ with $\rho = b - a$:

$$\begin{aligned} \max_{x,y\in[a,b]} |f(x) - f(y)| &= \max_{\sigma,\tau\in[0,1]} |f_{a,\rho}(\sigma) - f_{a,\rho}(\tau)| \leq C \llbracket f_{a,\rho} \rrbracket_{s,p,[0,1]} \\ &= C\rho^{s-\frac{1}{p}} \llbracket f \rrbracket_{s,p,[a,b]} = C(b-a)^{s-\frac{1}{p}} \llbracket f \rrbracket_{s,p,[a,b]}. \end{aligned}$$

Inserting this in the left hand side of (B.5) gives

$$\begin{aligned} &\int_{0\leq x\leq y\leq 1} \frac{\max_{\sigma,\tau\in[x,y]} |f(\sigma) - f(\tau)|^p}{|x-y|^{1+ps'}}\, d(x,y) \\ &\leq C \int_{0\leq x\leq y\leq 1} \frac{(y-x)^{ps-1}}{(y-x)^{1+ps}} \int_{x\leq\sigma\leq\tau\leq 1} \frac{|f(\sigma) - f(\tau)|^p}{(\tau-\sigma)^{1+ps}}\, d(\tau,\sigma)\, d(x,y) \\ &\leq C \int_{0\leq\sigma\leq\tau\leq 1} \left(\int_0^\sigma \int_\tau^1 (y-x)^{p(s-s')-2} d(y,x) \right) \frac{|f(\sigma) - f(\tau)|^p}{(\tau-\sigma)^{1+ps}}\, d(\tau,\sigma) \\ &\leq C \int_{0\leq\sigma\leq\tau\leq 1} \frac{|f(\sigma) - f(\tau)|^p}{(\tau-\sigma)^{1+ps}}\, d(\tau,\sigma). \end{aligned}$$

The constant C is determined by

$$\int_0^\sigma \int_\tau^1 (y-x)^{\delta-2} dy dx \leq \int_0^\sigma \int_\sigma^1 (y-x)^{\delta-2} dy dx \leq \begin{cases} \frac{1-2^{1-\delta}}{\delta(\delta-1)} & \text{if } \delta = p(s-s') \neq 1 \\ \ln(2) & \text{if } \delta = p(s-s') = 1. \end{cases}$$

Making use of Almgren's bilipschtiz embedding $\boldsymbol{\xi}$ we deduce that (B.5) holds as well for multivalued functions, *i.e.*, for any $u \in W^{s,p}([0,1], \mathcal{A}_Q(\mathbb{R}^n))$

$$\begin{aligned}&\int_{0\le x\le y\le 1} \frac{\max_{\sigma,\tau\in[x,y]} \mathcal{G}(u(\sigma),u(\tau))^p}{|x-y|^{1+ps'}} d(x,y)\\ &\le C\int_{0\le\sigma\le\tau\le 1} \frac{\mathcal{G}(u(\sigma),u(\tau))^p}{|\sigma-\tau|^{1+ps}} d(\sigma,\tau).\end{aligned} \tag{B.7}$$

We observed $W^{s,p}([0,1], \mathcal{A}_Q(\mathbb{R}^n)) \subset C^{0,s-\frac{1}{p}}([0,1], \mathcal{A}_Q(\mathbb{R}^n))$, so that we may apply the theory of regular selections developed in [5]. Especially we use the proof of [5, Theorem 1.1]. For a given $u \in W^{s,p}([0,1], \mathcal{A}_Q(\mathbb{R}^n)$ we can find $v = (v_1, \dots, v_Q) : [0,1] \to (\mathbb{R}^n)^Q$ continuous with the property that $[v(t)] = \sum_{i=1}^Q [\![v_i(t)]\!] = u(t)$ on $[0,1]$ and there is a constant $C_Q > 0$ so that for any $0 \le x \le y \le 1$

$$|v(x) - v(y)| \le C_Q \max_{\sigma,\tau\in[x,y]} \mathcal{G}(u(\sigma),u(\tau)).$$

Combining this with (B.7) gives the remaining part (ii) of the lemma. □

C. Construction of bilipschitz maps between B_{1+} and $\Omega_F \cap B_1$

Before showing the general situation, $\Omega_F \cap B_1$ with $\Omega_F = \{(x', x_N) \in \mathbb{R}^N : x_N > F(x')\}$, $F \in C^1(\mathbb{R}^{N-1})$, we consider the similar case of a bilipschitz map between B_1 and the upper half ball $B_{1+} = B_1 \cap \{x_N > 0\}$ that preserves "radial" homogeneity.

It is of interest for us to preserve "radial" homogeneity in the context of constructing competitors. We want to make use of the interpolation lemma on annuli, Lemma A.6. We cannot use a generic bilipschitz map between B_1 and B_{1+}, because in general it is not true that if $G : U \to V$ is bilipschitz and $\psi_k : U \to U$ a sequence of diffeomorphisms that satisfy $\psi_k \to id$ then $G \circ \psi_k \circ G^{-1} \to 1$ with $\mathrm{Lip}(G \circ \psi_k \circ G^{-1}) \to 1$ as $k \to \infty$.

Lemma C.1. *There is a bilipschitz map $G : \overline{B_1} \to \overline{B_{1+}}$ that preserves "radial" homogeneity in the sense that*

$$G \circ \frac{1}{R} \circ G^{-1}(y) = \left(1 - \frac{1}{R}\right) c + \frac{1}{R} y;$$

where $c = \frac{e_N}{2} = \left(0, \dots, 0, \frac{1}{2}\right)$ and $0 < R$.

Proof. We make the ansatz $G(x) = c + s(\widehat{x})x$ for a piecewise C^1 function $s : \mathcal{S}^{N-1} \to \partial B_{1+}$ with bounded derivative, where $\widehat{x} = \frac{x}{|x|}$. The constraints $|c +$

$s(x)x|^2 = 1$ for $x \in \mathcal{S}^{N-1} \cap \{x_N \geq a\}$ and $\langle e_N, c + s(x)x\rangle = 0$ for $x \in \mathcal{S}^{N-1} \cap \{x_N \leq a\}$ for some $-1 < a < 0$ determine s and a uniquely to $a = -\frac{1}{\sqrt{5}}$ and

$$s(x) = s(x_N) = \begin{cases} \frac{1}{2}\left(-x_N + \sqrt{x_N^2 + 3}\right) & \text{if } x_N \geq -\frac{1}{\sqrt{5}} \\ -\frac{1}{2x_N} & \text{if } x_N \leq -\frac{1}{\sqrt{5}}. \end{cases}$$

The derivative is

$$s'(x_N) = \begin{cases} -\frac{1}{2}\left(1 - \frac{x_N}{\sqrt{x_N^2+3}}\right) & \text{if } x_N > -\frac{1}{\sqrt{5}} \\ \frac{1}{2x_N^2} & \text{if } x_N < -\frac{1}{\sqrt{5}}. \end{cases}$$

So we may check the bounds $|s'| < 3$ and $\frac{1}{2} \leq s(x_N) \leq \frac{\sqrt{5}}{2}$. Furthermore we got $\operatorname{grad} s(x) = \operatorname{grad}_{\mathcal{S}^{N-1}} s(x) = s'(x_N)(\mathbf{1} - x \otimes x)e_N$.

The inverse is explicitly given by $G^{-1}(y) = \frac{1}{s(\widehat{y-c})}(y - c)$. G and G^{-1} are almost everywhere C^1 with

$$\begin{aligned} DG(x) &= s(\widehat{x})\,\mathbf{1} + \widehat{x} \otimes \operatorname{grad} s(\widehat{x}) \\ DG^{-1}(y) &= \frac{1}{s(\widehat{y-c})}\,\mathbf{1} - \widehat{y-c} \otimes \frac{\operatorname{grad} s(\widehat{y-c})}{s^2(\widehat{y-c})}. \end{aligned}$$

The "radial" homogeneity follows, *i.e.*, $G \circ \frac{1}{R} \circ G^{-1}(y) = G(\frac{1}{s(\widehat{y-c})}\frac{y-c}{R}) = \left(1 - \frac{1}{R}\right)c + \frac{1}{R}y$. Therefore $DG \circ \frac{1}{R} \circ G^{-1} = \frac{1}{R}\mathbf{1}$ converging to $\mathbf{1}$ as $R \to 1$. □

Lemma C.2. *For any $F \in C^1(\mathbb{R}^{N-1})$ that satisfies $F(0) = 0$, $\operatorname{grad} F(0) = 0$ and $\|\operatorname{grad} F\|_\infty < \frac{1}{4}$ there exists a C^1-diffeomorphism*

$$G_F : \overline{B_{1+}} \to \overline{\Omega_F \cap B_1}$$

with bounds $\|DG_F - \mathbf{1}\|_\infty, \left\|DG_F^{-1} - \mathbf{1}\right\|_\infty < 10\,\|\operatorname{grad} F\|_\infty$.

Furthermore if F_k is a sequence of admissible maps with $F_k \to F$ in C^1 then $G_{F_k} \to G_F$ in C^1.

Proof. Let F be fixed, then $\psi : (x', x_N) \mapsto (x', x_N + F(x'))$ is a C^1-diffeomorphism between $\mathbb{R}^N_+$ and Ω_F. Its inverse is $\psi^{-1}(x', x_N) = (x', x_N - F(x'))$. We make again an ansatz for $G = G_F$. Set $G(x) = \psi(s(\widehat{x})\,x)$ where $s : \mathcal{S}^{N-1} \to \mathbb{R}_+$ satisfies $\psi(s(y)\,y) \in \Omega_F \cap \mathcal{S}^{N-1}$ for all $y \in \mathcal{S}^{N-1}_+$. The inverse for such a G is $G^{-1}(x) = \frac{1}{s(\widehat{\psi^{-1}(x)})}\,\psi^{-1}(x)$.

As a consequence of the implicit function theorem applied to the level set at 1 of the auxiliary function

$$h(y,s) = |\psi(s\,y)|^2,$$

$s \in C^1(\mathcal{S}_+^{N-1}, \mathbb{R}_+)$ has the desired properties. Note that $s(e_N) = 1$ because $h(e_N, 1) = 1$.

Existence: to every $y \in \mathcal{S}_+^{N-1}$ there exists $s(y) \in \mathbb{R}_+$ so that $h(y, s(y)) = 1$ and $1 - \|\mathrm{grad}\, F\|_\infty \le \frac{1}{s} \le 1 + \|\mathrm{grad}\, F\|_\infty$, because

$$\begin{aligned} h(y,s) &= s^2\,|y + \frac{F(s\,y')}{s}\,e_N|^2 \\ &\le s^2\left(1 + \|\mathrm{grad}\,F\|_\infty\right)^2 < 1 \text{ if } s < \frac{1}{1+\|\mathrm{grad}\,F\|_\infty} \\ &\ge s^2\left(1 - \|\mathrm{grad}\,F\|_\infty\right)^2 > 1 \text{ if } s > \frac{1}{1-\|\mathrm{grad}\,F\|_\infty}. \end{aligned}$$

C^1_{loc} *homeomorphism*: every tuple (y_0, s_0) with $h(y_0, s_0) = 1$ has a neighbourhood $U \times I$ in $\mathcal{S}_+^{N-1} \times \mathbb{R}_+$ and a C^1 map $s : U \to I, C^1$ with $h(y, s(y)) = 1$ on U. This follows from the implicit function theorem, because at $x_0 = s_0\, y_0$

$$\begin{aligned} \frac{1}{2} s \frac{\partial h}{\partial s} &= 1 - \langle \psi(x_0), \psi(x_0) - d\psi(x_0)x_0\rangle \\ &= 1 - \psi_N(x_0)\left(F(x_0') - \langle \mathrm{grad}\, F(x_0'), x_0'\rangle\right) \ge 1 - 2\,\|\mathrm{grad}\,F\|_\infty \ge \frac{1}{2}. \end{aligned}$$

Uniqueness/well-definition: this is a consequence of $\frac{\partial h}{\partial s} > 0$ for each such tuple (y_0, s_0), so there cannot be two $s_1 < s_2$ with $h(y_0, s_1) = 1 = h(y_0, s_2)$.

Bounds on $\mathrm{grad}\, s = \mathrm{grad}_{\mathcal{S}^{N-1}}\, s$: fix any generic $\tau \in T_y\mathcal{S}^{N-1}$ and so $0 = \left(D_\tau h + \frac{\partial h}{\partial s}\, D_\tau s\right)(y, s(y))$. Furthermore writing $x = s(y)y$ we have

$$\frac{1}{2s} D_\tau h(y,s) = \frac{1}{s}\langle \psi(x), d\psi(x)s\tau\rangle = \tau_N F(x') + \psi_N(x')\langle \mathrm{grad}\, F(x'), \tau'\rangle,$$

that gives

$$\left|\frac{1}{2s} D_\tau h(y,s)\right| \le \sqrt{2}\,\|\mathrm{grad}\,F\|_\infty\,.$$

We conclude

$$|D_\tau s(y)| = s^2 \frac{|\frac{1}{2s} D_\tau h|}{|\frac{1}{2} s \frac{\partial h}{\partial s}|} \le 3s^2\,\|\mathrm{grad}\,F\|_\infty \le 16\,\|\mathrm{grad}\,F\|_\infty\,.$$

Bounds on DG, DG^{-1}: One calculates explicitly that

$$\begin{aligned} DG(x) &= d\psi(s(\widehat{x})x)\,(s(\widehat{x})\mathbf{1} + \widehat{x} \otimes \mathrm{grad}\, s(\widehat{x})) \\ &= s(\widehat{x})\mathbf{1} + \widehat{x}\otimes \mathrm{grad}\, s(\widehat{x}) + (e_N \otimes \mathrm{grad}\, F)\,(s(\widehat{x})\mathbf{1} + \widehat{x}\otimes \mathrm{grad}\, s(\widehat{x}))\,. \end{aligned}$$

As we have seen $|s(\widehat{x})-1| \le \frac{\|\operatorname{grad} F\|_\infty}{1-\|\operatorname{grad} F\|_\infty}$. Combining all obtained bounds one can conclude $\|DG(x)-\mathbf{1}\|_\infty \le 10\,\|\operatorname{grad} F\|_\infty$. DG^{-1} is given explicitly by

$$\begin{aligned} DG^{-1}(x) &= \frac{1}{s(\widehat{\psi^{-1}(x)})} d\psi^{-1}(x) - \widehat{\psi^{-1}(x)} \otimes \frac{\operatorname{grad} s(\widehat{\psi^{-1}(x)})}{s^2(\widehat{\psi^{-1}(x)})} \\ &= \frac{1}{s(\widehat{\psi^{-1}(x)})}\mathbf{1} - \frac{1}{s(\widehat{\psi^{-1}(x)})} e_N \otimes \operatorname{grad} F - \widehat{\psi^{-1}(x)} \otimes \frac{\operatorname{grad} s(\widehat{\psi^{-1}(x)})}{s^2(\widehat{\psi^{-1}(x)})}. \end{aligned}$$

Combing as before all obtained bounds especially $|\frac{1}{s(\widehat{\psi^{-1}(x)})}-1| \le \|\operatorname{grad} F\|_\infty$ one can get $\left\|DG^{-1}(x)-\mathbf{1}\right\|_\infty \le 6\,\|\operatorname{grad} F\|_\infty$.

The convergence statement follows as a consequence of the implicit function theorem, because $F_k \to F$ in C^1 then implies $s_{F_k} \to s_F$ in C^1. □

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