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Local homogeneity and dimensions of measures

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Abstract. We introduce two new concepts, local homogeneity and local L^q-spectrum, both of which are tools that can be used to study the local structure of measures. Combining homogeneity and L^q-spectrum estimates, we introduce a new method to bound the local dimensions of measures in general doubling metric spaces. As an application, we reach a new level of generality and obtain many new results in the study of conical densities and porous measures in Euclidean spaces and also in general doubling metric spaces.

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1. Introduction

In geometric measure theory and fractal geometry, it is common to encounter problems of the following type: given a measure μ and a set A of positive/full μ-measure, we have some local geometric information (on various densities, porosity, tangent measures, sceneries *etc.*) around all points of the set (or in a set of positive/full measure) and we want to gain some global information (on dimension, measure, rectifiability, and other geometric properties) from this. For example, if the set A is porous in the sense that for some $\varrho > 0$, for all $x \in A$, and all small $r > 0$, there is a ball $B(y, \varrho) \subset B(x, r) \setminus A$, one is lead to ask how the dimension of A depends on the value of ϱ. This question and its many variations have been under careful investigations in the past [6, 13, 25, 26, 32, 33, 39–41, 46, 52, 55, 59] also due to intrinsic interest, but mostly since many important sets and measures in analysis and dynamics satisfy such, or related, porosity conditions. These sets and measures may arise in geometric analysis [33,39,46,56,62,64], in the study of Julia sets [50,51], singular integrals [10], infinite iterated function systems [63], or as random sets [7,27]. The survey [58] gives some further background and motivation.

More generally, if we know how the set (or the measure) is distributed in small balls, we may use this information to bound its dimension from above. The notions

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of average homogeneity [24] and local entropy averages [22, 59] have been used to obtain quantitative dimension bounds for a measure based on its distribution in small balls or cubes; see also [6]. On the other hand, if μ is a measure of given dimension on a Euclidean space, it is a classical problem in geometric measure theory to estimate how it is distributed in different directions or cones. This leads to the study of conical densities [8, 9, 11, 18, 31, 32, 35, 38, 40, 44, 54, 60], which originate from the dichotomy between rectifiable and purely unrectifiable sets, and are intimately connected to the existence of tangents, see [15, 41]. This theory has been further applied to various porosity questions [31, 32, 40] and in the study of removability of Lipschitz harmonic functions [35, 42]. In the study of fractals and dynamical systems, it is natural to analyse properties of measures using globally observable parameters arising from the asymptotic behaviour of the system, such as the Lyapunov exponent. The entropy dimension and L^q-dimensions are concepts that measure the average distribution of the measure. In many cases, these global characteristics can then be related to the local regularity properties of the measure such as exact dimensionality and also to the values of the local dimension maps; see [4, 12, 17, 20, 45, 68].

In this article, the major objects of interest are the upper and lower local dimensions of measures defined as

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$
$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

whenever μ is a measure on a metric space X and $x \in X$. Large part of the analysis on measures aims at estimating these dimensions. The essential suprema and infima of the local dimensions are the upper and lower packing and Hausdorff dimensions of the measure whereas investigating the level set structure of the local dimension maps leads to an important branch of multifractal analysis. The main purpose of this article is to introduce two new concepts, local homogeneity and local L^q-spectrum. Both of these are shown to be useful tools in the study of local structure of measures.

We will next describe our main results. For notation and definitions of the basic concepts, we refer to Section 2 below. In Subsection 2.3, we introduce local versions of the classical L^q-spectra and dimensions. Using these concepts, in Subsections 3.1–3.3, we generalize the results of [17, 20, 45, 47] on the relations between the Hausdorff, entropy, packing, and L^q-dimensions for measures to metric spaces. The most important aspect of this generalization is that we consider local spectrum and dimension in place of the global counterparts. This is an advantage, when one wants to use these to estimate the values of $\underline{\dim}_{\mathrm{loc}}(\mu, x)$ and $\overline{\dim}_{\mathrm{loc}}(\mu, x)$, as will be apparent in the proofs of the main results, and as illustrated by various examples. In [29], we apply this new local spectrum to develop some multifractal analysis in metric spaces. Furthermore, in [28], the local spectrum is applied as a tool to show that certain doubling measures have small packing dimension, generalizing a result of Wu [67]. In Subsection 3.2, we will turn to the local homogeneity of measures. In our main result, Theorem 3.5, we show that, for any locally finite Borel regular measure μ, the upper local dimension $\overline{\dim}_{\mathrm{loc}}(\mu, x)$ is bounded from above by the local homogeneity dimension $\dim_{\mathrm{hom}}(\mu, x)$ at μ-almost all points. The value of $\dim_{\mathrm{hom}}(\mu, x)$ is the infimum of exponents s so that "large parts" of $B(x, r)$ in terms of μ can be covered by δ^{-s} balls of radius δr for all small $r, \delta > 0$; see (2.9) for a detailed definition. We believe that this relation between $\overline{\dim}_{\mathrm{loc}}$ and $\dim_{\mathrm{hom}}$ will turn out to be a useful tool in analysing fractal measures in general metric spaces. To illustrate the applicability of this result, we obtain new estimates on the dimension of porous measures; see Theorems 4.2 and 4.7. For instance, in Theorem 4.7, we show that if X is a metric space carrying a measure ν such that $ar^s \leq \nu(B(x, r)) \leq br^s$ for all balls $B(x, r) \subset X$ with $0 < r < \operatorname{diam}(X)$, then there is a uniform constant $c > 0$ such that for any measure μ on X, we have the estimate

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq s - c \operatorname{por}_1(\mu, x)^s$$

for μ-almost all $x \in X$. Here $\operatorname{por}_1(\mu, x)$ is the porosity of the measure μ at x analogous to the set porosity discussed above. See Section 4 for the exact definition. To the best of our knowledge, this is the first nontrivial result related to the dimension of porous measures in a non-Euclidean setting. As another application of the local homogeneity estimates, we obtain in Theorem 4.1 a new upper conical density result for measures with large packing dimension. This improves a result of [11] where a corresponding statement was proved for the Hausdorff dimension. The method of [11] is based on the use of the average homogeneity from [24], and it only works for the Hausdorff dimension.

Although the definitions of the local homogeneity and L^q-dimensions are somewhat technical and giving rigorous proofs for our results requires care, the advantage of our method is that it gives unified approach to various problems that have before required the use of separate ideas. As a by-product of our results, we also simplify and generalize the proofs of many existing results. For instance using a quantitative version of the statement $\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq \dim_{\mathrm{hom}}(\mu, x)$ (see Theorem 3.8), the proof of the Euclidean porosity result, Theorem 4.2, reduces to a rather straightforward geometric problem and is much simpler than the proofs of the earlier partial results in [5,6,23].

It turns out that the definitions of local homogeneity and local L^q-spectrum are of different nature since the order of taking limits is different. In defining the local homogeneity, we first let the scale tend to zero and only after that increase the resolution. This allows us to handle non-uniform properties, like porosity, with ease. On the other hand, the local L^q-spectrum sees some slight differences in the behaviour of the measure to which the local homogeneity is blind. This difference is made manifest in examples in Section 5. Despite these differences, in the proof of our main results in Subsections 3.1–3.2, we are able to combine the L^q-spectrum and homogeneity estimates in order to obtain an upper bound for $\overline{\dim}_{\mathrm{loc}}(\mu, x)$. This is the most important new method introduced in the paper. It builds on a simple use of a covering theorem for balls of equal radii which is valid on any doubling metric space, see Lemma 2.1. This makes the approach fairly general and, in our opinion, makes the proofs more natural and straightforward in the Euclidean setting as well.

In many recent studies the relations between the dimension and geometry of measures in Euclidean spaces are studied using a probabilistic approach and the dyadic self-similar structure of $\mathbb{R}^d$. For instance, see [21,22,24,53,58,59]. Since we work in a general doubling metric space, our approach is more robust and slightly less probabilistic. We remark that the paper [53], that appeared after an earlier version of this paper was made available, was partly inspired by the present work. In that paper, the authors use the method of local entropy averages to bound the local homogeneity, leading to further applications. Furthermore, in a recent paper [30], ergodic-theoretical methods were employed to study similar geometric properties of measures. The paper [2] (which as well appeared subsequent to an earlier version of this paper) gives a treatment of a local spectrum from a different perspective. There the local spectrum is applied for functions, measures, and more general distributions in the setting of Euclidean spaces, see also [3].

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2. Preliminaries

2.1. Basic notation

In writing down constants we often use notation such as $c = c(\cdots)$ to emphasize that the constant depends only on the parameters listed inside the parentheses.

We work on a metric space (X, d) which we usually assume to be *doubling*. This means that there is $N = N(X) \in \mathbb{N}$ (*the doubling constant of*) X such that any closed ball $B(x,r) = \{y \in X : d(x,y) \le r\}$ with centre $x \in X$ and radius $r > 0$ can be covered by N balls of radius $r/2$. Since we use only one distance d in the space X, we simply denote (X, d) by X.

Notice that even if $x \neq y$ or $r \neq t$, it may happen that $B(x,r) = B(y,t)$. For notational convenience, we keep to the convention that each ball comes with a fixed center and radius. This makes it possible to use notation such as $5B = B(x, 5r)$ without referring to the centre or radius of the ball $B = B(x,r)$.

In this article, *a measure* exclusively refers to a nontrivial Borel regular (outer) measure defined on all subsets of X so that bounded sets have finite measure.

We call any countable collection $\mathcal{B}$ of pairwise disjoint closed balls a *packing*. We also call $\mathcal{B}$ a *packing of* the subset $A \subset X$ if the centres of the balls of $\mathcal{B}$ lie in the set A, and a δ-*packing* if all the balls in $\mathcal{B}$ have radius δ. A δ-packing $\mathcal{B}$ of A is termed *maximal* if for every $x \in A$ there is $B \in \mathcal{B}$ so that $B(x,\delta) \cap B \neq \emptyset$. Note that if $\mathcal{B}$ is a maximal δ-packing of A, then $2\mathcal{B}$ covers A. Here $2\mathcal{B} = \{2B : B \in \mathcal{B}\}$.

Observe that a doubling metric space is separable. Hence for each $\delta > 0$ and $A \subset X$ there exists a maximal δ-packing of A. Moreover, the $5r$-covering theorem is applicable in every doubling metric space; see [41, Theorem 2.1].

Instead of δ-packings defined above, the theory developed in this paper could be presented by using δ-separated sets, *i.e.* sets $\{x_i\} \subset A$ for which $d(x_i, x_j) > \delta$ whenever $x_i \neq x_j$. Yet another option would be to define the necessary concepts using partitions or generalized dyadic cubes. We chose the packing approach mainly because of personal taste and since we wanted our packing balls to be geometrically (and not only algebraically) disjoint. The partition definition is sometimes more useful in computations. In [29], we use that approach to develop some multifractal analysis in metric spaces.

The doubling property can be stated in several equivalent ways. For instance, the following formulations are sometimes convenient. The proof is a simple exercise (see, *e.g.*, [19,36]).

Lemma 2.1. *For a metric space X, the following claims* (1)-(4) *are equivalent and each of them implies* (5)*:*

(1) *X is doubling;*
(2) *There are $s > 0$ and $c > 0$ such that for all $R > r > 0$ any ball of radius R can be covered by $c(r/R)^{-s}$ balls of radius r;*
(3) *There are $s > 0$ and $c > 0$ such that if $R > r > 0$ and $\mathcal{B}$ is an r-packing of a closed ball of radius R, then the cardinality of $\mathcal{B}$ is at most $c(r/R)^{-s}$;*
(4) *For every $0 < \lambda < 1$ there is a constant $M = M(X, \lambda) \in \mathbb{N}$, satisfying the following: if $\mathcal{B}$ is a collection of closed balls of radius $\delta > 0$ so that $\lambda\mathcal{B}$ is pairwise disjoint, then there are δ-packings $\{\mathcal{B}_1, \ldots, \mathcal{B}_M\}$ so that $\mathcal{B} = \bigcup_{i=1}^{M} \mathcal{B}_i$;*
(5) *There is $M = M(X) \in \mathbb{N}$ such that if $A \subset X$ and $\delta > 0$, then there are δ-packings of A, $\mathcal{B}_1, \ldots, \mathcal{B}_M$ whose union covers A.*

Remark 2.2. (1) It follows by elementary arguments that $s = \log_2 N$ will do in (2) and (3). The infimum over all admissible exponents s in (2) and (3) is usually called the *Assouad dimension* of X (see [19,36]). Thus, doubling metric spaces are precisely the metric spaces with finite Assouad dimension.

(2) Observe that (5) is Besicovitch's covering theorem ([41, Subsection 2.7]) for balls with equal radius. The following consequence of (5) is sometimes very useful: if μ is a measure on X and $A \subset X$, then there is a δ-packing $\mathcal{B}$ of A for $\delta > 0$, such that

$$\sum_{B \in \mathcal{B}} \mu(B) \geq c\mu(A). \tag{2.1}$$

Here $c > 0$ depends only on the doubling constant N.

We say that a measure μ on X is *doubling* if there is a constant $c \geq 1$ so that

$$0 < \mu\big(B(x, 2r)\big) \leq c\mu\big(B(x, r)\big) < \infty$$

for all $x \in X$ and $r > 0$. A complete doubling metric space always supports doubling measures; see [28,37,65–67]. Recall that the *support* of a measure μ, denoted by $\operatorname{spt}(\mu)$, is the smallest closed subset of X with full μ-measure. Furthermore, we say that a measure μ on X is *s-regular* (for $s > 0$) if there are constants $a, b > 0$ so that

$$ar^s \le \mu\big(B(x,r)\big) \le br^s$$

for all $x \in \operatorname{spt}(\mu)$ and $0 < r \le \operatorname{diam}(X)$. It is clear that each s-regular measure is doubling. A metric space X is called *s-regular* if it carries an s-regular measure μ with $\operatorname{spt}(\mu) = X$. In this case, a simple volume argument can be used to verify the conditions (2) and (3) of Lemma 2.1. Therefore an s-regular metric space is doubling. More generally, each metric space carrying a doubling measure is a doubling metric space.

A measure μ on X has the *density point property* if

$$\lim_{r \downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1 \tag{2.2}$$

for μ-almost all $x \in A$ whenever $A \subset X$ is μ-measurable. In general, the density point property is not necessarily valid for all measures in a doubling metric space; see Example 5.6. Nevertheless, in the proofs, it can be often replaced by the following weaker result.

Lemma 2.3. *If μ is a measure on a separable metric space X and $A \subset X$ is μ-measurable, then*

$$\lim_{r \downarrow 0} \frac{\mu\big(B(x,r) \setminus A\big)}{\mu\big(B(x,5r)\big)} = 0$$

for μ-almost all $x \in A$.

Proof. Define $E_\varepsilon = \{x \in A : \limsup_{r \downarrow 0} \mu\big(B(x,r) \setminus A\big)/\mu\big(B(x,5r)\big) > \varepsilon\}$ for all $\varepsilon > 0$. The claim follows if we can show that $\mu(E_\varepsilon) = 0$ for all $\varepsilon > 0$. Fix $\varepsilon > 0$ and for $\eta > 0$, let G_η be an open set containing E_ε such that $\mu(G_\eta \setminus E_\varepsilon) < \eta$. Applying the $5r$-covering theorem for the collection $\{B(x,r) : x \in E_\varepsilon$ and $r > 0$ such that $B(x,r) \subset G_\eta$ and $\mu\big(B(x,r) \setminus A\big) > \varepsilon\mu\big(B(x,5r)\big)\}$, we obtain a disjoint subcollection $\mathcal{B}$ such that $5\mathcal{B}$ covers E_ε. Thus

$$\varepsilon\mu(E_\varepsilon) \le \varepsilon \sum_{B \in \mathcal{B}} \mu(5B) < \sum_{B \in \mathcal{B}} \mu(B \setminus A) \le \mu(G_\eta \setminus A) \le \mu(G_\eta \setminus E_\varepsilon) < \eta.$$

Letting $\eta \downarrow 0$ implies $\mu(E_\varepsilon) = 0$, as required. □

Remark 2.4. (1) The constant 5 in Lemma 2.3 can be replaced by any constant $C > 2$. This is because in the $5r$-covering theorem, we may replace 5 by any such C. Furthermore, if Besicovitch's covering theorem holds in X, then the constant 5 in Lemma 2.3 can be replaced by 1. This can be seen just by applying Besicovitch's covering theorem (instead of the $5r$-covering theorem) in the proof of Lemma 2.3. In particular, this observation shows that in Euclidean spaces every measure has the density point property.

(2) The following upper density point property is true for all measures in any doubling metric space X: if μ is a measure on X and $A \subset X$ is μ-measurable, then

$$\limsup_{r\downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1$$

for μ-almost all $x \in A$. This follows from Lemma 2.3 and the fact that even if a measure is not doubling, it has arbitrary small "doubling scales" at each typical point; see *e.g.* [11, Lemma 2.2].

2.2. Local dimensions

We are mostly interested in estimating the *upper and lower local dimensions of the measure μ at x* defined by

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$
$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

respectively. If the upper and lower dimensions agree, we call their mutual value the *local dimension of the measure μ at x* and write $\dim_{\mathrm{loc}}(\mu, x)$ for this common value.

Remark 2.5. (1) If μ is an s-regular measure, then trivially $\dim_{\mathrm{loc}}(\mu, x) = s$ for all $x \in \mathrm{spt}(\mu)$.

(2) If A is a Borel set, then $\overline{\dim}_{\mathrm{loc}}(\mu|_A, x) = \overline{\dim}_{\mathrm{loc}}(\mu, x)$ and $\underline{\dim}_{\mathrm{loc}}(\mu|_A, x) = \underline{\dim}_{\mathrm{loc}}(\mu, x)$ for μ-almost all $x \in A$. This can be proven similarly as Lemma 2.3 once we observe that if the statement fails, then there is $\varepsilon > 0$ such that $\limsup_{r\downarrow 0} r^{\varepsilon}\mu\big(B(x,r)\big)/\mu\big(A \cap B(x,5r)\big) > 0$ in a set of positive measure.

2.3. Local L^q-spectrum and L^q-dimensions

Let μ be a measure on X, the set $A \subset X$ a bounded set, $q \in \mathbb{R}$, and $r > 0$. The *(global) L^q-spectrum of μ on A* is defined by

$$\tau_q(\mu, A) = \liminf_{\delta\downarrow 0} \frac{\log S_q(\mu, A, \delta)}{\log \delta}, \tag{2.3}$$

where

$$S_q(\mu, A, \delta) = \sup\Big\{ \sum_{B\in\mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } A \cap \mathrm{spt}(\mu) \Big\}$$

is the *L^q-moment sum of μ on A at the scale δ*. Furthermore, the *local L^q-spectrum of μ at x* is

$$\tau_q(\mu, x) = \lim_{r\downarrow 0} \tau_q\big(\mu, B(x,r)\big). \tag{2.4}$$

Given $A \subset X$ and $q \neq 1$, we define the *(global) L^q-dimension of μ on A* by setting

$$\dim_q(\mu, A) = \tau_q(\mu, A)/(q-1)$$

and the *local L^q-dimension of μ at x* by

$$\dim_q(\mu, x) = \lim_{r \downarrow 0} \dim_q\big(\mu, B(x,r)\big) = \tau_q(\mu, x)/(q-1).$$

We also denote $\tau_q(\mu) = \tau_q(\mu, X)$ and $\dim_q(\mu) = \dim_q(\mu, X)$ in the case $\operatorname{spt}(\mu)$ is bounded.

Remark 2.6. (1) The limit in (2.4) exists as $S_q(\mu, A, \delta) \leq S_q(\mu, B, \delta)$ whenever $\delta > 0$ and $A \subset B$. The use of $\liminf$ in (2.3) guarantees the concavity of the L^q-spectrum; see Lemma 2.7 (4).

(2) If $q \geq 0$ and A is closed, then the definition of $\tau_q(\mu, \cdot)$ does not change if we ignore $\operatorname{spt}(\mu)$ in the definition of $S_q(\mu, \cdot)$. That is, we can repeat the definition with

$$S_q(\mu, A, \delta) = \sup\left\{ \sum_{B \in \mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } A \right\}$$

(if $q = 0$, we interpret $0^q = 0$). Also, if $(\delta_n)_{n=1}^\infty$ is a decreasing sequence tending to 0 with $\lim_{n\to\infty} \log \delta_{n+1}/\log \delta_n = 1$, then it follows from Lemma 2.1(5) that the $\liminf$ in the definition of τ_q may be taken along the sequence $(\delta_n)_{n=1}^\infty$. These simple facts will be used frequently in what follows.

(3) If μ is an s-regular measure on X with $\operatorname{spt}(\mu) = X$, then $\dim_q(\mu, A) = s$ for all bounded $A \subset X$ with $\mu(A) > 0$ and, consequently, $\dim_q(\mu, x) = \dim_{\mathrm{loc}}(\mu, x)$ for all $x \in X$. Indeed, given $q \in \mathbb{R}$, we find constants $0 < c_1(A) < c_2(A) < \infty$ so that $c_1\delta^{s(q-1)} \leq S_q(\mu, A, \delta) \leq c_2\delta^{s(q-1)}$ for all $0 < \delta < 1$. This implies $\tau_q(\mu, A) = s(q-1)$ and thus $\dim_q(\mu, A) = s$.

(4) There are measures for which $\dim_q(\mu, x)$ is constant almost everywhere, but this constant is not the same as $\dim_q(\mu)$; see Examples 5.1 and 5.2.

(5) Recall from Remark 2.5 (2) that for any Borel set A the restriction measure $\mu|_A$ has the same upper and lower local dimension as the original measure μ for μ-almost all points in A. This is not true for the L^q-dimension. As an example in the case $q < 1$, take $\mu = \mathcal{L}^2 + \mathcal{H}^1|_L$ on $[0,1]^2$, where $\mathcal{L}^2$ is the Lebesgue measure and $\mathcal{H}^1|_L$ is the length measure on a line $L \subset [0,1]^2$. Now there exist constants $c_1, c_2 > 0$ so that for every $r > 0$ we have $S_q\big(\mu, B(x,r), \delta\big) = c_1 r^2 \delta^{2(q-1)}$ and $S_q\big(\mu|_L, B(x,r), \delta\big) = c_2 r \delta^{q-1}$ for all $\delta > 0$ small enough. Thus $\tau_q(\mu, x) = \tau_q(\mu)$ and $\tau_q(\mu|_L, x) = \tau_q(\mu|_L)$ for all $x \in L$. Since $\operatorname{spt}(\mu|_L) = \operatorname{spt}(\mu) \cap L$, we also have $\tau_q(\mu|_L) = \tau_q(\mu, L)$. Therefore,

$$\tau_q(\mu, x) = \tau_q(\mu) = 2(q-1) < q - 1 = \tau_q(\mu|_L) = \tau_q(\mu, L)$$

and

$$\dim_q(\mu, x) = 2 > 1 = \dim_q(\mu|_L, x)$$

for all $x \in L$. For $q > 1$ we can instead define a measure on the real line by letting $\nu = \mathcal{L}^2 + \sum_{n\in\mathbb{N}} 2^{-n}\delta_{q_n}$, where $\{q_1, q_2, \ldots\}$ is an enumeration of the rationals. Then $\dim_q(\nu, x) = 0$ while $\dim_q(\nu|_{\mathbb{R}\setminus\mathbb{Q}}, x) = 1$ for all $x \in \mathbb{R}$.

We list some of the basic properties of the L^q-spectrum in the following lemmas.

Lemma 2.7. *If μ is a measure on a doubling metric space X, the set $A \subset X$ is a bounded set with $\mu(A) > 0$, setting $q_0 = \inf\{q \in \mathbb{R} : \tau_q(\mu, A) > -\infty\}$, and s as in Lemma* 2.1(2)–(3)*, then:*

(1) $\tau_1(\mu, A) = 0$*;*
(2) $\min\{0, (q-1)s\} \le \tau_q(\mu, A) \le \max\{0, (q-1)s\}$ *for all* $0 \le q < \infty$*;*
(3) $0 \le \dim_q(\mu, A) \le s$ *for all* $0 \le q < \infty$ *with* $q \ne 1$*;*
(4) *the mapping* $q \mapsto \tau_q(\mu, A)$ *is concave on* (q_0, ∞)*;*
(5) *the mapping* $q \mapsto \dim_q(\mu, A)$ *is continuous and decreasing on both* $(q_0, 1)$ *and* $(1, \infty)$.

Furthermore, if $x \in \mathrm{spt}(\mu)$, then all the claims above remain true if $\tau_q(\mu, A)$ is replaced by $\tau_q(\mu, x)$ and $\dim_q(\mu, A)$ by $\dim_q(\mu, x)$.

Proof. We prove the claims for $\tau_q(\mu, A)$. The statements for $\tau_q(\mu, x)$ follow by simply taking $A = B(x, r)$ and letting $r \downarrow 0$. It suffices to show (2) and (4) since the other claims follow easily from these two. Fix $a \in A$ and define $U = B\big(a, \mathrm{diam}(A) + 1\big)$.

If $0 < \delta < 1$ and $\mathcal{B}$ is any δ-packing of A, then Lemma 2.1(3) gives $M \le C\delta^{-s}$, where M is the cardinality of $\mathcal{B}$. Therefore Hölder's inequality implies

$$\sum_{B\in\mathcal{B}} \mu(B)^q \le \begin{cases} \mu(U)^q M^{1-q} \le C^{1-q}\mu(U)^q\delta^{s(q-1)} & \text{if } 0 \le q \le 1 \\ \mu(U)^q & \text{if } q \ge 1. \end{cases}$$

In addition, if $\mathcal{B}$ satisfies (2.1), then we estimate

$$\sum_{B\in\mathcal{B}} \mu(B)^q \ge \begin{cases} c^q\mu\big(A\big)^q & \text{if } q \le 1 \\ c^q\mu(A)^q M^{1-q} \ge c^q C^{1-q}\mu(A)^q\delta^{s(q-1)} & \text{if } q \ge 1. \end{cases}$$

The claim (2) follows by taking logarithms and letting $\delta \downarrow 0$. To show (4), let $\mathcal{B}$ be a δ-packing of $A \cap \mathrm{spt}(\mu)$. For every $q, p \ge q_0$ and $\lambda \in (0, 1)$ we have

$$\sum_{B\in\mathcal{B}} \mu(B)^{\lambda q+(1-\lambda)p} \le \Big(\sum_{B\in\mathcal{B}} \mu(B)^q\Big)^{\lambda}\Big(\sum_{B\in\mathcal{B}} \mu(B)^p\Big)^{1-\lambda} \tag{2.5}$$

by Hölder's inequality. The claim follows. □

Lemma 2.8. *If μ is a measure on a compact doubling metric space X, then*

$$\tau_q(\mu) = \min\{\tau_q(\mu, x) : x \in \operatorname{spt}(\mu)\}$$

for every $q \in \mathbb{R}$. In particular,

$$\dim_q(\mu) = \begin{cases} \max\{\dim_q(\mu, x) : x \in \operatorname{spt}(\mu)\} & \text{if } q < 1 \\ \min\{\dim_q(\mu, x) : x \in \operatorname{spt}(\mu)\} & \text{if } q > 1. \end{cases}$$

Proof. According to Remark 2.6(1), we have $\tau_q(\mu) \le \tau_q(\mu, x)$ for every $x \in \operatorname{spt}(\mu)$. Since the second claim follows immediately from the first one, it remains to show that there exists $x \in \operatorname{spt}(\mu)$ for which $\tau_q(\mu, x) \le \tau_q(\mu)$. First we cover $\operatorname{spt}(\mu)$ with finitely many balls $\{B(y_i, \frac{1}{2})\}_{i=1}^{k_1}$, $y_i \in \operatorname{spt}(\mu)$. Then, for every j and $\delta > 0$, we have

$$\begin{aligned} S_q\big(\mu, B(y_j, \tfrac{1}{2}), \delta\big) \le S_q(\mu, X, \delta) &\le \sum_{i=1}^{k_1} S_q\big(\mu, B(y_i, \tfrac{1}{2}), \delta\big) \\ &\le k_1 \max_{i \in \{1,\dots,k_1\}} S_q\big(\mu, B(y_i, \tfrac{1}{2}), \delta\big). \end{aligned} \tag{2.6}$$

Let $(\delta_j)_{j=1}^{\infty}$ be a decreasing sequence tending to zero so that

$$\lim_{j\to\infty} \frac{\log S_q(\mu, X, \delta_j)}{\log \delta_j} = \liminf_{\delta \downarrow 0} \frac{\log S_q(\mu, X, \delta)}{\log \delta} = \tau_q(\mu).$$

Then, for every $j \in \mathbb{N}$, choose $i_j \in \{1, \dots, k_1\}$ so that

$$S_q\big(\mu, B(y_{i_j}, \tfrac{1}{2}), \delta_j\big) = \max_{i \in \{1,\dots,k_1\}} S_q\big(\mu, B(y_i, \tfrac{1}{2}), \delta_j\big).$$

Now for some $i \in \{1, \dots, k_1\}$ the set $\{j \in \mathbb{N} : i_j = i\}$ is infinite. Considering a suitable subsequence of $(\delta_j)_{j=1}^{\infty}$ and using (2.6), we get

$$\liminf_{\delta \downarrow 0} \frac{\log S_q\big(\mu, B(x_1, \frac{1}{2}), \delta\big)}{\log \delta} = \tau_q(\mu),$$

where $x_1 = y_i$.

Next we repeat the above argument by replacing $\frac{1}{2}$ with $\frac{1}{4}$ and $\operatorname{spt}(\mu)$ by $\operatorname{spt}(\mu) \cap B(x_1, \frac{1}{2})$. Then we find $x_2 \in B(x_1, \frac{1}{2})$ so that

$$\liminf_{\delta \downarrow 0} \frac{\log S_q\big(\mu, B(x_2, \frac{1}{4}), \delta\big)}{\log \delta} = \liminf_{\delta \downarrow 0} \frac{\log S_q\big(\mu, B(x_1, \frac{1}{2}), \delta\big)}{\log \delta} = \tau_q(\mu).$$

Continuing inductively, we find a sequence $x_i \in \operatorname{spt}(\mu)$ with $d(x_{i+1}, x_i) \le 2^{-i}$ and

$$\liminf_{\delta \downarrow 0} \frac{\log S_q\big(\mu, B(x_i, 2^{-i}), \delta\big)}{\log \delta} = \tau_q(\mu)$$

for every $i \in \mathbb{N}$. Since $\operatorname{spt}(\mu)$ is compact, for $x = \lim_{i\to\infty} x_i$, we eventually get

$$\liminf_{\delta\downarrow 0} \frac{\log S_q\big(\mu, B(x, 2^{-i+2}), \delta\big)}{\log\delta} \le \liminf_{\delta\downarrow 0} \frac{\log S_q\big(\mu, B(x_i, 2^{-i}), \delta\big)}{\log\delta}$$

for all $i \in \mathbb{N}$ and thus $\tau_q(\mu, x) \le \tau_q(\mu)$. $\square$

Remark 2.9. If μ is a measure on a doubling metric space X and $A \subset X$ is compact, then an easy modification of the above proof shows that for each $q \in \mathbb{R}$ there exists $x \in A \cap \operatorname{spt}(\mu)$ so that $\tau_q(\mu, x) \le \tau_q(\mu, A)$. Then $\dim_q(\mu, A) \le \max\{\dim_q(\mu, x) : x \in A \cap \operatorname{spt}(\mu)\}$ for $q < 1$ and $\dim_q(\mu, A) \ge \min\{\dim_q(\mu, x) : x \in A \cap \operatorname{spt}(\mu)\}$ for $q > 1$. The sets where these maximums and minimums are obtained can be extremely small in terms of μ measure, see Examples 5.1, 5.2.

2.4. Local homogeneity and homogeneity dimension

Let μ be a measure on X, $x \in X$, and $\delta, \varepsilon, r > 0$. Define for all $\Lambda > 1$

$$\begin{aligned}\hom^{\Lambda}_{\delta,\varepsilon,r}(\mu, x) = \sup\Big\{\#\mathcal{B} :\ &\mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x, r) \\ &\text{so that } \mu(B) > \varepsilon\mu\big(B(x, \Lambda r)\big) \text{ for all } B \in \mathcal{B}\Big\}.\end{aligned} \tag{2.7}$$

The *local δ-homogeneity* and the *local homogeneity dimension* (with a parameter Λ) of a measure μ at x are defined as

$$\hom^{\Lambda}_{\delta}(\mu, x) = \lim_{\varepsilon\downarrow 0} \limsup_{r\downarrow 0} \hom^{\Lambda}_{\delta,\varepsilon,r}(\mu, x), \tag{2.8}$$

$$\dim^{\Lambda}_{\hom}(\mu, x) = \liminf_{\delta\downarrow 0} \frac{\log \hom^{\Lambda}_{\delta}(\mu, x)}{-\log\delta}, \tag{2.9}$$

respectively. We interpret $\log 0 = 0$ to ensure $\dim^{\Lambda}_{\hom}(\mu, x) \ge 0$.

Remark 2.10. (1) The limit in (2.8) exists as $\hom^{\Lambda}_{\delta,\varepsilon_2,r}(\mu, x) \le \hom^{\Gamma}_{\delta,\varepsilon_1,r}(\mu, x)$ for all $0 < \varepsilon_1 < \varepsilon_2$ and $\Lambda \ge \Gamma > 1$.

(2) The definition of $\dim^{\Lambda}_{\hom}$ is quite technical. It may be helpful to compare it to the definition of the Assouad dimension given in Remark 2.2(1). The local homogeneity dimension may be considered as a kind of local Assouad dimension for the measure μ around x: it is the least possible exponent s so that for all small $\delta, r > 0$ the ball $B(x, r)$ has a δ-packing of size δ^{-s} such that the μ measure of the packing balls is comparable to $\mu\big(B(x, \Lambda r)\big)$.

(3) If μ is an s-regular measure on X with $\operatorname{spt}(\mu) = X$, then

$$\dim^{\Lambda}_{\hom}(\mu, x) = \dim_{\mathrm{loc}}(\mu, x) = s$$

for all $x \in X$. Indeed, a simple volume argument implies that for all $x \in X$, $r > 0$ and $0 < \delta < 1$, we have $c_1\delta^{-s} \le \sup\{\#\mathcal{B} : \mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x, r)\} \le c_2\delta^{-s}$. On the other hand, if $\varepsilon = \varepsilon(\delta) > 0$ is small, we have $\mu\big(B(y, \delta r)\big) > \varepsilon\mu\big(B(x, r)\big)$.

The next lemma shows that, at a typical point, the choice of the parameter Λ in the definition of homogeneity does not play any role. Therefore, in the applications, we may choose a convenient value for Λ.

Lemma 2.11. *If μ is a measure on a doubling metric space X and $\Lambda > \Gamma > 1$, then* $\dim_{\text{hom}}^{\Lambda}(\mu, x) = \dim_{\text{hom}}^{\Gamma}(\mu, x)$ *for μ-almost every $x \in X$.*

Proof. According to Remark 2.10(1), we have $\dim_{\text{hom}}^{\Lambda}(\mu, x) \leq \dim_{\text{hom}}^{\Gamma}(\mu, x)$ for all $x \in X$. The main point in the proof of the opposite inequality is the observation that if $\mathcal{B}$ is a δ-packing of $B(x, r)$, then for constants $c_1 = c_1(\Lambda, \Gamma) > 0$ and $c_2 = c_2(N, \Lambda, \Gamma) > 0$ there are $y \in B(x, r)$ and a δ-packing $\mathcal{B}' \subset \mathcal{B}$ of $B(y, cr)$ such that $B(y, \Lambda c_1 r) \subset B(x, \Gamma r)$ and $\#\mathcal{B}' \geq c_2 \#\mathcal{B}$.

In order to deliver full details of the proof, we assume to the contarary that there exists a set $A \subset X$ with $\mu(A) > 0$ and $t > 0$ so that

$$\dim_{\text{hom}}^{\Lambda}(\mu, x) < t < \dim_{\text{hom}}^{\Gamma}(\mu, x)$$

for all $x \in A$. Let $c = (\Gamma - 1)/2\Lambda\Gamma^q$ where $q \in \mathbb{N}$ is chosen so that $\Gamma^{q-1} \geq 5/(\Gamma - 1)$. According to Lemma 2.1(2) there exists $M \in \mathbb{N}$ such that a ball of radius r can be covered by M balls of radius $\min\{c, \Gamma^{-q}\}r$ for all $r > 0$. Going into a subset of A, if necessary, we find $r_0, \varepsilon, \delta > 0$ so that $\delta < \Gamma^{-q}$,

$$\text{hom}_{\delta,\varepsilon,r}^{\Lambda}(\mu, x) < \delta^{-t}/M^2$$

for every $0 < r < r_0$ and $x \in A$, and

$$\limsup_{r \downarrow 0} \text{hom}_{c\delta,\varepsilon,\Gamma^q r}^{\Gamma}(\mu, x) > \delta^{-t}$$

for all $x \in A$. Recalling Lemma 2.3, we may also assume that

$$\mu\big(B(x, \Gamma r) \setminus A\big) < \varepsilon\delta^{-t}\mu\big(B(x, 5\Gamma r)\big)/M^2 \tag{2.10}$$

for all $0 < r < r_0$ and $x \in A$. Next we fix $x \in A$ and choose $0 < r < r_0/\Gamma^q c$ so that

$$\text{hom}_{c\delta,\varepsilon,\Gamma^q r}^{\Gamma}(\mu, x) > \delta^{-t}.$$

Since $A \cap B(x, \Gamma^q r)$ can be covered by M balls of radius r with centers in $A \cap B(x, \Gamma^q r)$, we find $w \in A \cap B(x, \Gamma^q r)$ and a $(\Gamma^q c\delta r)$-packing $\mathcal{B}'$ of $B(w, r)$ so that $\#\mathcal{B}' \geq \delta^{-t}/M$ and

$$\mu(B) > \varepsilon\mu\big(B(x, \Gamma^{q+1} r)\big) \geq \varepsilon\mu\big(B(w, 5\Gamma r)\big) \tag{2.11}$$

for all $B \in \mathcal{B}'$. Covering $B(w, r)$ by M balls of radius cr, we see that at least one of the balls, say $B(y, cr)$, has a $(\Gamma^q c\delta r)$-packing $\mathcal{B} \subset \mathcal{B}'$ so that

$$\#\mathcal{B} \geq \delta^{-t}/M^2. \tag{2.12}$$

Since $B(z, \Lambda\Gamma^q cr) \subset B(w, 5\Gamma r)$ for $z \in B(y, 2cr)$ we now have

$$\hom^{\Lambda}_{\delta,\varepsilon,\Gamma^q cr}(\mu, z) > \delta^{-t}/M^2$$

for all $z \in B(y, 2cr)$, and, consequently, $A \cap B(y, 2cr) = \emptyset$. Using (2.10)–(2.12), we estimate

$$\begin{aligned}\mu\big(B(w, \Gamma r) \setminus A\big) < \#\mathcal{B}\varepsilon\mu\big(B(w, 5\Gamma r)\big) \le \sum_{B\in\mathcal{B}} \mu(B) &\le \mu\big(B(y, 2cr)\big)\\ &= \mu\big(B(y, 2cr) \setminus A\big).\end{aligned}$$

Since $B(y, 2cr) \subset B(w, \Gamma r)$ we arrive at a contradiction. □

Remark 2.12. (1) Let μ be a measure on X. Then, for every μ-measurable $A \subset X$, we have

$$\dim^{\Lambda}_{\hom}(\mu|_A, x) = \dim^{\Lambda}_{\hom}(\mu, x)$$

for μ-almost all $x \in A$. This is easily seen by combining Lemma 2.3 and Lemma 2.11 with the estimates $\mu\big(A \cap B(x, 5\Lambda r)\big) \ge \varepsilon\mu\big(B(x, \Lambda r)\big)$ and $\mu(B_i \cap A) \ge \mu(B_i) - \varepsilon\mu\big(B(x, 5r)\big)$ for $B_i \subset B(x, r)$ and $r, \varepsilon > 0$ small enough.

(2) The equality of Lemma 2.11 does not have to hold at every point $x \in X$ even when $X = \mathbb{R}^2$. To see this take

$$\mu = \sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{H}^1|_{S^1(0,2^{-k})},$$

where $\mathcal{H}^1|_{S^1(0,2^{-k})}$ is the length measure on $S^1(0, 2^{-k}) = \{y \in \mathbb{R}^2 : |y| = 2^{-k}\}$. Then we have $\dim^{3/2}_{\hom}\big(\mu, (0, 0)\big) = 1$, but $\dim^{5/2}_{\hom}\big(\mu, (0, 0)\big) = 0$.

(3) The definitions of local homogeneity and local L^q-spectrum make sense in any metric space in which balls are totally bounded. However, we will consider only doubling metric spaces since the doubling condition is needed in most of our proofs.

3. Main results

3.1. Relating L^q-dimensions with local dimensions

The L^q-spectrum of a measure is an essential tool in multifractal analysis and it has been investigated in many works, see *e.g.* [1,17,20,34,45,48,57] and [14,16,49] and references therein. It turns out that the well known Hausdorff and packing dimension estimates for a measure via its global L^q-spectrum generalize to the setting of local spectrum in doubling metric spaces. The following theorem is a local metric space version of the results obtained *e.g.* in [20, Theorem 1.3], [45, Theorem 1.1], and [17, Theorem 1.4]. See also [47, Corollary 1.3].

Theorem 3.1. *If μ is a measure on a doubling metric space X, then*

$$\lim_{q\downarrow 1}\dim_q(\mu,x) \le \underline{\dim}_{\mathrm{loc}}(\mu,x) \le \overline{\dim}_{\mathrm{loc}}(\mu,x) \le \lim_{q\uparrow 1}\dim_q(\mu,x) \tag{3.1}$$

for μ-almost all $x \in X$.

The proof of Theorem 3.1 is postponed until the end of this section. We remark that all the inequalities in (3.1) can be strict; see *e.g.* Remark 5.5.

Lemma 3.2. *If X is a doubling metric space, $A \subset X$ bounded, $r > 0$, μ a measure on X, $q \in \mathbb{R}$ and $0 < \delta < r$, then there is an r-packing $\mathcal{B}$ of A so that*

$$S_q(\mu, A, \delta) \le c \sum_{B\in\mathcal{B}} S_q(\mu, B, \delta),$$

where $c = c(N) \in \mathbb{N}$.

Proof. Using Lemma 2.1(5), we choose r-packings of A, say $\mathcal{B}_1, \ldots, \mathcal{B}_M$ where $M = M(N) \in \mathbb{N}$, whose union covers A. Fix $0 < \delta < r$ and let $\mathcal{B}'$ be a δ-packing of $A \cap \mathrm{spt}(\mu)$ such that $2\sum_{B\in\mathcal{B}'}\mu(B)^q > S_q(\mu, A, \delta)$. If $\mathcal{B}'_B = \{B' \in \mathcal{B}' :$ the center point of B' is in $B\}$ for all $B \in \bigcup_{i=1}^M \mathcal{B}_i$, then

$$\sum_{B'\in\mathcal{B}'} \mu(B')^q \le \sum_{i=1}^M \sum_{B\in\mathcal{B}_i} \sum_{B'\in\mathcal{B}'_B} \mu(B')^q \le \sum_{i=1}^M \sum_{B\in\mathcal{B}_i} S_q\big(\mu, B, \delta\big).$$

Thus $2M\sum_{B\in\mathcal{B}_i} S_q\big(\mu, B, \delta\big) \ge S_q(\mu, A, \delta)$ for some i. □

Lemma 3.3. *If μ is a measure on a doubling metric space X, then for any $q \ge 0$ and $\varepsilon > 0$, there is a countable covering of X by bounded sets A for which $\sup_{x\in A}\tau_q(\mu,x) \le \tau_q(\mu,A) + \varepsilon$.*

Proof. We may cover X by countably many sets of the form

$$A_\alpha = \{x \in X : \alpha < \tau_q(\mu,x) < \alpha + \varepsilon\}.$$

If $x \in A_\alpha$, then there exist $r > 0$ and $\delta_0 > 0$ such that $S_q\big(\mu, B(x,r), \delta\big) < \delta^\alpha$ for all $0 < \delta < \delta_0$. Thus, A_α can be covered by countably many sets of the form

$$A_{\alpha,r,\delta_0,R} = \{x \in A_\alpha \cap B(x_0,R) : S_q\big(\mu, B(x,r), \delta\big) < \delta^\alpha \text{ for all } 0 < \delta < \delta_0\}.$$

By Lemma 3.2, we find an r-packing $\mathcal{B}$ of $A_{\alpha,r,\delta_0,R}$ so that

$$\frac{\log S_q(\mu, A_{\alpha,r_0,\delta_0,R}, \delta)}{\log\delta} \ge \frac{\log c\sum_{B\in\mathcal{B}} S_q\big(\mu, B, \delta\big)}{\log\delta} \ge \frac{\log(\#\mathcal{B}c\delta^\alpha)}{\log\delta},$$

where $c = c(N) \in \mathbb{N}$. Since $\mathcal{B}$ has at most $M = M(r, R, N) \in \mathbb{N}$ elements by Lemma 2.1(3), we get $\tau_q(\mu, A_{\alpha,r,\delta_0,R}) \ge \alpha$ by letting $\delta \downarrow 0$. □

The following lemma can be considered as a global version of Theorem 3.1.

Lemma 3.4. *If μ is a measure on a doubling metric space X and $A \subset X$ is bounded, then*

$$\begin{aligned}\dim_q(\mu, A) &\le \mu\text{-}\operatorname{ess\,inf}_{x\in A} \underline{\dim}_{\mathrm{loc}}(\mu, x)\\ &\le \mu\text{-}\operatorname{ess\,sup}_{x\in A} \overline{\dim}_{\mathrm{loc}}(\mu, x) \le \dim_p(\mu, A)\end{aligned}$$

for all $0 < p < 1 < q$.

Proof. Let $q > 1$. If $s > \mu\text{-}\operatorname{ess\,inf}\{\underline{\dim}_{\mathrm{loc}}(\mu, x) : x \in A\}$ and $A_n = \{x \in A \cap \operatorname{spt}(\mu) : \mu\big(B(x, 2^{-n})\big) > 2^{-ns}\}$, then $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ by the Borel-Cantelli Lemma. Thus, there are arbitrarily large n such that $\mu(A_n) > n^{-2}$. Fix such an n and let $\mathcal{B}$ be a (2^{-n})-packing of A_n satisfying (2.1). Then

$$\begin{aligned}S_q(\mu, A, 2^{-n}) &\ge \sum_{B\in\mathcal{B}} \mu(B)^q = \sum_{B\in\mathcal{B}} \mu(B)\mu(B)^{q-1}\\ &\ge \sum_{B\in\mathcal{B}} \mu(B)2^{-ns(q-1)} \ge c\mu(A_n)2^{-ns(q-1)} \ge cn^{-2}2^{-ns(q-1)}.\end{aligned}$$

Taking logarithms and letting $n \to \infty$, this implies $\tau_q(\mu, A) \le s(q-1)$ and, consequently, $\dim_q(\mu, A) \le s$ as required.

If $0 < p < 1$, then we complete the proof by repeating the above argument with q replaced by p, $s < \mu\text{-}\operatorname{ess\,sup}\{\overline{\dim}_{\mathrm{loc}}(\mu, x) : x \in A\}$, and $A_n = \{x \in A \cap \operatorname{spt}(\mu) : \mu\big(B(x, 2^{-n})\big) < 2^{-ns}\}$. □

Proof of Theorem 3.1. The proof follows simply by combining Lemmas 3.3 and 3.4. Indeed, for $q > 1$ and $\varepsilon > 0$, decompose X into countably many bounded sets A for which $\sup_{x\in A} \tau_q(\mu, x) \le \tau_q(\mu, A) + \varepsilon$. Lemma 3.4 then implies that

$$\dim_q(\mu, x) - \varepsilon/(q-1) \le \dim_q(\mu, A) \le \underline{\dim}_{\mathrm{loc}}(\mu, x)$$

for μ-almost all $x \in A$. The leftmost inequality of (3.1) follows now by letting $\varepsilon \downarrow 0$. Recall that the limit exists by Lemma 2.7(5). The proof in the case $0 < q < 1$ is similar. □

3.2. Upper bound for local dimensions via local homogeneity dimension

In this section, we prove our main result showing that the local homogeneity dimension is almost everywhere at least as large as the upper local dimension.

Theorem 3.5. *If μ is a measure on a doubling metric space X and $\Lambda > 1$, then*

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le \dim_{\mathrm{hom}}^{\Lambda}(\mu, x)$$

for μ-almost all $x \in X$.

Theorem 3.5 is obtained as a corollary to a more quantitative result, Theorem 3.8, which will be essential in our applications in Section 4. Before we turn to Theorem 3.8, we exhibit some auxiliary results. We first observe that the homogeneity can be used to bound the L^q-moment sums.

Lemma 3.6. *If X is a doubling metric space and $0 < \delta < 1$, then there is $M = M(\delta, N) \in \mathbb{N}$ so that for every measure μ on X and for all $x \in X$, and $r, \varepsilon > 0$ we have*

$$S_q\big(\mu, B(x,r), \delta r\big) \le \big(\mathrm{hom}^{\Lambda}_{\delta,\varepsilon,r}(\mu, x)^{1-q} + M\varepsilon^q\big)\mu\big(B(x, \Lambda r)\big)^q,$$

with any $\Lambda > 1$ and $0 < q < 1$,.

Proof. If $\mathcal{B}$ is a (δr)-packing of $B(x,r)$ and $\mathcal{B}' = \{B \in \mathcal{B} : \mu(B) > \varepsilon\mu\big(B(x, \Lambda r)\big)\}$, then Hölder's inequality implies

$$\sum_{B \in \mathcal{B}'} \mu(B)^q \le \mathrm{hom}^{\Lambda}_{\delta,\varepsilon,r}(\mu, x)^{1-q}\mu\big(B(x, \Lambda r)\big)^q.$$

On the other hand, since

$$\sum_{B \in \mathcal{B}\setminus\mathcal{B}'} \mu(B)^q \le \#\mathcal{B}\varepsilon^q\mu\big(B(x, \Lambda r)\big)^q$$

and $\#\mathcal{B} \le M(\delta, N)$ by Lemma 2.1(3), the claim follows. □

Lemma 3.7. *If X is a doubling metric space, q, δ are positive and smaller than 1, and $m > 0$, then there exists a constant $\varepsilon = \varepsilon(q, \delta, m, N) > 0$ satisfying the following: if μ is a measure on X, $\Lambda > 1$, and $A \subset X$ is bounded so that $\mathrm{hom}^{\Lambda}_{\delta,\varepsilon,r}(\mu, x) \le \delta^{-m}$ for all $x \in A$ and $0 < r < r_0$, then there is a constant $c = c(N, \Lambda) \ge 1$ so that*

$$S_q(\mu, A, \delta r) \le c\delta^{m(q-1)}S_q(\mu, A, \Lambda r)$$

for all $0 < r < r_0$.

Proof. Let $\varepsilon > 0$ be so small that $M\varepsilon^q \le \delta^{m(q-1)}$, where M is as in Lemma 3.6. According to Lemma 3.2, there are $c = c(N) \in \mathbb{N}$ and an r-packing $\mathcal{B}$ of A so that

$$S_q(\mu, A, \delta r) \le c\sum_{B \in \mathcal{B}} S_q(\mu, B, \delta r) \le c\sum_{B \in \mathcal{B}} 2\delta^{m(q-1)}\mu(\Lambda B)^q$$

by Lemma 3.6, the homogeneity assumption and the choice of ε. The claim now follows since $\sum_{B\in\mathcal{B}} \mu(\Lambda B)^q \le c(N, \Lambda)S_q(\mu, A, \Lambda r)$ by Lemma 2.1(4). □

The following theorem is our main quantitative result concerning local homogeneity of measures.

Theorem 3.8. *If X is a doubling metric space, $0 < m < s$, and $\Lambda > 1$, then there exists a constant $\delta_0 = \delta_0(m, s, N, \Lambda) > 0$ satisfying the following: for every $0 < \delta < \delta_0$ there is $\varepsilon_0 = \varepsilon_0(\delta, m, N) > 0$ so that for each measure μ on X we have*

$$\limsup_{r\downarrow 0} \operatorname{hom}^{\Lambda}_{\delta,\varepsilon,r}(\mu, x) \geq \delta^{-m} \tag{3.2}$$

for all $0 < \varepsilon \leq \varepsilon_0$ and for μ-almost all $x \in X$ that satisfy $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$.

Proof. Fix $0<q<1$. Let $\delta_0>0$ be so small that $\log(c\Lambda^{m(q-1)})/\log((\delta_0/\Lambda)^{q-1}) > (s-m)(q-1)$, where $c = c(N, \Lambda) > 0$ is as in Lemma 3.7. Fix $0 < \delta < \delta_0$ and let $\varepsilon = \varepsilon(q, \delta, m, N) > 0$ be as in Lemma 3.7. Given $x_0 \in X$ and positive R, and r_0, it suffices to show that $\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq s$ for μ-almost every point in the set

$$A = \{x \in B(x_0, R) : \operatorname{hom}^{\Lambda}_{\delta,\varepsilon,r}(\mu, x) < \delta^{-m} \text{ for all } 0 < r < r_0\}.$$

According to Lemma 3.7, we have $S_q(\mu, A, \delta r/\Lambda) \leq c\delta^{m(q-1)} S_q(\mu, A, r)$ for all $0<r<r_0$. A simple induction gives $S_q\big(\mu, A, (\delta/\Lambda)^n r_0\big) \leq c^n \delta^{nm(q-1)} S_q(\mu, A, r_0)$ for all $n \in \mathbb{N}$. Therefore

$$\tau_q(\mu, A) = \liminf_{n\to\infty} \frac{\log S_q\big(\mu, A, (\delta/\Lambda)^n r_0\big)}{\log\big((\delta/\Lambda)^n r_0\big)} \geq m(q-1) + \frac{\log(c\Lambda^{m(q-1)})}{\log(\delta/\Lambda)}$$

and so $\dim_q(\mu, A) \leq s$ by the choice of δ_0. Lemma 3.4 then gives $\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq s$ at μ-almost all points $x \in A$. □

Remark 3.9. In general, it is possible that $\dim_q(\mu, x) > c > 0$ for all $0 < q < 1$ almost everywhere while $\dim_{\mathrm{hom}}(\mu, x) = 0$; see Example 5.4. It is essential in the proof of Theorem 3.8 that in the set A, where we have uniform estimates for $\operatorname{hom}^{\Lambda}_{\delta,\varepsilon,r}(\mu, x)$, we can use $\dim_{\mathrm{hom}}(\mu, x)$ to bound $\dim_q(\mu, A)$ from above.

Proof of Theorem 3.5. Assume to the contrary that there are $A \subset X$ with $\mu(A) > 0$ and $0 < m < s$ such that $\dim^{\Lambda}_{\mathrm{hom}}(\mu, x) < m < s < \overline{\dim}_{\mathrm{loc}}(\mu, x)$ for all $x \in A$. It follows from Theorem 3.8 that there is $\delta_0 = \delta_0(m, s, N, \Lambda) > 0$ so that $\operatorname{hom}^{\Lambda}_{\delta}(\mu, x) \geq \delta^{-m}$ for every $0 < \delta < \delta_0$ and for μ-almost all $x \in A$. Thus $\dim^{\Lambda}_{\mathrm{hom}}(\mu, x) \geq m$ for μ-almost all $x \in A$ giving a contradiction. □

3.3. Entropy dimension

We complete the discussion on $\dim_q$ by treating the case $q = 1$. This is done by defining for $A \subset X$ with $\mu(A) > 0$ the *(global) upper and lower entropy dimensions* of μ on A as

$$\overline{\dim}_1(\mu, A) = \limsup_{\delta\downarrow 0} \fint_A \frac{\log \mu\big(B(y, \delta)\big)}{\log \delta}\, \mathrm{d}\mu(y),$$

$$\underline{\dim}_1(\mu, A) = \liminf_{\delta\downarrow 0} \fint_A \frac{\log \mu\big(B(y, \delta)\big)}{\log \delta}\, \mathrm{d}\mu(y),$$

respectively. If they agree, then their common value is denoted by $\dim_1(\mu, A)$. Here and hereafter, for $A \subset X$ and a μ-measurable $f\colon X \to \overline{\mathbb{R}}$, we use notation $\fint_A f(y)\,\mathrm{d}\mu(y) = \mu(A)^{-1}\int_A f(y)\,\mathrm{d}\mu(y)$ whenever the integral is well defined. The *local upper and lower entropy dimensions at $x \in \mathrm{spt}(\mu)$* are then defined as

$$\begin{aligned}\overline{\dim}_1(\mu, x) &= \limsup_{r\downarrow 0} \overline{\dim}_1\bigl(\mu, B(x,r)\bigr),\\ \underline{\dim}_1(\mu, x) &= \liminf_{r\downarrow 0} \underline{\dim}_1\bigl(\mu, B(x,r)\bigr).\end{aligned} \tag{3.3}$$

Our results on $\dim_1(\mu, x)$ are local metric space versions of the corresponding global Euclidean results. For instance, see [20, Theorem 4.1] and [17, Theorem 1.4]. The case $q = 1$ is different from $q \neq 1$ in the sense that it cannot be studied solely by using Borel-Cantelli type arguments. Also, in the main result of this section, Theorem 3.11, the density point property is a crucial assumption and it cannot be replaced by the weaker and more general condition given by Lemma 2.3 as is the case $q \neq 1$.

The following proposition shows that the definition of $\dim_1$ is consistent with the basic properties of $\dim_q$.

Proposition 3.10. *If μ is a measure on a doubling metric space X, then*

$$\lim_{q\downarrow 1} \dim_q(\mu, x) \le \underline{\dim}_1(\mu, x) \le \overline{\dim}_1(\mu, x) \le \lim_{q\uparrow 1} \dim_q(\mu, x)$$

for all $x \in \mathrm{spt}(\mu)$.

The proof of the proposition involves the partition definition of $\dim_q$. Since we do not need the result in this article, we will omit the proof.

A detailed proof can be found in an earlier arXiv version of the manuscript, see `http://arxiv.org/abs/1003.2895v1`.

Theorem 3.11. *If μ is a measure on a doubling metric space X so that it satisfies the density point property, then*

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) \le \underline{\dim}_1(\mu, x) \le \overline{\dim}_1(\mu, x) \le \overline{\dim}_{\mathrm{loc}}(\mu, x) \tag{3.4}$$

for μ-almost all $x \in X$.

Proof. We may assume that the measure is non-atomic as the claim is obvious if $\mu(\{x\}) > 0$. Given $\varepsilon > 0$, we may cover μ-almost all of X by countably many sets of the form $A' = \{y \in X : t < \underline{\dim}_{\mathrm{loc}}(\mu, y) < t + \varepsilon\}$ and each of these can be covered by countably many sets of the form $A = \{y \in A' : \mu\bigl(B(x,r)\bigr) < r^t$ for all $0 < r < q\}$. For $x \in \mathrm{spt}(\mu)$ and $0 < \delta < q$, we have

$$\begin{aligned}\fint_{B(x,r)} \frac{\log\mu\bigl(B(y,\delta)\bigr)}{\log\delta}\,\mathrm{d}\mu(y) &\ge \frac{1}{\mu\bigl(B(x,r)\bigr)} \int_{A\cap B(x,r)} \frac{\log\mu\bigl(B(y,\delta)\bigr)}{\log\delta}\,\mathrm{d}\mu(y)\\ &\ge t\,\frac{\mu\bigl(A\cap B(x,r)\bigr)}{\mu\bigl(B(x,r)\bigr)}.\end{aligned}$$

Here we have assumed that $\mu\big(B(x, r+\delta)\big) < 1$. For small r and δ this is the case since μ has no atoms. Since almost all points $x \in A$ are density points, we get

$$\underline{\dim}_1(\mu, x) = \liminf_{r\downarrow 0} \liminf_{\delta\downarrow 0} \fint_{B(x,r)} \frac{\log \mu\big(B(y,\delta)\big)}{\log \delta}\, \mathrm{d}\mu(y) \geq t$$

for μ-almost all $x \in A$ and consequently $\underline{\dim}_1(\mu, x) \geq \underline{\dim}_{\mathrm{loc}}(\mu, x) - \varepsilon$ for μ-almost all $x \in X$.

To prove the estimates for the upper dimension, a similar covering argument as above implies that it suffices to show that if $0 < q, t < \infty$, then $\overline{\dim}_1(\mu, x) \leq t$ for μ-almost all $x \in A = \{y \in X : \mu\big(B(y,r)\big) > r^t$ for all $0 < r < q\}$. Let $x \in X$ and $0 < r < q$. For $0 < \delta < q$ and $t \leq \alpha < \infty$, define $E_{\delta,\alpha} = \{y \in B(x,r) : \mu\big(B(y,\delta)\big) < \delta^\alpha\}$. By Lemma 2.1(2), $E_{\delta,\alpha}$ can be covered by $C\delta^{-s}$ balls of radius δ with centres in $E_{\delta,\alpha}$, where $s = s(N) > 0$ and $C = C(N, r) > 0$. Thus $\mu(E_{\delta,\alpha}) \leq C\delta^{\alpha-s}$. Let $J = \{y \in B(x,r) : \mu\big(B(y,\delta)\big) > \delta^t\}$, $K = \{y \in B(x,r) : \delta^{2s} \leq \mu\big(B(y,\delta)\big) \leq \delta^t\}$, and $L = \{y \in B(x,r) : \mu\big(B(y,\delta)\big) < \delta^{2s}\}$. Then $B(x,r) = J \cup K \cup L$. Moreover,

$$\begin{aligned}
&\int_J \frac{\log \mu\big(B(y,\delta)\big)}{\log \delta}\, \mathrm{d}\mu(y) \leq t\mu(J) \leq t\mu\big(B(x,r)\big),\\
&\int_K \frac{\log \mu\big(B(y,\delta)\big)}{\log \delta}\, \mathrm{d}\mu(y) \leq 2s\mu(K) \leq 2s\mu\big(B(x,r) \setminus A\big),\\
&\int_L \frac{\log \mu\big(B(y,\delta)\big)}{\log \delta}\, \mathrm{d}\mu(y) = \int_{2s}^\infty \mu(E_{\delta,\alpha})\, \mathrm{d}\alpha \leq C\delta^{-s} \int_{2s}^\infty \delta^\alpha\, \mathrm{d}\alpha = \frac{C\delta^s}{-\log \delta}.
\end{aligned}$$

Putting these together and letting $\delta \downarrow 0$ in the last estimate, we get

$$\overline{\dim}_1\big(\mu, B(x,r)\big) = \limsup_{\delta\downarrow 0} \fint_{B(x,r)} \frac{\log \mu\big(B(y,\delta)\big)}{\log \delta}\, \mathrm{d}\mu(y) \leq t + 2s\frac{\mu\big(B(x,r) \setminus A\big)}{\mu\big(B(x,r)\big)}$$

and, consequently,

$$\overline{\dim}_1(\mu, x) = \limsup_{r\downarrow 0} \overline{\dim}_1\big(\mu, B(x,r)\big) \leq t$$

for all density points of A. The claim follows since μ has the density point property. $\square$

Remark 3.12. (1) By inspecting the above proof, we easily get a global analogue of Theorem 3.11: if $A \subset X$ is bounded and $\mu(A) > 0$, then it holds that

$$\begin{aligned}
\mu\text{-}\operatorname{ess\,sup}_{x\in A} \overline{\dim}_{\mathrm{loc}}(\mu, x) &\geq \overline{\dim}_1(\mu, A),\\
\mu\text{-}\operatorname{ess\,inf}_{x\in A} \underline{\dim}_{\mathrm{loc}}(\mu, x) &\leq \underline{\dim}_1(\mu, A).
\end{aligned}$$

It is worthwhile to notice that the density point property is not needed in this case.

(2) In Examples 5.7 and 5.8, we show that Theorem 3.11 does not hold without the density point property. This is a remarkable difference between the global and local entropy dimensions.

4. Applications

In this section, we use the local homogeneity estimate of Theorem 3.8 as the final step in proving various new results. In fact, understanding the conical density and porosity questions in Section 4.1–Section 4.3 below was our main motivation for investigating the local homogeneity. In addition to Theorem 3.8, the proofs will be based on already known geometric conclusions.

4.1. Upper conical densities in Euclidean spaces

Let $G(d, n)$ be the Grasmann manifold of all n-dimensional linear subspaces of $\mathbb{R}^d$ and $S^{d-1} = \{y \in \mathbb{R}^d : |y| = 1\}$ the unit sphere in $\mathbb{R}^d$. Then, given real numbers $0 < \alpha \le 1$, and $r > 0$, for $V \in G(d, d-k)$, and $x \in \mathbb{R}^d$ we define cones

$$X(x, r, V, \alpha) = \{y \in B(x, r) : \operatorname{dist}(y - x, V) < \alpha|y - x|\}$$

and, for $\theta \in S^{d-1}$,

$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^d : (y - x) \cdot \theta > \alpha|y - x|\}.$$

With small α the cones $X(x, r, V, \alpha)$ are small cones around the translate of the subspace V by x, whereas the cone $H(x, \theta, \alpha)$ is almost a half-space from the point x to the direction θ.

The distribution of Hausdorff and packing type measures inside cones is well studied and understood; see, for example, [31,38,40,54,60]. For general measures the following theorem was proved in [11, Theorem 4.1] under the assumption that the Hausdorff dimension of the measure is greater than s. We improve this result by showing that the theorem is true even if we assume a lower bound only for the packing (*i.e.* the upper local) dimension of the measure.

Theorem 4.1. *If $d \in \mathbb{N}$ and $k \in \{0, \dots, d-1\}$ with $s > k$, then there exists a constant $c = c(d, k, s, \alpha) > 0$, for $0 < \alpha \le 1$, so that for every measure μ on $\mathbb{R}^d$ we have*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{d-1} \\ V \in G(d,d-k)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{\mu\big(B(x, r)\big)} > c$$

for μ-almost all $x \in \mathbb{R}^d$ that satisfy $\overline{\dim}_{\text{loc}}(\mu, x) > s$.

Proof. We can reduce the proof to verifying the following condition (see [11, Proposition 4.5]): for a given $q, K \in \mathbb{N}$ and $1 < t < \infty$ there exists a constant $\varepsilon = \varepsilon(d, k, s, q, K, t) > 0$ so that for μ-almost all $x \in \{y \in \mathbb{R}^d : \overline{\dim}_{\text{loc}}(\mu, y) > s\}$ we may find arbitrarily small radii $r > 0$ and ball families $\mathcal{B}$ with the following properties:

(1) $B \subset B(x, r)$ for all $B \in \mathcal{B}$.
(2) The collection $t\mathcal{B} = \{tB : B \in \mathcal{B}\}$ is a packing.

(3) $\mu(B) > \varepsilon\mu\big(B(x, 3r)\big)$ for all $B \in \mathcal{B}$.
(4) If $\mathcal{B}' \subset \mathcal{B}$ with $\#\mathcal{B}' \geq \#\mathcal{B}/K$ and $V \in G(d, d-k)$, then there is a translate of V intersecting at least q balls from the collection $\mathcal{B}'$.

We will construct the families $\mathcal{B}$ with the help of Theorem 3.8. Let $M = M(N_d, t^{-1})$ be the constant from Lemma 2.1(4), where N_d is the doubling constant of $\mathbb{R}^d$. Let $m = (s+k)/2$ and choose $0 < \delta < \min\{\delta_0, \frac{1}{4}\}$ so that $4^{-k}\delta^{k-m} \geq 2KMq$, where δ_0 is as in Theorem 3.8. By Theorem 3.8 there is $\varepsilon = \varepsilon(m, s, N_d, \delta) > 0$ so that $\limsup_{r\downarrow 0} \hom^5_{\delta,\varepsilon,r}(\mu, x) \geq \delta^{-m}$ for μ-almost all $x \in \mathbb{R}^d$ that satisfy $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$. Fix such a point x and let $r > 0$ so that $\hom_{\delta,\varepsilon,\frac{3}{4}r}(\mu, x) > \delta^{-m}/2$. Now there is a $(\frac{3}{4}\delta r)$-packing of $B(x, \frac{3}{4}r)$, say $\mathcal{B}_0$, with $\#\mathcal{B}_0 > \delta^{-m}/2$ so that $\mu(B) > \varepsilon\mu\big(B(x, \frac{15}{4}r)\big) \geq \varepsilon\mu\big(B(x, 3r)\big)$ for all $B \in \mathcal{B}_0$.

Lemma 2.1(4) gives a subcollection $\mathcal{B} \subset \mathcal{B}_0$ for which $t\mathcal{B}$ is also a packing and $\#\mathcal{B} \geq \#\mathcal{B}/M \geq \delta^{-m}/(2M)$. Now, because $\delta \leq \frac{1}{4}$, $B \subset B(x, r)$ for each $B \in \mathcal{B}$. Thus conditions (1)–(3) hold. The only property we need to verify is the condition (4). Suppose that $\mathcal{B}' \subset \mathcal{B}$ with $\#\mathcal{B}' \geq \#\mathcal{B}/K \geq \delta^{-m}/(2KM)$, and let $V \in G(d, d-k)$. The orthogonal projection of $B(x, r)$ into the orthogonal complement of V can be covered by $4^k\delta^{-k}$ balls of radius $\frac{3}{4}\delta r$ and so some translate of V must intersect at least

$$4^{-k}\delta^k\#\mathcal{B}' \geq \frac{4^{-k}\delta^{k-m}}{2KM} \geq q$$

balls from the collection $\mathcal{B}'$. Thus also (4) holds and the proof is finished. □

4.2. Porous measures on Euclidean spaces

We first define porosity for sets. Let $A \subset \mathbb{R}^d$, let $k \in \{1, \dots, d\}$ for $x \in A$, and $r > 0$. We define

$$\begin{aligned}\mathrm{por}_k(A, x, r) = \sup\{\varrho \geq 0 : {} & \text{there are } y_1, \dots, y_k \in \mathbb{R}^d \text{ such that for every } i \\ & A \cap B(y_i, \varrho r) = \emptyset \text{ and } \varrho r + |x - y_i| \leq r, \\ & \text{and } (y_i - x)\cdot(y_j - x) = 0 \text{ if } j \neq i\}\end{aligned}$$

and from this the *k-porosity of A at x* as

$$\mathrm{por}_k(A, x) = \liminf_{r\downarrow 0} \mathrm{por}_k(A, x, r).$$

We refer to the balls $B(y_i, \varrho r)$ in the definition as "holes". The notion of k-porosity was introduced in [32].

When we combine this definition with the porosity for measures, defined for the first time in [13], we obtain k-porosity for measures: let μ be a measure on $\mathbb{R}^d$, for $k \in \{1, \dots, d\}$, and $x \in \mathbb{R}^d$, with $r > 0$, and $\varepsilon > 0$. We set

$$\begin{aligned}\mathrm{por}_k(\mu, x, r, \varepsilon) = \sup\{\varrho \geq 0 : {} & \text{there are } y_1, \dots, y_k \in \mathbb{R}^d \text{ such that for every } i \\ & \mu\big(B(y_i, \varrho r)\big) \leq \varepsilon\mu\big(B(x, r)\big) \text{ and } \varrho r + |x - y_i| \leq r, \\ & \text{and } (y_i - x)\cdot(y_j - x) = 0 \text{ if } j \neq i\}\end{aligned}$$

and the *k-porosity of the measure μ at x* is defined to be

$$\operatorname{por}_k(\mu, x) = \lim_{\varepsilon\downarrow 0}\liminf_{r\downarrow 0}\operatorname{por}_k(\mu, x, r, \varepsilon).$$

It follows from [13, Section 2] that $\operatorname{por}_k(\mu, x) \le \frac{1}{2}$ for μ-almost all $x \in \mathbb{R}^d$. We remark that a more precise name for the porosity just defined would be lower porosity, to distinguish this notion from the upper porosity of sets and measures; see, *e.g.*, [43,61].

We provide an upper bound for the upper local dimension of measures with k-porosity close to the maximum value $\frac{1}{2}$. In [5], this result was proved for $k = 1$. The first estimates for the dimension of sets with 1-porosity close to $\frac{1}{2}$ are from [40] and [55]. For more recent results on the dimension of porous sets and measures; see [10,26,32,52] and [5,6,13,23,31]. It is important to notice both here and in Theorem 4.7 that even if $\operatorname{por}_1(\mu, x) > 0$ in a set of positive μ-measure, it is possible that $\mu(A) = 0$ for all $A \subset X$ with $\inf_{x\in A}\operatorname{por}_1(A, x) > 0$; see [5, Theorem 4.1].

Theorem 4.2. *If $d \in \mathbb{N}$, then there exists a constant $c = c(d) > 0$ so that for every measure μ on $\mathbb{R}^d$ we have*

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le d - k + \frac{c}{-\log\bigl(1 - 2\operatorname{por}_k(\mu, x)\bigr)}$$

for μ-almost all $x \in \mathbb{R}^d$.

Remark 4.3. (1) It is rather easy to see that the upper bound in Theorem 4.2 is asymptotically sharp as $\operatorname{por}_k(\mu, x) \uparrow \frac{1}{2}$: for each $\varrho < \frac{1}{2}$ there exists a measure μ on $\mathbb{R}^d$ with $\operatorname{por}_k(\mu, x) \ge \varrho$ while $\underline{\dim}_{\mathrm{loc}}(\mu, x) \ge d - k - c/\log(1 - 2\varrho)$ for μ-almost all $x \in \mathbb{R}^d$. The easiest way to see this is to consider a regular Cantor set $C \subset \mathbb{R}$ with 1-porosity ϱ and to let μ be the natural measure on $C^k \times [0, 1]^{d-k}$.

(2) The proof of Theorem 4.2 in the case $k = 1$ given in [5] is based on an extensive use of dyadic cubes. The interplay between cubes and balls caused many technical problems, which were finally solved by considering the boundary regions of cubes separately. The method used there does not work for k-porosity when $k \ge 2$ although the statement itself has nothing to do with co-dimension being one.

Before proving Theorem 4.2, we will exhibit a couple of geometric lemmas concerning k-porous sets.

Lemma 4.4. *If $A \subset B(x_0, r) \subset \mathbb{R}^d$ is so that $\operatorname{por}_k(A, z, r) \ge \varrho$ for every $z \in A$, then the set A can be covered with $c(1 - 2\varrho)^{k-d}$ balls of radius $(1 - 2\varrho)r$, where $c > 0$ depends only on d.*

Proof. The proof is based on geometric arguments similar to these used in [26, Theorem 2.5], [5, Lemmas 3.4 and 3.5], and [52, Lemma 5.1]. In the proof, we will omit some of the elementary, if tedious, details.

Let $c_1, c_2, c_3 > 0$ be small constants. We may assume that $\varrho > \frac{1}{2} - c_1$. A simple compactness argument implies that $\mathbb{R}^d$ can be covered by $m = m(d, c_2)$ cones $\{H(0, \theta_i, 1 - c_2)\}_{i=1}^m$. Observe that $H(0, \theta_i, 1 - c_2)$ is a cone to the direction $\theta_i \in S^{d-1}$ with a small opening angle.

For each point $y \in A$, denote the centres of the holes obtained from the k-porosity on the scale r by $y_1, \dots, y_k$. Thus, $A \cap B(y_i, \varrho r) = \emptyset$ and $|y_i - y| + \varrho r \le r$ for every i, and $(y_i - y) \cdot (y_j - y) = 0$ whenever $i \ne j$. We observe that A may be divided into m^k sets of the form

$$A_{\mathtt{i}} = \big\{ y \in A : y_j - y \in H(0, \theta_{i_j}, 1 - c_2) \text{ for every } j \in \{1, \dots, k\} \big\}.$$

where $\mathtt{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k$. Since $(y_i - y) \cdot (y_j - y) = 0$ for all $y \in A$ and all $i \ne j$, it follows that actually most of the sets $A_{\mathtt{i}}$ are empty. Fix $\mathtt{i}$ so that $A_{\mathtt{i}} \ne \emptyset$ and choose x so that $A_{\mathtt{i}} \cap B(x, c_3 r) \ne \emptyset$. Define

$$M_j = B(x, 2c_3 r) \cap \partial \Bigg(\bigcup_{y \in A_{\mathtt{i}} \cap B(x, c_3 r)} B(y_j, \varrho r) \Bigg)$$

for all $j \in \{1, \dots, k\}$ and let

$$M = \bigcap_{j=1}^{k} M_j.$$

Here ∂C is the topological boundary of a given set C.

By simple (but rather technical) geometric inspections, we observe that if c_1, c_2, and c_3 are chosen small enough (depending only on d), then the following assertions are true: if f is the orthogonal projection from M to the k-dimensional linear subspace $\bigcap_{j=1}^k \theta_{i_j}^{\perp}$, then

$$|f(y) - f(z)| \le |y - z| \le 2|f(y) - f(z)|$$

for all $y, z \in M$, so f is bi-Lipschitz with constant 2. Moreover, $\operatorname{dist}(y, M) \le 2\sqrt{d}(1 - 2\varrho)r$ for all $y \in A_{\mathtt{i}} \cap B(x, c_3 r)$. These estimates easily imply that $B(x, c_3 r) \cap A_{\mathtt{i}}$ may be covered by $c_4(1 - 2\varrho)^{k-d}$ balls of radius $(1 - 2\varrho)r$, where c_4 depends only on d and the choice of c_3. On the other hand, the set $A_{\mathtt{i}} \cap B(x, r)$ is clearly covered by $2^{2d} c_3^{-d}$ balls of radius $c_3 r$ and finally A is covered by less than $m^k 2^{2d} c_3^{-d} c_4 (1 - 2\varrho)^{k-d}$ balls of radius $(1 - 2\varrho)r$. $\square$

Next we turn the previous lemma into a homogeneity estimate.

Lemma 4.5. *If* $0 < \varrho < \frac{1}{2}$ *and* μ *is a measure on* $\mathbb{R}^d$ *such that* $\mu(A) > 0$, *where* $A \subset \{x \in \mathbb{R}^d : \operatorname{por}_k(\mu, x) > \varrho\}$, *then for each* $\varepsilon > 0$ *there is a Borel set* $A_\varepsilon \subset A$ *with* $\mu(A_\varepsilon) > 0$ *such that*

$$\limsup_{r \downarrow 0} \hom^5_{1-2\varrho, \varepsilon, r}(\mu, x) < c(1 - 2\varrho)^{k-d}$$

for every $x \in A_\varepsilon$, *where* $c > 0$ *depends only on* d.

Proof. Let $\varepsilon > 0$ and take $r_0 > 0$ so that the set

$$A_\varepsilon = \{x \in A : \mathrm{por}_k(\mu, x, r, \varepsilon/2) \geq \varrho \text{ for all } 0 < r < r_0\}$$

has positive μ-measure. Now take a density point $x \in A_\varepsilon$ and a radius $0 < r \leq r_0/5$ for which

$$\frac{\mu\big(A_\varepsilon \cap B(x, 5r)\big)}{\mu\big(B(x, 5r)\big)} > 1 - \varepsilon. \tag{4.1}$$

Let $\mathcal{B}$ be a $((1 - 2\varrho)r)$-packing of $B(x, r)$ so that $\mu(B) > \varepsilon\mu\big(B(x, 5r)\big)$ for all $B \in \mathcal{B}$. Write $A_\mathcal{B}$ for the centres of the balls in $\mathcal{B}$. For each $B \in \mathcal{B}$ choose $y \in A_\varepsilon \cap B$. Because of (4.1), such a point y exists. A direct calculation using the k-porosity at y on the scale r implies that

$$\mathrm{por}_k(A_\mathcal{B}, x, r) \geq \varrho - 2(1 - 2\varrho),$$

where x is the centre of B. Since this holds for all $x \in A_\mathcal{B}$, Lemma 4.4 implies that $A_\mathcal{B}$ may be covered by $c\big(1 - 2\big(\varrho - 2(1 - 2\varrho)\big)\big)^{k-d} = 5^{k-d}c(1 - 2\varrho)^{k-d}$ balls of radius $5(1 - 2\varrho)r$. Here $c = c(d)$ is the constant of Lemma 4.4. It now follows that $\#\mathcal{B} = \#A_\mathcal{B} \leq 10^d 5^{k-d} c(1 - 2\varrho)^{k-d}$ yielding the claim. It is important to note here that we are not covering the set A_ε as it generally is not even porous. □

Proof of Theorem 4.2. Let $1 < c = c(d) < \infty$ be the constant of Lemma 4.5 and let $0 < \varrho < \frac{1}{2}$. From the proof of Theorem 3.8 we observe that there exists a constant $0 < c_1 = c_1(d) < 1$ so that for any $0 < m < s$ the choice $\delta_0 = c_1^{1/(m-s)}$ will suite as $\delta_0 = \delta_0 = (m, s, 5, N_d)$ in the claim of Theorem 3.8. Our aim is then to apply Theorem 3.8 with

$$m = d - k + \frac{\log c}{-\log(1 - 2\varrho)}, \quad s = m + \frac{\log c_1}{\log(1 - 2\varrho)},$$

and $\delta = 1 - 2\varrho$. Let $t = (m + s)/2$ and take $M = M(N_d, \frac{1}{10})$ from Lemma 2.1(4). Here N_d is the doubling constant of $\mathbb{R}^d$.

Let $\delta_0 = \delta_0(m, s, N_d)$ be the constant in Theorem 3.8. Because we chose the parameters so that

$$\delta_0 \geq c_1^{\frac{1}{m-s}} = 1 - 2\rho = \delta,$$

we may apply Theorem 3.8. Let ε to be the constant $\varepsilon_0 = \varepsilon_0(m, s, N_d, \delta)$ of Theorem 3.8.

Proving the theorem now easily reduces to showing that $\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq s$ almost everywhere on the set $A = \{y \in \mathbb{R}^d : \mathrm{por}_k(\mu, y) > \varrho\}$. We may assume that $\mu(A) > 0$ since otherwise there is nothing to prove. Suppose to the contrary that there exists a set $A' \subset A$ with positive measure such that $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$ for all $x \in A'$. Using Lemma 4.5, we find a set $A_\varepsilon \subset A'$ with $\mu(A_\varepsilon) > 0$ so that

$$\limsup_{r \downarrow 0} \mathrm{hom}^5_{1-2\varrho,\varepsilon,r}(\mu, x) < c(1 - 2\varrho)^{-d+k} = (1 - 2\varrho)^{-m}$$

for all $x \in A_\varepsilon$. Now Theorem 3.8 implies that $\overline{\dim}_{\mathrm{loc}}(\mu, x) \le s$ for μ-almost all $x \in A_\varepsilon$. This contradiction finishes the proof. □

Remark 4.6. A measure μ is called (ϱ, p)-mean k-porous at x if for all $\varepsilon > 0$ and for all sufficiently large n, there are at least pn values $l \in \{1, \dots, n\}$ with $\mathrm{por}_k(\mu, x, 2^{-l}, \varepsilon) \ge \varrho$. It follows from [5] that for any measure μ on $\mathbb{R}^d$, one has $\overline{\dim}_{\mathrm{loc}}(\mu, x) \le d - p - c(d)/\log(1-2\varrho)$ for μ-almost all $x \in \{y \in \mathbb{R}^d : \mu \text{ is } (\varrho, p)\text{-mean 1-porous at } y\}$. In light of Theorem 4.2 it is natural to ask whether this holds also for mean k-porous measures: if μ is a measure on $\mathbb{R}^d$, with $k \in \{1, \dots, d\}$, for $0 < \varrho < 1/2$, and $0 < p < 1$, is it true that

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le d - pk - c/\log(1-2\varrho)$$

for μ-almost all $x \in \{y \in X : \mu \text{ is } (\varrho, p)\text{-mean } k\text{-porous at } y\}$? An affirmative answer to this question was recently obtained in [53].

4.3. Porous measures on s-regular metric spaces

If we consider k-porosity with $k = 1$ there is no orthogonality condition on the direction of holes. By replacing the Euclidean distance $|x - y_1|$ by $d(x, y_1)$ in the definition, it makes perfect sense to investigate 1-porosity, which we simply call porosity, in a general metric space (X, d).

If X is an s-regular metric space, then for any $A \subset X$ with $\inf_{x \in A} \mathrm{por}_1(A, x) \ge \varrho$, the packing dimension of A is at most $s - c\varrho^s$; see [25, Theorem 4.7]. Recall that X is s-regular if there exists a measure ν on X and constants $a, b > 0$ so that

$$ar^s \le \nu\big(B(x, r)\big) \le br^s \tag{4.2}$$

for all $x \in X$ and $0 < r \le \mathrm{diam}(X)$. Our result for measures in this direction is the following.

Theorem 4.7. *If X is an s-regular metric space and μ is a measure on X, then*

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le s - c\,\mathrm{por}_1(\mu, x)^s$$

for μ-almost all $x \in X$, where $c > 0$ depends only on s and the constants a and b of (4.2).

In the proof of Theorem 4.2, we used a known estimate for k-porous sets via a density point argument. In the proof of Theorem 4.7 we will only be able to use Lemma 2.3 as the density point property is not true in every s-regular metric space. To prove Theorem 4.7, we recall the following estimate from [25, Corollary 4.6].

Lemma 4.8. *If X is an s-regular metric space with an s-regular measure ν, then there exist constants $c_1, c_2, c_3 > 0$ depending only on s and the constants a and b*

of (4.2) *that satisfy the following: if* $x \in X$, $r_p > 0$, $0 < r < c_3 \min\{r_p, \operatorname{diam}(X)\}$, $A \subset B(x,r)$, *and* $\operatorname{por}_1(A, y, r') \ge \varrho > 0$ *for all* $y \in A$ *and* $0 < r' < r_p$, *then*

$$\nu\big(A(r'')\big) \le c_1 \nu\big(B(x,r)\big)\Big(\frac{r''}{r}\Big)^{c_2\varrho^s}$$

for all $0 < r'' < r$.

Now we are ready to prove Theorem 4.7.

Proof of Theorem 4.7. Let ν be an s-regular measure on X with $\operatorname{spt}(\nu) = X$ and let the constants $c_1, c_2, c_3 > 0$ be as in Lemma 4.8. Let $0 < \varrho < \frac{1}{2}$ and choose $\delta' > 0$ so small that $\log(c_1 b/a)/\log(1/\delta) < (ac_2\varrho^s)/(4^s b)$ for all $0 < \delta \le \delta'$. We are going to apply Theorem 3.8 with

$$m' = s - \frac{c_2 a}{2b}(\varrho/4)^s + \frac{\log(c_1 b/a)}{-\log\delta'}, \quad s' = s - \frac{c_2 a}{4b}(\varrho/4)^s,$$

and $0 < \delta < \min\{1, \varrho \operatorname{diam}(X)/2, \delta', \delta_0\}$, where $\delta_0 = \delta_0(m', s', N, 10) > 0$ is as in Theorem 3.8. Let $\varepsilon > 0$ be the constant $\varepsilon_0 = \varepsilon_0(m', s', N, \delta) > 0$ from Theorem 3.8.

It is clearly sufficient to prove that we have $\overline{\dim}_{\mathrm{loc}}(\mu, x) \le s - c\varrho^s$ for almost all $x \in A_\varepsilon$, where

$$A_\varepsilon = \big\{x \in X : \operatorname{por}_1(\mu, x, r, \varepsilon/2) \ge \varrho \text{ for all } 0 < r < r_0\big\}.$$

We note that A_ε is a Borel set (a careful inspection of the definitions shows that it is in fact closed). Let $x \in A_\varepsilon$ be such that

$$\lim_{r\to 0} \frac{\mu\big(B(x,r) \setminus A_\varepsilon\big)}{\mu\big(B(x,5r)\big)} = 0.$$

Recall that by Lemma 2.3 this is true for μ-almost every $x \in A_\varepsilon$.

Take $0 < r < \min\{1, r_0/8\}$ so small that

$$\frac{\mu\big(B(x,2r) \setminus A_\varepsilon\big)}{\mu\big(B(x,10r)\big)} < \varepsilon. \tag{4.3}$$

Our goal is to show that for any (δr)-packing $\mathcal{B}$ of

$$A = \big\{y \in B(x,r) : \mu\big(B(y,\delta r)\big) > \varepsilon\mu\big(B(x,10r)\big)\big\}$$

the set $A_{\mathcal{B}} = \{y \in A : y \text{ is the centre point of some } B \in \mathcal{B}\}$ satisfies the assumptions of Lemma 4.8. Using Lemma 4.8, we are able to estimate the cardinality of $\mathcal{B}$ and hence also $\hom^{10}_{\delta,\varepsilon,r}(\mu, x)$. The desired upper bound for $\overline{\dim}_{\mathrm{loc}}(\mu, x)$ then follows from Theorem 3.8.

Fix a (δr)-packing $\mathcal{B}$ of A and $y \in A_{\mathcal{B}}$. Assume first that $0 < r' < 2\delta r/\varrho$. If $B(y, \varrho' r/4) \setminus B\big(y, (\frac{a}{2b})^{1/s}\varrho r'/4\big) = \emptyset$, then it follows from the s-regularity of ν that

$$a(\varrho r'/4)^s \le \nu\big(B(y, \varrho r'/4)\big) = \nu\big(B(y, (\tfrac{a}{2b})^{1/s}\varrho r'/4)\big) \le \tfrac{a}{2}(\varrho r'/4)^s$$

which is impossible. Hence there exists a point $z \in B(y, \varrho r'/4) \setminus B\big(y, (\frac{a}{2b})^{1/s}\varrho r'/4\big)$. Since $\varrho r'/4 < \delta r$, we have $A_{\mathcal{B}} \cap B\big(z, (\frac{a}{2b})^{1/s}\varrho r'/4\big) = \emptyset$ and as $(\frac{a}{2b})^{1/s}\varrho r'/4 + d(y, z) \le \varrho r'/2 < r'$, it follows that $\mathrm{por}_1(A_{\mathcal{B}}, y, r') \ge (\frac{a}{2b})^{1/s}\varrho/4$ for all $0 < r' < 2\delta r/\varrho$.

Let us next assume that $2\delta r/\varrho \le r' \le 8r$. If $A_\varepsilon \cap B(y, \delta r) = \emptyset$, then (4.3) and the definition of A would imply that

$$\mu\big(B(y, \delta r)\big) \le \mu\big(B(x, 2r) \setminus A_\varepsilon\big) < \varepsilon\mu\big(B(x, 10r)\big) \le \mu\big(B(y, \delta r)\big).$$

Hence there must be a point $z \in A_\varepsilon \cap B(y, \delta r)$. The definition of A_ε in turn guarantees the existence of a point $w \in X$ such that $\mu\big(B(w, \varrho r')\big) \le \frac{\varepsilon}{2}\mu\big(B(z, r')\big)$ and $\varrho r' + d(z, w) \le r'$. Now

$$\varrho r'/2 + d(y, w) \le \varrho r'/2 + d(y, z) + d(z, w) \le \varrho r' + d(z, w) \le r'$$

and $A_{\mathcal{B}} \cap B(w, \varrho r'/2) = \emptyset$ because for any $w' \in B(w, \varrho r'/2)$ we have $\mu\big(B(w', \delta r)\big) \le \mu\big(B(w, \varrho r')\big) \le \frac{\varepsilon}{2}\mu\big(B(z, r')\big) < \varepsilon\mu\big(B(x, 10r)\big)$, as $B(z, r') \subset B(x, 10r)$. Therefore $\mathrm{por}_1(A_{\mathcal{B}}, y, r') \ge \varrho/2$ for $2\delta r/\varrho \le r' \le 8r$ and consequently, for $2\delta r/\varrho \le r' \le 4(\frac{b}{a})^{1/s} r$ we have $\mathrm{por}_1(A_{\mathcal{B}}, y, r') \ge \min\{1, 2(\frac{a}{b})^{1/s}\}\varrho/2$.

Now let $4(\frac{b}{a})^{1/s} r < r' < \mathrm{diam}(X)$ and put $t = \frac{1}{4}(\frac{a}{b})^{1/s} r' + 2r$. Then $t < \frac{3}{4}(\frac{a}{b})^{1/s} r'$ and thus

$$\nu\big(B(y, t)\big) \le bt^s < a\big(\tfrac{3r'}{4}\big)^s \le \nu\big(B(y, \tfrac{3}{4}r')\big).$$

So there exists $w \in B(y, \frac{3}{4}r') \setminus B(y, t)$. Now $A_{\mathcal{B}} \cap B\big(w, \frac{1}{4}(\frac{a}{b})^{1/s} r'\big) \subset B(x, r) \cap B\big(w, \frac{1}{4}(\frac{a}{b})^{1/s} r'\big) = \emptyset$ and thus $\mathrm{por}_1(A_{\mathcal{B}}, y, r') \ge \frac{1}{4}(\frac{a}{b})^{1/s}$.

Putting the three estimates together, we have

$$\mathrm{por}_1(A_{\mathcal{B}}, y, r') \ge (\tfrac{a}{2b})^{1/s}\varrho/4$$

for all $y \in A_{\mathcal{B}}$ and $0 < r' < \mathrm{diam}(X)$. We can now use Lemma 4.8 to conclude

$$\#\mathcal{B}a(\delta r)^s \le \sum_{B \in \mathcal{B}} \nu(B) = \nu\big(A_{\mathcal{B}}(\delta r)\big) \le c_1 \nu\big(B(x, r)\big)\delta^{\frac{c_2 a}{2b}(\varrho/4)^s} \le c_1 b r^s \delta^{\frac{c_2 a}{2b}(\varrho/4)^s}$$

for all $0 < r < c_3 \,\mathrm{diam}(X)$. Since this is true for all (δr)-packings $\mathcal{B}$ of A, and (4.3) is true for all small $r > 0$, we get

$$\limsup_{r \downarrow 0} \mathrm{hom}^{10}_{\delta,\varepsilon,r}(\mu, x) \le \tfrac{c_1 b}{a}\delta^{\frac{c_2 a}{2b}(\varrho/4)^s - s} < \delta^{-m'}$$

for μ-almost every $x \in A_\varepsilon$. Therefore, by Theorem 3.8, we have $\overline{\dim}_{\mathrm{loc}}(\mu, x) \le s' = s - \frac{c_2 a}{4b}(\varrho/4)^s$ for μ-almost every $x \in A_\varepsilon$. This completes the proof. $\square$

Remark 4.9. Independently of our work, based on probabilistic ideas introduced in [22], it was recently proved in [59] that $\overline{\dim}_{\mathrm{loc}}(\mu, x) \le d - c(d) p \varrho^d$ for μ-almost all $x \in \{y \in \mathbb{R}^d : \mu$ is (ϱ, p)-mean 1-porous at $y\}$ for measures in $\mathbb{R}^d$. It is natural to ask whether an analogous estimate

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le s - c p \varrho^s$$

for μ-almost all $x \in \{y \in X : \mu$ is (ϱ, p)-mean 1-porous at $y\}$ is valid in the setting of Theorem 4.7 with a constant $c > 0$ depending only on the s-regularity data. This remains an open problem.

5. Examples and further remarks

So far in this article we have studied the relations between the local versions of L^q-spectrum, dimension and homogeneity, and shown how these concepts can be used in estimating the dimension of measures. Below, we give few straightforward examples of situations where the local L^q-dimension seem to be more reasonable than the global one. In [29], we show how the local L^q-spectrum can be used to develop local multifractal formalism; see also [2] for another account on local multifractal analysis.

In Examples 5.1–5.4, we use the fact that the L^q-spectrum can be defined using the dyadic cubes. For the global spectrum this is well known in the Euclidean setting and it is easy to see that this remains valid for the local spectrum. A detailed proof in the general metric setting can be found in an earlier arXiv version of the manuscript, see `http://arxiv.org/abs/1003.2895v1`.

Example 5.1. We construct a probability measure μ on $\mathbb{R}^d$ so that for all $0 \le q < 1$ we have $\dim_q(\mu) = d$ while $\dim_q(\mu, x) = 0 = \dim_{\mathrm{loc}}(\mu, x)$ for μ-almost all $x \in \mathbb{R}^d$.

Our measure μ will be a countable sum of weighted Dirac measures on $[0, 1]^d$. Let us denote by $\mathcal{Q}^n$ the dyadic subcubes of $[0, 1]^d$ of side-length 2^{-n}. At step 1, we let $n_1 = 1$ and attach a point mass of size 2^{-d} to the centre point of all but one dyadic subcubes of $[0, 1)^d$ in $Q \in \mathcal{Q}^1$. Let $Q_1 \in \mathcal{Q}^1$ be the one remaining cube of measure 2^{-d}. At step 2 we choose a large integer $n_2 \in \mathbb{N}$ and attach a point mass of magnitude $2^{-n_2 d}\mu(Q_1)$ to all but one of its dyadic subcubes in $\mathcal{Q}^{n_1+n_2}$. We continue inductively, at the k-th stage we choose the one remaining cube $Q_{k-1} \in \mathcal{Q}^{n_1+\cdots+n_{k-1}}$, choose a large integer n_k and attach a point mass of size $2^{-n_k d}\mu(Q_{k-1})$ to the centre points of all but one dyadic subcubes of Q_{k-1} in the collection $\mathcal{Q}^{n_1+\cdots+n_k}$.

At the k-th stage we have for all $0 < q < 1$ that

$$\frac{1}{\log 2^{-n_k}} \log \sum_{Q \in \mathcal{Q}^{n_k}} \mu(Q)^q \le \frac{1}{\log 2^{-n_k}} \log\Bigl(\sum_{\substack{Q \in \mathcal{Q}^{n_k} \\ Q \subset Q_{k-1}}} \mu(Q)^q \Bigr)$$
$$= \frac{\log\bigl(2^{n_k d(1-q)} \mu(Q_{k-1})^q\bigr)}{\log 2^{-n_k}} = (q-1)d + q \frac{\log \mu(Q_{k-1})}{\log 2^{-n_k}}.$$

Thus, choosing the numbers n_k large enough, we can ensure that

$$\tau_q(\mu) = \liminf_{n\to\infty} \frac{1}{\log 2^{-n_k}} \log \sum_{Q\in\mathcal{Q}^{n_k}} \mu(Q)^q \le (q-1)d.$$

On the other hand, it is well known and easy to see that $\tau_q(\mu) \ge (q-1)d$ for all measures μ on $\mathbb{R}^d$ with bounded support; see Lemma 2.7(2). Therefore it follows that $\dim_q(\mu) = d$. Furthermore, it is clear from the construction that $\tau_q(\mu, x) = \dim_q(\mu, x) = \dim_{\mathrm{loc}}(\mu, x) = 0$ for μ-almost all $x \in \mathbb{R}^d$.

Example 5.2. If μ is the sum of a Dirac point mass at the origin and the Lebesgue measure on the unit cube of $\mathbb{R}^d$, we see that $\dim_q(\mu) = 0$ whereas $\dim_q(\mu, x) = d = \dim_{\mathrm{loc}}(\mu, x)$ for all $q > 1$ and all $x \in [0,1]^d \setminus \{0\}$.

In $\mathbb{R}^d$, the L^q-spectrum estimates can be used directly to gain information on the dimension of porous measures, but the results obtained this way are somewhat weaker than the results obtained from the local homogeneity estimates in Section 4.2 above; see Remark 4.3(3). One motivation for investigating the local L^q-spectrum in metric spaces was to find out which are of these two methods, if any, is stronger. Also, in the view of Theorems 3.5 and 3.1, it is interesting to compare $\dim_{\mathrm{hom}}^{\Lambda}(\mu, x)$ to $\lim_{q\uparrow 1} \dim_q(\mu, x)$. In the following two examples we show that there is no general relationship between these two values. We present the examples in $\mathbb{R}$ but similar constructions work in any dimension. The first example also shows that a measure may have large homogeneity even if it is of packing dimension zero.

Example 5.3. We construct an example in $\mathbb{R}$ so that $\lim_{q\uparrow 1} \dim_q(\mu, x) = 0$ while $\dim_{\mathrm{hom}}(\mu, x) = 1$ for μ-almost all $x \in \mathbb{R}$. The idea is to apply a construction resulting to a zero dimensional measure on a Cantor set. The large homogeneity is obtained by performing infinitely many (but extremely seldom so that it does not affect the value of $\dim_q$) construction steps where the measure is distributed almost uniformly inside the construction intervals of that level.

We first pick a sequence $0 < \varepsilon_i \downarrow 0$ and then choose integers $m_i, n_i \to \infty$ so that

$$\frac{k + \sum_{j=1}^{k} m_j}{\sum_{j=1}^{k} (n_j + m_j)} < \varepsilon_k \tag{5.1}$$

for all $k \in \mathbb{N}$. In the first step of the construction, we put $\mu([0, 2^{-n_1}]) = \mu([1 - 2^{-n_1}, 1]) = \frac{1}{2}$. Then we divide both intervals $[0, 2^{-n_1}]$ and $[1 - 2^{-n_1}, 1]$ into 2^{m_1} dyadic subintervals of length $2^{-n_1-m_1}$ each getting 2^{-m_1} portion of their parent's measure.

We continue the construction inductively. In the k-th step, we perform the first step construction inside each of the construction intervals of level k just by replacing n_1 and m_1 with n_k and m_k, respectively.

As $m_k \to \infty$ it is clear that $\hom_{\delta}^{\Lambda}(\mu, x) \approx \frac{1}{\delta}$ for all $x \in \mathrm{spt}(\mu)$ and all small $\delta > 0$. Thus $\dim_{\mathrm{hom}}^{\Lambda}(\mu, x) = 1$ for all $x \in \mathrm{spt}(\mu)$. On the other hand, it follows easily from (5.1), that $\tau_q(\mu, x) = \dim_q(\mu, x) = 0 = \dim_q(\mu)$ for all $x \in \operatorname{spt}(\mu)$ and $0 < q < 1$.

Example 5.4. We construct an example in $\mathbb{R}$ so that $\lim_{q\uparrow 1}\dim_q(\mu, x) = 1$ but $\dim_{\mathrm{hom}}^{\Lambda}(\mu, x) = 0$ for μ-almost all $x \in \mathbb{R}$. The idea is to perform a Cantor type construction resulting to a zero dimensional measure, but add "one-dimensional" perturbation which affects only a dense set of measure zero, but nevertheless, guarantees that the $\dim_q(\mu, x)$ is large for all $x \in \operatorname{spt}(\mu)$.

Fix numbers $0 < q_k \uparrow 1$ and integers $n_k, l_k \in \mathbb{N}$ so that $n_k \to \infty$ and $\sum_{k=1}^{\infty} 2^{-l_k} < \infty$. In what follows, we choose a sequence of integers $m_k \to \infty$. First of these, m_1, is taken so that

$$\frac{m_1(1-q_1) - l_1 q_1}{n_1 l_1 + m_1} > \tfrac{1}{2}(1-q_1).$$

The numbers $m_2, m_3, \ldots$ will be defined inductively below.

We begin the step 1 of the construction by setting $\mu([0, 2^{-n_1}]) = \mu([1 - 2^{-n_1}, 1]) = \frac{1}{2}$. Iterating this in a self-similar manner for l_1 steps, we get 2^{l_1} dyadic subintervals of $[0, 1]$ of length $2^{-n_1 l_1}$ each of measure 2^{-l_1}. We choose one of these intervals, say I, and divide it into 2^{m_1} dyadic subintervals of length $2^{-m_1}|I|$ and of measure $2^{-m_1}\mu(I)$. Inside the other $2^{l_1} - 1$ construction intervals of length $2^{-n_1 l_1}$ we choose just the outermost subintervals of length $2^{-l_1 n_1 - m_1}$ and let both of these intervals have the same measure (half of the measure of their parent).

In the beginning of the step k, for $k \geq 2$, we have some dyadic intervals say $I_1, \ldots, I_{N_k}$ of equal length, denoted by 2^{-M_k}. We perform the step 1 construction inside each of these intervals, but replace n_1, l_1, and m_1 by n_k, l_k, and m_k, respectively. We choose m_k so large that for each $I = I_j$, the dyadic subintervals J_i of I of size $2^{-M_k - n_k l_k - m_k}$ chosen in the construction satisfy

$$\frac{\log\bigl(\sum_i \mu(J_i)^{q_k}\bigr)}{\log(2^{M_k + n_k l_l + m_k})} \geq \frac{\log\bigl(2^{m_k(1-q_k)}\bigl(2^{-l_k}\mu(I)\bigr)^{q_k}\bigr)}{\log(2^{M_k + n_k l_k + m_k})} > \frac{k}{k+1}(1-q_k).$$

The former estimate is obtained by summing over the range of intervals where the measure was distributed uniformly. As $q_k \uparrow 1$, we clearly get $\lim_{q\uparrow 1}\dim_q(\mu, x) \geq 1$ for all $x \in \operatorname{spt}(\mu)$. On the other hand, as $n_k \to \infty$, and $\sum_k 2^{-l_k} < \infty$, it follows that for μ-almost all $x \in \mathbb{R}$, we have $\hom_\delta^{\Lambda}(\mu, x) \leq C$ for all $0 < \delta < 1$ with some universal constant $C > 0$. Thus, in particular, $\dim_{\mathrm{hom}}^{\Lambda}(\mu, x) = 0$ for almost all x.

Remark 5.5. (1) From the previous example, it follows that a strict inequality $\overline{\dim}_{\mathrm{loc}}(\mu, x) < \lim_{q\uparrow 1}\dim_q(\mu, x)$ is possible almost everywhere in Theorem 3.1. We note that also

$$\lim_{q\downarrow 1}\dim_q(\mu, x) < \underline{\dim}_{\mathrm{loc}}(\mu, x) \tag{5.2}$$

is possible in a set of positive measure. A simple example is given by letting $\mu = \mathcal{L}^1|_{[0,1]} + \sum_{n\in\mathbb{N}} 2^{-n}\delta_{q_n}$ where $\mathcal{L}^1$ is the Lebesgue measure and $\{q_1, q_2, q_3, \ldots\}$ is dense in $[0, 1]$. In order to get an example where (5.2) holds almost everywhere, one can use a similar idea as in Example 5.4 but this time one has to construct a one dimensional measure with a dense zero dimensional perturbation.

(2) We note that also the other inequalities in Theorem 3.1 can be strict. For instance, see [4, Proposition 3.1].

We finish the article by constructing a doubling metric space in which the density point property does not hold. This space is then further modified in Examples 5.7 and 5.8 to show that the inequalities in Theorem 3.11 may fail in a set of positive measure without the density point property; see Remark 3.12(2).

Example 5.6. Let N_n be a sequence of integers and set $I_n = \{0, \dots, N_n\}$. We define an auxiliary function $f\colon (\mathbb{N} \cup \{0\})^2 \to [0, \infty)$ by setting

$$f(i, j) = f(j, i) = \begin{cases} 0 & \text{if } i = j \\ 2^{-i} & \text{if } i \neq 0 \text{ and } j = 0 \\ (2^{-i} + 2^{-j}) & \text{if } i, j \neq 0 \text{ and } i \neq j. \end{cases} \tag{5.3}$$

We now set $\Sigma = \prod_{n=1}^{\infty} I_n$ and denote its elements by $\mathtt{i} = i_1 i_2 \cdots$, $\mathtt{j} = j_1 j_2 \cdots$, and so on. We also denote $\Sigma_0 = \{\varnothing\}$ and $\Sigma_n = \prod_{j=1}^{n} I_j$ for all $n \in \mathbb{N}$. If $\mathtt{i} \in \Sigma$ and $n \in \mathbb{N}$, then we let $\mathtt{i}|_n = i_1 \cdots i_n \in \Sigma_n$. For $n \in \mathbb{N}$ and $\mathtt{i} \in \Sigma_n$ we denote $[\mathtt{i}] = \{\mathtt{j} \in \Sigma : \mathtt{j}|_n = \mathtt{i}\}$. If $\mathtt{i}, \mathtt{j} \in \Sigma$ so that $\mathtt{i} \neq \mathtt{j}$, then we let $\mathtt{i} \wedge \mathtt{j}$ denote their longest common beginning. Let $|\mathtt{i}|$ denote the length of a word $\mathtt{i}$ (with the convention $|\varnothing| = 0$) and $\mathtt{ij}$ the concatenation of two words $\mathtt{i}, \mathtt{j}$ with $|\mathtt{i}| < \infty$.

Let $\varepsilon_{\varnothing} = 1$ and for $\mathtt{i} \in \bigcup_n \Sigma_n$, let

$$\begin{cases} 0 < \varepsilon_{\mathtt{i}0} \leq 2^{-N_n} \varepsilon_{\mathtt{i}} \\ 0 < \varepsilon_{\mathtt{i}i} \leq 2^{-i} \varepsilon_{\mathtt{i}} \text{ if } 0 \neq i \in I_{n+1}. \end{cases} \tag{5.4}$$

With these parameters we now define a distance $e\colon \Sigma \times \Sigma \to [0, \infty)$ on Σ by setting

$$e(\mathtt{i}, \mathtt{j}) = \begin{cases} 0 & \text{if } \mathtt{i}, \mathtt{j} \in \Sigma \text{ so that } \mathtt{i} = \mathtt{j} \\ \varepsilon_{\mathtt{i} \wedge \mathtt{j}} f(i_{|\mathtt{i} \wedge \mathtt{j}|+1}, j_{|\mathtt{i} \wedge \mathtt{j}|+1}) & \text{if } \mathtt{i}, \mathtt{j} \in \Sigma \text{ so that } \mathtt{i} \neq \mathtt{j}. \end{cases}$$

This is indeed a distance: the triangle inequality follows easily from (5.4) and the definition of f.

Let us next show that Σ is doubling. For this, we choose $\mathtt{i} \in \Sigma$, a real number $0 < r < \operatorname{diam}(\Sigma) \leq 1$ and fix n so that $\varepsilon_{\mathtt{i}|_{n+1}} \leq r < \varepsilon_{\mathtt{i}|_n}$. We also choose $k \in \mathbb{N}$ so that $2^{-k} \varepsilon_{\mathtt{i}|_n} \leq r < 2^{-k+1} \varepsilon_{\mathtt{i}|_n}$. If $k > 1$, we get $B(\mathtt{i}, 2r) \subset B(\mathtt{i}, r) \cup B(\mathtt{i}_0, r) \cup B(\mathtt{i}_1, r)$, where $\mathtt{i}_0 = i_1 \cdots i_n 0 i_{n+2} \cdots$ and $\mathtt{i}_1 = i_1 \cdots i_n (k-1) i_{n+2} \cdots$. If $k = 1$, then $B(\mathtt{i}, 2r) \subset B(\mathtt{i}, r) \cup B(\mathtt{i}_2, r) \cup B(\mathtt{i}_3, r)$, where $\mathtt{i}_2 = i_1 \cdots i_{n-1} 0 i_{n+1} \cdots$ and $\mathtt{i}_3 = i_1 \cdots i_{n-1} N_n i_{n+1} \cdots$. In any case, we see that Σ is doubling with a doubling constant 3.

To finish the construction, fix $N_n = n^3$ and let μ be a probability measure on Σ that satisfies

$$\begin{aligned} \mu([\mathtt{i}0]) &= n^{-2}\mu([\mathtt{i}]), \\ \mu([\mathtt{i}j]) &= N_n^{-1}(1-n^{-2})\mu([\mathtt{i}]), \end{aligned} \tag{5.5}$$

for all $j \in \{1, \ldots, N_n\}$, $\mathtt{i} \in \Sigma_n$, and $n \geq 2$. If $A = \{\mathtt{i} \in \Sigma : i_j \neq 0 \text{ for all } j \in \mathbb{N}\}$, then $\mu(A) > 0$ since $\prod_{n=2}^{\infty}(1-n^{-2}) > 0$.

Let $\mathtt{i} \in A$ and define $r_{\mathtt{i},n} = \varepsilon_{\mathtt{i}|_n} 2^{-i_{n+1}}$ for all $n \in \mathbb{N}$. For each $\mathtt{i} \in A$ it follows that $B(\mathtt{i}, r_{\mathtt{i},n}) = [\mathtt{i}|_{n+1}] \cup [\mathtt{i}']$ for all $n \in \mathbb{N}$, where $\mathtt{i}' = i_1 \cdots i_n 0 \in \Sigma_{n+1}$. Thus we get

$$\begin{aligned} \frac{\mu\big(A \cap B(\mathtt{i}, r_{\mathtt{i},n})\big)}{\mu(B(\mathtt{i}, r_{\mathtt{i},n}))} &\leq \frac{\mu([\mathtt{i}|_{n+1}])}{\mu([\mathtt{i}|_{n+1}]) + \mu([\mathtt{i}'])} \\ &= \frac{N_n^{-1}(1-n^{-2})\mu([\mathtt{i}_n])}{\big(N_n^{-1}(1-n^{-2}) + n^{-2}\big)\mu([\mathtt{i}_n])} = \frac{1-n^{-2}}{1-n^{-2}+n}. \end{aligned} \tag{5.6}$$

In particular, as $n \to \infty$, we see that the density point property is not valid for μ.

Example 5.7. In this example, we modify the previous example to obtain $\overline{\dim}_1(\nu, \mathtt{i}) > \overline{\dim}_{\mathrm{loc}}(\nu, \mathtt{i})$ in a set of positive measure. We continue with the same notation as in Example 5.6. The space Σ is modified by gluing infinitely many small metric spaces into A: Denote by S the collection of all finite words $\mathtt{i} \in \bigcup_{n=0}^{\infty} \Sigma_n$ that contain no zeros. For each $\mathtt{i} \in S$, let $(X_\mathtt{i}, d_\mathtt{i})$ be a doubling metric space with diameter at most $\mathrm{diam}_e([\mathtt{i}0])$ and with a uniform doubling constant (independent of $\mathtt{i}$). Let $X = A \cup \bigcup_{\mathtt{i} \in S} X_\mathtt{i}$ and define a distance d on X by

$$d(x,y) = d(y,x) = \begin{cases} e(x,y) & \text{if } x, y \in A \\ d_\mathtt{i}(x,y) & \text{if } x, y \in X_\mathtt{i} \\ e(x, \mathtt{i}000\cdots) & \text{if } x \in A \text{ and } y \in X_\mathtt{i} \\ e(\mathtt{i}000\cdots, \mathtt{j}000\cdots) & \text{if } x \in X_\mathtt{i},\ y \in X_\mathtt{j} \text{ and } \mathtt{i} \neq \mathtt{j}. \end{cases} \tag{5.7}$$

Since $\mathrm{diam}_{d_\mathtt{i}}(X_\mathtt{i}) \leq \mathrm{diam}_e([\mathtt{i}0])$ and the doubling constant of $X_\mathtt{i}$ is uniformly bounded, it is readily checked that (X, d) is a doubling metric space.

If μ is a measure on Σ and $\nu_\mathtt{i}$ are measures on $X_\mathtt{i}$ with $\nu_\mathtt{i}(X_\mathtt{i}) = \mu([\mathtt{i}0])$, we define a measure ν on X by setting

$$\nu = \mu|_A + \sum_{\mathtt{i} \in S} \nu_\mathtt{i}\,. \tag{5.8}$$

Then $\nu|_A = \mu|_A$ and $\nu(X) = \mu(\Sigma)$. Moreover, since $X_{\mathtt{i}|_n} \subset B_X(\mathtt{i}, r_{\mathtt{i},n}))$, $\nu(B_X(\mathtt{i}, r_{\mathtt{i},n})) = \mu(B_\Sigma(\mathtt{i}, r_{\mathtt{i},n}))$, and $\nu(X_{\mathtt{i}|_n}) = \mu([\mathtt{i}'])$, (5.6) yields

$$\frac{\nu(X_{\mathtt{i}|_n})}{\nu(B(\mathtt{i}, r_{\mathtt{i},n}))} \longrightarrow 1, \tag{5.9}$$

as $n \to \infty$ for all $\mathtt{i} \in A \subset X$.

We now specify $X_{\mathtt{i}}$ and $\nu_{\mathtt{i}}$: let $X_{\mathtt{i}}$ be a Euclidean interval of length $\operatorname{diam}_\Sigma([\mathtt{i}0])$ and let $\nu_{\mathtt{i}}$ be the length measure on $X_{\mathtt{i}}$ normalized so that $\nu_{\mathtt{i}}(X_{\mathtt{i}}) = \mu([\mathtt{i}0])$. Then

$$\lim_{\delta\downarrow 0}\int_{X_{\mathtt{i}}} \frac{\log\nu\big(B(y,\delta)\big)}{\log\delta}\,\mathrm{d}\nu(y) = \nu(X_{\mathtt{i}})$$

and combined with (5.9), this yields

$$\overline{\dim}_1(\nu,\mathtt{i}) \geq \lim_{n\to\infty}\limsup_{\delta\downarrow 0} ⨍_{B(\mathtt{i},r_{\mathtt{i},n})} \frac{\log\nu\big(B(y,\delta)\big)}{\log\delta}\,\mathrm{d}\nu(y) \geq 1.$$

All the above is valid for any choice of the $\varepsilon_{\mathtt{i}}$, and by choosing them small enough, we can easily guarantee that $\overline{\dim}_{\mathrm{loc}}(\nu,\mathtt{i}) = 0$ for all $\mathtt{i}\in A$. This proves that the latter estimate of Theorem 3.11 may fail if the density point property is not satisfied.

Example 5.8. In this example, we modify the above examples to show that the density point property is needed also for the first estimate of Theorem 3.11. To obtain $\underline{\dim}_1(\nu,\mathtt{i}) = 0$ for $\mathtt{i}\in A$ we simply can replace the glued pieces $X_{\mathtt{i}}$ in the previous example by singletons. But since we simultaneously want $\underline{\dim}_{\mathrm{loc}}(\nu,\mathtt{i}) > 0$, we have to modify the construction such that on most scales, the measure ν is very uniformly distributed.

Let k_n be a strictly increasing sequence of integers and $J = \{k_n : n\in\mathbb{N}\}$. Let $N_{k_n} = n^3$ and $N_n = 2$ if $n\notin J$. For $n\in J$, let $I_n = \{0,\ldots,N_n\}$ and for $n\in\{0,1,2,\ldots\}\setminus J$, let $I_n = \{1,2\}$ (so that $0\in I_n$ if and only if $n\in J$). Define Σ as in Example 5.6 and for $\mathtt{i}\in\Sigma_n$, let

$$\begin{cases}\varepsilon_{\mathtt{i}0} = 2^{-N_n}\varepsilon_{\mathtt{i}}\\ \varepsilon_{\mathtt{i}i} = 2^{-i}\varepsilon_{\mathtt{i}} \text{ for } 0\neq i\in I_{n+1}\end{cases} \tag{5.10}$$

if $n\in J$ and

$$\varepsilon_{\mathtt{i}1} = \varepsilon_{\mathtt{i}2} = \frac{\varepsilon_{\mathtt{i}}}{2} \tag{5.11}$$

otherwise.

Define a distance e on Σ by $e(\mathtt{i},\mathtt{i}) = 0$, and for $\mathtt{i}\neq\mathtt{j}$, let

$$e(\mathtt{i},\mathtt{j}) = \varepsilon_{\mathtt{i}\wedge\mathtt{j}} f(i_{|\mathtt{i}\wedge\mathtt{j}|+1}, j_{|\mathtt{i}\wedge\mathtt{j}|+1}),$$

provided that $|\mathtt{i}\wedge\mathtt{j}|\in J$ (f is as in (5.3)) and

$$e(\mathtt{i},\mathtt{j}) = \frac{\varepsilon_{\mathtt{i}\wedge\mathtt{j}}}{2}$$

otherwise. Again, it is a direct consequence of (5.3) and (5.10)–(5.11) that e is a distance.

Let μ be a probability measure on Σ such that for $\mathtt{i}\in\Sigma_{k_n}$, $n\geq 2$,

$$\mu([\mathtt{i}0]) = n^{-2}\mu([\mathtt{i}]),$$
$$\mu([\mathtt{i}j]) = N_n^{-1}(1-n^{-2})\mu([\mathtt{i}]) \text{ for } 0\neq j\in I_{k_n+1},$$

and

$$\mu([\mathtt{i}1]) = \mu([\mathtt{i}2]) = \frac{\mu([\mathtt{i}])}{2}$$

if $|\mathtt{i}| \notin J$.

As in the previous example, let A (resp. S) be the collection of all infinite (resp. finite) words that contain no zeros. For each $n \in \mathbb{N}$ and $\mathtt{i} \in S \cap \Sigma_{k_n}$, let $X_\mathtt{i} = \{x_\mathtt{i}\}$ be a metric space consisting solely of one point. Define $X = A \cup \bigcup_{n\in\mathbb{N}} \bigcup_{\mathtt{i}\in S\cap\Sigma_{k_n}} X_\mathtt{i}$ and $\nu = \mu|_A + \sum_{n\in\mathbb{N},\mathtt{i}\in S\cap\Sigma_{k_n}} \mu([\mathtt{i}0])\delta_{x_\mathtt{i}}$. Let d be a distance on X defined via (5.7).

As in Example 5.7 above, it follows that $\nu(A) > 0$, $X_{\mathtt{i}|_n} \subset B(\mathtt{i}, r_{\mathtt{i},n})$, and that (5.9) holds for $\mathtt{i} \in A$, and $r_{\mathtt{i},n} = \varepsilon_{\mathtt{i}|_{k_n}} 2^{-i_{k_n+1}}$. Moreover, a simple calculation implies

$$\lim_{\delta\downarrow 0} \int_{X_{\mathtt{i}_{k_n}}} \frac{\log \nu\bigl(B(y,\delta)\bigr)}{\log \delta}\,\mathrm{d}\nu(y) = 0,$$

$$\limsup_{\delta\downarrow 0} \int_{B(\mathtt{i},r_{\mathtt{i},n})\setminus X_{\mathtt{i}_{k_n}}} \frac{\log \nu\bigl(B(y,\delta)\bigr)}{\log \delta}\,\mathrm{d}\nu(y) \le C\nu(B(\mathtt{i}_{k_n}, r_{\mathtt{i},n}) \setminus X_{\mathtt{i}_{k_n}}),$$

where $C > 0$ depends only on the doubling constant of X. These estimates, together with (5.9) imply that $\underline{\dim}_1(\nu,\mathtt{i}) = 0$ for all $\mathtt{i} \in A$.

Again, the above holds regardless of the choice of k_n and thus we can choose the sequence (k_n) so that

$$\underline{\dim}_{\mathrm{loc}}(\nu,\mathtt{i}) = 1 \tag{5.12}$$

for all $\mathtt{i} \in A$. To see this, observe first that if there were no sequence (k_n), *i.e.* if $J = \emptyset$, then it would be clear that

$$\underline{\dim}_{\mathrm{loc}}(\nu,\mathtt{i}) = \underline{\dim}_{\mathrm{loc}}(\mu,\mathtt{i}) = \liminf_{n\to\infty} \frac{\log \mu([\mathtt{i}|_n])}{\log \varepsilon_{\mathtt{i}|_n}} = 1. \tag{5.13}$$

and since N_{k_n}, and the ratios $0 < \operatorname{diam}([\mathtt{i}i])/\operatorname{diam}([\mathtt{i}]) = \varepsilon_{\mathtt{i}i}/\varepsilon_\mathtt{i}$ for $\mathtt{i} \in \Sigma_{k_n}$, and $i \in I_{k_n}$ do not depend on the choice of k_n, we can choose $k_n \gg k_{n-1}$ inductively such that (5.12) remains true.

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