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Construction of a stable blow-up solution for a class of strongly perturbed semilinear heat equations

VAN TIEN NGUYEN AND HATEM ZAAG

Abstract. We construct a solution for a class of perturbed semilinear heat equations which blows up in finite time with a prescribed blow-up profile. The construction relies on the reduction of the problem to a finite-dimensional one, and on the use of index theory for the conclusion. When the perturbation is in some sense weak, say polynomial, the construction initiated by Bricmont and Kupiainen [5], then parsued by Merle and Zaag [25], works with very minor adaptations. However, when the perturbation is stronger, say in logarithmic scales with respect to the main nonlinear term, a direct application of the methods of [5] and [25] is not successful. Truly new ideas are needed to perform the construction, in which the substantial novelty of our paper resides. As in earlier works, a geometric interpretation of the parameters of the finite-dimensional problem yields the stability of the constructed solution.

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1. Introduction

1.1. Motivation of the problem

The construction of solutions for partial differential equations with some prescribed behavior has attracted a lot of attention in the last three decades. For the semilinear heat equation

$$\partial_t U = \Delta U + |U|^{p-1} U, \tag{1.1}$$

where $p > 1$, $U = U(x,t)$, and $x \in \mathbb{R}^n$, one of the first contributions goes back to Bricmont and Kupiainen [5], who constructed a solution to equation (1.1) which

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blows up in some finite time T at only one blow-up point $x = \hat{a}$ and satisfies

$$U(x,t) \sim (T-t)^{-\frac{1}{p-1}} f\left(\frac{x-\hat{a}}{\sqrt{|\log(T-t)|(T-t)}}\right) \quad \text{as } t \to T, \tag{1.2}$$

where

$$f(\xi) = \left(p - 1 + \frac{(p-1)^2}{4p}|\xi|^2\right)^{-\frac{1}{p-1}}. \tag{1.3}$$

The proof was performed in the framework of similarity variables defined by

$$y = \frac{x-\hat{a}}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad W_{\hat{a}}(y,s) = (T-t)^{\frac{1}{p-1}} U(x,t). \tag{1.4}$$

In this setting, equation (1.1) yields the following equation for $W_{\hat{a}}$: for all $(y,s) \in \mathbb{R}^n \times [-\log T, +\infty)$,

$$\partial_s W_{\hat{a}} = \Delta W_{\hat{a}} - \frac{1}{2} y \cdot \nabla W_{\hat{a}} - \frac{p}{p-1} W_{\hat{a}} + |W_{\hat{a}}|^{p-1} W_{\hat{a}}, \tag{1.5}$$

and (1.2) reduces to the construction of a solution to (1.5) such that

$$Q(y,s) = W_{\hat{a}}(y,s) - f\left(\frac{y}{\sqrt{s}}\right) \to 0 \quad \text{as } s \to +\infty.$$

This property is guaranteed by the condition that $Q(s)$ belongs to some set $V_A(s) \subset L^\infty(\mathbb{R}^n)$ which shrinks to 0 as $s \to +\infty$ (see Proposition 3.1 below for a definition). Since the linearization of equation (1.5) around $f\left(\frac{y}{\sqrt{s}}\right)$ gives $n+1$ positive modes, a zero mode, then an infinite-dimensional negative spectrum, the method of Bricmont and Kupiainen [5] relies on two arguments:

i) The use of the bounding effect of the heat kernel to reduce the problem of the control of Q in V_A to the control of its positive modes;
ii) The control of the $n+1$ positives modes thanks to a topological argument based on index theory.

Later in [25] (see also [24]), Merle and Zaag suggested a modification of the argument of Bricmont and Kupiainen [5], allowing one a geometric interpretation of the parameters of the finite-dimensional problem, which implies the stability of the constructed solution with respect to initial data.

Much later, Ebde and Zaag [7] asked the question whether the methods of Bricmont-Kupiainen [5], and Merle-Zaag [25] would work for perturbations of equation (1.1). Consider the following equation

$$\partial_t U = \Delta U + |U|^{p-1} U + h(U, \nabla U), \tag{1.6}$$

where

$$|h(U, \nabla U)| \le M(1 + |U|^q + |\nabla U|^r) \tag{1.7}$$

with $M > 0, 0 \le q < p, 0 \le r < \frac{2p}{p+1}$. In some sense, the perturbation $h(U, \nabla U)$ has a subcritical size in the sense that in the similarity variables setting (1.4) one as

$$\left| e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} W_{\hat{a}}, e^{\frac{(p+1)s}{2(p-1)}} \nabla W_{\hat{a}} \right) \right| \le C e^{-\delta s} \left(|W_{\hat{a}}|^q + |\nabla W_{\hat{a}}|^r + 1 \right), \tag{1.8}$$

for some $\delta > 0$. Ebde and Zaag were able to prove the same result for equation (1.6) (construction and stability of a solution with the prescribed behavior (1.2)) thanks to the same technique as in [5] and [25], though the presence of the nonlinear gradient terms requested the use of some involved and delicate parabolic regularity arguments. Since the shrinking set $V_A(s)$ used in [25] involves polynomial decays in $\frac{1}{s}$, and the perturbation term of [7] turns to be exponentially small in the sense of (1.8), there was no need to change the definition of the shrinking set when handling the perturbed equation (1.6).

Following that result, we wanted to see how robust was the method of [5,7,25], and whether it would work with "strong" perturbations of (1.1), namely for the equation

$$\partial_t U = \Delta U + |U|^{p-1} U + h(U), \tag{1.9}$$

where

$$|h(\xi)| \le M \left(\frac{|\xi|^p}{\log^a(2 + \xi^2)} + 1 \right) \quad \text{with } a > 0.$$

Indeed, when moving to the similarity variables setting (1.4), this term turns out to have a polynomial decay, namely

$$\left| e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} W_{\hat{a}} \right) \right| \le \frac{C}{s^a} (|W_{\hat{a}}|^p + 1).$$

Since the definition of $V_A(s)$ involved polynomial terms of the type $\frac{1}{s^\alpha}$ with $\frac{1}{2} \le \alpha \le 2$, we would see immediately that the perturbative method of [7] works with no difficulties, and with the same definition of $V_A(s)$, provided that a is large enough. In fact, it is not difficult to see that the method of [7] works *verbatim* with $a \ge 3$. But for $0 < a < 3$ the strategy of [7] breaks down, which makes our problem completely meaningful. Fortunately, we were able to handle this case ($0 < a < 3$) and prove the existence of a solution to (1.9) with the behavior given in (1.2), thanks to three major ideas:

i) We no longer linearize the equation in the similarity variables (1.4) around $f\left(\frac{y}{\sqrt{s}}\right)$ as in the unperturbed case, since this would give birth to terms of order $\frac{1}{s^a}$, much larger than the bounds in the definition of $V_A(s)$; a modification of the profile is needed;

ii) Because of the perturbation terms, we need to change the definition of the shrinking set $V_A(s)$, in a very delicate way, allowing us to handle the whole range $a > 0$;

iii) We better understand the dynamics of the linearized operator of equation (1.5) around $f\left(\frac{y}{\sqrt{s}}\right)$ defined in (1.3) — see Lemma 3.2 below.

We would like to mention that Masmoudi and Zaag [21] adapted the method of [25] to the Ginzburg-Landau equation

$$\partial_t U = (1 + \imath\beta)\Delta U + (1 + \imath\delta)|U|^{p-1}U, \tag{1.10}$$

where $p - \delta^2 - \beta\delta(p+1) > 0$ and $U : \mathbb{R}^n \times [0, T) \to \mathbb{C}$. Note that the case $\beta = 0$ and $\delta \in \mathbb{R}$ small has been studied earlier by Zaag [34]. The same technique was successfully used by Nouaili and Zaag [28] for the non-variational complex-valued semilinear heat equation

$$\partial_t U = \Delta U + U^2,$$

where $U : \mathbb{R}^n \times [0, T) \to \mathbb{C}$.

For other equations of heat type, we would like to mention the works of Bressan [3,4] where the author considered the following semilinear heat equation with exponential source

$$\partial_t U = \Delta U + e^U \tag{1.11}$$

in a bounded open convex set of $\mathbb{R}^n$. Relying on the same kind of topological techniques as in [5,25], he showed the existence of solutions for equation (1.11) which blow up in finite time T exactly at $x = \hat{a}$ and whose final profile has the form

$$U(x, T) \approx -2\ln|x - \hat{a}| + \ln|\ln|x - \hat{a}|| + \ln 8, \quad \text{as } x \to \hat{a}.$$

He also proved that this asymptotic profile is stable with respect to small perturbations of the initial data.

Surprisingly enough, the kind of topological arguments introduced in [5] and [25] has proved successful in various situations, including hyperbolic and parabolic equations, in particular with energy-critical exponents. This was the case for the construction of multi-solitons for the semilinear wave equation in one space dimension by Côte and Zaag [6], the wave maps by Raphaël and Rodnianski [29], the Schrödinger maps by Merle, Raphaël and Rodnianski [23], the critical harmonic heat flow by Schweyer [31] and the two-dimensional Keller-Segel equation by Raphaël and Schweyer [30].

1.2. Problem setting and statement of results

We are interested in the following nonlinear parabolic equation:

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u + h(u) \\ u(0) = u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \tag{1.12}$$

where u is defined for $(x,t) \in \mathbb{R}^n \times [0,T)$, $1 < p$ and $p < \frac{n+2}{n-2}$ if $n \geq 3$, the function h is in $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfying

$$j = 0, 1, \quad \left|h^{(j)}(z)\right| \leq M\left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1\right) \quad \text{with } a > 1,\ M > 0, \tag{1.13}$$

or

$$h(z) = \mu \frac{|z|^{p-1}z}{\log^a(2+z^2)} \quad \text{with } a > 0,\ \mu \in \mathbb{R}. \tag{1.14}$$

By standard results, the Cauchy problem for equation (1.12) can be solved in $L^\infty(\mathbb{R}^n)$. The solution $u(t)$ of (1.12) would exist either on $[0,+\infty)$ (global existence) or only on $[0,T)$, with $0 < T < +\infty$. In this case, we say that $u(t)$ blows up in finite time T, namely

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

Here T is called the blow-up time, and a point $x_0 \in \mathbb{R}^n$ is called a blow-up point if and only if there exist $(x_n, t_n) \to (x_0, T)$ such that $|u(x_n,t_n)| \to +\infty$ as $n \to +\infty$. In this paper, we are interested in the finite time blow-up for equation (1.12).

We consider equation (1.12) as a perturbation of the semilinear heat equation (1.1), which has been carefully studied in the last decades. The existence of blow-up solutions for equation (1.1) has been proved by several authors (see, for example, Fujita [12], Ball [2], Levine [20]). We have a lots of results concerning the asymptotic blow-up behavior, locally near a given blow-up point. Giga and Kohn showed first in [13–15] that if $\hat{a}$ is a blow-up point of u, then

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} U\left(\hat{a} + y\sqrt{T-t}, t\right) = \pm(p-1)^{-\frac{1}{p-1}}, \tag{1.15}$$

uniformly on compact sets $|y| \leq R$.

The estimate (1.15) has been refined up to a higher order by Filippas and Liu [11] (see also Filippas and Kohn [10]) and Herrero and Velázquez [16,17,32,33] who established that in the (supposedly) generic case,

$$\sup_{|x-\hat{a}|\leq K\sqrt{|\log(T-t)|(T-t)}} \left|(T-t)^{\frac{1}{p-1}} U(x,t) - f(\xi)\right| \to 0, \tag{1.16}$$

where ξ is the hot-spot rescaled spatial variable

$$\xi = \frac{x-\hat{a}}{\sqrt{|\log(T-t)|(T-t)}},$$

and f is given in (1.3).

Note that Herrero and Velázquez [17] (see also [19]) proved the genericity of the blow-up behavior (1.16) only in one space dimension with nonnegative initial data. The question remains open in the higher dimensional case or with no positivity condition. From Bricmont and Kupiainen [5], Herrero and Velázquez [18], we have examples for initial data leading to the above asymptotic behavior. The stability of the blow-up profile (1.16) with respect to initial data has been proved by Merle and Zaag in [25], Fermanian Kammerer, Merle and Zaag in [9] and [8]. Note that the equation (1.1) admits also highly unstable blow-up behavior which are nonmonotone in the space variables and can be constructed by the technique of Amadori [1].

Given a blow-up point b of u, we study the behavior of u near the singularity (b, T) through the similarity variables (1.4) introduced by Giga and Kohn [14,15]:

$$y = \frac{x-b}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad w_b(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{1.17}$$

and w_b satisfies for all $(y,s) \in \mathbb{R}^n \times [-\log T, +\infty)$,

$$\partial_s w_b = \left(\Delta - \frac{y}{2}\cdot\nabla + 1\right) w_b - \frac{p}{p-1} w_b + |w_b|^{p-1} w_b + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w_b\right). \tag{1.18}$$

Note that the last term in (1.18) satisfies

$$\forall z \in \mathbb{R}, \quad \left|e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} z\right)\right| \le \frac{C}{s^a}(|z|^p + 1), \quad \forall s \ge s_0, \tag{1.19}$$

for some $s_0 > 0$ (see Lemma A.1 for the proof of this fact).

In [26] and [27], the author showed that if w_b does not approach ϕ exponentially fast, where ϕ is the positive solution of the associated ordinary differential equation of equation (1.18),

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} \phi\right) \tag{1.20}$$

such that $\phi(s) \to \kappa$ as $s \to +\infty$, then the solution u of (1.12) would approach an explicit universal profile

$$(T-t)^{\frac{1}{p-1}} u\left(b + \xi\sqrt{(T-t)|\log(T-t)|}, t\right) \to f(\xi) \quad \text{as} \quad t \to T, \tag{1.21}$$

in L^∞_{loc}, where f is defined in (1.3).

The aim of this work is to show that the behavior (1.21) does occur. More precisely, we construct a blow-up solution of equation (1.12) satisfying the behavior described in (1.21). This is our main result:

Theorem 1.1 (Existence of a blow-up solution for equation (1.12) **with description of its profile).** *There exists* $T > 0$ *such that equation* (1.12) *has a solution* $u(x,t)$ *in* $\mathbb{R}^n \times [0,T)$ *satisfying:*

i) *the solution* u *blows up in finite time* T *only at the point* $b = 0$,

ii) $$\left\| (T-t)^{\frac{1}{p-1}} u\left(\cdot\sqrt{T-t}, t\right) - f\left(\frac{\cdot}{\sqrt{|\log(T-t)|}}\right)\right\|_{W^{1,\infty}(\mathbb{R}^n)} \leq \frac{C}{|\log(T-t)|^{\varrho}}, \quad (1.22)$$

for all $\varrho \in (0,\nu)$ *with* $\nu = \min\{a-1, \frac{1}{2}\}$ *in the case* (1.13) *and* $\nu = \min\{a, \frac{1}{2}\}$ *in the case* (1.14), C *is some positive constant and* f *is defined in* (1.3).

iii) *There exists* $u_* \in \mathcal{C}(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *such that* $u(x,t) \to u_*(x)$ *as* $t \to T$ *uniformly on compact subsets of* $\mathbb{R}^n \setminus \{0\}$, *where*

$$u_*(x) \sim \left(\frac{8p|\log|x||}{(p-1)^2|x|^2}\right)^{\frac{1}{p-1}} \quad \text{as} \quad x \to 0.$$

Remark 1.2. Note that i) directly follows from ii) and iii). Indeed, ii) implies that $u(0,t) \sim \kappa(T-t)^{-\frac{1}{p-1}} \to +\infty$ as $t \to T$, which means that u blows up in finite time T at the point 0. From iii), we see that u blows up only at the point $b = 0$.

Remark 1.3. Note that the profile f is the same as in the nonlinear heat equation without the perturbation ($h \equiv 0$), see Bricmont and Kupiainen [5], Merle and Zaag [25].

The estimate (1.22) holds in $W^{1,\infty}$ and uniformly in $z \in \mathbb{R}^n$. In the previous work, Ebde and Zaag [7] give such a uniform convergence in the case h involving a nonlinear gradient term. In fact, the convergence in $W^{1,\infty}$ comes from a parabolic regularity estimate for equation (1.18) (see Proposition 3.4 below). Dealing with the case $h \equiv 0$, Bricmont and Kupiainen [5], Merle and Zaag [25] also give such a uniform convergence but only in $L^\infty(\mathbb{R}^n)$. In most papers, the same kind of convergence is proved, but only uniformly on a smaller subsets, $|z| \leq K\sqrt{|\log(T-t)|}$ (see Velázquez [32]).

As mentioned above, the proof of Theorem 1.2 bases on techniques developed by Bricmont and Kupiainen in [5] and Merle and Zaag in [25] for the semilinear heat equation (1.1). Because the perturbation term h certainly impacts on the construction of solutions of (1.12) satisfying (1.22), this causes some crucial modifications in [25] in order to totally control the term h. Although these modifications do not affect the general framework developed in [25], they lay in 3 crucial places:

i) We modify the profile around which we study equation (1.18), so that we go beyond the order $\frac{1}{s^a}$ generated by the perturbation term (see (1.19)). Indeed, for small $a > 0$ and with the same profile as in [25], the order $\frac{1}{s^a}$ will become too strong and will not allow us to close our estimates. See Section 2 below, particularly the definition (2.1) of the profile to be linearized around, which enables us to reach the order $\frac{1}{s^{a+1}}$;

ii) In order to handle the order $\frac{1}{s^{a+1}}$, we need to modify the definition of the shrinking set near the profile. See Section 3 and particularly Proposition 3.1 below;

iii) An improved understanding of the dynamics of the linearized operator of (1.18) around the profile (2.1), and which allows to handle the new definition of the shrinking set (see Lemma 3.2 below). Note that in [5] and [25] the dynamics of the linearized operator were understood only for initial data in the old-style shrinking set.

We would like to emphasize the fact that the strategy of the non-perturbed equation (1.1) with no adaptations would work only when $a \geq 3$. Therefore having a strong perturbation, namely $0 < a < 3$, is challenging and makes our problem completely meaningful. Furthermore, because of the difference in the definition of the profile to be linearized around and the difference in the definition of the shrinking set, the proof is far from being an adaptation of the proof written in [25]. We therefore need some involved arguments in order to overcome technical difficulties in the proof to get the conclusion. In particular, the proof of Theorem 1.2 relies on the understanding of the dynamics of the self-similar version of equation (1.18) around the profile (1.3). Following the work by Merle and Zaag [25], the proof will be divided into 2 steps:

- Thanks to a dynamical system formulation, we show that controlling the similarity variable version $w(y, s)$ (1.17) around the expected behavior (1.3) reduces to the control of the unstable directions, whose number is finite;
- Then, we solve the finite-dimensional problem thanks to a topological argument based on index theory.

As in [21,25,34], it is possible to make the interpretation of the finite-dimensional variable in terms of the blow-up time and the blow-up point. This allows us to derive the stability of the profile f in Theorem 1.2 with respect to perturbations in the initial data. More precisely, we have the following:

Theorem 1.4 (Stability of the solution constructed in Theorem 1.2). *Let us denote by $\hat{u}(x, t)$ the solution constructed in Theorem* 1.2 *and by $\hat{T}$ its blow-up time. Then there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}(x, 0)$ in $W^{1,\infty}$ such that for any $u_0 \in \mathcal{V}_0$, equation* (1.12) *has a unique solution $u(x, t)$ with initial data u_0, and $u(x, t)$ blows up in finite time $T(u_0)$ at one single blow-up point $b(u_0)$. Moreover, estimate* (1.21) *is satisfied by $u(x - b, t)$ and*

$$T(u_0) \to \hat{T}, \quad b(u_0) \to 0 \quad \text{as } u_0 \to \hat{u}_0 \text{ in } W^{1,\infty}(\mathbb{R}^n).$$

Remark 1.5. We will not give the proof of Theorem 1.4 because the stability result follows from the reduction to a finite-dimensional case as in [25] with the same proof. Hence, we only prove the reduction and refer to [25] for the stability. Note that, from the parabolic regularity, our stability result holds in the larger space $L^\infty(\mathbb{R}^n)$.

2. Formulation of the problem

As in [5,25], we give the proof in one dimension ($n = 1$). The proof remains the same for higher dimensions ($n \geq 2$). We would like to find initial data u_0 such that the solution u of equation (1.12) blows up in finite time T and satisfies the estimate (1.22). Using similarity variables (1.17), this is equivalent to finding $s_0 > 0$ and $w_0(y) \equiv w(y, s_0)$ such that the solution w of equation (1.18) with initial data w_0 satisfies

$$\lim_{s\to+\infty} \left\| w(s) - f\left(\frac{\cdot}{\sqrt{s}}\right)\right\|_{W^{1,\infty}} = 0,$$

where f is given in (1.3). In order to prove this, we will not linearize equation (1.18) around $f + \frac{\kappa}{2ps}$ as in [25]. We will instead introduce

$$q = w - \varphi, \quad \text{where} \quad \varphi = \frac{\phi(s)}{\kappa}\left(f\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa}{2ps}\right), \tag{2.1}$$

with ϕ and f are introduced in (1.20) and (1.3). As we explain below after (2.6), this is one of the major innovations in our work. With the introduction of q in (2.1), the problem is then reduced to constructing a function q such that

$$\lim_{s\to+\infty} \|q(s)\|_{W^{1,\infty}} = 0$$

and q is a solution of the following equation for all $(y, s) \in \mathbb{R} \times [s_0, +\infty)$,

$$q_s = (\mathcal{L} + V)q + B(q) + R(y, s) + N(y, s), \tag{2.2}$$

where $\mathcal{L} = \Delta - \frac{y}{2} \cdot \nabla + 1$ and

$$V(y, s) = p\left(\varphi(y, s)^{p-1} - \frac{1}{p-1}\right) + \imath e^{-s} h'(e^{\frac{s}{p-1}}\varphi), \tag{2.3}$$

$$B(q) = |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \tag{2.4}$$

$$R(y, s) = -\varphi_s + \Delta\varphi - \frac{y}{2} \cdot \nabla\varphi - \frac{\varphi}{p-1} + \varphi^p + e^{\frac{-ps}{p-1}} h\left(e^{\frac{s}{p-1}}\varphi\right), \tag{2.5}$$

$$N(q, s) = e^{\frac{-ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(\varphi + q)\right) - h\left(e^{\frac{s}{p-1}}\varphi\right) - \imath e^{\frac{s}{p-1}} h'\left(e^{\frac{s}{p-1}}\varphi\right) q\right], \tag{2.6}$$

with $\imath = 0$ in the case (1.13) and $\imath = 1$ in the case (1.14).

One can remark that we do not linearize (1.18) around $\tilde{\varphi} = f(\frac{y}{\sqrt{s}}) + \frac{\kappa}{2ps}$ as in the case of equation (1.1) treated in [25]. In fact, if we do the same, we may obtain some terms like $\frac{1}{s^a}$ coming from the strong perturbation h in equation (1.18), and we may not be able to control these terms in the case $a < 3$. To extend the range of a, we multiply the factor $\frac{\phi(s)}{\kappa}$ to $\tilde{\varphi}$ in order to go beyond the order $\frac{1}{s^a}$ and reach at the order $\frac{1}{s^{a+1}}$. Linearizing around φ given in (2.1) is a major novelty in our approach.

In following analysis, we will use the following integral form of equation (2.2): for each $s \geq \sigma \geq s_0$:

$$q(s) = \mathcal{K}(s, \sigma)q(\sigma) + \int_\sigma^s \mathcal{K}(s, \tau)\left[B(q(\tau)) + R(\tau) + N(q(\tau), \tau)\right] d\tau, \quad (2.7)$$

where $\mathcal{K}$ is the fundamental solution of the linear operator $\mathcal{L} + V$ defined for each $\sigma > 0$ and for each $s \geq \sigma$,

$$\partial_s \mathcal{K}(s, \sigma) = (\mathcal{L} + V)\mathcal{K}(s, \sigma), \quad \mathcal{K}(\sigma, \sigma) = \text{Identity}. \quad (2.8)$$

Since the dynamics of equation (2.2) are influenced by the linear part, we first need to recall some properties of the operator $\mathcal{L}$ from Bricmont and Kupiainen [5]. The operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^n)$, where L^2_ρ is the weighted L^2 space associated with the weight ρ defined by

$$\rho(y) = \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|y|^2}{4}}.$$

Its spectrum is given by

$$\text{spec}(\mathcal{L}) = \left\{1 - \frac{m}{2},\ m \in \mathbb{N}\right\},$$

and its eigenfunctions are derived from Hermite polynomials. If $n = 1$, the eigenfunction corresponding to $1 - \frac{m}{2}$ is

$$h_m(y) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{m!}{k!(m-2k)!}(-1)^k y^{m-2k}. \quad (2.9)$$

We also set $k_m(y) = \frac{h_m(y)}{\|h_m\|^2_{L^2_\rho}}$. If $n \geq 2$, we write the spectrum of $\mathcal{L}$ as $\text{spec}(\mathcal{L}) = \{1 - \frac{|m|}{2},\ |m| = m_1 + \cdots + m_n,\ (m_1, \ldots, m_n) \in \mathbb{N}^n\}$. Given $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, the eigenfunction corresponding to $1 - \frac{|m|}{2}$ is

$$H_m(y) = h_{m_1}(y_1) \ldots h_{m_n}(y_n), \quad \text{where } h_m \text{ is defined in (2.9)}. \quad (2.10)$$

The potential $V(y,s)$ has two fundamental properties:

i) $V(\cdot,s) \to 0$ in L^2_ρ as $s \to +\infty$. In particular, the effect of V on the bounded sets or in the "blow-up" region ($|y| \le K\sqrt{s}$) is regarded as a perturbation of the effect of $\mathcal{L}$;
ii) outside of the "blow-up" region, we have the following property: for all $\epsilon > 0$, there exist $C_\epsilon > 0$ and s_ϵ such that

$$\sup_{s\ge s_\epsilon, |y|\ge C_\epsilon\sqrt{s}} |V(y,s) - \left(-\frac{p}{p-1}\right)| \le \epsilon. \tag{2.11}$$

This means that $\mathcal{L}+V$ behaves like $\mathcal{L}-\frac{p}{p-1}$ in the region $|y| \ge K\sqrt{s}$. Because 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L} - \frac{p}{p-1}$ has purely negative spectrum. Therefore, the control of $q(y,s)$ in L^∞ outside of the "blow-up" region will be done without difficulties.

Since the behavior of V inside and outside of the "blow-up" region are different, let us decompose q as following: Let $\chi_0 \in \mathcal{C}_0^\infty([0,+\infty))$ with $\mathrm{supp}(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$ on $[0,1]$. We define

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\sqrt{s}}\right), \tag{2.12}$$

where $K > 0$ to be fixed large enough, and write

$$q(y,s) = q_b(y,s) + q_e(y,s), \tag{2.13}$$

where $q_b(y,s) = \chi(y,s)q(y,s)$ and $q_e(y,s) = (1-\chi(y,s))q(y,s)$.

In order to control q_b, we expand it with respect to the spectrum of $\mathcal{L}$ in L^2_ρ. More precisely, we write q into 5 components as follows:

$$q(y,s) = \sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s) + q_e(y,s), \tag{2.14}$$

where q_m, q_- are coordinates of q_b (not of q), namely that q_m is the projection of q_b in h_m and $q_- = P_-(q_b)$ with P_- being the projector on the negative subspace of $\mathcal{L}$.

3. Proof of Theorem 1.2

In this section we use the framework developed in [25] in order to prove Theorem 1.2. We proceed in 5 steps which are presented in 5 separate subsections:

– In the first step, we define a shrinking set $V_A(s)$ and translate our goal of making $q(s)$ go to 0 in $L^\infty(\mathbb{R})$ in terms of belonging to $V_A(s)$. We also exhibit a two-parameter initial data family for equation (2.2) whose coordinates are very small (with respect to the requirements of $V_A(s)$), except the two first q_0 and q_1. Note that the set $V_A(s)$ is different from the corresponding one in [25], and this makes the second major novelty of our work, in addition to the modification of the profile in (2.1);

- In the second step, using the spectral properties of equation (2.2), we reduce our goal from the control of $q(s)$ (an infinite-dimensional variable) in $V_A(s)$ to the control of its two first components $(q_0(s), q_1(s))$ (a two-dimensional variable) in $\left[-\frac{A}{s^{1+\nu}}, \frac{A}{s^{1+\nu}}\right]^2$ with $\nu > 0$;
- In the third step, we solve the local in time Cauchy problem for equation (2.2);
- In the fourth step, we solve the finite-dimensional problem using index theory and conclude the proof of ii) of Theorem 1.2;
- In the last step, we derive conclusion iii) of Theorem 1.2 from ii).

In what follows, the constant C denotes a universal one independent of variables, only depending upon constants of the problems such as a, p, M, μ and K in (2.12).

3.1. Definition of a shrinking set $V_A(s)$ and preparation of initial data

Let us first give the following statement:

Proposition 3.1 (A set shrinking to zero). *Let* $\nu = \min\left\{a - 1, \frac{1}{2}\right\}$ *in case* (1.13) *and* $\nu = \min\left\{a, \frac{1}{2}\right\}$ *in case* (1.14)*, and fix* $\varrho \in (0, \nu)$. *For each* $A > 0$*, and* $s > 0$*, we define* $V_A(s)$ *as the set of all functions* g *in* L^∞ *such that*

$$m = 0, 1, \quad |g_m(s)| \le \frac{A}{s^{1+\nu}}, \quad |g_2(s)| \le \frac{A^2}{s^{1+\nu}},$$
$$\forall y \in \mathbb{R}, \quad |g_-(y, s)| \le \frac{A}{s^{3/2+\varrho}}\left(1 + |y|^3\right), \quad \|g_e(s)\|_{L^\infty} \le \frac{A^2}{s^\varrho},$$

where g_m, g_- *and* g_e *are defined in* (2.14)*. Then, for all* $g \in V_A(s)$*, we have for all* $A > 1$, $s \ge 2$ *and* $y \in \mathbb{R}$,

$$|g(y, s)| \le \frac{CA^2}{s^{3/2+\varrho}}\left(1 + |y|^3\right) + \frac{CA^2}{s^{1+\nu}}\left(1 + |y|^2\right) \quad \text{and} \quad \|g(s)\|_{L^\infty} \le \frac{CA^2}{s^\varrho}. \tag{3.1}$$

Remark 3.2. Note that this new shrinking set is the second innovation of our paper with respect to [5] and [25].

Proof. The conclusion simply follows from the definition of $V_A(s)$ and the fact that $\left|\frac{1-\chi(y,s)}{1+|y|^3}\right| \le \frac{C}{s^{3/2}}$. □

Initial data (at time $s_0 = -\log T$) for the equation (2.2) will depend on two real parameters d_0 and d_1 as given in the following proposition:

Lemma 3.3 (Decomposition of initial data on the different components). *For each $A > 1$, there exists $\delta_1(A) > 0$ such that for all $s_0 \geq \delta_1(A)$: If we consider the following function as initial data for equation* (2.2)*:*

$$q_{d_0,d_1}(y,s_0) = \frac{\phi(s_0)}{\kappa}\left(f^p(z)(d_0 + d_1 z) - \frac{\kappa}{2ps_0}\right), \tag{3.2}$$

where $z = \frac{y}{\sqrt{s_0}}$, f and ϕ are defined in (1.3) *and* (1.20)*, then*

i) *There exists a constant $C = C(p) > 0$ such that the components of $q_{d_0,d_1}(s_0)$ (or $q(s_0)$ for short) satisfy:*

$$q_0(s_0) = d_0 a_0(s_0) + b_0(s_0), \quad \textit{with} \quad a_0(s_0) \sim C, \quad |b_0(s_0)| \leq \frac{C}{s_0}, \tag{3.3}$$

$$q_1(s_0) = d_1 a_1(s_0) + b_1(s_0), \quad \textit{with} \quad a_1(s_0) \sim \frac{C}{\sqrt{s_0}}, \quad |b_1(s_0)| \leq \frac{C}{s_0^2}, \tag{3.4}$$

and

$$|q_2(s_0)| \leq \frac{C|d_0|}{s_0} + Ce^{-s_0}, \quad |q_-(y,s_0)| \leq \left(\frac{C|d_0|}{s_0} + \frac{C|d_1|}{s_0\sqrt{s_0}}\right)\left(1 + |y|^3\right),$$

$$\|q_e(s_0)\|_{L^\infty} \leq C|d_0| + \frac{C|d_1|}{\sqrt{s_0}}, \quad \|\nabla q(s_0)\|_{L^\infty} \leq \frac{C(|d_0| + |d_1|)}{\sqrt{s_0}}.$$

ii) *For each $A > 0$, if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}}\right]$, then*

$$|d_0| + |d_1| \leq \frac{C}{s_0},$$

$$|q_2(s_0)| \leq \frac{C}{s_0^2}, \quad \left\|\frac{q_-(y,s_0)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s_0^2}, \quad \|q_e(s_0)\|_{L^\infty} \leq \frac{C}{s_0},$$

$$q(s_0) \in V_A(s_0), \quad \|\nabla q(s_0)\|_{L^\infty} \leq \frac{C}{s_0\sqrt{s_0}},$$

where the statement $q(s_0) \in V_A(s_0)$ holds with "strict inequalities", except for $(q_0, q_1)(s_0)$, in the sense that

$$m = 0, 1, \quad |q_m(s)| \leq \frac{A}{s^{1+\nu}}, \quad |q_2(s)| < \frac{A^2}{s^{1+\nu}},$$

$$\forall y \in \mathbb{R}, \quad |q_-(y,s)| < \frac{A}{s^{3/2+\varrho}}\left(1 + |y|^3\right), \quad \|q_e(s)\|_{L^\infty} < \frac{A^2}{s^\varrho}.$$

iii) *There exists a rectangle* $\mathcal{D}_{s_0} \subset \left[-\frac{C}{s_0}, \frac{C}{s_0}\right]^2$ *such that the mapping* $(d_0, d_1) \mapsto (q_0, q_1)(s_0)$ *is linear and one to one from* $\mathcal{D}_{s_0}$ *onto* $\left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}}\right]^2$ *and maps* $\partial\mathcal{D}_{s_0}$ *into* $\partial\left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}}\right]^2$. *Moreover, it is of degree one on the boundary and the following equivalence holds:*

$$q(s_0) \in V_A(s_0) \;\; \textit{if and only if} \;\; (d_0, d_1) \in \mathcal{D}_{s_0}.$$

Proof. i) Since we have the similar expression of initial data (3.2) as in [25], we refer the reader to [25, Lemma 3.5], except for the bound on $\|\nabla q(s_0)\|_{L^\infty}$. Note that although i) is not stated explicitly in [25, Lemma 3.5], they are clearly written in its proof. For $\|\nabla q(s_0)\|_{L^\infty}$, we use (3.2) and the fact that $f'(z) = -\frac{p-1}{2p} z f^p(z)$, $f^p(z)$, $zf^{p-1}(z)$ and $z^2 f^{p-1}(z)$ are in $L^\infty(\mathbb{R})$ to derive

$$\begin{aligned}|\nabla q(y, s_0)| &\leq \left|\frac{\phi(s_0)}{\kappa}\right| \left|\frac{f^p(z)}{\sqrt{s_0}}\left(pd_0 z f^{p-1}(z) + d_1 + pd_1 z^2 f^{p-1}(z)\right)\right| \\ &\leq \frac{C}{\sqrt{s_0}}(|d_0| + |d_1|).\end{aligned}$$

ii) We see from (3.3) and (3.4) that if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}}\right]^2$, then $|d_0|$ and $|d_1|$ are bounded by $\frac{C}{s_0}$. Substituting these bounds into the estimates stated in i), we immediately derive ii).

iii) It follows from (3.3) and (3.4), part ii) and the definition of V_A given in Proposition 3.1. This ends the proof of Lemma 3.3. □

As stated in Theorem 1.2, the convergence holds in $W^{1,\infty}(\mathbb{R})$, we need the following parabolic regularity estimate for equation (2.2), with $q(s_0)$ given by (2.7) and $q(s) \in V_A(s)$. More precisely, we have the following:

Proposition 3.4. *For each* $A \geq 1$, *there exists* $\delta_2(A) > 0$ *such that for all* $s_0 \geq \delta_2(A)$: *if* $q(s)$ *is a solution of equation* (2.2) *on* $[s_0, s_1]$ *with initial data at* $s = s_0$, $q_{d_0,d_1}(s_0)$ *given in* (2.7) *where* $(d_0, d_1) \in \mathcal{D}_{s_0}$, *and* $q(s) \in V_A(s)$ *for* $s \in [s_0, s_1]$, *then*

$$\|\nabla q(s)\|_{L^\infty} \leq \frac{CA^2}{s^\varrho}, \quad \forall s \in [s_0, s_1],$$

for some positive constant C.

Proof. The proof is the same as [7, Proposition 3.3]. We would like to mention that the proof bases on a Gronwall's argument and the following properties of the kernel $e^{\theta\mathcal{L}}$ defined in (B.1):

$$\forall g \in L^\infty, \quad \left\|\nabla(e^{\theta\mathcal{L}} g)\right\|_{L^\infty} \leq \frac{Ce^{\theta/2}\|g\|_{L^\infty}}{\sqrt{1 - e^{-\theta}}},$$

and

$$\forall f \in W^{1,\infty}, \quad \left\| \nabla (e^{\theta \mathcal{L}} f) \right\|_{L^\infty} \le C e^{\theta/2} \| \nabla f \|_{L^\infty}.$$

Although the definition of V_A is slightly different from the one defined in [7], the readers will have absolutely no difficulty to adapt their proof to the new situation. For that reason, we refer the readers to [7] for details of the proof. □

3.2. Reduction to a finite-dimensional problem

We are going to the crucial step of the proof of Theorem 1.2. In this step, we will show that through a priori estimates, the control of $q(s)$ in V_A reduces to the control of $(q_0, q_1)(s)$ in $\left[-\frac{A}{s^{1+\nu}}, \frac{A}{s^{1+\nu}}\right]$. As presented in [25] (see also [21,34]), we would like to emphasize that this step makes the heart of the contribution. Even more, here lays another major contribution of ours, in the sense that we understand better the dynamics of the fundamental solution $\mathcal{K}(s, \sigma)$ defined in (2.8). Our sharper estimates are given in Lemma 3.2 below. In fact all that we do is to rewrite the corresponding estimates of Bricmont and Kupiainen [5] without taking into account the particular form of the shrinking set they used. Furthermore, because of the difference in the definition (2.1) of φ and the difference in the definition of V_A, the proof is far from being an adaptation of the proof written in [25]. We therefore need some involved arguments to control the components of q and conclude the reduction to a finite-dimensional problem.

We mainly claim the following:

Proposition 3.5 (Control of $q(s)$ by $(q_0, q_1)(s)$ in $V_A(s)$). *For each $A > 0$ and $s > 0$, we define $\hat{V}_A(s) = \left[-\frac{A}{s^{1+\nu}}, \frac{A}{s^{1+\nu}}\right]^2 \subset \mathbb{R}^2$. Then, there exist $A_3 > 0$ such that for each $A \ge A_3$, there exists $\delta_3(A) > 0$ such that for each $s_0 \ge \delta_3(A)$, we have the following properties:*

- *if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, and*
- *if for all $s \in [s_0, s_1]$, $q(s) \in V_A(s)$ and $q(s_1) \in \partial V_A(s_1)$ for some $s_1 \ge s_0$, then:*
  i) (***Reduction to a finite-dimensional problem***) $(q_0, q_1)(s_1) \in \partial \hat{V}_A(s_1)$,
  ii) (***Transversality***) *there exists $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, $(q_0, q_1)(s_1 + \eta) \notin \partial \hat{V}_A(s_1 + \eta)$ (hence, $q(s_1 + \eta) \notin V_A(s_1 + \eta)$).*

The proof follows the general ideas of [25] and we proceed in three steps:

- Step 1: we give a priori estimates on $q(s)$ in $V_A(s)$: assume that for given $A > 0$ larger, $\lambda > 0$ and an initial time $s_0 \ge \sigma_2(A, \lambda) \ge 1$, we have $q(s) \in V_A(s)$ for each $s \in [\tau, \tau + \lambda]$ where $\tau \ge s_0$. Then using the integral form (2.7) of $q(s)$, we derive new bounds on $q_2(s)$, $q_-(s)$ and $q_e(s)$ for $s \in [\tau, \tau + \lambda]$.

– Step 2: we show that these new bounds are better than those defining $V_A(s)$. It then remains to control $q_0(s)$ and $q_1(s)$. This means that the problem is reduced to the control of a two dimensional variable $(q_0, q_1)(s)$ and we then conclude i) of Proposition 3.5.
– Step 3: we use dynamics of $(q_0, q_1)(s)$ to show its transversality on $\partial V_A(s)$, which corresponds to part ii) of Proposition 3.5.

Step 1: (A priori estimates on $q(s)$ in $V_A(s)$). As indicated above, the derivation of the new bounds on the components of $q(s)$ bases on the integral formula (2.7). It is clear to see the strong influence of the kernel $\mathcal{K}$ in this formula. Therefore, it is convenient to give the following result from Bricmont and Kupiainen in [5] which gives the dynamics of the linear operator $\mathcal{L} + V$:

Lemma 3.6 (Refined understanding of the linearized operator in the decomposition (2.14)**).** *For all $\lambda > 0$, there exists $\sigma_0 = \sigma_0(\lambda)$ such that if $\sigma \geq \sigma_0 \geq 1$ and $\psi(\sigma)$ satisfies*

$$\sum_{m=0}^{2} |\psi_m(\sigma)| + \left\| \frac{\psi_-(y,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \|\psi_e(\sigma)\|_{L^\infty} < +\infty, \tag{3.5}$$

then $\theta(s) = \mathcal{K}(s,\sigma)\psi(\sigma)$ satisfies, for all $s \in [\sigma, \sigma+\lambda]$,

$$\begin{aligned} |\theta_2(s)| \leq & \left(\frac{\sigma}{s}\right)^2 |\psi_2(\sigma)| + \frac{C(s-\sigma)}{s}\left(\sum_{l=0}^{2} |\psi_l(\sigma)| + \left\| \frac{\psi_-(y,\sigma)}{1+|y|^3} \right\|_{L^\infty}\right) \\ & + C(s-\sigma)e^{-s/2}\|\psi_e(\sigma)\|_{L^\infty}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \left\| \frac{\theta_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq & \frac{Ce^{s-\sigma}\left((s-\sigma)^2+1\right)}{s}\left(|\psi_0(\sigma)| + |\psi_1(\sigma)| + \sqrt{s}|\psi_2(\sigma)|\right) \\ & + Ce^{-\frac{(s-\sigma)}{2}} \left\| \frac{\psi_-(y,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \frac{Ce^{-(s-\sigma)^2}}{s^{3/2}}\|\psi_e(\sigma)\|_{L^\infty}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|\theta_e(s)\|_{L^\infty} \leq & Ce^{s-\sigma}\left(\sum_{l=0}^{2} s^{l/2}|\psi_l(\sigma)| + s^{3/2}\left\| \frac{\psi_-(y,\sigma)}{1+|y|^3} \right\|_{L^\infty}\right), \\ & + Ce^{-\frac{(s-\sigma)}{p}}\|\psi_e(\sigma)\|_{L^\infty} \end{aligned} \tag{3.8}$$

where $C = C(\lambda, K) > 0$ (K is given in (2.12)*), ψ_m, ψ_-, ψ_e and $\theta_m, \theta_-, \theta_e$ are defined by* (2.13) *and* (2.14).

Remark 3.7. Lemma 3.2 is the corner stone of our paper. Indeed, it gives a sharp understanding of the fundamental solution $\mathcal{K}(s,\sigma)$, regardless of the size of initial data, whereas in [5] and [25], such estimates were obtained only for initial data in $V_A(\sigma)$. In view of the formula (2.7), we see that Lemma 3.2 will play an important role in deriving the new bounds on the components of $q(s)$ and making our proof simpler. This means that, given bounds on the components of $q(\sigma)$, $B(q(\tau))$, $R(\tau)$,

$N(q(\tau), \tau)$, we directly apply Lemma 3.2 with $\mathcal{K}(s,\sigma)$ replaced by $\mathcal{K}(s,\tau)$ and then integrate over τ to obtain estimates on the components of q.

Proof. Let us mention that Lemma 3.2 relays mainly on the understanding of the behavior of the kernel $\mathcal{K}(s,\sigma)$. The proof is essentially the same as in [5], but those estimates did not present explicitly the dependence on all the components of $\psi(\sigma)$ which is less convenient for our analysis below. Because the proof is long and technical, we leave it to Appendix B. □

We now assume that for each $\lambda > 0$, for each $s \in [\sigma, \sigma+\lambda]$, we have $q(s) \in V_A(s)$ with $\sigma \geq s_0$. Applying Lemma 3.2, we get new bounds on all terms in the right-hand side of (2.7), and then on q. More precisely, we claim the following:

Lemma 3.8. *There exists $A_2 > 0$ such that for each $A \geq A_2$, $\lambda^* > 0$, there exists $\sigma_2(A,\lambda^*) > 0$ with the following property: for all $s_0 \geq \sigma_2(A,\lambda^*)$, for all $\lambda \leq \lambda^*$, assume that for all $s \in [\sigma, \sigma+\lambda]$, $q(s) \in V_A(s)$ with $\sigma \geq s_0$, then we have for all $s \in [\sigma, \sigma+\lambda]$,*
i) (***linear term***)

$$|\alpha_2(s)| \leq \left(\frac{\sigma}{s}\right)^{1-\nu}\frac{A^2}{s^{1+\nu}} + \frac{CA^2(s-\sigma)}{s^{2+\nu}},$$
$$\left\|\frac{\alpha_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}} + \frac{C}{s^{3/2+\varrho}}\left(Ae^{-\frac{s-\sigma}{2}} + A^2e^{-(s-\sigma)^2}\right),$$
$$\|\alpha_e(s)\|_{L^\infty} \leq \frac{C}{s^{\varrho}} + \frac{C}{s^{\varrho}}\left(Ae^{s-\sigma} + A^2e^{-\frac{s-\sigma}{p}}\right),$$

where

$$\mathcal{K}(s,\sigma)q(\sigma) = \alpha(y,s) = \sum_{m=0}^{2}\alpha_m(s)h_m(y) + \alpha_-(y,s) + \alpha_e(y,s).$$

If $\sigma = s_0$, we assume in addition that (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$. Then for all $s \in [s_0, s_0+\lambda]$, we have

$$|\alpha_2(s)| \leq \frac{C}{s^2}, \quad \left\|\frac{\alpha_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s^2}, \quad \|\alpha_e(s)\|_{L^\infty} \leq \frac{Ce^{s-s_0}}{\sqrt{s}},$$

ii) (***remaining terms***)

$$|\beta_2(s)| \leq \frac{C(s-\sigma)}{s^{2+\nu}}, \quad \left\|\frac{\beta_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}}, \quad \|\beta_e(s)\|_{L^\infty} \leq \frac{C}{s^{\varrho}},$$

where

$$\int_\sigma^s \mathcal{K}(s,\tau)\left[B(q(\tau)) + R(\tau) + N(q(\tau),\tau)\right]d\tau$$
$$= \beta(y,s) = \sum_{m=0}^{2}\beta_m(s)h_m(y) + \beta_-(y,s) + \beta_e(y,s).$$

Proof. i) It immediately follows from the definition of $V_A(\sigma)$ and Lemma 3.2. Note that in the case $\sigma = s_0$, we use in addition part ii) of Lemma 3.3 to have the conclusion. For part ii), all what we need to do is to find the estimates on the components of different terms appearing in equation (2.2), then we use Lemma 3.2 and the linearity to have the conclusion. We claim the following:

Lemma 3.9. *We have the following properties:*

i) (***Estimates on*** $B(q)$) *For all* $A > 0$, *there exists* $\sigma_3(A)$ *such that for all* $\tau \geq \sigma_3(A)$, $q(\tau) \in V_A(\tau)$ *implies*

$$m = 0, 1, 2, \quad |B_m(\tau)| \leq \frac{CA^4}{\tau^{2+2\nu}},$$
$$\left|\frac{B_-(y,\tau)}{1+|y|^3}\right| \leq \frac{CA^4}{\tau^{3/2+2\varrho}}, \quad \|B_e(\tau)\|_{L^\infty} \leq \frac{CA^{2p'}}{\tau^{\varrho p'}}, \tag{3.9}$$

where $p' = \min\{p, 2\}$.

ii) (***Estimates on*** R) *There exists* $\sigma_4 > 0$ *such that for all* $\tau \geq \sigma_4$,

$$m = 0, 1, \ |R_m(\tau)| \leq \frac{C}{\tau^2}, \quad |R_2(\tau)| \leq \frac{C}{\tau^{2+\nu}},$$
$$\text{and} \ \left\|\frac{R_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{\tau^2}, \quad \|R_e(\tau)\|_{L^\infty} \leq \frac{C}{\tau^\nu}. \tag{3.10}$$

iii) (***Estimates on*** $N(q,\tau)$) *For all* $A > 0$, *there exists* $\sigma_5(A)$ *such that for all* $\tau \geq \sigma_5(A)$, $q(\tau) \in V_A(\tau)$ *implies*

$$m = 0, 1, 2, \quad |N_m(\tau)| \leq \frac{CA^4}{\tau^{2+2\nu}},$$
$$\left\|\frac{N_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{CA^4}{\tau^{2+2\varrho}}, \quad \|N_e(\tau)\|_{L^\infty} \leq \frac{CA^4}{\tau^{2\varrho}}. \tag{3.11}$$

Proof. Since the proof is technical, we leave it to Appendix C. □

Substituting the estimates stated in Lemma 3.9 into Lemma 3.2, then integrating over $[\sigma, s]$ with respect to τ, and taking $\sigma_2(A, \lambda^*) \geq \max\{\sigma_3, \sigma_4, \sigma_5\}$ such that

$$\forall s \geq \sigma_2(A, \lambda^*), \quad (A^4+1)e^{\lambda^*}\left((\lambda^*)^3+1\right)\left(s^{-\varrho(p'-1)} + s^{-(\nu-\varrho)}\right) \leq 1,$$

with $p' = \min\{p, 2\}$, we have the conclusion. This ends the proof of Lemma 3.8. □

Thanks to Lemma 3.9, we obtain the following equations satisfied by the expanding modes:

Lemma 3.10 (ODE satisfied by the expanding modes). *For all $A > 0$, there exists $\sigma_6(A)$ such that for all $s \geq \sigma_6(A)$, $q(s) \in V_A(s)$ implies that for all $s \geq \sigma_6(A)$,*

$$m = 0, 1, \quad \left| q_m'(s) - \left(1 - \frac{m}{2}\right) q_m(s) \right| \leq \frac{C}{s^{3/2+\nu}}, \tag{3.12}$$

and

$$\left| q_2'(s) + \frac{2}{s}\, q_2(s) \right| \leq \frac{C}{s^{2+\nu}}. \tag{3.13}$$

Proof. The proof is very close to that in [25]. We therefore give the sketch of the proof. By the definition (2.14), we write

$$m = 0, 1, 2, \; \frac{dq_m(s)}{ds} = \int \frac{\partial \chi(y,s)}{\partial s} q(s) k_m \rho dy + \int \chi(y,s) \frac{\partial q(s)}{\partial s} k_m \rho dy := I + II.$$

Since the support of $\frac{\partial \chi(y,s)}{\partial s}$ is the set $K\sqrt{s} \leq |y| \leq 2K\sqrt{s}$ (see (2.12)), using the fact that $\|q(s)\|_{L^\infty} \leq \frac{CA^2}{s^\varrho}$ (see (3.1)), we obtain

$$|I| \leq \int \left| \frac{\partial \chi(y,s)}{\partial s} \right| |q(s)||k_m| \rho dy \leq CA^2 e^{-s} s^{-\varrho},$$

for K large enough.

For II, we have by equation (2.2),

$$II = \int \chi(y,s) \mathcal{L} q(s) k_m \rho dy + \int \chi(y,s) V(s) q(s) k_m \rho dy$$
$$+ \int \chi(y,s) \left[B(q(s)) + R(s) + N(q(s), s) \right] k_m \rho dy := IIa + IIb + IIc.$$

Since $\mathcal{L}$ is self-adjoint on L^2_ρ and $\mathcal{L}(\chi(y,s)k_m) = (1 - \frac{m}{2})\chi(y,s)k_m + \frac{\partial^2 \chi(y,s)}{\partial s^2} k_m + \frac{\partial \chi(y,s)}{\partial s}(2\frac{\partial k_m}{dy} - \frac{y}{2} k_m)$, we obtain

$$IIa = \int \mathcal{L}(\chi(y,s)k_m) q(s) \rho dy = \left(1 - \frac{m}{2}\right) q_m(s) + \mathcal{O}(CA^2 e^{-s}),$$

where $\mathcal{O}(r)$ stands for a quantity whose absolute value is bounded precisely by r and not Cr.

Recalling from part c) of Lemma B.1 that $|V(y,s)| \leq \frac{C}{s}(1 + |y|^2)$ and from (3.1) that $|q(y,s)| \leq \frac{CA^2}{s^{1+\nu}}(1 + |y|^3)$, we derive

$$m = 0, 1, \quad |IIb| \leq \frac{CA^2}{s^{2+\nu}} \int (1 + |y|^5)|k_m| \rho dy \leq \frac{CA^2}{s^{2+\nu}}.$$

For $m = 2$, using the second estimate in part c) of Lemma B.1, namely that $V(y,s) = -\frac{h_2(y)}{4s} + \mathcal{O}\left(\frac{C(1+|y|^4)}{s^{1+\bar{a}}}\right)$ with $\bar{a} = \min\{a-1, a\}$ in the case (1.13) and $\bar{a} = \min\{a, 1\}$ in the case (1.14), simultaneously noting that $\int h_2^2 \rho dy = 8$, $\int h_2^3 \rho dy = 64$ and $2 + \bar{a} + \nu \geq 2 + 2\nu$, we obtain

$$m = 2, \quad IIb = -\frac{2}{s} q_2(s) + \mathcal{O}\left(\frac{CA^2}{s^{2+2\nu}}\right).$$

The bound for *IIc* already obtained from (3.9), (3.10) and (3.11). Adding all these bounds and taking $\sigma_6(A)$ large enough such that for all $s \geq \sigma_6(A)$, $A^4 s^{-\nu} + A^2 s^{2+\nu} e^{-s} \leq 1$, we then have the conclusion. This ends the proof of Lemma 3.10. □

Step 2: Deriving conclusion i) of Proposition 3.5. Here we use Lemma 3.8 in order to derive conclusion i) of Proposition 3.5. Indeed, from equation (2.7) and Lemma 3.8, we derive new bounds on $|q_2(s)|$, $\left\|\frac{q_-(y,s)}{1+|y|^3}\right\|_{L^\infty}$ and $\|q_e(s)\|_{L^\infty}$, assuming that for all $s \in [\sigma, \sigma + \lambda]$, $q(s) \in V_A(s)$, for $\lambda \leq \lambda^*$ and $\sigma \geq s_0 \geq \sigma_1(A, \lambda^*)$ (σ_1 is given in Lemma 3.8). The key estimate is to show that for $s = \sigma + \lambda$ (or $s \in [\sigma, \sigma + \lambda]$ if $\sigma = s_0$), these bounds are better than those defining $V_A(s)$, provided that $\lambda \leq \lambda^*(A)$. More precisely, we claim the following proposition which directly follows i) of Proposition 3.5:

Proposition 3.11 (Control of $q(s)$ by $(q_0, q_1)(s)$ in $V_A(s)$). *There exist $A_4 > 1$ such that for each $A \geq A_4$, there exists $\delta_4(A) > 0$ such that for each $s_0 \geq \delta_4(A)$, we have the following properties:*

- *if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, where $\hat{V}_A$ is introduced in Proposition* 3.5*, and*
- *if for all $s \in [s_0, s_1]$, $q(s) \in V_A(s)$ for some $s_1 \geq s_0$, then: for all $s \in [s_0, s_1]$,*

$$|q_2(s)| < \frac{A^2}{s^{1+\nu}}, \quad \left\|\frac{q_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{A}{2s^{3/2+\varrho}}, \quad \|q_e(s)\|_{L^\infty} \leq \frac{A^2}{2s^\varrho}. \tag{3.14}$$

Let us now derive the conclusion i) of Proposition 3.5 from Proposition 3.11, and we then prove it later.

Proof of i) *of Proposition* 3.5. Indeed, if $q(s_1) \in \partial V_A(s_1)$, we see from (3.14) and the definition of $V_A(s)$ given in Proposition 3.1 that the first two components of $q(s_1)$ must be in $\partial \hat{V}_A(s_1)$, which is the conclusion of part i) of Proposition 3.5, assuming Proposition 3.11 holds. □

We now give the proof of Proposition 3.11 in order to conclude the proof of part i) of Proposition 3.5.

Proof of Proposition 3.11. Note that the conclusion of this proposition is very similar to [25, Proposition 3.7, page 157]. However, its proof is far from being an adaptation of the proof given in the case of the semilinear heat equation treated in [25] because of the difference of the definition of $V_A(s)$ and the presence of the strong perturbation term. In fact, the argument given in [25] does not work here to control $|q_2(s)|$ in this new situation, we use instead equation (3.13) to handle this term.

Let $\lambda_1 \geq \lambda_2$ be two positive numbers which will be fixed in term of A later. It is enough to show that (3.14) holds in two cases: $s - s_0 \leq \lambda_1$ and $s - s_0 \geq \lambda_2$. In both cases, we use Lemma 3.8 formula (2.7), and suppose $A \geq A_2 > 0$, $s_0 \geq \max\{\sigma_2(A, \lambda_1), \sigma_2(A, \lambda_2), \sigma_6(A), 1\}$.

Case $s - s_0 \leq \lambda_1$: Since we have for all $\tau \in [s_0, s]$, $q(\tau) \in V_A(\tau)$, we apply Lemma 3.8 with A and $\lambda^* = \lambda_1$, and $\lambda = s - s_0$. From (2.7) and Lemma 3.8, we have

$$|q_2(s)| \leq \frac{C}{s^2} + \frac{C\lambda_1}{s^{2+\nu}}, \qquad \left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}}, \quad \|q_e(s)\|_{L^\infty} \leq \frac{Ce^{\lambda_1}}{\sqrt{s}} + \frac{C}{s^\varrho}.$$

If we fix $\lambda_1 = \frac{3}{2}\log A$ and A large enough, then (3.14) is satisfied.

Case $s - s_0 \geq \lambda_2$: Since we have for all $\tau \in [\sigma, s]$, $q(\tau) \in V_A(\tau)$, we apply Lemma 3.8 with $A, \lambda = \lambda^* = \lambda_2, \sigma = s - \lambda_2$. From (2.7) and Lemma 3.8, we have

$$\left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}} \left(1 + Ae^{-\frac{\lambda_2}{2}} + A^2 e^{-\lambda_2^2}\right),$$
$$\|q_e(s)\|_{L^\infty} \leq \frac{C}{s^\varrho}\left(1 + Ae^{\lambda_2} + A^2 e^{-\frac{\lambda_2}{p}}\right).$$

To obtain (3.14), except for $|q_2(s)|$ which will be treated later, it is enough to have $A \geq 4C$ and

$$C\left(Ae^{-\frac{\lambda_2}{2}} + A^2 e^{-\lambda_2^2}\right) \leq \frac{A}{4},$$
$$C\left(Ae^{\lambda_2} + A^2 e^{-\frac{\lambda_2}{p}}\right) \leq \frac{A^2}{4}.$$

If we fix $\lambda_2 = \log(A/8C)$ and take A large enough, we see that these requests are satisfied. There follow the last two estimates in (3.14).

It remains to show that if $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ then $|q_2(s)| < \frac{A^2}{s^{1+\nu}}$ for all $s \in [s_0, s_1]$. We proceed by contradiction, assume that for all $s \in [s_0, s_*)$, $|q_2(s)| < \frac{A^2}{s^{1+\nu}}$ and $|q_2(s_*)| = \frac{A^2}{s_*^{1+\nu}}$. Considering the case $q_2(s_*) = -\frac{A^2}{s_*^{1+\nu}}$, we have

$$q_2'(s_*) \leq \frac{d}{ds}\left(\frac{-A^2}{s_*^{1+\nu}}\right) \leq \frac{(1+\nu)A^2}{s_*^{2+\nu}}. \tag{3.15}$$

On the other hand, we have from (3.13),

$$q_2'(s_*) \geq -\frac{2}{s}\, q_2(s_*) - \frac{C}{s_*^{2+\nu}} = \frac{2A^2 - C}{s_*^{2+\nu}},$$

which contradicts (3.15) if we take A large enough.

Using the same argument in the case where $q_2(s_*) = \frac{A^2}{s_*^{1+\nu}}$, we also have a contradiction. This completes the proof of Proposition 3.11 and part i) of Proposition 3.5 too. □

Step 3: Deriving conclusion ii) of Proposition 3.5 We prove part ii) of Proposition 3.5 here. In order to prove this, we follow the ideas of [25] to show that for each $m \in \{0, 1\}$ and each $\iota \in \{-1, 1\}$, if $q_m(s_1) = \frac{\iota A}{s_1^{3/2+\nu}}$, then $\frac{dq_m}{ds}(s_1)$ has the opposite sign of $\frac{d}{ds}\left(\frac{\iota A}{s^{3/2+\nu}}\right)(s_1)$ so that $(q_0, q_1)(s_1)$ actually leaves $\hat{V}_A$ at s_1 for $s_1 \geq s_0$ where s_0 will be large enough. Indeed, from equation (3.12), we take $A = 2C + 1$. If $\iota = 1$, then $\frac{dq_m}{ds}(s_1) > 0$ and if $\iota = -1$, then $\frac{dq_m}{ds}(s_1) < 0$. This implies that $(q_0, q_1)(s_1 + \eta) \notin \partial \hat{V}_A(s_1 + \eta)$ which yields conclusion ii) of Proposition 3.5.

3.3. Local in time solution of equation (2.2)

In the following, we find a local in time solution for equation (2.2).

Proposition 3.12 (Local in time solution of equation (2.2)). *For all $A > 1$, there exists $\delta_5(A)$ such that for all $s_0 \geq \delta_5(A)$, the following holds: For all $(d_0, d_1) \in \mathcal{D}_{s_0}$, there exists $s_{\max}(d_0, d_1) > s_0$ such that equation (2.2) with initial data $q(s_0)$ given in (3.2) has a unique solution satisfying $q(s) \in V_{A+1}(s)$ for all $s \in [s_0, s_{\max})$.*

Proof. Using the definition (2.1) of q and the equivalent formulation (1.17), we see that the Cauchy problem (2.2) is equivalent to the Cauchy problem of equation (1.12). Note that from the initial data for (1.12) is derived the initial data for (2.2) at $s = s_0$ given in (3.2), namely

$$u_{d_0,d_1}(x) = \frac{T^{-\frac{1}{p-1}}\phi(-\log T)}{\kappa}\left\{ f(z)\left(1 + \frac{d_0 + d_1 z}{p - 1 + \frac{(p-1)^2}{4p} z^2}\right)\right\},$$

where f is defined in (1.3), $T = e^{-s_0}$ and $z = \frac{x}{\sqrt{T|\log T|}}$.

This initial data belongs to $L^\infty(\mathbb{R})$ which insures the local existence of u in $L^\infty(\mathbb{R})$ (see the introduction). From part iii) of Lemma 3.3, we have $q_{d_0,d_1}(s_0) \in V_A(s_0) \subseteq V_{A+1}(s_0)$. Then there exists $s_{\max}$ such that for all $s \in [s_0, s_{\max})$, we have $q(s) \in V_{A+1}(s)$. This concludes the proof of Proposition 3.12. □

3.4. Deriving conclusion ii) of Theorem 1.2

In this subsection, we derive conclusion ii) of Theorem 1.2 using the previous subsections. Although the derivation of the conclusion is the same as in [25], we would like to give details of its proof for the reader's convenience and to explain the two-point strategy: reduction to a finite-dimensional problem and the conclusion ii) of Theorem 1.2 using index theory.

Proof of ii) *of Theorem* 1.2. We first solve the finite-dimensional problem and show the existence of $A > 1$, $s_0 > 0$ and $(d_0, d_1) \in \mathcal{D}_{s_0}$ such that problem (2.2) with initial data at $s = s_0$, $q_{d_0,d_1}(s_0)$ given in (3.2) has a solution $q(s)$ defined for all $s \in [s_0, +\infty)$ such that

$$q(s) \in V_A(s), \quad \forall s \in [s_0, +\infty). \tag{3.16}$$

For this purpose, let us take $A \geq A_1$ and $s_0 \geq \delta_3$, where A_1 and δ_3 are given in Proposition 3.5; we will find the parameter (d_0, d_1) in the set $\mathcal{D}_{s_0}$ defined in Lemma 3.3 such that (3.16) holds. We proceed by contradiction and assume from iii) of Lemma 3.3 that for all $(d_0, d_1) \in \mathcal{D}_{s_0}$, there exists $s_*(d_0, d_1) \geq s_0$ such that $q_{d_0,d_1}(s) \in V_A(s)$ for all $s \in [s_0, s_*]$ and $q_{d_0,d_1}(s_*) \in \partial V_A(s_*)$. Applying Proposition 3.5, we see that $q_{d_0,d_1}(s_*)$ can leave $V_A(s_*)$ only by its first two components, hence,

$$(q_0, q_1)(s_*) \in \partial \hat{V}_A(s_*),$$

(see Proposition 3.1 for the definition of $\hat{V}_A$). Therefore, we can define the following function:

$$\begin{aligned} \Phi\ &: \mathcal{D}_{s_0} \mapsto \partial([-1, 1]^2) \\ (d_0, d_1) &\to \frac{s_*^{1+\nu}}{A}(q_0, q_1)(s_*). \end{aligned}$$

Since $q(y, s_0)$ is continuous in (d_0, d_1) (see Lemma 3.3), we have that $(q_0, q_1)(s)$ is continuous with respect to (d_0, d_1, s). Then using the transversality property of (q_0, q_1) on $\partial \hat{V}_A$ (part ii) of Proposition 3.5), we claim that $s_*(d_0, d_1)$ is continuous. Therefore, Φ is continuous.

If we manage to prove that Φ is of degree one on the boundary, then we have a contradiction from the degree theory. Let us prove that. From Lemma 3.3, we see that if (d_0, d_1) is on the boundary of $\mathcal{D}_{s_0}$, then

$$(q_0, q_1)(s_0) \in \partial \hat{V}_A(s_0), \quad \text{and} \quad q(s_0) \in V_A(s_0),$$

where the statement $q(s) \in V_A(s)$ holds with strict inequalities for q_2, q_- and q_e. Using again ii) of Proposition 3.5, we see that $q(s)$ can leave $V_A(s)$ at $s = s_0$, hence $s_* = s_0$. Using iii) of Lemma 3.3, we have that the restriction of Φ to the boundary is of degree 1. This gives us a contradiction (by the index theory). Thus, there exists

$(d_0, d_1) \in \mathcal{D}_{s_0}$ such that for all $s \geq s_0$, $q_{d_0,d_1}(s) \in V_A(s)$, which is the conclusion (3.16).

Since $q_{d_0,d_1}(s)$ satisfies (3.16), we use the parabolic estimate in Proposition 3.4, the transformations (2.1) and (1.17) and the fact that $\frac{\phi(s)}{\kappa} = 1 + \mathcal{O}(s^{-a})$ with $a > 0$ to derive estimate (1.22). This concludes the proof of ii) of Theorem 1.2. □

3.5. Deriving conclusion iii) of Theorem 1.2

We give the proof of part iii) of Theorem 1.2 in this subsection. We consider $u(t)$ solution of equation (1.12) which blows-up in finite time $T > 0$ at only one blow-up point $x = 0$ and satisfies (1.22). Adapting the techniques used by Merle in [22] to equation (1.12) without the perturbation ($h \equiv 0$), we show the existence of a profile $u_* \in \mathcal{C}(\mathbb{R} \setminus \{0\}, \mathbb{R})$ such that $u(x,t) \to u_*(x)$ as $t \to T$ uniformly on compact subsets of $\mathbb{R} \setminus \{0\}$, where u_* is given in iii) of Theorem 1.2. Note that Zaag [34], Masmoudi and Zaag [21] successfully applied these techniques to equation (1.10). Since the proof is very similar to that written in [34] and [21], and no new idea is needed, we just give the key argument and kindly refer the reader to see [34, Section 4] for details.

For each $x_0 \in \mathbb{R}\setminus\{0\}$ small enough, we define for all $(\xi, \tau) \in \mathbb{R} \times \left[-\frac{t(x_0)}{T - t(x_0)}, 1\right)$ the following function:

$$v(x_0, \xi, \tau) = (T - t(x_0))^{\frac{1}{p-1}} u(x_0 + \xi\sqrt{T - t(x_0)}, t(x_0) + (T - t(x_0))\tau), \quad (3.17)$$

where $t(x_0)$ is uniquely defined by

$$|x_0| = K_0\sqrt{(T - t(x_0))|\log(T - t(x_0))|}, \quad (3.18)$$

with $K_0 > 0$ to be fixed large enough later.

Note that v blows up at time $\tau = 1$ at only one blow-up point $x_0 = 0$. From (1.12) and (3.17), we see that $v(x_0, \xi, \tau)$ satisfies the following equation: for all $\tau \in \left[-\frac{t(x_0)}{T - t(x_0)}, 1\right)$,

$$\frac{\partial v}{\partial \tau} = \Delta_\xi v + |v|^{p-1} v + (T - t(x_0))^{\frac{p}{p-1}} h\left((T - t(x_0))^{-\frac{1}{p-1}} v\right). \quad (3.19)$$

From estimate (1.22), the definition (3.17) of v and (3.18), we have the following:

$$\sup_{|\xi| < |\log(T - t(x_0))|^{\frac{\varrho}{2}}} |v(x_0, \xi, 0) - f(K_0)| \leq \frac{C}{|\log(T - t(x_0))|^{\frac{\varrho}{2}}} \to 0 \quad \text{as } x_0 \to 0,$$

where f is given in (1.3).

Using the continuity with respect to initial data for equation (1.12) (also for equation (3.19)) associated to a space-localization in the ball $B(0, |\xi| < |\log(T - t(x_0))|^{\frac{\varrho}{2}})$, it is showed in [34, Section 4] that

$$\sup_{|\xi|<|\log(T-t(x_0))|^{\frac{\varrho}{2}}, 0\le\tau<1} |v(x_0, \xi, \tau) - \hat{f}_{K_0}(\tau)| \le \epsilon(x_0) \to 0 \quad \text{as } x_0 \to 0,$$

where $\hat{f}_{K_0}(\tau) = \kappa(1 - \tau + \frac{p-1}{4p} K_0^2)^{-\frac{1}{p-1}}$.

Then letting $\tau \to 1$ and using the definition (3.17) of v, we have

$$\begin{aligned} u_*(x_0) = \lim_{t'\to T} u(x, t') &= (T - t(x_0))^{-\frac{1}{p-1}} \lim_{\tau\to 1} v(x_0, 0, \tau) \\ &\sim (T - t(x_0))^{-\frac{1}{p-1}} \hat{f}_{K_0}(1) \quad \text{as } x_0 \to 0. \end{aligned}$$

From (3.18), we have

$$(T - t(x_0))^{-\frac{1}{p-1}} \sim \left(\frac{|x_0|^2}{2K_0^2 |\log x_0|}\right)^{-\frac{1}{p-1}} \quad \text{as } x_0 \to 0.$$

Hence,

$$u_*(x_0) \sim \left(\frac{8p|\log x_0|}{(p-1)^2 |x_0|^2}\right)^{\frac{1}{p-1}} \quad \text{as } x_0 \to 0,$$

which is the conclusion iii) of Theorem 1.2 and completes the proof of Theorem 1.2.

Appendix

A. Some elementary lemmas

We claim the following:

Lemma A.1. *Let $\varepsilon \in (0, p]$, there exist $C = C(a, p, \mu, M) > 0$ and $s_0 = s_0(a, \varepsilon) > 0$ such that for all $s \ge s_0$,*

i) *if h is given by* (1.13)*,*

$$j = 0, 1, \quad e^{-\frac{(p-j)s}{p-1}} \left| h^{(j)}\left(e^{\frac{s}{p-1}} w\right)\right| \le C s^{-a} \left(|w|^{p-j} + 1\right),$$

ii) *if h is given by* (1.14)*,*

$$\sum_{j=0}^{3} e^{-\frac{(p-j)s}{p-1}} |w|^j \left| h^{(j)}\left(e^{\frac{s}{p-1}} w\right)\right| \le C s^{-a} (|w|^p + |w|^{p-\varepsilon}).$$

Proof. We see that the proof directly follows from the following key estimate:

$$\frac{|w|^\varepsilon}{\log^a\left(2+e^{\frac{2s}{p-1}}w^2\right)} \le \frac{C}{s^a}(|w|^\varepsilon+1), \quad \forall s \ge s_0(a,\varepsilon). \tag{A.1}$$

Indeed, considering the case $w^2 e^{\frac{s}{p-1}} \ge 4$, we have

$$\frac{|w|^\varepsilon}{\log^a\left(2+e^{\frac{2s}{p-1}}w^2\right)} \le \frac{|w|^\varepsilon}{\log^a\left(4e^{\frac{s}{p-1}}\right)} \le \frac{C|w|^\varepsilon}{s^a},$$

while in the case $w^2 e^{\frac{s}{p-1}} \le 4$ it follows that

$$\frac{|w|^\varepsilon}{\log^a\left(2+e^{\frac{2s}{p-1}}w^2\right)} \le \frac{|w|^\varepsilon}{\log^a(2)} \le Ce^{-\frac{\varepsilon s}{p-1}} \le Cs^{-a}.$$

This concludes the proof of (A.1) and the proof of Lemma A.1 also. □

The following lemma shows the existence of solutions of the associated ODE of equation (1.18):

Lemma A.2. *Let ϕ be a positive solution of the following ordinary differential equation:*

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right). \tag{A.2}$$

Then as $s \to +\infty$, $\phi(s) \to \kappa$ and

$$\phi(s) = \kappa(1+\eta_a(s))^{-\frac{1}{p-1}}, \quad \textit{where} \quad \eta_a(s) = \mathcal{O}\left(\frac{1}{s^a}\right). \tag{A.3}$$

If $h(x) = \mu\frac{|x|^{p-1}x}{\log^a(2+x^2)}$, then for all $k \in \mathbb{N}$,

$$\eta_a(s) \sim C_0 \int_s^{+\infty} \frac{e^{s-\tau}}{\tau^a} d\tau = \frac{C_0}{s^a}\left(1 + \sum_{j=1}^k \frac{b_j}{s^j}\right) + \mathcal{O}\left(\frac{1}{s^{a+k+1}}\right),$$

where $C_0 = \mu\left(\frac{p-1}{2}\right)^a$ and $b_j = (-1)^j \prod_{i=0}^{j-1}(a+i)$.

Proof. See [26, Lemma A.2]. □

B. Proof of Lemma 3.2

In this appendix, we give the proof of Lemma 3.2. The proof follows from the techniques of Bricmont and Kupiainen [5] with some additional care, since we have a different profile function φ defined in (2.1), and since we give the explicit dependence of the bounds in terms of all the components of initial data. As mentioned early, the proof relies mainly on the understanding of the behavior of the kernel $\mathcal{K}(s,\sigma,y,x)$ (see (2.8)). This behavior follows from a perturbation method around $e^{(s-\sigma)\mathcal{L}}(y,s)$, where the kernel of $e^{t\mathcal{L}}$ is given by Mehler's formula:

$$e^{t\mathcal{L}}(y,x) = \frac{e^t}{\sqrt{4\pi(1-e^{-t})}} \exp\left[-\frac{(ye^{-t/2}-x)^2}{4(1-e^{-t})}\right]. \tag{B.1}$$

By definition (2.8) of $\mathcal{K}$, we use a Feynman-Kac representation for $\mathcal{K}$:

$$\mathcal{K}(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)\int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma} V(\omega(\tau),\sigma+\tau)d\tau}, \tag{B.2}$$

where $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0, s-\sigma] \to \mathbb{R}$ with $\omega(0) = x$, $\omega(s-\sigma) = y$, *i.e.*, the Gaussian probability measure with covariance kernel

$$\begin{aligned}\Gamma(\tau,\tau') &= \omega_0(\tau)\omega_0(\tau')\\ &\quad + 2\left(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)+\tau-\tau'|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau-\tau'|}\right),\end{aligned}$$

which yields $\int d\mu_{yx}^{s-\sigma}\omega(\tau) = \omega_0(\tau)$, with

$$\omega_0(\tau) = (\sinh((s-\sigma)/2))^{-1}\left(y\sinh\left(\frac{\tau}{2}\right) + x\sinh\left(\frac{s-\sigma-\tau}{2}\right)\right).$$

In view of (B.2), we can consider the expression for $\mathcal{K}$ as a perturbation of $e^{(s-\sigma)\mathcal{L}}$. Since our profile φ defined in (2.1) is different from the one defined in [5], we have here a potential V defined in (2.3) which is different as well. Thus, we first estimate the potential V, then we restate some basic properties of the kernel $\mathcal{K}$.

Lemma B.1 (Estimates on the potential V). *For s large enough, we have*

a) $V(y,s) \le \frac{C}{s^{a'}}$ *with* $a' = \min\{a, 1\}$.

b) $\left|\frac{d^m V(y,s)}{dy^m}\right| \le \frac{C}{s^{m/2}}$ *for* $m = 0, 1, 2$.

c) $|V(y,s)| \le \frac{C}{s}(1+|y|^2)$, $V(y,s) = -\frac{h_2(y)}{4s} + \tilde{V}(y,s)$, *where* $|\tilde{V}(y,s)| = \mathcal{O}\left(\frac{1+|y|^4}{s^2}\right) + \mathcal{O}\left(\frac{1}{s^a}\right)$ *in the case* (1.13) *and* $|\tilde{V}(y,s)| = \mathcal{O}\left(\frac{1+|y|^4}{s^2}\right) + \mathcal{O}\left(\frac{1+|y|^2}{s^{a+1}}\right)$ *in the case* (1.14). *In particular, in both cases* $|\tilde{V}(y,s)| \le \frac{C(1+|y|^4)}{s^{1+\bar{a}}}$, *where* $\bar{a} = \min\{a-1, 1\}$ *in the case* (1.13) *and* $\bar{a} = \min\{a, 1\}$ *in the case* (1.14).

Proof. a) From the definition (2.3) of V, we see that

$$V(y,s) \le p(\varphi(0,s)^{p-1} - \kappa^{p-1}) + \imath \left| e^{-s} h'\left(e^{\frac{s}{p-1}}\varphi\right)\right|,$$

where $\imath$ is defined in (2.3). From Lemma A.2, we have

$$\varphi(0,s)^{p-1} - \kappa^{p-1} = \kappa^{p-1}\left[(1+\eta_a(s))^{-1}\left(1+\frac{1}{2ps}\right)^{p-1} - 1\right] \le \frac{C}{s^{a'}}.$$

Since $|\varphi|$ is bounded, Lemma A.1 yields $\imath \left| e^{-s} h'\left(e^{\frac{s}{p-1}}\varphi\right)\right| \le \frac{\imath C}{s^a}$. This concludes part a).

b) We introduce $W(z,s) = V(y,s)$ with $z = \frac{y}{\sqrt{s}}$. In order to derive part b), it is enough to show that $|\frac{d^m W}{dz^m}| \le C$ for $m = 0, 1, 2$, which follows easily from Lemma A.1 and the following key estimate

$$\frac{\partial f(z)}{\partial z} = -\frac{z f(z)}{2p(1+c_p z^2)},$$

where f and c_p are defined in (1.3).

c) Since $|V(y,s)| \le C$ for all $y \in \mathbb{R}$ and $s \ge 1$, considering the cases $|y| \le \sqrt{s}$, then $|y| \ge \sqrt{s}$, we directly see that the first estimate follows from the second. Hence, we only prove the second. To do so, we introduce $\tilde{W}(Z,s) = V(y,s)$ with $Z = \frac{|y|^2}{s}$. By the definition (2.1) and by a direct calculation, we find that

$$\begin{aligned}\frac{d^2\tilde{W}(Z,s)}{dZ^2} &= p(p-1)(p-2)\varphi^{p-3}(Z,s)\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \\ &\quad + \imath e^{-\frac{(p-3)s}{p-1}} h'''\left(e^{\frac{s}{p-1}}\varphi(Z,s)\right)\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \\ &\quad + \left[p(p-1)\varphi^{p-2}(Z,s) + \imath e^{-\frac{(p-2)s}{p-1}} h''\left(e^{\frac{s}{p-1}}\varphi(Z,s)\right)\right]\frac{d^2\varphi(Z,s)}{dZ^2}.\end{aligned}$$

Applying Lemma A.1 with $\varepsilon = \frac{p-1}{2}$, we see that

$$\begin{aligned}\left|\frac{d^2\tilde{W}(Z,s)}{dZ^2}\right| &\le C\left(\varphi^{p-3}(Z,s) + \imath\varphi^{p-3-\frac{p-1}{2}}(Z,s)\right)\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \\ &\quad + C\left(\varphi^{p-2}(Z,s) + \imath\varphi^{p-2-\frac{p-1}{2}}(Z,s)\right)\left|\frac{d^2\varphi(Z,s)}{dZ^2}\right| \quad \forall s \ge s_0.\end{aligned}$$

From the definition (2.1) of φ, we note that $\frac{d\varphi}{dZ} = -\frac{c_p\phi}{\kappa}F^p(Z)$, where $c_p = \frac{p-1}{4p}$ and $F(Z) = \kappa(1+c_p Z)^{-\frac{1}{p-1}}$, and derive

$$\varphi(Z,s)^{p-3-\frac{p-1}{2}}\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \le C\left(F + \frac{\kappa}{2ps}\right)^{p-3-\frac{p-1}{2}} F^{2p} \le 2C,$$

and

$$\varphi(Z,s)^{p-2-\frac{p-1}{2}}\left|\frac{d^2\varphi(Z,s)}{dZ^2}\right| \le C\left(F+\frac{\kappa}{2ps}\right)^{p-2-\frac{p-1}{2}} F^{2p} \le 2C.$$

Hence, $\left|\frac{d^2\tilde{W}(Z,s)}{dZ^2}\right|$ is bounded for all $Z \in [0,+\infty)$ and for all $s \ge s_0$. Then, by a Taylor expansion, we have

$$\left|\tilde{W}(Z,s)-\tilde{W}(0,s)-Z\frac{\partial \tilde{W}(0,s)}{\partial Z}\right| \le CZ^2 \quad \forall Z \in [0,+\infty), \quad \forall s \ge s_0.$$

From the definition (2.1) of φ and from Lemma A.2, we have

$$\begin{aligned} W(0,s) &= \frac{p}{p-1}\left[\frac{1}{(1+\eta_a)}\left(1+\frac{1}{2ps}\right)^{p-1}-1\right]+\imath e^{-s}h'\left(e^{\frac{s}{p-1}}(\phi+\frac{\phi}{2ps})\right)\\ &= \frac{1}{2s}-\frac{p}{p-1}\left(\frac{\eta_a(s)}{1+\eta_a(s)}\right)+\imath e^{-s}h'\left(e^{\frac{s}{p-1}}\phi\right)+\mathcal{O}\left(\frac{1}{s^{a+1}}\right)+\mathcal{O}\left(\frac{1}{s^2}\right). \end{aligned}$$

Recalling from Lemma A.2 that $\eta_a(s)=\mathcal{O}\left(\frac{1}{s^a}\right)$, this immediately yields $W(0,s) = \frac{1}{2s}+\mathcal{O}\left(\frac{1}{s^a}\right)+\mathcal{O}\left(\frac{1}{s^2}\right)$ in the case (1.13). In the case (1.14), we obtain by a direct calculation,

$$\begin{aligned} &\left|-\frac{p}{p-1}\left(\frac{\eta_a(s)}{1+\eta_a(s)}\right)+\imath e^{-s}h'\left(e^{\frac{s}{p-1}}\phi\right)\right|\\ &= \left|-\frac{p}{(p-1)(1+\eta_a(s))}\left(\eta_a(s)-\frac{C_0}{s^a}\right)\right|+\mathcal{O}\left(\frac{1}{s^{a+1}}\right)=\mathcal{O}\left(\frac{1}{s^{a+1}}\right). \end{aligned}$$

In the last estimate, we used that fact that $\eta_a(s) = \frac{C_0}{s^a}+\mathcal{O}\left(\frac{1}{s^{a+1}}\right)$ in the case (1.14) (see Lemma A.2). Hence, $W(0,s) = \frac{1}{2s}+\mathcal{O}\left(\frac{1}{s^{a+1}}\right)+\mathcal{O}\left(\frac{1}{s^2}\right)$ in the case (1.14).

For $\frac{\partial W(0,s)}{\partial Z}$, we use Lemmas A.1 and A.2 to derive

$$\begin{aligned} \frac{\partial W(0,s)}{\partial Z} &= -\frac{1}{4(1+\eta_a(s))}\left(1+\frac{1}{2ps}\right)^{p-2}-\imath e^{-\frac{(p-2)s}{p-1}}\frac{\phi}{4p}h''\left(e^{\frac{s}{p-1}}\left(\phi+\frac{\phi}{2ps}\right)\right)\\ &= -\frac{1}{4}+\mathcal{O}\left(\frac{1}{s^a}\right)+\mathcal{O}\left(\frac{1}{s}\right). \end{aligned}$$

Returning to V, we conclude part c). This ends the proof of Lemma B.1. □

In what follows, we denote $\int f(y)g(y)\rho(y)dy$ by $\langle f,g\rangle$ and write $\chi(y,s) = \chi(s)$ (χ is defined in (2.12)). Let us now recall some basic properties of the kernel $\mathcal{K}$ stated in [5]:

Lemma B.2 (Bricmont and Kupiainen [5]). *For all* $s \geq \sigma \geq \max\{s_0, 1\}$ *with* $s \leq 2\sigma$ *and* s_0 *given in Lemma A.1, for all* $(y, x) \in \mathbb{R}^2$,

a) $|\mathcal{K}(s, \sigma, y, x)| \leq Ce^{(s-\sigma)\mathcal{L}}(y, x)$.
b) $\mathcal{K}(s, \sigma, y, x) = e^{(s-\sigma)\mathcal{L}}(y, x)\,(1 + P_2(y, x) + P_4(y, x))$, *where*

$$|P_2(y, x)| \leq \frac{C(s-\sigma)}{s}(1 + |y| + |x|)^2,$$
$$\text{and } |P_4(y, x)| \leq \frac{C(s-\sigma)(1+s-\sigma)}{s^2}(1 + |y| + |x|)^4.$$

Moreover, $\left|\left\langle k_2, \left(\mathcal{K}(s, \sigma) - \left(\frac{\sigma}{s}\right)^2\right) h_2\right\rangle\right| \leq \frac{C(s-\sigma)(1+s-\sigma)}{s^{1+\bar{a}}}$, *with* $\bar{a} = \min\{a - 1, 1\}$ *in the case* (1.13) *and* $\bar{a} = \min\{a, 1\}$ *in the case* (1.14).

c) $\|\mathcal{K}(s, \sigma)(1 - \chi)\|_{L^\infty} \leq Ce^{-\frac{(s-\sigma)}{p}}$.

Proof. a) From part a) of Lemma B.1 and the definition (B.2) of $\mathcal{K}$, we have

$$\begin{aligned} |\mathcal{K}(s, \sigma, y, x)| &\leq e^{(s-\sigma)\mathcal{L}}(y, x) \int d\mu_{yx}^{s-\sigma}(\omega) e^{\int_0^{s-\sigma} C(\sigma+\tau)^{-a'} d\tau} \\ &\leq Ce^{(s-\sigma)\mathcal{L}}(y, x) \int d\mu_{yx}^{s-\sigma}(\omega) \leq Ce^{(s-\sigma)\mathcal{L}}(y, x), \end{aligned}$$

since $s \leq 2\sigma$ and $d\mu_{yx}^{s-\sigma}$ is a probability.

b) The proof is exactly the same as the corresponding one written in [5]. Although there is the difference of $\tilde{V}(y, s)$ given in part c) of Lemma B.1, this change does not affect the argument given in [5]. For that reason, we refer the reader to [5, Lemma 5, page 555] for details of the proof.

c) Our potential V given in (2.3) has the same behavior as the potential in [5] for $\frac{|y|^2}{s}$ and s large (see (2.11)). For that reason, we refer to [5, Lemma, page 559] for its proof. □

Before going to the proof of Lemma 3.2, we would like to state some basic estimates which will be used frequently in the proof.

Lemma B.3. *For K large enough, we have the following estimates:*

a) *For any polynomial* P,

$$\int P(y)\mathbf{1}_{\{|y| \geq K\sqrt{s}\}}\rho(y)dy \leq C(P)e^{-s}. \quad \text{(B.3)}$$

b) *Let* $p \geq 0$ *and* $|f(x)| \leq (1 + |x|)^p$, *then*

$$|(e^{t\mathcal{L}} f)(y)| \leq Ce^t(1 + e^{-t/2}|y|)^p. \quad \text{(B.4)}$$

Proof. i) follows from a direct calculation. ii) follows from the explicit expression (B.1) by a simple change of variable. □

Let us now give the proof of Lemma 3.2.

Proof of Lemma 3.2. We consider $\lambda > 0$, let $\sigma_0 \geq \lambda$, $\sigma \geq \sigma_0$ and $\psi(\sigma)$ satisfying (3.5). We want to estimate some components of $\theta(y,s) = \mathcal{K}(s,\sigma)\psi(\sigma)$ for each $s \in [\sigma, \sigma+\lambda]$. Since $\sigma \geq \sigma_0 \geq \lambda$, we have

$$\sigma \leq s \leq 2\sigma. \tag{B.5}$$

Therefore, up to a multiplying constant, any power of any $\tau \in [\sigma, s]$ will be bounded systematically by the same power of s.

a) **Estimate of $\boldsymbol{\theta_2}$**: We first write

$$\begin{aligned}\theta_2(s) &= \langle k_2, \chi(s)\mathcal{K}(s,\sigma)\psi(\sigma)\rangle\\ &= \sigma^2 s^{-2}\psi_2(\sigma) + \left\langle k_2, (\chi(s)-\chi(\sigma))\sigma^2 s^{-2}\psi(\sigma)\right\rangle\\ &\quad + \left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma)-\sigma^2 s^{-2})\psi(\sigma)\right\rangle := \sigma^2 s^{-2}\psi_2(\sigma) + Ib + IIb.\end{aligned}$$

To bound Ib, we write $\psi(x,\sigma) = \sum_{l=0}^{2}\psi_l(\sigma)h_l(x) + \frac{\psi_-(x,\sigma)}{1+|x|^3}(1+|x|^3) + \psi_e(x,\sigma)$ and use (B.3) to derive

$$|Ib| \leq C(s-\sigma)e^{-s}\sigma^2 s^{-2}\left(\sum_{l=0}^{2}|\psi_l(\sigma)| + \left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty} + \|\psi_e(\sigma)\|_{L^\infty}\right).$$

For IIb, we write

$$\begin{aligned}IIb &= \sum_{l=0}^{2}\left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma)-\sigma^2 s^{-2})h_l\right\rangle\psi_l(\sigma)\\ &\quad + \left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma)-\sigma^2 s^{-2})\psi_-(\sigma)\right\rangle\\ &\quad + \left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma)-\sigma^2 s^{-2})\psi_e(\sigma)\right\rangle := IIb.1 + IIb.2 + IIb.3.\end{aligned}$$

Let us bound *IIb.1*. For $l = 2$, we already get from part b) of Lemma B.2 and (B.3) that

$$\left|\left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma)-\sigma^2 s^{-2})h_2\right\rangle\psi_2(\sigma)\right| \leq \frac{C(s-\sigma)(1+s-\sigma)}{s^{1+\bar a}}|\psi_2(\sigma)|,$$

with $\bar a > 0$.

For $l = 0$ or 1, we use b) of Lemma B.2, (B.4), (B.3) and the fact that $\langle k_2, h_l\rangle = 0$ and $e^{(s-\sigma)\mathcal{L}}h_l = e^{(1-l/2)(s-\sigma)}h_l$ to find that

$$\begin{aligned}\left|\left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma)-\sigma^2 s^{-2})h_l\right\rangle\psi_l(\sigma)\right| &\leq \left|\left\langle k_2, \chi(s)\left(\mathcal{K}(s,\sigma)-e^{(s-\sigma)\mathcal{L}}\right)h_l\right\rangle\right||\psi_l(\sigma)|\\ &\quad + \left|\left\langle k_2, \chi(s)\left(e^{(s-\sigma)\mathcal{L}}-\sigma^2 s^{-2}\right)h_l\right\rangle\right||\psi_l(\sigma)|\\ &\leq C(s-\sigma)\left(s^{-1}+e^{-s}\right)|\psi_l(\sigma)|\\ &\leq \frac{C(s-\sigma)}{s}|\psi_l(\sigma)|.\end{aligned}$$

This yields

$$|IIb.1| \le \frac{C(s-\sigma)}{s}\sum_{l=0}^{2}|\psi_l(\sigma)|.$$

If we write $\psi_-(x,\sigma) = \frac{\psi_-(x,\sigma)}{1+|x|^3}(1+|x|^3)$ and use the same arguments as for $l=0$, we obtain

$$|IIb.2| \le \frac{C(s-\sigma)}{s}\left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty}.$$

For $IIb.3$, we write

$$\begin{aligned}IIb.3 = &\left\langle k_2, \chi(s)\left(\mathcal{K}(s,\sigma) - e^{(s-\sigma)\mathcal{L}}\right)\psi_e(\sigma)\right\rangle \\ &+ \left\langle k_2, \chi(s)\left(e^{(s-\sigma)\mathcal{L}} - 1\right)\psi_e(\sigma)\right\rangle + \left\langle k_2, \chi(s)(1-\sigma^2 s^{-2})\psi_e(\sigma)\right\rangle.\end{aligned}$$

Using (B.3), we bound the last term by $C(s-\sigma)e^{-\sigma}\|\psi_e(\sigma)\|_{L^\infty} \le C(s-\sigma)e^{-s/2}$ $\cdot\|\psi_e(\sigma)\|_{L^\infty}$ from (B.5). For the second term, we write $e^{(s-\sigma)\mathcal{L}}-1 = \int_0^{s-\sigma} d\tau\mathcal{L}e^{\tau\mathcal{L}}$ and use the fact that

$$\sup_{|y|\le 2K\sqrt{s}, |x|\ge K\sqrt{\sigma}} e^{-\frac{|y|^2}{4} - \frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}} \le e^{-2s}, \tag{B.6}$$

for K large enough, then it is also bounded by $C(s-\sigma)e^{-s}\|\psi_e(\sigma)\|_{L^\infty}$. For the first term, we use b) of Lemma B.2, (B.4) and again (B.6) to bound it by $C(s-\sigma)s^{-1}e^{-s}\|\psi_e(\sigma)\|_{L^\infty}$. This yields

$$|IIb.3| \le C(s-\sigma)e^{-s/2}\|\psi_e(\sigma)\|_{L^\infty}.$$

Collecting all these bounds yields the bound for $\theta_2(s)$ as stated in (3.6).

b) **Estimate of $\boldsymbol{\theta_-}$:** By definition,

$$\begin{aligned}\theta_-(y,s) = P_-\left[\chi(s)\mathcal{K}(s,\sigma)\psi(\sigma)\right] = &\sum_{l=0}^{2}\psi_l(\sigma)P_-\left[\chi(s)\mathcal{K}(s,\sigma)h_l\right] \\ &+ P_-\left[\chi(s)\mathcal{K}(s,\sigma)\psi_-(\sigma)\right] + P_-\left[\chi(s)\mathcal{K}(s,\sigma)\psi_e(\sigma)\right] := Ic + IIc + IIIc.\end{aligned}$$

In order to bound Ic, we write $\mathcal{K}(s,\sigma) = \mathcal{K}(s,\sigma) - e^{(s-\sigma)\mathcal{L}} + e^{(s-\sigma)\mathcal{L}}$, then we use the fact that $e^{(s-\sigma)\mathcal{L}}h_l = e^{(1-l/2)(s-\sigma)}h_l$, part b) of Lemma B.2 and (B.4) to derive for $l = 0, 1, 2$,

$$\begin{aligned}\left|\left(\mathcal{K}(s,\sigma) - e^{(s-\sigma)(1-l/2)}\right)h_l\right| &= \left|e^{(s-\sigma)\mathcal{L}}(P_2+P_4)h_l\right| \\ &\le \frac{Ce^{s-\sigma}(s-\sigma)}{s}\left(1+e^{-(s-\sigma)/2}|y|\right)^{2+l} \\ &\quad + \frac{Ce^{s-\sigma}(s-\sigma)(1+s-\sigma)}{s^2}\left(1+e^{-(s-\sigma)/2}|y|\right)^{4+l}.\end{aligned}$$

On the support of $\chi(s)$, namely when $|y| \le 2K\sqrt{s}$, we can bound $s^{-k/2}|y|^k$ by C for $k \in \mathbb{N}$. Then, from the easy-to-check fact that

$$\text{if } |f(y)| \le m\big(1+|y|^3\big), \text{ then } P_-[f(y)] \le Cm\big(1+|y|^3\big), \tag{B.7}$$

we obtain

$$l=0,1,\quad P_-\Big[\psi_l(\sigma)\chi(s)\mathcal{K}(s,\sigma)h_l - \psi_l(\sigma)e^{(s-\sigma)(1-l/2)}(\chi(s)h_l)\Big] \le \frac{Ce^{s-\sigma}(s-\sigma)(1+s-\sigma)}{s}(1+|y|^3)|\psi_l(\sigma)|,$$

and

$$P_-\Big[\psi_2(\sigma)\chi(s)\mathcal{K}(s,\sigma)h_2 - \psi_2(\sigma)e^{(s-\sigma)(1-l/2)}(\chi(s)h_2)\Big] \le \frac{Ce^{s-\sigma}(s-\sigma)(1+s-\sigma)}{\sqrt{s}}(1+|y|^3)|\psi_2(\sigma)|.$$

Since $P_-(h_l)=0$ and $|(1-\chi(y,s))h_l(y)| \le Cs^{-3/2+l/2}(1+|y|^3)$, we have

$$l=0,1,2,\quad \Big|\psi_l(\sigma)e^{(s-\sigma)(1-l/2)}P_-[\chi(s)h_l(y)]\Big| \le \frac{Ce^{s-\sigma}}{s^{3/2-l/2}}|\psi_l(\sigma)|\big(1+|y|^3\big).$$

Hence,

$$|Ic| \le \frac{Ce^{s-\sigma}\big((s-\sigma)^2+1\big)}{s}\big(|\psi_0(\sigma)|+|\psi_1(\sigma)|+\sqrt{s}|\psi_2(\sigma)|\big)\big(1+|y|^3\big).$$

To bound $IIIc$, we use a) of Lemma B.2 and the definition (B.1) of $e^{(s-\sigma)\mathcal{L}}$ to write

$$\left\|\frac{\chi(y,s)\mathcal{K}(s,\sigma)\psi_e(x,\sigma)}{1+|y|^3}\right\|_{L^\infty} \le Ce^{s-\sigma}\|\psi_e(\sigma)\|_{L^\infty} \times \sup_{|y|\le 2K\sqrt{s},|x|\ge K\sqrt{\sigma}} e^{-\frac{1}{2}\frac{(ye^{-(s-\sigma)/2}-x)^2}{4(1-e^{-(s-\sigma)})}}\big(1+|y|^3\big)^{-1} \le \begin{cases} Cs^{-3/2}\|\psi_e(\sigma)\|_{L^\infty} & \text{if } s-\sigma\le s_* \\ Ce^{-s}\|\psi_e(\sigma)\|_{L^\infty} & \text{if } s-\sigma\ge s_* \end{cases}$$

for a suitable constant s_*.

Exploiting again (B.7), we obtain the bound on this term which can be written as

$$|IIIc| \le Cs^{-3/2}e^{-(s-\sigma)^2}\|\psi_e(\sigma)\|_{L^\infty}(1+|y|^3) \text{ for } \sigma \text{ large enough.}$$

We still have to consider IIc. In order to bound this term, we proceed as in [5]. We write

$$\mathcal{K}(s,\sigma)\psi_-(\sigma) = \int dx e^{x^2/4}\mathcal{K}(s,\sigma)(\cdot,x)f(x) = \int dx N(\cdot,x)E(\cdot,x)f(x), \tag{B.8}$$

where $f(x) = e^{-x^2/4}\psi_-(x,\sigma)$ and

$$N(y,x) = \frac{e^{s-\sigma}e^{x^2/4}}{\sqrt{4\pi(1-e^{-(s-\sigma)})}}e^{-\frac{(ye^{-(s-\sigma)/2}-x)^2}{4(1-e^{-(s-\sigma)})}},$$

$$E(y,x) = \int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma}V(\omega(\tau),\sigma+\tau)d\tau}.$$

Let $f^0 = f$ and for $m \geq 1$, $f^{(-m-1)}(y) = \int_{-\infty}^{y} dx f^{(-m)}(x)$, then we have the following:

Lemma B.4. $|f^{(-m)}(y)| \leq C\left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty}(1+|y|)^{(3-m)}e^{-y^2/4}$ *for* $m \leq 3$.

Proof. See [5, Lemma 6, page 557]. □

We now rewrite (B.8) by integrating by parts as follows:

$$\begin{aligned}\mathcal{K}(s,\sigma)\psi_-(\sigma) = \sum_{l=0}^{2}(-1)^{l+1}\int dx\partial_x^l N(y,x)\partial_x E(y,x)f^{(-l-1)}(x)\\ +\int dx\partial_x^3 N(y,x)E(y,x)f^{-3}(x).\end{aligned} \tag{B.9}$$

From the definition of $N(y,x)$, we have for $l = 0, 1, 2, 3$,

$$|\partial_x^l N(y,x)| \leq Ce^{-l(s-\sigma)/2}(1+|y|+|x|)^l e^{x^2/4}e^{(s-\sigma)\mathcal{L}}(y,x).$$

Now using the integration by parts formula for Gaussian measures to write

$$\partial_x E(y,x) = \frac{1}{2}\int_0^{s-\sigma}\int_0^{s-\sigma} d\tau d\tau'\partial_x\Gamma(\tau,\tau')\int d\mu_{yx}^{s-\sigma}(\omega)V'(\omega(\tau),\sigma+\tau)$$
$$V'(\omega(\tau'),\sigma+\tau')e^{\int_0^{s-\sigma}d\tau'' V(\omega(\tau''),\sigma+\tau'')}$$
$$+\frac{1}{2}\int_0^{s-\sigma} d\tau\partial_x\Gamma(\tau,\tau)\int d\mu_{yx}^{s-\sigma}(\omega)V''(\omega(\tau),\sigma+\tau)e^{\int_0^{s-\sigma}d\tau'' V(\omega(\tau''),\sigma+\tau'')}.$$

Recalling from Lemma B.1 that $V(y,s) \leq \frac{C}{s^{a'}}$ with $a' > 0$ and $\left|\frac{d^m V(y,s)}{dy^m}\right| \leq \frac{C}{s^{m/2}}$ for $m = 0, 1, 2$. Since $s \leq 2\sigma$, this yields $\int_0^{s-\sigma} V(\omega(\tau),\sigma+\tau)d\tau \leq C$. Because $d\mu_{yx}^{s-\sigma}$ is a probability, we then obtain

$$|E(y,x)| \leq C \quad \text{and} \quad |\partial_x E(y,x)| \leq \frac{C}{s}(s-\sigma)(1+s-\sigma)(|y|+|x|).$$

Substituting all these bounds into (B.9), then using (B.4), Lemma B.4, the fact that $s^{-1}(s-\sigma)(1+s-\sigma) \leq e^{-3/2(s-\sigma)}$ for s large and then (B.7), we derive

$$|IIc| \leq Ce^{-(s-\sigma)/2}\left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty}\left(1+|y|^3\right).$$

Collecting all the bounds for Ic, IIc and $IIIc$, we obtain the bound (3.7).

c) **Estimate for θ_e**: By definition, we write

$$\theta_e(y,s) = (1-\chi(y,s))\mathcal{K}(s,\sigma)\psi(\sigma) = (1-\chi(y,s))\mathcal{K}(s,\sigma)\,(\psi_b(\sigma)+\psi_e(\sigma)).$$

Using c) of Lemma B.2, we have

$$\|(1-\chi(y,s))\mathcal{K}(s,\sigma)\psi_e(\sigma)\|_{L^\infty} \le Ce^{-(s-\sigma)/p}\|\psi_e(\sigma)\|_{L^\infty}.$$

It remains to bound $(1-\chi(y,s))\mathcal{K}(s,\sigma)\psi_b(\sigma)$. To this end, we write

$$\psi_b(x,\sigma) = \sum_{l=0}^{2}\psi_l(\sigma)h_l(x) + \frac{\psi_-(x,\sigma)}{1+|x|^3}\big(1+|x|^3\big),$$

then we use $\chi(x,\sigma)|x|^k \le C\sigma^{k/2} \le Cs^{k/2}$ for $k\in\mathbb{N}$, and a) of Lemma B.2 to derive

$$\|(1-\chi(y,s))\mathcal{K}(s,\sigma)\psi_b(x,\sigma)\|_{L^\infty} \le Ce^{s-\sigma}\sum_{l=0}^{2}s^{l/2}|\psi_l(\sigma)| + Ce^{s-\sigma}s^{3/2}\left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty}.$$

This yields the bound (3.8) and concludes the proof of Lemma 3.2. □

C. Proof of Lemma 3.9

We give the proof of Lemma 3.9 here.

Proof of Lemma 3.9. i) From the definition (2.4) of B, we use a Taylor expansion and the boundedness of $|\varphi|$ and $|q|$ to find that

$$|\chi(\tau)B(q(\tau))| \le C|q(\tau)|^2 \quad\text{and}\quad |B(q(\tau))| \le C|q(\tau)|^{p'}, \tag{C.1}$$

where $p' = \min\{2, p\}$.

(Since we have the same definition of B as in [25], we do not give the proof of (C.1) and kindly refer the reader to Lemma 3.15, page 168 of [25] for its proof.)

Using (C.1) and (3.1), we have

$$|\chi(\tau)B(q(\tau))| \le \frac{CA^4}{\tau^{3+2\varrho}}\big(1+|y|^6\big) + \frac{CA^4}{\tau^{2+2\nu}}\big(1+|y|^4\big). \tag{C.2}$$

From (C.2), we then derive for $m = 0, 1, 2$,

$$|B_m(\tau)| = \left|\int \chi(\tau)B(q(\tau))k_m\rho dy\right| \le \frac{CA^4}{\tau^{2+2\nu}}. \tag{C.3}$$

Since $B_-(y,\tau) = \chi(\tau)B(q(\tau)) - \sum_{m=0}^{2} B_m(\tau)h_m(y)$, we have from (C.2) and (C.3),

$$\left|\frac{B_-(y,\tau)}{1+|y|^3}\right| \le \left|\frac{\chi(\tau)B(q(\tau))}{1+|y|^3}\right| + \left|\frac{\sum\limits_{m=0}^{2} B_m(\tau)h_m(y)}{1+|y|^3}\right|$$

$$\le \chi(\tau)\left[\frac{CA^4}{\tau^{3+2\varrho}}\left(1+|y|^3\right) + \frac{CA^4}{\tau^{2+2\nu}}(1+|y|)\right] + \frac{CA^4}{\tau^{2+2\nu}}\left(\frac{\left|\sum\limits_{m=0}^{2} h_m(y)\right|}{1+|y|^3}\right).$$

If we use $|y|^l\chi(y,\tau) \le C\tau^{l/2}$ for $l \in \mathbb{N}$, and $|\sum_{m=0}^{2} h_m(y)| \le C(1+|y|^2)$, then we obtain

$$\left\|\frac{B_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \le \frac{CA^4}{\tau^{3/2+2\varrho}}.$$

Using the second estimates in (C.1) and (3.1), we obviously obtain $\|B(\tau)\|_{L^\infty} \le \frac{CA^{2p'}}{\tau^{\varrho p'}}$ which yields $\|B_e(\tau)\|_{L^\infty} \le \frac{CA^{2p'}}{\tau^{\varrho p'}}$. This ends the proof of part i).

ii) From the definition (2.5) of R, we write $\varphi(y,\tau) = \frac{\phi(\tau)}{\kappa}\vartheta(y,\tau)$ and $R(y,\tau) = \frac{\phi(\tau)}{\kappa}Q + G$, where $\vartheta(y,\tau) = f(\frac{y}{\sqrt{\tau}}) + \frac{\kappa}{2p\tau}$ and

$$Q(y,\tau) = -\vartheta_\tau + \Delta\vartheta - \frac{y}{2}\nabla\vartheta - \frac{\vartheta}{p-1} + \vartheta^p, \tag{C.4}$$

$$G(y,\tau) = -\frac{\phi'}{\kappa}\vartheta - \frac{\phi}{\kappa}\vartheta^p + \phi^p\left(\frac{\vartheta}{\kappa}\right)^p + e^{\frac{-p\tau}{p-1}}h\left(e^{\frac{\tau}{p-1}}\frac{\phi}{\kappa}\vartheta\right). \tag{C.5}$$

The conclusion of part ii) is a direct consequence of the following:

Lemma C.1. *There exists $\sigma_7 > 0$ such that for all $\tau \ge \sigma_7$, we have*

i) (***Estimates on Q***)

$$m = 0, 1,\ |Q_m(\tau)| \le \frac{C}{\tau^2}, \quad |Q_2(\tau)| \le \frac{C}{\tau^3},$$
$$\left\|\frac{Q_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{\tau^2}, \quad \|Q_e(\tau)\|_{L^\infty} \le \frac{C}{\sqrt{\tau}}. \tag{C.6}$$

ii) (***Estimates on G***)

$$m=0,1,2,\ |G_m(\tau)|\le\frac{C}{\tau^{1+a'}},\ \left\|\frac{G_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty}\le\frac{C}{\tau^{1+a'}},\ \|G_e(\tau)\|_{L^\infty}\le\frac{C}{\tau^a},\quad \text{(C.7)}$$

where $a'=a>1$ *in the case* (1.13) *and* $a'=a+1>1$ *in the case* (1.14).

Proof. i) See [5, page 563]. For part ii), one can see that it is a direct consequence of the following:

$$|G(y,\tau)|\le\frac{C}{\tau^a}\quad\text{and}\quad|\chi(\tau)G(y,\tau)|\le\frac{C}{\tau^{1+a'}}\big(1+|y|^2\big).\qquad\text{(C.8)}$$

By the definition of G_m, G_- and G_e, part ii) simply follows from (C.8). By the linearity, this also concludes the proof of part ii) of Lemma 3.9.

Let us now give the proof of (C.8). For the first estimate, we use the definition (C.5) of G, Lemmas A.1 and A.2,

$$|G(y,\tau)|\le\left|\frac{\phi'\vartheta}{\kappa}\right|+\left|\frac{\phi\vartheta}{\kappa}\right|\left|1-\frac{\phi^{p-1}}{\kappa^{p-1}}\right|+\left|e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}\frac{\phi\vartheta}{\kappa}\right)\right|\le\frac{C}{s^a}.$$

For the second estimate in (C.8), we use the fact that ϕ satisfies (1.20) and write

$$\begin{aligned}G(y,\tau)&=\frac{\vartheta\phi}{\kappa^p}\big(\kappa^{p-1}-\phi^{p-1}\big)\big(\kappa^{p-1}-\vartheta^{p-1}\big)\\&\quad+e^{-\frac{p\tau}{p-1}}\left[h\left(e^{\frac{\tau}{p-1}}\frac{\phi\vartheta}{\kappa}\right)-h\left(e^{\frac{\tau}{p-1}}\phi\right)\right]\\&\quad+\left(1-\frac{\vartheta}{\kappa}\right)e^{-\frac{p\tau}{p-1}}h\left(e^{\frac{\tau}{p-1}}\phi\right):=\bar G+\tilde G+\hat G.\end{aligned}$$

Noting that $\vartheta(y,\tau)=\kappa\left(1-\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right)$ uniformly for $y\in\mathbb{R}$ and $\tau\ge1$, and recalling from Lemma A.2 that $\phi(\tau)=\kappa(1+\eta_a(\tau))^{-\frac{1}{p-1}}$ where $\eta_a(\tau)=\mathcal{O}(\tau^{-a})$, then using a Taylor expansion, we derive

$$\begin{aligned}\bar G(y,\tau)&=\frac{\phi\eta_a(\tau)}{1+\eta_a(\tau)}\left(\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right),\\ \tilde G(y,\tau)&=-\phi e^{-\tau}h'\left(e^{\frac{\tau}{p-1}}\phi\right)\left(\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right),\\ \hat G(y,\tau)&=e^{-\frac{p\tau}{p-1}}h\left(e^{\frac{\tau}{p-1}}\phi\right)\left(\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right).\end{aligned}$$

This yields the second estimate in (C.8) in the case (1.13). If h is given by (1.14), we have furthermore

$$\begin{aligned}&\left|\frac{\phi\eta_a(\tau)}{1+\eta_a(\tau)} - e^{-s}h'(e^{\frac{s}{p-1}}\phi)\phi + e^{-\frac{ps}{p-1}}h(e^{\frac{s}{p-1}}\phi)\right|\\ &\le \left|\frac{\phi}{1+\eta_a(\tau)}\right|\left|\eta_a(\tau) - \frac{\mu}{\log^a\left(2+e^{\frac{2\tau}{p-1}}\phi^2(\tau)\right)}\right| + \frac{C}{\tau^{a+1}} \le \frac{2C}{\tau^{1+a}},\end{aligned}$$

which yields the second estimate in (C.8) in the case (1.14). This concludes the proof of (C.8) and the proof of part ii) of Lemma 3.9 also. □

iii) From the definition (2.6) of N, we use a Taylor expansion for N to find that in the case (1.13),

$$N(q(\tau),\tau) = e^{-\tau}h'\left(e^{\frac{\tau}{p-1}}(\phi(\tau)+\theta_1 q(\tau))\right)q(\tau) \quad \text{with} \quad \theta_1 \in [0,1],$$

and in the case (1.14),

$$N(q(\tau),\tau) = e^{-\frac{(p-2)\tau}{p-1}}h''\left(e^{\frac{\tau}{p-1}}(\phi(\tau)+\theta_2 q(\tau))\right)q^2(\tau) \quad \text{with} \quad \theta_2 \in [0,1].$$

Since $\varphi(\tau) \to \kappa$ and $\|q(\tau)\|_{L^\infty(\mathbb{R})} \to 0$ as $\tau \to +\infty$, this implies that there exists τ_0 large enough such that $\frac{\kappa}{2} \le |\phi(\tau) + \theta_i q(\tau)| \le \frac{3\kappa}{2}$ for all $\tau \ge \tau_0$ and $y \in \mathbb{R}$. Then by Lemma A.1, we have $|N(q(\tau),\tau)| \le \frac{C|q|^\beta}{\tau^a}$ where $\beta = 1$ in the case (1.13) and $\beta = 2$ in the case (1.14), which implies part iii) of Lemma 3.9. This concludes the proof of Lemma 3.9. □

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Université Paris 13
Sorbonne Paris Cité
LAGA, CNRS (UMR 7539)
F-93430, Villetaneuse, France
and
New York University Abu Dhabi
Saadiyat Island, PO Box 129188
Abu Dhabi, UAE
Tien.Nguyen@nyu.edu

Université Paris 13
Institut Galilée
LAGA 99
Avenue Jean-Baptiste Clément
93430 Villetaneuse, France
Hatem.Zaag@univ-paris13.fr