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# An over-determined boundary value problem arising from neutrally coated inclusions in three dimensions

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*Dedicated to the memory of Professor Kenjiro Okubo*

**Abstract.** We consider the neutral inclusion problem in three dimensions: prove that if a coated inclusion consisting of a core and a shell is neutral to all uniform fields, then the core and the whole inclusion must be concentric balls, if the matrix is isotropic, or confocal ellipsoids if the matrix is anisotropic. We first derive an over-determined boundary value problem in the shell of the neutral inclusion, and then prove in the isotropic case that if the over-determined problem admits a solution, then the core and the whole inclusion must be concentric balls. As a consequence it is proved that the structure is neutral to all uniform fields if and only if it consists of concentric balls provided that the coefficient of the core is larger than that of the shell.

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## 1. Introduction

The purpose of this paper is to prove that the coated inclusions neutral to all uniform fields in the isotropic medium have structures consisting of concentric balls in three dimensions. The coated inclusion is denoted by $(D, \Omega)$ where $D$ and $\Omega$ are bounded domains with Lipschitz boundaries in $\mathbb{R}^d$ for $d = 2, 3$, such that $\overline{D} \subset \Omega$. Here, $D$ represents the core and $\Omega \setminus D$ the shell. The conductivity (or the dielectric constant) is $\sigma_c$ in the core and $\sigma_s$ in the shell ($\sigma_c \neq \sigma_s$). If the structure $(D, \Omega)$ is inserted into the free space $\mathbb{R}^d$ with conductivity $\sigma_m$ where there is a uniform field $-\nabla(\mathbf{a} \cdot \mathbf{x}) = -\mathbf{a}$ for some constant vector $\mathbf{a}$, then, in general, the field is perturbed. But for certain inclusions the field is not perturbed, in other words, the field does not recognize the existence of the inclusion. In particular, if the coated inclusion is made of concentric balls with specially chosen conductivities

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(respectively, confocal ellipsoids if $\sigma_m$ is anisotropic), one can see that the uniform field is not perturbed. An inclusion with this property is called a neutral inclusion (or neutrally coated inclusion) and the neutral inclusion problem is to show that the inclusions of concentric balls (or confocal ellipsoids) are the only coated inclusions neutral to all uniform fields.

Let $\sigma$ denote the conductivity distribution of the medium so that

$$\sigma = \begin{cases} \sigma_c & \text{in } D \\ \sigma_s & \text{in } \Omega \setminus D \\ \sigma_m & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \tag{1.1}$$

Here we assume that $\sigma_c$ and $\sigma_s$ are constants (or isotropic matrices), but $\sigma_m$ is allowed to be an anisotropic symmetric matrix. We consider the following problem:

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^d \\ u(\mathbf{x}) - \mathbf{a} \cdot \mathbf{x} = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{1.2}$$

where $\mathbf{a}$ is a constant vector. The term $u(\mathbf{x}) - \mathbf{a} \cdot \mathbf{x}$ describes the perturbation of the potential due to insertion of the coated inclusion $(D, \Omega)$. If the potential is not perturbed, namely,

$$u(\mathbf{x}) - \mathbf{a} \cdot \mathbf{x} \equiv 0 \quad \text{in } \mathbb{R}^d \setminus \Omega, \tag{1.3}$$

the coated inclusion $(D, \Omega)$ is said to be neutral to the field $\mathbf{a}$. If $(D, \Omega)$ is neutral to the field $\mathbf{e}_j$ for $j = 1, \ldots, d$, where $\mathbf{e}_j$ is the standard basis of $\mathbb{R}^d$, then $(D, \Omega)$ is neutral to all uniform fields.

Great interest in neutrally coated inclusions was aroused by the work of Hashin and Shtrikman [7] and Hashin [6]. They showed that since the insertion of neutral inclusions does not perturb the outside uniform field, the effective conductivity of the assemblage filled with coated inclusions of many different scales is $\sigma_m$. We refer to [14] for developments on neutral inclusions in relation to the theory of composites. Interest in neutral inclusions has been aroused also in relation to invisibility cloaking. The neutral inclusion is invisible to uniform probe fields as observed in [12]. Recently, the idea of neutrally coated inclusions has been extended to multi-coated circular structures which are neutral not only to uniform fields but also to fields of higher order up to $N$ for a given integer $N$ [2]. It was proved there that the multi-coated structure combined with a transformation dramatically enhances the near cloaking of [13]. Cloaking by transformation optics was proposed in [17] (and [5]).

As mentioned before, concentric balls (or disks) are made neutral to all uniform fields by choosing $\sigma_c$, $\sigma_s$ and $\sigma_m$ properly ($\sigma_m$ is isotropic). Confocal ellipsoids (or ellipses) are also neutral to all uniform fields if $\sigma_m$ is anisotropic [12] (see also Section 3). Then a question naturally arises: are there any other shapes which are neutral to all uniform fields? In two dimensions there are no other shapes: if a coated inclusion $(D, \Omega)$ is neutral to all uniform fields in two dimensions, then $D$ and $\Omega$ are concentric disks (confocal ellipses if $\sigma_m$ is anisotropic). This is proved when $\sigma_c = 0$ or $\infty$ in [15] and when $\sigma_c$ is finite in [10]. In this paper we consider the neutral inclusion problem in three dimensions. We emphasize that the methods in [10, 15] use powerful tools from complex analysis such as conformal mappings and harmonic conjugates, which cannot be applied to three dimensions. It is worth mentioning that there are many different shapes of coated inclusions neutral to a single uniform field as shown in two dimensions in [8, 15].

We first show that if $(D, \Omega)$ is neutral to all uniform fields in three dimensions and if $\sigma_c > \sigma_s$, then the following problem admits a solution:

$$\begin{cases} \Delta w = k & \text{in } \Omega \setminus \overline{D} \\ \nabla w = 0 & \text{on } \partial\Omega \\ \nabla w(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{d} & \text{on } \partial D, \end{cases} \tag{1.4}$$

where $k (\neq 0)$ is a constant, $\mathbf{A}$ is a symmetric matrix, and $\mathbf{d}$ is a constant vector. We emphasize that this is an over-determined problem because $\nabla w$ is prescribed on the boundaries. The problem, which is of independent interest, is to prove that if (1.4) admits a solution in three dimensions, then $D$ and $\Omega$ are confocal ellipsoids. If $D$ and $\Omega$ are confocal ellipsoids, then (1.4) admits a solution and $\mathbf{A}$ should be either positive or negative-definite depending on the sign of $k$ (see Section 3). So a part of the problem is to show that $\mathbf{A}$ is either positive or negative-definite. In two dimensions it is proved in [10] that if (1.4) admits a solution then $D$ and $\Omega$ are confocal ellipses (concentric disks if $\mathbf{A}$ is isotropic). However, the proof there is based on the powerful result that there is a conformal mapping from $\Omega \setminus \overline{D}$ onto an annulus. So it cannot be extended to three dimensions. The condition $\sigma_c > \sigma_s$, which is not natural, is required because of a technical reason for the derivation of (1.4) in Subsection 2.2. Even though we do not know how to do so, it is likely that the condition can be removed.

In this paper we solve the problem partially as the following theorem shows:

**Theorem 1.1.** *Let $D$ and $\Omega$ be bounded domains with Lipschitz boundaries in $\mathbb{R}^3$ with $\overline{D} \subset \Omega$. Suppose that $\Omega \setminus \overline{D}$ is connected. If (1.4) admits a solution for $\mathbf{A} = c\mathbf{I}$ for some constant $c$ where $\mathbf{I}$ is the identity matrix in three dimensions, then $D$ and $\Omega$ are concentric balls whose radii, denoted by $r_e$ (for $\Omega$) and $r_i$ (for $D$), satisfy*

$$k(r_e^3 - r_i^3) = -3cr_i^3. \tag{1.5}$$

We emphasize that formula (1.5) can be generalized to anisotropic cases: if (1.4) admits a solution, then

$$k|\Omega \setminus D| = -\mathrm{Tr}\,\mathbf{A}|D|, \tag{1.6}$$

where $|D|$ indicates the volume of $D$. In fact, (1.6) can be obtained by integrating the first equation in (1.4) over $\Omega \setminus \overline{D}$ and applying the divergence theorem.

As a consequence of Theorem 1.1, we obtain the following theorem:

**Theorem 1.2.** *Let $D$ and $\Omega$ be bounded domains with Lipschitz boundaries in $\mathbb{R}^3$ with $\overline{D} \subset \Omega$. Suppose that $\partial D$ is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected. If $\sigma_m$ is isotropic, $\sigma_c > \sigma_s$ and $(D, \Omega)$ is neutral to all uniform fields, then $D$ and $\Omega$ are concentric balls.*

This paper is organized as follows. In Section 2 we show that if $(D, \Omega)$ is neutral to all uniform fields then (1.4) admits a solution. In Section 3 we construct a solution to (1.4) when $D$ and $\Omega$ are confocal ellipsoids. Section 4 is to prove Theorem 1.1. In Section 5 we formulate the problem (1.4) using Newtonian potentials and relate the problem with a known characterization of ellipsoids.

## 2. Derivation of the over-determined problem

In this section we derive (1.4) from the neutral inclusion problem. We will do so only in three dimensions since (1.4) has already been derived in two dimensions [10]. Here we assume that $\partial D$ is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected.

Suppose, after diagonalization, that

$$\sigma_m = \mathrm{diag}[\sigma_{m,1}, \sigma_{m,2}, \sigma_{m,3}]. \tag{2.1}$$

Let $u_j$ for $j = 1, 2, 3$ be the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u_j = 0 & \text{in } \mathbb{R}^3 \\ u_j(x) - x_j = O(|x|^{-2}) & \text{as } |x| \to \infty. \end{cases} \tag{2.2}$$

The structure being neutral to all three fields means that $u_j(x) - x_j = 0$ in $\mathbb{R}^3 \setminus \Omega$ for $j = 1, 2, 3$. Let

$$w_j = \frac{1}{\beta_j} u_j \tag{2.3}$$

where

$$\beta_j := \frac{\sigma_{m,j}}{\sigma_s} - 1 \quad \text{for} \quad j = 1, 2, 3,$$

and $\mathbf{w} = (w_1, w_2, w_3)^T$ ($T$ stands for transpose). Set also

$$\mathbf{B} = \mathrm{diag}\,[1/\beta_1, 1/\beta_2, 1/\beta_3]. \tag{2.4}$$

We will show the following:

(i) $\nabla \mathbf{w}$ is symmetric and $\mathrm{div}\, \mathbf{w}$ is constant, and hence there is a function $\psi$ in $\overline{\Omega} \setminus D$ such that

$$\mathbf{w} = \nabla \psi \ \text{ and } \ \Delta \psi = \mathrm{Tr}\, \mathbf{B} + 1 \quad \text{in } \Omega \setminus \overline{D}; \tag{2.5}$$

(ii) $\mathbf{w}(\mathbf{x}) = c_0 \mathbf{x} + \mathbf{d}$, for $\mathbf{x} \in \partial D$, some constant $c_0$ and constant vector $\mathbf{d}$ (under the assumption that $\sigma_c > \sigma_s$).

We emphasize that it is in (ii) where the condition $\sigma_c > \sigma_s$ is required. Once we have (i) and (ii), then we can show that (1.4) has a solution. In fact, since $u_j = x_j$ on $\partial \Omega$, we have

$$\nabla \psi(\mathbf{x}) = \mathbf{B}\mathbf{x} \quad \text{on } \partial \Omega.$$

Note that $\nabla\psi(\mathbf{x}) = c_0\mathbf{x} + \mathbf{d}$ on $\partial D$. Now define

$$w(\mathbf{x}) := \psi(\mathbf{x}) - \frac{1}{2}\mathbf{x}\cdot\mathbf{B}\mathbf{x}. \tag{2.6}$$

Then $w$ satisfies (1.4) with $k = 1$ and $\mathbf{A} = c_0\mathbf{I} - \mathbf{B}$. We emphasize that if $\sigma_m$ is isotropic, so are $\mathbf{B}$ and $\mathbf{A}$.

## 2.1. Proof of (i)

Let us first deal with the case where $0 < \sigma_c < \infty$. Denote by $\nu = (n_1, n_2, n_3)^T$ the outward unit normal vector field to $\partial\Omega$ or $\partial D$. Note that the solution $u_j$ for $j = 1, 2, 3$ to (2.2) satisfies the following transmission conditions on the two interfaces:

$$u_j|_+ - u_j|_- = 0, \quad \sum_{i=1}^{3}\sigma_{m,i}n_i\frac{\partial u_j}{\partial x_i}\Big|_+ - \sigma_s\frac{\partial u_j}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial\Omega \tag{2.7}$$

and

$$u_j|_+ - u_j|_- = 0, \quad \sigma_s\frac{\partial u_j}{\partial \nu}\Big|_+ - \sigma_c\frac{\partial u_j}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial D \tag{2.8}$$

where $+$ denotes the limit from the outside and $-$ that from the inside of $\Omega$ or $D$. If $(D, \Omega)$ is neutral to $\mathbf{e}_j$, then $u_j(\mathbf{x}) - x_j = 0$ in $\mathbb{R}^3 \setminus \Omega$, so we see from (2.7) that

$$u_j|_- = x_j, \quad \sigma_s\frac{\partial u_j}{\partial \nu}\Big|_- = \sigma_{m,j}n_j \quad \text{on } \partial\Omega. \tag{2.9}$$

In other words, $u_j$ is the solution to the following over-determined problem:

$$\begin{cases} \nabla\cdot\sigma\nabla u_j = 0 & \text{in } \Omega \\ u_j = x_j, \quad \dfrac{\partial u_j}{\partial \nu} = \dfrac{\sigma_{m,j}}{\sigma_s}n_j & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Let $v_j \in C^2(\overline{\Omega})$. Then we see from the divergence theorem and (2.8) that

$$\begin{aligned} \int_{\partial\Omega}\frac{\partial u_j}{\partial \nu}\Big|_- v_j - u_j\frac{\partial v_j}{\partial \nu} &= -\int_{\Omega\setminus D}u_j\Delta v_j + \int_{\partial D}\frac{\partial u_j}{\partial \nu}\Big|_+ v_j - u_j\frac{\partial v_j}{\partial \nu} \\ &= -\int_{\Omega\setminus D}u_j\Delta v_j + \left(\frac{\sigma_c}{\sigma_s} - 1\right)\int_{\partial D}\frac{\partial u_j}{\partial \nu}\Big|_- v_j \\ &\quad + \int_{\partial D}\frac{\partial u_j}{\partial \nu}\Big|_- v_j - u_j\frac{\partial v_j}{\partial \nu} \\ &= -\int_{\Omega\setminus D}u_j\Delta v_j + \left(\frac{\sigma_c}{\sigma_s} - 1\right)\int_D \nabla u_j\cdot\nabla v_j - \int_D u_j\Delta v_j \\ &= -\int_{\Omega}u_j\Delta v_j + \left(\frac{\sigma_c}{\sigma_s} - 1\right)\int_D \nabla u_j\cdot\nabla v_j. \end{aligned}$$

On the other hand, we see from (2.9) that

$$\int_{\partial\Omega} \frac{\partial u_j}{\partial \nu}\Big|_- v_j - u_j \frac{\partial v_j}{\partial \nu} = \int_{\partial\Omega} \frac{\sigma_{m,j}}{\sigma_s} n_j v_j - y_j \frac{\partial v_j}{\partial \nu}$$
$$= \left(\frac{\sigma_{m,j}}{\sigma_s} - 1\right) \int_\Omega \frac{\partial v_j}{\partial y_j} - \int_\Omega y_j \Delta v_j.$$

Equating two identities above we obtain

$$\int_\Omega (y_j - u_j)\Delta v_j + \alpha \int_D \nabla u_j \cdot \nabla v_j - \beta_j \int_\Omega \frac{\partial v_j}{\partial y_j} = 0 \quad \text{for} \quad j = 1, 2, 3 \tag{2.11}$$

for $v_j \in C^2(\overline{\Omega})$, where $\alpha$ and $\beta_j$ are defined for ease of notation to be

$$\alpha = \frac{\sigma_c}{\sigma_s} - 1 \quad \text{and} \quad \beta_j = \frac{\sigma_{m,j}}{\sigma_s} - 1. \tag{2.12}$$

Let $w_j$ be defined by $w_j := \frac{1}{\beta_j} u_j$ as in (2.3). Then (2.11) can be rephrased as

$$\int_\Omega \left(\frac{1}{\beta_j} y_j - w_j\right) \Delta v_j + \alpha \int_D \nabla w_j \cdot \nabla v_j - \int_\Omega \frac{\partial v_j}{\partial y_j} = 0 \text{ for } \ j = 1, 2, 3. \tag{2.13}$$

Summing (2.13) over $j = 1, 2, 3$ we have

$$\int_\Omega \sum_{j=1}^3 \left(\frac{1}{\beta_j} y_j - w_j\right) \Delta v_j + \alpha \int_D \sum_{j=1}^3 \nabla w_j \cdot \nabla v_j - \int_\Omega \sum_{j=1}^3 \frac{\partial v_j}{\partial y_j} = 0$$

for $v_j \in C^2(\overline{\Omega})$. If we use vector notation $\mathbf{w} = (w_1, w_2, w_3)^T$ and $\mathbf{v} = (v_1, v_2, v_3)^T$ ($T$ stands for transpose), then the above identity can be rewritten as

$$\int_\Omega (\mathbf{B}\mathbf{y} - \mathbf{w}) \cdot \Delta \mathbf{v} + \alpha \int_D \nabla \mathbf{w} : \nabla \mathbf{v} - \int_\Omega \operatorname{div} \mathbf{v} = 0. \tag{2.14}$$

Here and afterwards $\mathbf{A} : \mathbf{B}$ denotes the contraction of two matrices $\mathbf{A}$ and $\mathbf{B}$, *i.e.*, $\mathbf{A} : \mathbf{B} = \sum a_{ij} b_{ij} = \mathrm{Tr}(\mathbf{A}^T \mathbf{B})$.

Let $\Gamma$ be the fundamental solution of the Laplace operator in $\mathbb{R}^3$, *i.e.*,

$$\Gamma(\mathbf{x}) := -\frac{1}{4\pi |\mathbf{x}|} \quad \text{for} \quad \mathbf{x} \neq 0. \tag{2.15}$$

Let $v_j(\mathbf{y}) = \Gamma(\mathbf{x} - \mathbf{y})$ for a fixed $\mathbf{x} \in \Omega$. Since $\Delta v_j(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$, by applying the divergence theorem over $\Omega \setminus B_\epsilon(\mathbf{x})$ for sufficiently small $\epsilon$ (where $B_\epsilon(\mathbf{x})$ is the ball of radius $\epsilon$ centered at $\mathbf{x}$) we see from (2.13) that

$$w_j(\mathbf{x}) = \frac{1}{\beta_j} x_j + \alpha \int_D \nabla w_j(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \Gamma(\mathbf{x} - \mathbf{y}) d\mathbf{y} + \frac{\partial}{\partial x_j} N_\Omega(\mathbf{x}), \text{ for } \mathbf{x} \in \Omega, \text{ and } j = 1,2,3, \tag{2.16}$$

where $N_\Omega$ is the Newtonian potential on a domain $\Omega$, *i.e.*,

$$N_\Omega(\mathbf{x}) := \int_\Omega \Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.17}$$

Let

$$f_j(\mathbf{x}) := \int_D \nabla w_j(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} \quad \text{for} \quad j = 1, 2, 3,$$

and let $\mathbf{f} = (f_1, f_2, f_3)^T$. Note that $f_j$ is harmonic in $\mathbb{R}^3 \setminus \overline{D}$, and (2.16) can be rewritten as

$$\mathbf{w}(\mathbf{x}) = \alpha\mathbf{f}(\mathbf{x}) + \nabla\left(\frac{1}{2}\mathbf{x}\cdot\mathbf{B}\mathbf{x} + N_\Omega(\mathbf{x})\right) \text{ for } \mathbf{x} \in \Omega. \tag{2.18}$$

For any fixed $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$, let

$$v_j(\mathbf{y}) = \frac{\partial}{\partial x_j}\Gamma(\mathbf{x}-\mathbf{y}) \quad \text{for} \quad j = 1, 2, 3.$$

Then $\operatorname{div}\mathbf{v}(\mathbf{y}) = -\Delta_{\mathbf{y}}\Gamma(\mathbf{x}-\mathbf{y}) = 0$ and $\Delta\mathbf{v}(\mathbf{y}) = 0$ for $\mathbf{y} \in \Omega$. So we see from (2.14) that

$$\int_D \nabla\mathbf{w} : \nabla\mathbf{v} = 0,$$

and hence

$$\begin{aligned} \operatorname{div}\mathbf{f}(\mathbf{x}) &= \int_D \sum_j \nabla w_j(\mathbf{y}) \cdot \nabla\frac{\partial}{\partial x_j}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} \\ &= \int_D \nabla\mathbf{w} : \nabla\mathbf{v} = 0 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{aligned} \tag{2.19}$$

Since $f_j$ is harmonic in $\mathbb{R}^3 \setminus \overline{D}$, (2.19) holds for all $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}$.

Again fix $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$ and let

$$v_i(\mathbf{y}) = \frac{\partial}{\partial x_j}\Gamma(\mathbf{x}-\mathbf{y}), \quad v_j(\mathbf{y}) = -\frac{\partial}{\partial x_i}\Gamma(\mathbf{x}-\mathbf{y}), \quad v_k = 0 \quad \text{for} \quad \mathbf{y} \in \Omega.$$

Then, $\Delta\mathbf{v} = 0$ and $\operatorname{div}\mathbf{v} = 0$ in $\Omega$. So we have from (2.14)

$$\int_D \nabla w_i(\mathbf{y}) \cdot \nabla\frac{\partial}{\partial x_j}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} - \int_D \nabla w_j(\mathbf{y}) \cdot \nabla\frac{\partial}{\partial x_i}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} = 0,$$

which implies that

$$\partial_i f_j(\mathbf{x}) = \partial_j f_i(\mathbf{x}) \tag{2.20}$$

for all $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$ and hence for all $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}$. Moreover, since $\mathbb{R}^3 \setminus \overline{D}$ is simply connected, by the Stokes theorem there is $\varphi$ such that

$$\mathbf{f}(\mathbf{x}) = \nabla\varphi(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}. \tag{2.21}$$

Because of (2.19), we have

$$\Delta\varphi(\mathbf{x}) = 0 \ \text{ for } \ \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}. \tag{2.22}$$

Let

$$\psi(\mathbf{x}) = \alpha\varphi(\mathbf{x}) + \frac{1}{2}\mathbf{x}\cdot\mathbf{B}\mathbf{x} + N_\Omega(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Omega \setminus \overline{D}. \tag{2.23}$$

Then, we have from (2.18) and (2.21)

$$\mathbf{w}(\mathbf{x}) = \nabla\psi(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Omega \setminus \overline{D}. \tag{2.24}$$

Since $\Delta N_\Omega(\mathbf{x}) = 1$ for $\mathbf{x} \in \Omega$, we have from (2.22) that

$$\Delta\psi(\mathbf{x}) = \operatorname{Tr}\mathbf{B} + 1 \ \text{ for } \ \mathbf{x} \in \Omega \setminus \overline{D}. \tag{2.25}$$

So far we have shown that $\nabla\mathbf{w}$ is symmetric, $\operatorname{div}\mathbf{w}$ is constant, and (2.5) holds when $\sigma_c$ is finite.

We now assume that $\sigma_c = 0$. In this case the problem (2.10) becomes

$$\begin{cases} \Delta u_j = 0 & \text{in } \Omega \setminus \overline{D}, \\ \dfrac{\partial u_j}{\partial \nu} = 0 & \text{on } \partial D, \\ u_j = x_j, \ \ \dfrac{\partial u_j}{\partial \nu} = \dfrac{\sigma_{m,j}}{\sigma_s} n_j & \text{on } \partial\Omega. \end{cases} \tag{2.26}$$

So, we see in a way similar to (2.11) that

$$\int_\Omega y_j \Delta v_j - \int_{\Omega\setminus D} u_j \Delta v_j - \int_{\partial D} u_j \frac{\partial v_j}{\partial \nu} - \beta_j \int_\Omega \frac{\partial v_j}{\partial y_j} = 0 \tag{2.27}$$

for all $v_j \in C^2(\overline{\Omega})$. So we obtain a representation of the solution similar to (2.16):

$$w_j(\mathbf{x}) = \frac{1}{\beta_j} x_j - \int_{\partial D} w_j(\mathbf{y}) \frac{\partial}{\partial \nu}\Gamma(\mathbf{x}-\mathbf{y})d\sigma(\mathbf{y}) + \frac{\partial}{\partial x_j} N_\Omega(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Omega\setminus D. \tag{2.28}$$

So, we infer in exactly the same way as in the previous sections that $\nabla\mathbf{w}$ is symmetric and $\operatorname{div}\mathbf{w}$ is constant, and there is a function $\psi$ such that (2.5) holds. Suppose that $\sigma_c = \infty$. In this case the problem (2.10) becomes

$$\begin{cases} \Delta u_j = 0 & \text{in } \Omega \setminus \overline{D} \\ u_j = \gamma_j \ \text{(constant)} & \text{on } \partial D \\ u_j = x_j, \ \ \dfrac{\partial u_j}{\partial \nu} = \dfrac{\sigma_{m,j}}{\sigma_s} n_j & \text{on } \partial\Omega. \end{cases} \tag{2.29}$$

The constant $\gamma_j$ is determined by the condition

$$\int_{\partial D} \frac{\partial u_j}{\partial \nu}\bigg|_+ = 0.$$

We then obtain similarly to (2.11)

$$\int_\Omega y_j \Delta v_j - \int_{\Omega\setminus D} u_j \Delta v_j + \int_{\partial D} \frac{\partial u_j}{\partial \nu} v_j - \gamma_j \int_D \Delta v_j - \beta_j \int_\Omega \frac{\partial v_j}{\partial y_j} = 0 \quad (2.30)$$

for all $v_j \in C^2(\overline{\Omega})$. We then obtain a representation of the solution similar to (2.16):

$$w_j(\mathbf{x}) = \frac{1}{\beta_j} x_j + \int_{\partial D} \frac{\partial w_j}{\partial \nu}(\mathbf{y}) \Gamma(\mathbf{x}-\mathbf{y}) d\sigma(\mathbf{y}) + \frac{\partial}{\partial x_j} N_\Omega(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \setminus D. \quad (2.31)$$

So, we infer that $\nabla \mathbf{w}$ is symmetric, $\operatorname{div} \mathbf{w}$ is constant, and there is a function $\psi$ such that (2.5) holds.

## 2.2. Proof of (ii)

The transmission conditions (2.8) on $\partial D$ can be rephrased as

$$\mathbf{w}|_+ = \mathbf{w}|_-, \quad \sigma_s \nabla \mathbf{w}|_+ \nu = \sigma_c \nabla \mathbf{w}|_- \nu. \quad (2.32)$$

Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two orthonormal tangent vector fields to $\partial D$. Then, we have

$$(\operatorname{div} \mathbf{w})_- = \langle (\nabla \mathbf{w})_- \nu, \nu \rangle + \langle (\nabla \mathbf{w})_- \mathbf{t}_1, \mathbf{t}_1 \rangle + \langle (\nabla \mathbf{w})_- \mathbf{t}_2, \mathbf{t}_2 \rangle,$$

and

$$(\operatorname{div} \mathbf{w})_+ = \langle (\nabla \mathbf{w})_+ \nu, \nu \rangle + \langle (\nabla \mathbf{w})_+ \mathbf{t}_1, \mathbf{t}_1 \rangle + \langle (\nabla \mathbf{w})_+ \mathbf{t}_2, \mathbf{t}_2 \rangle.$$

Here $(\operatorname{div} \mathbf{w})_-$ denotes the limit of $\operatorname{div} \mathbf{w}$ to $\partial D$ from the inside $D$, and $(\operatorname{div} \mathbf{w})_+$ denotes that from the outside $D$. Since

$$\langle (\nabla \mathbf{w})_- \mathbf{t}_j, \mathbf{t}_j \rangle = \langle (\nabla \mathbf{w})_+ \mathbf{t}_j, \mathbf{t}_j \rangle \quad \text{for} \quad j = 1, 2,$$

we have

$$(\operatorname{div} \mathbf{w})_- - (\operatorname{div} \mathbf{w})_+ = \langle (\nabla \mathbf{w})_- \nu, \nu \rangle - \langle (\nabla \mathbf{w})_+ \nu, \nu \rangle.$$

It then follows from the second identity in (2.32) that

$$\left\langle \left( (\nabla \mathbf{w})_-^T - \frac{\sigma_c}{\sigma_s} (\nabla \mathbf{w})_- \right) \nu, \nu \right\rangle = (\operatorname{div} \mathbf{w})_- - (\operatorname{div} \mathbf{w})_+. \quad (2.33)$$

On the other hand, since $(\nabla \mathbf{w})_+$ is symmetric, we obtain

$$\begin{aligned} \left\langle \left( (\nabla \mathbf{w})_-^T - \frac{\sigma_c}{\sigma_s} (\nabla \mathbf{w})_- \right) \nu, \mathbf{t}_j \right\rangle &= \langle \nu, (\nabla \mathbf{w})_- \mathbf{t}_j \rangle - \frac{\sigma_c}{\sigma_s} \langle (\nabla \mathbf{w})_- \nu, \mathbf{t}_j \rangle \\ &= \langle \nu, (\nabla \mathbf{w})_+ \mathbf{t}_j \rangle - \langle (\nabla \mathbf{w})_+ \nu, \mathbf{t}_j \rangle = 0. \end{aligned} \quad (2.34)$$

We then infer from (2.33) and (2.34) that

$$\left( (\nabla \mathbf{w})_-^T - \frac{\sigma_c}{\sigma_s} (\nabla \mathbf{w})_- \right) \nu = (\operatorname{div} \mathbf{w})_- \nu - (\operatorname{div} \mathbf{w})_+ \nu. \quad (2.35)$$

Recall that $\operatorname{div}\mathbf{w}$ is constant in $\Omega \setminus D$. Let

$$\mathbf{v}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \frac{(\operatorname{div}\mathbf{w})_+}{2+\frac{\sigma_c}{\sigma_s}}\mathbf{x} \quad \text{for} \quad \mathbf{x} \in D. \tag{2.36}$$

Then one can see from (2.35) that

$$\left((\nabla\mathbf{v})^T - \frac{\sigma_c}{\sigma_s}(\nabla\mathbf{v})\right)\nu - (\operatorname{div}\mathbf{v})\nu = 0 \quad \text{on } \partial D. \tag{2.37}$$

Let $\mathbf{g}$ be a smooth vector field on $\overline{D}$. It follows from (2.37) and the divergence theorem that

$$\begin{aligned} 0 &= \int_{\partial D} \nu \cdot \left((\nabla\mathbf{v})\mathbf{g} - \frac{\sigma_c}{\sigma_s}(\nabla\mathbf{v})^T\mathbf{g} - (\operatorname{div}\mathbf{v})\mathbf{g}\right) d\sigma \\ &= \int_D \operatorname{div}\left((\nabla\mathbf{v})\mathbf{g} - \frac{\sigma_c}{\sigma_s}(\nabla\mathbf{v})^T\mathbf{g} - (\operatorname{div}\mathbf{v})\mathbf{g}\right) d\mathbf{x}. \end{aligned}$$

One can easily show that

$$\operatorname{div}\left((\nabla\mathbf{v})\mathbf{g} - \frac{\sigma_c}{\sigma_s}(\nabla\mathbf{v})^T\mathbf{g} - (\operatorname{div}\mathbf{v})\mathbf{g}\right) = \nabla\mathbf{v}^T : \nabla\mathbf{g} - \frac{\sigma_c}{\sigma_s}\nabla\mathbf{v} : \nabla\mathbf{g} - (\operatorname{div}\mathbf{v})(\operatorname{div}\mathbf{g}),$$

and so we obtain

$$\int_D \nabla\mathbf{v}^T : \nabla\mathbf{g} - \frac{\sigma_c}{\sigma_s}\nabla\mathbf{v} : \nabla\mathbf{g} - (\operatorname{div}\mathbf{v})(\operatorname{div}\mathbf{g}) = 0. \tag{2.38}$$

Using the notation

$$\widehat{\nabla}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \quad \text{and} \quad \breve{\nabla}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} - \nabla\mathbf{v}^T),$$

it can be rewritten as

$$\left(1 - \frac{\sigma_c}{\sigma_s}\right)\int_D \widehat{\nabla}\mathbf{v} : \widehat{\nabla}\mathbf{g} - \left(1 + \frac{\sigma_c}{\sigma_s}\right)\int_D \breve{\nabla}\mathbf{v} : \breve{\nabla}\mathbf{g} - \int_D (\operatorname{div}\mathbf{v})(\operatorname{div}\mathbf{g}) = 0. \tag{2.39}$$

If $\sigma_c > \sigma_s$, then we take $\mathbf{g} = \mathbf{v}$ so that

$$\left(1 - \frac{\sigma_c}{\sigma_s}\right)\int_D |\widehat{\nabla}\mathbf{v}|^2 - (1 + \frac{\sigma_c}{\sigma_s})\int_D |\breve{\nabla}\mathbf{v}|^2 - \int_D (\operatorname{div}\mathbf{v})^2 = 0. \tag{2.40}$$

Thus, we infer that $\mathbf{v}$ is constant in $D$ and hence

$$\mathbf{w}(\mathbf{x}) = \frac{(\operatorname{div}\mathbf{w})_+}{2+\frac{\sigma_c}{\sigma_s}}\mathbf{x} + \text{a constant vector}, \quad \mathbf{x} \in D. \tag{2.41}$$

If $\sigma_c = \infty$, then by (2.29) $\mathbf{u}$ is constant on $\partial D$, and hence by (2.3) $\mathbf{w}$ is constant on $\partial D$. So, we can see that (ii) holds with $c_0 = 0$.

## 3. Existence of solutions on confocal ellipsoids

We first mention that the solution $w$ to (1.4) is unique in the sense that if $w_1$ and $w_2$ are two solutions (with different $k$, $\mathbf{A}$'s, and $\mathbf{d}$'s), then $w_1 = Cw_2 + E$ for some constants $C$ and $E$. In fact, if $w_j$ is a solution to (1.4) with $k = k_j \neq 0, \mathbf{A} = \mathbf{A}_j$ and $\mathbf{d} = \mathbf{d}_j$ (for $j = 1, 2$), then $w = w_1 - \frac{k_1}{k_2} w_2$ satisfies $\Delta w = 0$ in $\Omega \setminus \overline{D}$ and $\nabla w = 0$ on $\partial\Omega$, so we have that $w$ must be a constant. We now construct a solution to (1.4) when $D$ and $\Omega$ are confocal ellipsoids. To do so, assume that $\partial D$ is given by

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_3^2}{c_3^2} = 1. \tag{3.1}$$

We then use the confocal ellipsoidal coordinates $\rho,\ \mu,\ \xi$ such that

$$\begin{aligned} \frac{x_1^2}{c_1^2+\rho} + \frac{x_2^2}{c_2^2+\rho} + \frac{x_3^2}{c_3^2+\rho} &= 1, \\ \frac{x_1^2}{c_1^2+\mu} + \frac{x_2^2}{c_2^2+\mu} + \frac{x_3^2}{c_3^2+\mu} &= 1, \\ \frac{x_1^2}{c_1^2+\xi} + \frac{x_2^2}{c_2^2+\xi} + \frac{x_3^2}{c_3^2+\xi} &= 1, \end{aligned}$$

subject to the conditions $-c_3^2 < \xi < -c_2^2 < \mu < -c_1^2 < \rho$. Then the confocal ellipsoid $\partial\Omega$ is given by $\rho = \rho_0$ for some $\rho_0 > 0$. Let

$$g(\rho) = (c_1^2+\rho)(c_2^2+\rho)(c_3^2+\rho), \tag{3.2}$$

and define

$$\varphi_j(\rho) = \int_\rho^\infty \frac{1}{(c_j^2+s)\sqrt{g(s)}} ds \quad \text{for} \quad j = 1, 2, 3. \tag{3.3}$$

Then the function $w$ defined by

$$w(\mathbf{x}) = \frac{1}{2}\int_\rho^\infty \frac{1}{\sqrt{g(s)}} ds - \frac{1}{2}\sum_{j=1}^{3} \varphi_j(\rho) x_j^2 + \frac{1}{2}\sum_{j=1}^{3} \varphi_j(\rho_0) x_j^2 \tag{3.4}$$

is a solution of (1.4). In fact, we can see that

$$\frac{\partial}{\partial x_i}\left[\frac{1}{2}\int_\rho^\infty \frac{1}{\sqrt{g(s)}} ds - \frac{1}{2}\sum_{j=1}^{3} \varphi_j(\rho) x_j^2\right] = \left(-\frac{1}{\sqrt{g(\rho)}} - \sum_{j=1}^{3} \varphi_j'(\rho) x_j^2\right)\frac{\partial\rho}{\partial x_i} - \varphi_i(\rho) x_i.$$

Since

$$\sum_{j=1}^{3} \varphi_j'(\rho) x_j^2 = -\sum_{j=1}^{3} \frac{x_j^2}{(c_j^2+\rho)\sqrt{g(\rho)}} = -\frac{1}{\sqrt{g(\rho)}},$$

we have

$$\frac{\partial}{\partial x_i}\left[\frac{1}{2}\int_\rho^\infty \frac{1}{\sqrt{g(s)}}ds - \frac{1}{2}\sum_{j=1}^{3}\varphi_j(\rho)x_j^2\right] = -\varphi_i(\rho)x_i,$$

from which we see that

$$\nabla w(\mathbf{x}) = -(\varphi_1(\rho)x_1, \varphi_2(\rho)x_2, \varphi_3(\rho)x_3)+(\varphi_1(\rho_0)x_1, \varphi_2(\rho_0)x_2, \varphi_3(\rho_0)x_3). \quad (3.5)$$

Using the relation

$$\frac{\partial\rho}{\partial x_i} = \frac{2x_i}{c_i^2+\rho}\left[\frac{x_1^2}{(c_1^2+\rho)^2}+\frac{x_2^2}{(c_2^2+\rho)^2}+\frac{x_3^2}{(c_3^2+\rho)^2}\right]^{-1}, \quad (3.6)$$

we obtain that $\Delta w$ is constant. Note that $\nabla w = 0$ on $\partial\Omega$ ($\rho = \rho_0$) and $\nabla w = \mathbf{Ax}$ on $\partial D$ where

$$\mathbf{A} = \text{diag}[\varphi_1(\rho_0) - \varphi_1(0), \varphi_2(\rho_0) - \varphi_2(0), \varphi_3(\rho_0) - \varphi_3(0)]. \quad (3.7)$$

We emphasize that $\mathbf{A}$ is negative-definite.

## 4. Proof of Theorem 1.1

Let $w$ be the solution to (1.4) with $\mathbf{A} = c\mathbf{I}$. We notice that $c \neq 0$. Indeed, if $c = 0$, then we have

$$0 \neq k|\Omega\setminus\overline{D}| = \int_{\Omega\setminus\overline{D}}\Delta w\,dx = \int_{\partial\Omega}\frac{\partial w}{\partial\nu}\,d\sigma - \int_{\partial D}\frac{\partial w}{\partial\nu}\,d\sigma = 0 - \int_{\partial D}\nu\cdot\mathbf{d}\,d\sigma = 0,$$

which is a contradiction. Since $c \neq 0$, by introducing new variables

$$\mathbf{y} = \mathbf{x} + \frac{1}{c}\mathbf{d},$$

we may assume that $\mathbf{d} = \mathbf{0}$. Set

$$A_{ij} = x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j} \quad \text{for } i \neq j. \quad (4.1)$$

It is worth mentioning that $A_{ij}$ is the angular derivative. Observe that $A_{ij}$ commutes with $\Delta$, namely, $A_{ij}\Delta = \Delta A_{ij}$. So, we have $\Delta A_{ij}w = 0$ in $\Omega\setminus\overline{D}$. Note that $A_{ij}w = 0$ on $\partial\Omega$. Since $\nabla w(\mathbf{x}) = c\mathbf{x}$ on $\partial D$, we see that $A_{ij}w = 0$ on $\partial D$. Then the maximum principle yields that

$$A_{ij}w = 0 \quad \text{in } \Omega\setminus\overline{D}. \quad (4.2)$$

Since $\Delta w = k$ in $\Omega \setminus \overline{D}$, we see that $w$ satisfies the ordinary differential equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r} = k \quad \text{in } \Omega \setminus \overline{D} \tag{4.3}$$

for $r = |\mathbf{x}|$. Choose a ball $B$ with $\overline{B} \subset \Omega \setminus \overline{D}$. By (4.3), $w$ is of the form

$$w(r) = \frac{k}{6}r^2 + \frac{k_1}{r} + k_2 \quad \text{in } \overline{B} \tag{4.4}$$

for some real constants $k_1$ and $k_2$. Since $\Omega \setminus \overline{D}$ is connected and

$$\Delta\left(w - \frac{k}{6}r^2 - \frac{k_1}{r} - k_2\right) = 0 \quad \text{in } \Omega \setminus \overline{D},$$

we have, from (4.4),

$$w(r) = \frac{k}{6}r^2 + \frac{k_1}{r} + k_2 \quad \text{in } \Omega \setminus \overline{D}. \tag{4.5}$$

Since $\frac{\partial w}{\partial r} = 0$ on $\partial\Omega$, we must have

$$\frac{k}{3}r - \frac{k_1}{r^2} = 0 \quad \text{on } \partial\Omega,$$

and hence

$$r^3 = \frac{3k_1}{k} \quad \text{on } \partial\Omega.$$

This means that $\partial\Omega = \partial B_R(\mathbf{0})$ for some $R > 0$. Therefore we have

$$\nabla w(\mathbf{x}) = \frac{k}{3}\mathbf{x} - \frac{kR^3}{3}\frac{\mathbf{x}}{r^3}, \quad \mathbf{x} \in \Omega \setminus \overline{D}.$$

Since $\nabla w(\mathbf{x}) = c\mathbf{x}$ for all $\mathbf{x} \in \partial D$, we must have

$$\frac{k}{3} - \frac{kR^3}{3}\frac{1}{r^3} = c \quad \text{on } \partial D,$$

or $r =$ constant for all $\mathbf{x} \in \partial D$. It means that $\partial D$ is a sphere centered at $\mathbf{0}$. This completes the proof. □

## 5. Newtonian potential formulation

In this section we reformulate the problem (1.4) in terms of the Newtonian potentials and relate it with the known characterization of ellipsoids using the property of the Newtonian potential. Here, as in Theorem 1.1, we assume this $\overline{D} \subset \Omega$ and that

$\Omega \setminus \overline{D}$ is connected. We emphasize that under the assumptions $\partial D$ is connected, though $\partial\Omega$ is not necessarily connected.

Suppose that (1.4) admits a solution, which we denote by $w$. By the second equation of (1.4) one can see that $w$ is constant on each connected component of $\partial\Omega$. Since $\partial D$ is connected, one can also infer from the third equation of (1.4) that

$$w(\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot \mathbf{A}\mathbf{x} + \mathbf{d} \cdot \mathbf{x} + C \quad \text{for } \mathbf{x} \in \partial D, \tag{5.1}$$

for some constant $C$.

Fix $\mathbf{x} \notin \overline{\Omega} \setminus D$. We obtain from the divergence theorem that

$$\begin{aligned}
&k \int_{\Omega\setminus D} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y} \\
&= \int_{\Omega\setminus D} \big[\Delta w(\mathbf{y})\Gamma(\mathbf{x}-\mathbf{y}) - w(\mathbf{y})\Delta_{\mathbf{y}}\Gamma(\mathbf{x}-\mathbf{y})\big] d\mathbf{y} \\
&= -\int_{\partial D} \left[\frac{\partial w}{\partial \nu}(\mathbf{y})\Gamma(\mathbf{x}-\mathbf{y}) - w(\mathbf{y})\frac{\partial}{\partial \nu_{\mathbf{y}}}\Gamma(\mathbf{x}-\mathbf{y})\right] d\sigma(\mathbf{y}) \\
&\quad - \int_{\partial\Omega} w(\mathbf{y})\frac{\partial}{\partial \nu_{\mathbf{y}}}\Gamma(\mathbf{x}-\mathbf{y}) d\sigma(\mathbf{y}) \qquad (5.2) \\
&= -\int_{\partial D} \bigg[(\nu \cdot \mathbf{A}\mathbf{y} + \nu \cdot \mathbf{d})\Gamma(\mathbf{x}-\mathbf{y}) \\
&\qquad\qquad - \left(\frac{1}{2}\mathbf{y} \cdot \mathbf{A}\mathbf{y} + \mathbf{d} \cdot \mathbf{y} + C\right)\frac{\partial}{\partial \nu_{\mathbf{y}}}\Gamma(\mathbf{x}-\mathbf{y})\bigg] d\sigma(\mathbf{y}) \\
&\quad - \int_{\partial\Omega} w(\mathbf{y})\frac{\partial}{\partial \nu_{\mathbf{y}}}\Gamma(\mathbf{x}-\mathbf{y}) d\sigma(\mathbf{y}).
\end{aligned}$$

Since $w$ is constant on each connected component of $\partial\Omega$, we see from the divergence theorem that the integral

$$\int_{\partial\Omega} w(\mathbf{y})\frac{\partial}{\partial \nu_{\mathbf{y}}}\Gamma(\mathbf{x}-\mathbf{y}) d\sigma(\mathbf{y})$$

is constant on each connected component of the open set $\left(\mathbb{R}^3 \setminus \overline{\Omega}\right) \cup D$. So we infer that the above integral vanishes in the unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. Suppose that $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$: we also have

$$\begin{aligned}
&\int_{\partial D} \left[(\nu \cdot \mathbf{A}\mathbf{y} + \nu \cdot \mathbf{d})\Gamma(\mathbf{x}-\mathbf{y}) - \left(\frac{1}{2}\mathbf{y} \cdot \mathbf{A}\mathbf{y} + \mathbf{d} \cdot \mathbf{y} + C\right)\frac{\partial}{\partial \nu_{\mathbf{y}}}\Gamma(\mathbf{x}-\mathbf{y})\right] d\sigma(\mathbf{y}) \\
&= \operatorname{Tr}\mathbf{A} \int_{D} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y}.
\end{aligned}$$

Therefore we have from (1.6) and (5.2) that the quantity

$$k \int_{\Omega \setminus D} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y} - \frac{k|\Omega \setminus D|}{|D|} \int_D \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

is constant on each connected component of the open set $\mathbb{R}^3 \setminus \overline{\Omega}$. This can be rephrased as the statement that $\widehat{N}_\Omega(\mathbf{x}) - \widehat{N}_D(\mathbf{x})$ is constant on each connected component of the open set $\mathbb{R}^3 \setminus \overline{\Omega}$, where $\widehat{N}_\Omega$ and $\widehat{N}_D$ are the (averaged) Newtonian potentials on $\Omega$ and $D$, respectively, namely,

$$\widehat{N}_\Omega(\mathbf{x}) := \frac{1}{|\Omega|} \int_\Omega \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \tag{5.3}$$

and similarly for $\widehat{N}_D$. If $\mathbf{x} \in D$, then we have from (5.2) that

$$k \int_{\Omega \setminus D} \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y} = -\operatorname{Tr} \mathbf{A} \int_D \Gamma(\mathbf{x}-\mathbf{y}) d\mathbf{y} + \frac{1}{2} \mathbf{x} \cdot \mathbf{A}\mathbf{x} + \mathbf{d} \cdot \mathbf{x} + C^*,$$

for some constant $C^*$, and from (1.6) that

$$k|\Omega| \left[\widehat{N}_\Omega(\mathbf{x}) - \widehat{N}_D(\mathbf{x})\right] = \frac{1}{2} \mathbf{x} \cdot \mathbf{A}\mathbf{x} + \mathbf{d} \cdot \mathbf{x} + C^*. \tag{5.4}$$

In conclusion, we have shown that if (1.4) admits a solution, then

$$\widehat{N}_\Omega(\mathbf{x}) - \widehat{N}_D(\mathbf{x}) = \begin{cases} \text{constant on each connected component of } \mathbb{R}^3 \setminus \overline{\Omega} \\ \text{a quadratic polynomial in } D, \end{cases} \tag{5.5}$$

and moreover, $\widehat{N}_\Omega(\mathbf{x}) - \widehat{N}_D(\mathbf{x}) = 0$ in the unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

One can easily see that the converse is also valid: if (5.5) holds, then (1.4) admits a solution. So we may reformulate the statement: if (5.5) holds, then $D$ and $\Omega$ are confocal ellipsoids. This is reminiscent of a question related to the Newton potential problem: if a Newtonian potential of a simply connected domain is a quadratic polynomial in the domain, then the domain must be an ellipsoid (and vice versa). This problem has been solved by Dive [4] and Nikliborc [16] (see also [3]). It is worth mentioning that this characterization of ellipsoids by their Newtonian potentials is an essential ingredient in resolving conjectures of Polya-Szegö and Eshelby in [11].

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