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Bergman kernel and projection on the unbounded Diederich–Fornæss worm domain

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Abstract. In this paper we study the Bergman kernel and projection on the unbounded worm domain

$$\mathcal{W}_\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left|z_1 - e^{i \log|z_2|^2}\right|^2 < 1 \text{ for } z_2 \neq 0\right\}.$$

We first show that the Bergman space of $\mathcal{W}_\infty$ is infinite dimensional. Then we study the Bergman kernel K and the Bergman projection $\mathcal{P}$ for $\mathcal{W}_\infty$. We prove that $K(z, w)$ extends holomorphically in z (and antiholomorphically in w) near each point of the boundary except for a specific subset that we study in detail. By means of an appropriate asymptotic expansion for K, we prove that the Bergman projection $\mathcal{P} : W^s \not\to W^s$ if $s > 0$ and $\mathcal{P} : L^p \not\to L^p$ if $p \neq 2$, where W^s and L^p denote the classic Sobolev space, and the Lebesgue space, respectively, on $\mathcal{W}_\infty$.

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1. Introduction

In this paper we study the Bergman kernel and projection on the unbounded domain

$$\mathcal{W}_\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left|z_1 - e^{i \log|z_2|^2}\right|^2 < 1 \text{ for } z_2 \neq 0\right\} \tag{1.1}$$

(see Figure 1.1). Recall that, for $\mu > 0$, the Diederich–Fornæss worm domain $\mathcal{W}_\mu$ is defined by

$$\mathcal{W}_\mu = \left\{(z_1, z_2) \in \mathbb{C}^2 : \left|z_1 - e^{i \log|z_2|^2}\right|^2 < 1 - \eta\big(\log|z_2|^2\big)\right\}, \tag{1.2}$$

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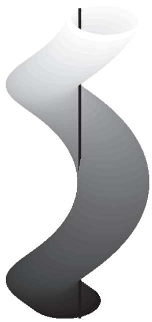

Figure 1.1. A portrait in $\mathbb{C} \times \mathbb{R}$ of a section of $\mathcal{W}$. The first variable z_1 spans in the horizontal plane $\mathbb{C}$, while $\log |z_2|^2$ spans along the vertical line $\mathbb{R}$ (drawn in black).

where η is a smooth, even, convex, non-negative function on the real line, chosen so that $\eta^{-1}(0) = [-\mu, \mu]$ and so that $\mathcal{W}_\mu$ is bounded, smooth, and pseudoconvex. Its boundary is strongly pseudoconvex except at the points $\{(0, z_2) : \big|\log |z_2|^2\big| \le \mu\}$. The worm domain $\mathcal{W}_\mu$ was introduced in [16] by K. Diederich and J. E. Fornæss and turned out to be of great interest as it provides (*counter-*)examples for many important phenomena.

Diederich and Fornæss showed that the worm is the first example of a smoothly bounded domain with nontrivial Nebenhülle. Moreover, it gives an example of a smoothly bounded, pseudoconvex domain which lacks a global plurisubharmonic defining function. Furthermore, nearly 15 years after its introduction, the worm domain showed another feature that is of great interest. In order to describe this feature of $\mathcal{W}_\mu$ and to motivate our present work on $\mathcal{W}_\infty = \bigcup_{\mu>0} \mathcal{W}_\mu$, let us first recall some preliminary material concerning the Bergman space of a complex domain and the associated Bergman projection, as well as its role in the study of the geometry of the domain.

If Ω is a given domain in $\mathbb{C}^n$, denote by $A^2(\Omega)$ the space of holomorphic functions on Ω that are square integrable with respect to the Lebesgue measure. Then, $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and the Hilbert space projection

$$P : L^2(\Omega) \longrightarrow A^2(\Omega)$$

can be represented by an integration formula

$$Pf(z) = \int_\Omega K(z, \zeta) f(\zeta)\, dV(\zeta)\,.$$

The kernel $K(z, \zeta) = K_\Omega(z, \zeta)$ is called the *Bergman kernel*. There exists a vast literature on the Bergman kernel and projection, and on their role in geometric analysis in one and several variables; here we only mention [15,29,32] for the basic ideas and a general overview.

Clearly the Bergman projection P is bounded on $L^2(\Omega)$. Its regularity, or irregularity, in other norms or more general topologies is of great interest.

When Ω is assumed to be smooth, bounded and pseudoconvex, S. R. Bell [5] formulated the notion of *Condition R*, which is the requirement that $P : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ is bounded. The work of Bell and that of Bell and E. Ligocka [6] led to the following fundamental result: if $\Phi : \Omega_1 \longrightarrow \Omega_2$ is a biholomorphic mapping between smoothly bounded, pseudoconvex domains of $\mathbb{C}^n$, one of which satisfies Condition R, then Φ extends to be a C^∞ diffeomorphism of $\overline{\Omega}_1$ to $\overline{\Omega}_2$.

Many different classes of domains are known to satisfy Condition R: *e.g.*, strongly pseudoconvex domains and domains of finite type, domains with real-analytic boundary, complete Hartogs domains in $\mathbb{C}^2$, domains that admit a defining function that is plurisubharmonic on the boundary, see [7,9,12,13,17], respectively. On the other hand, considerable effort has been put into the search for examples of domains that do not satisfy Condition R. Among the first works on this matter we might mention [1], where D. Barrett showed that there exists a smoothly bounded, non-pseudoconvex domain Ω in $\mathbb{C}^2$ on which Condition R fails. In particular, Barrett's work provides some insight on the problem caused by rapidly varying normals to the boundary; see also [2].

Clearly, one way to try to measure whether a domain Ω satisfies or not Condition R is to determine the Sobolev regularity of P; namely, whether or not, for $s > 0$, the projection P preserves the Sobolev space $W^s(\Omega)$ (see, *e.g.* [21,28]). In this direction, J. J. Kohn [24] and B. Berndtsson and P. Charpentier [4] proved (independently and with completely different approaches) that for each smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$ there exists $s_\Omega > 0$ such that $P : W^s(\Omega) \to W^s(\Omega)$ is bounded for $0 < s < s_\Omega$. In [4] it is shown that $s_\Omega \geq \mathrm{DF}(\Omega)/2$, where DF denotes the Diederich-Fornæss exponent of the given domain Ω

$$\mathrm{DF}(\Omega) = \sup\Big\{ 0 < \delta \leq 1 : \exists \text{ defining function } \varrho \text{ for } \Omega,\ -(-\varrho)^\delta \text{ plurisubharmonic on } \partial\Omega \Big\}. \tag{1.3}$$

The lower bound obtained in [24] is not explicit; one way to obtain such a lower bound is described in [31].

An alternative method to establish regularity is via the Neumann operator $\mathcal{N}$, that is, the solution operator of the complex Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on square-integrable (0, 1)-forms. In fact H. P. Boas and E. J. Straube [8] established a connection between regularity of $\mathcal{N}$ and P; see also [32] and the references therein.

Another interesting result in this context is [20] where A.-K. Herbig, J. D. McNeal and Straube address the problem of studying on which subspace of $C^\infty(\overline{\Omega})$ the Bergman projection is bounded as a map into $C^\infty(\overline{\Omega})$.

Consider now the worm domain $\mathcal{W}_\mu$. Let $\mathcal{P}_\mu$ denote the Bergman projection on $\mathcal{W}_\mu$ and set $\nu = \pi/(2\mu)$. Boas and Straube [10] showed that the Bergman projection on $\mathcal{W}_\mu$ maps W^k into itself if k is an integer and $k \geq \nu$, or if $k = \frac{1}{2}$. Furthermore, the result of [4] applies to $\mathcal{W}_\mu$ so that W^s must be preserved by $\mathcal{P}_\mu$ for all $s < \mathrm{DF}(\mathcal{W}_\mu)/2$. We point out, though, that in [16] Diederich and Fornæss showed that $\mathrm{DF}(\mathcal{W}_\mu) \leq \nu$ (see also [27] for details).

In the direction of understanding *irregularity* of the Bergman projection, it was C. O. Kiselman [23] who established an important connection between the worm domain and Condition R. He proved that, for a certain non-smooth version of the worm, a form of Condition R fails.

Stemming from the ideas developed in [23], Barrett proved in [3] the groundbreaking fact that

(i) $\mathcal{P}_\mu : W^s(\mathcal{W}_\mu) \not\to W^s(\mathcal{W}_\mu)$ when $s \geq \nu$

where $W^s(\mathcal{W}_\mu)$ denotes the standard Sobolev space. By the same proof, see also [27], it also follows that

(ii) $\mathcal{P}_\mu : L^p(\mathcal{W}_\mu) \not\to L^p(\mathcal{W}_\mu)$ for $\left|\frac{1}{p} - \frac{1}{2}\right| \geq \nu/2$.

Based on Barrett's result on the irregularity of $\mathcal{P}_\mu$, the work of M. Christ [14] showed that the worm domain is a counterexample to Condition R. After decomposing the space of square-integrable (0, 1)-forms as $L^2_{(0,1)}(\mathcal{W}_\mu) = \oplus_{j\in\mathbf{Z}}\mathcal{H}^1_j$, where $\mathcal{H}^1_j = \left\{u \in L^2_{(0,1)}(\mathcal{W}_\mu) : u(w_1, e^{i\theta}w_2) = e^{ij\theta}u(w_1, w_2)\right\}$, he showed that for all $s > 0$ (apart from a discrete set of exceptions) the Neumann operator $\mathcal{N}$ satisfies an a priori estimate $||\mathcal{N}u||_{W^s} \leq C_{s,j}||u||_{W^s}$ valid for every $u \in \mathcal{H}^1_j \cap C^\infty(\overline{\mathcal{W}_\mu})$ such that $\mathcal{N}u \in C^\infty(\overline{\mathcal{W}_\mu})$. If $\mathcal{N} : C^\infty(\overline{\mathcal{W}_\mu}) \to C^\infty(\overline{\mathcal{W}_\mu})$ were bounded, such estimates would contradict the irregularity of $\mathcal{P}_\mu$.

The peculiar properties of the worm domain $\mathcal{W}_\mu$ have already earned it considerable attention as a counterexample to many important phenomena and they motivate a deeper study of the Bergman space of $\mathcal{W}_\mu$. This study is extremely challenging: for instance, writing down a basis or even a complete system for $A^2(\mathcal{W}_\mu)$ is still an open problem. As a step towards the study of $\mathcal{W}_\mu$, in this paper we study the unbounded worm domain $\mathcal{W}_\infty$ defined in (1.1), which can be thought of as the limit of the smoothly bounded worm domains $\mathcal{W}_\mu$ as $\mu \to +\infty$. This makes it an easier domain to study than the original $\mathcal{W}_\mu$, as we are about to see. We will explain in our Concluding Remarks how the technique applied here may shed some light on the study of the original smoothly bounded worm domains. For simplicity of notation, we are going to write $\mathcal{W}$ instead of $\mathcal{W}_\infty$ and $\mathcal{P}$ for $\mathcal{P}_\infty$ in the remainder of this paper.

The domain $\mathcal{W}$ is clearly unbounded. Denote by $\partial\mathcal{W}$ its boundary. It is well known (see [15,18], and the next section for details) that:

- $\mathcal{W}$ is pseudoconvex;
- $\partial\mathcal{W}$ is smooth except at the points $\mathcal{N} := \{(z_1, 0) : |z_1| \leq 2\}$;
- $\mathcal{W}$ has nontrivial Nebenhülle;
- the smooth part of $\partial\mathcal{W}$ is strongly pseudoconvex except at the points of the critical annulus $\mathcal{A} := \{0\} \times \mathbb{C}^*$.

Here, and in what follows, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

In this work we first show that the Bergman space of $\mathcal{W}$ is not trivial, showing in particular that it is infinite dimensional. Then we consider a biholomorphically equivalent domain $\mathcal{U}$ that we call the *unwound* worm, which is also unbounded, but has the property that the fibers in the second component, that is, the sets $\{z_2 \in \mathbb{C} : (z_1, z_2) \in \mathcal{W}\}$, are connected. This allows us to reduce our study to a family of weighted Bergman spaces $\{A^2(\mathbf{U}, \alpha_j)\}_{j\in\mathbb{Z}}$ on the upper half-plane $\mathbf{U}$ and to the corresponding kernels $\{K_j\}_{j\in\mathbb{Z}}$. At each point of $\mathbf{U} \times \mathbf{U}$, we compute the value of K_j as $\widehat{\phi_\lambda}(j+1)$, where λ is a number in the right half-plane $\mathbf{H}$, associated to the given point of $\mathbf{U} \times \mathbf{U}$ and $\widehat{\phi_\lambda}$ denotes the Fourier transform of the function

$$\phi_\lambda(s) = \frac{1}{2\pi^3}\frac{1}{\cosh^2 s}\Big[\big(2\log(\cosh s)+\lambda\big)^{-2} + 4\big(2\log(\cosh s)+\lambda\big)^{-3}\Big].$$

Altogether, we express the Bergman kernel K of $\mathcal{W}$ as a series of functions, each of which is explicitly computed in terms of the aforementioned K_j.

By means of this machinery, we prove that $K(z, w)$ extends holomorphically in z (and antiholomorphically in w) near each point of the boundary except for a specific subset, which includes the critical set $(\mathcal{A} \times \mathcal{W}) \cup (\mathcal{W} \times \mathcal{A})$. We then find an asymptotic expansion for K near the critical set that allows us to prove:

Theorem 1.1. *For all $s > 0$, the Bergman projection $\mathcal{P}$ does not map the Sobolev space $W^s(\mathcal{W})$ into itself; nor does it map $L^p(\mathcal{W})$ into itself for any p other than* 2.

We point out again that the domain is unbounded and non-smooth. However, the analysis of the singularities of the Bergman kernel shows that the irregularity of the projection is caused by the pathological behavior of $K(\cdot, w)$ near each point of the critical annulus $\mathcal{A}$, where the boundary of the domain is smooth.

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2. Basic facts about $\mathcal{W}$ and $\mathcal{U}$

We begin with the following well-known result —see, *e.g.*, [18].

Proposition 2.1. *The domain $\mathcal{W}$ is pseudoconvex and has nontrivial Nebenhülle. Moreover, the boundary $\partial\mathcal{W}$ is smooth except at the points $\mathcal{N} = \{(z_1, 0) : |z_1| \leq 2\}$ and the smooth part of $\partial\mathcal{W}$ is strongly pseudoconvex except at the points of the critical annulus $\mathcal{A} = \{0\} \times \mathbb{C}^*$.*

We write $\Delta(\zeta, r)$ to denote the disk of center ζ and radius r in $\mathbb{C}$ and $\mathbf{H}$ to denote the right half-plane in the complex plane. Observe that

$$\mathcal{W} = \bigcup_{z_2 \in \mathbb{C}^*} \Delta(e^{i \log |z_2|^2}, 1) \times \{z_2\}.$$

In particular, the projection of $\mathcal{W}$ onto the first variable is $\Delta(0, 2) \setminus \{0\}$.

We denote by $\log \zeta$ the principal branch of logarithm for $\zeta \in \mathbb{C} \setminus (-\infty, 0]$ and use it to define some useful functions on $\mathcal{W}$.

Lemma 2.2. *The formula*

$$L(z) = \log\left(z_1 e^{-i \log |z_2|^2}\right) + i \log |z_2|^2 \tag{2.1}$$

defines a complex-valued holomorphic function in the variable $z = (z_1, z_2) \in \mathbb{C}^2$ on the domain $\mathcal{D} = \bigcup_{z_2 \in \mathbb{C}^*} \{e^{i \log |z_2|^2} \mathbf{H}\} \times \{z_2\} \subset \mathbb{C}^2$. The same is true for

$$E_\eta(z) := e^{\eta L(z)} = \left(z_1 e^{-i \log |z_2|^2}\right)^\eta e^{i \eta \log |z_2|^2} \tag{2.2}$$

for each $\eta \in \mathbb{C}$.

Proof. It is elementary to check that $L(z)$ is well defined on $\mathcal{D} \supseteq \mathcal{W}$ and that it is annihilated by $\bar{\partial}$. □

We point out that the fiber of $\mathcal{W}$ over each $z_1 \in \Delta(0, 2) \setminus \{0\}$ is not connected and that $L(z)$ is locally constant in z_2, but not constant. The same happens with $E_\eta(z)$ for $\eta \in \mathbb{C} \setminus \mathbb{Z}$ (while $E_k(z) = z_1^k$ for all $k \in \mathbb{Z}$ and $z \in \mathcal{W}$).

We can next explicitly construct elements of the Bergman space $A^2(\mathcal{W})$, proving in particular that it is infinite dimensional.

Proposition 2.3. *Let $\mu \in (0, +\infty)$, $c \in (\log 2, +\infty)$ and $j \in \mathbb{Z}$. Then:*

(i) *the function $E_\eta(z) z_2^j$ for $\eta \in \mathbb{C}$ belongs to $A^2(\mathcal{W}_\mu)$ if and only if* $\operatorname{Re} \eta > -1$*;*
(ii) *the function*

$$F_{\eta,c,j,m}(z) = \frac{E_\eta(z) z_2^j}{\left(L(z) - c\right)^m},$$

belongs to $A^2(\mathcal{W}_\mu)$ if and only if $\operatorname{Re} \eta > -1$, *for any $m \in \mathbb{R}$, or* $\operatorname{Re} \eta = -1$, *for $m > 1$.*

Finally,

(iii) *if* $\operatorname{Re} \eta > -1$, $\operatorname{Im} \eta = \frac{j+1}{2}$ *and $m > \frac{1}{2}$, then $F_{\eta,c,j,m} \in A^2(\mathcal{W})$, and if $\eta = -1 + i \frac{j+1}{2}$ and $m > 1$, then $F_{\eta,c,j,m} \in A^2(\mathcal{W})$.*

Proof. We write dV to denote the Lebesgue measure both in $\mathbb{C}$ and in $\mathbb{C}^2$ and $\arg\zeta$ to denote the principal branch of the argument of $\zeta \in \mathbb{C} \setminus (-\infty, 0]$. We have

$$\|F_{a+ib,c,j,m}\|^2_{A^2(\mathcal{W}_\mu)} = \int_{\mathcal{W}_\mu} \left| \frac{E_{a+ib}(z) z_2^j}{\big(L(z)-c\big)^m} \right|^2 dV(z)$$

$$= \int_{-\mu<\log|z_2|^2<\mu} \int_{\Delta(e^{i\log|z_2|^2},1)}$$

$$\times \frac{|z_1|^{2a}|z_2|^{2j} \exp\big\{-2b[\arg(z_1 e^{-i\log|z_2|^2}) + \log|z_2|^2]\big\}}{\big[(\log|z_1|-c)^2 + \big(\arg(z_1 e^{-i\log|z_2|^2}) + \log|z_2|^2\big)^2\big]^m} dV(z_1) dV(z_2)$$

$$= \int_{-\mu<\log|z_2|^2<\mu} \int_{\Delta(1,1)} \frac{|\zeta|^{2a}|z_2|^{2j} \exp\big\{-2b(\arg(\zeta) + \log|z_2|^2)\big\}}{\big[(\log|\zeta|-c)^2 + (\arg(\zeta)+\log|z_2|^2)^2\big]^m} dV(\zeta)\, dV(z_2)$$

$$= 2\pi \int_{e^{-\mu/2}}^{e^{\mu/2}} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r^{2a+1}\rho^{2j+1} e^{-2b(\theta+\log\rho^2)}}{\big[(\log r - c)^2 + (\theta+\log\rho^2)^2\big]^m} dr\, d\theta\, d\rho$$

$$= \pi \int_0^{\frac{\pi}{2}} \int_{\theta-\mu}^{\theta+\mu} \int_{-\infty}^{\log(2\cos\theta)} \frac{e^{2(a+1)s} e^{(t-\theta)(j+1)} e^{-2bt}}{\big[(s-c)^2+t^2\big]^m} ds\, dt\, d\theta$$

$$= \pi \int_0^{\frac{\pi}{2}} \int_{\theta-\mu}^{\theta+\mu} \int_{-\infty}^{\log(2\cos\theta)} \frac{e^{2(a+1)s} ds}{\big[(s-c)^2+t^2\big]^m} e^{t(j+1-2b)}\, dt\, e^{-\theta(j+1)}\, d\theta\,.$$

For $\mu \in (0, +\infty)$, the above integral converges if and only if

$$\int_0^{\frac{\pi}{2}} \int_{\theta-\mu}^{\theta+\mu} \int_{-\infty}^{\log(2\cos\theta)} \frac{e^{2(a+1)s}}{\big[(s-c)^2+t^2\big]^m} ds\, dt\, d\theta$$

is finite, that is, if and only if

$$\int_{\frac{\pi}{2}-\mu}^{\frac{\pi}{2}+\mu} \int_{-\infty}^{0} \frac{e^{2(a+1)s}}{\big[s^2+\varepsilon^2+t^2\big]^m} ds\, dt$$

is finite, where $\varepsilon = c - \log 2 > 0$. Now, assertions (i) and (ii) follow at once. Next, if μ is taken to be $+\infty$ and $b = \frac{j+1}{2}$, we have

$$\|F_{a+ib,c,j,m}\|^2_{A^2(\mathcal{W})} \le C \int_{-\infty}^{\log 2} \int_{\mathbb{R}} \frac{e^{2(a+1)s}}{\big[(s-c)^2+t^2\big]^m} dt\, ds$$

and again (iii) follows easily. □

In order to study the Bergman space it is convenient to "unwind" the domain $\mathcal{W}$ as follows.

Proposition 2.4. *For $z = (z_1, z_2) \in \mathcal{W}$ set*

$$\Phi(z) = \big(-i(L(z) - \log 2), z_2 \big) . \tag{2.3}$$

Moreover, let

$$\mathcal{U} = \Big\{ (u+iv, w_2) \in \mathbb{C}^2 : v > 0, \ \text{and} \ \big| u - \log |w_2|^2 \big| < \arccos(e^{-v}), \ \text{for} \ w_2 \neq 0 \Big\}. \tag{2.4}$$

Then, $\mathcal{U}$ is pseudoconvex, $\Phi : \mathcal{W} \to \mathcal{U}$ is a biholomorphism with $\Phi^{-1}(w_1, w_2) = (2e^{iw_1}, w_2)$ for $(w_1, w_2) \in \mathcal{U}$ and $A^2(\mathcal{U})$ is infinite dimensional.

Proof. It is easily checked that Φ is holomorphic and injective. Moreover, we observe that

$$\begin{aligned} \mathcal{W} &= \big\{ (z_1, z_2) : \operatorname{Re}\big(z_1 e^{-i \log |z_2|^2} \big) > |z_1|^2/2, \ \text{for} \ z_2 \neq 0 \big\} \\ &= \big\{ (re^{i\theta}, z_2) : r < 2, \big| \theta - \log |z_2|^2 \big| < \arccos(r/2), \ \text{for} \ z_2 \neq 0 \big\} . \end{aligned}$$

The conclusion $\Phi(\mathcal{W}) = \mathcal{U}$ now follows easily. Hence, $\mathcal{U}$ is pseudoconvex. Additionally, $\Phi(2e^{iw_1}, w_2) = (w_1, w_2)$ by direct computation. Finally, setting $Tf(w_1, w_2) = 2ie^{iw_1} f(2e^{iw_1}, w_2)$, then we obtain an isometric isomorphism

$$T : A^2(\mathcal{W}) \to A^2(\mathcal{U}) ,$$

so that $A^2(\mathcal{U})$ is nontrivial by Proposition 2.3. □

It is interesting to compare $\mathcal{W}$ with the domain

$$D_\infty = \Big\{ (z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}\big(z_1 e^{-\log |z_2|^2} \big) > 0 \ \text{for} \ z_2 \neq 0 \Big\} , \tag{2.5}$$

and $\mathcal{U}$ with the domain

$$D'_\infty = \Big\{ (z_1, z_2) \in \mathbb{C}^2 : |\operatorname{Im} z_1 - \log |z_2|^2| < \frac{\pi}{2} \ \text{for} \ z_2 \neq 0 \Big\} ,$$

biholomorphic to D_∞ via the mapping $D'_\infty \ni (z_1, z_2) \mapsto (e^{z_1}, z_2) \in D_\infty$. We can think of D_∞ and D'_∞ as the limits as $\mu \to +\infty$ of the domains

$$D_\mu = \Big\{ (z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}\big(z_1 e^{-\log |z_2|^2} \big) > 0 \ \text{ and } \ |\log |z_2|^2| \leq \mu \Big\}$$

and

$$D'_\mu = \Big\{ (z_1, z_2) \in \mathbb{C}^2 : |\operatorname{Im} z_1 - \log |z_2|^2| < \frac{\pi}{2} \ \text{ and } \ |\log |z_2|^2| \leq \mu \Big\} ,$$

studied in [3,23,26–28].

Proposition 2.5. *The spaces $A^2(D_\infty)$ and $A^2(D'_\infty)$ are trivial.*

We postpone the proof to the end of the next section.

3. Reduction to one variable

If Ω denotes either $\mathcal{W}$ or $\mathcal{U}$, the Bergman space $A^2(\Omega)$ decomposes as $\bigoplus_{j\in\mathbb{Z}} \mathcal{H}^j(\Omega)$ where

$$\begin{aligned}\mathcal{H}^j(\Omega) &= \left\{F \in A^2(\Omega) : \; F(w_1, e^{i\theta} w_2) = e^{ij\theta} F(w_1, w_2) \; \text{ for } \theta \in \mathbb{R}\right\} \\ &= \left\{F \in A^2(\Omega) : F(w_1, w_2) w_2^{-j} \text{ is locally constant in } w_2\right\}.\end{aligned}$$

Proposition 2.3 shows that, for every $j \in \mathbb{Z}$, $\mathcal{H}^j(\mathcal{W})$ is nontrivial. Furthermore, $T(\mathcal{H}^j(\mathcal{W})) = \mathcal{H}^j(\mathcal{U})$ and the restriction $T : \mathcal{H}^j(\mathcal{W}) \to \mathcal{H}^j(\mathcal{U})$ is an isometric isomorphism.

We recall that the projection $Q_j : A^2(\Omega) \to \mathcal{H}^j(\Omega)$ is given by

$$Q_j F(z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} F(z_1, e^{i\theta} z_2) e^{-ij\theta} \, d\theta \, .$$

For more details, see [3].

Let $\pi_1 : \mathcal{U} \to \mathbb{C}$ be the projection map onto the first variable. Then $\pi_1(\mathcal{U})$ equals the upper half-plane $\mathbf{U} = \{w_1 = u + iv : v > 0\}$.

The fiber over each point $w_1 \in \mathbf{U}$ is connected (contrary to the case of $\mathcal{W}$). Indeed, the fiber over $w_1 = u + iv$, $v > 0$, is the annulus

$$\begin{aligned}\pi_1^{-1}(u + iv) &= \left\{w_2 \in \mathbb{C} : \; \left|u - \log |w_2|^2\right| < \arccos(e^{-v})\right\} \\ &= \left\{w_2 \in \mathbb{C} : \; e^{[u - \arccos(e^{-v})]/2} < |w_2| < e^{[u + \arccos(e^{-v})]/2}\right\}.\end{aligned}$$

Hence $F \in \mathcal{H}^j(\mathcal{U})$ if and only if (F is square integrable and) $F(w_1, w_2) = f(w_1) w_2^j$ for some holomorphic function $f : \mathbf{U} \to \mathbb{C}$. In the next lemma, and in the rest of the paper, we denote by $A^2(\Omega, \alpha)$ the weighted Bergman space on the domain Ω with respect to the continuous, positive weight α.

Lemma 3.1. *For $F \in \mathcal{H}^j(\mathcal{U})$ set $L_j F(w_1, w_2) = F(w_1, w_2) w_2^{-j}$. Then L_j is an isometric isomorphism from $\mathcal{H}^j(\mathcal{U})$ to the weighted Bergman space $A^2(\mathbf{U}, \omega_j)$, where the weight ω_j is defined as*

$$\omega_{-1}(u + iv) = 2\pi \arccos(e^{-v}) \tag{3.1}$$

for $j = -1$ and as

$$\omega_j(u + iv) = \frac{2\pi}{j+1} e^{(j+1)u} \sinh\left[(j+1)\arccos(e^{-v})\right] \tag{3.2}$$

for all other $j \in \mathbb{Z}$.

Proof. Let $F, G \in \mathcal{H}^j$, and let f, g be holomorphic on U such that $F(w_1, w_2) = f(w_1)w_2^j$ and $G(w_1, w_2) = g(w_1)w_2^j$, $w_1 \in \mathbf{U}$. We have

$$\begin{aligned}\langle F, G\rangle &= \int_{\mathbf{U}} f(w_1)\overline{g(w_1)} \int_{\pi_1^{-1}(w_1)} |w_2|^{2j}\, dV(w_2)dV(w_1)\\ &= \int_{\mathbf{U}} f(w_1)\overline{g(w_1)}\omega_j(w_1)\, dV(w_1)\,,\end{aligned}$$

where

$$\omega_j(u+iv) = 2\pi \int_{e^{[u-\arccos(e^{-v})]/2}}^{e^{[u+\arccos(e^{-v})]/2}} \rho^{2j+1}\, d\rho.$$

The conclusion now follows. □

Taking into account that $e^{(j+1)u} = \left|e^{\frac{j+1}{2}w_1}\right|^2$ for all $w_1 = u + iv \in \mathbf{U}$, if we set

$$M_j f(\zeta) = f(\zeta)e^{\frac{j+1}{2}\zeta}\,, \tag{3.3}$$

we obtain an isometric isomorphism $M_j : A^2(\mathbf{U}, \omega_j) \to A^2(\mathbf{U}, \alpha_j)$. Here

$$\alpha_j(u+iv) = \frac{2\pi}{j+1} \sinh\left[(j+1)\arccos(e^{-v})\right] \tag{3.4}$$

if $j \neq -1$, and $\alpha_{-1}(u + iv) = 2\pi \arccos(e^{-v})$.

Hence we have the following.

Corollary 3.2. *The mapping $M_j f(\zeta) = f(\zeta)e^{[(j+1)\zeta]/2}$ defines an isometric isomorphism $M_j : A^2(\mathbf{U}, \omega_j) \to A^2(\mathbf{U}, \alpha_j)$.*

Notice that $\alpha_j(u + iv)$ is independent of u and that, with an abuse of notation, we may write $\alpha_j(u + iv) = \alpha_j(v)$, $v > 0$. Moreover,

$$0 < \alpha_j(v) < \frac{2\pi}{j+1} \sinh\left[(j+1)\pi/2\right]$$

for all $v > 0$. This implies that $A^2(\mathbf{U}, \alpha_j)$ contains the unweighted Bergman space $A^2(\mathbf{U})$. However, $\alpha_j(v)$ is asymptotic to $\sqrt{v}$ as $v \to 0^+$, so the reverse inclusion does not hold.

We also point out that the mapping $j \mapsto \alpha_j$ is even in $j+1$, that is, $\alpha_j = \alpha_{-2-j}$ for all $j \in \mathbb{Z}$.

We conclude this section with a proof of Proposition 2.5.

Proof of Proposition 2.5. By holomorphic invariance, it suffices to show that $A^2(D'_\infty) = \{0\}$. Arguing as we did for $A^2(\mathcal{U})$, we obtain that $A^2(D'_\infty) = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}^j(D'_\infty)$, where

$$\mathcal{H}^j(D'_\infty) = \left\{F \in A^2(D'_\infty) :\ F(z_1, z_2) = f(z_1)z_2^j,\ f \text{ entire}\right\}.$$

For $F \in A^2(D'_\infty)$ with $F(z_1, z_2) = f(z_1)z_2^j$, we have

$$\begin{aligned}\|F\|^2_{A^2(D'_\infty)} &= 2\pi \int_0^{+\infty} \int_{|\operatorname{Im} z_1 - \log r^2| < \frac{\pi}{2}} |f(z_1)|^2 \, dV(z_1)\, r^{2j+1}\, dr \\ &= \pi \int_{\mathbb{C}} |f(z_1)|^2 \int_{|\operatorname{Im} z_1 - s| < \pi/2} e^{(j+1)s}\, ds dV(z_1) \\ &= 2\pi \frac{\sinh\big[(j+1)\pi/2\big]}{j+1} \int_{\mathbb{C}} \big|e^{-\frac{i}{2}(j+1)z_1} f(z_1)\big|^2 \, dV(z_1)\,,\end{aligned}$$

if $j \neq -1$, and with the obvious modification otherwise. Thus, $F \in A^2(D'_\infty)$ forces the entire function $e^{-\frac{i}{2}(j+1)z_1} f(z_1)$ to be identically zero; hence the conclusion. □

4. The Bergman kernel of $A^2(\mathbf{U}, \alpha_j)$

We now study the kernel of $A^2(\mathbf{U}, \alpha_j)$. In order to do so, we adapt the technique of [3]. For each $f \in A^2(\mathbf{U}, \alpha_j)$, owing to the fact that α_j is bounded and that it depends only on v, and since $f(\cdot + iv) \in L^2(\mathbb{R})$ for every v fixed, we can consider the partial Fourier transform and set

$$\widehat{f}(\xi, v) = \int_{\mathbb{R}} f(u + iv) e^{-iu\xi}\, du\,.$$

For our current purposes, we need the following simple version of the Paley–Wiener theorem for weighted Bergman spaces. The equality

$$\widehat{\alpha}_j(-2i\xi) = \int_0^{+\infty} e^{-2v\xi} \alpha_j(v)\, dv$$

is clearly well defined for any $\xi > 0$, and it is the Fourier transform of α_j, defined to be zero on the negative reals, extended to the lower half-plane and computed at $-2i\xi$.

Proposition 4.1. (1) *Let $f \in A^2(\mathbf{U}, \alpha_j)$. Then, for all $v > 0$, we have* supp $\widehat{f}(\cdot, v) \subseteq (0, +\infty)$, *with $\widehat{f}(\cdot, v) \in L^2\big((0, +\infty), \widehat{\alpha}_j(-2i\xi)d\xi\big)$, and there exists $g \in L^2\big((0, +\infty), \widehat{\alpha}_j(-2i\xi)d\xi\big)$ such that*

$$\widehat{f}(\cdot, v) \to g \quad \text{in } L^2\left((0, +\infty), \widehat{\alpha}_j(-2i\xi)d\xi\right) \tag{4.1}$$

as $v \to 0^+$. Moreover,

$$f(w) = \frac{1}{2\pi} \int_0^{+\infty} e^{iw\xi} g(\xi)\, d\xi \tag{4.2}$$

and

$$\|f\|_{A^2(\mathbf{U},\alpha_j)} = \frac{1}{2\pi}\|g\|_{L^2((0,+\infty),\widehat{\alpha}_j(-2i\xi)d\xi)} \,. \tag{4.3}$$

(2) *Conversely, if* $g \in L^2\big((0,+\infty),\widehat{\alpha}_j(-2i\xi)d\xi\big)$ *then* (4.2) *defines a function* $f \in A^2(\mathbf{U},\alpha_j)$ *such that* (4.3) *holds.*

Proof. For simplicity we write $\alpha_j = \alpha$. Let $f \in A^2(\mathbf{U},\alpha)$. For every $\varepsilon > 0$ the function $\mathbf{U} \ni \zeta \mapsto f(\zeta + i\varepsilon)$ is in the Hardy space $H^2(\mathbf{U})$. By the Paley–Wiener theorem, there exists a function $g_\varepsilon \in L^2(0,+\infty)$ such that

$$f(\zeta + i\varepsilon) = \frac{1}{2\pi}\int_0^{+\infty} e^{i\zeta\xi} g_\varepsilon(\xi)\,d\xi \,. \tag{4.4}$$

Moreover, the Fourier transform $\mathcal{F}(f(\cdot + i\varepsilon))$ is supported in $(0,+\infty)$ and it coincides with g_ε. Now

$$\begin{aligned} f(u + i\varepsilon' + i\varepsilon) &= \frac{1}{2\pi}\int_0^{+\infty} e^{iu\xi} e^{-\varepsilon'\xi} g_\varepsilon(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_0^{+\infty} e^{iu\xi} e^{-\varepsilon\xi} g_{\varepsilon'}(\xi)\,d\xi \,, \end{aligned}$$

so that $e^{\varepsilon\xi} g_\varepsilon(\xi) = e^{\varepsilon'\xi} g_{\varepsilon'}(\xi)$ for every $\varepsilon, \varepsilon' > 0$. We are thus able to set $g(\xi) = e^{\varepsilon\xi} g_\varepsilon(\xi)$ without ambiguity. For every $u + iv \in \mathbf{U}$, observing that the integrals below converge absolutely, we have

$$\begin{aligned} \mathcal{F}^{-1}(g_v)(u) &= \frac{1}{2\pi}\int_0^{+\infty} e^{iu\xi} e^{-v\xi} g(\xi)\,d\xi = \frac{1}{2\pi}\int_0^{+\infty} e^{i(u+iv)\xi} g(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_0^{+\infty} e^{i(u+iv-i\varepsilon)\xi} g_\varepsilon(\xi)\,d\xi = f(u + iv - i\varepsilon + i\varepsilon) \\ &= f(u+iv) \end{aligned}$$

by (4.4). This proves both (4.2) and the equality $\widehat{f}(\cdot, v) = g_v$, from which (4.1) immediately follows. Moreover, by Plancherel's theorem,

$$\begin{aligned} \|f\|^2_{A^2(\mathbf{U},\alpha)} &= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{+\infty} \left|e^{-v\xi} g(\xi)\right|^2 d\xi\,\alpha(v)\,dv \\ &= \int_0^{+\infty} |g(\xi)|^2 \int_0^{+\infty} e^{-2v\xi}\alpha(v)\,dv\,d\xi \\ &= \int_0^{+\infty} |g(\xi)|^2 \widehat{\alpha}(-2i\xi)\,d\xi \,. \end{aligned}$$

This proves (4.3). The proof of part (2) follows the same lines. □

Notice that in particular we have that, for $w \in \mathbf{U}$,

$$f(w) = \frac{1}{2\pi} \int_0^{+\infty} \widehat{f}(\xi, 0)\, e^{iw\xi}\, d\xi\, .$$

The previous lemma allows us to prove the following result, where B and Γ denote the classical beta function and gamma function.

Proposition 4.2. *The kernel K_j of $A^2(\mathbf{U}, \alpha_j)$ can be computed as*

$$K_j(z, w) = \frac{1}{2\pi} \int_0^{+\infty} \frac{e^{i(z-\overline{w})\xi}}{\widehat{\alpha}_j(-2i\xi)} d\xi, \tag{4.5}$$

for $z, w \in \mathbf{U}$, where for $\xi > 0$ we have

$$\frac{1}{\widehat{\alpha}_j(-2i\xi)} = \frac{2^{2\xi+1}\xi(2\xi+1)}{\pi^2} B\left(\xi + 1 + i\frac{j+1}{2}, \xi + 1 - i\frac{j+1}{2}\right) \tag{4.6}$$

$$= \frac{1}{\pi^2} \frac{2^{2\xi}}{\Gamma(2\xi)} \left|\Gamma\left(\xi + 1 + i\frac{j+1}{2}\right)\right|^2 . \tag{4.7}$$

Proof. Fix $v_0 > 0$ and let $K_j^w(z) = K_j(z, w)$. Then, for $f \in A^2(\mathbf{U}, \alpha_j)$ and $w \in \mathbf{U}$, we have

$$\begin{aligned}
f(w) = \langle f, K_j^w \rangle_{\alpha_j} &= \int_0^{+\infty} \int_{\mathbb{R}} f(x+iy) \overline{K_j^w(x+iy)}\, dx\, \alpha_j(y)\, dy \\
&= \frac{1}{2\pi} \int_0^{+\infty} \int_{\mathbb{R}} \widehat{f}(x, \xi) \overline{\widehat{K}_j^w(\xi, y)}\, d\xi\, \alpha_j(y) dy \\
&= \frac{1}{2\pi} \int_0^{+\infty} \int_{\mathbb{R}} e^{-2y\xi}\, \widehat{f}(\xi, 0) \overline{\widehat{K}_j^w(\xi, 0)} d\xi\, \alpha_j(y)\, dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi, 0) \overline{\widehat{K}_j^w(\xi, 0)} \int_0^{+\infty} e^{-2y\xi} \alpha_j(y)\, dy\, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi, 0) \overline{\widehat{K}_j^w(\xi, 0)} \widehat{\alpha}_j(-2i\xi)\, d\xi\, .
\end{aligned}$$

Coupling this with (4.2), we conclude that, on the support of $\widehat{K}_j^w(\cdot, 0)$,

$$e^{iw\xi} = \overline{\widehat{K}_j^w(\xi, 0)} \widehat{\alpha}_j(-2i\xi) = \overline{\widehat{K}_j^w(\xi, y)} e^{y\xi} \widehat{\alpha}_j(-2i\xi)$$

for all $y \geq 0$. Therefore

$$\widehat{K}_j^w(\xi, y) = \frac{e^{i(iy-\overline{w})\xi}}{\widehat{\alpha}_j(-2i\xi)}\, ,$$

which gives

$$K_j^w(z) = \frac{1}{2\pi}\int_0^{+\infty} \frac{e^{i(z-\overline{w})\xi}}{\widehat{\alpha}_j(-2i\xi)} d\xi$$

provided the integral converges absolutely. Let us compute $\widehat{\alpha}_j(-2i\xi)$. We have

$$\begin{aligned}
\widehat{\alpha}_j(-2i\xi) &= \frac{2\pi}{j+1}\int_0^{+\infty} e^{-2y\xi} \sinh\big[(j+1)\arccos(e^{-y})\big]\, dy \\
&= \frac{2\pi}{j+1}\int_0^{1} t^{2\xi} \sinh\big[(j+1)\arccos(t)\big]\, \frac{dt}{t} \\
&= \frac{2\pi}{j+1}\int_0^{\pi/2} (\cos s)^{2\xi-1} \sinh\big[(j+1)s\big] \sin s\, ds \\
&= \frac{\pi}{\xi}\int_0^{\pi/2} (\cos s)^{2\xi} \cosh\big[(j+1)s\big]\, ds\,.
\end{aligned}$$

Since $\cosh\big[(j+1)s\big] = \cos(\theta s)$ with $\theta := i(j+1)$ and since $\tau := 2\xi > 0$, we may use [19, formula 3.631(9)] to obtain

$$\begin{aligned}
\frac{2\pi}{\tau}\int_0^{\pi/2} (\cos s)^{\tau} \cos(\theta s)\, ds &= \frac{\pi^2}{2^{\tau}\tau(\tau+1)} \frac{1}{B\left(\frac{\tau+2+\theta}{2}, \frac{\tau+2+\overline{\theta}}{2}\right)} \\
&= \frac{\pi^2}{2^{\tau}} \frac{\Gamma(\tau)}{\Gamma\left(\frac{\tau+2+\theta}{2}\right)\Gamma\left(\frac{\tau+2+\overline{\theta}}{2}\right)}\,.
\end{aligned}$$

Formulas (4.6) and (4.7) now follow.

We are now in a position to prove the absolute convergence of the integral in (4.5) by means of estimates for the weight function $\big[\widehat{\alpha}_j(-2i\xi)\big]^{-1}$. We set

$$\eta = \frac{j+1}{2} \quad \text{and} \quad \beta_\eta(\xi) = \frac{1}{2\pi\widehat{\alpha}_j(-2i\xi)} = c\,\frac{\xi 2^{2\xi}\big|\Gamma\big(\xi+1+i\eta\big)\big|^2}{\Gamma(2\xi+1)}\,.$$

According to Stirling's formula,

$$\begin{aligned}
&\big|\Gamma(\xi+1+i\eta)\big|^2 \\
&= \Big|\sqrt{2\pi}\exp\big\{(\xi+1/2+i\eta)\log(\xi+1+i\eta)-(\xi+1+i\eta)\big\}\Big|^2\Big[1+O\Big(\frac{1}{\xi+1+i\eta}\Big)\Big] \quad (4.8)\\
&\le c\,\exp\big\{2(\xi+1/2)\log|\xi+1+i\eta| - 2\eta\arg(\xi+1+i\eta) - 2(\xi+1)\big\}\,,
\end{aligned}$$

for some constant c, independent of ξ and η. Also

$$\begin{aligned}
\frac{2^{2\xi}}{\Gamma(2\xi+1)} &\le c\,\exp\big\{(2\log 2)\xi - (2\xi+1/2)\log(2\xi+1) + (2\xi+1)\big\} \\
&= c\exp\big\{-(2\xi+1/2)\log(\xi+1/2) + (2\xi+1)\big\}\,.
\end{aligned}$$

Putting together (4.8) and (4.9) we obtain that

$$
\begin{aligned}
&|\beta_\eta(\xi)| \\
&\le c\,\xi \exp\left\{2(\xi+1/2)\log\left(\frac{|\xi+1+i\eta|}{\xi+1/2}\right)-2\eta\arg(\xi+1+i\eta)+1/2\log(2\xi+1)\right\} \\
&\le c\,\xi^{3/2}\exp\left\{2(\xi+1/2)\log\left(\frac{|\xi+1+i\eta|}{\xi+1/2}\right)-2\eta\arg(\xi+1+i\eta)\right\} \\
&\le c\,\xi^{3/2}\exp\left\{2(\xi+1/2)\log\left(1+\frac{|\eta|+1/2}{\xi+1/2}\right)-2\eta\arg(\xi+1+i\eta)\right\}.
\end{aligned}
$$

Observing that $\eta\arg(\xi+1+i\eta)>0$ for $\xi>0$ and that $\operatorname{Re}(i(z-\overline{w}))<0$, the absolute convergence of the integral in (4.5) follows. Moreover, for any fixed $\varepsilon>0$, the absolute convergence of the integral is uniform for $\operatorname{Re}(i(z-\overline{w}))\le-\varepsilon$. □

We now show that for fixed (z,w) all the values $K_j(z,w)$ can be obtained by evaluating a single function at the integer points. This further representation allows us to describe the behavior of $K_j(z,w)$ as $\operatorname{Re}(i(z-\overline{w}))\to 0^-$. Recall that we denote by $\mathbf{H}$ the right half-plane in $\mathbb{C}$.

Proposition 4.3. *The kernel K_j of $A^2(\mathbf{U},\alpha_j)$ is given by $K_j(z,w)=\widehat{\phi_\lambda}(j+1)$, where $\lambda:=-i(z-\overline{w})\in\mathbf{H}$ and*

$$
\phi_\lambda(s)=\frac{1}{2\pi^3}\frac{1}{\cosh^2 s}\Big[\big(2\log(\cosh s)+\lambda\big)^{-2}+4\big(2\log(\cosh s)+\lambda\big)^{-3}\Big]. \quad (4.9)
$$

The mapping $\lambda\mapsto\phi_\lambda$ is holomorphic in $\mathbf{H}$ and it takes its values in the Schwartz space $\mathcal{S}(\mathbb{R})$. The same is true for the Fourier transform $\widehat{\phi_\lambda}(\xi)=\int_{\mathbb{R}}e^{-i\xi s}\phi_\lambda(s)ds$. Moreover, for every $j\in\mathbb{Z}$,

$$
K_j:\mathbf{U}\times\mathbf{U}\to\mathbb{C}
$$

extends holomorphically in z and anti-holomorphically in w to $\overline{\mathbf{U}}\times\overline{\mathbf{U}}\setminus\Delta$, where Δ denotes the boundary diagonal and the "bar" the topological closure.

Proof. From (4.5) and (4.6), having set $\lambda=-i(z-\overline{w})$, we have that

$$
\begin{aligned}
K_j(z,w)&=\frac{1}{\pi^3}\int_0^{+\infty}2^{2\xi}e^{-\lambda\xi}\xi(2\xi+1)B\Big(\xi+1+i(j+1)/2,\xi+1-i(j+1)/2\Big)d\xi \\
&=\frac{1}{\pi^3}\int_0^{+\infty}2^{2\xi}e^{-\lambda\xi}\xi(2\xi+1)\int_0^{+\infty}\frac{t^{\xi+i(j+1)/2}}{(1+t)^{2\xi+2}}\,dt\,d\xi \\
&=\frac{1}{\pi^3}\int_0^{+\infty}t^{i(j+1)/2}\int_0^{+\infty}\frac{2^{2\xi}e^{-\lambda\xi}t^{\xi}}{(1+t)^{2\xi+2}}\xi(2\xi+1)d\xi\,dt \\
&=\frac{1}{\pi^3}\int_0^{+\infty}\frac{t^{i(j+1)/2}}{(1+t)^2}\int_0^{+\infty}\xi(2\xi+1)\exp\big\{\xi\big(\log\chi(t)-\lambda\big)\big\}\,d\xi\,dt\,,
\end{aligned}
$$

where $\chi(t) = 4t/(1+t)^2$. Therefore

$$K_j(z,w) = \frac{1}{\pi^3}\int_0^{+\infty} \frac{t^{i(j+1)/2}}{(1+t)^2}\Big[\big(\log\chi(t)-\lambda\big)^{-2} - 4\big(\log\chi(t)-\lambda\big)^{-3}\Big]\,dt$$
$$= \frac{1}{2\pi^3}\int_0^{+\infty} t^{i(j+1)/2}\chi(t)\Big[\big(\log\chi(t)-\lambda\big)^{-2} - 4\big(\log\chi(t)-\lambda\big)^{-3}\Big]\,\frac{dt}{2t}\,.$$

Setting $t = e^{2s}$ and observing that $\chi(e^{2s}) = \big(2e^s/(1+e^{2s})\big)^2 = \cosh^{-2} s$, we have

$$K_j(z,w) = \frac{1}{2\pi^3}\int_{\mathbb{R}} \frac{e^{i(j+1)s}}{\cosh^2 s}\Big[\big(2\log\cosh s+\lambda\big)^{-2} + 4\big(2\log\cosh s+\lambda\big)^{-3}\Big]\,ds$$
$$= \widehat{\phi_\lambda}(j+1)\,,$$

as claimed, taking into account that ϕ_λ is even.

Finally, it is clear that $\phi_\lambda(s)$ is a Schwartz function in s when λ is bounded away from the set $(-\infty, 0]$. It is also easy to see that the mapping $\lambda \mapsto \phi_\lambda \in \mathcal{S}(\mathbb{R})$ is holomorphic in λ in the slit plane $\mathbb{C}\setminus(-\infty,0]$. Therefore $K_j(z,w)$ extends holomorphically in z and anti-holomorphically in w in a neighborhood of each point (z,w) of $\overline{\mathbf{U}}\times\overline{\mathbf{U}}$ except those for which $\lambda = -i(z-\overline{w}) = 0$, that is, $z-\overline{w} = 0$. This implies that $z = w \in \partial\mathbf{U}$ so that $K_j(z,w)$ extends holomorphically in z and anti-holomorphically in w to a neighborhood of each point (z,w) in $\overline{\mathbf{U}}\times\overline{\mathbf{U}}\setminus\Delta$. □

We now study the dependence of K_j on the index j. Recall that we have set $\lambda = -i(z-\overline{w})$.

Corollary 4.4. *Let*

$$b_\lambda = \max\Big\{\arccos\big(e^{-\operatorname{Re}\lambda/2}\big),\ \min\big\{|\operatorname{Im}\lambda|/2,\ \pi/2\big\}\Big\}. \tag{4.10}$$

Then, for $0 < b < b_\lambda$ *and for* $(z,w)\in\overline{\mathbf{U}}\times\overline{\mathbf{U}}\setminus\Delta$ *we have*

$$\lim_{j\to\pm\infty} |K_j(z,w)|e^{b|j+1|} = 0\,. \tag{4.11}$$

As a consequence, for $(z,w)\in\overline{\mathbf{U}}\times\overline{\mathbf{U}}\setminus\Delta$,

$$\limsup_{j\to\pm\infty} |K_j(z,w)|^{1/|j+1|} \le e^{-b_\lambda}. \tag{4.12}$$

Proof. We set $S_b = \{s+it : |t| < b\}$, and $I_+ = i\left(\frac{\pi}{2},\pi\right)$, $I_- = i\left(-\pi,-\frac{\pi}{2}\right)$ to denote two intervals on the imaginary axis.

The function $\log\cosh s$ extends holomorphically to $S_\pi\setminus(I_+\cup I_-)$, since the function $\cosh(s+it) = \cosh s\cos t + i\sinh s\sin t$ maps $S_\pi\setminus(I_+\cup I_-)$ to $\mathbb{C}\setminus(-\infty,0]$. For each $\lambda\in\overline{\mathbf{H}}\setminus\{0\}$, the functions $s\mapsto\phi_\lambda(s)$ and $s\mapsto s\phi_\lambda(s) = \widetilde{\phi_\lambda}(s)$ extend holomorphically to $S_{\pi/2}$. We still denote by ϕ_λ and $\widetilde{\phi_\lambda}$ such extensions.

We claim that ϕ_λ and $\widetilde{\phi}_\lambda$ belong to the Hardy space $H^2(S_b)$, for every $b < b_\lambda$. Assuming the claim, we complete the proof.

By the classical Paley–Wiener theorem for $H^2(S_b)$, the functions $e^{\pm b\xi}\widehat{\phi}_\lambda(\xi)$ and $e^{\pm b\xi}\frac{d}{d\xi}\widehat{\phi}_\lambda(\xi)$ belong to $L^2(\mathbb{R})$. If we set $f_\pm(\xi) = e^{\pm b\xi}\widehat{\phi}_\lambda(\xi)$, then $f_\pm \in W^1(\mathbb{R})$. By the Sobolev embedding theorem it follows that $f_\pm$ is a continuous function vanishing at infinity. Hence

$$\lim_{\xi\to\pm\infty} e^{b|\xi|}\widehat{\phi}_\lambda(\xi) = 0\,,$$

which gives (4.11). It only remains to prove the claim. Notice that, assuming $|t| < \pi/2$, we have that

$$\left|\operatorname{Re}\big(2\log\cosh(s+it)+\lambda\big)\right| = \log\big(\sinh^2 s+\cos^2 t\big) + \operatorname{Re}\lambda \geq \varepsilon_0$$

if $|\cos t| \geq e^{\varepsilon_0/2}e^{-\operatorname{Re}\lambda/2}$, and that

$$\left|\operatorname{Im}\big(2\log\cosh(s+it)+\lambda\big)\right| \geq |\operatorname{Im}\lambda| - 2\big|\arctan(\tanh s\tan t)\big| \geq |\operatorname{Im}\lambda| - 2|t| \geq \varepsilon_0,$$

for some $\varepsilon_0 > 0$, if $|t| < |\operatorname{Im}\lambda|/2$. The claim now follows easily by Plancherel's theorem and the last two inequalities. □

We conclude this section by describing the behavior of K_j near the extended boundary of $\mathbf{U}\times\mathbf{U}$. In order to do so, we first expand at infinity and then restrict to a special case that allows explicit computations. Recall that we denote by $\mathbf{H}$ the right half-plane and we write $\lambda = -i(z-\overline{w})$.

Lemma 4.5. *Let K_j be the Bergman kernel for $A^2(\mathbf{U},\alpha_j)$. Let $N \geq 2$ and $\varepsilon > 0$ be fixed. Then there exist*

(i) *Schwartz functions $\psi_1,\dots,\psi_N$*
(ii) *a Schwartz function $\Psi_{N,\lambda}$ holomorphic in $\lambda \in \mathbf{H}$ and converging to ψ_N in $\mathcal{S}(\mathbb{R})$ as $\lambda \to \infty$ within the half-plane $\overline{\mathbf{H}}_\varepsilon = \{\lambda : Re(\lambda) \geq \varepsilon\} \subset \mathbf{H}$*

such that

$$K_j(z,w) = \sum_{n=2}^{N-1}\frac{\psi_n(j+1)}{(z-\overline{w})^n} + \frac{\Psi_{N,\lambda}(j+1)}{(z-\overline{w})^N}\,, \tag{4.13}$$

for $z, w \in \mathbf{U}$. Explicitly,

$$\psi_n(\xi) = \frac{(-i)^n(n-1)}{2\pi^3}\big[I_{n-2}(\xi) - 2(n-2)I_{n-3}(\xi)\big],$$

$$\textit{where}\quad I_m(\xi) = \int_{\mathbb{R}} e^{-i\xi s}\frac{\big(2\log\cosh s\big)^m}{\cosh^2 s}\,ds\,.$$

Proof. For $s \in \mathbb{R}$ set $D_s = \big(2\sinh s\big)^{-1}\dfrac{\partial}{\partial s}$. We use (4.9) and the expansion $(1+x)^{-1} = \sum_{n=0}^{N-1}(-x)^n + (-x)^N(1+x)^{-1}$ to obtain that

$$\begin{aligned}\phi_\lambda(s) &= \frac{1}{\pi^3}D_s^2\big(2\log\cosh s+\lambda\big)^{-1}\\ &= \frac{1}{\pi^3\lambda}D_s^2\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-1}\\ &= \sum_{n=2}^{N}\frac{a_n(s)}{\lambda^n}+\frac{A_{N+1,\lambda}(s)}{\lambda^{N+1}},\end{aligned}\tag{4.14}$$

where

$$\begin{aligned}a_n(s) &= \frac{(-1)^{n-1}}{\pi^3}D_s^2\Big[\big(2\log\cosh s\big)^{n-1}\Big]\\ &= \frac{(-1)^n(n-1)}{2\pi^3\cosh^2 s}\Big[\big(2\log\cosh s\big)^{n-2}-2(n-2)\big(2\log\cosh s\big)^{n-3}\Big],\end{aligned}$$

and

$$\begin{aligned}A_{N+1,\lambda}(s) &= \frac{\lambda^N}{\pi^3}D_s^2\Big[\Big(\frac{-2\log\cosh s}{\lambda}\Big)^N\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-1}\Big]\\ &= \frac{(-1)^N}{\pi^3}D_s^2\Big[\big(2\log\cosh s\big)^N\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-1}\Big]\\ &= \frac{P_{N+1}\Big(1+[2\log\cosh s]/\lambda\Big)}{\cosh^2 s\Big(1+[2\log\cosh s]/\lambda\Big)^3}.\end{aligned}$$

Here $P_{N+1}(\zeta)$ is a polynomial of degree 2 with coefficients integral powers of $\log\cosh s$ such that

$$P_{N+1}(1) = \frac{(-1)^{N+1}N}{2\pi^3}\Big[\big(2\log\cosh s\big)^{N-1}-2(N-1)\big(2\log\cosh s\big)^{N-2}\Big].$$

For $N \geq 1$, we have $A_{N+1,\lambda} \to a_{N+1}$ in $\mathcal{S}(\mathbb{R})$ as $\lambda \to \infty$ within the closed half-plane $\overline{\mathbf{H}}_\varepsilon$.

Therefore, taking the Fourier transform in (4.14) and recalling (4.9), we obtain (4.13), where

$$\begin{aligned}\psi_n(\xi) &= i^n\widehat{a}_n(\xi) = \frac{(-i)^n(n-1)}{2\pi^3}\big[I_{n-2}(\xi)-2(n-2)I_{n-3}(\xi)\big],\\ I_m(\xi) &= \int_{\mathbb{R}} e^{-i\xi s}\frac{\big(2\log\cosh s\big)^m}{\cosh^2 s}\,ds.\end{aligned}$$

Moreover, $\Psi_{N,\lambda} = i^N\widehat{A}_{N,\lambda}$ are again Schwartz functions such that, for each $N \geq 2$, $\Psi_{N,\lambda} \to \psi_N$ in $\mathcal{S}(\mathbb{R})$ as $\lambda \to \infty$ within a half-plane $\mathbf{H}_\varepsilon$. □

Theorem 4.6. *Let K_j be the Bergman kernel for $A^2(\mathbf{U}, \alpha_j)$. There exists a holomorphic function $f_j : \mathbf{H} \to \mathbb{C}$ such that*

$$K_j(z, w) = \frac{f_j\big(-i(z - \overline{w})\big)}{(z - \overline{w})^2} \tag{4.15}$$

and

$$\lim_{\overline{\mathbf{H}}_\varepsilon \ni \lambda \to \infty} f_j(\lambda) = \frac{1}{\pi^3} \frac{\pi \frac{j+1}{2}}{\sinh(\pi \frac{j+1}{2})} \tag{4.16}$$

for all $\varepsilon > 0$. Moreover, f_j extends holomorphically to a neighborhood of each point of $\overline{\mathbf{H}} \setminus \{0\}$. The product $\sqrt{\lambda}\, f_j(\lambda)$ is bounded near 0 in $\overline{\mathbf{H}}$ and $\lim_{\mathbb{R}^+ \ni \lambda \to 0} \sqrt{\lambda}\, f_{-1}(\lambda) < 0$.

As a consequence:

(1) *the function $(z, w) \mapsto K_j(z, \overline{w})$ extends holomorphically to a neighborhood of each point $(z, w) \in \partial\mathbf{U} \times \partial\mathbf{U}$ with $z \neq \overline{w}$;*
(2) *the product $\big(-i(z - \overline{w})\big)^{5/2} K_j(z, w)$ remains bounded as $z - \overline{w} \to 0$ in $\overline{\mathbf{U}}$ and, for $j = -1$, its limit as $z - \overline{w} \to 0$ in $i\mathbb{R}^+$ is a strictly positive real number;*
(3) *for all $w \in \mathbf{U}$,* $\lim_{\mathbf{U} \ni z \to \infty} K_j(z, w) = 0$ *and, for all $w \in \partial\mathbf{U}$ and $\varepsilon > 0$,* $\lim_{\mathbf{U}_\varepsilon \ni z \to \infty} K_j(z, w) = 0$*; similar considerations apply to the limits as $w \to \infty$ with $z \in \overline{\mathbf{U}}$ fixed.*

Remark. Statement (1) above was already obtained in Proposition 4.3 and we repeated it here for the sake of completeness. Statement (2) shows that K_{-1} is singular as z and w tend to the same point on the boundary of $\mathbf{U}$ and that for each j the (possible) singularity of $K_j(z, w)$ is not worse that $\big(-i(z - \overline{w})\big)^{-5/2}$. Finally, (3) describes the behavior of $K_j(z, w)$ as $\mathbf{U} \ni z \to \infty$.

Proof. Owing to Lemma 4.5, in order to prove the first statement it suffices to set $f_j(\lambda) = \Psi_{2,\lambda}(j + 1)$ and to compute $\psi_2(\xi) = -(1/[2\pi^3]) I_0(\xi)$. We observe that $I_0(0) = \int_{\mathbb{R}} 1/[\cosh^2 s]\, ds = 2$. For all $\xi \in \mathbb{R}$ other than 0, we make use of the fact that the integrand in $I_0(\xi)$ extends to $\mathbb{C}$ except the points $\{ik\frac{\pi}{2}\}_{k \in \mathbb{Z}}$. If we integrate along the rectangle through $-R$, R, $R + i\pi$, $-R + i\pi$ and we let $R \to +\infty$ in $\mathbb{R}$ we may conclude that

$$I_0(\xi) = \int_{\mathbb{R}} \frac{e^{-i\xi s}}{\cosh^2 s} ds = \frac{2\pi i}{1 - e^{\xi\pi}} \operatorname{Res}_{i\pi/2} \left(\frac{e^{-i\xi s}}{\cosh^2 s} \right).$$

Taking into account that $\cosh(z + i\frac{\pi}{2}) = i \sinh z$ and that $1/\sinh^2 z - 1/z^2$ is holomorphic near $z = 0$, we obtain that

$$\operatorname{Res}_{i\pi/2} \left(\frac{e^{-i\xi s}}{\cosh^2 s} \right) = -e^{\xi\pi/2} \operatorname{Res}_0 \left(\frac{e^{-i\xi z}}{\sinh^2 z} \right) = -e^{\xi\pi/2} \operatorname{Res}_0 \left(\frac{e^{-i\xi z}}{z^2} \right) = e^{\xi\pi/2} i\xi\,.$$

Therefore

$$\psi_2(\xi) = -\frac{1}{2\pi^3} I_0(\xi) = -\frac{1}{\pi^3}\frac{\pi e^{\xi\pi/2}\xi}{e^{\xi\pi}-1} = -\frac{1}{\pi^3}\frac{\xi\pi/2}{\sinh(\xi\pi/2)}$$

for all $\xi \in \mathbb{R}$. As for the behavior of $f_j(\lambda) = \Psi_{2,\lambda}(j+1) = -\widehat{A}_{2,\lambda}(j+1)$ near the finite boundary, we observe that

$$A_{2,\lambda}(s) = \frac{1}{2\pi^3\cosh^2 s}\left[\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-2} + \frac{4}{\lambda}\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-3}\right]$$

admits a transform even if $\operatorname{Re}\lambda = 0$, and $\operatorname{Im}\lambda \neq 0$. Moreover, we shall prove that $\sqrt{\lambda}\,\widehat{A}_{2,\lambda}(\xi)$ stays bounded as $\lambda \to 0$ and that $\lim_{\mathbb{R}^+\ni\lambda\to 0}\sqrt{\lambda}\,A_{2,\lambda}(0) > 0$. As $\lambda \to 0$, the only relevant part in $\sqrt{\lambda}\,\widehat{A}_{2,\lambda}(\xi)$ is

$$\begin{aligned}
&\frac{2}{\pi^3\sqrt{\lambda}}\int_{\mathbb{R}}\frac{e^{-i\xi s}}{\cosh^2 s}\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-3} ds\\
=&\frac{4}{\pi^3\sqrt{\lambda}}\int_0^{+\infty}\frac{\cos(\xi s)}{\cosh^2 s}\Big(1+\frac{2\log\cosh s}{\lambda}\Big)^{-3} ds\\
=&\frac{4}{\pi^3\sqrt{\lambda}}\int_0^1 \cos(\xi\,\operatorname{arctanh} t)\Big(1-\log(1-t^2)/\lambda\Big)^{-3} dt.
\end{aligned}$$

Now $-\log(1-t^2) = \sum_{n\geq 1}\frac{t^{2n}}{n} \geq t^2$ implies that

$$\Big|1-\log(1-t^2)/\lambda\Big|^2 \geq \left(1+t^2\frac{\operatorname{Re}\lambda}{|\lambda|^2}\right)^2 + \left(t^2\frac{\operatorname{Im}\lambda}{|\lambda|^2}\right)^2 \geq 1+\frac{t^4}{|\lambda|^2}$$

for all $t \in (0,1)$. Hence, for appropriate positive constants,

$$\begin{aligned}
\Big|\sqrt{\lambda}\,\widehat{A}_{2,\lambda}(\xi)\Big| &\leq \frac{C}{\sqrt{\lambda}}\int_0^1\left(1+\frac{t^4}{|\lambda|^2}\right)^{-\frac{3}{2}} dt\\
&\leq \frac{C}{\sqrt{\lambda}}\int_0^{\sqrt{|\lambda|}} dt + \frac{C}{\sqrt{\lambda}}\int_{\sqrt{|\lambda|}}^1\left(1+\frac{t^4}{|\lambda|^2}\right)^{-\frac{3}{2}}\Big(\frac{t}{\sqrt{|\lambda|}}\Big)^3 dt\\
&\leq C + \frac{C}{4}\int_0^1\Big(1+\tau\Big)^{-\frac{3}{2}} d\tau\\
&\leq C\,.
\end{aligned}$$

Moreover, for $\lambda \in \mathbb{R}^+$ sufficiently small and $t \in (0,\sqrt{\lambda})$, the function

$$1-\log(1-t^2)/\lambda = 1+\frac{1}{\lambda}\sum_{n\geq 1}\frac{t^{2n}}{n} \leq 1+\sum_{n\geq 1}\frac{\lambda^{n-1}}{n}$$

takes values in an interval $(0, \varepsilon)$ with $\varepsilon > 0$, so that

$$\begin{aligned}\left|\sqrt{\lambda}\,\widehat{A}_{2,\lambda}(0)\right| &= \frac{4}{\pi^3\sqrt{\lambda}}\int_0^1 \Big(1-\log(1-t^2)/\lambda\Big)^{-3} dt + o(\sqrt{\lambda}) \\ &\geq \frac{4}{\pi^3\sqrt{\lambda}}\int_0^{\sqrt{\lambda}} \Big(1-\log(1-t^2)/\lambda\Big)^{-3} dt + o(\sqrt{\lambda}) \\ &\geq C\,,\end{aligned}$$

for an appropriate positive constant C. □

5. Back to the worm domain

We can now express the Bergman kernel of the "unwound" worm $\mathcal{U}$ as a series. In this part of the paper we write $z = (z_1, z_2)$, $w = (w_1, w_2)$ to denote points in $\mathbb{C}^2$. This change of notation with respect to the previous sections should cause no confusion. Recall that $\mathcal{U}$ is defined in (2.4).

Proposition 5.1. *The Bergman kernel of $\mathcal{U}$ is given by*

$$K_{\mathcal{U}}(z, w) = \frac{1}{z_2\overline{w}_2}\sum_{j\in\mathbb{Z}} K_j(z_1, w_1)\Big(e^{-\frac{1}{2}(z_1+\overline{w}_1)}z_2\overline{w}_2\Big)^{j+1}, \tag{5.1}$$

for $z = (z_1, z_2)$, $w = (w_1, w_2)$ in $\mathcal{U}$, where for each $w \in \mathcal{U}$ fixed (or $z \in \mathcal{U}$ fixed) the series converges in the $L^2(\mathcal{U})$-norm, absolutely and uniformly on compact subsets of $\mathcal{U}$.

Proof. Considering the decomposition $A^2(\mathcal{U}) = \bigoplus_{j\in\mathbb{Z}}\mathcal{H}^j(\mathcal{U})$ and the isometry $M_jL_j : \mathcal{H}^j(\mathcal{U}) \to A^2(\mathbf{U}, \alpha_j)$ given by

$$M_jL_jF(w_1, w_2) = F(w_1, w_2)w_2^{-j}e^{[(j+1)w_1]/2}\,,$$

we obtain that the Bergman kernel of $\mathcal{H}^j(\mathcal{U})$ is given by

$$U_j(z, w) = K_j(z_1, w_1)e^{-[(j+1)/2](z_1+\overline{w}_1)}(z_2\overline{w}_2)^j\,.$$

We are going to show that the sum $\sum_{j\in\mathbb{Z}} U_j(\cdot, w)$ converges to $K_{\mathcal{U}}(\cdot, w)$ in $L^2(\mathcal{U})$ for any $w \in \mathcal{U}$ fixed. This will imply that the series converges also absolutely and uniformly on compact subsets.

It is easy to see that $\sum_{|j|\leq n} U_j(\cdot, w)$ weakly converges to $K_{\mathcal{U}}(\cdot, w)$, as $n \to +\infty$, for $w \in \mathcal{U}$ fixed. Indeed, let $\mathcal{P}_{\mathcal{U}}$ denote the Bergman projection on $\mathcal{U}$ and let $f \in L^2(\mathcal{U})$. Then its projection on $A^2(\mathcal{U})$ is given by

$$\mathcal{P}_{\mathcal{U}}f(w) = \langle f, K_{\mathcal{U}}(\cdot, w)\rangle = \sum_{j\in\mathbb{Z}} f_j(w)$$

with $f_j \in \mathcal{H}^j$. Now

$$\Big\langle f, \sum_{|j|\le n} U_j(\cdot, w)\Big\rangle = \sum_{|j|\le n} \langle f, U_j(\cdot, w)\rangle = \sum_{|j|\le n} f_j(w) \to \mathcal{P}_{\mathcal{U}} f(w)$$

as $n \to +\infty$. Hence there exists $C > 0$ independent of n such that

$$\sum_{|j|\le n} \|U_j(\cdot, w)\|^2_{L^2(\mathcal{U})} = \Big\| \sum_{|j|\le n} U_j(\cdot, w)\Big\|^2_{L^2(\mathcal{U})} \le C\,.$$

Therefore $\sum_{|j|\le n} U_j(\cdot, w)$ converges in $L^2(\mathcal{U})$, necessarily to $K_{\mathcal{U}}(\cdot, w)$. □

We now study the pointwise regularity of $K_{\mathcal{U}}$ at the boundary. In the statement, $\mathbf{U}_\varepsilon = \{\zeta : \operatorname{Im}\zeta > \varepsilon\}$ with $\varepsilon > 0$. Moreover, we set

$$\begin{aligned}\Sigma = \Big\{&(z, w) \in \partial\mathcal{U} \times \partial\mathcal{U} : \exists\, v \ge 0 \text{ such that } \operatorname{Im} z_1 = \operatorname{Im} w_1 = v,\\ &\operatorname{Re} z_1 - \log|z_2|^2 = \operatorname{Re} w_1 - \log|w_2|^2 \\ &= \pm \arccos(e^{-v}),\ |\log|z_2|^2 - \log|w_2|^2| \le 2\arccos(e^{-v})\Big\}.\end{aligned} \tag{5.2}$$

Remark. The set Σ contains the diagonal Δ of $\partial\mathcal{U} \times \partial\mathcal{U}$; but also by other points (z, w) of $\partial\mathcal{U} \times \partial\mathcal{U}$, *e.g.*, those such that $z_1 = w_1 \in \partial\mathbf{U}$ and $|z_2| = |w_2|$. See Figure 5.1 for other cases.

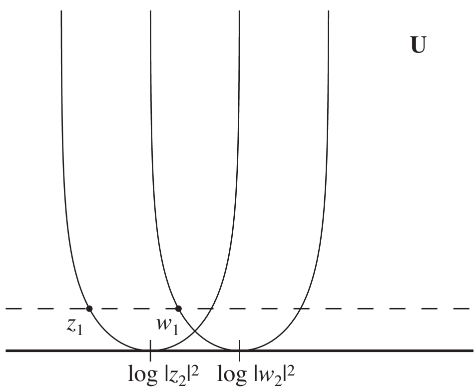

Figure 5.1. The set Σ is defined to include (z, w) if and only if: $z_1, w_1 \in \overline{\mathbf{U}}$ lie on the same horizontal line; z_1, w_1 belong both to the left arcs (or both to the right arcs) of the boundaries of $\pi_1(\pi_2^{-1}(z_2)), \pi_1(\pi_2^{-1}(w_2))$; w_1 belongs to $\pi_1(\pi_2^{-1}(z_2))$ or $z_1 \in \pi_1(\pi_2^{-1}(w_2))$.

Theorem 5.2. (1) *The kernel function $K_{\mathcal{U}}(z, w)$ extends holomorphically in z and antiholomorphically in w near each point (z, w) in $\overline{\mathcal{U}} \times \overline{\mathcal{U}} \setminus \Sigma$.*
(2) *There exist a holomorphic function $G : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$ with*

$$K_{\mathcal{U}}(z, w) = \frac{G(z, w)}{z_2 \overline{w}_2 (z_1 - \overline{w}_1)^2}, \tag{5.3}$$

and a holomorphic function g on $A := \{\zeta : e^{-\pi/2} < |\zeta| < e^{\pi/2}\}$ such that:

(a) *$G(z, w)$ stays bounded as either z_1 or w_1 tends to ∞;*
(b) *if $z_1 - \overline{w}_1 \to \infty$ within a half-plane $\mathbf{U}_\varepsilon$ and if $e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2 \overline{w}_2 \to \zeta \in A$ then $G(z, w) \to g(\zeta)$;*
(c) *$g(\zeta) - [e^{-\pi/2}\zeta]/[\pi^2(1-e^{-\pi/2}\zeta)^2] - [e^{\pi/2}\zeta]/[\pi^2(1-e^{\pi/2}\zeta)^2]$ extends holomorphically to a neighborhood of $\overline{A}$.*

As a consequence, $K_{\mathcal{U}}$ tends to 0 near each point (z, w) or (w, z) with $z_1 = \infty$, $z_2 \in \mathbb{C}^* \cup \{\infty\}$, $w \in \mathcal{U}$.

Proof. We wish to study the behavior of

$$\sum_{j \in \mathbb{Z}} K_j(z_1, w_1) \Big(e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2 \overline{w}_2 \Big)^{j+1}$$

as z and w in $\mathcal{U}$ approach the boundary. It follows from Corollary 4.4 that for all $(z_1, w_1) \in \overline{\mathbf{U}} \times \overline{\mathbf{U}} \setminus \Delta$

$$\limsup_{j \to \pm\infty} |K_j(z_1, w_1)|^{1/|j+1|} \le e^{-b_\lambda},$$

where $\lambda = -i(z_1 - \overline{w}_1)$ and b_λ is as in (4.10). We will now complete the study of convergence, proving that

$$e^{-b_\lambda} < \Big| e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2 \overline{w}_2 \Big| < e^{b_\lambda}, \tag{5.4}$$

for all $(z, w) \in \overline{\mathcal{U}} \times \overline{\mathcal{U}} \setminus \Sigma$. For $(z, w) \in \mathcal{U} \times \mathcal{U}$ we have that

$$\begin{aligned} e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2 \overline{w}_2 &= e^{-\frac{1}{2}(z_1+\overline{w}_1)} e^{\frac{1}{2}(\log|z_2|^2 + \log|w_2|^2)} \frac{z_2 \overline{w}_2}{|z_2 \overline{w}_2|} \\ &= \exp\Big\{ \tfrac{1}{2}\big(\log|z_2|^2 - \operatorname{Re} z_1 + \log|w_2|^2 - \operatorname{Re} w_1 \big) \\ &\qquad - \tfrac{i}{2}\big(\operatorname{Im} z_1 - \operatorname{Im} w_1 \big) \Big\} \frac{z_2 \overline{w}_2}{|z_2 \overline{w}_2|}, \end{aligned}$$

where $\left|\log|z_2|^2 - \operatorname{Re} z_1\right| < \arccos(e^{-\operatorname{Im} z_1})$ and $\left|\log|w_2|^2 - \operatorname{Re} w_1\right| < \arccos(e^{-\operatorname{Im} w_1})$. Hence, using the concavity of the function $r \mapsto \arccos(e^r)$ we obtain

$$
\begin{aligned}
\left|e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2\overline{w}_2\right| &< \exp\left\{\tfrac{1}{2}\left(\arccos(e^{-\operatorname{Im} z_1}) + \arccos(e^{-\operatorname{Im} w_1})\right)\right\} \\
&\le \exp\left\{\arccos\left(e^{-\frac{1}{2}(\operatorname{Im} z_1 + \operatorname{Im} w_1)}\right)\right\} \\
&= \exp\left\{\arccos\left(e^{-\frac{1}{2}\operatorname{Re}\lambda}\right)\right\} \\
&\le e^{b_\lambda};
\end{aligned}
\tag{5.5}
$$

and similarly $\left|e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2\overline{w}_2\right| > \exp\left\{-\arccos\left(e^{-\frac{1}{2}\operatorname{Re}\lambda}\right)\right\} \ge e^{-b_\lambda}$ for all $(z, w) \in \mathcal{U}\times\mathcal{U}$. The first inequality in the display above remains strict as either z or w tends to $\partial\mathcal{U}$ and if either z_1 or w_1 tends to infinity. Now let us consider z and w in $\partial\mathcal{U}$. The equality

$$
\left|e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2\overline{w}_2\right| = \exp\left\{\pm\arccos\left(e^{-\frac{1}{2}\operatorname{Re}\lambda}\right)\right\}
$$

holds if and only if there exists $v \ge 0$ such that

$$
\operatorname{Im} z_1 = \operatorname{Im} w_1 = v \text{ and } \log|z_2|^2 - \operatorname{Re} z_1 = \log|w_2|^2 - \operatorname{Re} w_1 = \pm\arccos(e^{-v}). \tag{5.6}
$$

According to formula (4.10), $\arccos\left(e^{-[1/2]\operatorname{Re}\lambda}\right) = b_\lambda$ if and only if $\operatorname{Im}|\lambda|/2 \le \arccos\left(e^{-[1/2]\operatorname{Re}\lambda}\right)$, which is equivalent in the special case (5.6) to $\left|\log|z_2|^2 - \log|w_2|^2\right| \le 2\arccos\left(e^{-v}\right)$. This proves (5.4) and also part (1) of the statement.

In order to prove (2) we further study the points at infinity by means of the expansion

$$
K_{\mathcal{U}}(z, w) = \sum_{j\in\mathbb{Z}} \frac{f_j(-i(z_1 - \overline{w}_1))}{(z_1 - \overline{w}_1)^2 z_2\overline{w}_2} \left(e^{-1/2(z_1+\overline{w}_1)} z_2\overline{w}_2\right)^{j+1},
$$

where $f_j(\lambda) \to [k\pi/2]/[\pi^3\sinh(k\pi/2)]$ as $\lambda \to \infty$ within a half-plane $\mathbf{H}_\varepsilon$. If we set $G(z, w) = z_2\overline{w}_2(z_1 - \overline{w}_1)^2 K_{\mathcal{U}}(z, w)$, then

$$
\lim_{e^{-\frac{1}{2}(z_1+\overline{w}_1)} z_2\overline{w}_2 \to \zeta} G(z, w) = \sum_{j\in\mathbb{Z}} f_j(-i(z_1 - \overline{w}_1))\zeta^{j+1}
$$

for

$$
\exp\left\{-\arccos\left(e^{-(\operatorname{Im} z_1 + \operatorname{Im} w_1)/2}\right)\right\} < |\zeta| < \exp\left\{\arccos\left(e^{-(\operatorname{Im} z_1 + \operatorname{Im} w_1)/2}\right)\right\}.
$$

Moreover, $\sum_{j\in\mathbb{Z}} f_j(\lambda)\zeta^{j+1}$ tends to $g(\zeta)=\frac{1}{\pi^3}\sum_{k\in\mathbb{Z}}\frac{k\pi/2}{\sinh(k\frac{\pi}{2})}\zeta^k$ as $\lambda\to\infty$ within a half-plane $\mathbf{H}_\varepsilon$. We have that

$$\begin{aligned}\sum_{k>0}\frac{k\pi/2}{\sinh(k\pi/2)}\zeta^k &= \pi\zeta\frac{\partial}{\partial\zeta}\sum_{k>0}\frac{1}{e^{k\pi/2}-e^{-k\pi/2}}\zeta^k\\ &= \pi\zeta\frac{\partial}{\partial\zeta}\sum_{k>0}\frac{1}{1-e^{-k\pi}}(e^{-\pi/2}\zeta)^k\\ &= \pi\zeta\frac{\partial}{\partial\zeta}\sum_{k>0,m\geq 0}e^{-km\pi}(e^{-\pi/2}\zeta)^k\\ &= \pi\zeta\frac{\partial}{\partial\zeta}\sum_{m\geq 0}\frac{1}{1-e^{-(m+1/2)\pi}\zeta}\\ &= \pi\zeta\sum_{m\geq 0}\frac{e^{-(m+1/2)\pi}}{(1-e^{-(m+1/2)\pi}\zeta)^2}\\ &= \frac{\pi e^{-\pi/2}\zeta}{(1-e^{-\pi/2}\zeta)^2}+f(\zeta),\end{aligned}$$

where all the series converge absolutely and uniformly on compact sets in the annulus A and f is holomorphic in a neighborhood of $\overline{A}$. Thus

$$\begin{aligned}g(\zeta) &= \frac{e^{-\pi/2}\zeta}{\pi^2(1-e^{-\pi/2}\zeta)^2}+\frac{f(\zeta)}{\pi^3}+\frac{1}{\pi^3}+\frac{e^{-\pi/2}\zeta^{-1}}{\pi^2(1-e^{-\pi/2}\zeta^{-1})^2}+\frac{f(\zeta^{-1})}{\pi^3}\\ &= \frac{e^{-\pi/2}\zeta}{\pi^2(1-e^{-\pi/2}\zeta)^2}+\frac{e^{\pi/2}\zeta}{\pi^2(1-e^{\pi/2}\zeta)^2}+\frac{f(\zeta)+1+f(\zeta^{-1})}{\pi^3},\end{aligned}$$

which concludes the proof. □

Now we turn back to the unbounded worm domain $\mathcal{W}$ via the biholomorphism $\Phi(z)=(\ell(z),z_2)$, where $\ell(z)=-i(L(z)-\log 2)$ and $L(z)$ is given by (2.1), and via the isometric isomorphism

$$\begin{aligned}T^{-1}&: A^2(\mathcal{U})\to A^2(\mathcal{W})\\ T^{-1}f(z)&=\frac{1}{iz_1}f\big(\ell(z),z_2\big).\end{aligned}$$

Recall also that we set $E_\eta(z)=e^{\eta L(z)}$ in (2.2). The next result follows at once from Proposition 5.1.

Theorem 5.3. *The Bergman kernel K of $A^2(\mathcal{W})$ can be computed at each $(z,w)\in\mathcal{W}\times\mathcal{W}$ as*

$$K(z,w)=(z_1\overline{w}_1z_2\overline{w}_2)^{-1}\sum_{j\in\mathbb{Z}}K_j(\ell(z),\ell(w))\Big(E_{i/2}(z)z_2\overline{E_{i/2}(w)w_2}\Big)^{j+1}. \quad (5.7)$$

In particular, when $(z, w) \in \mathcal{W}_{\pi/2} \times \mathcal{W}_{\pi/2}$*, the kernel function takes the form*

$$K(z, w) = (z_1\overline{w}_1 z_2\overline{w}_2)^{-1} \sum_{j\in\mathbb{Z}} K_j\Big(-i \log z_1/2, -i \log w_1/2\Big)\Big(z_1^{i/2} z_2 \overline{w_1^{i/2} w_2}\Big)^{j+1} .$$

As in the case of $\mathcal{U}$ we study the boundary behavior of K.

Proposition 5.4. *The Bergman kernel* $K(z, w)$ *of* $A^2(\mathcal{W})$ *extends holomorphically in* z *and antiholomorphically in* w *near each point* (z, w) *of the boundary of* $\mathcal{W}\times\mathcal{W}$ *except:*

(i) *when* $z_1 = 0$ *or* $w_1 = 0$*;*
(ii) *when* $z_2 = 0$ *or* $w_2 = 0$*;*
(iii) *when, for some* $r \in (0, 2]$*, we have*

$$z_1 = re^{i \log|z_2|^2 \pm i \arccos(r/2)}, \quad w_1 = re^{i \log|w_2|^2 \pm i \arccos(r/2)} \quad \text{and}$$
$$\Big|\log|z_2|^2 - \log|w_2|^2\Big| \le 2\arccos(r/2) .$$

For case (i)*, we note that there exist a holomorphic function* $H : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ *with*

$$K(z, w) = \frac{H(z, w)}{z_1\overline{w}_1 z_2\overline{w}_2\big(\ell(z) - \overline{\ell(w)}\big)^2} \tag{5.8}$$

and a holomorphic function g *on* $A := \{\zeta : e^{-\pi/2} < |\zeta| < e^{\pi/2}\}$ *such that:*

(a) $H(z, w)$ *stays bounded as either* z_1 *or* w_1 *tends to* 0*;*
(b) *if* $z_1 \to 0$ *or* $w_1 \to 0$ *and if* $E_{i/2}(z)z_2\overline{E_{i/2}(w)w_2} \to \zeta \in A$ *then* $H(z, w) \to g(\zeta)$*;*
(c) $g(\zeta) - [e^{-\pi/2}\zeta]/[\pi^2(1 - e^{-\pi/2}\zeta)^2] - [e^{\pi/2}\zeta]/[\pi^2(1 - e^{\pi/2}\zeta)^2]$ *extends holomorphically to a neighborhood of* $\overline{A}$*.*

As a consequence, K *is singular at all points* (z, w) *of the boundary with* $z_1 = 0, z_2 \in \mathbb{C}$ *or* $w_1 = 0, w_2 \in \mathbb{C}$*.*

Remark. Case (iii) of Proposition 5.4 comprises all points (z, z) of the diagonal of $\partial\mathcal{W} \times \partial\mathcal{W}$; but also other points (z, w) of $\partial\mathcal{W} \times \partial\mathcal{W}$, *e.g.*, those such that $z_1 = w_1 \in \partial\Delta(0, 2)$ and $|z_2| = |w_2|$. See Figure 5.2 for other cases.

Proof of Proposition 5.4. The first and second statements are direct consequences of Theorem 5.2, taking into account that ℓ extends holomorphically to a neighborhood of each point z of $\overline{\mathcal{W}}$ except for those with vanishing z_1 or z_2.

As for the last statement, we begin by noting that the function $z_1\overline{w}_1 z_2\overline{w}_2(\ell(z) - \overline{\ell(w)})^2$ tends to 0 as $z_1\overline{w}_1$ approaches 0 while $z_2\overline{w}_2$ stays bounded; and that $|z_1\overline{w}_1 z_2\overline{w}_2||\ell(z) - \overline{\ell(w)}|^2$ tends to $+\infty$ as $z_2\overline{w}_2 \to \infty$. Furthermore, since g extends to a meromorphic function on a neighborhood of $\overline{A}$, it can only have finitely

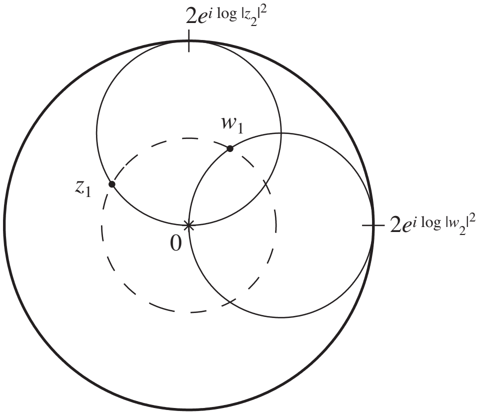

Figure 5.2. Case (iii) of Proposition 5.4 regards those points (z, w) such that: (1) the points $z_1, w_1 \in \overline{\Delta(0, 2)} \setminus \{0\}$ both lie on some circle $\mathscr{C} = \{\zeta \in \mathbb{C} : |\zeta| = r\}$ (dashed) and, respectively, on the boundaries $\mathscr{C}_{z_2}, \mathscr{C}_{w_2}$ of the discs $\pi_1(\pi_2^{-1}(z_2)), \pi_1(\pi_2^{-1}(w_2))$ (solid); (2) when circling along $\mathscr{C}$ from point r with an orientation such that z_1 is the first point of $\mathscr{C}_{z_2}$ encountered, then w_1 is the first point of $\mathscr{C}_{w_2}$ encountered; (3) $\big| \log |z_2|^2 - \log |w_2|^2 \big| \leq 2 \arccos(r/2)$ (which implies that, but is not equivalent to, $w_1 \in \pi_1(\pi_2^{-1}(z_2))$ or $z_1 \in \pi_1(\pi_2^{-1}(w_2))$).

many zeros in $\overline{A}$. Let $t \in (-\pi/2, \pi/2)$ be such that the circle $|\zeta| = e^t$ does not include any zero of g. For every (z, w) with $z_1 = 0$ or $w_1 = 0$, one can easily construct a sequence of points tending to (z, w) such that the corresponding values of H tend to $g(\zeta)$ with $|\zeta| = e^t$ (hence with $g(\zeta) \neq 0$). □

Corollary 5.5. *For $\mu \in (0, \infty]$ and fixed $w \in \mathcal{W}$, the following properties hold:*

(1) *$K(\cdot, w) \notin L^p(\mathcal{W}_\mu)$ for any $p > 2$;*
(2) *$K(\cdot, w) \notin W^s(\mathcal{W}_\mu)$ for any $s > 0$.*

Proof. We begin by refining our remarks concerning the function g that appears in the previous proposition. As we mentioned in the previous proof, g can only have finitely many zeros in $\overline{A}$. Fix $w \in \mathcal{W}$ and set $a := E_{i/2}(w) w_2$. For some $-\pi/2 < \alpha < \beta < \pi/2$, the function $z \mapsto \big| g\big(E_{i/2}(z) z_2 \overline{a}\big) \big|$ is bounded from below by a constant for z_1 in the sector $S(e^{i \log |z_2|^2}, \varepsilon) = \{r e^{i(t + \log |z_2|^2)} : \alpha < t < \beta, 0 < r < \varepsilon\}$ for all ε small enough that $S(e^{i \log |z_2|^2}, \varepsilon) \subset \Delta(e^{i \log |z_2|^2}, 1)$.

Now, for fixed $\mu \in (0, +\infty)$, let us consider the smooth worm $\mathcal{W}_\mu$. We recall that a defining function for $\mathcal{W}_\mu$ is $\rho(z) = \big| z_1 - e^{i \log |z_2|^2} \big|^2 - 1 + \eta_\mu(\log |z_2|^2) = |z_1|^2 - 2 \operatorname{Re}(z_1 e^{-i \log |z_2|^2}) + \eta_\mu(\log |z_2|^2)$, where η_μ is an appropriately chosen function such that $\eta_\mu^{-1}(0) = [-\mu, \mu]$. As a consequence, $\mathcal{W}_\mu$ always includes

$\bigcup_{-\mu<\log|z_2|^2<\mu} \Delta(e^{i\log|z_2|^2}, 1)$. Notice that

$$
\begin{aligned}
|\ell(z)-\overline{\ell(w)}|^2 = |L(z)+\overline{L(w)}-2\log 2|^2 &= \Big(\log(|z_1|/2)+\log(|w_1|/2)\Big)^2 \\
&\quad + \Big(\arg\big(z_1 e^{-i\log|z_2|^2}\big) + \log|z_2|^2 - \arg\big(w_1 e^{-i\log|w_2|^2}\big) - \log|w_2|^2\Big)^2 \\
&\le \big(\log(|z_1|/2)+c_1\big)^2 + c_2\,,
\end{aligned}
$$

where $c_1=\log(|w_1|/2)<0$ and $c_2\le(\pi+2\mu)^2$.

Owing to formula (5.8), there exist ε, and $C>0$ so that, for all $z\in \bigcup_{-\mu<\log|z_2|^2<\mu} S(e^{i\log|z_2|^2},\varepsilon)\times\{z_2\}$,

$$
|K(z,w)| \ge \frac{C}{|z_1||\ell(z)-\overline{\ell(w)}|^2} \ge \frac{C}{|z_1|}\,\frac{1}{\big(\log(|z_1|/2)+c_1\big)^2+c_2}\,.
$$

Therefore

$$
\begin{aligned}
&\|K(\cdot,w)\|^p_{L^p(\mathcal{W}_\mu)} \\
&\ge \int_{-\mu<\log|z_2|^2<\mu}\int_{S(e^{i\log|z_2|^2},\varepsilon)} \frac{C^p}{|z_1|^p\big[\big(\log(|z_1|/2)+c_1\big)^2+c_2\big]^p}\,dV(z_1)dV(z_2) \\
&= \int_{-\mu<\log|z_2|^2<\mu}\int_{S(1,\varepsilon)} \frac{C^p}{|\zeta|^p\big[\big(\log(|\zeta|/2)+c_1\big)^2+c_2\big]^p}\,dV(\zeta)dV(z_2) \\
&= C_\mu\int_0^\varepsilon \frac{1}{r^{p-1}\big[\big(\log(r/2)+c_1\big)^2+c_2\big]^p}\,dr\,,
\end{aligned}
$$

where the inner integral diverges when $p>2$.

The last statement will be proved for all $s>0$ if we can prove it for all $s\in(0,\frac{1}{2})$. In the latter case, according to [30], the function $K(\cdot,w)$ belongs to the Sobolev space $W^s(\mathcal{W}_\mu)$ if and only if $\rho(\cdot)^{-s}K(\cdot,w)$ is in $L^2(\mathcal{W}_\mu)$. But

$$
\begin{aligned}
&\|\rho(\cdot)^{-s}K(\cdot,w)\|^2_{L^2(\mathcal{W}_\mu)} \\
&\ge \int_{-\mu<\log|z_2|^2<\mu}\int_{S(e^{i\log|z_2|^2},\varepsilon)} \\
&\qquad\times \frac{C^2}{\big||z_1|^2-2\,\mathrm{Re}(z_1e^{-i\log|z_2|^2})\big|^s|z_1|^2\big[\big(\log(|z_1|/2)+c_1\big)^2+c_2\big]^2}\,dV(z_1)dV(z_2) \\
&= \int_{-\mu<\log|z_2|^2<\mu}\int_{S(1,\varepsilon)} \frac{C^2}{\big||\zeta|^2-2\,\mathrm{Re}(\zeta)\big|^s|\zeta|^2\big[\big(\log(|\zeta|/2)+c_1\big)^2+c_2\big]^2}\,dV(\zeta)dV(z_2) \\
&= C\int_\alpha^\beta\int_0^\varepsilon \frac{1}{\big|r-2\cos t\big|^s r^{1+s}\big[\big(\log(r/2)+c_1\big)^2+c_2\big]^2}\,dr\,dt
\end{aligned}
$$

where the inner integral diverges when $s>0$, for all $t\in(\alpha,\beta)$. □

Proof of Theorem 1.1. We saw in the previous theorem that $K_w = K(\cdot, w)$ does not belong to $W^s(\mathcal{W})$ nor to $L^p(\mathcal{W})$ for any $s > 0$ or $p > 2$. Since K_w can be obtained as the projection $\mathcal{P}(\chi_w)$ of a smooth cut-off function $\chi_w \in C_0^\infty$ supported in a compact neighborhood of w (see [22]), the inclusion $\mathcal{P}(W^s(\mathcal{W})) \subseteq W^s(\mathcal{W})$ implies $s \le 0$ and $\mathcal{P}(L^p(\mathcal{W})) \subseteq L^p(\mathcal{W})$ implies $p \le 2$.

We complete the proof by showing that $\mathcal{P}(L^p(\mathcal{W})) \subseteq L^p(\mathcal{W})$ implies $p \ge 2$. This part of the proof makes use of the duality between L^p and $L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. We observe that, since $\mathcal{P}f(w) = \langle f, K_w \rangle$,

$$\begin{aligned}\|K_w\|_{L^{p'}} &= \sup_{\|f\|_{L^p}=1} \Big| \int_{\mathcal{W}} f(z)\overline{K_w(z)}\,dV(z)\Big| = \sup_{\|f\|_{L^p}=1} |\mathcal{P}f(w)| \\ &\le \sup_{\|f\|_{L^p}=1} \Big|\frac{1}{V(B)} \int_B \mathcal{P}f(z)\,dV(z)\Big| \le C \sup_{\|f\|_{L^p}=1} \|\mathcal{P}f\|_{L^p} \le C',\end{aligned}$$

which implies $p' \le 2$, hence that $p \ge 2$ as desired. □

6. Concluding remarks

We have studied the worm now for several years and achieved some success in analyzing the unbounded (sometimes non-smooth) worm. See for instance [25–27]. Our ultimate goal, however, is to study the original worm domain $\mathcal{W}_\mu$ of Diederich and Fornæss [16].

The approach used in the present paper allows, even in the case of $\mathcal{W}_\mu$, to reduce the study of the Bergman space to a family of weighted Bergman spaces on a planar domain. In this case the planar domain is not a half-plane anymore and the weight depends on both real variables, two facts which prevent from computing the kernel with the technique used for $\mathcal{W}$. However, the reduction to a planar domain may shed some light on the challenging problem of writing down a complete system for the Bergman space of $\mathcal{W}_\mu$. We intend to explore these matters in a forthcoming paper.

We also intend to apply the approach used in the present paper to the higher-dimensional version of the worm domain introduced and studied by Barrett and S. Şahutoğlu in [11]. Namely, for $n \ge 3$ they defined the domain

$$\Omega_{\alpha\beta} = \big\{(z_1, z', z_n) \in \mathbb{C}^n : r(z_1, z', z_n) < 0\big\} \tag{6.1}$$

where

$$r(z_1, z', z_n) = \big|z_1 - e^{i\alpha \log |z_n|^2}\big|^2 + |z'|^2 - 1 + \sigma(|z_n|^2 - \beta) + \sigma(1 - |z_n|^2),$$

$z_1, z_n \in \mathbb{C}$, $z' \in \mathbb{C}^{n-2}$, the parameters α and β satisfy $\alpha > 0$, $\beta > 1$ and $\sigma(t) = M\chi_{(0,+\infty)}(t)e^{-1/t}$, for some $M > 0$. They proved that the Bergman projection on $\Omega_{\alpha\beta}$ is irregular on the Sobolev space $W^{s,p}(\Omega_{\alpha\beta})$ when $1 \le p < \infty$ and $s \ge \frac{\pi}{2\alpha \log \beta} + n\big(\frac{1}{p} - \frac{1}{2}\big)$. Here $W^{s,p}(\Omega_{\alpha\beta})$ denotes the space of functions whose derivatives up to order s are L^p-integrable. In particular, our approach may apply to study the unbounded domain obtained from $\Omega_{\alpha\beta}$ by letting $\beta \to +\infty$.

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