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On Runge-curved domains in Stein spaces

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Abstract. We prove the following result: if X is a Stein complex space and $D \subset X$ is an open subset, then D is Runge-curved in X if and only if the canonical map $H^1_c(D, \mathcal{F}) \to H^1_c(X, \mathcal{F})$ is injective for every $\mathcal{F} \in \mathrm{Coh}(X)$. We also show that a Runge-curved open subset of a Stein manifold is necessarily Stein.

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1. Introduction

Let X be a (not necessarily reduced) Stein space and $D \subset X$ be an open subset. Such a D is called a Runge domain if D is Stein and the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(D, \mathcal{O}_X)$ has dense image in the topology of uniform convergence of compact subsets of D. The domain D (not necessarily Stein) is called Runge-curved if for any complex curve $\Gamma \subset X$ (closed complex space of dimension 1) the pair $(\Gamma, \Gamma \cap D)$ is a Runge pair.

For Stein spaces of dimension 1, a complete characterization has been given by N. Mihalache [10]. If $n = \dim X > 1$ the notions of Runge domain and Runge-curved domain do not coincide. G. Stolzenberg [14] has given an example of a domain $D \subset \mathbb{C}^2$ that is Runge-curved and is not Runge.

The main purpose of this paper is to continue the study of Runge-curved domains. As a first step we prove in Theorem 2.4 that a Runge-curved domain in a Stein manifold is Stein. It is an open question if the same result holds in the singular case. The main result of the paper is Theorem 3.7 which gives a characterization of Runge-curved domains in Stein spaces using cohomology with compact support. More precisely we prove that the following two conditions are equivalent:

1) D is Runge-curved;
2) The natural map between cohomology groups with compact support $H^1_c(D, \mathcal{F}) \to H^1_c(X, \mathcal{F})$ is injective for every $\mathcal{F} \in \mathrm{Coh}(X)$.

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The proof is done by induction on $\dim(X)$, and because of this one needs to use complex spaces with nilpotent elements and properties of the torsion of a coherent sheaf defined on them. If one uses only torsion free sheaves, condition 2) is replaced by the weaker condition that $X \setminus D$ has no holes, see [6]. Therefore the use of torsion sheaves on complex spaces with nilpotents is essential for our proof.

2. Steiness of Runge curved domains in Stein manifolds

For the following statement see [1].

Lemma 2.1. *Let X be a Stein manifold of dimension n. Then there exists a finite number of holomorphic maps $\Phi_j : X \to \mathbb{C}^n$ with discrete fibers $j = 1, 2 \dots, m$ such that if A_j is the branch locus of Φ_j then $\bigcap_{j=1}^m A_j = \emptyset$.*

The following statement is part of [10, Theorem 6.8]:

Lemma 2.2. *Let X be a pure 1-dimensional Stein space and $D \subset X$ an open subset. Then the following are equivalent:*

1) *(X, D) is a Runge pair;*
2) *For any open neighborhood U of $X \setminus D$ and any irreducible component C of $Red(U)$, $C \setminus D$ is not compact;*
3) *$H^1_c(D, \mathcal{O}) \to H^1_c(X, \mathcal{O})$ is injective.*

The proof of the following lemma is similar to the proof of [12, Proposition 5.5].

Lemma 2.3. *Let D be an open subset of $\mathbb{C}^n$. If for every closed complex curve $\Gamma \subset \mathbb{C}^n$ biholomorphic to $\mathbb{C}$ we have that $(\Gamma, \Gamma \cap D)$ is a Runge pair, then D is Stein.*

Proof. We may assume, of course, that $D \neq \mathbb{C}^n$ and we denote by $\delta : D \to (0, \infty)$ the distance to the boundary of D which is a continuous function. For every $u \in \mathbb{C}^n$ with $u \neq 0$ we define $\delta_u : D \to (0, \infty]$ by

$$\delta_u(z) := \sup\{\tau : z + \eta u \in D \text{ for every } \eta \in \mathbb{C},\ |\eta| \leq \tau\}.$$

We have that $\delta = \inf\{\delta_u : u \in \mathbb{C}^n,\ \|u\| = 1\}$. By Oka's theorem we must prove that $-\log \delta = \sup\{-\log \delta_u : u \in \mathbb{C}^n,\ \|u\| = 1\}$ is plurisubharmonic. Therefore it suffices to prove that $-\log \delta_u$ is plurisubharmonic for every u. Let $u \in \mathbb{C}^n$ and $\|u\| = 1$. To prove that $-\log \delta_u$ is plurisubharmonic we have to show that if $z \in D$ and $w \in \mathbb{C}^n \setminus \{0\}$, and $g : \mathbb{C} \to \mathbb{C}$ is a polynomial function of one complex variable such that

$$-\log \delta_u(z + \lambda w) \leq \operatorname{Re} g(\lambda) \quad \forall \lambda \text{ such that } |\lambda| = r,$$

where $r > 0$, then we have

$$-\log \delta_u(z + \lambda w) \leq \operatorname{Re} g(\lambda) \quad \forall \lambda \text{ such that } |\lambda| \leq r.$$

This is equivalent to proving that if

$$z + \lambda w + \mu e^{-g(\lambda)} u \in D \quad \forall \lambda, \mu \in \mathbb{C} \text{ such that } |\lambda| = r \text{ and } |\mu| < 1$$

then we have

$$z + \lambda w + \mu e^{-g(\lambda)} u \in D \quad \forall \lambda, \mu \in \mathbb{C} \text{ such that } |\lambda| \leq r \text{ and } |\mu| < 1.$$

Note that if u and w are linearly dependent over $\mathbb{C}$ then this is just a standard 1-dimensional statement. Let us assume that u and w are linearly independent and let's fix $\mu \in \mathbb{C}$ with $|\mu| < 1$. We consider $h : \mathbb{C} \to \mathbb{C}^n$ given by $h(\lambda) = z + \lambda w + \mu e^{-g(\lambda)} u$. Then h is an embedding and therefore $\Gamma := h(\mathbb{C})$ is a closed complex curve in $\mathbb{C}^n$, biholomorphic to $\mathbb{C}$. Therefore $(\Gamma, \Gamma \cap D)$ is a Runge pair. At the same time we have that $\{\lambda \in \mathbb{C} : |\lambda| = r\} \subset h^{-1}(D)$. We deduce that $\{\lambda \in \mathbb{C} : |\lambda| \leq r\} \subset h^{-1}(D)$ and hence $z + \lambda w + \mu e^{-g(\lambda)} u \in D$, $\forall \lambda$ with $|\lambda| \leq r$. □

Theorem 2.4. *Let X be a n-dimensional Stein manifold and D an open subset of X. If $(C, C \cap D)$ is a Runge pair for every closed complex curve $C \subset X$, then D is Stein.*

Proof. Since X is a Stein manifold, it suffices to show that D is locally Stein in X. Let $x_0 \in \partial D$. By Lemma 2.1 there exists a holomorphic map with discrete fibers, $\Phi : X \to \mathbb{C}^n$, such that Φ is locally a biholomorphism around x_0. Let U be a Stein open neighborhood of x_0 and V be an open neighborhood of $y_0 := \Phi(x_0)$ such that $\Phi : U \to V$ is a biholomorphism. Moreover, we choose U to be Runge in X. Replacing V by a small ball $B(y_0, \epsilon)$ centered at y_0 and U by the connected component of $\Phi^{-1}(B(y_0, \epsilon)) \cap U$ that contains x_0, we may assume that U is Runge in X and V is Runge in $\mathbb{C}^n$.

In this setting, we would like to show that $U \cap D$ is Stein, which is equivalent to $\Phi(D \cap U)$ being Stein. If Γ is a closed complex curve in $\mathbb{C}^n$ then we let $C := \Phi^{-1}(\Gamma)$, which is a closed complex curve in X. Therefore $C \cap D$ is Runge in C. It follows that $C \cap D \cap U$ is Runge in $C \cap U$ and therefore $\Gamma \cap \Phi(D \cap U)$ is Runge in $\Gamma \cap V$. Since $\Gamma \cap V$ is Runge in Γ we deduce that $\Gamma \cap \Phi(D \cap U)$ is Runge in Γ. In order to finish the proof, we apply Lemma 2.3. □

Remark 2.5. We do not know if the above theorem is true when X is a Stein complex space.

Remark 2.6. If D is a Runge open subset of Stein manifold X, it was proved by A. Cassa in [4] that for every closed irreducible complex curve C in D one can find a sequence of closed irreducible curves $\{C_k\}_{k \geq 1}$ in X such that $C = \lim_{k \to \infty}(C_k \cap D)$. The convergence is the one induced by the currents. See also [5].

It was proved in [7] that if X is an $(n-1)$-convex irreducible complex space of $\dim X > 1$, the map $\rho : X \to \mathbb{R}$ is a smooth exhaustion function which is $(n-1)$-convex on $X_c = \{x \in X : \rho(x) > c\}$, with c a real number, D is a bordered Riemann surface, and $f : \overline{D} \to X$ is a $\mathcal{C}^2$ function which is holomorphic in D and satisfies $f(D) \not\subset X_{\text{sing}}$ and $f(bD) \subset X_c$, then there is a sequence of proper holomorphic maps $g_\nu : D \to X$ converging to f uniformly on compact subsets of D.

Example 2.7. Suppose that B_1 and B_2 are two balls in $\mathbb{C}^n$ such that $B_1 \not\subset B_2$ and $B_2 \not\subset B_1$ and $B_1 \cap B_2 \neq \emptyset$. Then $D := B_1 \cup B_2$ is not Stein. At the same time for every complex affine line, L in $\mathbb{C}^n$, the intersection $L \cap D$ is simply connected because $L \cap B_1, L \cap B_2, L \cap B_1 \cap B_2$ are connected and simply connected. Therefore $L \cap D$ is Runge in L. According to our theorem there must exist a curve C in $\mathbb{C}^n$ such that $C \cap D$ is not Runge in C. We are going to exhibit such a curve. For simplicity we work with $n = 2$, assume B_1 is the open ball of radius 3 centered at $(-2, 0) \in \mathbb{C}^2$ and B_2 is the open ball of radius 3 centered at $(2, 0)$. Let $C = \{(z, w) \in \mathbb{C}^2 : w = z^2 + \sqrt{5}\}$. We choose $r > 0$ small enough such that

$$r^3 + \left[2\sqrt{5}(2\lambda^2 - 1) + 1\right] r - 4\lambda < 0 \ \text{ for all } \lambda \text{ such that } \lambda \in [0, 1]. \tag{2.1}$$

Note that this is equivalent to

$$r^3 + \left[2\sqrt{5}(2\lambda^2 - 1) + 1\right] r + 4\lambda < 0 \ \text{ for all } \lambda \text{ such that } \lambda \in [-1, 0].$$

Then clearly $\left(0, \sqrt{5}\right) \in \partial D$ (hence $\notin D$) where $D := B_1 \cup B_2$. At the same time, we have $\{(z, w) \in C : z = re^{i\theta} \text{ for } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\} \subset B_1$. Indeed, we have to check that, for $z = re^{i\theta}$ with $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we have that $\left|z + 2\right|^2 + |z^2 + \sqrt{5}\right|^2 < 9$.

We set $\lambda = \cos\theta$ and hence $\lambda \in [-1, 0]$ and $\cos 2\theta = 2\lambda^2 - 1$. We get that $|z + 2|^2 = r^2 + 4r\lambda + 4$ and $|z^2 + \sqrt{5}|^2 = r^4 + 2\sqrt{5}(2\lambda^2 - 1)r^2 + 5$. We need to check that $r^2 + 4r\lambda + 4 + r^4 + 2\sqrt{5}(2\lambda^2 - 1)r^2 + 5 < 9$. Which is the same as $r(r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r + 4\lambda) < 0$ and this exactly the above inequality.

The exact same argument shows that $\{(z, w) \in C : z = re^{i\theta} \text{ for } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\} \subset B_2$. It follows then that $\{(z, w) \in C : z = re^{i\theta}\} \subset D$ and we deduce that $C \cap D$ is not Runge in C.

It remains to show that if $r > 0$ is small enough, then (2.1) is satisfied. Indeed, let $\delta > 0$ be such that $2\sqrt{5}(2\delta^2 - 1) + 1 < 0$ (e.g $\delta = \frac{1}{2}$). If $\lambda \in [0, \delta]$ then $r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r - 4\lambda < 0$ for every $r > 0$. On the other hand, for r small enough $r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r \leq 2\delta$ for every $\lambda \in [\delta, 1]$ and therefore $r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r - 4\lambda < 0$ for every $\lambda \in [\delta, 1]$.

3. A characterization of Runge-curved domains in Stein spaces

Definition 3.1. Let X be a complex space, $\text{red}(X)$ its reduction and $\nu : \tilde{X} \to \text{red}(X)$ be the normalization of $\text{red}(X)$. If $A \subset X$ is a closed set we say that A has no irreducible compact component if $\tilde{A} := \nu^{-1}(A)$ has no compact connected component.

Lemma 3.2. *Let X be a Stein space and $D \subset X$ an open subset. If $(\Gamma, \Gamma \cap D)$ is a Runge pair for every closed complex curve $\Gamma \subset X$, then $A := X \setminus D$ has no compact irreducible component.*

Proof. Let $\nu : \tilde{X} \to \mathrm{red}(X)$ be the normalization of $\mathrm{red}(X)$ and let $\tilde{A} := \nu^{-1}(A)$ and $\tilde{D} := \nu^{-1}(D)$. We assume, by reductio ad absurdum, that $\tilde{A}$ has a compact connected component K_1. Let $K := \nu(K_1)$.

Let $\Gamma \subset X$ be a closed complex curve such that $\Gamma \cap K \neq \emptyset$ and let $\tilde{\Gamma} := \nu^{-1}(\Gamma)$. Let $U_2 \Subset \tilde{X}$ be an open, relatively compact neighborhood of K_1 such that $\partial U_2 \cap \tilde{A} = \emptyset$. For the existence of U_2 see, for example, [11, Proposition 2]. Let U_1 be an open neighborhood of K_1 such that $U_1 \Subset U_2$ and $(U_2 \setminus U_1) \cap \tilde{A} = \emptyset$.

If we put $U := \tilde{\Gamma} \cap U_1$ and $U' = \tilde{\Gamma} \cap (\tilde{X} \setminus \overline{U_2})$ we have that $U \bigcup U'$ is an open neighborhood of $\tilde{\Gamma} \setminus \tilde{D}$ in $\tilde{\Gamma}$, the intersection $U \cap U' = \emptyset$ and $U \setminus (\tilde{\Gamma} \cap \tilde{D})$ is compact. Therefore, for any irreducible component C of U, we have that C is an irreducible component of $U \bigcup U'$ and $C \setminus (\tilde{\Gamma} \cap \tilde{D})$ is compact. By Lemma 2.2 we deduce that the $(\tilde{\Gamma}, \tilde{\Gamma} \cap \tilde{D})$ is not a Runge pair and therefore $(\Gamma, \Gamma \cap D)$ is not a Runge pair, which contradicts our assumption. □

For basic facts regarding the torsion of sheaves on complex spaces we refer to [13].

Suppose that X is a complex space and $\mathcal{F}$ is a coherent analytic sheaf on X. We denote by $t\mathcal{F}$ the torsion sheaf of $\mathcal{F}$. More precisely, for $x \in X$, $t\mathcal{F}_x$ is the set of all germs $s_x \in \mathcal{F}_x$ for which there exists $g_x \in \mathcal{O}_{X,x}$ such that $\mathrm{red}(g_x)$ is a non zero divisor (g_x is called an active germ) and $g_x s_x = 0$. We have that $t\mathcal{F}$ is also a coherent sheaf and $\mathrm{supp}(t\mathcal{F})$ is a thin analytic subset of X. If $t\mathcal{F} = 0$ the sheaf $\mathcal{F}$ is called torsion free.

For the proof of the following see for example [9, page 67].

Theorem 3.3 (Rückert Nullstellensatz). *Suppose that X is a complex space and $\mathcal{S}$ is a coherent analytic sheaf on X. Let $f \in \Gamma(X, \mathcal{O}_X)$ vanish on $\mathrm{supp}(\mathcal{S})$. Then for each point $x \in X$ there exists an open neighborhood U of x and a positive integer k such that $f^k \mathcal{S}_U = 0$.*

Lemma 3.4. *Suppose that X is a complex space, $\mathcal{F}$ is a coherent sheaf on X and $s \in \Gamma(X, \mathcal{F})$ is a section. If $\mathrm{supp}(s)$ is a thin analytic subset of X then $s \in \Gamma(X, t\mathcal{F})$.*

Proof. Let $S = \mathrm{supp}(s)$ and $x \in S$. We choose $V \subset X$ an open Stein neighborhood of x and $f \in \mathcal{O}_X(U)$ such that $\mathrm{red}(f|_{S\cap V}) = 0$ and $\mathrm{red}(f_x)$ is not a zero divisor. Such an f exists because S is thin. Applying Theorem 3.3 to the subsheaf of $\mathcal{F}$ generated by s we deduce that there exists a positive integer k and an open neighborhood $U \subset V$ of x such that $f^k s|_U = 0$. It follows that $s_x \in t\mathcal{F}_x$. □

Corollary 3.5. *If $\mathcal{F}$ is a torsion free coherent sheaf on X and $s \in \Gamma(X, \mathcal{F})$ is a section then $\mathrm{supp}(s)$ is a union of irreducible components of $\mathrm{red}(X)$.*

This means that the identity principle holds for torsion free coherent sheaves: if $U \subset X$ is an open subset and $s \in \Gamma(X, \mathcal{F})$ is a section such that $s \equiv 0$ on U then $s \equiv 0$ on every irreducible component of $\mathrm{red}(X)$ that intersects U.

Proposition 3.6. *Suppose that X is a complex space and $A \subset X$ is a closed subset without compact irreducible components. Then, for every torsion free coherent analytic sheaf $\mathcal{F}$ on X, we have $H^0_c(A, \mathcal{F}) = 0$.*

Proof. We assume that there exists $s \in H^0_c(A, \mathcal{F})$ with $s \neq 0$. We choose a representative $\tilde{s} \in \Gamma(U, \mathcal{F})$ of s, where U is an open neighborhood of A and $\mathrm{supp}(\tilde{s}) \cap A$ is compact and non empty. Since $\mathcal{F}$ is torsion free it follows that $\mathrm{supp}(\tilde{s})$ is an union of irreducible components of $\mathrm{red}(U)$.

Let $\nu : \tilde{X} \to \mathrm{red}(X)$ be the normalization map and let $\tilde{U} := \nu^{-1}(U)$ and $\tilde{A} := \nu^{-1}(A)$. We consider B to be the union of the connected components of $\tilde{U}$ whose images are the irreducible components of $\mathrm{supp}(\tilde{s})$. The intersection $B \cap \tilde{A}$ is compact.

It follows that $\tilde{A}$ has compact connected components and therefore A has compact irreducible components, which contradicts our assumption. □

The following theorem gives a characterization of Runge-curved domains using cohomology with compact supports. For basic definitions and results for cohomology with compact supports with values in a sheaf on a paracompact topological space, see [8] or [3].

Theorem 3.7. *Let X be a Stein complex space and $D \subset X$ an open subset. Then the following are equivalent:*

1) *for every closed complex curve $\Gamma \subset X$, we have that $(\Gamma, \Gamma \cap D)$ is a Runge pair;*
2) *the canonical map $H^1_c(D, \mathcal{F}) \to H^1_c(X, \mathcal{F})$ is injective for every $\mathcal{F} \in \mathrm{Coh}(X)$.*

Proof. 2) $\Longrightarrow$ 1) is straightforward: By choosing $\mathcal{F}$ to be $\mathcal{O}_X/\mathcal{I}_\Gamma$, where $\mathcal{I}_\Gamma$ is the ideal sheaf determined by Γ, we obtain that $H^1_c(\Gamma \cap D, \mathcal{O}_\Gamma) \to H^1_c(\Gamma, \mathcal{O}_\Gamma)$ is injective. It follows then from Lemma 2.2 that $(\Gamma, \Gamma \cap D)$ is a Runge pair.

1) $\Longrightarrow$ 2) We will prove this implication by induction on $\dim X$. If $\dim X = 1$, the statement follows from Lemma 2.2. We assume that the statement is true for all Stein complex spaces of dimension $\leq n-1$ and we let X be a Stein complex space with $\dim X = n$.

Let $\mathcal{F}$ be a coherent sheaf on X and let $A := X \setminus D$. It follows from Lemma 3.2 that A has no compact irreducible component. We consider the exact sequence

$$0 \to t\mathcal{F} \to \mathcal{F} \to \mathcal{F}/t\mathcal{F} \to 0$$

which gives us the following commutative diagram with exact rows and columns, the rows being long exact sequences for the inclusion of an open set:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & H_c^0(X,\mathcal{F}/t\mathcal{F}) \\
 & & & & & & & & & & \downarrow \\
0 & \to & H_c^0(D,t\mathcal{F}) & \to & H_c^0(X,t\mathcal{F}) & \to & H_c^0(A,t\mathcal{F}) & \to & H_{\mathrm{c}}^1(D,t\mathcal{F}) & \xrightarrow{\beta} & H_{\mathrm{c}}^1(X,t\mathcal{F}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta} \\
0 & \to & H_c^0(D,\mathcal{F}) & \to & H_c^0(X,\mathcal{F}) & \to & H_c^0(A,\mathcal{F}) & \to & H_{\mathrm{c}}^1(D,\mathcal{F}) & \xrightarrow{\alpha} & H_{\mathrm{c}}^1(X,\mathcal{F}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_c^0(D,\mathcal{F}/t\mathcal{F}) & \to & H_c^0(X,\mathcal{F}/t\mathcal{F}) & \to & H_c^0(A,\mathcal{F}/t\mathcal{F}) & \to & H_{\mathrm{c}}^1(D,\mathcal{F}/t\mathcal{F}) & \xrightarrow{\gamma} & H_{\mathrm{c}}^1(X,\mathcal{F}/t\mathcal{F}).
\end{array}
$$

Notice now that:

- $\mathrm{supp}(t\mathcal{F})$ is a closed analytic subspace of X of dimension $< \dim X$ and $t\mathcal{F}_{|\,\mathrm{supp}(t\mathcal{F})}$ is a coherent sheaf. On the other hand, we have that $H_{\mathrm{c}}^1(D,t\mathcal{F}) = H_{\mathrm{c}}^1(D \cap \mathrm{supp}(t\mathcal{F}), t\mathcal{F})$ and $H_{\mathrm{c}}^1(X,t\mathcal{F}) = H_{\mathrm{c}}^1(\mathrm{supp}(t\mathcal{F}), t\mathcal{F})$. Hence, according to our induction hypothesis, the map $\beta : H_{\mathrm{c}}^1(D,t\mathcal{F}) \to H_{\mathrm{c}}^1(X,t\mathcal{F})$ is injective;
- since A has no compact irreducible components and $\mathcal{F}/t\mathcal{F}$ has no torsion it follows that $H_c^0(A,\mathcal{F}/t\mathcal{F})=0$ and therefore $\gamma : H_{\mathrm{c}}^1(D,\mathcal{F}/t\mathcal{F}) \to H_{\mathrm{c}}^1(X,\mathcal{F}/t\mathcal{F})$ is injective;
- since $\mathcal{F}/t\mathcal{F}$ has no torsion it follows that $H_c^0(X,\mathcal{F}/t\mathcal{F})$ vanishes and therefore the map $\delta : H_{\mathrm{c}}^1(X,t\mathcal{F}) \to H_{\mathrm{c}}^1(X,\mathcal{F})$ is injective.

From the injectivity of β, γ and δ, it follows that α is injective as well. □

Remark 3.8. It is proved in [2] that if X is a Stein space and $D \subset X$ is an open Runge subset then the map $H_c^i(D,\mathcal{F}) \to H_c^i(X,\mathcal{F})$ is injective for every $i \geq 0$ and every coherent sheaf $\mathcal{F}$.

Remark 3.9. Note that the ideal sheaf that defines $\mathrm{supp}(t\mathcal{F})$ is $\mathrm{Ann}(t\mathcal{F})$ and $\mathrm{supp}(t\mathcal{F})$ might have nilpotent elements even if X is reduced. Hence, even if we wanted to prove Theorem 3.7 only for reduced complex spaces, in order to use an induction argument we would have to consider non-reduced complex spaces as well.

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