

On rational maps from the product of two general curves

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Abstract. This paper treats dominant rational maps from the product of two very general curves to nonsingular projective surfaces. Combining the result in [5] we prove that the product of two very general curves of genus $g \geq 7$ and $g' \geq 3$ does not admit dominant rational maps of degree > 1 if the image surface is non-ruled. We also treat the case of the 2-symmetric product of a curve.

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1. Introduction

Let $X \subset \mathbb{C}\mathbb{P}^3$ be a very general smooth surface of degree ≥ 5 , and $\mathbb{C}(X)$ be its function field. In [9] we proved that every proper subfield $\mathbb{C} \subset K \subset \mathbb{C}(X)$ is a pure transcendental extension of \mathbb{C} . This means that if X is very general and $F : X \dashrightarrow S$ is a dominant map which is not birational, then S is a point, a projective line or a rational surface. We recall that a very general element of a variety U has the property P if P holds in the complement of a union of countably many proper subvarieties of U . Let X be smooth complex projective variety of general type. The dominant rational maps of finite degree $X \dashrightarrow Y$ to smooth varieties of general type, up to birational equivalence of Y form a finite set $M(X)$. The proof follows from the approach of Maehara [10], combined with the results of Hacon and McKernan [8], of Takayama [11], and of Tsuji [12].

Motivated by this finiteness theorem for dominant rational maps on a variety of general type, and by the results obtained in [5] and [9], in this paper we study the case of the product of two very general smooth curves $X = C \times D$ of genus g_C and g_D respectively. The 2-symmetric product $X = C_2$ of C , is also treated.

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The product case has previously been studied in [5]. It was proved there that if $g_C \geq 7$, $g_D \geq 2$, and $S \neq C \times D$ is of general type then a dominant rational map $F : C \times D \dashrightarrow S$ does not exist. Here we complete the analysis by considering surfaces S of Kodaira dimension $\text{kod}(S) = 0$ and 1. Our main result is:

Theorem 1.1 (=Theorem 3.1). *Let C and D be very general curves of genus $g_C \geq 7$ and $g_D \geq 3$ respectively. Let $F : C \times D \dashrightarrow S$ be a dominant rational map of degree > 1 where S is a smooth projective surface. Then S is a ruled surface, that is $\text{kod}(S) = -\infty$.*

We recall the Riemann-Hurwitz Theorem [3, Chapter XXI], which says that if $\phi : C \rightarrow \bar{C}$ is a non-constant morphism from a very general curve C of genus $g > 1$ onto a curve \bar{C} , then either ϕ is birational, or \bar{C} is rational. As in [7, 9] Hodge theory and deformation theory are the two main methods used to handle our problem. The analysis of the Hodge structure of [5] allows to reduce the problem to the case where the geometric genus and the irregularity of S are zero: $p_g(S) = q(S) = 0$. Therefore we have to deal with the case where S is an elliptic surface. The main new technical obstacle comes out from the fact that the fundamental group $C \times D$ is not Abelian. When S has Kodaira dimension 1, we cannot infer directly neither that the elliptic surface S has bounded topology nor that its moduli space has bounded dimension. To overcome this problem we begin by proving that the first homology group $H_1(S, \mathbb{Z})$ of S , vanishes. Then we use the result of the deformation of curves on elliptic surfaces with multiple fibers proved in [9] to obtain a contradiction. The main ingredient we use to show that $H_1(S, \mathbb{Z}) = 0$ is Theorem 2.6 which provides some restriction on the Hodge structure of a certain Abelian covering of $C \times D$. Theorem 2.6 seems to be of independent interest and to deserve further development.

The last section deals with the case of the 2-symmetric product of a curve. This case is simpler, by using a slightly improvement on the deformation of curves on elliptic surfaces S we can prove (we do not attempt to find the optimal genus)

Theorem 1.2 (=Theorem 4.6). *Let C be a very general curve of genus ≥ 10 . If $f : C_2 \dashrightarrow S$ is a dominant rational map of degree > 1 then S is a rational surface.*

In this paper we work on the field of complex numbers. We use the customary notation in algebraic geometry. For a smooth projective surface S , we let $p_g(S) = h^{2,0}(S)$ be the geometric genus, $q(S) = h^{1,0}(S)$ be the irregularity of S , and $alb(S)$ be the Albanese variety of S . We use the notation \equiv for a linear equivalence of divisors.

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2. Product of curves

This section considers the product of two very general curves $C \times D$ of genus $g \geq 3$ and the dominant rational map $f : C \times D \dashrightarrow S$ where S is a smooth projective surface with $0 \leq \text{kod}(S) \leq 1$.

2.1. Hodge structure on Abelian étale covering of curves

In this subsection we let C and D be curves of genus $g_C \geq 3$ and $g_D \geq 3$ respectively. We provide a proof of Proposition 2.5, which is the main technical result of this paper. We will deal here with the infinitesimal variation of the Hodge structure of Abelian étale covering of $C' \times D' \rightarrow C \times D$. We will show the existence of decomposable elements in any irreducible sub-Hodge structure of $H^2(C' \times D')$ orthogonal to the classes of the algebraic cycles. These decomposable elements provide the role of the *highest weights* and allow us to understand these sub-hodge structures.

Definition 2.1. We call a curve C pairwise torsion free (PTF) if $nA \not\cong nB$ for any pair of distinct points $A, B \in C$ with $A \neq B$, and for any nonzero integer n , i.e., $\mathcal{O}_C(A - B)$ is not a torsion element of $\text{Pic}^0(C)$.

Remark 2.2. A PTF curve C is not hyperelliptic, and its genus $g_C \geq 3$. From the computation of the moduli of the Hurwitz schemes of a curve having a map to \mathbb{P}^1 with 2 total ramification points, we see that the moduli of curves which are not PTF depends on $2g - 1$ parameters. Then a very general curve of genus ≥ 3 is a PTF curve.

We now prove two simple algebraic lemmas that will be helpful in studying the infinitesimal variation of Hodge structure.

Lemma 2.3. *Let C be a PTF curve of genus $g \geq 3$, and let L and M be torsion line bundles in $\text{Pic}^0(C)$. Let $\alpha \in H^0(C, \omega_C(L))$ and $\beta \in H^0(C, \omega_C(M))$ be non trivial sections. Assume that either $L \neq M$ or $L = M$ with $\alpha \neq \lambda\beta$ for any $\lambda \in \mathbb{C}$. Then there exists $\eta \in H^1(C, T_C)$ such that the cup-product $\eta \cdot \alpha \neq 0 \in H^1(C, L)$ and $\eta \cdot \beta = 0 \in H^1(C, M)$.*

Proof. We consider the following subspaces H_β and H_α of $H^0(C, \omega_C^2)$:

$$H_\beta = \beta \cdot H^0(C, \omega_C(-M)); \quad H_\alpha = \alpha \cdot H^0(C, \omega_C(-L)).$$

If we can prove that H_α is not contained in H_β , then by Serre duality we can find an element $\eta \in H^1(T_C) \cong H^0(C, \omega_C^2)^\vee$ such that for $\eta : H^0(C, \omega_C^2) \rightarrow \mathbb{C}$ there holds $\ker(\eta) \not\supset H_\alpha$, but $\ker(\eta) \supset H_\beta$.

Let E and F be the divisor of α and β respectively. Let G be an effective divisor of the maximal degree (with multiplicity) contained in E and F . Since α and β are not proportional $\deg G < 2g - 2$. Moreover, since C is a PTF curve $\deg G < 2g - 3$. In fact if $\deg G = 2g - 3$ then there exist two distinct points P

and Q such that $G + P \equiv \omega_C(L)$, $G + Q \equiv \omega_C(M)$, and $L - M \equiv P - Q$. Since L and M are torsion line bundles, $L - M$ is also a torsion line bundle and we get a contradiction. Next we consider the following exact sequence:

$$0 \rightarrow G - L - M \rightarrow \omega_C(-L) \oplus \omega_C(-M) \xrightarrow{(\alpha, \beta)} \omega_C^2(-G) \rightarrow 0 \quad (2.1)$$

It follows that $H_\alpha + H_\beta$ is the image of the cohomology map

$$H^0(C, \omega_C(-L)) \oplus H^0(C, \omega_C(-M)) \xrightarrow{(\alpha, \beta)} H^0(C, \omega_C^2(-G)) \subset H^0(C, \omega_C^2)$$

and we can identify $H_\beta \cap H_\alpha$ and $H^0(C, G - L - M)$. Since $\deg G - L - M = \deg G \leq 2g - 4$, and C is not hyperelliptic we obtain

$$\dim H_\alpha \cap H_\beta \leq g - 2 < \min(\dim H_\alpha, \dim H_\beta),$$

which proves our lemma. □

Lemma 2.4. *Let C be a curve of genus ≥ 3 . Let $L \in \text{Pic}^0(C)$, and let $\alpha \in H^0(C, \omega_C(L))$, with $\alpha \neq 0$. Then*

- (1) *for any $\gamma \in H^1(C, L)$, there is an element $\eta \in H^1(C, T_C)$ such that $\eta \cdot \alpha = \gamma$;*
- (2) *there is an element $\eta \in H^1(T_C)$ such that the map $H^0(C, \omega_C(L)) \xrightarrow{\eta} H^1(C, L)$ is an isomorphism.*

Proof. Let E be again the divisor of α and consider the exact sequence:

$$0 \rightarrow T_C \xrightarrow{\alpha} L \rightarrow L_E \rightarrow 0 \quad \text{where } L_E = \mathcal{O}_E(L).$$

Since $h^1(L_E) = 0$ it follows that $H^1(T_C) \xrightarrow{\alpha} H^1(C, L)$ is surjective. This proves the first part of the lemma. For the second part we consider the coboundary map:

$$\partial : H^0(C, L_E) \rightarrow H^1(C, T_C).$$

We have that $h^0(L_E) = \deg E = 2g - 2$. Therefore $\Gamma_\alpha = \partial(H^0(C, L_E))$ has dimension $2g - 3$ if $L = \mathcal{O}_C$ or $2g - 2$ if $L \neq \mathcal{O}_C$. We remark that

$$\eta \in \Gamma_\alpha \iff \alpha \in \ker \eta.$$

So we have to show that $\bigcup_\alpha \Gamma_\alpha \neq H^1(C, T_C)$. In fact every element in $H^1(T_C) \setminus \bigcup_\alpha \Gamma_\alpha$ defines an isomorphism $H^0(C, \omega_C(L)) \xrightarrow{\eta} H^1(C, L)$.

This will be done by a dimension count passing to the associated projective spaces. Consider the projective space $\mathbb{P} = \mathbb{P}H^1(T_C)$, $\dim \mathbb{P} = 3g - 4$, we let $\mathbb{P}_\alpha \subset \mathbb{P}$ be the sub-projective space associated to Γ_α . We have to show that $\bigcup_\alpha \mathbb{P}_\alpha \neq \mathbb{P}$. Let \mathbb{P}_L be the projective space associated to $H^0(C, \omega_C(L))$. Since $h^0(C, \omega_C(L)) =$

$g - 1$ if L is not trivial and g if it is trivial, we have that $\dim \mathbb{P}_L = g - 2$ and $g - 1$ respectively. Now consider the incidence correspondence $\mathcal{I} \subset \mathbb{P} \times \mathbb{P}_L$:

$$\mathcal{I} = \{((\eta), (\alpha)) \in \mathbb{P} \times \mathbb{P}_L : \eta \cdot \alpha = 0\}.$$

Let π_i for $i = 1, 2$ be the projections. Since π_2 is surjective and the fibers are the \mathbb{P}_α we get $\dim \mathcal{I} = 3g - 5$. Set $Y_L = \pi_1(\mathcal{I}) \subset \mathbb{P}$. We notice that

$$Y_L = \bigcup_{(\alpha) \in \mathbb{P}_L} \mathbb{P}_\alpha = \{(\zeta) : \zeta \in \Gamma_\alpha, (\alpha) \in \mathbb{P}_L\}.$$

Therefore we obtain $\dim Y_L \leq 3g - 5$ in all cases (and the equality must hold since the Y_L can be defined by the vanishing of a determinant). This proves the lemma. \square

Now we set our notation. We let $p > 1$ be a prime number, L_1 be a line bundle on C and L_2 be a line bundle on D , such that $L_1^p = \mathcal{O}_C$ and $L_2^p = \mathcal{O}_D$. We assume that L_1 and L_2 are not trivial, but the case $L_1 = \mathcal{O}_C$ could be considered and it is simpler. Let $f_1 : C' \rightarrow C$ and $f_2 : D' \rightarrow D$ be the étale covering associated to L_1 and L_2 . One has:

$$f_{1*} \mathcal{O}_{C'} = \sum_{i=0}^{p-1} L_1^i \quad f_{2*} \mathcal{O}_{D'} = \sum_{i=0}^{p-1} L_2^i. \tag{2.2}$$

Set $X = C' \times D'$. We have from the Kunnet decomposition $H^{2,0}(X) \cong H^{1,0}(C') \otimes H^{1,0}(D')$, then using the above decomposition (2.2) :

$$\begin{aligned} H^{2,0}(X) &\cong H^{1,0}(C') \otimes H^{1,0}(D'); & H^{0,2}(X) &\cong H^{0,1}(C') \otimes H^{0,1}(D') \\ H^{1,0}(C') &\cong \bigoplus_{i=0}^{p-1} H^0(C, \omega_C(L_1^i)); & H^{0,1}(C') &\cong \bigoplus_{i=0}^{p-1} H^1(C, L_1^{-i}) \\ H^{1,0}(D') &\cong \bigoplus_{i=0}^{p-1} H^0(D, \omega_D(L_2^i)); & H^{0,1}(D') &\cong \bigoplus_{i=0}^{p-1} H^1(D, L_2^{-i}). \end{aligned}$$

We set

$$\begin{aligned} V_{i,j} &\cong H^0(C, \omega_C(L_1^i)) \otimes H^0(D, \omega_D(L_2^j)), \\ V'_{i,j} &\cong H^1(C, L_1^i) \otimes H^1(D, L_2^j). \end{aligned}$$

For the sake of notation we will use the above isomorphism as an identification. Then we will often omit the pull-backs, for instance we write $H^0(C, \omega_C(L_1^i))$ instead of $f_1^* H^0(C, \omega_C(L_1^i))$, etc.

$$H^{2,0}(X) = \bigoplus_{i,j=0}^{p-1} V_{i,j}; \quad H^{0,2}(X) = \bigoplus_{i,j=0}^{p-1} V'_{i,j}. \tag{2.3}$$

One has $V_{i,j} = V_{i+p,j} = V_{i,j+p}$ and by the complex conjugation

$$\overline{V_{i,j}} = V'_{-i,-j} = V'_{p-i,p-j}.$$

Then we get the following proposition.

Proposition 2.5. *Let C and D be very general curves of genus ≥ 3 . Let $\Lambda \subset H^2(C' \times D')$ be a Hodge substructure. If $\Lambda^{2,0} \neq 0$ then for some index a, b , we have $\Lambda^{2,0} \supset V_{a,b}$.*

Proof. We remark that the deformations of $C' \times D'$ that preserve the coverings correspond to the deformations of C and of D . In fact let $\pi_1 : \mathcal{C} \rightarrow \Delta$ and $\pi_2 : \mathcal{D} \rightarrow \Delta'$ be Kuranishi families of C and D respectively, then $\pi : \mathcal{C} \times \mathcal{D} \rightarrow \Delta \times \Delta'$ is a Kuranishi family of $C \times D$. We have $\Delta \times \Delta'$ contractible and a point $p \in \Delta \times \Delta'$ such that $\pi^{-1}(p) = C \times D$. The tangent space T of $\Delta \times \Delta'$ at p is identified with $H^1(C, T_C) \oplus H^1(D, T_D)$ via the Kodaira-Spencer map. Since $\Delta \times \Delta'$ is contractible, the coverings f_1 and f_2 define a family $\pi' : \mathcal{C}' \times \mathcal{D}' \rightarrow \Delta \times \Delta'$ and an étale covering $\mathcal{C}' \times \mathcal{D}' \rightarrow \mathcal{C} \times \mathcal{D}$. This gives the Kuranishi family of the covering map $f_1 \times f_2$. In particular this induces an injective map $T = H^1(C, T_C) \oplus H^1(D, T_D) \rightarrow H^1(C', T_{C'}) \oplus H^1(D', T_{D'})$ that gives an identification of T with the infinitesimal deformations of $C' \times D'$ preserving the covering. We shall use a basic result from the infinitesimal variation of Hodge structures (cf. [13]). Let $\mathcal{H} = R^2\pi_*\mathbb{C}$ be the variation of Hodge structures defined by the Kuranishi family on the second complex cohomology. Let ∇ be the Gauss Manin connection defined on the sections of \mathcal{H} . Let $\mathcal{L} \subset \mathcal{H}$ be a Hodge substructure. It follows that ∇ must preserve the sections of \mathcal{L} . Consider the Hodge decomposition

$$H^2(C' \times D', \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

The infinitesimal variations $\nabla^{1,1} : H^{2,0} \times T \rightarrow H^{1,1}$ and $\nabla^{0,2} : H^{1,1} \times T \rightarrow H^{0,2}$ are defined via the Kodaira-Spencer map. It means that if s is a section of $\mathcal{H}^{p,q} \subset \mathcal{H}$ and $\phi \in T$ then

$$(\nabla_{\phi}s)^{p-1,q+1}(p) = \phi \cdot s(p),$$

where \cdot is the cup-product. This follows from the Griffiths transversality theorem (cf. [13, Section 2.2]). Let $\Lambda = \mathcal{L}_p \subseteq H^2(C' \times D', \mathbb{C})$ be the Hodge substructure. Then we have $\Lambda^{p,q} \subset H^{p,q}$, and then $\nabla^{1,1} : \Lambda^{2,0} \times T \rightarrow \Lambda^{1,1}$ and $\nabla^{0,2} : \Lambda^{1,1} \times T \rightarrow \Lambda^{0,2}$.

Therefore for any pair of the infinitesimal deformations, η and $\zeta \in T$, we get:

$$\eta \cdot \zeta \cdot \Lambda^{2,0} \subset \Lambda^{0,2}. \tag{2.4}$$

We will consider $\eta \in H^1(T_C) \subset T$ and $\zeta \in H^1(T_D) \subset T$. Our first aim is to show that if the space $\Lambda^{2,0}$ is nonzero then it contains some nontrivial decomposable element. For any $\Gamma \in \Lambda^{2,0} \subset H^{2,0}(X)$, we write $\Gamma = \sum \gamma_{i,j}$, $\gamma_{i,j} \in V_{i,j}$. First we will show that there is an element $\Gamma \in \Lambda^{2,0}$ such that the components $\gamma_{i,j} \in V_{i,j}$ are all decomposable: $\gamma_{i,j} = \alpha_{ij} \wedge \beta_{ij}$. (Since our elements are forms we use the wedge product instead of \otimes). Starting with any $\Omega = \sum \omega_{i,j} \neq 0$ and take an index i, j such that $\omega_{i,j} \neq 0$. Write

$$\omega_{i,j} = \sum_{k=1}^s \alpha_k \wedge \beta_k.$$

We assume that the α_k and the β_k are independent, that is the rank of tensor $\omega_{i,j}$ to be s . If $s = 1$ there is nothing to do for the index i, j . Assume $s > 1$. In particular α_1 and α_2 are not proportional. Now by Lemma 2.3 we find $\eta \in H^1(C, T_C)$ such that $\eta \cdot \alpha_1 = 0$ and $\eta \cdot \alpha_2 \neq 0$. And by Lemma 2.4 we have $\zeta \in H^1(D, T_D)$ such that $\zeta : H^0(D, \omega_D(L_2^j)) \rightarrow H^1(D, L_2^j)$ is an isomorphism. Since the Hodge structure Λ must deform with $C \times D$, $\Theta = \zeta \cdot (\eta \cdot \Omega) = \eta \cdot (\zeta \cdot \Omega) \in \Lambda^{0,2}$. The infinitesimal variation of Hodge structure is given by the cup-product, and the cup product commutes with the decomposition (2.3) we get that $\Theta_{i,j} = \zeta \cdot (\eta \cdot \omega_{i,j})$. Then

$$\Theta_{i,j} = \sum_{k=1}^s \eta \cdot \alpha_k \wedge \zeta \cdot \beta_k. \tag{2.5}$$

We remark that the rank cannot increase; moreover since $\eta \cdot \alpha_1 = 0$, one concludes that Θ has rank $\leq s - 1$:

$$\Theta_{i,j} = \sum_{k=2}^s \eta \cdot \alpha_k \wedge \zeta \cdot \beta_k.$$

Since ζ is an isomorphism the vectors $\zeta \cdot \beta_k$ are all independent and $\eta \cdot \alpha_2 \neq 0$, and then we obtain that $\Theta_{i,j} \neq 0$. Now we use the complex conjugation. We define $\tilde{\Omega} = \overline{\Theta} \in \Lambda^{2,0}$, moreover $\Omega'_{p-i,p-j} :$

$$\tilde{\Omega}_{p-i,p-j} = \sum_{k=2}^s \overline{(\eta \cdot \alpha_k)} \wedge \overline{(\zeta \cdot \beta_k)}.$$

is non trivial of rank $\leq s - 1$. As formula (2.5) shows the rank of the component of our tensor cannot increase under the cup product action. We can repeat the above operation for any index i', j' . Finally we find $0 \neq \Omega' \in \Lambda^{2,0}$ such that $\Omega' = \sum \alpha_{ij} \wedge \beta_{ij}$. Next we would like to show that we can find a decomposable non trivial element $\alpha_{ij} \wedge \beta_{ij} \in \Lambda^{2,0}$. We start this time from $\Omega' = \sum_{ij} \alpha_{ij} \wedge \beta_{ij}$. If its rank is one then we have done. We may assume the rank is $r \geq 2$, and then that for two pairs of different indices ij and $i'j'$ $\alpha_{ij} \wedge \beta_{ij} \neq 0$ and $\alpha_{i'j'} \wedge \beta_{i'j'} \neq 0$. We have $i \neq i'$ or $j \neq j'$. By the symmetry of the hypothesis on C and D we may assume $i \neq i'$. By Lemma 2.3 we can find $\eta \in H^1(C, T_C)$ such that $\eta \cdot \alpha_{ij} \neq 0$ and $\eta \cdot \alpha_{i'j'} = 0$, and $\zeta \in H^1(D, T_D)$ such that $\zeta \cdot \beta_{ij} \neq 0$. It follows then $\Theta = \zeta \cdot (\eta \cdot \Omega) \in \Lambda^{0,2} :$

$$\Theta'_{i,j} = \zeta \cdot (\eta \cdot \Omega')_{i,j} = \eta \cdot \alpha_{ij} \wedge \zeta \cdot \beta_{ij} \neq 0$$

and $\Theta'_{i',j'} = 0$. Taking $\Omega'' = \overline{\Theta'}$ we get a non trivial element in $\Lambda^{2,0}$ with rank $0 < r' < r$. Repeating the operation we can find an element

$$0 \neq \alpha \wedge \beta \in V_{a,b} \cap \Lambda^{2,0}.$$

Finally we see that $\eta \cdot \alpha \wedge \zeta \cdot \beta$ belongs to $\Lambda^{0,2}$ for all $\eta \in H^1(C, T_C)$ and $\zeta \in H^1(D, T_D)$. Using the first part of Lemma 2.4 we get that all the decomposable elements in $V'_{a,b}$ are in $\Lambda^{0,2}$. This implies $V'_{a,b} \subset \Lambda^{0,2}$. Therefore $V_{p-a,p-b} \subset \Lambda^{2,0}$. \square

Now we can prove the following:

Theorem 2.6. *Let C and D be very general curves of genus $g \geq 3$. Let $f_1 : C' \rightarrow C$ and $f_2 : D' \rightarrow D$ as above where f_i for $i = 1, 2$ are étale coverings of prime order p . Let $X = C' \times D'$. Assume that $\Lambda \subset H^2(X)$ is a Hodge substructure such that $\Lambda^{2,0} \neq 0$ ($\Lambda^{2,0} \subset H^{2,0} = H^0(X, K_X)$). Let $|\Lambda^{2,0}|$ be the corresponding sublinear series of $H^0(X, K_X)$. Then the image of the rational map $X = C' \times D' \dashrightarrow |\Lambda^{2,0}|$ has dimension 2.*

Proof. From Proposition 2.5 we can find indices i, j such that $\Lambda^{2,0} \supset V_{i,j} = H^0(C, \omega_C(L_1^i)) \otimes H^0(D, \omega_D(L_2^j))$. It is enough to show that the image of

$$X \dashrightarrow |V_{i,j}|$$

has dimensions 2. This is clear since it factorizes through

$$C \times D \dashrightarrow |H^0(C, \omega_C(L_1^i)) \otimes H^0(D, \omega_D(L_2^j))|.$$

Since C and D have both genus ≥ 3 , then $h^0(C, \omega_C(L_1^i)) \geq 2$ and $h^0(D, \omega_D(L_2^j)) \geq 2$. The result follows by using the Segre embedding. \square

For the sake of completeness, we can consider $P(C', C) \subset H^1(C)$, and $P(D', D) \subset H^1(D)$ respectively. These are the Hodge structures corresponding to the kernel of the norm mappings $H^1(C') \rightarrow H^1(C)$, and $H^1(D') \rightarrow H^1(D)$ respectively. Clearly $P(C', C)$ corresponds to the Prym variety of the covering $C' \rightarrow C$. The result of Theorem 2.1 and the examination of the monodromy on the torsion line bundle imply the following proposition.

Proposition 2.7. *Let $C, D, C'D'$ be as above. Assume that the covering $C' \rightarrow C$ and $D' \rightarrow D$ are non-trivial. There are exactly 4 irreducible Hodge substructures of $H^2(C' \times D')$ with non trivial $(2, 0)$ part: $H^1(C) \otimes H^1(D)$, $P(C', C) \otimes H^1(D)$, $H^1(C) \otimes P(D', D)$ and $P(C', C) \otimes P(D', D)$.*

We remark that

$$(P(C', C) \otimes P(D', D))^{2,0} = \oplus V_{i,j}, \text{ for } 0 < i, j < p.$$

2.2. Torsion freeness

This subsection proves our main technical result, which is Theorem 2.9. Let C and D be two very general curves of genus $g \geq 3$. Suppose there is a dominant rational map $f : C \times D \dashrightarrow S$ where S is a smooth projective surface with $0 \leq \text{kod}(S) \leq 1$.

Lemma 2.8. *We have $p_g(S) = q(S) = 0$ and S has an elliptic fibration $\pi : S \rightarrow \mathbb{P}^1$.*

Proof. We first prove that the geometric genus vanishes. The lemma follows from by the irreducibility of the transcendental Hodge structure of H^2 of a very general product. That is for a very general product, the Néron Severi group $NS(C \times D)$ of $C \times D$ has rank two generated by the classes of $C \times \{x\}$ and $\{y\} \times D$ for $(y, x) \in C \times D$. The Hodge structure orthogonal to the image of the algebraic cycle is irreducible (see for instance [5, Lemma 2.7 and Lemma 4.1] (or apply the argument of the previous subsection to $L_1 = \mathcal{O}_C$ and $L_2 = \mathcal{O}_D$). Since the map $f^* : H^{2,0}(S) \rightarrow H^{2,0}(C \times D)$ is injective, then either $H^{2,0}(S) = 0$ or f^* is an isomorphism. The second case is impossible, otherwise the canonical map would factorize through f and the canonical map of S . Now the canonical map of $C \times D$ gives an embedding, but the canonical image of S is not a surface since Kodaira dimension of S is < 2 . Now we prove that $q(S) = 0$. Assume by contradiction that $q(S) > 0$. Let $a : S \rightarrow \text{alb}(S)$ be the albanese map. Since $p_g(S) = 0$, the image of a must be a smooth curve E of genus $q(S) > 0$. The composition of f with a define a map $C \times D \rightarrow \text{alb}(S)$ and therefore $f_1 : C \rightarrow E$ and $f_2 : D \rightarrow E$. Since C and D are very general it follows that one of these is an isomorphism and the other one is constant. Suppose that $E \cong C$ and f_2 is constant, we have $q(S) = g_C > 1$. It follows that Kodaira dimension S is 1 and that $a : S \rightarrow E$ is an elliptic fibration. Set $E = C$. Since f_2 is constant, we have a commutative fibration diagram

$$\begin{array}{ccc}
 C \times D & \xrightarrow{f} & S \\
 \searrow g & & \swarrow a \\
 & C &
 \end{array}$$

It follows that the curve D maps onto to elliptic fibration a . Since D is very general this is impossible, and we obtain a contradiction.

The final statement follows from the theory of classification of the surfaces (see for instance [4]). □

Since the fundamental group $C \times D$ is not Abelian, the fundamental group of S need not be Abelian. But we will prove the following theorem by using Theorem 2.6.

Theorem 2.9. *We have $H_1(S, \mathbb{Z}) = 0$, that is $H^2(S, \mathbb{Z})$ is torsion free and $q(S) = 0$.*

Proof. Let S be a minimal surface of Kodaira dimension 0 or 1, and K_S be the canonical divisor of S . Then we have $K_S^2 = 0$. We assume by contradiction that

there is a dominant rational map

$$f : C \times D \dashrightarrow S$$

where C and D are very general curves and $H_1(S, \mathbb{Z}) \neq 0$. Since $H_1(S, \mathbb{Z})$ is a finite generated Abelian group we can find a prime p , and a surjection $H_1(S, \mathbb{Z}) \rightarrow \mathbb{Z}/p\mathbb{Z}$. Therefore we have a surjection $\psi : \pi_1(S) \rightarrow \mathbb{Z}/p\mathbb{Z}$. Let $q : S' \rightarrow S$ be the étale covering associated to $\ker(\psi)$. Let $K_{S'}$ be the canonical class of S' we have $K_{S'}^2 = f^*(K_S)^2 = 0$, then it follows that S' has Kodaira dimension 0 or 1. We also have the following proportionality result for the topological and the holomorphic Euler characteristic: $c_2(S') = p \cdot c_2(S)$ and $\chi(\mathcal{O}_{S'}) = p \cdot \chi(\mathcal{O}_S)$.

By composition with f (after a suitable resolution) we get

$$\psi' : \pi_1(C \times D) = \pi_1(C) \times \pi_1(D) \rightarrow \pi_1(S) \rightarrow \mathbb{Z}/p\mathbb{Z}.$$

Claim: The map ψ' is surjective.

First note that either ψ' is a surjection or it is the zero map. Assume by contradiction $\psi' = 0$. In this case we have a lifting map $f' : C \times D \dashrightarrow S'$. Then by Lemma 2.8 either f' is birational or $p_g(S') = q(S') = 0$. The first case is not possible since S' has Kodaira dimension ≤ 1 . Now assume $p_g(S') = q(S') = 0$. We get $\chi(\mathcal{O}_{S'}) = 1 - q(S') - p_g(S') = 1 = p \cdot \chi(\mathcal{O}_S) = p$. This also gives a contradiction.

We have proven $\psi' : \pi_1(C \times D) \rightarrow \mathbb{Z}/p\mathbb{Z}$ is surjective. This gives two maps $\pi_1(C) \rightarrow \mathbb{Z}/p\mathbb{Z}$ and $\pi_1(D) \rightarrow \mathbb{Z}/p\mathbb{Z}$. They produce two étale coverings $C' \rightarrow C$ and $D' \rightarrow D$ (one of the two coverings can be trivial).

We have remarked that $\text{kod}(S') \leq 1$ since $K_{S'}^2 = 0$, then we have $\chi(\mathcal{O}_S) = p_g(S) - q(S) + 1 = 1$ by Lemma 2.8 and therefore $\chi(\mathcal{O}_{S'}) = p\chi(\mathcal{O}_S) = 1$. It follows then $\dim H^{2,0}(S') = p_g(S') = \chi(\mathcal{O}_{S'}) + q(S') - 1 \geq p - 1 \geq 1$.

Now we take a resolution f'' of f' , that is a smooth projective surface Y , a birational map $\epsilon : Y \rightarrow C' \times D'$ and a morphism $f'' : Y \rightarrow S'$ such that $f'' \circ \epsilon = f'$. We define then the substructure $\Lambda \subset H^2(C' \times D')$ as follows :

$$\Lambda = \epsilon_* f''^*(H^2(S'))$$

is a Hodge substructure of $H^2(C' \times D')$ with non zero $(2, 0)$ part, $\Lambda^{2,0} \neq 0$. In fact $\epsilon_* : H^{2,0}(Y) \rightarrow H^{2,0}(C' \times D')$ is injective. By Theorem 2.6 we get that the image $\kappa : C' \times D' \dashrightarrow |\Lambda^{2,0}|$ has dimension 2. On the other hand $\Lambda^{2,0} = \epsilon_* f''^*(H^{2,0}(S')) = f'^*(H^{2,0}(S'))$, then κ must factorize: $\kappa = f' \circ \gamma$ where γ is the canonical map of S' . Since Kodaira dimension of S' is ≤ 1 we get a contradiction. □

3. Dominant rational map

Combining Theorem 2.9 with the result in [5] we will prove our main theorem. We also recall the fact that there is no map from a very general curve C of genus

$g(C) \geq 7$ to S which is birational on its image if S is an elliptic surface with $p_g(S) = q(S) = 0$ of Kodaira dimension 1 and if $\text{Pic}(S)$ is torsion free [9, Remark 4.4.].

Theorem 3.1. *Let C and D be very general curves of genus $g_C \geq 7$ and $g_D \geq 3$ respectively. Then there is no dominant rational map of degree > 1 from $C \times D$ to S where S is a smooth projective surface of $\text{kod}(S) \geq 0$.*

Proof. Let S be a smooth projective surface of $\text{kod}(S) \geq 0$ and let $F : C \times D \dashrightarrow S$ be a dominant map. We have to show that $\text{kod}(S) = -\infty$. In [5] it is shown that there is no dominant rational map from $C \times D \dashrightarrow S$ where C and D are generic and S is of general type. By Theorem 2.9 we may assume that $p_g(S) = 0$ and $H_1(S, \mathbb{Z}) = 0$, this takes care of the case of surfaces with Kodaira dimension 0, because no such surface exists. So we have only to consider the case where $\text{kod}(S) = 1$, where $p_g = q = 0$, and $\text{Pic}(S)$ is torsion free in force of Theorem 2.9. Remark 4.4 in [9] shows that there is no map from a very general curve C of genus $g(C) \geq 7$ to S which is birational on its image. This implies the map F restricted to the fiber $C \times t$ for a general $t \in D$ cannot be birational. Therefore $F(C \times t)$ is a rational curve since C is very general. Therefore S is a ruled surface because a nonempty Zariski open set of S is covered by rational curves, this gives a contradiction. \square

4. Curves on an elliptic surface and symmetric products

4.1. Curves on an elliptic surface

In this subsection we will slightly improve the result of [9] on deformation on curves on elliptic surface. We will give an application of this result in the next subsection.

Let $\pi : S \rightarrow \mathbb{P}^1$ be an elliptic surface (relatively minimal) with $p_g = q = 0$ and of Kodaira dimension 1. Let C be a smooth projective curve of genus $g > 1$. Let

$$\kappa : C \rightarrow S$$

be a birational immersion, that is, the map $C \rightarrow \kappa(C)$ is birational. Let U be an irreducible component of the Kuranishi family of κ .

Proposition 4.1. *Assume that C is neither a hyperelliptic nor a trigonal curve of genus $g > 2$. Then $\dim U \leq g - 2$.*

Proof. We assume that C is a general curve in the family. Since K_S is nef, non-trivial and semi-ample, we have that $\deg \kappa^*(K_S) \geq 1$. The differential of κ induces an exact sequence

$$0 \rightarrow T_C \rightarrow \kappa^*T_S \rightarrow N \rightarrow 0.$$

Let N_{tors} be the torsion of the normal bundle N and $N' = N/N_{\text{tors}}$ be the quotient. We have ([1], [3, Chapter XXI]) $\dim(U) \leq h^0(N')$, $N' \subset K_C \otimes \kappa^*K_S^{-1}$. Since the

curve C is not hyperelliptic, we have to consider only the case where $\deg \kappa^*(K_S) = 1$, by [9, Proposition 2.2]. By examination of the multiple fibers we see that K_S is numerically a multiple $K_S = \rho\lambda$ where $\rho \geq 1$ of the line bundle λ . Then $\deg \kappa^*(K_S) = 1$ implies $\rho = 1$, this is possible only in the following two cases, both with only two multiple fibers (cf. [9, Remark 4.2]):

- (1) the (2, 3) case,
- (2) the (2, 4) case.

In the (2, 3) case, $S = S_{2,3}$ has multiple fibers F_1 and F_2 in the elliptic fibration where $2F_1 = F = 3F_2$ for a general fiber F . Then $F_1 = 3K_S$ and $F_2 = 2K_S$, and it implies $K_S = F_1 - F_2$. If $h^0(\kappa^*(K_S)) = 1$ then $\kappa^*(K_S) = \mathcal{O}_C(P)$ for $P \in C$. On the other hand since $\kappa(C)$ is not contained in the fiber $\kappa^*(F_1) = \mathcal{O}_C(G+E+F)$ and $\kappa^*(F_2) = \mathcal{O}_C(R+S)$ where G, E, F, R, S are point of C . Therefore $R+S+P \equiv G+E+F$, but since C is not trigonal we would have then $\{R, S, P\} = \{G, E, F\}$, which is impossible since the two fibers are distinct.

In the second case we have $K_S = F - F_1 - F_2 = F_1 - F_2 = 3F_2 - F_1$, $2F_1 = F = 4F_2$. If $\kappa^*(K_S) = \mathcal{O}_C(P)$ we have $\kappa^*(F_1) = \mathcal{O}_C(A+B)$ and $\kappa^*(F_2) = \mathcal{O}_C(Q)$. Then $A+B \equiv P+Q$, which is also impossible. \square

Example 4.2. We note that the above result is sharp. Let us consider the (2, 3) case, that is, $S = S_{2,3}$. The elliptic surface $S = S_{2,3}$ is simply connected with $p_g = q = 0$ [6]. We have $6K_S = F$ and $3K_S = F_1$ and $2K_S = F_2$. By Poincaré duality the coset $H = \{\alpha \in H^2(S, \mathbb{Z}) : K_S \cdot \alpha = 1\}$ is not empty. We notice that

- (1) if $\alpha \in H$ then $\alpha^2 = 2s + 1$ is odd,
- (2) if $\alpha \in H$ then $\alpha + K_S \in H$,
- (3) if $\alpha \in H$ then $(\alpha + K_S)^2 = \alpha^2 + 2$.

We choose $\alpha \in H$ such that $\alpha^2 = -3$ and consider the line bundle $L_r = \alpha \otimes rK_S$. Then $L_r^2 = 2r - 3$, and we have from the Riemann Roch theorem $\chi(L_r) = r - 1$ and $L_r(L_r + K_S) = 2(r - 1)$.

By duality $h^2(L_r) = h^0(K_S - L_r) = 0$ since $K_S - L_r = (1 - r)K_S - \alpha$, $-\alpha \cdot F = -6 < 0$, and F is semi-ample. It follows that for $g > 1$ L_g has a global section of arithmetic genus g and the family depends upon $h^0(L_g) - 1 \geq g - 2$ parameters.

4.2. Symmetric product

This subsection is devoted to prove the following.

Proposition 4.3. *Let C be a very general curve of genus $g \geq 6$. Let C_2 be the 2-symmetric product of C . Let S be a surface of Kodaira dimension 1. Then there is no dominant rational map $f : C_2 \dashrightarrow S$.*

Proof. Assume that $f : C_2 \dashrightarrow S$ is a dominant rational map and let $f' : Y \rightarrow S$ be a regular morphism, where f' is a resolution of f . That is there is a birational morphism $\epsilon : Y \rightarrow C_2$. By the irreducibility of the transcendental Hodge structure of $H^2(C_2)$ (see also Subsection 2.1) and the fact that the canonical map is birational

for C_2 we get $p_g(S) = q(S) = 0$. Since $\text{kod}(S) = 1$ we may assume that $\pi : S \rightarrow \mathbb{P}^1$ is a relatively minimal elliptic fibration. We claim that the map on the fundamental groups $f'_* : \pi_1(Y) \rightarrow \pi_1(S)$ is surjective.

We assume by contradiction that $f'_* : \pi_1(Y) \rightarrow \pi_1(S)$ is not surjective. In fact we know that the image Γ of f'_* has a finite index. Assume by contradiction that $\Gamma \neq \pi_1(S)$. Let $\pi : S' \rightarrow S$ be the associated covering of degree $m = [\pi_1(S) : \Gamma]$. By construction we can find a lift $f'' : Y \rightarrow S'$ such that $\pi \circ f'' = f'$. By proportionality we get $\chi(\mathcal{O}_{S'}) = m\chi(\mathcal{O}_S) = m$. Therefore $p_g \geq m - 1$. But again we have that $p_g(S') = 0$ or $S' \cong Y$. In the first case we obtain $m = 1$ and therefore a contradiction, in the second one case we may assume that C_2 is birational to S' so we may assume $S' = Y$ and $f'' = f'$ the resolution of f . Since the map π is étale, $K_{S'}$ is nef and $K_{S'}^2 = K_S^2 = 0$. But since Y is birational to C_2 , which is a surface of general type, we find a contradiction.

Since $\pi_1(Y) \cong \pi_1(C_2) \cong H_1(C, \mathbb{Z})$, then $\pi_1(S)$ is Abelian. Therefore $\pi : S \rightarrow \mathbb{P}^1$ must have exactly two multiple fibers since it is non-rational [6]. We have $H^2(T_S) = 0$, and the deformation of S depend upon 10 parameters (see [9, Proposition 3.4]). Let C be a very general curve of genus $g \geq 6$. For any point P of C , let $c_P : C \rightarrow C_2$ be the embedding (called a coordinate curve)

$$c_P(Q) = P + Q.$$

Set $C_P = c_P(C)$. We remark that $C_P^2 = 1$ and this gives an ample divisor of C_P .

Consider the composition map $f_P = f \circ c_P : C \rightarrow S$. Then for general P , f_P is birational onto its image: otherwise, C can be mapped non-trivially only on \mathbb{P}^1 since C is a very general curve in the moduli space of curves of genus g . But f is dominant and S is not ruled.

It follows that we have a family of birational immersions of dimension $3g - 3 + 1$ (the dimension of the moduli plus the one due to the coordinates curves $C_{P \in C}$ in C_2). We remark that C is not trigonal. Since the deformations of our surfaces depend on 10 dimensional moduli, in some fixed surface S we must find at least a $3g - 2 - 10$ dimensional family of birational immersions. Therefore by Proposition 4.1 we have

$$3g - 2 - 10 \leq g - 2 \text{ and } 2g \leq 10$$

we get $g \leq 5$. □

Remark 4.4. The same argument for the surjection of the map on the fundamental groups in the proof of Proposition 4.3 works in the proof of Theorem 2.9.

Remark 4.5. When $g = 5$ the above example 4.2 shows that there is a three dimensional family in $S_{2,3}$. Then there is a family of $3g - 2 = 13$ dimensional deformations into the family of elliptic surfaces $S_{2,3}$. It could be however that on a general $S_{2,3}$ we have no isotrivial deformation that is deformation of constant moduli.

For the maps between C_2 and surfaces of general type, we have the inequality $3g - 2 - 19 \leq g - 2$ [7]: that is, $g < 10$. The case $g = 9, 8$ can be also excluded.

When $\text{kod}(S) = 0$ with $p_g = q = 0$, the case of Enriques surfaces can be also excluded by the dimensional count as above or by using the method in the proof of Theorem 2.9.

In conclusion we get the following.

Theorem 4.6. *Let C be a very general curve of genus ≥ 10 and let $f : C_2 \dashrightarrow S$ be a dominant rational map of degree ≥ 2 . Then S is a rational surface.*

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