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N-Laplacian problems with critical Trudinger-Moser nonlinearities

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Abstract. We prove existence and multiplicity results for an N-Laplacian problem with a critical exponential nonlinearity that is a natural analog of the Brezis-Nirenberg problem for the borderline case of the Sobolev inequality. This extends results in the literature for the semilinear case $N = 2$ to all $N \geq 2$. When $N > 2$ the nonlinear operator $-\Delta_N$ has no linear eigenspaces and hence this extension requires new abstract critical point theorems that are not based on linear subspaces. We prove new abstract results based on the $\mathbb{Z}_2$-cohomological index and a related pseudo-index that are applicable here.

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1. Introduction and main results

Elliptic problems with critical nonlinearities have been widely studied in the literature. Let Ω be a bounded domain in $\mathbb{R}^N$, for $N \geq 2$. In a celebrated paper [6], Brezis and Nirenberg considered the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

when $N \geq 3$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Among other things, they proved that this problem has a positive solution when $N \geq 4$ and $0 < \lambda < \lambda_1$, where $\lambda_1 > 0$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω. Capozzi *et al*. [7] extended this result by proving the existence of a nontrivial solution when $N = 4$ and $\lambda > \lambda_1$ is not an eigenvalue, and when $N \geq 5$ and $\lambda \geq \lambda_1$. García Azorero and Peral Alonso [15], Egnell [13], and Guedda and Véron [16] studied

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the corresponding problem for the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

when $N > p > 1$, where $p^* = Np/(N-p)$. They proved that this problem has a positive solution when $N \geq p^2$ and $0 < \lambda < \lambda_1(p)$, where $\lambda_1(p) > 0$ is the first Dirichlet eigenvalue of $-\Delta_p$ in Ω. Degiovanni and Lancelotti [12] extended their result by proving the existence of a nontrivial solution when $N \geq p^2$ and $\lambda > \lambda_1(p)$ is not an eigenvalue, and when $N^2/(N+1) > p^2$ and $\lambda \geq \lambda_1(p)$ (see also Arioli and Gazzola [3]).

In the borderline case $N = p \geq 2$, the critical growth is of exponential type and is governed by the Trudinger-Moser inequality

$$\sup_{u \in W_0^{1,N}(\Omega),\ \|\nabla u\|_N \leq 1} \int_\Omega e^{\alpha_N |u|^{N'}} dx < \infty, \tag{1.3}$$

where $W_0^{1,N}(\Omega)$ is the usual Sobolev space with the norm $\|\nabla u\|_N = \left(\int_\Omega |\nabla u|^N dx\right)^{1/N}$, we set $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, denote by ω_{N-1} the area of the unit sphere in $\mathbb{R}^N$, and $N' = N/(N-1)$ (see Trudinger [23] and Moser [18]). A natural analog of problem (1.2) for this case is

$$\begin{cases} -\Delta_N u = \lambda |u|^{N-2} u e^{|u|^{N'}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Delta_N u = \operatorname{div}\left(|\nabla u|^{N-2} \nabla u\right)$ is the N-Laplacian of u. A result by Adimurthi [1] implies that this problem has a nonnegative and nontrivial solution when $0 < \lambda < \lambda_1(N)$, where $\lambda_1(N) > 0$ is the first Dirichlet eigenvalue of $-\Delta_N$ in Ω (see also do Ó [17]). Theorem 1.4 in de Figueiredo *et al*. [9, 10] implies that the semilinear problem

$$\begin{cases} -\Delta u = \lambda u e^{u^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

has a nontrivial solution when $N = 2$ and $\lambda \geq \lambda_1$. In the present paper we first prove the existence of a nontrivial solution of problem (1.4) when $N \geq 3$ and $\lambda > \lambda_1(N)$ is not an eigenvalue. We have the following theorem:

Theorem 1.1. *If $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta_N$ in Ω, then problem* (1.4) *has a nontrivial solution.*

This extension to the quasilinear case is nontrivial. Indeed, the linking argument based on eigenspaces of $-\Delta$ in de Figueiredo *et al.* [9,10] does not work when $N \geq 3$ since the nonlinear operator $-\Delta_N$ does not have linear eigenspaces. We will use a more general construction based on sublevel sets as in Perera and Szulkin [21] (see also Perera *et al.* [20, Proposition 3.23]). Moreover, the standard sequence of eigenvalues of $-\Delta_N$ based on the genus does not give enough information about the structure of the sublevel sets to carry out this linking construction. Therefore we will use a different sequence of eigenvalues introduced in Perera [19] that is based on a cohomological index, and show that problem (1.4) has a nontrivial solution if $\lambda > 0$ is not an eigenvalue from this particular sequence.

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [14] is defined as follows. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup \left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\} & A \neq \emptyset \\ 0 & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, for $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

The Dirichlet spectrum of $-\Delta_N$ in Ω consists of those $\lambda \in \mathbb{R}$ for which the problem

$$\begin{cases} -\Delta_N\, u = \lambda\, |u|^{N-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

has a nontrivial solution. Although a complete description of the spectrum is not yet known when $N \geq 3$, we can define an increasing and unbounded sequence of eigenvalues via a suitable minimax scheme. The standard scheme based on the genus does not give the index information necessary to prove Theorem 1.1, so we will use the following scheme based on the cohomological index as in Perera [19]. Let

$$\Psi(u) = \frac{1}{\displaystyle\int_\Omega |u|^N\, dx}, \quad \text{for } u \in \mathcal{M} = \left\{u \in W_0^{1,N}(\Omega) : \int_\Omega |\nabla u|^N\, dx = 1\right\}. \tag{1.7}$$

Then eigenvalues of problem (1.6) on $\mathcal{M}$ coincide with critical values of Ψ. We use the standard notation

$$\Psi^a = \{u \in \mathcal{M} : \Psi(u) \leq a\}, \text{ and } \Psi_a = \{u \in \mathcal{M} : \Psi(u) \geq a\}, \text{ for } a \in \mathbb{R}$$

for the sublevel sets and superlevel sets, respectively. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$ and set

$$\lambda_k(N) := \inf_{M \in \mathcal{F},\ i(M) \geq k} \ \sup_{u \in M} \Psi(u), \text{ for } k \in \mathbb{N}.$$

Then $0 < \lambda_1(N) < \lambda_2(N) \leq \lambda_3(N) \leq \cdots \to +\infty$ is a sequence of eigenvalues of problem (1.6) and

$$\lambda_k(N) < \lambda_{k+1}(N) \implies i(\Psi^{\lambda_k(N)}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}(N)}) = k \tag{1.8}$$

(see Perera *et al*. [20, Propositions 3.52 and 3.53]). Proof of Theorem 1.1 will make essential use of (1.8).

Now we turn to the question of the multiplicity of solutions to problem (1.4). Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ be the Dirichlet eigenvalues of $-\Delta$ in Ω, repeated according to multiplicity, let

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$$

be the best constant for the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ when $N \geq 3$, and let $|\cdot|$ denote the Lebesgue measure in $\mathbb{R}^N$. Cerami *et al*. [8] proved that if $\lambda_k \leq \lambda < \lambda_{k+1}$ and

$$\lambda > \lambda_{k+1} - \frac{S}{|\Omega|^{2/N}},$$

and m denotes the multiplicity of λ_{k+1}, then problem (1.1) has m distinct pairs of nontrivial solutions $\pm u_j^\lambda$, for $j = 1, \ldots, m$ such that $u_j^\lambda \to 0$ as $\lambda \nearrow \lambda_{k+1}$. A result of Adimurthi and Yadava [2] implies that there exists a constant $\mu_k \in [\lambda_k, \lambda_{k+1})$ such that if $\mu_k < \lambda < \lambda_{k+1}$, then the same conclusion holds for problem (1.5) when $N = 2$. We prove a similar bifurcation result for problem (1.4) when $N \geq 3$. We have the following theorem:

Theorem 1.2. *If $N \geq 3$, and $\lambda_k(N) < \lambda < \lambda_{k+1}(N) = \cdots = \lambda_{k+m}(N)$ for some $k, m \in \mathbb{N}$, and*

$$\lambda > \lambda_{k+1}(N) - \left(\frac{N\alpha_N^{N-1}}{|\Omega|}\right)^{1/N} \lambda_k(N)^{1/N'}, \tag{1.9}$$

then problem (1.4) *has m distinct pairs of nontrivial solutions $\pm u_j^\lambda$, for $j = 1, \ldots, m$ such that $u_j^\lambda \to 0$ as $\lambda \nearrow \lambda_{k+1}(N)$.*

The abstract result of Bartolo *et al*. [4] used in Cerami *et al*. [8] and Adimurthi and Yadava [2] is based on linear subspaces and therefore cannot be used to prove Theorem 1.2. We will prove a more general critical point theorem based on a pseudo-index related to the cohomological index that is applicable here (see also Perera *et al*. [20, Proposition 3.44]).

In closing the introduction we remark that we have confined ourselves to the model problem (1.4) only for the sake of simplicity. The methods developed in this paper can be easily adapted to treat nonlinearities more general than $|u|^{N-2}\, ue^{|u|^{N'}}$ as in Adimurthi [1], Adimurthi and Yadava [2], de Figueiredo *et al.* [9,10], and do Ó [17].

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2. Abstract critical point theorems

In this section we prove two abstract critical point theorems based on the cohomological index that we will use to prove Theorems 1.1 and 1.2. The following proposition summarizes the basic properties of the cohomological index:

Proposition 2.1 (Fadell-Rabinowitz [14]). *The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:*

- (i_1) *Definiteness: $i(A) = 0$ if and only if $A = \emptyset$;*
- (i_2) *Monotonicity: If there is an odd continuous map from A to B (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism;*
- (i_3) *Dimension: $i(A) \leq \dim W$;*
- (i_4) *Continuity: If A is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of A such that $i(N) = i(A)$. When A is compact, N may be chosen to be a δ-neighborhood $N_\delta(A) = \{u \in W : dist\,(u, A) \leq \delta\}$;*
- (i_5) *Subadditivity: If A and B are closed, then $i(A \cup B) \leq i(A) + i(B)$;*
- (i_6) *Stability: If SA is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$;*
- (i_7) *Piercing property: If A, A_0 and A_1 are closed, and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous map such that $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, the image $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$ and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A)$;*
- (i_8) *Neighborhood of zero: If U is a bounded closed symmetric neighborhood of 0, then $i(\partial U) = \dim W$.*

Let

$$S = \{u \in W : \|u\| = 1\}$$

be the unit sphere in W and let

$$\pi : W \setminus \{0\} \to S$$
$$u \mapsto \frac{u}{\|u\|}$$

be the radial projection onto S. The following abstract result generalizes the linking theorem of Rabinowitz [22].

Theorem 2.2. *Let Φ be a C^1-functional on W and let $A_0,\ B_0$ be disjoint nonempty closed symmetric subsets of S such that*

$$i(A_0) = i(S \setminus B_0) < \infty. \tag{2.1}$$

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup \Phi(A) \le \inf \Phi(B), \qquad \sup \Phi(X) < \infty,$$

where

$$A = \{tu : u \in A_0,\ 0 \le t \le R\} \cup \{R\,\pi((1-t)\,u + tv) : u \in A_0,\ 0 \le t \le 1\},$$

$$B = \{ru : u \in B_0\},$$

$$X = \{tu : u \in A,\ \|u\| = R,\ 0 \le t \le 1\}.$$

Let $\Gamma = \left\{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\right\}$ and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u).$$

Then

$$\inf \Phi(B) \le c \le \sup \Phi(X), \tag{2.2}$$

in particular, c is finite. If, in addition, Φ satisfies the $(\mathrm{PS})_c$ *condition, then c is a critical value of Φ.*

Proof. First we show that A (homotopically) links B with respect to X in the sense that

$$\gamma(X) \cap B \ne \emptyset \quad \forall \gamma \in \Gamma. \tag{2.3}$$

If (2.3) does not hold, then there is a map $\gamma \in C(X, W \setminus B)$ such that $\gamma(X)$ is closed and $\gamma|_A = id_A$. Let

$$\widetilde{A} = \{R\,\pi((1-|t|)\,u + tv) : u \in A_0, \text{ and } -1 \le t \le 1\}$$

and note that $\widetilde{A}$ is closed since A_0 is closed (here $(1-|t|)\,u + tv \ne 0$ since v is not in the symmetric set A_0). Since

$$\begin{aligned} SA_0 &\to \widetilde{A}, \\ (u,t) &\mapsto R\,\pi((1-|t|)\,u + tv) \end{aligned}$$

is an odd continuous map,

$$i(\widetilde{A}) \ge i(SA_0) = i(A_0) + 1 \tag{2.4}$$

by (i_2) and (i_6) of Proposition 2.1. Consider the map $\varphi : \widetilde{A} \times [0, 1] \to W \setminus B$, given by

$$\varphi(u, t) = \begin{cases} \gamma(tu) & u \in \widetilde{A} \cap A \\ -\gamma(-tu) & u \in \widetilde{A} \setminus A, \end{cases}$$

which is continuous since γ is the identity on the symmetric set $\{tu : u \in A_0,$ and $0 \le t \le R\}$. We have $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in \widetilde{A} \times [0, 1]$, and $\varphi(\widetilde{A} \times [0, 1]) = \gamma(X) \cup -\gamma(X)$ is closed, and $\varphi(\widetilde{A} \times \{0\}) = \{0\}$ and $\varphi(\widetilde{A} \times \{1\}) = \widetilde{A}$ since $\gamma|_A = id_A$. Applying (i_7) with $\widetilde{A}_0 = \{u \in W : \|u\| \le r\}$ and $\widetilde{A}_1 = \{u \in W : \|u\| \ge r\}$ gives

$$i(\widetilde{A}) \le i(\varphi(\widetilde{A} \times [0, 1]) \cap \widetilde{A}_0 \cap \widetilde{A}_1) \le i((W \setminus B) \cap S_r) = i(S_r \setminus B) = i(S \setminus B_0), \quad (2.5)$$

where $S_r = \{u \in W : \|u\| = r\}$. By (2.4) and (2.5), $i(A_0) < i(S \setminus B_0)$, contradicting (2.1). Hence (2.3) holds.

It follows from (2.3) that $c \ge \inf \Phi(B)$, and $c \le \sup \Phi(X)$ since $id_X \in \Gamma$. If Φ satisfies the $(\mathrm{PS})_c$ condition, then c is a critical value of Φ by the classical minimax principle (see, *e.g.*, Perera *et al.* [20]). □

Remark 2.3. The linking construction in the proof of Theorem 2.2 was used in Perera and Szulkin [21] to obtain nontrivial solutions of p-Laplacian problems with nonlinearities that interact with the spectrum. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [11]. See also Perera *et al.* [20, Proposition 3.23].

Now let Φ be an even C^1-functional on W and let $\mathcal{A}^*$ denote the class of symmetric subsets of W. Let $r > 0$, let $S_r = \{u \in W : \|u\| = r\}$, let $0 < b \le +\infty$, and let Γ denote the group of odd homeomorphisms of W that are the identity outside $\Phi^{-1}(0, b)$. The pseudo-index of $M \in \mathcal{A}^*$ related to i, S_r, and Γ is defined by

$$i^*(M) = \min_{\gamma \in \Gamma} i(\gamma(M) \cap S_r)$$

(see Benci [5]). The following critical point theorem generalizes Bartolo *et al.* [4, Theorem 2.4].

Theorem 2.4. *Let A_0, B_0 be symmetric subsets of S such that A_0 is compact, B_0 is closed, and*

$$i(A_0) \ge k + m, \qquad i(S \setminus B_0) \le k$$

for some $k, m \in \mathbb{N}$. Assume that there exists $R > r$ such that

$$\sup \Phi(A) \le 0 < \inf \Phi(B), \qquad \sup \Phi(X) < b,$$

where $A = \{Ru : u \in A_0\}$, $B = \{ru : u \in B_0\}$, and $X = \{tu : u \in A, 0 \le t \le 1\}$. For $j = k + 1, \dots, k + m$, let

$$\mathcal{A}_j^* = \{M \in \mathcal{A}^* : M \text{ is compact and } i^*(M) \ge j\}$$

and set

$$c_j^* := \inf_{M \in \mathcal{A}_j^*} \max_{u \in M} \Phi(u).$$

Then

$$\inf \Phi(B) \le c_{k+1}^* \le \cdots \le c_{k+m}^* \le \sup \Phi(X),$$

in particular, $0 < c_j^* < b$. *If, in addition,* Φ *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c \in (0, b)$, *then each* c_j^* *is a critical value of* Φ *and there are* m *distinct pairs of associated critical points.*

Proof. If $M \in \mathcal{A}_{k+1}^*$,

$$i(S_r \setminus B) = i(S \setminus B_0) \le k < k + 1 \le i^*(M) \le i(M \cap S_r)$$

since $id_W \in \Gamma$. Hence M intersects B by (i_2) of Proposition 2.1. It follows that $c_{k+1}^* \ge \inf \Phi(B)$. If $\gamma \in \Gamma$, consider the continuous map $\varphi : A \times [0,1] \to W$, given by

$$\varphi(u, t) = \gamma(tu).$$

We have $\varphi(A \times [0,1]) = \gamma(X)$, which is compact. Since γ is odd, $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0,1]$ and $\varphi(A \times \{0\}) = \{\gamma(0)\} = \{0\}$. Since $\Phi \le 0$ on A, we have $\gamma|_A = id_A$ and hence $\varphi(A \times \{1\}) = A$. Applying ($i_7$) with $\widetilde{A}_0 = \{u \in W : \|u\| \le r\}$ and $\widetilde{A}_1 = \{u \in W : \|u\| \ge r\}$ gives

$$i(\gamma(X) \cap S_r) = i(\varphi(A \times [0,1]) \cap \widetilde{A}_0 \cap \widetilde{A}_1) \ge i(A) = i(A_0) \ge k + m.$$

It follows that $i^*(X) \ge k + m$. So $X \in \mathcal{A}_{k+m}^*$ and hence $c_{k+m}^* \le \sup \Phi(X)$. The rest now follows from standard results in critical point theory (see, *e.g.*, Perera *et al*. [20]). □

Remark 2.5. A similar construction was used in Perera and Szulkin [21]. See also Perera *et al*. [20, Proposition 3.44].

3. Variational setting

Solutions of problem (1.4) coincide with critical points of the C^1-functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{N} |\nabla u|^N - \lambda F(u) \right] dx \text{ for } u \in W_0^{1,N}(\Omega),$$

where

$$F(t) = \int_0^t |s|^{N-2} s e^{|s|^{N'}} ds = \int_0^{|t|} s^{N-1} e^{s^{N'}} ds.$$

The following lemma is a special case of a result of Adimurthi [1]:

Lemma 3.1. *Φ satisfies the* $(\mathrm{PS})_c$ *condition for all* $c < \alpha_N^{N-1}/N$.

Let $\mathcal{M}$ and Ψ be as in (1.7). The following lemma implies that for any subset A of $\mathcal{M}$ on which Ψ is bounded, there exists $R > 0$ such that $\Phi(tu) \le 0$ for all $u \in A$ and $t \ge R$.

Lemma 3.2. *For all $u \in \mathcal{M}$ and $t \ge 0$,*

$$\Phi(tu) \le \frac{t^N}{N}\left[1 - \frac{\lambda}{N'\,|\Omega|^{1/(N-1)}}\left(\frac{t}{\Psi(u)}\right)^{N'}\right].$$

Proof. Since $e^t \ge t$ for all $t \ge 0$,

$$F(t) \ge \frac{|t|^{N+N'}}{N+N'} \quad \forall t \in \mathbb{R},$$

so

$$\Phi(tu) \le t^N\left(\frac{1}{N} - \frac{\lambda t^{N'}}{N+N'}\int_\Omega |u|^{N+N'}\,dx\right).$$

By the Hölder inequality,

$$|\Omega|^{1/(N-1)}\int_\Omega |u|^{N+N'}\,dx \ge \left(\int_\Omega |u|^N\,dx\right)^{N'} = \frac{1}{\Psi(u)^{N'}}.$$ □

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Our strategy is to apply Theorem 2.2 with suitable sets defined in terms of the eigenvalues of $-\Delta_N$, for which the minimax level c is below the threshold for compactness given by Lemma 3.1.

Since problem (1.4) has a nontrivial solution when $0 < \lambda < \lambda_1(N)$ by Adimurthi [1], we may assume that $\lambda > \lambda_1(N)$. Then

$$\lambda_k(N) < \lambda < \lambda_{k+1}(N) \tag{4.1}$$

for some k. By Degiovanni and Lancelotti [12, Theorem 2.3], the sublevel set $\Psi^{\lambda_k(N)}$ has a compact symmetric subset C of index k that is bounded in $C^1(\Omega)$. Without loss of generality we may assume that $0 \in \Omega$. For all $m \in \mathbb{N}$ so large that $B_{1/m}(0) \subset \Omega$, let

$$\eta_m(x) = \begin{cases} 0 & |x| \le 1/2\,m^{m+1} \\ 2\,m^m\left(|x| - \dfrac{1}{2\,m^{m+1}}\right) & 1/2\,m^{m+1} < |x| \le 1/m^{m+1} \\ (m\,|x|)^{1/m} & 1/m^{m+1} < |x| \le 1/m \\ 1 & |x| > 1/m \end{cases}$$

(see Zhang *et al*. [24]). Let

$$\pi(u) = \frac{u}{\|u\|}, \text{ for } u \in W_0^{1,N}(\Omega) \setminus \{0\}$$

be the radial projection onto $\mathcal{M}$.

Lemma 4.1. *As* $m \to \infty$,

$$\int_\Omega |\eta_m u|^N \, dx = \int_\Omega |u|^N \, dx + O\left(\frac{1}{m^N}\right); \tag{4.2}$$

$$\int_\Omega |\nabla(\eta_m u)|^N \, dx = 1 + O\left(\frac{1}{m^{N-1}}\right); \tag{4.3}$$

$$\Psi(\pi(\eta_m u)) = \Psi(u) + O\left(\frac{1}{m^{N-1}}\right) \tag{4.4}$$

uniformly in $u \in C$.

Proof. We have

$$\left|\int_\Omega |\eta_m u|^N \, dx - \int_\Omega |u|^N \, dx\right| \leq \int_{B_{1/m}(0)} \left(|\eta_m u|^N + |u|^N\right) dx = \mathrm{O}\left(\frac{1}{m^N}\right)$$

since u is bounded on C and $|\eta_m| \leq 1$, so (4.2) holds. Next

$$\left|\int_\Omega |\nabla(\eta_m u)|^N \, dx - \int_\Omega |\nabla u|^N \, dx\right| \leq \int_{B_{1/m}(0)} \left(|\nabla(\eta_m u)|^N + |\nabla u|^N\right) dx$$

and

$$\int_{B_{1/m}(0)} |\nabla(\eta_m u)|^N \, dx \leq \sum_{j=0}^{N} \binom{N}{j} \int_{B_{1/m}(0)} |\nabla u|^{N-j} \, |u|^j \, |\nabla \eta_m|^j \, dx.$$

Since C is bounded in $C^1(\Omega)$, u and ∇u are bounded, and a direct calculation shows that

$$\int_{B_{1/m}(0)} |\nabla \eta_m|^j \, dx = \mathrm{O}\left(\frac{1}{m^{N-1}}\right), \text{ for } j = 0, \ldots, N,$$

so (4.3) follows. Since

$$\Psi(\pi(\eta_m u)) = \frac{\displaystyle\int_\Omega |\nabla(\eta_m u)|^N \, dx}{\displaystyle\int_\Omega |\eta_m u|^N \, dx},$$

(4.4) is immediate from (4.2) and (4.3). □

Set $C_m = \{\pi(\eta_m u) : u \in C\}$. Since $C \subset \Psi^{\lambda_k(N)}$,

$$\Psi(\pi(\eta_m u)) \le \lambda_k(N) + \mathrm{O}\left(\frac{1}{m^{N-1}}\right) \quad \forall u \in C$$

by (4.4). Using $\lambda_k(N) < \lambda$, we fix m so large that

$$\Psi(u) \le \lambda \quad \forall u \in C_m. \tag{4.5}$$

Then $C_m \subset \mathcal{M} \setminus \Psi_{\lambda_{k+1}(N)}$ since $\lambda < \lambda_{k+1}(N)$, so

$$i(C_m) \le i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}(N)}) = k$$

by (i_2) of Proposition 2.1 and (1.8). On the other hand, $C \to C_m$, and $u \mapsto \pi(\eta_m u)$ is an odd continuous map, hence

$$i(C_m) \ge i(C) = k$$

by (i_2) again. Thus,

$$i(C_m) = k. \tag{4.6}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We apply Theorem 2.2 to our functional Φ with

$$A_0 = C_m, \qquad B_0 = \Psi_{\lambda_{k+1}(N)},$$

noting that (2.1) follows from (4.6), (4.1), and (1.8). Let $R > r > 0$, let $v \in \mathcal{M} \setminus C_m$, and let A, B and X be as in Theorem 2.2. First we show that $\inf \Phi(B) > 0$ if r is sufficiently small. Since $e^t \le 1 + te^t$ for all $t \ge 0$,

$$F(t) \le \frac{|t|^N}{N} + |t|^{N+N'} e^{|t|^{N'}} \quad \forall t \in \mathbb{R},$$

so for $u \in \Psi_{\lambda_{k+1}(N)}$,

$$\begin{aligned} \Phi(ru) &\ge \int_\Omega \left[\frac{r^N}{N} |\nabla u|^N - \frac{\lambda r^N}{N} |u|^N - \lambda r^{N+N'} |u|^{N+N'} e^{r^{N'}|u|^{N'}}\right] dx \\ &\ge \frac{r^N}{N}\left(1 - \frac{\lambda}{\lambda_{k+1}(N)}\right) - \lambda r^{N+N'} \left(\int_\Omega e^{2r^{N'}|u|^{N'}} dx\right)^{1/2} \|u\|_{2(N+N')}^{N+N'}. \end{aligned}$$

If $2\,r^{N'} \le \alpha_N$, then

$$\int_\Omega e^{2r^{N'}|u|^{N'}} dx \le \int_\Omega e^{\alpha_N |u|^{N'}} dx,$$

which is bounded by (1.3). Since $W_0^{1,N}(\Omega) \hookrightarrow L^{2(N+N')}(\Omega)$ and $\lambda < \lambda_{k+1}(N)$, it follows that $\inf \Phi(B) > 0$ if r is sufficiently small. Next we show that $\sup \Phi(A) \leq 0$ if R is sufficiently large. Since $e^t \geq 1$ for all $t \geq 0$,

$$F(t) \geq \frac{|t|^N}{N} \quad \forall t \in \mathbb{R},$$

so for $u \in C_m$ and any $t \geq 0$,

$$\begin{aligned} \Phi(tu) &\leq \int_\Omega \left[\frac{t^N}{N} |\nabla u|^N - \frac{\lambda t^N}{N} |u|^N\right] dx \\ &\leq \frac{t^N}{N}\left(1 - \frac{\lambda}{\Psi(u)}\right) \\ &\leq 0 \end{aligned}$$

by (4.5). Since C is compact and the map $C \to C_m$, given by $u \mapsto \pi(\eta_m u)$ is continuous, C_m is compact, and hence so is the set $\{\pi((1-t)\,u + tv) : u \in C_m, 0 \leq t \leq 1\}$. So Ψ is bounded on this set, and there exists $R > r$ such that $\Phi \leq 0$ on $\{R\,\pi((1-t)\,u + tv) : u \in C_m,\ 0 \leq t \leq 1\}$ by Lemma 3.2.

Now we show that $\sup \Phi(X) < \alpha_N^{N-1}/N$ for a suitably chosen v. Let

$$v_j(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log j)^{(N-1)/N} & |x| \leq 1/j \\ \dfrac{\log |x|^{-1}}{(\log j)^{1/N}} & 1/j < |x| \leq 1 \\ 0 & |x| > 1. \end{cases}$$

Then $v_j \in W^{1,N}(\mathbb{R}^N)$, it satisfies $\|\nabla v_j\|_N = 1$, and $\|v_j\|_N^N = \mathrm{O}(1/\log j)$ as $j \to \infty$. We take $v(x) = \widetilde{v}_j(x) := v_j(x/r_m)$ with $r_m = 1/2\,m^{m+1}$ and j sufficiently large. Since $B_{r_m}(0) \subset \Omega$, $\widetilde{v}_j \in W_0^{1,N}(\Omega)$ and $\|\nabla \widetilde{v}_j\|_N = 1$. For sufficiently large j,

$$\Psi(\widetilde{v}_j) = \frac{1}{r_m^N\ \|v_j\|_N^N} > \lambda$$

and hence $\widetilde{v}_j \notin C_m$ by (4.5). For $u \in C_m$ and $s,\ t \geq 0$,

$$\Phi(su + t\widetilde{v}_j) = \Phi(su) + \Phi(t\widetilde{v}_j)$$

since $u = 0$ on $B_{r_m}(0)$ and $\widetilde{v}_j = 0$ on $\Omega \setminus B_{r_m}(0)$. Since $\Phi(su) \leq 0$, it suffices to show that $\sup_{t \geq 0} \Phi(t\widetilde{v}_j) < \alpha_N^{N-1}/N$ for arbitrarily large j. Since $\Phi(t\widetilde{v}_j) \to -\infty$ as $t \to +\infty$ by Lemma 3.2, there exists $t_j \geq 0$ such that

$$\Phi(t_j\widetilde{v}_j) = \frac{t_j^N}{N} - \lambda \int_{B_{r_m}(0)} F(t_j\widetilde{v}_j)\,dx = \sup_{t \geq 0} \Phi(t\widetilde{v}_j) \tag{4.7}$$

and

$$\Phi'(t_j\widetilde{v}_j)\,\widetilde{v}_j = t_j^{N-1}\left(1-\lambda\int_{B_{rm}(0)}\widetilde{v}_j^N e^{t_j^{N'}\widetilde{v}_j^{N'}}\,dx\right)=0. \tag{4.8}$$

Suppose $\Phi(t_j\widetilde{v}_j) \geq \alpha_N^{N-1}/N$ for all sufficiently large j. Since $F(t) \geq 0$ for all $t \in \mathbb{R}$, then (4.7) gives $t_j^{N'} \geq \alpha_N$, and then (4.8) gives

$$\frac{1}{\lambda} = \int_{B_{rm}(0)}\widetilde{v}_j^N e^{t_j^{N'}\widetilde{v}_j^{N'}}\,dx \geq \int_{B_{rm}(0)}\widetilde{v}_j^N e^{\alpha_N\,\widetilde{v}_j^{N'}}\,dx$$

$$= r_m^N\int_{B_1(0)} v_j^N e^{\alpha_N\,v_j^{N'}}\,dx \geq r_m^N\int_{B_{1/j}(0)} v_j^N e^{\alpha_N\,v_j^{N'}}\,dx = \frac{r_m^N}{N}(\log j)^{N-1},$$

which is impossible for large j.

Now

$$c \leq \sup\Phi(X) < \frac{\alpha_N^{N-1}}{N}$$

by (2.2), so Φ satisfies the $(\mathrm{PS})_c$ condition by Lemma 3.1. Thus, Φ has a critical point u at the level c by Theorem 2.2. Since

$$c \geq \inf\Phi(B) > 0$$

by (2.2) again, u is nontrivial. □

5. Proof of Theorem 1.2

Lemma 5.1. *For all $t \in \mathbb{R}$,*

$$F(t) \leq \frac{|t|^N}{N}\,e^{|t|^{N'}} - \frac{|t|^{N+N'}}{N^2}; \tag{5.1}$$

$$F(t) \leq \frac{|t|^{N-N'}}{N'}\,e^{|t|^{N'}}. \tag{5.2}$$

Proof. Integrating by parts gives

$$F(t) = \frac{|t|^N}{N}\,e^{|t|^{N'}} - \frac{N'}{N}\int_0^{|t|} s^{N+N'-1}e^{s^{N'}}\,ds$$

$$\leq \frac{|t|^N}{N}\,e^{|t|^{N'}} - \frac{N'}{N}\int_0^{|t|} s^{N+N'-1}\,ds = \frac{|t|^N}{N}\,e^{|t|^{N'}} - \frac{|t|^{N+N'}}{N^2}$$

and

$$F(t) = \frac{|t|^{N-N'}}{N'}\,e^{|t|^{N'}} - \frac{N-N'}{N'}\int_0^{|t|} s^{N-N'-1}e^{s^{N'}}\,ds \leq \frac{|t|^{N-N'}}{N'}\,e^{|t|^{N'}}.$$ □

Proof of Theorem 1.2. In view of Lemma 3.1, we apply Theorem 2.4 with $b = \alpha_N^{N-1}/N$. By Degiovanni and Lancelotti [12, Theorem 2.3], the sublevel set $\Psi^{\lambda_{k+m}(N)}$ has a compact symmetric subset A_0 with

$$i(A_0) = k + m.$$

We take $B_0 = \Psi_{\lambda_{k+1}(N)}$, so that

$$i(S \setminus B_0) = k$$

by (1.8). Let $R > r > 0$ and let A, B and X be as in Theorem 2.4. As in the proof of Theorem 1.1, $\inf \Phi(B) > 0$ if r is sufficiently small. Since $A_0 \subset \Psi^{\lambda_{k+1}(N)}$, there exists $R > r$ such that $\Phi \leq 0$ on A by Lemma 3.2. Since $e^t \geq 1 + t$ for all $t \geq 0$,

$$F(t) \geq \frac{|t|^N}{N} + \frac{|t|^{N+N'}}{N+N'} \quad \forall t \in \mathbb{R},$$

so for $u \in X$,

$$\begin{aligned} \Phi(u) &\leq \int_\Omega \left[\frac{1}{N}\,|\nabla u|^N - \frac{\lambda}{N}\,|u|^N - \frac{\lambda}{N+N'}\,|u|^{N+N'}\right] dx \\ &\leq \frac{\lambda_{k+1}(N) - \lambda}{N} \int_\Omega |u|^N\,dx - \frac{\lambda_k(N)}{(N+N')\,|\Omega|^{1/(N-1)}} \left(\int_\Omega |u|^N\,dx\right)^{N'} \\ &\leq \sup_{\rho \geq 0} \left[\frac{(\lambda_{k+1}(N) - \lambda)\,\rho}{N} - \frac{\lambda_k(N)\,\rho^{N'}}{(N+N')\,|\Omega|^{1/(N-1)}}\right] \\ &= \frac{(\lambda_{k+1}(N) - \lambda)^N\,|\Omega|}{N^2\,\lambda_k(N)^{N-1}}. \end{aligned}$$

So

$$\sup \Phi(X) \leq \frac{(\lambda_{k+1}(N) - \lambda)^N\,|\Omega|}{N^2\,\lambda_k(N)^{N-1}} < \frac{\alpha_N^{N-1}}{N}$$

by (1.9). Thus, problem (1.4) has m distinct pairs of nontrivial solutions $\pm\, u_j^\lambda$, for $j = 1, \ldots, m$ such that

$$0 < \Phi(u_j^\lambda) \leq \frac{(\lambda_{k+1}(N) - \lambda)^N\,|\Omega|}{N^2\,\lambda_k(N)^{N-1}}. \tag{5.3}$$

To prove that $u_j^\lambda \to 0$ as $\lambda \nearrow \lambda_{k+1}(N)$, it suffices to show that for every sequence $\nu_n \nearrow \lambda_{k+1}(N)$, a subsequence of $v_n := u_j^{\nu_n}$ converges to zero. We have

$$\Phi(v_n) = \int_\Omega \left[\frac{1}{N}\,|\nabla v_n|^N - \nu_n\,F(v_n)\right] dx \to 0 \tag{5.4}$$

by (5.3) and

$$\Phi'(v_n)\, v_n = \int_\Omega \left[|\nabla v_n|^N - \nu_n\, |v_n|^N e^{|v_n|^{N'}} \right] dx = 0. \tag{5.5}$$

By (5.1), (5.4), and (5.5),

$$\frac{1}{N^2} \int_\Omega |v_n|^{N+N'}\, dx \le \int_\Omega \left[\frac{1}{N}\, |v_n|^N e^{|v_n|^{N'}} - F(v_n) \right] dx = \frac{\Phi(v_n)}{\nu_n} \le \frac{\Phi(v_n)}{\lambda_k(N)} \to 0,$$

so $v_n \to 0$ a.e. in Ω for a renamed subsequence. By (5.2),

$$N' \int_\Omega F(v_n)\, dx \le \int_\Omega |v_n|^{N-N'} e^{|v_n|^{N'}}\, dx =: I_1 + I_2, \tag{5.6}$$

where

$$\begin{aligned} I_1 &= \int_{\left\{|v_n| > (2N/N')^{1/N'}\right\}} |v_n|^{N-N'} e^{|v_n|^{N'}}\, dx \\ &\le \frac{N'}{2N} \int_\Omega |v_n|^N e^{|v_n|^{N'}}\, dx = \frac{N'}{2N\nu_n}\, \|v_n\|^N \end{aligned} \tag{5.7}$$

by (5.5) and

$$I_2 = \int_\Omega \chi_{\left\{|v_n| \le (2N/N')^{1/N'}\right\}}(x)\, |v_n|^{N-N'} e^{|v_n|^{N'}}\, dx \to 0 \tag{5.8}$$

by the Lebesgue dominated convergence theorem. Combining (5.4), (5.6), and (5.7) gives

$$\frac{1}{2N}\, \|v_n\|^N \le \Phi(v_n) + \frac{\lambda_{k+1}(N)}{N'}\, I_2 \to 0$$

by (5.4) and (5.8). □

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