Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVI (2016), 1019-1047

Propagation of strong singularities in semilinear parabolic equations with degenerate absorption

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Abstract. We study equations of the form $(*)$ $u_t - \Delta u + h(x)|u|^{q-1}u = 0$ in a half space $\mathbb{R}^{N+1}_+$. Here $q > 1$ and h is a continuous function in $\mathbb{R}^N$, vanishing at the origin and positive elsewhere. Let $\bar{h}(s) = e^{-\omega(s)/s^2}$ and assume that $\omega(s)/s^2$ is monotone on $(0, 1)$ and tends to infinity as $s \to 0$. We show that, if ω satisfies the Dini condition and $h(x) \geq \bar{h}(|x|)$ then there exists a maximal solution of $(*)$. This solution tends to infinity as $t \to 0$. On the contrary, if the Dini condition in the half space fails and $h(x) \leq \bar{h}(x)$, we construct a sequence of solutions whose initial data shrinks to the Dirac measure with infinite mass at the origin, but the limit of the sequence blows up everywhere on the positive time axis.

Mathematics Subject Classification (2010): 35K58 (primary); 35K15, 35B44 (secondary).

1. Introduction and statement of results

In this paper we study the propagation in time of strong singularities in the initial data, for positive solutions of equations of the form

$$u_t - \Delta u + h(x)|u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^{N+1}_+ := \mathbb{R}_+ \times \mathbb{R}^N. \tag{1.1}$$

We assume that $q > 1, h \in C(\mathbb{R}^N), h(0) = 0$ and $h > 0$ when $x \neq 0$. By a 'strong singularity' we mean a singularity that cannot be realized by a positive solution of the heat equation. In contrast, an 'ordinary' singularity is any singularity that can be described by a finite Borel measure μ, singular relative to Lebesgue measure, *e.g.* δ_ξ (= the Dirac measure at a point $\xi \in \mathbb{R}^N$).

Solutions with isolated singularities in the initial data, for equation

$$u_t - \Delta u + |u|^{q-1}u = 0, \tag{1.2}$$

The first author (MM) was supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities, through grant 91/10.

Received February 11, 2015; accepted in revised form June 11, 2015.

Published online September 2016.

have been thoroughly investigated. Brezis and Friedman [3] proved that: (i) if $1 < q < \frac{N+2}{N}$ then, for every $c \in \mathbb{R}$, (1.2) has a unique solution with initial data $c\delta_0$; (ii) if $q \geq \frac{N+2}{N}$ there is no such solution; (iii) the first statement holds whenever the initial data is given by a finite Borel measure. The proof employs an extension of the Keller-Osserman estimates to (1.2). Assuming that $1 < q < \frac{N+2}{N}$, Brezis, Peletier and Terman [4] provided a precise description of a positive solution with a strong isolated singularity at the origin. Kamin and Peletier [7] showed that this solution is in fact the limit as $c \to \infty$ of solutions of (1.2) with initial data $c\delta_0$. Marcus and Veron [8] proved that, in the class of positive solutions, there is a unique solution with a strong isolated singularity at a given point. In fact they proved a more general result: the initial data problem for (1.2) with initial data ν has a unique solution for every positive Borel measure ν such that $\nu = \infty$ on a compact set $F \subset \mathbb{R}^N$ (possibly empty) and ν is locally finite in the complement of F. For further results on solutions of (1.2) with strong singularities see [9,10,17] and references therein. For a survey of results on positive solutions with strong singularities for the corresponding elliptic equation see [12] and its references.

If equation (1.2) possesses a solution in $\mathbb{R}_+^{N+1}$ for some initial data ν – a positive Radon measure in $\mathbb{R}^N$ – then the same holds with respect to equation (1.1). Necessary and sufficient conditions on ν in order that such a solution exists have been provided by Baras and Pierre [1]. However if ν has an atom with infinite mass at the origin then it may happen that (1.1) does not have a solution in $\mathbb{R}_+^{N+1}$ although it would still have a solution in $\mathbb{R}_+^{N+1} \setminus [x = 0]$ satisfying the prescribed data on $\{(x, 0) : x \neq 0\}$. In such a case the singularity at the origin propagates along the time axis. This phenomenon may occur if the absorption term is too weak at zero or, in other words, if $h(x)$ tends to zero as $x \to 0$ sufficiently fast.

We consider coefficients h comparable to a function $H : x \mapsto \bar{h}(|x|)$ where

$$\bar{h}(s) := \exp(-\mu(s)) \quad \forall s > 0, \quad \mu(s) := \frac{\omega(s)}{s^2} \tag{1.3}$$

and

$$\begin{aligned} &\text{(i)} \quad \omega \in C(0, \infty) \quad \text{is positive, nondecreasing;} \\ &\text{(ii)} \quad \lim_{s \to 0} \mu(s) = \infty. \end{aligned} \tag{1.4}$$

Our first result provides a sufficient condition for the existence of a maximal solution of (1.1) in $\mathbb{R}_+^{N+1}$. Since the absorption term is positive for $x \neq 0$, it follows that, if a maximal solution exists, it is necessarily a 'large' solution, *i.e.* it blows up everywhere as $t \to 0$. In continuation it will be shown that this condition is sharp.

Theorem 1.1. *Suppose that*

$$h \geq cH \tag{1.5}$$

where c is a positive constant, $H(x) = \bar{h}(|x|)$, ω satisfies (1.4) *and*

$$s\omega'(s) \leq \delta\omega(s), \quad \textit{for } s \in (0, \infty) \textit{ and for some } \delta \in (0, 2). \tag{1.6}$$

If

$$\int_0^1 \frac{\omega(s)}{s}\,ds < \infty, \tag{1.7}$$

then every sequence of positive solutions of (1.1) *in* $\mathbb{R}_+^{N+1}$ *is locally bounded in this domain. In particular,* (1.1) *possesses a maximal solution* U *in* $\mathbb{R}_+^{N+1}$ *and* U *is a large solution.*

The next result shows that, under some additional assumptions on μ, the Dini condition (1.7) is necessary as well as sufficient for the existence of a large solution. In fact, if the Dini condition fails we exhibit a sequence of solutions that tends to zero as $t \to 0$, except at the origin, and tends to infinity as $x \to 0$ for every $t > 0$. The result applies to any $q > 1$.

Theorem 1.2. *Suppose that*

$$h \le cH \quad \text{in} \quad \mathbb{R}_+^{N+1} \tag{1.8}$$

where c *is a positive constant and* $H(x) = \bar{h}(|x|)$. *Assume that:*
(i) ω *satisfies* (1.4).
(ii) *The function* μ, *defined in* (1.3), *is non-increasing and satisfies*

$$\limsup_{j\to\infty} \frac{\mu(a^{-j+1})}{\mu(a^{-j})} < 1 \quad \text{for some } a > 1. \tag{1.9}$$

(iii) ω *does not satisfy the Dini condition,* i.e.,

$$\int_0^1 \frac{\omega(s)}{s}\,ds = \infty. \tag{1.10}$$

Let u_j *denote the solution of* (1.1) *with initial data*

$$u_j(0,x) = \gamma_j(x)$$

where

$$\gamma_j(x) := \begin{cases} A_j^{-1}\varphi_1\left(\frac{x}{r_{j+1}}\right) & \text{if} \quad |x| < r_{j+1} \\ 0 & \text{if} \quad |x| \geqslant r_{j+1}, \end{cases} \tag{1.11}$$

$$r_j = 2^{-j}, \quad A_j = r_j^2\bar{h}(r_j))^{\frac{1}{q-1}} \tag{1.12}$$

and $\varphi_1(x)$ *is the first eigenfunction of* $-\Delta_x$ *in* B_1 *such that* $\varphi(0) = 1$.
Let

$$u_\infty = \lim_{j\to\infty} u_j \quad \text{in } \mathbb{R}_+^{N+1}.$$

Under the above assumptions,

$$u_\infty(t,0) = \infty \quad \forall t > 0, \qquad u_\infty(0,x) = 0 \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

If $1 < q < \frac{N+2}{N}$ we show that the previous theorem remains valid even if u_j is a solution with initial data concentrated at the origin.

Theorem 1.3. *Suppose that* $1 < q < \frac{N+2}{N}$. *Let* u_k *denote the solution of* (1.1) *with initial data* $k\delta_0$ *and put* $u_\infty = \lim_{k\to\infty} u_k$. *Under the assumptions of Theorem* 1.2

$$u_\infty(t,0) = \infty \quad \forall t > 0.$$

Examples. Given $\beta > 0$, let $\omega_{1,\beta} \in C^1(0,\infty)$ be the function given by

$$\omega_{1,\beta}(s) = \left(\ln \frac{1}{s}\right)^{-\beta}$$

for $0 < s \leq \exp(-\beta)$ and is linear for $\exp(-\beta) < s$. Furthermore let $\omega_{2,\beta} \in C^1(0,\infty)$ be the function given by

$$\omega_{2,\beta}(s) = \left(\ln \frac{1}{s}\right)^{-1} \left(\ln\left(\ln \frac{1}{s}\right)\right)^{-\beta}$$

for $0 < s \leq \exp(-\exp(\beta))$ and is linear for $\exp(-\exp(\beta)) < s$.

The conditions of Theorem 1.1 are satisfied, for instance, if $\omega(s) = s^\delta$ for some $\delta \in (0,2)$ or $\omega = \omega_{1,\beta}$ or $\omega = \omega_{2,\beta}$ for some $\beta > 1$. The conditions of Theorem 1.2 are satisfied, for instance, if $\omega(s) \equiv 1$ or $\omega = \omega_{1,\beta}$ or $\omega = \omega_{2,\beta}$ for some $\beta \leq 1$.

The problem of propagation of singularities with respect to semilinear elliptic or parabolic equations with 'fading absorption' has been studied quite intensively in the last ten years (see [10,11,13,18–21]). We refer the reader to [13] for a brief description of these works. In [13] the authors treated the elliptic counterpart of the present problem. In that case too it was shown that a Dini condition is necessary and sufficient for non-propagation of strong singularities from the boundary data into the interior of the domain.

In [20] Shishkov and Veron considered the same parabolic problem that is studied here and proved that under the assumptions of Theorem 1.1 there exists a solution of (1.1) in $\mathbb{R}^{N+1}_+$ with a strong isolated singularity at the origin. The fact that the singularity does not propagate even in the case of a solution that blows up everywhere on the boundary – as in Theorem 1.1 – requires a more delicate argument. The necessity of the Dini condition was not discussed in [20], but it was shown that, in the special case where ω is a constant, a solution that blows up everywhere at $t = 0$ must also blow up on the axis $x = 0$. To prove the sufficiency of the Dini condition we estimate a sequence of supersolutions of (1.1) leading to a large solution in a finite cylinder. The proof is based on an iteration method leading to progressively improved local energy estimates. This basic approach has been used, in various forms, in papers going back to the work of Oleinik and Iosifyan [16] and, more recently, Galaktionov and Shishkov [5] dealing with certain analogues of Saint Venant's principle. The proof of the necessity of the Dini condition employs a result of [6] on the asymptotic behavior of positive solutions of semilinear parabolic equations as $t \to \infty$ and similarity transformations. We construct a sequence of subsolutions of (1.1), say $\{v_k\}$, such that $v_k \le u_k$ and show that $v_k(t,0) \to \infty$. The sequence $\{v_k\}$ is estimated by an iterative scheme similar to the one used in [13] (see also [14]).

The paper is organized as follows. In Sections 2, 3 we derive a priori estimates of Saint-Venant principle type that are needed for the proof of Theorem 1.1. The theorem itself is proved in Section 4. Finally Theorems 1.2 and 1.3 are proved in Section 5.

2. Energy estimates outside the set of absorption degeneration

For every $R > 0$ denote $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and

$$Q_R = \{(t,x) : 0 < t < R,\ x \in B_R\}, \quad \bar{\partial} Q_R = \{(t,x) \in \partial Q_R : 0 \le t < R\}.$$

Consider the following auxiliary parabolic problem

$$\begin{aligned} u_t - \Delta u + h(x)u^q = 0 \quad \text{in} \quad Q_R, \\ u\lfloor_{\bar{\partial} Q_R} = M, \end{aligned} \tag{2.1}$$

where $q > 1$, $M \ge 1$, $h(\cdot) \in C(\mathbb{R})$ and $h(0) = 0$, whereas $h(x) > 0\ \ \forall x \ne 0$.

We derive several estimates for solutions of this problem. We start with a standard *a priori* estimate.

Lemma 2.1. *Let u be a solution of* (2.1) *with $M \ge 1$ and let $\alpha > 1$. Then there exists $c > 0$ independent of M, R such that*

$$\begin{aligned} &\int_{B_R} u(\tau,x)^{\alpha+1} dx + \int_0^\tau \int_{B_R} (|\nabla_x (u^{\frac{\alpha+1}{2}})|^2 + hu^{q+\alpha}) dx dt \\ &\le cR^N M^{q+\alpha}(1 + R \sup_{B_R} h) \qquad \forall \tau \in (0, R]. \end{aligned} \tag{2.2}$$

Proof. Multiplying equation (1.1) by $u^\alpha - M^\alpha$ ($\alpha > 1$) and integrating over $(0,\tau) \times B_R(0)$, for $0 < \tau < R$, we obtain,

$$\begin{aligned} 0 &= \int_0^\tau \int_{B_R} (u_t - \Delta u + hu^q)(u^\alpha - M^\alpha) dx dt \\ &= \int_0^\tau \int_{B_R} u_t(u^\alpha - M^\alpha) dx dt + \int_0^\tau \int_{B_R} (\alpha |\nabla u|^2 u^{\alpha-1} + hu^{q+\alpha} - hu^q M^\alpha) dx dt \\ &=: I_1 + I_2. \end{aligned} \tag{2.3}$$

Due to the maximum principle we have $0 < u \le M$ in Q_R. Therefore,

$$\begin{aligned} I_1 =& \frac{1}{\alpha+1}\int_{B_R}\left(u^{\alpha+1}(\tau,x)-u(0,x)^{\alpha+1}\right)dx - M^\alpha\int_{B_R}(u(\tau,x)-u(0,x))\,dx \\ \ge& \frac{1}{1+\alpha}\int_{B_R}\left(u(\tau,x)^{\alpha+1}-M^{\alpha+1}\right)dx \end{aligned}$$

and

$$\int_0^\tau\int_{B_R} hu^q M^\alpha\,dxdt \le \tau M^{\alpha+q}|B_R|(\max h).$$

Hence

$$\begin{aligned} 0 = I_1 + I_2 \ge& \frac{1}{1+\alpha}\int_{B_R}\left(u(\tau,x)^{\alpha+1}-M^{\alpha+1}\right)dx \\ &+ \int_0^\tau\int_{B_R}\left(\alpha|\nabla_x u|^2u^{\alpha-1} + hu^{q+\alpha}\right)dxdt - \tau(\max h)|B_R|\,M^{\alpha+q}. \end{aligned}$$

This inequality and (2.3) imply (2.2). □

In the following estimates we consider functions h such that

$$h(x) \ge \overline{h}(|x|) := \exp\left(-\frac{\omega(|x|)}{|x|^2}\right), \tag{2.4}$$

where ω is a continuous positive function on $(0,\infty)$ and both $\bar{h}$ and ω are nondecreasing. Keeping R fixed we denote,

$$\Omega_s := \{x \in \mathbb{R}^N : s < |x| < R - s\} \qquad \forall s \in (0, R/2), \tag{2.5}$$

$$Q_{\tau,s} := \{(t,x) :\ 0 < \tau < t < R,\ x \in \Omega_s\}. \tag{2.6}$$

Lemma 2.2. *Assume that h satisfies* (2.4) *and that, for some* $0 < \delta < 2$,

$$0 < s\omega'(s) \le \delta\omega(s) \quad \forall s > 0. \tag{2.7}$$

Let u be a solution of (2.1) *and let* $\alpha > 0$. *Put*

$$v = u^{(\alpha+1)/2}, \quad p = p(\alpha) := \frac{\alpha+2q-1}{\alpha+1} \tag{2.8}$$

and define the following energy function:

$$J_u(\tau,s) = \int_{Q_{\tau,s}}(|\nabla_x v|^2 + hv^{p+1})\,dxdt \quad \forall\tau\in(0,R), \quad \forall s\in\left(0,\frac{R}{2}\right).$$

Let $\tau = \tau(s)$ be a positive function in $C^1(0, R)$ such that $\tau' > 0$ and denote,

$$I_u(s) = J_u(\tau(s), s) = \int_{\tau(s)}^{R} \int_{\Omega_s} \big(|\nabla_x u^{\frac{\alpha+1}{2}}|^2 + hu^{\alpha+q}\big)dxdt. \tag{2.9}$$

Then $I_u(\cdot)$ satisfies the following ordinary differential inequality:

$$I_u(s) \le \psi_1(s)(-I_u'(s))^{\frac{1}{1+\lambda_1}} + \psi_2(s)(-I_u'(s))^{\frac{1}{1+\lambda_2}} \quad \forall s \in \left(0, \frac{R}{2}\right), \tag{2.10}$$

where

$$\psi_1(s) = cR^{\sigma} s^{(N-1)\sigma} \overline{h}(s)^{-\frac{1}{p+1}}, \quad \psi_2(s) = c' R^{2N\sigma} \overline{h}(s)^{-\frac{2}{p+1}} \tau'(s)^{-\frac{2}{p+1}}, \tag{2.11}$$

c, c' are constants depending only on α, q, N and

$$\lambda_1 = \frac{p-1}{p+3}, \quad \lambda_2 = \frac{p-1}{2}, \quad \sigma = \frac{p-1}{2(p+1)}. \tag{2.12}$$

Remark. In the course of the proof, u is fixed; therefore we drop the index in I_u.

Proof. Multiplying equation (2.1) by u^α and integrating by parts over $Q_{\tau,s}$ we obtain:

$$\int_{\Omega_s} \int_{\tau}^{R} u_t u^\alpha \, dtdx = \int_{\tau}^{R} \int_{\Omega_s} u^\alpha (\Delta u - hu^q) dxdt;$$

$$\int_{\Omega_s} \int_{\tau}^{R} u_t u^\alpha \, dtdx = \frac{1}{\alpha+1} \int_{\Omega_s} (v^2(R, x) - v^2(\tau, x))dx,$$

$$\begin{aligned}
\int_{\Omega_s} u^\alpha (\Delta u - hu^q)dx &= -\int_{\Omega_s} (\nabla_x u \cdot \nabla_x u^\alpha + hu^{\alpha+q})dx + \int_{\partial\Omega_s} u^\alpha \frac{\partial u}{\partial \mathbf{n}} dS \\
&= -\int_{\Omega_s} \left(\frac{4\alpha}{(\alpha+1)^2} |\nabla_x v|^2 + hv^{p+1}\right) dx \\
&\quad + \frac{2}{\alpha+1} \int_{\partial\Omega_s} v \frac{\partial v}{\partial \mathbf{n}} dS.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\alpha+1} \int_{\Omega_s} v(R, x)^2 dx + \int\int_{Q_{\tau,s}} \left(\frac{4\alpha}{(\alpha+1)^2} |\nabla_x v|^2 + h(x) v^{1+p}\right) dxdt \\
=&\frac{2}{\alpha+1} \int_{\tau}^{R} \int_{\partial\Omega_s} \frac{\partial v}{\partial n} \cdot v dSdt + \frac{1}{\alpha+1} \int_{\Omega_s} v(\tau, x)^2 dx
\end{aligned} \tag{2.13}$$

We estimate the first term on the right using Hölder's inequality.

$$\begin{aligned}
&\int_\tau^R \int_{\partial\Omega_s} v \left|\frac{\partial v}{\partial n}\right| dSdt \\
&\leq \left(\int_\tau^R \int_{\partial\Omega_s} |\nabla_x v|^2 dSdt\right)^{\frac{1}{2}} \left(\int_\tau^R \int_{\partial\Omega_s} h(x) v^{p+1} dSdt\right)^{\frac{1}{p+1}} \\
&\quad \times \left(\int_\tau^R \int_{\partial\Omega_s} h(x)^{-\frac{2}{p-1}} dSdt\right)^{\frac{p-1}{2(p+1)}} \\
&\leq c_1(\alpha, q)(R-\tau)^{\frac{p-1}{2(p+1)}} \varphi_1(s) \left(\int_\tau^R \int_{\partial\Omega_s} (|\nabla_x v|^2 + h(x) v^{p+1}) dSdt\right)^{\frac{p+3}{2(p+1)}},
\end{aligned} \tag{2.14}$$

where

$$\varphi_1(s) := \left(\int_{\partial\Omega_s} h(x)^{-\frac{2}{p-1}} dS\right)^{\frac{p-1}{2(p+1)}}.$$

Next we estimate the second term on the right hand side of (2.13):

$$\begin{aligned}
\int_{\Omega_s} v(\tau,x)^2 dx &= \int_{\Omega_s} h(x)^{-\frac{2}{p+1}} h(x)^{\frac{2}{p+1}} v^2(\tau,x) dx \\
&\leq c_2 (\operatorname{meas} \Omega_s)^{\frac{p-1}{p+1}} \max h(x)^{-\frac{2}{p+1}} \left(\int_{\Omega_s} h(x) v(\tau,x)^{p+1} dx\right)^{\frac{2}{p+1}} \\
&\leq c_3 (R-s)^{\frac{N(p-1)}{p+1}} \varphi_2(s) \left(\int_{\Omega_s} h(x) v(\tau,x)^{p+1} dx\right)^{\frac{2}{p+1}},
\end{aligned} \tag{2.15}$$

where $c_2 = c_2(\alpha), c_3 = c_3(\alpha, N)$ and

$$\varphi_2(s) = \max_{s \leq |x| \leq R-s} h(x)^{-\frac{2}{p+1}} = \bar{h}(s)^{-\frac{2}{p+1}}. \tag{2.16}$$

Combining (2.13), (2.14) and (2.15) we get:

$$\begin{aligned}
&\int_{\Omega_s} v(R,x)^2 dx + c_0 J_{(\tau,s)} \\
&\leq 2c_1 R^{\frac{p-1}{2(p+1)}} \varphi_1(s) \left(\int_\tau^R \int_{\partial\Omega_s} (|\nabla_x v|^2 + h v^{p+1}) dSdt\right)^{\frac{p+3}{2(p+1)}} \\
&\quad + c_3 R^{\frac{N(p-1)}{p+1}} \varphi_2(s) \left(\int_{\Omega_s} h v(\tau,x)^{p+1} dx\right)^{\frac{2}{p+1}},
\end{aligned} \tag{2.17}$$

where $c_0 = \min\{\frac{4\alpha}{\alpha+1}, \alpha + 1\}$. Observe that by (2.9):

$$-\frac{d}{ds}I(s) = \int_{\tau(s)}^{R}\int_{\partial\Omega_s}(|\nabla_x v|^2 + hv^{p+1})dSdt + \tau'(s)\int_{\Omega_s}(|\nabla_x v(\tau(s),x)|^2 + hv(\tau(s),x)^{p+1})dx.$$

Since $\tau' > 0$ it follows that,

$$\int_{\tau(s)}^{R}\int_{\partial\Omega_s}(|\nabla_x v|^2 + hv^{p+1})dSdt \le -I'(s).$$

Therefore, by (2.17),

$$c_0 I(s) \le 2c_1 R^{\frac{p-1}{2(p+1)}}\varphi_1(s)(-I'(s))^{\frac{p+3}{2(p+1)}} + c_3 R^{\frac{N(p-1)}{p+1}}\varphi_2(s)(-I'(s)/\tau'(s))^{\frac{2}{p+1}}. \tag{2.18}$$

Condition (2.7) implies that $\omega(s)/s^{\delta}$ is non-increasing. Therefore,

$$\omega(s) \geqslant \omega_0 s^{\delta} \qquad \forall s > 0, \tag{2.19}$$

where ω_0 is a positive constant. By assumption $\delta < 2$ so that $\omega(s)s^{-2} \ge \omega_0 s^{-\epsilon}$ where $\epsilon = 2 - \delta$. It follows that, for any $\gamma > 0$,

$$s^{N-1}\exp\Big(\gamma\omega(s)s^{-2}\Big) \ge s^{N-1}\exp(\gamma s^{-\epsilon}) \uparrow \infty \text{ as } \omega_0 s \downarrow 0 \text{ for } \omega_0 s < (\gamma\epsilon/(N-1))^{1/\epsilon}.$$

Therefore we obtain the following estimate for $s \le \min(\frac{R}{4}, (\frac{2\epsilon\omega_0}{(p-1)(N-1)})^{1/\epsilon})$:

$$\begin{aligned}\varphi_1(s) &= \left(\int_{\partial\Omega_s} h(x)^{-\frac{2}{p-1}}dS\right)^{\frac{p-1}{2(p+1)}} \le \left(\int_{\partial\Omega_s}\overline{h}(|x|)^{-\frac{2}{p-1}}dS\right)^{\frac{p-1}{2(p+1)}} \\ &= c_4\left(s^{N-1}\overline{h}(s)^{-\frac{2}{p-1}} + (R-s)^{N-1}\overline{h}(R-s)^{-\frac{2}{p-1}}\right)^{\frac{p-1}{2(p+1)}} \\ &\le c_5 s^{\frac{(N-1)(p-1)}{2(p+1)}}\overline{h}(s)^{-\frac{1}{p+1}}\end{aligned} \tag{2.20}$$

Finally (2.16), (2.20) and (2.18) imply (2.10) with $c = 2c_0^{-1}c_1c_5, c' = c_0^{-1}c_3$. □

Proposition 2.3. *Assume the conditions of Lemma 2.2 employing the notation introduced there. Then there exists $\alpha_0 > 0$ (depending only on N, q) and, for every $\alpha \ge \alpha_0$, there exist positive numbers ν and γ (depending on α, q, N) such that the following assertion holds. Let*

$$\tau(s) = s^{\gamma}\omega^{-\nu}(s), \quad \textit{for } s > 0 \tag{2.21}$$

Then $\tau'(s) > 0$ *and*

$$\begin{aligned} I_u(s) &= \int_{\tau(s)}^{R} \int_{\Omega_s} \left(|\nabla_x u^{\frac{\alpha+1}{2}}|^2 + hu^{\alpha+q}\right) dx dt \\ &\le C s^{N-1-\frac{3(p+3)}{p-1}} \omega(s)^{\frac{p+3}{p-1}} \exp\left(\frac{2\omega(s)}{(p-1)s^2}\right) \end{aligned} \tag{2.22}$$

for $0 < s < \bar{s}(R)$, *with* C *depending only on* α, q, N, R.

Proof. As before, we drop the index in I_u.

PART 1. Let γ, ν be positive numbers such that

$$2\nu + 1 < \gamma. \tag{2.23}$$

Then, by (2.7),

$$\tau'(s) = s^{\gamma-1}\omega(s)^{-\nu-1}(\gamma\omega(s) - \nu s\omega'(s)) \ge s^{\gamma-1}\omega(s)^{-\nu}(\gamma - \nu\delta).$$

Recall that $\delta \in (0, 2)$. Therefore, by (2.23) $\gamma - \nu\delta \ge \gamma - 2\nu > 1$. Consequently

$$\tau'(s) \ge s^{\gamma-1}\omega(s)^{-\nu} > 0. \tag{2.24}$$

By (2.24), the function $\psi_2(s)$ defined in (2.11) satisfies

$$\psi_2(s) \le c' R^{\frac{N(p-1)}{(p+1)}} \bar{h}(s)^{-\frac{2}{p+1}} (s^{\gamma-1}\omega(s)^{-\nu})^{-\frac{2}{p+1}}. \tag{2.25}$$

Denote,

$$\begin{aligned} \tilde{\psi}_1(s) &= 2c(Rs^{N-1})^{\frac{p-1}{2(p+1)}} \bar{h}(s)^{-\frac{1}{p+1}} = 2\psi_1(s), \\ \tilde{\psi}_2(s) &= 2c' R^{\frac{N(p-1)}{(p+1)}} \bar{h}(s)^{-\frac{2}{p+1}} s^{-\frac{(\gamma-1)2}{p+1}} \omega(s)^{\frac{2\nu}{p+1}}. \end{aligned} \tag{2.26}$$

By (2.19) with respect to $\tilde{\psi}_1$ and (2.23) with respect to $\tilde{\psi}_2(s)$,

$$\tilde{\psi}_j(s) \to \infty \quad \text{as } s \to 0, \quad \text{for } j = 1, 2. \tag{2.27}$$

By (2.10),

$$I(s) \le \max\left\{ \tilde{\psi}_1(s) \left(-I'(s)\right)^{\frac{1}{1+\lambda_1}}, \tilde{\psi}_2(s) \left(-I'(s)\right)^{\frac{1}{1+\lambda_2}} \right\} \tag{2.28}$$

with λ_1, λ_2 as in (2.12). Inequality (2.28) is equivalent to

$$I'(s) \le -\min\{\tilde{\psi}_1(s)^{-(1+\lambda_1)} I(s)^{1+\lambda_1}, \tilde{\psi}_2(s)^{-(1+\lambda_2)} I(s)^{1+\lambda_2}\}. \tag{2.29}$$

In the remainder of the proof we derive (2.22) from (2.29).

Put

$$I_k(s) := \lambda_k^{-\frac{1}{\lambda_k}} \left(\int_0^s \tilde{\psi}_k(r)^{-(1+\lambda_k)} dr \right)^{-\frac{1}{\lambda_k}}, \quad \text{for } k = 1, 2. \tag{2.30}$$

Then,

$$I_k'(s) = -\tilde{\psi}_k(s)^{-(1+\lambda_k)} I_k(s)^{1+\lambda_k} \ \forall s > 0, \quad \lim_{s \to 0} I_k(s) = \infty, \quad \text{for } k = 1, 2. \tag{2.31}$$

By (2.11) and (2.26)

$$\begin{aligned} I_1(s) &= c_7 \left(\int_0^s \eta^{-\frac{(N-1)(p-1)}{(p+3)}} \exp\left(-\frac{2\omega(\eta)}{(p+3)\eta^2} \right) d\eta \right)^{-\frac{p+3}{p-1}}, \\ I_2(s) &= c_8 \left(\int_0^s \eta^{\gamma-1} \omega(\eta)^{-\nu} \exp\left(-\frac{\omega(\eta)}{\eta^2} \right) d\eta \right)^{-\frac{2}{p-1}}, \end{aligned} \tag{2.32}$$

where $c_7 = \lambda_1^{-\frac{1}{\lambda_1}} (2c)^{\frac{1+\lambda_1}{\lambda_1}} R^{\frac{(p-1)(1+\lambda_1)}{2(p+1)\lambda_1}}$ and $c_8 = \lambda_2^{-\frac{1}{\lambda_2}} (2c')^{\frac{1+\lambda_2}{\lambda_2}} R^{\frac{N(p-1)(1+\lambda_2)}{(p+1)\lambda_2}}$. (Here c, c' are the constants in (2.11).)

By [2, Lemma A.1] — using (2.7) — it follows that there exists $\bar{s} \in (0, R/4)$ such that,

$$\begin{aligned} &\int_0^s \eta^{-\frac{(N-1)(p-1)}{p+3}} \exp\left(-\frac{2\omega(\eta)}{(p+3)\eta^2} \right) d\eta \approx s^{-\frac{(N-1)(p-1)}{p+3}+3} \omega(s)^{-1} \exp\left(-\frac{2\omega(s)}{(p+3)s^2} \right), \\ &\int_0^s \eta^{\gamma-1} \omega(\eta)^{-\nu} \exp\left(-\frac{\omega(\eta)}{\eta^2} \right) d\eta \approx s^{\gamma+2} \omega^{-\nu-1}(s) \cdot \exp\left(-\frac{\omega(s)}{s^2} \right), \text{ for } 0 < s \leq \bar{s}. \end{aligned} \tag{2.33}$$

Therefore, by (2.32),

$$\begin{aligned} I_1(s) &\approx s^{N-1-\frac{3(p+3)}{p-1}} \omega(s)^{\frac{p+3}{p-1}} \exp\left(\frac{2\omega(s)}{(p-1)s^2} \right) \\ I_2(s) &\approx s^{-\frac{(\gamma+2)2}{p-1}} \omega(s)^{\frac{2(\nu+1)}{p-1}} \exp\left(\frac{2\omega(s)}{(p-1)s^2} \right), \quad \text{for } 0 < s \leq \bar{s}. \end{aligned} \tag{2.34}$$

PART 2. Now we choose α, γ and ν so that $I_1 \approx I_2$. Let

$$\gamma := \frac{3p + 5 - (N-1)(p-1)}{2} \tag{2.35}$$

and

$$\nu = \frac{p+1}{2}. \tag{2.36}$$

Recall that

$$p = p(\alpha) = 1 + 2\frac{q-1}{\alpha+1} \to 1 \quad \text{as} \quad \alpha \to \infty. \tag{2.37}$$

Therefore, if $\alpha_0 = (N-2)(q-1)-1$ then (2.23) holds for every $\alpha \geq \alpha_0$. In addition,(2.35) and (2.36) imply,

$$\frac{3(p+3)}{p-1} - (N-1) = \frac{2(\gamma+2)}{p-1} \quad \text{and} \quad p+3 = 2(\nu+1). \tag{2.38}$$

Hence, by (2.34), $I_1 \approx I_2$ in the interval $(0, \bar{s})$, *i.e.*,

$$c_9^{-1} I_2(s) \leq I_1(s) \leq c_9 I_2(s) \qquad \forall s \in (0, \bar{s}) \tag{2.39}$$

for some constant c_9 depending only on α, q, N, R.

Put

$$g_k(s, K) = \tilde{\psi}_k(s)^{-(1+\lambda_k)} K^{1+\lambda_k}, \quad \text{for } k = 1, 2 \tag{2.40}$$

and denote

$$\begin{aligned} D_0 &= \{(s, K) : s \geq 0, K > 0\}, \\ D_1 &= \{(s, K) : g_1(s, K) < g_2(s, K), \ s > 0, K > 0\}, \\ D_2 &= \{(s, K) : g_2(s, K) < g_1(s, K), \ s > 0, K > 0\}, \\ \Gamma &= \{(s, K) : g_1(s, K) = g_2(s, K), \ s > 0, K > 0\}. \end{aligned} \tag{2.41}$$

Then Γ is the locus of the function,

$$K = \bar{I}(s) = \tilde{\psi}_2(s)^{\frac{1+\lambda_2}{\lambda_2-\lambda_1}} \cdot \tilde{\psi}_1(s)^{-\frac{1+\lambda_1}{\lambda_2-\lambda_1}} = C_1 \bar{h}(s)^{-\frac{2}{p-1}} \cdot s^{-\theta} \omega(s)^{\beta} \quad \forall s > 0, \tag{2.42}$$

where, for γ and ν as in (2.35) and (2.36),

$$\theta = \frac{3(p+3)}{p-1} - (N-1) = \frac{2(\gamma+2)}{p-1}, \quad \beta = \frac{p+3}{p-1} = \frac{2(\nu+1)}{p-1} \tag{2.43}$$

and $C_1 = (2c')^{\frac{p+3}{p-1}} (2c)^{-\frac{4}{p-1}} R^{\frac{2N\sigma(p+3)-4\sigma}{p-1}}$. Hence, by (2.34),

$$c_{10}^{-1} I_1(s) \leqslant \bar{I}(s) \leqslant c_{10} I_1(s) \qquad \forall s \in (0, \bar{s}) \tag{2.44}$$

for some constant c_{10} depending only on α, q, N, R. Observe that the sets D_1, D_2 defined in (2.41) can be described in terms of $\bar{I}$ as follows:

$$D_1 = \left\{(s, K) : s > 0, \quad K > \bar{I}(s)\right\}, \quad D_2 = \left\{(s, K) : s > 0, \quad K < \bar{I}(s)\right\}. \tag{2.45}$$

By (2.38), (2.42) and (2.43),

$$\bar{I}(s) = C_1 \left(\bar{h}(s)^{-1} s^{-(\gamma+2)} \omega(s)^{\nu+1}\right)^{\frac{2}{p-1}} = C_1 \left(e^{\mu(s)} s^{2\nu-\gamma} \mu(s)^{\nu+1}\right)^{\frac{2}{p-1}}. \tag{2.46}$$

Therefore by (1.3), (1.4), (2.23) and (2.42),

$$\bar{I}(s) \to \infty \quad \text{as } s \to 0. \tag{2.47}$$

In fact, this also follows from (2.31) and (2.44).

PART 3. For every $a > 0$ denote by J_a the solution of the initial value problem

$$J'(s) = -\min(g_1(s, J), g_2(s, J)) \quad \text{in } D_0, \quad J(0) = a. \tag{2.48}$$

We note that $g_j \in C(D_0)$ and is Lipschitz continuous w.r. to J in compact subsets of D_0. Therefore the equation in (2.48) has a unique solution through any given point in D_0. As $g_j > 0$ when $s > 0$ and $J > 0$, the mapping $s \mapsto J_a(s)$ is strictly decreasing. Furthermore, the mapping $a \mapsto J_a(s)$ is strictly increasing for every $s > 0$. If $J_a > 0$ for every $s > 0$ put $z_a = \infty$; otherwise let z_a be the point where $J_a(z_a) = 0$.

Next we prove:

ASSERTION 1. Let $0 < b < z_1$. (Note that by the definition of z_1, $J_a(b) > 0$, for $a \geq 1$.) Then there exists a number M_b such that

$$J_a(s) \leq M_b \quad \forall s \geq b,\ \forall a \geq 1. \tag{2.49}$$

Because of monotonicity, it is enough to show that

$$J_a(b) \leq M_b < \infty \quad \forall a \geq 1,\ \forall b \in (0, z_1).$$

Suppose instead that, for some $b \in (0, z_1)$, $J_a(b) \to \infty$ as $a \to \infty$. Pick $a_b > 1$ such that $J_a(b) > \bar{I}(b)$ for $a \geq a_b$. It follows that there exists $b' \in (0, b)$ such that

$$J_a(s) > \bar{I}(s) \quad \forall s \in [b', b],\ \forall a \geq a_b.$$

This is equivalent to

$$(s, J_a(s)) \in D_1 \quad \forall s \in [b', b],\ \forall a \geq a_b. \tag{2.50}$$

Therefore,

$$J_a' = -g_1(s, J_a) \quad \text{in } (b', b), \quad \forall a \geq a_b.$$

Consequently, for every $a \geq a_b$,

$$J_a(s) = \left(J_a(b)^{-\lambda_1} - \lambda_1 \int_s^b \tilde{\psi}_1(r)^{-(1+\lambda_1)} dr \right)^{-\frac{1}{\lambda_1}} \quad \forall\, s \in (b', b). \tag{2.51}$$

It follows that, $\lambda_1 \int_s^b \tilde{\psi}_1(r)^{-(1+\lambda_1)} dr \leq J_a(b)^{-\lambda_1}$, *i.e.*,

$$J_a(b) \leq \left(\lambda_1 \int_s^b \tilde{\psi}_1(r)^{-(1+\lambda_1)} dr \right)^{-\frac{1}{\lambda_1}}, \quad \forall s \in (b', b),\ \text{and for } a \geq a_b. \tag{2.52}$$

This proves Assertion 1 which in turn implies that $J_\infty := \lim_{a \to \infty} J_a$ is well defined in $(0, z_1)$ and satisfies

$$J'(s) = -\min(g_1(s, J), g_2(s, J)), \quad \text{for } 0 < s < z_1, \text{ and } \lim_{s \to 0} J(s) = \infty. \tag{2.53}$$

PART 4. We complete the proof of the proposition. In view of (2.42), inequality (2.22) is equivalent to the following:

$$J_\infty(s) \le M\bar{I}(s) \quad \forall s \in (0, z_1), \tag{2.54}$$

for some constant $M > 0$. Suppose that for some $b \in (0, z_1)$ and $M \ge 1$, we have $J_\infty(b) > M\bar{I}(b)$. Then, either $J_\infty(s) > \bar{I}(s)$ for all $s \in (0, b)$, or there exists $b' \in (0, b)$ such that

$$\bar{I}(b') = J_\infty(b') \quad \text{and} \quad J_\infty(s) > \bar{I}(s), \text{ for } b' < s \le b. \tag{2.55}$$

In the second case $J'_\infty = -g_1(s, J_\infty)$ in (b', b). Therefore

$$J_\infty(s) = \left(J_\infty(b)^{-\lambda_1} - \lambda_1 \int_s^b \tilde{\psi}_1(r)^{-(1+\lambda_1)} dr\right)^{-\frac{1}{\lambda_1}} \quad \forall s \in (b', b). \tag{2.56}$$

Therefore, since $J_\infty(b) > M\bar{I}(b)$, we obtain — using (2.44) —

$$\begin{aligned} J_\infty(b') &> \left((M\bar{I}(b))^{-\lambda_1} - \lambda_1 \int_{b'}^b \tilde{\psi}_1^{-(1+\lambda_1)}(r) dr\right)^{-\frac{1}{\lambda_1}} \\ &\ge \left((c_{10}/M)^{\lambda_1} I_1(b)^{-\lambda_1} - \lambda_1 \int_{b'}^b \tilde{\psi}_1^{-(1+\lambda_1)}(r) dr\right)^{-\frac{1}{\lambda_1}} \\ &\ge \frac{M}{c_{10}} \left(I_1(b)^{-\lambda_1} - \lambda_1 \int_{b'}^b \tilde{\psi}_1^{-(1+\lambda_1)}(r) dr\right)^{-\frac{1}{\lambda_1}} = \frac{M}{c_{10}} I_1(b') \end{aligned} \tag{2.57}$$

where c_{10} is the constant in (2.44). Hence, by (2.55) and the second inequality in (2.44),

$$J_\infty(b') > \frac{M}{c_{10}^2} \bar{I}(b') = \frac{M}{c_{10}^2} J_\infty(b').$$

If $M > c_{10}^2$ this is not possible. Therefore, assuming $M > c_{10}^2$, we reach the following alternative:

Either there exists $b^* \in (0, z_1)$ such that $J_\infty(s) > \bar{I}(s)$ for all $s \in (0, b^*)$ or (2.54) *holds*.

In the first case $J'_\infty = -g_1(s, J_\infty)$ in $(0, b^*)$ and consequently (2.56) holds for all s, b such that $0 < s < b \le b^*$. Therefore

$$J_\infty(s)^{-\lambda_1} = J_\infty(b)^{-\lambda_1} - \lambda_1 \int_s^b \tilde{\psi}_1(r)^{-(1+\lambda_1)} dr.$$

Letting s tend to zero we obtain,

$$J_\infty(b)^{-\lambda_1} = \lambda_1 \int_0^b \tilde{\psi}_1(r)^{-(1+\lambda_1)} dr.$$

Thus $J_\infty = I_1$ (see (2.30)) and (2.54) follows by (2.44). In conclusion, (2.54) holds for any $M > c_{10}^2$. □

The following is a slightly stronger version of Proposition 2.3.

Proposition 2.4. *Let u be a solution of problem* (2.1) *and let $\alpha > 0$. Let v and p be as in* (2.8)*. Denote,*

$$E_u(s,T) := \int_{\Omega_s} v(T,x)^2 dx, \quad A_u(s,T) := E_u(s,T) + I_u(s)$$

for $T \in (0,R)$ and $s \in (0,\bar{s}(R))$ such that $\tau(s) \le T$ ($\tau, \bar{s}(R)$ as in Proposition 2.3*). Then, under the assumptions of Proposition* 2.3*,*

$$A_u(s,T) \le C s^{N-1-\frac{3(p+1)}{p-1}} \omega(s)^{\frac{p+3}{p-1}} \exp\left(\frac{2\omega(s)}{(p-1)s^2}\right) =: F(s), \tag{2.58}$$

for every s, T as above, where C is a constant independent of R, M.

Proof. Multiplying equation (2.1) by u^α and integrating by parts over $Q_{\tau,s} \setminus Q_{T,s}$ we obtain:

$$\frac{1}{\alpha+1}\int_{\Omega_s} v(T,x)^2 dx + \int\int_{Q_{\tau,s}\setminus Q_{T,s}} \left(\frac{4\alpha}{(\alpha+1)^2}|\nabla_x v|^2 + h(x)v^{p+1}\right) dxdt$$
$$= \frac{2}{\alpha+1}\int_\tau^T \int_{\partial\Omega_s} \frac{\partial v}{\partial n} v dSdt + (\alpha+1)^{-1}\int_{\Omega_s} v(\tau,x)^2 dx \qquad \forall T : \tau \le T < R. \tag{2.59}$$

Summing (2.13) and (2.59) we get:

$$(\alpha+1)^{-1}\int_{\Omega_s} v(T,x)^2 dx + \int\int_{Q_{\tau,s}} \left(\frac{4\alpha}{(\alpha+1)^2}|\nabla_x v|^2 + h(x)v^{1+p}\right) dxdt$$
$$\le \frac{4}{\alpha+1}\int_\tau^R \int_{\partial\Omega_s} \left|\frac{\partial v}{\partial n}\right| \cdot v dSdt + 2(\alpha+1)^{-1}\int_{\Omega_s} v(\tau,x)^2 dx \quad \forall T : \tau \leqslant T < R. \tag{2.60}$$

Using now (2.60) instead of (2.13), by the same argument as in the proof of Lemma 2.2, we obtain the following analogue of (2.10):

$$A_u(s,T) = E_u(s,T) + I_u(s) \le 2\psi_1(s)(-I_u'(s))^{\frac{1}{1+\lambda_1}} + 2\psi_2(-I_u'(s))^{\frac{1}{1+\lambda_2}}, \tag{2.61}$$

with ψ_k and λ_k as in Lemma 2.2. Since $\frac{\partial E_u(s,T)}{\partial s} \le 0$ we have $-I_u'(s) \le -A_u'(s,T)$ so as (2.61) yields,

$$A_u(s,T) \le 2\psi_1(s)(-A_u'(s,T))^{\frac{1}{1+\lambda_1}} + 2\psi_2(s)(-A_u'(s,T))^{\frac{1}{1+\lambda_2}}. \tag{2.62}$$

Using this inequality we obtain (2.58) precisely in the same way as (2.22) was derived from (2.28). □

3. Integral estimates in the neighborhood of the set of degeneration of the absorption potential

Given $M > 0$ denote by $U_M = u_{\bar{M}}$ a solution of (2.1) with initial-boundary data

$$\bar{M} := \left(cR^N(1 + R\sup_{B_R} h)\right)^{-\frac{1}{q+\alpha}} M^{\frac{1}{q+\alpha}}, \tag{3.1}$$

and c as in inequality (2.2). Then, by Lemma 2.1 and (3.1),

$$\begin{aligned}\int_{B_R} U_M(\tau, x)^{\alpha+1}dx + \int_0^\tau \int_{B_R} \left(\left|\nabla_x \left(U_M^{\frac{\alpha+1}{2}}\right)\right|^2 + hU_M^{q+\alpha}\right) dxdt \\ \leq M \quad \forall \tau \in (0, R].\end{aligned} \tag{3.2}$$

By (2.58)

$$\int_{\Omega_s} U_M(T, x)^{\alpha+1}dx + \int_{\tau(s)}^R \int_{\Omega_s} \left(\left|\nabla_x U_M^{\frac{\alpha+1}{2}}\right|^2 + hU_M(t, x)^{\alpha+q}\right) dxdt \leq F(s) \tag{3.3}$$

for $T \in (0, R)$ and $s \in (0, \bar{s}(R))$ such that $\tau(s) \leq T$, where $\tau, \bar{s}(R)$ are as in Proposition 2.3. Note that $F(s) \to \infty$ as $s \to 0$. Let β be a number in $(0, 1)$ (to be determined later on) and denote

$$s_M = s_M(\beta) := \inf\{s \in (0, R] : F(s) \leq M^\beta\}. \tag{3.4}$$

Then $s_M \to 0$ as $M \to \infty$ and there exists $M(\beta) > 0$ such that $M \mapsto s_M$ is decreasing in $(M(\beta), \infty)$. In view of condition (2.7) and the definition of F in (2.58),

$$\exp\left(\frac{2\omega(s)}{(p-1)s^2}(1-\epsilon)\right) \leq F(s) \leq \exp\left(\frac{2\omega(s)}{(p-1)s^2}(1+\epsilon)\right) \tag{3.5}$$

for every $\epsilon \in (0, 1)$, $s \in (0, s_0(\epsilon))$ and $s_0(\epsilon) \to 0$ as $\epsilon \to 0$. By (3.4) and (3.5),

$$\frac{(p-1)\beta \ln M}{2(1+a)} \leq \frac{\omega(s_M)}{s_M^2} \leq \frac{(p-1)\beta \ln M}{2(1-a)}, \tag{3.6}$$

where a is a constant in $(0, 1/2)$. The inequality on the left yields

$$s_M \leq (\ln M)^{-1/2} \cdot \left(\frac{2(1+a)\omega_R}{(p-1)\beta}\right)^{1/2}, \quad \text{where } \omega_R = \max_{0<s\leq R} \omega(s). \tag{3.7}$$

From (3.4) and (3.3) we obtain,

$$\begin{aligned}\int_{\Omega_{s_M}} U_M(T, x)^{\alpha+1}dx \leq F(s_M) = M^\beta \\ \forall T \geq \tau_M := \tau(s_M), \ \tau(\cdot) \text{ is from (2.21)},\end{aligned} \tag{3.8}$$

and (see definition of I_u in Lemma 2.2)

$$I_{U_M}(s_M) = \int_{\tau_M}^{R} \int_{\Omega_{s_M}} \left(\left| \nabla_x U_M \right|^2 U_M^{(\alpha-1)} + h U_M^{q+\alpha} \right) dx dt \leq F(s_M) = M^{\beta}. \quad (3.9)$$

Remark also that by (2.19), (2.21) and (2.23)

$$s^{\gamma} < \tau(s) < \omega_0^{-\nu} \delta^{\gamma - \delta\nu} \qquad \forall s \in (0, 1) \text{ such that } \omega(s) < 1. \quad (3.10)$$

Next we estimate an energy integral associated with U_M in the domain $(\tau_M, R) \times B_{s_M}$.

Lemma 3.1. *Given $\sigma \in (0, R/2)$ let $\varphi = \varphi_\sigma \in C^1(\mathbb{R}^N)$ be a radially symmetric function such that:*

$$\varphi_\sigma(x) = 1 \text{ if } \quad |x| < \sigma, \;\; \varphi_\sigma(x) = 0 \text{ if } \quad |x| > 2\sigma, \text{ with } |\nabla\varphi(x)| \leq 2\sigma^{-1}. \quad (3.11)$$

Put,

$$\Phi_{\sigma,M}(t) := \int_{B_R} (U_M(t, x)\varphi_\sigma)^{\alpha+1} \, dx. \quad (3.12)$$

Then there exist positive constants c_0, c_1 independent of M such that

$$\frac{d}{dt}\Phi_{\sigma,M}(t) + c_0\sigma^{-2}\Phi_{\sigma,M}(t) \leq c_1\sigma^{-2}F(\sigma) \qquad \forall t \in (\tau_M, R], \quad (3.13)$$

$$\Phi_{\sigma,M}(t) \leq M \qquad \forall t > 0. \quad (3.14)$$

Proof. Put $V_M := (U_M\varphi_\sigma)^{(\alpha+1)/2}$. Since U_M satisfies (2.1) and $h \geq 0$ we have,

$$\frac{\partial U_M}{\partial t} - \Delta U_M \leq 0.$$

Multiplying this inequality by $U_M^{\alpha}\varphi_\sigma^{\alpha+1}$, α as in Proposition 2.4, and integrating over B_R we obtain

$$\begin{aligned}
&\frac{1}{\alpha+1}\frac{d}{dt}\int_{B_R} V_M^2 dx + \frac{4\alpha}{(\alpha+1)^2}\int_{B_R} |\nabla_x V_M|^2 dx \\
&\leq \frac{8\alpha}{(\alpha+1)^2}\int_{B_R} \nabla_x V_M \cdot \nabla\left(\varphi_\sigma^{\frac{\alpha+1}{2}}\right) U_M^{\frac{\alpha+1}{2}} dx \\
&\quad - \frac{4\alpha}{(\alpha+1)^2}\int_{B_R} U_M^{\alpha+1}\left|\nabla\varphi_\sigma^{\frac{\alpha+1}{2}}\right|^2 dx - (\alpha+1)\int_{B_R} (\nabla_x U_M \cdot \nabla\varphi_\sigma) U_M^{\alpha}\varphi_\sigma^{\alpha} \, dx.
\end{aligned}$$

Furthermore, for every $a > 0$,

$$\int_{B_R} \nabla_x V_M \cdot \nabla(\varphi_\sigma^{\frac{\alpha+1}{2}}) U_M^{\frac{\alpha+1}{2}} dx \leq a\int_{B_R} |\nabla_x V_M|^2 \, dx + C(a)\int_{B_R} \left|\nabla(\varphi_\sigma^{\frac{\alpha+1}{2}}\right|^2 U_M^{\alpha+1} dx,$$

where $C(a) \to \infty$ as $a \to 0$. We also have,

$$
\begin{aligned}
&(\alpha+1)\int_{B_R}\left|\nabla_x U_M\right|\left|\nabla\varphi_\sigma\right|U_M^\alpha\varphi_\sigma^\alpha\,dx = 2^{-1}\int_{B_R}\left|\nabla_x U_M^{\frac{\alpha+1}{2}}\right|U_M^{\frac{\alpha+1}{2}}\varphi_\sigma^\alpha\left|\nabla\varphi_\sigma\right|dx\\
&= 2^{-1}\int_{B_R}\left|\nabla V_M - U_M^{\frac{\alpha+1}{2}}\nabla\left(\varphi_\sigma^{\frac{\alpha+1}{2}}\right)\right|U_M^{\frac{\alpha+1}{2}}\varphi_\sigma^{\frac{\alpha-1}{2}}|\nabla\varphi_\sigma|dx\\
&= (1+\alpha)^{-1}\int_{B_R}\left|\nabla V_M - U_M^{\frac{\alpha+1}{2}}\nabla\left(\varphi_\sigma^{\frac{\alpha+1}{2}}\right)\right|U_M^{\frac{\alpha+1}{2}}\left|\nabla\varphi_\sigma^{\frac{\alpha+1}{2}}\right|dx\\
&\le (1+\alpha)^{-1}\left(a\int_{B_R}|\nabla V_M|^2\,dx + C(a)\int_{B_R}\left|\nabla\varphi_\sigma^{\frac{\alpha+1}{2}}\right|^2U_M^{\alpha+1}dx\right).
\end{aligned}
$$

and as consequence,

$$
\begin{aligned}
&\frac{1}{\alpha+1}\frac{d}{dt}\int_{B_R}V_M^2dx + \frac{4\alpha}{(\alpha+1)^2}\int_{B_R}|\nabla_x V_M|^2dx\\
&\le c(\alpha)\left(a\int_{B_R}\left|\nabla_x(U_M\varphi_\sigma)^{\frac{\alpha+1}{2}}\right|^2dx + C(a)\int_{B_R}U_M^{\alpha+1}\varphi_\sigma^{\alpha-1}|\nabla\varphi_\sigma|^2\,dx\right),
\end{aligned}
\tag{3.15}
$$

for any $a > 0$ with a constant $C(a)$ tending to infinity as $a \to 0$. Choosing $a = \frac{2\alpha}{c(\alpha)(\alpha+1)^2}$ we obtain,

$$
\begin{aligned}
&\frac{d}{dt}\int_{B_R}(U_M(t,x)\varphi_\sigma)^{\alpha+1}\,dx + c\int_{B_R}\left|\nabla_x(U_M\varphi_\sigma)^{\frac{\alpha+1}{2}}\right|^2dx\\
&\le c'\sigma^{-2}\int_{\sigma\le|x|\le 2\sigma}U_M(t,x)^{\alpha+1}dx \le c_1\sigma^{-2}F(\sigma) \qquad \forall\, t\in(\tau_M, R]
\end{aligned}
\tag{3.16}
$$

where c, c', c_1 are positive constants depending on α, q, R but not on M. Since $\operatorname{supp}\varphi_\sigma \in B_{2\sigma}(0) := \{x \in \mathbb{R}^N : |x| < 2\sigma\}$, it follows by the Poincaré inequality that

$$
\int_{B_R}|\nabla_x(U_M\varphi_\sigma)^{\frac{\alpha+1}{2}}|^2dx \ge d_0\sigma^{-2}\int_{B_{2\sigma}}(U_M\varphi_\sigma)^{\alpha+1}dx,
$$

where $d_0 = \text{const} > 0$. This inequality and (3.16) imply (3.13) with $c_0 = cd_0$. Finally (3.14) follows from (3.2). □

Lemma 3.2. *Suppose that, for some constant $\sigma > 0$, $T \geqslant 0$ and $M \geqslant M_0 > c_2^{-1} := \frac{c_0}{2c_1}$ where c_0, c_1 are from* (3.13)*, the following inequalities hold:*

$$
F(\sigma) \ge M_0^\beta, \quad \beta = const \in \left(0, (2e)^{-1})\right), \tag{3.17}
$$

$$
\Phi_{\sigma,M}(T) \le M_0. \tag{3.18}
$$

Then there exists a constant $\mu > 0$*, depending only on* β *from* (3.17) *and* p *from* (2.8)*, such that*

$$\Phi_{\sigma,M}(t) \le c_2 F(\sigma) \quad \forall t > t' := T + \mu\omega(\sigma). \tag{3.19}$$

Proof. To simplify notation we drop the indices for Φ. First we show that there exists $\tau' \in [T, T + 2^{-1}\mu\omega(\sigma)] =: J$ such that

$$\Phi(\tau') \le c_2 F(\sigma). \tag{3.20}$$

By contradiction, assume that for every $\mu > 0$ there exists $M > M_0 > c_2^{-1}$ such that

$$\Phi(t) > c_2 F(\sigma) \qquad \forall t \in J. \tag{3.21}$$

Then, by (3.13) and (3.21) it follows

$$\Phi'(t) + \frac{c_0}{2\sigma^2}\Phi(t) \le 0 \qquad \forall t \in J. \tag{3.22}$$

Solving this ordinary differential inequality with initial condition (3.18) we obtain

$$\Phi(T+t) \le M_0 \exp\left(-\frac{c_0 t}{2\sigma^2}\right) \qquad \forall\, t \in \left[0, \frac{1}{2}\mu\omega(\sigma)\right]. \tag{3.23}$$

By (3.21) and (3.23),

$$M_0 \exp\left(-\frac{c_0\mu\omega(\sigma)}{4\sigma^2}\right) \ge \Phi\left(T + \frac{1}{2}\mu\omega(\sigma)\right) \ge c_2 F(\sigma).$$

By (3.5),

$$\exp\left(-\frac{c_0\mu\omega(\sigma)}{4\sigma^2}\right) \le F(\sigma)^{-\frac{c_0\mu(p-1)}{8(1+\varepsilon)}}.$$

Hence, by (3.17)

$$M_0 \ge c_2 F(\sigma)^{1+\frac{c_0\mu(p-1)}{8(1+\varepsilon)}} \ge c_2 M_0^{\left(1+\frac{c_0\mu(p-1)}{8(1+\varepsilon)}\right)\beta}.$$

Choosing

$$\mu = 8(2\beta^{-1} - 1)(1+\varepsilon)(p-1)^{-1}, \tag{3.24}$$

we obtain $M_0 \le c_2^{-1}$. In conclusion, for every $M_0 > c_2^{-1}$, (3.20) holds for some $\tau' \in J$. It remains to prove (3.19). Again, by contradiction, suppose that there exists $t' > \tau'$ such that $\Phi(t') > c_2 F(\sigma)$. Then, in view of (3.20), there exists a point $t^* \ge \tau'$ such that $\Phi(t^*) = c_2 F(\sigma)$ and $\frac{d\Phi}{dt}(t^*) \ge 0$. On the other hand, by (3.13),

$$\frac{d\Phi}{dt}(t^*) + \frac{c_0}{2}\Phi(t^*) \le 0$$

so that $\frac{d\Phi}{dt}(t^*) < 0$. Contradiction! □

4. Existence of large solutions

Notation. Let $M_k := \exp(\exp k)$ for $k = 1, 2 \dots$.

To simplify notation we shall write s_k instead of s_{M_k} (see (3.4)) and denote $\tau_k = \tau(s_k)$ (see (2.21)). We define $T_{k,j}$, $1 \le j \le k$, by induction as follows

$$\begin{aligned} T_{k,0} &= \max\{\tau_k + \mu\omega(s_k), \tau_{k-1}\}, \\ T_{k,j} &= \max\left\{T_{k,j-1} + \mu\omega(s_{k-j}), \tau_{k-j-1}\right\}, \\ &\quad \text{for } j \ge 1, \quad \tau_0 = 0, \end{aligned} \tag{4.1}$$

with $\tau(\cdot)$ as in Proposition 2.3 and μ given by (3.24). From (3.4) and (2.24) it follows that the sequences $\{s_k\}$ and $\{\tau_k\}$ are non-increasing. Hence

$$T_{k,j} = \max\left\{\tau_k + \mu\sum_{i=0}^{j}\omega(s_{k-i}), \tau_{k-j-1}\right\}, \quad \text{for } j = 1, 2 \dots, k. \tag{4.2}$$

The next result provides the main estimate in the proof of the Theorem 1.1.

Proposition 4.1. *Let U_k be the solution of* (2.1) *with $M = \bar{M}_k$ (see* (3.1)*). Then there exists a number $\ell > 0$ such that*

$$\int_{\{|x| \le R - s_{k-j}\}} U_k(t,x)^{\alpha+1} dx \le M_{k-j-1}, \quad \forall k \ge j + \ell, \ t \ge T_{k,j}. \tag{4.3}$$

Proof. Put $L = 1 + c_2$ (c_2 as in (3.19)) and let k^* be an integer such that

$$e^{k^*} > \max(3(1 + \ln L), -\ln c_2).$$

First we prove (4.3) for $j = 0$. By (3.3), (3.4) and Lemma 3.2 with $M = M_0 = M_k$:

$$\begin{aligned} \int_{\{|x|<R-s_k\}} U_k(t,x)^{\alpha+1} dx &\le \int_{s_k \le |x| < R - s_k} U_k(t,x)^{\alpha+1} dx + \int_{B_R} (U_k(t,x)\varphi_k)^{\alpha+1} dx \\ &\le F(s_k) + c_2 F(s_k) = L M_k^{\beta}, \quad \forall k > k^*, \ t \ge T_{k,0}, \end{aligned} \tag{4.4}$$

(Here we use the fact that $T_{k,0} \ge \tau_k'$, where $\tau_k' \in \left(\tau_k, \tau_k + 2^{-1}\mu\omega(s_k)\right)$ is a point such that (3.20) holds for $\tau' = \tau_k'$, and $M_k > c_2^{-1}$.) Note that

$$L M_k^{\beta} = \exp\left(\left(\frac{\ln L}{\exp k} + \beta\right)\exp k\right). \tag{4.5}$$

Due to (3.17) $0 < \beta < (2e)^{-1}$. Therefore it follows from (4.5): $L M_k^{\beta} \le M_{k-1}$ $\forall k > \ln 2 + 1 + \ln\ln L$. Consequently, by (4.4),

$$\int_{\{|x| \le R - s_k\}} U_k(t,x)^{\alpha+1} dx \le M_{k-1} \qquad \forall k \ge k^*, \ t \ge T_{k,0}. \tag{4.6}$$

Now assume that, for some $j \geq 1$ and $k > 1 + j + k^*$,

$$\int_{\{|x| \leq R - s_{k-j}\}} U_k(t,x)^{\alpha+1} dx \leq M_{k-j-1} \quad \forall t \geq T_{k,j}.$$

By Lemma 3.2 with $T = T_{k,j}$, $M = M_k$, $M_0 = M_{k-j-1}$, $\sigma = s_{k-j-1}$, it follows (recall that $F(s_{k-j-1}) = M_{k-j-1}^{\beta}$),

$$\int_{B_R} (U_k(t,x)\varphi_{k-j-1})^{\alpha+1} dx \leq c_2 F(s_{k-j-1}) \quad \forall t \geq T_{k,j+1}.$$

Hence, using Proposition 2.4,

$$\begin{aligned}\int_{\{|x|<R-s_{k-j-1}\}} U_k(t,x)^{\alpha+1} dx &\leq \int_{s_{k-j-1} \leq |x| < R - s_{k-j-1}} U_k(t,x)^{\alpha+1} dx \\ &\quad + \int_{B_R} (U_k(t,x)\varphi_{k-j-1})^{\alpha+1} dx \\ &\leq L M_{k-j-1}^{\beta} \leq M_{k-j-2}, \quad \forall t \geq T_{k,j+1}.\end{aligned} \tag{4.7}$$

□

Completion of proof of Theorem 1.1. Put $m = k - j$. We show that

$$T_{k,j} \to 0 \quad \text{as} \quad m = k - j \to \infty. \tag{4.8}$$

Since, $\tau_m \to 0$ as $m \to \infty$ it remains to show (see (4.2))

$$\sum_{i=0}^{j} \omega(s_{k-i}) \to 0 \quad \text{as} \quad k - j \to \infty.$$

It is enough to show that $\sum_{n=0}^{\infty} \omega(s_n) < \infty$. By (3.7),

$$s_n \leq C \exp(-n/2), \quad C^2 = \frac{2(1+\epsilon)\omega_R}{(p-1)\beta}.$$

Therefore, by Dini condition,

$$\sum_{n=0}^{\infty} \omega(s_n) \leq \sum_{n=0}^{\infty} \omega(C\exp(-n/2)) \leq \int_0^{\infty} \omega(C\exp(-s/2)) ds \leq 2\int_0^1 \omega(y)\frac{dy}{y} < \infty.$$

This proves (4.8).

Recall that α can be chosen arbitrarily large ($\alpha > \alpha_0$). Therefore we may assume that $\alpha > q$. By (4.3),

$$\int_{\{|x|\le R-s_m\}} U_{m+j}(t,x)^{q+1}dx \le M_{m-1}, \quad \forall m \ge \ell,\ T_{m+j,j} \le t \le R.$$

By standard parabolic estimates this implies

$$U_{m+j}(t,x) \le C M_{m-1}^{\frac{1}{q+1}} \quad \forall t \ge 2T_{m+j,j}, \quad \forall x : |x| < \frac{R-s_m}{2}, \quad \forall m \ge \ell.$$

This, together with (4.8), implies that $U_\infty = \lim_{k\to\infty} U_k$ remains bounded as $x \to 0$ for every $t \in (0, R)$. Now let v_M denote the solution of the initial value problem

$$\begin{aligned} v_t - \Delta v + hv^q = 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^N \\ v(0,x) = M \quad \forall x \in \mathbb{R}^N. \end{aligned} \tag{4.9}$$

Put $v_\infty = \lim_{M\to\infty} v_M$. Clearly $U_\infty > v_\infty$ in $(0, R) \times B_R$, for every $R > 0$. Hence v_∞ is a large solution of the equation in (4.9). □

5. The propagation of an isolated singularity

Consider the initial value problem

$$\begin{aligned} U_t - \Delta U + \overline{h}|U|^{p-1}U = 0 \quad \text{in} \quad \mathbb{R}_+^{N+1}, \\ U(0,x) = f(x) \quad \forall x \in \mathbb{R}^N, \end{aligned} \tag{5.1}$$

where $\overline{h} = \overline{h}(|x|)$ is given by (2.4).

Denote

$$r_j = 2^{-j}, \quad A_j = (r_j^2 \overline{h}(r_j))^{\frac{1}{q-1}}, \quad a_j = \overline{h}(r_j). \tag{5.2}$$

Theorem 1.2 is an immediate consequence of the following result.

Proposition 5.1. *Let U_j be the solution of* (5.1) *with $f = \gamma_j$ given by* (1.11). *Then, under the assumptions of Theorem 1.2 the function $U_\infty = \lim_{j\to\infty} U_j$ in $\mathbb{R}_+^{N+1}$ satisfies*

$$\lim_{x\to 0} U_\infty(t,x) = \infty \quad \forall t > 0. \tag{5.3}$$

Proof. Let

$$\Omega_j = \left\{ x \in \mathbb{R}^N : |x| < r_j \right\}, \quad \text{where } r_j \text{ is from (5.2).}$$

Consider the following auxiliary initial–boundary value problems:

$$\begin{aligned} u_t - \Delta u + \bar{h}(r_j)u^q &= 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega_j, \\ u(t,x) &= 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega_j \\ u(0,x) &= \gamma_j(x) \quad \text{for} \quad x \in \Omega_j, \quad \text{where } j = 1, 2 \dots . \end{aligned} \tag{5.4}$$

The solution of this problem will be denoted by u_j. Since the function μ from (1.3) is non-increasing, the corresponding function $\bar{h}$ is non-decreasing, so that

$$\bar{h}(r_j) = \sup_{0<s<r_j} \bar{h}(s). \tag{5.5}$$

Consequently, due to the comparison principle we have:

$$u_j \leq U_j \quad \text{in } \mathbb{R}_+ \times \Omega_j. \tag{5.6}$$

Next we estimate the asymptotic behaviour of u_j as $t \to \infty$. Let,

$$\tau := \frac{t}{r_j^2}, \quad y := \frac{x}{r_j}, \quad v_j(\tau, y) := A_j u_j(r_j^2\tau, r_j y). \tag{5.7}$$

Then, for every $j \in \mathbb{N}$, v_j is a solution of the problem:

$$\begin{aligned} v_\tau - \Delta_y v + v^q &= 0 \quad \text{in} \quad \mathbb{R}_+ \times B_1, \quad B_1 = \{|y| < 1\}, \\ v(\tau, y) &= 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial B_1, \\ v(0, y) = \bar{\gamma}(y) &:= \begin{cases} \varphi_1(2y) & \text{if} \quad |y| < \frac{1}{2} \\ 0 & \text{if} \quad |y| > \frac{1}{2}. \end{cases} \end{aligned} \tag{5.8}$$

Since this problem has a unique solution v it follows that $v_j = v$, for $j = 1, 2, \dots$. By [6, Theorem 3.1 and Remark 4.1], there exists a positive constant α, such that

$$\lim_{\tau \to \infty} \exp(\lambda_1 \tau) v(\tau, y) = \alpha \varphi_1(y) \tag{5.9}$$

uniformly with respect to $y \in B_1$. Hence, choosing β sufficiently large,

$$\frac{\alpha}{2}\varphi_1(y)\exp(-\lambda_1\tau) \leq v(y) \leq 2\alpha\varphi_1(y)\exp(-\lambda_1\tau) \quad \forall \tau \geq \beta, \ y \in B_1.$$

Consequently, by (5.2),

$$\begin{gathered} \frac{\alpha}{2A_j}\varphi_1\left(\frac{x}{r_j}\right)\exp\left(-\lambda_1\frac{t}{r_j^2}\right) \leq u_j(t,x) \leq \frac{2\alpha}{A_j}\varphi_1\left(\frac{x}{r_j}\right)\exp\left(-\frac{\lambda_1 t}{r_j^2}\right) \\ \forall\, t \geqslant \beta r_j^2, \quad \forall\, x \in \Omega_j, \quad \forall\, j \in \mathbb{N}. \end{gathered} \tag{5.10}$$

Let $t_j > 0$ be the number determined by,

$$\frac{\alpha}{2A_j}\varphi_1\left(\frac{x}{r_j}\right)\exp\left(-\frac{\lambda_1 t_j}{r_j^2}\right) = \gamma_{j-1}(x) \qquad \forall x : |x| < r_j. \tag{5.11}$$

Due to (1.11) and (5.2) this is equivalent to,

$$\frac{\alpha}{2}\exp\left(-\frac{\lambda_1 t_j}{r_j^2}\right)=2^{-\frac{2}{q-1}}\left(\frac{a_j}{a_{j-1}}\right)^{\frac{1}{q-1}}=2^{-\frac{2}{q-1}}\exp\left(-\frac{\mu(r_j)-\mu(r_{j-1})}{q-1}\right). \tag{5.12}$$

Therefore,

$$\lambda_1\frac{t_j}{r_j^2} = \frac{\mu(r_j)-\mu(r_{j-1})}{q-1} + \bar{c}, \qquad \bar{c} = \frac{q-3}{q-1}\ln 2 + \ln\alpha. \tag{5.13}$$

By (1.9) there exists $\varkappa \in (0,1)$ such that

$$\mu(2^{-i}) - \mu(2^{-(i-1)}) \geq \varkappa\mu(2^{-i}) \qquad \forall i \in \mathbb{N}. \tag{5.14}$$

(Here we assume that (1.9) holds with $a = 2$; otherwise we redefine $r_j = a^{-j}$.) By (5.13) and (5.14),

$$\frac{\varkappa r_j^2}{\lambda_1(q-1)}(\mu(r_j)+(q-1)\bar{c}) \leq t_j < \frac{1}{\lambda_1(q-1)}r_j^2(\mu(r_j)+(q-1)\bar{c}). \tag{5.15}$$

Since $\mu(r_j) \to \infty$ as $j \to \infty$ it follows that, for any $\beta > 0$, there exist positive constants c_0, c_1 and j_0 such that

$$\beta r_j^2 \leq c_0\omega(r_j) \leq t_j \leq c_1\omega(r_j) \qquad \forall j \geqslant j_0. \tag{5.16}$$

This inequality will be used with β as in (5.10). Thus, we found a value t_j, satisfying (5.16), for which due to (5.10), (5.11) the following important intermediate inequality holds:

$$\gamma_{j-1}(x) \leq u_j(t_j, x) \qquad \forall x \in \Omega_j, \quad \forall j \geq j_0. \tag{5.17}$$

Now we return back to the solution U_j of the problem (5.1) with $f = \gamma_j$. By (5.17), (5.6) we have

$$u_{j-1}(0,x) = \gamma_{j-1}(x) \leq u_j(t_j,x) \leq U_j(t_j,x) \qquad \forall x \in \Omega_j, \quad \forall j \geq j_0. \tag{5.18}$$

Moreover, $0 = u_{j-1}(0,x) = \gamma_{j-1}(x) \leqslant U_j(t_j,x)$ for all $x : r_j \leqslant x \leqslant r_{j-1}$. Additionally we have

$$U_j|_{\mathbb{R}_+\times\partial\Omega_{j-1}} \geq 0 = u_{j-1}|_{\mathbb{R}_+\times\partial\Omega_{j-1}}$$

Therefore by the comparison principle,

$$u_{j-1}(t,x) \le U_j(t_j + t, x) \qquad \forall\, j \geqslant j_0, \quad \forall\, t > 0, \quad \forall\, x \in \Omega_{j-1}. \tag{5.19}$$

Fix now $j > j_0$. We claim that for every integer k such that $0 \le k \le j - j_0$ the following inequality holds:

$$u_{j-k-1}(t,x) \le U_j\left(t + \sum_{i=0}^{k} t_{j-i}, x\right) \quad \forall\, x \in \Omega_{j-k-1}, \quad \forall\, t > 0. \tag{5.20}$$

Indeed, by (5.19) this inequality holds for $k = 0$. Let $0 < k \le j - j_0$ and suppose that (5.20) holds when k is replaced by $k - 1$:

$$u_{j-k}(t,x) \le U_j\left(t + \sum_{i=0}^{k-1} t_{j-i}, x\right) \qquad \forall\, x \in \Omega_{j-k}, \quad \forall\, t > 0. \tag{5.21}$$

In particular for $t = t_{j-k}$ we have:

$$u_{j-k}(t_{j-k},x) \le U_j\left(\sum_{i=0}^{k} t_{j-i}, x\right) \qquad \forall\, x \in \Omega_{j-k}, \quad \forall\, t > 0. \tag{5.22}$$

By (5.17) and (5.22):

$$\gamma_{j-k-1}(x) \le u_{j-k}(t_{j-k},x) \le U_j\left(\sum_{i=0}^{k} t_{j-i}, x\right) \qquad \forall\, x \in \Omega_{j-k}.$$

Since $\gamma_{j-k-1}(x) = 0$ for all $x \in \Omega_{j-k-1} \setminus \Omega_{j-k}$ it follows that,

$$u_{j-k-1}(0,x) \le U_j\left(\sum_{i=0}^{k} t_{j-i}, x\right) \qquad \forall\, x \in \Omega_{j-k-1}. \tag{5.23}$$

In addition $u_{j-k-1}(t,x) = 0 < U_j(t,x)$ for every $(t,x) \in \mathbb{R}_+ \times \partial\Omega_{j-k-1}$. By the comparison principle, this implies (5.20).

Now we are ready to complete to proof of Proposition 5.1. By (1.10) we have:

$$\sum_{j=0}^{\infty} \omega(r_j) = \infty.$$

Therefore by (5.16)

$$\sum_{j=0}^{\infty} t_j = \infty. \tag{5.24}$$

Furthermore, since $\omega(s)$ is bounded in $(0, 1]$, (5.16) implies that

$$\bar{t} := \sup_{i>0} t_i < \infty.$$

By (5.24) for any $b \in (0, \infty)$ there exists a number j_b such that

$$\sum_{i=0}^{j} t_i \geq b \quad \forall\, j \geq j_b.$$

Pick $b > \bar{t}$ and for each $j \geq j_b$ denote

$$\ell_j = \sup \left\{ \ell \in \mathbb{N} :\ 0 \leq \ell \leq j,\ \sum_{i=\ell}^{j} t_i \geq b \right\}.$$

Then $\sum_{i=\ell_j+1}^{j} t_i < b$ and

$$b \leq b_j := \sum_{i=\ell_j}^{j} t_i \leq b + t_{\ell_j} \leq b + \bar{t} \tag{5.25}$$

Clearly the function $j \to \ell_j$ is non-decreasing and, in view of (5.24),

$$\lim_{j\to\infty} \ell_j = \infty. \tag{5.26}$$

By (5.20),

$$u_{\ell_j}(t_{\ell_j}, x) \leq U_j(b_j, x) \quad \forall\, x \in \Omega_{\ell_j}. \tag{5.27}$$

By (5.10) and (5.16),

$$\sup_{x\in\Omega_{\ell_j}} u_{\ell_j}(t_{\ell_j}, x) \leq 4u_{\ell_j}(t_{\ell_j}, 0). \tag{5.28}$$

Finally, by (5.17), (5.27) and (5.28) we obtain,

$$A_{\ell_j-1}^{-1} = \gamma_{\ell_j-1}(0) \leq 4u_{\ell_j}(t_{\ell_j}, 0) \leq 4U_j(b_j, 0). \tag{5.29}$$

Since $\ell_j \to \infty$ and $A_k \to 0$ as $k \to \infty$, it follows that

$$\lim_{j\to\infty} U_j(b_j, 0) = \infty. \tag{5.30}$$

Denote, $U_\infty := \lim_{k\to\infty} U_k$. Suppose that $U_\infty(t_0, \cdot) \in L^1(B_\epsilon)$, for some $t_0 > 0$ and some $\epsilon > 0$. Then $U_\infty(t_0, \cdot) \in L^1_{loc}(\mathbb{R}^N)$ and consequently U_∞ is locally bounded for $t > t_0$. Therefore, since b is arbitrarily large, (5.30) implies that $x \mapsto U_\infty(t_0, x)$ is unbounded in any neighborhood of zero for every $t_0 > 0$. □

Proof of Theorem 1.3. Let $\overline{U_j}$ be a solution of problem (5.1) with

$$f(x) = f_j(x) = A_{j+1}^{-1} r_{j+1}^{N} \delta_0(x), \tag{5.31}$$

where r_{j+1}, A_{j+1} are defined in (5.2). Additionally let v_j be the solution of the problem:

$$\begin{aligned} v_{j_t} - \Delta v_j + \bar{h}(r_{j+1}) v_j^q &= 0 && \text{in } \mathbb{R}_+ \times \Omega_{j+1}, \\ v_j &= 0 && \text{on } \mathbb{R}_+ \times \partial\Omega_{j+1}, \\ v_j(0,x) &= A_{j+1}^{-1} r_{j+1}^{N} \delta_0(x) && \text{in } \Omega_{j+1}. \end{aligned} \tag{5.32}$$

By the comparison principle we have

$$\overline{U_j} \geqslant v_j \qquad \forall (t,x) \in \mathbb{R}_+ \times \Omega_{j+1}. \tag{5.33}$$

By scaling

$$y = x \cdot r_{j+1}^{-1}, \quad \tau = t r_{j+1}^{-2}, \quad \tilde{v}_j(\tau, y) = A_{j+1} v_j(r_{j+1}^2 \tau, r_{j+1} y), \tag{5.34}$$

we find that, for each j, $\tilde{v}_j$ satisfies

$$\begin{aligned} \tilde{v}_\tau - \Delta_y \tilde{v} + \tilde{v}^q &= 0 \quad \text{in} \quad \mathbb{R}_+ \times B_1, \\ \tilde{v} &= 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial B_1, \\ \tilde{v}(0,y) &= \delta_0(y) \quad \text{in} \quad B_1. \end{aligned}$$

This boundary value problem has a unique solution. For every $t_0 > 0$, the function $\tilde{v}(t_0, \cdot)$ is bounded in B_1. Therefore, we can apply [6, Theorem 3.1] in $(t_0, \infty) \times B_1$ to obtain

$$2^{-1} \alpha \varphi_1(y) \exp(-\lambda_1 \tau) \le \tilde{v}(\tau, y) \le 2\alpha \varphi_1(y) \exp(-\lambda_1 \tau) \;\; \forall \tau > \beta_0, \;\forall y \in B_1, \tag{5.35}$$

for some positive α and β_0. Choose $\beta > \beta_0$ large enough so that $\alpha' = 2\alpha \exp(-\lambda_1 \beta) \le 1$. Then,

$$\frac{\alpha'}{4} \varphi_1(y) \le \tilde{v}(\beta, y) \le \varphi_1(y) \qquad \forall y \in B_1,$$

and consequently

$$\frac{\alpha'}{4} A_{j+1}^{-1} \varphi_1\left(\frac{x}{r_{j+1}}\right) \le v_j\left(r_{j+1}^2 \beta, x\right) \le A_{j+1}^{-1} \varphi_1\left(\frac{x}{r_{j+1}}\right) \\ \forall x \in \Omega_{j+1}, \quad \text{for } j = 1, 2, \dots . \tag{5.36}$$

Hence, by (5.33),

$$\overline{U_j}\left(r_{j+1}^2 \beta, x\right) \geqslant \frac{\alpha'}{4} A_{j+1}^{-1} \varphi_1\left(\frac{x}{r_{j+1}}\right) = \frac{\alpha'}{4} \frac{A_j}{A_{j+1}} \gamma_j(x) \qquad \forall x \in \Omega_{j+1}, \tag{5.37}$$

with γ_j as in (1.11). By (1.3), (1.4) $A_j A_{j+1}^{-1} \geqslant 2^{\frac{2}{q-1}}$. Therefore, as $\overline{U}_j \geq 0$,

$$\overline{U}_j\left(r_{j+1}^2\beta, x\right) \geqslant \bar{c}\gamma_j(x) \qquad \forall x \in \mathbb{R}^N, \quad \bar{c} = 4^{\frac{2-q}{q-1}}\alpha'. \tag{5.38}$$

Thus, the function $\overline{U}_j$ given by $V_j(t,x) = \bar{U}_j(t + r_{j+1}^2\beta, x)$ is the solution of problem (5.1) with initial data $f \geqslant \bar{c}\gamma_j$. If $\bar{c} \geqslant 1$ then, by the comparison principle $V_j \geqslant U_j$. If $\bar{c} < 1$ then $\bar{c}U_j$ is a subsolution of (5.1). Therefore, applying again the comparison principle we obtain $V_j \geqslant \bar{c}U_j$. If $V_\infty = \lim V_j$, Proposition 5.1 implies that $\lim_{x\to 0} V_\infty(t,x) = \infty$ for every $t > 0$. But $\lim V_j = \lim \bar{U}_j$ everywhere in $\mathbb{R}_+ \times (\mathbb{R}^N \setminus \{0\})$. This proves the theorem in the case that $h = H$. Obviously it remains true for h satisfying (1.8). □

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