

Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds

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Abstract. In this paper we prove Nikolskii's inequality (also known as the reverse Hölder inequality) on general compact Lie groups and on compact homogeneous spaces with the constant interpreted in terms of the eigenvalue counting function of the Laplacian on the space, giving the best constant for certain indices, attained on the Dirichlet kernel. Consequently, we establish embedding theorems between Besov spaces on compact homogeneous spaces, as well as embeddings between Besov spaces and Wiener and Beurling spaces. We also analyse Triebel-Lizorkin spaces and β -versions of Wiener and Beurling spaces and their embeddings, and interpolation properties of all these spaces.

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1. Introduction

In this paper we analyse the families of Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact Lie groups G and on compact homogeneous manifolds G/K . To a large extent, the analysis is based on establishing an appropriate version of Nikolskii's inequality (also known as the reverse Hölder inequality) in this setting and on working with discrete Lebesgue spaces on the unitary dual of the group and its class I representations.

The classical Nikolskii inequality for trigonometric polynomials T_L of degree at most L on the circle was given by [27]

$$\|T_L\|_{L^q(\mathbb{T})} \leq C L^{1/p-1/q} \|T_L\|_{L^p(\mathbb{T})},$$

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where $1 \leq p < q \leq \infty$ and the constant C can be taken to be 2. A similar result is also known ([27]) on the Euclidean space for entire functions of exponential type. Moreover, for $f \in L^p(\mathbb{R}^n)$ such that $\text{supp}(\widehat{f})$ is compact we have (see [30])

$$\|f\|_{L^q(\mathbb{R}^n)} \leq (C(p)\mu(\text{conv}[\text{supp}(\widehat{f})]))^{1/p-1/q} \|f\|_{L^p(\mathbb{R}^n)}, \quad (1.1)$$

where $1 \leq p \leq q \leq \infty$, $\mu(E)$ denotes the Lebesgue measure of E , and $\text{conv}[E]$ denotes the convex hull of E . Inequalities of type (1.1) are sometimes called the Plancherel-Polya-Nikolskii inequalities.

The Nikolskii inequality plays a key role in the investigations of properties of different function spaces (see, e.g., [28,44]), in approximation theory (see, e.g., [7]), or to obtain embedding theorems (see, e.g., [10,44]).

The Nikolskii inequality for the spherical polynomials on \mathbb{S}^n was proved in [23, Th. 1], see also [5,26]. Moreover, Pesenson in [35] obtained a very general version of Bernstein-Nikolskii type inequalities on non-compact symmetric spaces, and in [34] on compact homogeneous spaces.

In this paper, by a very different method of proof, we extend Pesenson's result [34] to a wider range of indices $0 < p < q \leq \infty$ and give an interpretation of the constant that we obtain in terms of the eigenvalue counting function for the Laplace operator on the group acting on the homogeneous manifold. For certain indices this gives the best constant in the Nikolskii inequality, and this constant is attained on the Dirichlet kernel.

Consequently, we use this to establish embedding properties between different families of function spaces. Besov spaces on Lie groups have been recently actively analysed, e.g., from the point of view of the heat kernel [40] and the Littlewood-Paley theory [14], see also [19], but apart from general definitions and certain properties no embedding properties for these spaces have been given. For functions on \mathbb{S}^n , the Besov spaces were studied in, e.g., [21]. In that paper the Nikolskii inequality was applied to obtain the Sobolev-type embedding theorems. For more general results see [11,12].

Apart from embeddings between Besov spaces that can be obtained by trivial arguments, the Nikolskii inequality allows us to derive a rather complete list of embeddings with respect to all three indices of the space. The norms we use for proofs depend on the number k_ξ of invariant vector fields in the representation space (for a representation $[\xi] \in \widehat{G}$ of dimension d_ξ) with respect to the subgroup K . In Section 9, as a consequence of the interpolation theorems, we show that for certain ranges of indices the Besov spaces defined in terms of the global Littlewood-Paley theory agree with the Besov spaces defined in local coordinates. Still, in this paper, Besov and Triebel-Lizorkin spaces are originally defined using dyadic decompositions on the Fourier transform side, also allowing to cover a wider range of indices, and for a connection with the bandlimited expansions we refer to Geller and Pesenson [16].

The setting of this paper provides a general environment where ideas resembling the classical analysis dealing with the Fourier series can still be carried out.

The overall analysis of this paper relies on working with discrete Lebesgue spaces $\ell^p(\widehat{G})$ on the unitary dual \widehat{G} of the compact Lie group G . Such spaces have been introduced and developed recently in [36] and they allow one to quantify the Fourier transforms of functions on G by fixing the Hilbert-Schmidt norms of the Fourier coefficients and working with weights expressed in terms of the dimensions of the representations of the group. These spaces $\ell^p(\widehat{G})$ have been already effective and were used in [9] to give the characterisation of Gevrey spaces (of Roumeau and Beurling types) and the corresponding spaces of ultradistributions on compact Lie groups and homogeneous spaces.

In this paper we modify this construction and extend it further to enable one to work with class I representations thus facilitating the analysis of functions which are constant on right cosets leading to the quantification of the Fourier coefficients on the compact homogeneous spaces G/K .

Typical examples of homogeneous spaces are the spheres $\mathbb{S}^n = \text{SO}(n+1)/\text{SO}(n)$ in which case we can take $k_\xi = 1$. Similarly, we can consider complex spheres (or complex projective spaces) $\mathbb{P}\mathbb{C}^n = \text{SU}(n+1)/\text{SU}(n)$, or quaternionic projective spaces $\mathbb{P}\mathbb{H}^n = \text{Sp}(n+1)/\text{Sp}(n) \times \text{Sp}(1)$. In the case of the trivial subgroup $K = \{e\}$ the homogenous space is the compact Lie group G itself, and we recover the original spaces in [36] by taking $k_\xi = d_\xi$.

Consequently, we look at the Wiener algebra A of functions with summable Fourier transforms and its ℓ^β -versions, the β -Wiener spaces A^β . In particular, we prove the embedding $B_{2,1}^{n/2} \hookrightarrow A$ as well as its β -version as the embeddings between A^β and $B_{p,\beta}^r$ (and their inverses depending on whether $1 < p \leq 2$ or $2 \leq p < \infty$). We note that our version of these spaces is based on the scale of ℓ^p spaces described above (given in (2.2)) and not on the Schatten norms. Spaces with Schatten norms have been considered as well as their weighted versions, see [25], and also [8] and [24] for rather extensive analysis. Such spaces go under the name of Beurling-Fourier spaces in the literature. To distinguish from Beurling spaces described below (which appear to be almost new in the recent literature especially since they are based on the different scale of ℓ^p spaces developed in [36]), we use the name of Wiener spaces for the spaces A^β referring to the original studies of Wiener of what is also known as the Wiener algebra.

The Beurling space A^* was introduced by Beurling [2] for establishing contraction properties of functions. In [4] it was shown that $A^*(\mathbb{T})$ is an algebra and its properties were investigated. The definition of the Beurling space in multi-dimensions, even on \mathbb{T}^n , is not straightforward since we would need to take into account the sums in different directions which can be done in different ways. For this, we first reformulate the norm of the space A^* in the (Littlewood-Paley) way which allows extension to spaces when the unitary dual \widehat{G} is discrete but is different from \mathbb{Z}^n . Consequently, we analyse the space A^* as well as its β -version $A^{*\beta}$ in the setting of general compact homogeneous spaces. These function spaces play an important role in the summability theory and in the Fourier synthesis (see, e.g., [42, Theorems 1.25 and 1.16] and [43, Theorem 8.1.3]). On the circle, the spaces $A^{*\beta}$ were studied in [43, Chapter 6]. Here, we analyse an analogue of these

spaces on general compact homogeneous manifolds and prove two-sided embedding properties between these spaces and appropriate Besov spaces. The analysis is again based on the scale of ℓ^p spaces in (2.2) which allow us to also establish their interpolation properties.

We note that the questions of estimating projectors to individual eigenspaces (rather than to eigenspaces corresponding to eigenvalues $\leq L$ as it arises in the Nikolskii inequality) have appeared naturally in different problems in harmonic analysis and have been also studied. For example, in the analysis related to the Carleson-Sjölin theorem for spherical harmonics on the 2-sphere, Sogge [41, Theorem 4.1] obtained L^2 - L^p estimates for harmonic projections on spheres. The same L^2 - L^p estimates but on compact Lie groups have been obtained by Giacalone and Ricci [17]. We note that on the one hand, estimates for the projection to the eigenspace corresponding to an eigenvalue L are better than those appearing when projecting to the span of eigenspaces corresponding to eigenvalues $\leq L$. But on the other hand, the Nikolskii inequality provides a better estimate compared to the one that can be obtained by summing up the individual ones. In Corollary 4.4 we show that for certain indices the power in the projection to the span of eigenspaces corresponding to eigenvalues $\leq L$ on a compact Lie group can be improved compared to the one in the Nikolskii's inequality (but which, in turn, is also sharp for certain indices, see Theorem 3.1).

The paper is organised as follows. Section 2 is devoted to introducing the spaces $\ell^p(\widehat{G}_0)$ for the class I representations of the compact Lie group G and for the corresponding Fourier analysis. Sections 3 and 4 are devoted to Nikolskii's inequality. There, in Section 3 we establish the Nikolskii inequality on general compact homogeneous spaces, and in Section 4 we give its refinement of compact Lie groups taking into account the number of non-zero Fourier coefficients in the constant. In Section 5 we analyse embedding properties between various smooth function spaces including Sobolev, Besov, and Triebel-Lizorkin spaces. Section 6 and Section 7 deal with Wiener and Beurling spaces, respectively. In Section 8 we establish interpolation properties of the introduced spaces. In Section 9 we show that, for certain ranges of indices, the Besov spaces introduced in this paper by the global Littlewood-Paley theory agree with the Besov spaces in local coordinates.

For an index $1 \leq p \leq \infty$ we will always write p' for its dual index defined by $\frac{1}{p} + \frac{1}{p'} = 1$. For $a, b \geq 0$, we write $a \lesssim b$ if there is a constant $C > 0$ such that $a \lesssim Cb$, and we write $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$.

2. Fourier analysis on homogeneous spaces

In this section we collect the necessary facts concerning the Fourier analysis on compact homogeneous spaces. We start with compact Lie groups.

Let G be a compact Lie group of dimension $\dim G$ with its unitary dual denoted by \widehat{G} . Fixing the basis in representation spaces, we can work with matrix representations $\xi : G \rightarrow \mathbb{C}^{d_\xi \times d_\xi}$ of degree d_ξ . We recall that by the Peter-Weyl

theorem the collection $\{\sqrt{d_\xi}\xi_{ij} : [\xi] \in \widehat{G}, \text{ for } 1 \leq i, j \leq d_\xi\}$ forms an orthonormal basis in $L^2(G)$ with respect to the normalised Haar measure on G . The integrals and the spaces $L^p(G)$ are always taken with respect to the normalised bi-invariant Haar measure on G . For $f \in C^\infty(G)$, its Fourier coefficient at $\xi \in [\xi] \in \widehat{G}$ is defined as

$$\widehat{f}(\xi) = \int_G f(x)\xi(x)^*dx.$$

Consequently, we have $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier series of f becomes

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)). \tag{2.1}$$

In [36], the Lebesgue spaces $\ell^p(\widehat{G})$ have been introduced on \widehat{G} by the following norms, which we now write for the Fourier coefficients of f , as

$$\|\widehat{f}\|_{\ell^p(\widehat{G})} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi^{p(\frac{2}{p}-\frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^p \right)^{1/p}, \text{ for } 1 \leq p < \infty, \tag{2.2}$$

and

$$\|\widehat{f}\|_{\ell^\infty(\widehat{G})} = \sup_{[\xi] \in \widehat{G}} d_\xi^{-\frac{1}{2}} \|\widehat{f}(\xi)\|_{\text{HS}}, \tag{2.3}$$

with $\|\widehat{f}(\xi)\|_{\text{HS}} = \text{Tr}(\widehat{f}(\xi)\widehat{f}(\xi)^*)^{1/2}$. These are interpolation spaces, and the Hausdorff-Young inequality holds in them in both directions depending on p . We refer to [36, Section 10.3.3] for these and other properties of these spaces, and to [36,37] for further operator analysis.

For $[\xi] \in \widehat{G}$, we denote by $\langle \xi \rangle$ the eigenvalue of the operator $(I - \mathcal{L}_G)^{1/2}$ corresponding to the representation class $[\xi] \in \widehat{G}$ where \mathcal{L}_G is the Laplace-Beltrami operator (Casimir element) on G .

We now give modifications of this construction for compact homogeneous spaces M . Let G be a compact motion group of M and K a stationary subgroup at some point, so that we can identify $M = G/K$. Typical examples are the spheres $\mathbb{S}^n = \text{SO}(n + 1)/\text{SO}(n)$ or complex spheres $\mathbb{P}\mathbb{C}^n = \text{SU}(n + 1)/\text{SU}(n)$, or quaternionic projective spaces $\mathbb{P}\mathbb{H}^n$. We normalise measures so that the measure on K is a probability one.

We denote by \widehat{G}_0 the subset of \widehat{G} of representations that are class I with respect to the subgroup K . This means that $[\xi] \in \widehat{G}_0$ if ξ has at least one non-zero invariant vector a with respect to K , *i.e.*, that $\xi(k)a = a$ for all $k \in K$. Let $\mathcal{H}_\xi \simeq \mathbb{C}^{d_\xi}$ denote the representation space of $\xi(x) : \mathcal{H}_\xi \rightarrow \mathcal{H}_\xi$ and let \mathcal{B}_ξ be the space of these invariant vectors. Let $k_\xi := \dim \mathcal{B}_\xi$. We fix an orthonormal basis of \mathcal{H}_ξ so that its first k_ξ vectors are the basis of \mathcal{B}_ξ . The matrix elements $\xi_{ij}(x)$, for $1 \leq j \leq k_\xi$, are invariant under the right shifts by K .

We note that if $K = \{e\}$ so that $M = G/K = G$ is the Lie group, we have $\widehat{G} = \widehat{G}_0$ and $k_\xi = d_\xi$ for all ξ . As the other extreme, if K is a massive subgroup of

G , i.e., if for every ξ there is precisely one invariant vector with respect to K , we have $k_\xi = 1$ for all $[\xi] \in \widehat{G}_0$. This is, for example, the case for the spheres $M = \mathbb{S}^n$. Other examples can be found in Vilenkin [47].

For a function $f \in C^\infty(G/K)$ we can write the Fourier series of its canonical lifting $\widetilde{f}(g) := f(gK)$ to G , $\widetilde{f} \in C^\infty(G)$, so that the Fourier coefficients satisfy $\widetilde{f}(\xi) = 0$ for all representations with $[\xi] \notin \widehat{G}_0$. Moreover, for class I representations we have $\widetilde{f}(\xi)_{jk} = 0$ for $j > k_\xi$. We will often drop writing tilde for simplicity, and agree that for a distribution $f \in \mathcal{D}'(G/K)$ we have $\widehat{f}(\xi) = 0$ for $[\xi] \notin \widehat{G}_0$ and $\widehat{f}(\xi)_{ij} = 0$ if $i > k_\xi$.

With this, we can write the Fourier series of f (or of \widetilde{f}) in terms of the spherical functions ξ_{ij} , $1 \leq j \leq k_\xi$, of the representations ξ , $[\xi] \in \widehat{G}_0$, with respect to the subgroup K . Namely, the Fourier series (2.1) becomes

$$f(x) = \sum_{[\xi] \in \widehat{G}_0} d_\xi \sum_{i=1}^{d_\xi} \sum_{j=1}^{k_\xi} \widehat{f}(\xi)_{ji} \xi_{ij}(x). \tag{2.4}$$

For the details of this construction we refer to [48].

In the case we work on the homogeneous space G/K , in order to shorten the notation, for $[\xi] \in \widehat{G}_0$, it makes sense to set $\xi(x)_{ij} := 0$ for all $j > k_\xi$. Indeed, this will not change the Fourier series expression (2.4) since these entries do not appear in the sum (2.4) for $f \in \mathcal{D}'(G/K)$. With this convention we can still write (2.4) in the compact form

$$f(x) = \sum_{[\xi] \in \widehat{G}_0} d_\xi \text{Tr}(\widehat{f}(\xi) \xi(x)), \text{ for } f \in C^\infty(G/K). \tag{2.5}$$

For future use, we note that with these conventions the matrix $\xi(x)\xi(x)^*$ is diagonal with the first k_ξ diagonal entries equal to one and others equal to zero, so that we have

$$\|\xi(x)\|_{\text{HS}} = k_\xi^{1/2} \text{ for all } [\xi] \in \widehat{G}_0 \text{ and } x \in G/K. \tag{2.6}$$

For the space of Fourier coefficients of functions on G/K we define

$$\Sigma(G/K) := \left\{ \sigma : \xi \mapsto \sigma(\xi) : [\xi] \in \widehat{G}_0, \sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}, \sigma(\xi)_{ij} = 0 \text{ for } i > k_\xi \right\}. \tag{2.7}$$

In analogy to (2.2), we can define the Lebesgue spaces $\ell^p(\widehat{G}_0)$ by the following norms which we will apply to Fourier coefficients $f \in \Sigma(G/K)$ of $f \in \mathcal{D}'(G/K)$. Thus, for $\sigma \in \Sigma(G/K)$ we set

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{[\xi] \in \widehat{G}_0} d_\xi k_\xi^{p(\frac{1}{p}-\frac{1}{2})} \|\sigma(\xi)\|_{\text{HS}}^p \right)^{1/p}, \text{ if } 1 \leq p < \infty, \tag{2.8}$$

and

$$\|\sigma\|_{\ell^\infty(\widehat{G}_0)} := \sup_{[\xi] \in \widehat{G}_0} k_\xi^{-\frac{1}{2}} \|\sigma(\xi)\|_{\text{HS}}.$$

Note that the choice of exponents in (2.8) is natural since in this case these spaces satisfy the classical properties: they are interpolation spaces having the duality property and satisfying the Hausdorff-Young inequality. In the case $K = \{e\}$, so that $G/K = G$, these spaces coincide with those defined by (2.2) since $k_\xi = d_\xi$ in this case. Again, by the same argument as that in [36], these spaces are interpolation spaces and the Hausdorff-Young inequality holds:

$$\|\widehat{f}\|_{\ell^{p'}(\widehat{G}_0)} \leq \|f\|_{L^p(G/K)}, \quad \|f\|_{L^{p'}(G/K)} \leq \|\widehat{f}\|_{\ell^p(\widehat{G}_0)}, \quad \text{for } 1 \leq p \leq 2, \quad (2.9)$$

where here and in the sequel we define the dual index p' by $\frac{1}{p'} + \frac{1}{p} = 1$.

We will need the following embedding property of these spaces:

$$\ell^p(\widehat{G}_0) \hookrightarrow \ell^q(\widehat{G}_0) \text{ and } \|\sigma\|_{\ell^q(\widehat{G}_0)} \leq \|\sigma\|_{\ell^p(\widehat{G}_0)}, \quad \text{for } 1 \leq p \leq q \leq \infty. \quad (2.10)$$

Indeed, we can assume $p < q$. Then, in the case $1 \leq p < \infty$ and $q = \infty$, we can estimate

$$\|\sigma\|_{\ell^\infty(\widehat{G}_0)}^p = \left(\sup_{[\xi] \in \widehat{G}_0} k_\xi^{-\frac{1}{2}} \|\sigma(\xi)\|_{\text{HS}} \right)^p \leq \sum_{[\xi] \in \widehat{G}_0} d_\xi k_\xi^{1-\frac{p}{2}} \|\sigma(\xi)\|_{\text{HS}}^p = \|\sigma\|_{\ell^p(\widehat{G}_0)}^p.$$

Let now $1 \leq p < q < \infty$. Denoting $a_\xi := d_\xi^{\frac{1}{q}} k_\xi^{\frac{1}{q}-\frac{1}{2}} \|\sigma(\xi)\|_{\text{HS}}$, we get

$$\begin{aligned} \|\sigma\|_{\ell^q(\widehat{G}_0)} &= \left(\sum_{[\xi] \in \widehat{G}_0} a_\xi^q \right)^{\frac{1}{q}} \leq \left(\sum_{[\xi] \in \widehat{G}_0} a_\xi^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{[\xi] \in \widehat{G}_0} d_\xi^{\frac{p}{q}} k_\xi^{\frac{p}{q}-\frac{p}{2}} \|\sigma(\pi)\|_{\text{HS}}^p \right)^{\frac{1}{p}} \leq \|\sigma\|_{\ell^p(\widehat{G}_0)}, \end{aligned}$$

completing the proof.

Let $\mathcal{L}_{G/K}$ be the differential operator on G/K obtained by \mathcal{L}_G acting on functions that are constant on right cosets of G , i.e., such that $\widetilde{\mathcal{L}_{G/K}} f = \mathcal{L}_G \widetilde{f}$ for $f \in C^\infty(G/K)$. For $[\xi] \in \widehat{G}_0$, we denote by $\langle \xi \rangle$ the eigenvalue of the operator $(1 - \mathcal{L}_{G/K})^{1/2}$ corresponding to $\xi(x)_{ij}$, for $1 \leq i \leq d_\xi$ and $1 \leq j \leq k_\xi$. The operator $(1 - \mathcal{L}_{G/K})^{1/2}$ is a classical first order elliptic pseudo-differential operator (see, e.g., Seeley [38]), with the same eigenspaces as the operator $-\mathcal{L}_{G/K}$. We refer to [36, 37] for further details of the Fourier and operator analysis on compact homogeneous spaces.

Let $N(L)$ be the Weyl counting function for the elliptic pseudo-differential operator $(1 - \mathcal{L}_{G/K})^{1/2}$, yielding the number of eigenvalues $\leq L$, counted with multiplicity. From the above, we have

$$N(L) = \sum_{\substack{\langle \xi \rangle \leq L \\ [\xi] \in \widehat{G}_0}} d_\xi k_\xi. \quad (2.11)$$

For sufficiently large L , the Weyl asymptotic formula asserts that

$$N(L) \sim C_0 L^n, \quad C_0 = (2\pi)^{-n} \int_{\sigma_1(x, \omega) < 1} dx d\omega, \tag{2.12}$$

where $n = \dim G/K$, and the integral is taken with respect to the measure on the cotangent bundle $T^*(G/K)$ induced by the canonical symplectic structure, with σ_1 being the principle symbol of the operator $(1 - \mathcal{L}_{G/K})^{1/2}$, see, e.g., Shubin [39].

In the sequel we will often use the counting function for eigenvalues in the dyadic annuli, and in these cases we can always assume that $\langle \xi \rangle$ is sufficiently large, so that (2.11)–(2.12) imply

$$\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi \asymp 2^{sn} \tag{2.13}$$

with a constant independent of s , in view of the estimates for the remainder in Weyl spectral asymptotics, see, e.g., [6]. This property will be often used in the sequel, and we may often write \leq instead of \lesssim in the corresponding estimates to emphasise that the constant is independent of s .

3. General Nikolskii inequality

In this section we establish the Nikolskii inequality for trigonometric functions on G/K . These are functions $T \in C^\infty(G/K)$ for which only finitely many Fourier coefficients are non-zero, so that the Fourier series (2.5) is finite. We will also use the notation T_L for trigonometric polynomials for which $\widehat{T}_L(\xi) = 0$ for $\langle \xi \rangle > L$. Thus, given the coefficients $\sigma = (\sigma(\xi))_{[\xi] \in \widehat{G}_0} \in \Sigma(G/K)$, we can define the trigonometric polynomial with these Fourier coefficients by

$$T_L(x) = \sum_{\langle \xi \rangle \leq L} d_\xi \text{Tr}(\sigma(\xi)\xi(x)), \tag{3.1}$$

where the sum is taken over all $[\xi] \in \widehat{G}_0$ with $\langle \xi \rangle \leq L$. In this case we have $\widehat{T}_L(\xi) = \sigma(\xi)$ for $\langle \xi \rangle \leq L$ and $\widehat{T}_L(\xi) = 0$ for $\langle \xi \rangle > L$.

Nikolskii’s inequality in this setting has been analysed by Pesenson [34]. Here we give a more elementary proof of the Nikolskii inequality for T_L , extending the range of indices p, q , and also with the constant interpreted in terms of the eigenvalue counting function $N(L)$ in (2.12).

Let $D \in C^\infty(G/K)$ be a Dirichlet-type kernel, defined by setting its Fourier coefficients to be

$$\widehat{D}(\xi) := \begin{pmatrix} I_{k_\xi} & 0 \\ 0 & 0 \end{pmatrix} \text{ for } [\xi] \in \widehat{G}_0 \text{ and } \langle \xi \rangle \leq L, \tag{3.2}$$

and zero otherwise, where $I_{k_\xi} \in \mathbb{C}^{k_\xi \times k_\xi}$ is the unit matrix.

Theorem 3.1. *Let $M = G/K$ be a compact homogeneous space of dimension n . Let $0 < p < q \leq \infty$. For $0 < p \leq 2$ set $\rho := 1$, and for $2 < p < \infty$ set ρ to be the smallest integer $\geq p/2$. Then for any L we have the estimate*

$$\|T_L\|_{L^q(G/K)} \leq N(\rho L)^{\frac{1}{p}-\frac{1}{q}} \|T_L\|_{L^p(G/K)}, \tag{3.3}$$

where $N(L)$ is the Weyl eigenvalue counting function for the elliptic pseudo-differential operator $(1 - \mathcal{L}_{G/K})^{1/2}$. Consequently, using (2.12), we have

$$\|T_L\|_{L^q(G/K)} \leq (C_1 \rho^n)^{\frac{1}{p}-\frac{1}{q}} L^{n(\frac{1}{p}-\frac{1}{q})} \|T_L\|_{L^p(G/K)}, \tag{3.4}$$

which holds for sufficiently large L for any constant $C_1 > C_0$, with C_0 given in (2.12).

Moreover, the inequality (3.3) is sharp for $p = 2$ and $q = \infty$, and it becomes equality for $T = D$, where D is the Dirichlet-type kernel.

In the case $p = 2$ and $q = \infty$ a similar inequality appeared in [15, Corollary 3.3].

Remark 3.2. For example, in the case of the real spheres $M = \mathbb{S}^n$, for $0 < p \leq 2$ and $p < q \leq \infty$, we have

$$\|T_L\|_{L^q(\mathbb{S}^n)} \leq C_1^{\frac{1}{p}-\frac{1}{q}} L^{n(\frac{1}{p}-\frac{1}{q})} \|T_L\|_{L^p(\mathbb{S}^n)}, \tag{3.5}$$

for any $C_1 > \frac{2}{n!}$, for all sufficiently large L . Here the value of C_0 in (2.12) follows from the explicit formulae for the Weyl counting function on \mathbb{S}^n , see, e.g., Shubin [39, Section 22]. In particular, considering $\mathbb{S}^1 = \mathbb{T}^1$ we obtain the constant $C_1^{\frac{1}{p}-\frac{1}{q}}$, where $C_1 > 2$.

Proof of Theorem 3.1. We first note that formula (3.4) follows from (3.3) by the asymptotics of $N(L)$ in (2.12), so it is sufficient to prove (3.3). We will give the proof of (3.3) and its sharpness in five steps.

Step 1. The case $p = 2$ and $q = \infty$. From formula (3.1) and using $\|\xi(x)\|_{\text{HS}} = k_\xi^{1/2}$ from (2.6), we can estimate

$$\begin{aligned} \|T_L\|_{L^\infty(G/K)} &\leq \sum_{\substack{\langle \xi \rangle \leq L \\ [\xi] \in \widehat{G}_0}} d_\xi \|\sigma(\xi)\|_{\text{HS}} \|\xi(x)\|_{\text{HS}} \\ &= \sum_{\substack{\langle \xi \rangle \leq L \\ [\xi] \in \widehat{G}_0}} d_\xi k_\xi^{1/2} \|\sigma(\xi)\|_{\text{HS}} \\ &\leq \left(\sum_{\substack{\langle \xi \rangle \leq L \\ [\xi] \in \widehat{G}_0}} d_\xi k_\xi \right)^{1/2} \left(\sum_{\substack{\langle \xi \rangle \leq L \\ [\xi] \in \widehat{G}_0}} d_\xi \|\sigma(\xi)\|_{\text{HS}}^2 \right)^{1/2} \\ &= N(L)^{1/2} \|T_L\|_{L^2(G/K)}, \end{aligned} \tag{3.6}$$

where in the last inequality we used (2.11) and the Plancherel identity.

Step 2. The case $p = 2$ and $2 < q \leq \infty$. We take $1 \leq q' < 2$ so that $\frac{1}{q} + \frac{1}{q'} = 1$. We set $r := \frac{2}{q'}$ so that its dual index satisfies $\frac{1}{r'} = 1 - \frac{q'}{2}$. By the Hausdorff-Young inequality in (2.9), and using (2.8) and Hölder's inequality, we obtain

$$\begin{aligned} \|T_L\|_{L^q(G/K)} &\leq \|\sigma\|_{\ell^{q'}(\widehat{G}_0)} = \left(\sum_{\langle \xi \rangle \leq L} d_\xi^{1-\frac{q'}{2}} k_\xi^{1-\frac{q'}{2}} d_\xi^{\frac{q'}{2}} \|\sigma(\xi)\|_{\text{HS}}^{q'} \right)^{\frac{1}{q'}} \\ &\leq \left(\sum_{\langle \xi \rangle \leq L} (d_\xi k_\xi)^{(1-\frac{q'}{2})r'} \right)^{\frac{1}{q'r'}} \left(\sum_{\langle \xi \rangle \leq L} d_\xi^{\frac{q'r}{2}} \|\sigma(\xi)\|_{\text{HS}}^{q'r} \right)^{\frac{1}{q'r}} \\ &= \left(\sum_{\langle \xi \rangle \leq L} d_\xi k_\xi \right)^{\frac{1}{q'} - \frac{1}{2}} \left(\sum_{\langle \xi \rangle \leq L} d_\xi \|\sigma(\xi)\|_{\text{HS}}^2 \right)^{\frac{1}{2}} \\ &= N(L)^{\frac{1}{2} - \frac{1}{q}} \|T_L\|_{L^2(G/K)}, \end{aligned}$$

where we have used that $\frac{q'r}{2} = 1$.

Step 3. The case $p > 2$. For $2 < p < q \leq \infty$ and an integer $\rho \geq p/2$, we claim to have

$$\|T_L\|_{L^q(G/K)} \leq N(\rho L)^{(1/p-1/q)} \|T_L\|_{L^p(G/K)}.$$

Indeed, if $q = \infty$, for $T_L \neq 0$, we get

$$\begin{aligned} \|T_L^\rho\|_{L^2} &= \| |T_L|^{\rho-p/2} |T_L|^{p/2} \|_{L^2} \leq \| |T_L|^{\rho-p/2} \|_{L^\infty} \| |T_L|^{p/2} \|_{L^2} \\ &= \|T_L\|_{L^\infty}^{\rho-p/2} \| |T_L|^{p/2} \|_{L^2} = \|T_L\|_{L^\infty}^\rho \|T_L\|_{L^\infty}^{-p/2} \| |T_L|^{p/2} \|_{L^2} \\ &= \|T_L\|_{L^\infty}^\rho \|T_L\|_{L^\infty}^{-p/2} \|T_L\|_{L^p}^{p/2} \\ &\leq N(\rho L)^{1/2} \|T_L\|_{L^2}^\rho \|T_L\|_{L^\infty}^{-p/2} \|T_L\|_{L^p}^{p/2}, \end{aligned} \tag{3.7}$$

where we have used (3.6) in the last line. We have also used the fact that T_L^ρ is a trigonometric polynomial of degree ρL , which follows from the decomposition of (Kronecker's) tensor products of representations into irreducible components by looking at the corresponding characters on the maximal torus of G . Therefore, using that $T_L^\rho \neq 0$, we have

$$\|T_L\|_{L^\infty} \leq N(\rho L)^{1/p} \|T_L\|_{L^p}. \tag{3.8}$$

For $p < q < \infty$ we obtain

$$\begin{aligned} \|T_L\|_{L^q} &= \| |T_L|^{1-p/q} |T_L|^{p/q} \|_{L^q} \leq \|T_L\|_{L^\infty}^{1-p/q} \|T_L\|_{L^p}^{p/q} \\ &\leq N(\rho L)^{1/p(1-p/q)} \|T_L\|_{L^p}^{1-p/q} \|T_L\|_{L^p}^{p/q}, \end{aligned} \tag{3.9}$$

where we have used (3.8), which implies (3.3) in this case.

Step 4. The case $0 < p < 2$. For $p < q \leq \infty$ and $0 < p \leq 2$, proceeding as we did for (3.7) with $\rho = 1$ (note that $p/2 \leq 1$), we get

$$\|T_L\|_{L^\infty} \leq N(L)^{1/p} \|T_L\|_{L^p}.$$

Then the estimate as in (3.9) implies (3.3) also in the case where $p < q \leq \infty$ and $0 < p \leq 2$.

Step 5. Sharpness. If D is the Dirichlet-type kernel (3.2), then using Plancherel's identity and the definition of $N(L)$ we can calculate

$$\|D\|_{L^2(G/K)} = \left(\sum_{[\xi] \in \widehat{G}_0} d_\xi \|\widehat{D}(\xi)\|_{\text{HS}}^2 \right)^{1/2} = \left(\sum_{[\xi] \in \widehat{G}_0} d_\xi k_\xi \right)^{1/2} = N(L)^{1/2}. \quad (3.10)$$

On the other hand, writing the Fourier series $D(x) = \sum_{[\xi] \in \widehat{G}_0} d_\xi \text{Tr}(\xi(x) \widehat{D}(\xi))$, and recalling our convention about zeros in the last rows of $\xi(x)$, we have

$$D(eK) = \sum_{[\xi] \in \widehat{G}_0} d_\xi \text{Tr}(\widehat{D}(\xi)) = \sum_{\langle \xi \rangle \leq L} d_\xi k_\xi = N(L).$$

Combining this with (3.10) in the Nikolskii inequality (3.3) we obtain

$$N(L) = D(eK) \leq \|D\|_{L^\infty} \leq N(L)^{1/2} \|D\|_{L^2} = N(L),$$

showing the sharpness of the constant in (3.3) in the case $p = 2$ and $q = \infty$. \square

4. Nikolskii inequalities on groups

In this section we prove that, in the case of the group G , for a given trigonometric polynomial T , a constant in the Nikolskii inequality depends on the number of non-zero Fourier coefficients of T , and this statement is sharp for $p = 2$ and $q = \infty$.

Lemma 4.1. *Let $1 \leq p < q \leq \infty$ be such that $\frac{1}{p} \geq \frac{1}{q} + \frac{1}{2}$. Let T be a trigonometric polynomial on the compact Lie group G . Then*

$$\|T\|_{L^q(G)} \leq \left(\sum_{\widehat{T}(\xi) \neq 0} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}} \|T\|_{L^p(G)}. \quad (4.1)$$

For $p = 2$ and $q = \infty$, this estimate is sharp.

Proof. Let us define the Dirichlet-type kernel D by setting its Fourier coefficients to be $\widehat{D}(\xi) := I_{d_\xi}$ if $\widehat{T}(\xi) \neq 0$, and $\widehat{D}(\xi) := 0$ if $\widehat{T}(\xi) = 0$. Then for any $1 \leq r' < \infty$, we can calculate

$$\|\widehat{D}\|_{\ell^{r'}(\widehat{G})} = \sum_{\widehat{T}(\xi) \neq 0} d_\xi^{r'(\frac{2}{r'} - \frac{1}{2})} \|\widehat{D}(\xi)\|_{\text{HS}}^{r'} = \sum_{\widehat{T}(\xi) \neq 0} d_\xi^2.$$

Consequently, for any $r \geq 2$, by the Hausdorff-Young inequality (2.9) with $k_\xi = d_\xi$, we get

$$\|D\|_{L^r(G)} \leq \|\widehat{D}\|_{\ell^{r'}(\widehat{G})} = \left(\sum_{\widehat{T}(\xi) \neq 0} d_\xi^2 \right)^{\frac{1}{r'}}.$$

Now, we take r so that $\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r} = \frac{1}{r'}$, and observe that the condition $\frac{1}{p} \geq \frac{1}{q} + \frac{1}{2}$ implies $r \geq 2$. On the other hand, we have $\widehat{T * D} = \widehat{D}\widehat{T} = \widehat{T}$, so that $T = T * D$, see, e.g., [36, Proposition 7.7.5]. Applying the Young inequality and using the estimates for D as above, we obtain

$$\|T\|_{L^q(G)} = \|T * D\|_{L^q(G)} \leq \|T\|_{L^p(G)} \|D\|_{L^r(G)} \leq \|T\|_{L^p(G)} \left(\sum_{\widehat{T}(\xi) \neq 0} d_\xi^2 \right)^{\frac{1}{r'}}, \tag{4.2}$$

implying (4.1). The sharpness follows by an argument similar to that in the proof of Theorem 3.1. □

The main result of this section is the following:

Theorem 4.2. *Let $0 < p < q \leq \infty$. For $0 < p \leq 2$ set $\rho := 1$, and for $2 < p < \infty$, set ρ to be the smallest integer $\geq p/2$. Let T be a trigonometric polynomial on the compact Lie group G . Then*

$$\|T\|_{L^q(G)} \leq \left(\sum_{\widehat{T}^\rho(\xi) \neq 0} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}} \|T\|_{L^p(G)}. \tag{4.3}$$

Moreover, this inequality is sharp for $p = 2$ and $q = \infty$ and it becomes equality for $T = D$, where D is the Dirichlet-type kernel.

Remark 4.3. We note that if $T = T_L$, i.e., if $\widehat{T}(\xi) = 0$ for $|\xi| > L$, we have $\sum_{\widehat{T}(\xi) \neq 0} d_\xi^2 \leq N(L)$, uniformly over such T , in agreement with the corresponding part of Theorem 3.1.

Proof of Theorem 4.2. From Lemma 4.1 we

$$\|T\|_{L^\infty(G)} \leq \left(\sum_{\widehat{T}(\xi) \neq 0} d_\xi^2 \right)^{\frac{1}{2}} \|T\|_{L^2(G)}.$$

Then following the proof of Theorem 3.1, we get (4.3) for $2 \leq p < q \leq \infty$ and $0 < p \leq 2, p \leq q \leq \infty$. The sharpness follows by an argument similar to that in the proof of Theorem 3.1. \square

From (the proof of) Lemma 4.1, for a function $f \in L^p(G)$, we immediately get the following estimate for the partial sums of its Fourier series: if

$$S_L f(x) := \sum_{\langle \xi \rangle \leq L} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)),$$

we have

$$\|S_L f\|_{L^q(G)} \leq N(L)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p(G)}, \text{ for } \frac{1}{p} \geq \frac{1}{q} + \frac{1}{2}. \tag{4.4}$$

Indeed, defining the Dirichlet-type kernel D by setting its Fourier coefficients to be $\widehat{D}(\xi) := I_{d_\xi}$ for $\langle \xi \rangle \leq L$, and $\widehat{D}(\xi) := 0$ for $\langle \xi \rangle > L$, from the identity $\widehat{S_L f} = \widehat{D} \widehat{f}$, we get $S_L = f * D$, so that applying the Young inequality with indices as in (4.2) and arguing as in the proof of Lemma 4.1, we obtain (4.4) since $k_\xi = d_\xi$ in this case.

But in fact we can prove a sharper estimate:

Corollary 4.4. *Let G be a compact Lie group and let $1 \leq p < q \leq \infty$ be such that $\frac{1}{p} > \frac{1}{q} + \frac{1}{2}$. For $f \in L^p(G)$ we have*

$$\left(\sum_{k=1}^{\infty} \frac{\left(k^{1-1/p+1/q} \sup_{N(L) \geq k} \frac{1}{N(L)} \|S_L f\|_{L^q(G)} \right)^p}{k} \right)^{1/p} \leq C \|f\|_{L^p(G)}. \tag{4.5}$$

In particular, we have

$$N(L)^{\frac{1}{q}-\frac{1}{p}} \|S_L f\|_{L^q(G)} = o(1) \text{ as } L \rightarrow \infty \tag{4.6}$$

and therefore

$$L^{n(\frac{1}{q}-\frac{1}{p})} \|S_L f\|_{L^q(G)} = o(1) \text{ as } L \rightarrow \infty, \tag{4.7}$$

with $n = \dim G$.

Proof. We have shown in (4.4) that

$$N(L)^{-\frac{1}{p} + \frac{1}{q}} \|S_L f\|_{L^q(G)} \leq \|f\|_{L^p(G)}. \tag{4.8}$$

Let us define a quasi-linear operator A by

$$Af := \left\{ \sup_{N(L) \geq k} \frac{1}{N(L)} \|S_L f\|_{L^q(G)} \right\}_{k=1}^{\infty}.$$

Then (4.8) implies that A is bounded from $L_p(G)$ to $l_{r,\infty}$, where $1/r = 1 - 1/p + 1/q$ and $l_{r,\infty}$ denotes the Lorentz sequence space. Indeed,

$$\begin{aligned} \|Af\|_{l_{r,\infty}} &= \sup_k k^{1/r} \sup_{N(L) > k} \frac{1}{N(L)} \|S_L f\|_{L^q(G)} \\ &\leq \sup_k \sup_{N(L) > k} \frac{N(L)^{1/r}}{N(L)} \|S_L f\|_{L^q(G)} \leq \|f\|_{L_p}. \end{aligned}$$

Let (p_0, r_0) and (p_1, r_1) be such that

$$p_0 < p < p_1, \quad r_0 < r < r_1, \quad \frac{1}{r_0} + \frac{1}{p_0} = \frac{1}{r_1} + \frac{1}{p_1} = \frac{1}{r} + \frac{1}{p}.$$

Then, using the interpolation theorem (see, e.g., [3]), since $A : L_{p_0}(G) \rightarrow l_{r_0,\infty}$ and $A : L_{p_1}(G) \rightarrow l_{r_1,\infty}$ are bounded, we get that

$$A : L_p(G) \rightarrow l_{r,p}$$

is bounded, which is (4.5).

Now, for any $\xi \in \mathbb{N}$ we choose $t \in \mathbb{N} : 2^t \leq \xi^n < 2^{t+1}$. Then (4.5) implies

$$\sum_{k=2^{t-1}}^{2^t-1} \frac{\left(k^{1-1/p+1/q} \sup_{N(L) \geq k} \frac{1}{N(L)} \|S_L f\|_{L^q(G)} \right)^p}{k} \rightarrow 0.$$

The left-hand side is greater than

$$\begin{aligned} &C \left(2^{t(1-1/p+1/q)} \sup_{N(L) \geq 2^t} \frac{1}{N(L)} \|S_L f\|_{L^q(G)} \right)^p \sum_{k=2^{t-1}}^{2^t-1} \frac{1}{k} \\ &\geq C \left(2^{t(1-1/p+1/q)} \sup_{L^n \geq 2^t} \frac{1}{L^n} \|S_L f\|_{L^q(G)} \right)^p \geq C \left(\frac{2^{t(1-1/p+1/q)}}{\xi^n} \|S_{\xi} f\|_{L^q(G)} \right)^p \\ &\geq C \left(N(\xi)^{1/q-1/p} \|S_{\xi} f\|_{L^q(G)} \right)^p, \end{aligned}$$

finishing the proof of (4.6).

Finally, (4.7) follows from (4.6), using (2.12)

□

Note that a similar result for periodic functions can be found in [3, 29]. Also, for certain values of p , there are low bounds results for estimates in Corollary 4.4, showing that norms of some characters (depending on L) should still go to infinity, see, *e.g.*, [20, Corollary 4].

5. Besov spaces

In this section we analyse embedding properties of Besov spaces on compact homogeneous spaces G/K . By using the Fourier series (2.5), we define the Besov space

$$B^r_{p,q} = B^r_{p,q}(G/K) = \left\{ f \in \mathcal{D}'(G/K) : \|f\|_{B^r_{p,q}} < \infty \right\}, \tag{5.1}$$

where

$$\|f\|_{B^r_{p,q}} := \left(\sum_{s=0}^{\infty} 2^{srq} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^q \right)^{1/q}, \text{ if } q < \infty, \tag{5.2}$$

and

$$\|f\|_{B^r_{p,q}} := \sup_{s \in \mathbb{N}} 2^{sr} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p, \text{ if } q = \infty. \tag{5.3}$$

Here we allow $r \in \mathbb{R}$ and $0 < p, q \leq \infty$. We also note that since we always have $\langle \xi \rangle \geq 1$, the trivial representation is included in this norm.

We first analyse these Besov spaces using the global Littlewood-Paley theory, and in Section 9 we show that for certain ranges of indices these spaces agree with the Besov spaces that could be defined on G/K as a manifold, using the standard Besov spaces on \mathbb{R}^n in local coordinates. For these indices the embeddings can be compared with those on \mathbb{R}^n in, *e.g.*, Triebel [45].

On unimodular Lie groups Besov spaces have been analysed in [40] in terms of the heat kernel, however no embedding theorems have been proved. In addition, using the Littlewood-Paley decomposition, one can also use the characterisation given in [14]. Using the Nikolskii inequality and the Fourier analysis in Section 2, we can establish embedding properties for these spaces.

First, let us prove that the norm $\|f\|_{B^r_{p,q}}$ is equivalent to a certain approximative characteristic of f . Results similar in spirit have been obtained also in [33].

Proposition 5.1. *Let $0 < p \leq \infty, 0 < q \leq \infty$, and $r \in \mathbb{R}$. We have*

$$\|f\|_{B^r_{p,q}} \asymp \left| \int_{G/K} f(x) dx \right| + \left(\sum_{s=0}^{\infty} 2^{srq} \left\| f - S_{2^s} f \right\|_p^q \right)^{1/q},$$

where $S_L f$ is the partial Fourier series of f , that is, $S_L f(x) = \sum_{\langle \xi \rangle \leq L} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x))$.

Proof. First we observe that if 1 is the trivial representation of G , we have $|S_1 f| = |\widehat{f}(1)| = |\int_{G/K} f(x) dx|$, and that without loss of generality we can assume that $\widehat{f}(1) = 0$. Denote

$$a_s := S_{2^{s+1}} f(x) - S_{2^s} f(x) = \sum_{2^s < \langle \xi \rangle \leq 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)).$$

With this notation we can write

$$\|f\|_{B_{p,q}^r} \asymp \left(\sum_{s=0}^\infty 2^{srq} \|a_s\|_p^q \right)^{1/q}.$$

We first show “ \gtrsim ”. If $0 < q < 1$, then

$$\begin{aligned} \left(\sum_{s=0}^\infty 2^{srq} \|f - S_{2^s} f\|_p^q \right)^{1/q} &\asymp \left(\sum_{s=0}^\infty 2^{srq} \left\| \sum_{k=s}^\infty a_k \right\|_p^q \right)^{1/q} \\ &\leq \left(\sum_{s=0}^\infty 2^{srq} \sum_{k=s}^\infty \|a_k\|_p^q \right)^{1/q} \\ &\leq \left(\sum_{k=0}^\infty \|a_k\|_p^q \sum_{s=0}^k 2^{srq} \right)^{1/q} \\ &\lesssim \left(\sum_{k=0}^\infty 2^{krq} \|a_k\|_p^q \right)^{1/q} \asymp \|f\|_{B_{p,q}^r}. \end{aligned}$$

If $q \geq 1$, using Hardy’s inequalities, we also get

$$\begin{aligned} \left(\sum_{s=0}^\infty 2^{srq} \|f - S_{2^s} f\|_p^q \right)^{1/q} &\leq \left(\sum_{s=0}^\infty 2^{srq} \left(\sum_{k=s}^\infty \|a_k\|_p \right)^q \right)^{1/q} \\ &\lesssim \left(\sum_{k=0}^\infty 2^{krq} \|a_k\|_p^q \right)^{1/q} \asymp \|f\|_{B_{p,q}^r}. \end{aligned}$$

Indeed, if $q \geq 1$, $\varepsilon > 0$ and $\alpha_k \geq 0$, the Hardy inequality asserts that

$$\sum_{s=0}^\infty 2^{\varepsilon s} \left(\sum_{k=s}^\infty \alpha_k \right)^q \leq C(q, \varepsilon) \sum_{s=0}^\infty 2^{\varepsilon s} \alpha_s^q.$$

To prove the part “ \lesssim ”, we write $a_k = a_k + f - f$ to obtain

$$\|f\|_{B_{p,q}^r} \asymp \left(\sum_{k=0}^\infty 2^{krq} \|a_k\|_p^q \right)^{1/q} \lesssim \left(\sum_{s=0}^\infty 2^{srq} \|f - S_{2^s} f\|_p^q \right)^{1/q},$$

completing the proof. □

For $r \in \mathbb{R}$, we denote by H_p^r the Sobolev space on G/K defined in local coordinates, *i.e.* the space of distributions such that in each local coordinate systems they belong to the usual Sobolev spaces $H_p^r(\mathbb{R}^n)$. By ellipticity it can be described as the set of all $f \in \mathcal{D}'(G/K)$ such that we have $(1 - \mathcal{L}_{G/K})^{r/2} f \in L_p(G/K)$. Writing the Fourier series for the lifting of f to G , we see from (2.5) that the Fourier series of $(1 - \mathcal{L}_{G/K})^{r/2} f$ is given by $\sum_{[\xi] \in \widehat{G}_0} d_\xi \langle \xi \rangle^r \text{Tr}(\widehat{f}(\xi)\xi(x))$, and hence we have

$$\|f\|_{H_p^r} \asymp \left\| \sum_{[\xi] \in \widehat{G}_0} d_\xi \langle \xi \rangle^r \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p. \tag{5.4}$$

We will often use Plancherel's identity in the following form:

$$\left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_2 = \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \|\widehat{f}(\xi)\|_{\mathbb{H}^S}^2 \right)^{1/2}, \tag{5.5}$$

which holds for $L^2(G/K)$ if we use our convention of having zeros in $\xi(x)$, the equality $\|f\|_{L^2(G)} = \|f\|_{L^2(G/K)}$ in our choice of normalisation of measures, and apply the Plancherel's identity on G .

We now collect the embedding properties of $B_{p,q}^r$ in the following theorem.

Theorem 5.2. *Let G/K be a compact homogeneous space of dimension n . Below, we allow $r \in \mathbb{R}$ unless stated otherwise. We have*

- (1) $B_{p,q_1}^{r+\varepsilon} \hookrightarrow B_{p,q_1}^r \hookrightarrow \|B_{p,q_2}^r \hookrightarrow B_{p,\infty}^r$, for $0 < \varepsilon, 0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty$;
- (2) $B_{p,q_1}^{r+\varepsilon} \hookrightarrow B_{p,q_2}^r$, for $0 < \varepsilon, 0 < p \leq \infty, 1 \leq q_2 < q_1 < \infty$;
- (3) $B_{2,2}^r = H^r$;
- (4) $H^{r+\varepsilon} \hookrightarrow B_{2,q}^r$, for $\varepsilon, q > 0$;
- (5) $B_{p_1,q}^{r_1} \hookrightarrow B_{p_2,q}^{r_2}$, for $0 < p_1 \leq p_2 \leq \infty, 0 < q < \infty, r_2 = r_1 - n \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$;
- (6) $B_{p,p}^r \hookrightarrow H_p^r \hookrightarrow B_{p,2}^r$, for $1 < p \leq 2$;
- (7) $B_{p,2}^r \hookrightarrow H_p^r \hookrightarrow B_{p,p}^r$, for $2 \leq p < \infty$;
- (8) $B_{p,1}^r \hookrightarrow L_q$, for $0 < p < q \leq \infty, r = n \left(\frac{1}{p} - \frac{1}{q} \right)$;
- (9) $B_{p,q}^r \hookrightarrow L_q$, for $1 < p < q < \infty, r = n \left(\frac{1}{p} - \frac{1}{q} \right)$.

We note that (6) and (7) can be rewritten as

$$(6') \ B_{p,\min\{p,2\}}^r \hookrightarrow H_p^r \hookrightarrow B_{p,\max\{p,2\}}^r, \quad \text{for } 1 < p < \infty.$$

Proof of Theorem 5.2. (1) These embeddings follow from

$$\sup_s a_s \leq \left(\sum_s a_s^{q_2} \right)^{1/q_2} \leq \left(\sum_s a_s^{q_1} \right)^{1/q_1} \leq \left(\sum_s 2^{s\varepsilon q_1} a_s^{q_1} \right)^{1/q_1}$$

for a sequence $a_s \geq 0$ and $\varepsilon > 0$.

(2) Using $q_2 < q_1$, by Hölder’s inequality, we get

$$\begin{aligned} \|f\|_{B_{p,q_2}^r} &\asymp \left(\sum_s \frac{2^{sq_2(r+\varepsilon)}}{2^{sq_2\varepsilon}} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^{q_2} \right)^{1/q_2} \\ &\leq \left(\sum_s 2^{sq_1(r+\varepsilon)} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^{q_1} \right)^{1/q_1} \\ &\quad \times \left(\sum_s \frac{1}{2^{sq_1q_2/(q_1-q_2)\varepsilon}} \right)^{(q_1-q_2)/(q_1q_2)} \\ &\leq C \|f\|_{B_{p,q_1}^{r+\varepsilon}}. \end{aligned}$$

(3) Using Plancherel’s identity (5.5) we get

$$\begin{aligned} \|f\|_{B_{2,2}^r} &= \left(\sum_s 2^{2sr} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_2^2 \right)^{1/2} \\ &\asymp \left(\sum_s \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} \langle \xi \rangle^{2r} d_\xi \|\widehat{f}(\xi)\|_{\mathbb{HS}}^2 \right) \right)^{1/2} \\ &\asymp \|\langle \xi \rangle^r \widehat{f}\|_{\ell^2(\widehat{G}_0)} = \|f\|_{H^r}. \end{aligned}$$

(4) This embedding follows from properties (1)–(3).

(5) Using Nikolskii’s inequality from Theorem 3.1,

$$\begin{aligned} \|f\|_{B_{p_2,q}^{r_2}} &\asymp \left(\sum_s 2^{sqr_2} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_{p_2}^q \right)^{1/q} \\ &\leq C \left(\sum_s 2^{sqr_2} 2^{sqn(\frac{1}{p_1}-\frac{1}{p_2})} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_{p_1}^q \right)^{1/q} \\ &\asymp \|f\|_{B_{p_1,q}^{r_1}}. \end{aligned}$$

(6) First, by the Littlewood-Paley theorem (see [14]), applied to functions which are constant on right cosets, we get

$$\begin{aligned} \|f\|_{H_p^r} &\asymp \left\| \sum_s 2^{sr} \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p \\ &\asymp \left\| \left[\sum_s 2^{2sr} \left| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right|^2 \right]^{1/2} \right\|_p. \end{aligned} \tag{5.6}$$

Since we have already shown the case $p = 2$ in (3), we can assume that $1 < p < 2$. Using the inequality $(\sum_k a_k^2)^{1/2} \leq (\sum_k a_k^p)^{1/p}$, for $a_k \geq 0$, we obtain

$$\begin{aligned} \|f\|_{H_p^r} &\lesssim \left\| \left[\sum_s 2^{psr} \left| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right|^p \right]^{1/p} \right\|_p \\ &\leq \left(\sum_s 2^{psr} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^p \right)^{1/p} = \|f\|_{B_{p,p}^r}. \end{aligned}$$

On the other hand, using Minkowski inequality $(\sum_j (f_j)^\alpha)^{1/\alpha} \leq f(\sum_j f_j^\alpha)^{1/\alpha}$ for $\alpha > 1$ and $f_j \geq 0$, we get for $\alpha = 2/p$,

$$\begin{aligned} \|f\|_{B_{p,2}^r} &= \left(\sum_s 2^{2sr} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_2^2 \right)^{1/2} \\ &\leq \left(\int \left[\sum_s 2^{2sr} \left| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right|^2 \right]^{p/2} dx \right)^{1/p} \\ &\leq \left\| \left[\sum_s 2^{2sr} \left| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right|^2 \right]^{1/2} \right\|_p \asymp \|f\|_{H_p^r}, \end{aligned}$$

using (5.6) in the last equivalence of norms again. The proof of (7) is similar.

(8) Using the Fourier series representation (2.5) and Nikolskii’s inequality from Theorem 3.1, we get

$$\begin{aligned} \|f\|_{L_q} &\leq \sum_{s=0}^{\infty} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_{L_q} \\ &\leq \sum_{s=0}^{\infty} 2^{sn\left(\frac{1}{p}-\frac{1}{q}\right)} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_{L_p} = \|f\|_{B_{p,1}^r}. \end{aligned}$$

(9) We show that (9) follows from (8). Let F be such that $F : B_{p,1}^r \hookrightarrow L_q$. Then for parameters p, q, r one can find couples $(q_0, r_0), (q_1, r_1)$ and $\theta \in (0, 1)$ such that

$$n\left(\frac{1}{p} - \frac{1}{q_0}\right) = r_0, \quad n\left(\frac{1}{p} - \frac{1}{q_1}\right) = r_1, \quad r_0 < r < r_1,$$

and

$$r = (1 - \theta)r_0 + \theta r_1, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

Since $F : B_{p,1}^{r_0} \rightarrow L_{q_0}$ and $F : B_{p,1}^{r_1} \rightarrow L_{q_1}$, then by the interpolation theorems

$$F : B_{p,q}^r = \left(B_{p,1}^{r_0}, B_{p,1}^{r_1}\right)_{\theta,q} \rightarrow \left(L_{q_0}, L_{q_1}\right)_{\theta,q} = L_q,$$

i.e., $B_{p,q}^r \hookrightarrow L_q$ with $n\left(\frac{1}{p} - \frac{1}{q}\right) = r$. We will discuss the interpolation properties of the Besov spaces in more detail in Section 8 below. □

5.1. Triebel-Lizorkin spaces

Similarly to Besov spaces one defines the Triebel-Lizorkin spaces as follows:

$$F_{p,q}^r = F_{p,q}^r(G/K) = \left\{ f \in \mathcal{D}'(G/K) : \|f\|_{F_{p,q}^r} < \infty \right\},$$

where

$$\|f\|_{F_{p,q}^r} := \left\| \left(\sum_{s=0}^{\infty} 2^{srq} \left| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right|^q \right)^{1/q} \right\|_p.$$

Let us mention several embedding properties of the spaces $F_{p,q}^r$:

Theorem 5.3. *Let G/K be a compact homogeneous space of dimension n . Below, we allow $r \in \mathbb{R}$ unless stated otherwise. We have*

- (1) $F_{p,q_1}^{r+\varepsilon} \hookrightarrow F_{p,q_1}^r \hookrightarrow F_{p,q_2}^r \hookrightarrow F_{p,\infty}^r$, $0 < \varepsilon$, for $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$;
- (2) $F_{p,q_1}^{r+\varepsilon} \hookrightarrow F_{p,q_2}^r$, for $0 < \varepsilon$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$;
- (3) $F_{p,p}^r = B_{p,p}^r$, for $0 < p < \infty$;
- (4) $B_{p,\min\{p,q\}}^r \hookrightarrow F_{p,q}^r \hookrightarrow B_{p,\max\{p,q\}}^r$, for $0 < p < \infty$, $0 < q \leq \infty$.

The proof of this Theorem is similar to the proof of Theorem 5.2, see also [44].

6. Wiener and β -Wiener spaces

Let us recall the definition of the Wiener algebra A of absolutely convergent Fourier series on the circle \mathbb{T}^1 (see, e.g., Kahane’s book [22]):

$$A(\mathbb{T}^1) = \left\{ f : \|f\|_{A(\mathbb{T}^1)} = \sum_{j=-\infty}^{\infty} |\widehat{f}(j)| < \infty \right\}.$$

With the norm $\ell^1(\widehat{G}_0)$ in (2.8) this corresponds to

$$A(G/K) = \left\{ f \in \mathcal{D}'(G/K) : \|f\|_{A(G/K)} := \|\widehat{f}\|_{\ell^1(\widehat{G}_0)} = \sum_{\xi \in \widehat{G}} d_\xi k_\xi^{1/2} \|\widehat{f}(\xi)\|_{\mathbb{H}^s} < \infty \right\}.$$

We will abbreviate this norm to $\|\cdot\|_A$. The main problem here is to determine which smoothness of f guarantees the absolute convergence of the Fourier series.

It is known that in the case of a unitary group G , if $f \in C^k(G)$ with an even $k > \frac{\dim G}{2}$, then $\widehat{f} \in \ell^1(\widehat{G})$ and hence $f \in A(G)$, see, e.g., Faraut [13] (in the classical case on the torus this is even more well-known, see, e.g., [18, Theorem 3.2.16, page 184]).

On the other hand, on general compact Lie groups, applying powers of the Laplacian to the Fourier series, it is also easy to show that for $s > \frac{\dim G}{2}$, we have $H^s(G) \hookrightarrow A(G)$. The following theorem sharpens this to the Besov space $B_{2,1}^{\dim G/2}$, since we can observe that by Theorem 5.2, Part (4), we have the embedding

$$\|f\|_{B_{2,1}^{\dim G/2}} \lesssim \|f\|_{H^s}, \text{ for } s > \frac{1}{2} \dim G.$$

Thus, we sharpen the above embeddings, also extending them to compact homogeneous spaces.

Theorem 6.1. *Let G/K be a compact homogeneous space of dimension n . Then*

$$\|f\|_A \lesssim \|f\|_{B_{2,1}^{n/2}}.$$

Proof. We write

$$\|f\|_A = \sum_{s=0}^{\infty} F_s, \quad \text{where} \quad F_s = \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{1/2} \|\widehat{f}(\xi)\|_{\text{HS}}.$$

By Hölder’s inequality, (2.13), and the Plancherel identity (5.5),

$$\begin{aligned} F_s &\leq \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi \right)^{1/2} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 \right)^{1/2} \\ &\leq 2^{sn/2} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_2, \end{aligned}$$

and the result follows. □

Let us now study the β -absolute convergence of the Fourier series, which on the circle would be

$$A^\beta(\mathbb{T}^1) = \left\{ f : \|f\|_{A^\beta} = \left(\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^\beta \right)^{1/\beta} < \infty \right\}.$$

This means $\|f\|_{A^\beta} = \|\widehat{f}\|_{\ell^\beta}$, so that its analogue on the homogeneous spaces for the family of ℓ^p -norms (2.2) becomes

$$\begin{aligned} &A^\beta(G/K) \\ &= \left\{ f \in \mathcal{D}'(G/K) : \|f\|_{A^\beta} := \|\widehat{f}\|_{\ell^\beta(\widehat{G}_0)} = \left(\sum_{[\xi] \in \widehat{G}_0} d_\xi k_\xi^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \right)^{1/\beta} < \infty \right\}, \end{aligned}$$

where we can allow any $0 < \beta < \infty$. We now analyse its embedding properties.

Theorem 6.2. *Let G/K be a compact homogeneous space of dimension n and let $1 < p \leq 2$. Then*

$$\|f\|_{A^\beta} \lesssim \|f\|_{B_{p,\beta}^{\alpha n}}$$

for any $\alpha > 0$ and $\beta = (\alpha + \frac{1}{p'})^{-1}$.

Remark 6.3. (i) In the classical case of functions on the torus, this result was proved by Szasz, see, e.g., [31, page 119].

(ii) Note that the strongest result is when $p = 2$, that is,

$$\|f\|_{A^\beta} \lesssim \|f\|_{B_{2,\beta}^{\alpha^*n}} \lesssim \|f\|_{B_{p,\beta}^{\alpha n}},$$

where $\alpha^*, \alpha > 0$ and $\beta = (\alpha^* + \frac{1}{2})^{-1} = (\alpha + \frac{1}{p'})^{-1}$. This follows from Theorem 5.2, Part (5).

Proof of Theorem 6.2. We can assume that $\widehat{f}(\xi) \neq 0$ only for sufficiently large $\langle \xi \rangle$. We write

$$\|f\|_{A^\beta}^\beta = \sum_{s=0}^\infty F_s, \quad \text{where} \quad F_s = \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta.$$

Let us first assume that $\beta \equiv (\alpha + \frac{1}{p'})^{-1} \geq 2$. Since $\beta \geq 2$, applying the Hausdorff-Young inequality (2.9), we get

$$F_s = \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \leq \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_{\beta'}^\beta.$$

Now by Nikolskii's inequality from Theorem 3.1 for $(L^{\beta'}, L^p)$ with $\beta < p'$, or equivalently, $p < \beta'$, we have

$$\begin{aligned} F_s &\leq 2^{sn\beta(\frac{1}{p} - \frac{1}{\beta'})} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^\beta \\ &= 2^{sn\beta(\frac{1}{\beta} - \frac{1}{p'})} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^\beta, \end{aligned}$$

which is the required result.

To prove Theorem 6.2 in the case of $\beta \equiv (\alpha + \frac{1}{p'})^{-1} < 2$, we put

$$\gamma := \frac{p'}{p' - \beta} > 1.$$

By Hölder’s inequality, taking $\delta := \frac{\beta}{p'} - \frac{\beta}{2}$, so that $\frac{1}{\gamma} + \delta = 1 - \frac{\beta}{2} = \beta(\frac{1}{\beta} - \frac{1}{2})$, using (2.13), we get

$$\begin{aligned}
 F_s &= \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \\
 &\leq \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} (d_\xi^\gamma k_\xi^{\frac{1}{\gamma}})^\gamma \right)^{1/\gamma} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi^{\frac{1}{\gamma'} \gamma'} k_\xi^{\delta \gamma'} \|\widehat{f}(\xi)\|_{\text{HS}}^{\beta \gamma'} \right)^{1/\gamma'} \\
 &\lesssim 2^{sn/\gamma} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{(\frac{\beta}{p'} - \frac{\beta}{2}) \frac{p'}{\beta}} \|\widehat{f}(\xi)\|_{\text{HS}}^{\beta \gamma'} \right)^{1/\gamma'} \\
 &= 2^{\frac{sn(p' - \beta)}{p'}} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{p'(\frac{1}{p'} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^{p'} \right)^{\beta/p'} . \tag{6.1}
 \end{aligned}$$

Since $p' \geq 2$, by the Hausdorff-Young inequality (2.9), we have

$$F_s \leq 2^{\frac{sn(p' - \beta)}{p'}} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^\beta ,$$

i.e.,

$$\|f\|_{A^\beta} \leq \left[\sum_{s=0}^\infty \left(2^{sn(\frac{1}{\beta} - \frac{1}{p'})} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \text{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p \right)^\beta \right]^{1/\beta} = \|f\|_{B_{p,\beta}^{n\alpha}} ,$$

completing the proof. □

The converse to Theorem 6.2 is as follows:

Theorem 6.4 (Inverse result). *Let $2 \leq p < \infty$, then*

$$\|f\|_{B_{p,\beta}^{\bar{\alpha}n}} \lesssim \|f\|_{A^\beta}$$

for $\bar{\alpha} := \min\{\alpha, 0\}$ and $\beta = (\alpha + \frac{1}{p'})^{-1} > 0$.

Remark 6.5. (i) If $\beta \geq 2$, then the strongest result is when $p = 2$, that is,

$$\|f\|_{B_{p,\beta}^{\alpha n}} \lesssim \|f\|_{B_{2,\beta}^{\alpha*n}} \lesssim \|f\|_{A^\beta} ,$$

where $\beta = (\alpha^* + \frac{1}{2})^{-1} = (\alpha + \frac{1}{p})^{-1}$. This follows from Theorem 5.2, Part (5). Note that in this case $\alpha^*, \alpha < 0$, i.e., $\bar{\alpha} = \alpha$ and $\bar{\alpha}^* = \alpha^*$.

(ii) If $\beta < 2$, then the strongest result is when $p = \beta'$, that is,

$$\|f\|_{B_{p,\beta}^{\alpha n}} \lesssim \|f\|_{B_{\beta',\beta}^0} \lesssim \|f\|_{A^\beta}.$$

This follows from Theorem 5.2, Part (5).

Proof of Theorem 6.4. Let first $\beta \leq p'$, or equivalently, $\alpha \geq 0$. In this case $\bar{\alpha} = 0$. Since $p \geq 2$, then applying the Hausdorff-Young inequality (2.9), we get

$$\begin{aligned} \|f\|_{B_{p,\beta}^0} &= \left[\sum_{s=0}^{\infty} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^\beta \right]^{1/\beta} \\ &\leq \left[\sum_{s=0}^{\infty} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{p'(\frac{1}{p'} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^{p'} \right)^{\beta/p'} \right]^{1/\beta} \\ &\lesssim \left[\sum_{s=0}^{\infty} \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \right]^{1/\beta} = \|f\|_{A^\beta}, \end{aligned}$$

where in the last line we have used the inequality $\|\widehat{f}\|_{\ell^{p'}(\widehat{G}_0)} \leq \|\widehat{f}\|_{\ell^\beta(\widehat{G}_0)}$ for $\beta \leq p'$, see (2.10).

Let now $\beta > p'$, or equivalently, $\beta' < p$, i.e., $\bar{\alpha} = \alpha < 0$. First, we observe that by the Hölder inequality, we have

$$\begin{aligned} &\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{p'(\frac{1}{p'} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^{p'} \\ &\leq \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} \left(d_\xi k_\xi \right)^{\frac{\beta - p'}{\beta}} \right)^{1 - \frac{p'}{\beta}} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} \left(d_\xi^{\frac{p'}{\beta}} k_\xi^{p'(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^{p'} \right)^{\frac{\beta}{p'}} \right)^{\frac{p'}{\beta}} \\ &\lesssim 2^{sn(1 - \frac{p'}{\beta})} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi k_\xi^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \right)^{\frac{p'}{\beta}}. \end{aligned}$$

Using this and $p \geq 2$, by the Hausdorff-Young inequality (2.9), we get

$$\begin{aligned} \|f\|_{B_{p,\beta}^{\alpha n}} &\leq \left[\sum_{s=0}^{\infty} 2^{s\alpha n\beta} \left(\sum_{2^s \leq |\xi| < 2^{s+1}} d_{\xi} k_{\xi}^{p'(\frac{1}{p'} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\mathbb{H}^S}^{p'} \right)^{\beta/p'} \right]^{1/\beta} \\ &\leq \left[\sum_{s=0}^{\infty} 2^{s\alpha n\beta} \left(2^{sn(\frac{\beta}{p'} - 1)} \sum_{2^s \leq |\xi| < 2^{s+1}} d_{\xi} k_{\xi}^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\mathbb{H}^S}^{\beta} \right) \right]^{1/\beta} \\ &= \left[\sum_{s=0}^{\infty} \sum_{2^s \leq |\xi| < 2^{s+1}} d_{\xi} k_{\xi}^{\beta(\frac{1}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\mathbb{H}^S}^{\beta} \right]^{1/\beta} = \|f\|_{A^{\beta}}. \end{aligned}$$

where in the second line we used (6.1) with β and p' interchanged. □

7. Beurling and β -Beurling spaces

Let us recall the definition of the Beurling space on the circle \mathbb{T}^1 :

$$A^*(\mathbb{T}^1) = \left\{ f : \|f\|_{A^*(\mathbb{T}^1)} = \sum_{j=0}^{\infty} \sup_{j \leq |k|} |\widehat{f}(k)| < \infty \right\}. \tag{7.1}$$

The space A^* was introduced by Beurling [2] for establishing contraction properties of functions. In [4] it was shown that $A^*(\mathbb{T}^1)$ is an algebra and its properties were investigated. We note that $\|f\|_{A^*}$ can be represented as follows:

$$\|f\|_{A^*(\mathbb{T}^1)} \asymp \sum_{s=0}^{\infty} 2^s \sup_{2^s \leq |k| < 2^{s+1}} |\widehat{f}(k)|. \tag{7.2}$$

Indeed, we have

$$\begin{aligned} \|f\|_{A^*(\mathbb{T}^1)} &= \sum_{j=0}^{\infty} \sup_{j \leq |k|} |\widehat{f}(k)| \asymp \sum_{s=0}^{\infty} \sum_{2^s \leq j < 2^{s+1}} \sup_{j \leq |k|} |\widehat{f}(k)| \asymp \sum_{s=0}^{\infty} 2^s \sup_{2^s \leq |k|} |\widehat{f}(k)| \\ &\asymp \sum_{s=0}^{\infty} 2^s \sup_{s \leq l} \sup_{2^l \leq |k| < 2^{l+1}} |\widehat{f}(k)| =: J. \end{aligned}$$

It is clear that $J \geq \sum_{s=0}^{\infty} 2^s \sup_{2^s \leq |k| < 2^{s+1}} |\widehat{f}(k)|$. On the other hand,

$$\begin{aligned} J &\leq \sum_{s=0}^{\infty} 2^s \sum_{l=s}^{\infty} \sup_{2^l \leq |k| < 2^{l+1}} |\widehat{f}(k)| = \sum_{l=0}^{\infty} \sup_{2^l \leq |k| < 2^{l+1}} |\widehat{f}(k)| \sum_{s=0}^l 2^s \\ &\asymp \sum_{l=0}^{\infty} 2^l \sup_{2^l \leq |k| < 2^{l+1}} |\widehat{f}(k)|. \end{aligned}$$

In our case the space $A^*(G/K)$ on a compact homogeneous space G/K of dimension n , analogous to that in (7.2) on the circle, for the ℓ^∞ -norm (2.3), becomes

$$A^*(G/K) = \left\{ f \in \mathcal{D}'(G/K) : \|f\|_{A^*(G/K)} := \sum_{s=0}^{\infty} 2^{ns} \sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} < \infty \right\}. \tag{7.3}$$

In fact, we can analyse a more general scale of spaces, the β -version of these spaces. Such function spaces play an important role in the summability theory and in the Fourier synthesis (see, e.g., [42, Theorems 1.25 and 1.16] and [43, Theorem 8.1.3, Chapter 6]).

Thus, for any $0 < \beta < \infty$, we define $A^{*,\beta}$ by

$$\|f\|_{A^{*,\beta}} := \left(\sum_{s=0}^{\infty} 2^{ns} \left(\sup_{2^s \leq \langle \xi \rangle} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^\beta \right)^{1/\beta} < \infty. \tag{7.4}$$

We note that for convenience we change the range of the supremum in ξ from a dyadic strip in (7.3) to an infinite set in (7.4), but we can show that this change produces equivalent norms. Thus, we show that we have $A^* = A^{*,1}$ and, more generally, for any $0 < \beta < \infty$ we have the equivalence

$$\sum_{s=0}^{\infty} 2^{ns} \left(\sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^\beta \asymp \sum_{s=0}^{\infty} 2^{ns} \left(\sup_{2^s \leq \langle \xi \rangle} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^\beta. \tag{7.5}$$

Indeed, the inequality \leq is trivial. Conversely, we write

$$\begin{aligned} \sum_{s=0}^{\infty} 2^{ns} \left(\sup_{2^s \leq \langle \xi \rangle} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^\beta &\asymp \sum_{s=0}^{\infty} 2^{ns} \sup_{s \leq l} \sup_{2^l \leq \langle \xi \rangle < 2^{l+1}} k_\xi^{-\beta/2} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \\ &\leq \sum_{s=0}^{\infty} 2^{ns} \sum_{l=s}^{\infty} \sup_{2^l \leq \langle \xi \rangle < 2^{l+1}} k_\xi^{-\beta/2} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \\ &= \sum_{l=0}^{\infty} \sup_{2^l \leq \langle \xi \rangle < 2^{l+1}} k_\xi^{-\beta/2} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \sum_{s=0}^l 2^{ns} \\ &\asymp \sum_{l=0}^{\infty} 2^{ls} \sup_{2^l \leq \langle \xi \rangle < 2^{l+1}} k_\xi^{-\beta/2} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta, \end{aligned}$$

proving (7.5). So, we can work with either of the equivalent expressions in (7.5).

We now prove the following embedding properties between Beurling’s and Besov spaces:

Theorem 7.1. *Let G/K be a compact homogeneous space of dimension n . Let $0 < \beta < \infty$ and $p \geq 2$. Then we have*

$$\|f\|_{B_{p,\beta}^{n(\frac{1}{\beta}-\frac{1}{p'})}} \lesssim \|f\|_{A^{*,\beta}} \lesssim \|f\|_{B_{1,\beta}^{n/\beta}}.$$

Proof. Again, we may assume that $\widehat{f}(\xi) = 0$ for small $\langle \xi \rangle$. To prove the left-hand side inequality, using (2.9) and $p \geq 2$, we get

$$\begin{aligned} & \|f\|_{B_{p,\beta}^{n(1/\beta-1/p')}} \\ &= \left(\sum_{s=0}^{\infty} 2^{sn(1-\beta/p')} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} \operatorname{Tr}(\widehat{f}(\xi)\xi(x)) \right\|_p^{\beta} \right)^{1/\beta} \\ &\leq \left(\sum_{s=0}^{\infty} 2^{sn(1-\beta/p')} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} k_{\xi}^{p'(\frac{1}{p'}-\frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^{p'} \right)^{\beta/p'} \right)^{1/\beta} \\ &\leq \left(\sum_{s=0}^{\infty} 2^{sn(1-\beta/p')} \left(\sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_{\xi} k_{\xi} \right)^{\beta/p'} \left(\sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\beta} \right)^{1/\beta} \\ &\leq \left(\sum_{s=0}^{\infty} 2^{sn} \left(\sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\beta} \right)^{1/\beta} = \|f\|_{A^{*,\beta}}. \end{aligned}$$

To show the right-hand side inequality, we denote

$$S_l f(x) := \sum_{\langle \eta \rangle \leq l-1} d_{\eta} \operatorname{Tr}(\widehat{f}(\eta)\eta(x)).$$

Then, by the orthogonality of representation coefficients, we have $\widehat{S_l f}(\xi) = 0$ for any $[\xi] \in \widehat{G}_0$ with $\langle \xi \rangle \geq l$. Consequently, we note that we have $\widehat{f}(\xi) = \widehat{(f - S_{2^s} f)}(\xi)$ if $\langle \xi \rangle \geq 2^s$. Using these observations and the Hausdorff-Young inequality, we can estimate

$$\begin{aligned} \sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} &= \sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_{\xi}^{-1/2} \|\widehat{f}(\xi) - \widehat{S_{2^s} f}(\xi)\|_{\text{HS}} \\ &\leq \|\widehat{f - S_{2^s} f}\|_{\ell^{\infty}(\widehat{G}_0)} \leq \|f - S_{2^s} f\|_{L^1(G/K)}. \end{aligned}$$

Then

$$\sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \leq \|f - S_{2^s} f\|_1 \leq \sum_{k=s}^{\infty} \|S_{2^{k+1}} f - S_{2^k} f\|_1.$$

Therefore,

$$\begin{aligned} \|f\|_{A^{*,\beta}} &\leq C \left(\sum_{s=0}^{\infty} 2^{sn} \left(\sup_{2^s \leq \langle \xi \rangle < 2^{s+1}} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\beta} \right)^{1/\beta} \\ &\leq C \left(\sum_{s=0}^{\infty} 2^{sn} \|f - S_{2^s}(f)\|_1^{\beta} \right)^{1/\beta}. \end{aligned}$$

Using Proposition 5.1 we get

$$\|f\|_{A^{*,\beta}} \lesssim \|f\|_{B_{1,\beta}^{n/\beta}}$$

completing the proof. □

Theorems 6.1 and 7.1 imply, taking $p = 2$:

Corollary 7.2. *For $0 < \beta < \infty$, we have*

$$\|f\|_{A^{\beta}} \lesssim \|f\|_{B_{2,\beta}^{n(1/\beta-1/2)}} \lesssim \|f\|_{A^{*,\beta}}.$$

In particular, taking $\beta = 1$, we have

$$\|f\|_A \lesssim \|f\|_{B_{2,1}^{n/2}} \lesssim \|f\|_{A^*}.$$

8. Interpolation

Let X_0, X_1 be two Banach spaces, with X_1 continuously embedded in $X_0 : X_1 \hookrightarrow X_0$. We define the K -functional for $f \in X_0 + X_1$ by

$$K(f, t; X_0, X_1) := \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1}), \text{ for } t \geq 0.$$

To investigate intermediate space X for the pair (X_0, X_1) , i.e., $X_0 \cap X_1 \subset X \subset X_0 + X_1$, one uses the θ, q -interpolation spaces.

For $0 < q < \infty$ and $0 < \theta < 1$, we define

$$(X_0, X_1)_{\theta,q} := \left\{ f \in X_0 + X_1 : \|f\|_{(X_0, X_1)_{\theta,q}} = \left(\int_0^{\infty} (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\},$$

and for $q = \infty$,

$$(X_0, X_1)_{\theta,\infty} := \left\{ f \in X_0 + X_1 : \|f\|_{(X_0, X_1)_{\theta,\infty}} = \sup_{0 < t < \infty} t^{-\theta} K(f, t) < \infty \right\}.$$

For technical convenience, we define the following Beurling-type spaces:

$$A_r^{*,\beta} = \left\{ f : \|f\|_{A_r^{*,\beta}} := \left(\sum_{s=0}^{\infty} \left(2^{rns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\beta} \right)^{\frac{1}{\beta}} < \infty \right\}.$$

Note that $A_{1/\beta}^{*,\beta} = A^{*,\beta}$, where $A^{*,\beta}$ is given by the norm (7.4).

Theorem 8.1. *Let $0 < r_1 < r_0 < \infty$, $0 < \beta_0, \beta_1, q \leq \infty$, and*

$$r = (1 - \theta)r_0 + \theta r_1.$$

(i) *We have*

$$(A_{r_0}^{*,\beta_0}, A_{r_1}^{*,\beta_1})_{\theta,q} = A_r^{*,q}.$$

In particular,

$$(A^{*,1/r_0}, A^{*,1/r_1})_{\theta,1/r} = A^{*,1/r}.$$

(ii) *If $0 < p \leq \infty$,*

$$(B_{p,\beta_0}^{r_0}, B_{p,\beta_1}^{r_1})_{\theta,q} = B_{p,q}^r.$$

(iii) *If $1 < p < \infty$,*

$$(H_p^{r_0}, H_p^{r_1})_{\theta,q} = B_{p,q}^r.$$

(iv) *If $0 < p < \infty$,*

$$(F_{p,\beta_0}^{r_0}, F_{p,\beta_1}^{r_1})_{\theta,q} = B_{p,q}^r.$$

Proof. Let $f \in (A_{r_0}^{*,\beta_0}, A_{r_1}^{*,\beta_1})_{\theta,q}$. Take any representation $f = f_0 + f_1$ such that $f_0 \in A_{r_0}^{*,\beta_0}$ and $f_1 \in A_{r_1}^{*,\beta_1}$. Then for any $s \in \mathbb{Z}_+$ we get

$$\begin{aligned} 2^{rns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} &\leq 2^{(r-r_0)ns} \left(2^{r_0ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f_0}(\xi)\|_{\text{HS}} \right. \\ &\quad \left. + 2^{(r_0-r_1)ns} 2^{r_1ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f_1}(\xi)\|_{\text{HS}} \right) \\ &\lesssim 2^{(r-r_0)ns} \left(\left(\sum_{r=0}^s 2^{r_0ns\beta_0} \right)^{1/\beta_0} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f_0}(\xi)\|_{\text{HS}} \right. \\ &\quad \left. + 2^{(r_0-r_1)ns} \left(\sum_{r=0}^s 2^{r_1nr\beta_1} \right)^{1/\beta_1} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f_1}(\xi)\|_{\text{HS}} \right) \\ &\lesssim 2^{(r-r_0)ns} \left(\|f_0\|_{A_{r_0}^{*,\beta_0}} + 2^{(r_0-r_1)ns} \|f_1\|_{A_{r_1}^{*,\beta_1}} \right). \end{aligned}$$

Taking into account our choice of f_0 and f_1 , by the definition of the K -functional $K(f, t) := K(f, t; A_{r_0}^{*,\beta_0}, A_{r_1}^{*,\beta_1})$, we have

$$\begin{aligned} 2^{rns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} &\lesssim 2^{(r-r_0)ns} K(f, 2^{(r_0-r_1)ns}) \\ &= 2^{-\theta(r_0-r_1)ns} K(f, 2^{(r_0-r_1)ns}). \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_{A_r^{*,q}} &\lesssim \left(\sum_{s=0}^{\infty} \left(2^{-\theta(r_0-r_1)ns} K(f, 2^{(r_0-r_1)ns}) \right)^q \right)^{\frac{1}{q}} \\ &\asymp \left(\int_1^{\infty} \left(t^{-\theta(r_0-r_1)n} K(f, t^{(r_0-r_1)n}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^{\infty} \left(t^{-\theta} K(f, t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \|f\|_{(A_{r_0}^{*,\beta_0}, A_{r_1}^{*,\beta_1})_{\theta,q}}, \end{aligned}$$

i.e.,

$$(A_{r_0}^{*,\beta_0}, A_{r_1}^{*,\beta_1})_{\theta,q} \hookrightarrow A_r^{*,q}.$$

Let us show the inverse embedding. Let $f \in A_r^{*,q}$, $\tau = \min(\beta_0, \beta_1, q)$, and $r \in \mathbb{Z}_+$. Define f_0 and f_1 as follows:

$$f_0(x) := S_{2^l} f(x) = \sum_{\langle \xi \rangle \leq 2^l} d_{\xi} \text{Tr}(\widehat{f}(\xi)\xi(x)),$$

and

$$f_1 := f - f_0.$$

Then

$$\begin{aligned} \|f_0\|_{A_{r_0}^{*,\beta_0}} &= \left(\sum_{s=0}^{\infty} \left(2^{r_0ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f_0}(\xi)\|_{\text{HS}} \right)^{\beta_0} \right)^{\frac{1}{\beta_0}} \\ &\leq \left(\sum_{s=0}^{\infty} \left(2^{r_0ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f_0}(\xi)\|_{\text{HS}} \right)^{\tau} \right)^{\frac{1}{\tau}} \\ &= \left(\sum_{s=0}^l \left(2^{r_0ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\tau} \right)^{\frac{1}{\tau}} \end{aligned}$$

and

$$\begin{aligned} \|f_1\|_{A_{r_1}^{*,\beta_1}} &\lesssim 2^{r_1nl} \sup_{2^l \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} + \left(\sum_{s=l+1}^{\infty} \left(2^{r_1ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\tau} \right)^{\frac{1}{\tau}} \\ &\lesssim 2^{(r_1-r_0)nl} \left(\sum_{s=0}^l \left(2^{r_0ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\tau} \right)^{\frac{1}{\tau}} \\ &\quad + \left(\sum_{s=l+1}^{\infty} \left(2^{r_1ns} \sup_{2^s \leq \langle \xi \rangle} k_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\tau} \right)^{\frac{1}{\tau}}. \end{aligned}$$

Consider now

$$\begin{aligned} \|f\|_{(A_{r_0}^{*\beta_0}, A_{r_1}^{*\beta_1})_{\theta, q}} &= \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \frac{1}{(r_0 - r_1)^{1/q}} \left(\int_0^\infty (t^{-\theta(r_0-r_1)n} K(f, t^{(r_0-r_1)n}))^q \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

Since $K(f, t^{(r_0-r_1)n}) = \inf_{f=f_0+f_1} (\|f_0\|_{A_{r_0}^{*\beta_0}} + t^{(r_0-r_1)n} \|f_1\|_{A_{r_1}^{*\beta_1}}) \leq t^{(r_0-r_1)n} \|f\|_{A_{r_1}^{*\beta_1}}$, we have

$$\begin{aligned} \|f\|_{(A_{r_0}^{*\beta_0}, A_{r_1}^{*\beta_1})_{\theta, q}} &\lesssim \left[\int_0^1 (t^{-\theta(r_0-r_1)n} t^{(r_0-r_1)n} \|f\|_{A_{r_1}^{*\beta_1}})^q \frac{dt}{t} \right. \\ &\quad \left. + \int_1^\infty (t^{-\theta(r_0-r_1)n} K(f, t^{(r_0-r_1)n}))^q \frac{dt}{t} \right]^{1/q} \\ &\asymp \|f\|_{A_{r_1}^{*\beta_1}} + \left(\sum_{l=0}^\infty (2^{-\theta(r_0-r_1)nl} K(f, 2^{(r_0-r_1)nl}))^q \right)^{\frac{1}{q}}. \end{aligned}$$

In view of $r_1 < r$, we get $\|f\|_{A_{r_1}^{*\beta_1}} \lesssim \|f\|_{A_r^{*\beta}}$. Then, using

$$K(f, 2^{(r_0-r_1)nl}) \leq \|f_0\|_{A_{r_0}^{*\beta_0}} + 2^{(r_0-r_1)nl} \|f_1\|_{A_{r_1}^{*\beta_1}}$$

and the above estimates, we have

$$\begin{aligned} \|f\|_{(A_{r_0}^{*\beta_0}, A_{r_1}^{*\beta_1})_{\theta, q}} &\lesssim \|f\|_{A_r^{*\beta}} \\ &\quad + \left(\sum_{l=0}^\infty 2^{-\theta(r_0-r_1)qnl} \left\{ \left[\sum_{s=0}^l (2^{r_0ns} \sup_{2^s \leq \xi} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\mathbb{HS}})^\tau \right]^{\frac{1}{\tau}} \right. \right. \\ &\quad \left. \left. + 2^{(r_0-r_1)nl} \left[\sum_{s=l+1}^\infty (2^{r_1ns} \sup_{2^s \leq \xi} k_\xi^{-1/2} \|\widehat{f}(\xi)\|_{\mathbb{HS}})^\tau \right]^{\frac{1}{\tau}} \right\}^q \right)^{\frac{1}{q}}. \end{aligned}$$

Further, since $\tau \leq q$ we apply Hardy’s inequality to get

$$\|f\|_{(A_{r_0}^{*\beta_0}, A_{r_1}^{*\beta_1})_{\theta, q}} \lesssim \|f\|_{A_r^{*\beta}}.$$

This completes the proof of (i).

To show (ii), we first note that $(B_{p, \beta_0}^{r_0}, B_{p, \beta_1}^{r_1})_{\theta, q} \hookrightarrow B_{p, q}^r$ can be proved similarly to the proof of (i) using the embedding $B_{p, \beta}^r \hookrightarrow B_{p, \infty}^r$ from Theorem 5.2 (1).

Let us verify the inverse embedding. We have

$$\begin{aligned} & \|f\|_{(B_{p,\beta_0}^{r_0}, B_{p,\beta_1}^{r_1})_{\theta,q}} \\ & \lesssim \|f\|_{B_{p,\beta_1}^{r_1}} + \left(\sum_{k=0}^{\infty} \left(2^{-\theta(r_0-r_1)k} K(f, 2^{(r_0-r_1)k}; B_{p,\beta_0}^{r_0}, B_{p,\beta_1}^{r_1}) \right)^q \right)^{\frac{1}{q}} \\ & \lesssim \|f\|_{B_{p,q}^{r_1}} + \left(\sum_{k=0}^{\infty} \left[2^{-\theta(r_0-r_1)k} \left(\|S_{2^k}(f)\|_{B_{p,\beta_0}^{r_0}} + 2^{(r_0-r_1)k} \|f - S_{2^k}(f)\|_{B_{p,\beta_1}^{r_1}} \right) \right]^q \right)^{\frac{1}{q}} \\ & \lesssim \|f\|_{B_{p,q}^{r_1}} + \left(\sum_{k=0}^{\infty} 2^{-\theta(r_0-r_1)qk} \left(\|S_{2^k}(f)\|_{B_{p,\tau}^{r_0}} + 2^{(r_0-r_1)k} \|f - S_{2^k}(f)\|_{B_{p,\tau}^{r_1}} \right)^q \right)^{\frac{1}{q}} \\ & \lesssim \|f\|_{B_{p,q}^{r_1}} + \left(\sum_{k=0}^{\infty} 2^{-\theta(r_0-r_1)qk} \left\{ \left[\sum_{s=0}^k (2^{r_0s} \|S_{2^k}(f) - S_{2^s}(f)\|_{L_p})^\tau \right]^{1/\tau} \right. \right. \\ & \quad \left. \left. + 2^{(r_0-r_1)k} \left[\sum_{s=k+1}^{\infty} (2^{r_1s} \|f - S_{2^s}(f)\|_{B_{p,\tau}^{r_1}})^\tau \right]^{1/\tau} \right\}^q \right)^{\frac{1}{q}} \lesssim \|f\|_{B_{p,q}^{r_1}}, \end{aligned}$$

where in the last estimate we have used Hardy's inequality.

Part (iii) follows from

$$B_{p,1}^r \hookrightarrow H_p^r \hookrightarrow B_{p,\infty}^r$$

and

$$B_{p,q}^r = (B_{p,1}^{r_0}, B_{p,1}^{r_1})_{\theta,q} \hookrightarrow (H_p^{r_0}, H_p^{r_1})_{\theta,q} \hookrightarrow (B_{p,\infty}^{r_0}, B_{p,\infty}^{r_1})_{\theta,q} = B_{p,q}^r;$$

see Theorem 5.2 (6)–(7).

Let us finally prove (iv). Since $B_{p,\min\{p,q\}}^r \hookrightarrow F_{p,q}^r \hookrightarrow B_{p,\max\{p,q\}}^r$, we have

$$\begin{aligned} B_{p,q}^r &= (B_{p,\min\{p,\beta_0\}}^{r_0}, B_{p,\min\{p,\beta_1\}}^{r_1})_{\theta,q} \hookrightarrow (F_{p,\beta_0}^{r_0}, F_{p,\beta_1}^{r_1})_{\theta,q} \\ &\hookrightarrow (B_{p,\max\{p,\beta_0\}}^{r_0}, B_{p,\max\{p,\beta_1\}}^{r_1})_{\theta,q} = B_{p,q}^r. \end{aligned}$$

The proof is complete. □

Above we have provided rather direct proofs of the interpolation theorems. Another proof of such results could be also obtained using other methods, see, e.g., [1, 3].

9. Localisation of Besov spaces

In this section we show that as a corollary of Theorem 8.1, (iii), for certain ranges of indices, the localisations of the Besov spaces (5.1)–(5.2) coincide with the usual

Besov spaces on \mathbb{R}^n , so that the norm (5.2) provides the global characterisation of Besov spaces defined on the space G/K considered as a smooth manifold. We note that for Sobolev spaces such a characterisation is much simpler and follows directly from the elliptic regularity, see (5.4) and the discussion before it.

For $x, h \in \mathbb{R}^n$, let us denote

$$\Delta_h^m f(x) := \sum_{k=0}^m C_m^k (-1)^{m-k} f(x + kh)$$

and

$$\omega_p^m(t, f) := \sup_{|h| \leq t} \|\Delta_h^m f\|_{L_p}.$$

Then it is known that for $r > 0$ and $1 \leq p, q \leq \infty$, the Besov space $B_{p,q}^r(\mathbb{R}^n)$ on \mathbb{R}^n can be characterised by the difference condition

$$\|f\|_{B_{p,q}^r(\mathbb{R}^n)} \asymp \|f\|_{L_p} + \sum_{j=1}^n \left(\int_0^\infty \left(t^{-r} \omega_p^m(t, f) \right)^q \frac{dt}{t} \right)^{1/q}, \tag{9.1}$$

with a natural modification for $q = \infty$, see, e.g., [49, Proposition 1.18], or [46, (1.13)] for a slightly different expression, with some generalisations in [32]. From (9.1) we see that for $r > 0$ and $1 \leq p, q \leq \infty$, the Besov spaces $B_{p,q}^r(\mathbb{R}^n)$ are invariant under smooth changes of variables and can, therefore, be defined on arbitrary smooth manifolds. Consequently, the same is true for any $r < 0$ by duality and, in fact, for any $r \leq 0$ by the property $(1 - \Delta)^{s/2} B_{p,q}^r = B_{p,q}^{r-s}$.

We say that such an extension is the Besov space on the manifold defined by localisations. For Sobolev spaces H_p^r we can use the same terminology, and the equivalence of norms (5.4) says that the Sobolev space H_p^r on G/K defined by localisations coincides with the Sobolev space defined by the norm on the right hand side of (5.4). We now formulate the analogue of this for Besov spaces.

Theorem 9.1. *Let G/K be the compact homogeneous space and let $1 < p < \infty$, $1 \leq q \leq \infty$ and $r \in \mathbb{R}$. Then the Besov space $B_{p,q}^r$ on G/K defined by localisations coincides with the Besov space $B_{p,q}^r(G/K)$ defined by (5.1)–(5.2), with the equivalence of norms.*

We note that Theorem 9.1 yields an alternative proof for some of the embeddings of Theorem 5.2.

Proof of Theorem 9.1. Let first $r > 0$. By Theorem 8.1, (iii), we know that the Besov space $B_{p,q}^r(G/K)$ defined by (5.1)–(5.2) is the interpolation space for Sobolev spaces with the norms given on the right hand side of (5.4). Since Sobolev spaces with such norms coincide with their localisation, the statement of Theorem 9.1 follows from the corresponding interpolation property of Sobolev spaces on \mathbb{R}^n , see [44]. The statement for $r < 0$ follows by duality or, in fact for any $r \leq 0$, since the property $(1 - \Delta)^{s/2} B_{p,q}^r = B_{p,q}^{r-s}$ holds for an elliptic operator Δ for both scales of spaces. □

References

- [1] V. BURENKOV, D. DARBAEVA and E. D. NURSULTANOV, *Description of interpolation spaces for general local Morrey-type spaces*, Eurasian Math. J. **1** (2013), 46–53.
- [2] A. BEURLING, *On the spectral synthesis of bounded functions*, Acta Math. **81** (1949), 225–238.
- [3] J. BERGH and J. LÖFSTRÖM, “Interpolation Spaces. An introduction”, Grundlehren der Mathematischen Wissenschaften, Vol. 223, Springer-Verlag, Berlin, 1976.
- [4] E. S. BELINSKII, E. R. LIFLYAND and R. M. TRIGUB, *The Banach algebra A^* and its properties*, J. Fourier Anal. Appl. **3** (1997), 103–129.
- [5] F. DAI, *Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights*, J. Funct. Anal. **235** (2006), 137–170.
- [6] J. J. DUISTERMAAT and V. W. GUILLEMIN, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.
- [7] R. A. DEVORE and G. G. LORENTZ, “Constructive Approximation”, Grundlehren der Mathematischen Wissenschaften, Vol. 303, Springer-Verlag, Berlin, 1993.
- [8] H. G. DALES and A. T.-M. LAU, “The Second Duals of Beurling Algebras”, Mem. Amer. Math. Soc., Vol. 177, 2005.
- [9] A. DASGUPTA and M. RUZHANSKY, *Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces*, Bull. Sci. Math. **138** (2014), 756–782.
- [10] Z. DITZIAN and S. TIKHONOV, *Ul’yanov and Nikol’skiĭ-type inequalities*, J. Approx. Theory **133** (2005), 100–133.
- [11] F. DAI and H. WANG, *Optimal cubature formulas in weighted besov spaces with A_∞ weights on multivariate domains*, Constr. Approx. **37** (2013), 167–194.
- [12] F. DAI and Y. XU, “Analysis on h-harmonics and Dunkl Transforms”, Birkhäuser Verlag, to appear.
- [13] J. FARAUT, “Analysis on Lie Groups. An Introduction”, Cambridge Studies in Advanced Mathematics, Vol. 110, Cambridge University Press, Cambridge, 2008.
- [14] G. FURIOLI, C. MELZI and A. VENERUSO, *Littlewood-Paley decompositions and Besov spaces on Lie groups of polynomial growth*, Math. Nachr. **279** (2006), 1028–1040.
- [15] E. GINÉ M, *The addition formula for the eigenfunctions of the Laplacian*, Adv. Math. **18** (1975), 102–107.
- [16] D. GELLER and I. Z. PESENSON, *Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds*, J. Geom. Anal. **21** (2011), 334–371.
- [17] E. GIACALONE and F. RICCI, *Norms of harmonic projection operators on compact Lie groups*, Math. Ann. **280** (1988), 21–31.
- [18] L. GRAFAKOS, “Classical Fourier Analysis”, Graduate Texts in Mathematics, Vol. 249, second edition, Springer, New York, 2008.
- [19] I. GALLAGHER and Y. SIRE, *Besov algebras on Lie groups of polynomial growth*, Studia Math. **212** (2012), 119–139.
- [20] S. GIULINI, P. M. SOARDI and G. TRAVAGLINI, *Norms of characters and Fourier series on compact Lie groups*, J. Funct. Anal. **46** (1982), 88–101.
- [21] K. HESSE, H. N. MHASKAR and I. H. SLOAN, *Quadrature in Besov spaces on the Euclidean sphere*, J. Complexity **23** (2007), 528–552.
- [22] J.-P. KAHANE, “Séries de Fourier absolument convergentes”, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 50, Springer-Verlag, Berlin, 1970.
- [23] A. I. KAMZOLOV, *Approximation of functions on the sphere S^n* , Serdica **10** (1984), 3–10.
- [24] H. H. LEE and E. SAMEI, *Beurling-Fourier algebras, operator amenability and Arens regularity*, J. Funct. Anal. **262** (2012), 167–209.
- [25] J. LUDWIG, N. SPRONK and L. TUROWSKA, *Beurling-Fourier algebras on compact groups: spectral theory*, J. Funct. Anal. **262** (2012), 463–499.

- [26] H. N. MHASKAR, F. J. NARCOWICH and J. D. WARD, *Approximation properties of zonal function networks using scattered data on the sphere. Radial basis functions and their applications*, Adv. Comput. Math. **11** (1999), 121–137.
- [27] S. M. NIKOLSKY, *Inequalities for entire functions of finite degree and their applications in the theory of differentiable functions of many variables*, Proc. Steklov Inst. Math. **38** (1951), 244–278.
- [28] S. M. NIKOL'SKIĬ, “Approximation of Functions of Several Variables and Imbedding Theorems”, Die Grundlehren der Mathematischen Wissenschaften, Band 205, Springer-Verlag, New York, 1975. Translated from the Russian by John M. Danskin, Jr.
- [29] E. D. NURSULTANOV, *S. M. Nikol'skiĭ's inequality for different metrics, and properties of the sequence of norms of Fourier sums of a function in the Lorentz space*, Tr. Mat. Inst. Steklova **255** (2006), 197–215.
- [30] R. J. NESSEL and G. WILMES, *Nikolskii-type inequalities for trigonometric polynomials and entire functions of exponential type*, J. Aust. Math. Soc. **25** (1978), 7–18.
- [31] J. PEETRE, “New Thoughts on Besov Spaces”, Duke University Mathematics Series, Vol. 1, Mathematics Department, Duke University, Durham, N.C., 1976.
- [32] I. Z. PESENSON, *Interpolation spaces on Lie groups*, Dokl. Akad. Nauk SSSR **246** (1979), 1298–1303.
- [33] I. Z. PESENSON, *Best approximations in a space of the representation of a Lie group*, Dokl. Akad. Nauk SSSR **302** (1988), 1055–1058.
- [34] I. PESENSON, *Bernstein-Nikolskii inequalities and Riesz interpolation formula on compact homogeneous manifolds*, J. Approx. Theory **150** (2008), 175–198.
- [35] I. PESENSON, *Bernstein-Nikolskii and Plancherel-Polya inequalities in L_p -norms on non-compact symmetric spaces*, Math. Nachr. **282** (2009), 253–269.
- [36] M. RUZHANSKY and V. TURUNEN, “Pseudo-differential Operators and Symmetries. Background Analysis and Advanced Topics”, Pseudo-Differential Operators. Theory and Applications, Vol. 2, Birkhäuser Verlag, Basel, 2010.
- [37] M. RUZHANSKY and V. TURUNEN, *Global quantization of pseudo-differential operators on compact Lie groups, $SU(2)$, 3-sphere, and homogeneous spaces*, Int. Math. Res. Not. **2013** (2013), 2439–2496.
- [38] R. T. SEELEY, *Complex powers of an elliptic operator*, In: “Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)”, A. P. Calderón (ed.), Amer. Math. Soc., Providence, R.I. (1967), 288–307.
- [39] M. A. SHUBIN, “Pseudodifferential Operators and Spectral Theory”, second edition, Springer-Verlag, Berlin, 2001. Translated from the 1978 Russian original by Stig I. Andersson.
- [40] L. SKRZYPCZAK, *Besov spaces and Hausdorff dimension for some Carnot-Carathéodory metric spaces*, Canad. J. Math. **54** (2002), 1280–1304.
- [41] C. D. SOGGE, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), 43–65.
- [42] E. M. STEIN and G. WEISS, “Introduction to Fourier Analysis on Euclidean Spaces”, Princeton Mathematical Series, Vol. 32, Princeton University Press, Princeton, N.J., 1971.
- [43] R. M. TRIGUB and E. S. BELLINSKY, “Fourier Analysis and Approximation of Functions”, Kluwer Academic Publishers, Dordrecht, 2004.
- [44] H. TRIEBEL, “Theory of Function Spaces”, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel, 1983.
- [45] H. TRIEBEL, “Theory of Function Spaces. II”, Monographs in Mathematics, Vol. 84, Birkhäuser Verlag, Basel, 1992.
- [46] H. TRIEBEL, “Theory of Function Spaces. III”, Monographs in Mathematics, Vol. 100, Birkhäuser Verlag, Basel, 2006.
- [47] N. J. VILENKIN, “Special Functions and the Theory of Group Representations”, Translations of Mathematical Monographs, Vol. 22, American Mathematical Society, Providence, R. I., 1968. Translated from the Russian by V. N. Singh.

- [48] N. J. VILENKIN and A. U. KLIMYK, “Representation of Lie Groups and Special Functions. Vol. 1. Simplest Lie Groups, Special Functions and Integral Transforms”, Mathematics and its Applications (Soviet Series), Vol. 72, Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the Russian by V. A. Groza and A. A. Groza.
- [49] B. WANG, Z. HUO, C. HAO and Z. GUO, “Harmonic Analysis Method for Nonlinear Evolution Equations. I”, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

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