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Existence of minimizers for the Reifenberg Plateau problem

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Abstract. Given a compact set $B \subset \mathbb{R}^n$ and a subgroup L of the Čech homology group $\check{H}_{d-1}(B; G)$ of dimension $d-1$ over some Abelian group G, we find a compact set $E \supset B$ such that the image of L by the natural map $\check{H}_{d-1}(B; G) \to \check{H}_{d-1}(S; G)$ induced by the inclusion $B \to E$, is reduced to $\{0\}$, and such that the Hausdorff measure $\mathcal{H}^d(E \setminus B)$ is minimal under these constraints. Thus we have no restriction on the group G or the dimensions $0 < d < n$. We can also replace the Hausdorff measure with the integral of a special integrand.

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1. Introduction

Plateau problems usually concern the existence of *surfaces* that minimize an *area* under some *boundary* constraints, but many different meanings can be given to the terms "surface" and "area", and many different boundary constraints can be considered.

In the present text, we shall prove an existence result for a minor variant of the homological Plateau problem considered by Reifenberg [20]. That is, we shall give ourselves dimensions $0 < d < n$, a compact set $B \subset \mathbb{R}^n$, an Abelian group G, and a subgroup L of the Čech homology group $\check{H}_{d-1}(B; G)$, and we shall find a compact set $E \supset B$ that minimizes the Hausdorff measure $\mathcal{H}^d(E \setminus B)$ under the constraint that the restriction to L of the natural map $\check{H}_{d-1}(B; G) \to \check{H}_{d-1}(S; G)$ induced by the inclusion $B \to E$ is trivial. See the slightly more precise definitions below.

This problem was first studied by Reifenberg [20], who gave a general existence result when the group G is a compact Abelian group and B is a $(d-1)$-dimensional compact set.

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Also, Almgren [1] announced an extension of Reifenberg's result, obtained in connection to varifolds, and where the Hausdorff measure $\mathcal{H}^d$ is no longer necessarily minimized alone, but integrated against an elliptic integrand.

More recently, De Pauw [18] proved the existence of minimizers also when $G = \mathbb{Z}$ is the group of integers, $n = 3$, $d = 2$, and B is a nice curve.

Here we remove these restrictions, and also use a quite different method of proof, based on a construction of quasiminimal sets introduced by Feuvrier [13].

Let us introduce some notation and definitions, and then we will rapidly discuss our main result and its background. We denote by $G(n, d)$ the Grassmann manifold of unoriented d-plane directions in $\mathbb{R}^n$. An integrand is a continuous function $F : \mathbb{R}^n \times G(n, d) \to \mathbb{R}^+$ which is bounded, *i.e.* there exist $0 < c \leq C < +\infty$ such that $c \leq F(x, \pi) \leq C$ for all $x \in \mathbb{R}^n$ and $\pi \in G(n, d)$. For any d-dimensional set E, we write $E = E_{\text{rec}} \cup E_{\text{irr}}$, where E_{rec} is d-rectifiable, E_{irr} is purely d-unrectifiable. For any integrand F and any positive function $f : \mathbb{R}^n \to \mathbb{R}^+$, we set

$$J_{F,f}(E) = \int_{x\in E_{\text{rec}}} F(x, T_x E_{\text{rec}}) d\mathcal{H}^d(x) + \int_{x\in E_{\text{irr}}} f(x) d\mathcal{H}^d(x).$$

We set $J_F(E) = J_{F,\tilde{F}}(E)$, where $\tilde{F}$ is the function $\mathbb{R}^n \to \mathbb{R}^+$ defined by

$$\tilde{F}(x) = \sup_{\pi\in G(n,d)} F(x, \pi).$$

We shall consider a class of integrands $\mathfrak{F}$ that are integrands F satisfying the following properties: there exists a nondecreasing function $h : \mathbb{R} \to [0, +\infty]$ with $\lim_{r\to 0} h(r) = 0$, such that for all $x \in \mathbb{R}^n$, $r > 0$ and $\pi \in G(n, d)$,

$$J_F(D_{\pi,r}) \leq J_F(S) + h(r)r^d$$

where $D_{\pi,r} = (x + \pi) \cap \overline{B(x, r)}$ and $S \subset \overline{B(x, r)}$ is a compact d-rectifiable set which cannot be mapped into $\partial D_{\pi,r} := \pi \cap \partial B(x, r)$ by any Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_{\partial D_{\pi,r}} = \text{id}_{\partial D_{\pi,r}}$.

It is easy to see that all continuous functions $g : \mathbb{R}^n \to [a, b]$ with $0 < a < b < +\infty$ are contained in the class $\mathfrak{F}$. One can also easily check that all elliptic integrands introduced by Almgren in [2, page 423] and [1, page 322] are contained in the class $\mathfrak{F}$.

Let $\mathscr{C}$ be a class of compact subsets in $\mathbb{R}^n$. We set

$$m(\mathscr{C}, F) = \inf\{J_F(E \setminus B) \mid E \in \mathscr{C}\}.$$

Theorem 1.1. *Let $F \in \mathfrak{F}$ be an integrand, $\mathscr{C}$ a class of compact subsets in $\mathbb{R}^n$. If $\mathscr{C}$ satisfies the following conditions:*

(1) *For any Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_B = \text{id}_B$ and any $E \in \mathscr{C}$, one has $\varphi(E) \in \mathscr{C}$;*

(2) *For any sequence* $\{E_k\}_{k=1}^{\infty} \subset \mathscr{C}$ *with* $E_k \to E$ *in Hausdorff distance, one has* $E \in \mathscr{C}$.

Then we can find $E \in \mathscr{C}$ *such that* $J_F(E \setminus B) = m(\mathscr{C}, F)$.

Of course the problem will only be interesting when $m(\mathscr{C}, F) < +\infty$, which is usually fairly easy to arrange. We subtracted B because this way we shall not need to assume that $\mathcal{H}^d(B) < +\infty$, but of course if $\mathcal{H}^d(B) < +\infty$ we could replace $J_F(S \setminus B)$ with $J_F(S)$ in the definition.

Now let us introduce Reifenberg's homological Plateau problem, which is our main concern. When $B \subset \mathbb{R}^n$ is a compact set, G is a commutative group, and $k \geq 0$ is an integer, we shall denote by $H_k(B; G)$ and $\check{H}_k(B; G)$ the singular and Čech homology groups on B, of order k and with the group G; we refer to [9] for a definition and basic properties.

If S is another compact set that contains B, we shall denote by $i_{B,S} : B \to S$ the natural inclusion, by $H_k(i_{B,S}) : H_k(B; G) \to H_k(S; G)$ the corresponding homomorphism between homology groups, and by $\check{H}_k(i_{B,S}) : \check{H}_k(B; G) \to \check{H}_k(S; G)$ the corresponding homomorphism between Čech homology groups.

Definition 1.2. Fix a compact set $B \subset \mathbb{R}^n$, an integer $0 < d < n$, a commutative group G, and a subgroup L of $\check{H}_{d-1}(B; G)$. We say that the compact set $S \supset B$ spans L in Čech homology if $L \subset \ker \check{H}_{d-1}(i_{B,S})$.

A simple case is when L is the full group $\check{H}_k(B; G)$; then $S \supset B$ spans L in Čech homology precisely when the mapping $\check{H}_k(i_{B,S})$ is trivial. But it may be interesting to study other subgroups L, and this will not make the proofs any harder.

We have a similar definition of "$S \supset B$ spans L in singular homology", where we just replace $\check{H}_{d-1}(i_{B,S})$ with $H_{d-1}(i_{B,S})$. It would be very nice if our main statement was in terms of singular homology, but unfortunately we cannot prove the corresponding statement at this time.

We shall denote by $\mathcal{H}^d(E)$ the d-dimensional Hausdorff measure of the Borel set $E \subset \mathbb{R}^n$. Recall that

$$\mathcal{H}^d(E) = \lim_{\delta \to 0+} \mathcal{H}^d_\delta(E),$$

where

$$\mathcal{H}^d_\delta(E) = \inf \left\{ \sum_j \operatorname{diam}(U_j)^d \,\middle|\, E \subset \bigcup_j U_j,\ \operatorname{diam}(U_j) < \delta \right\},$$

i.e., the infimum is over all the coverings of E by a countable collection of sets U_j with diameters less than δ. We refer to [10, 15] for the basic properties of $\mathcal{H}^d$; notice incidentally that we could also have used the spherical Hausdorff measure, or even some more exotic variants, essentially because the competition will rather fast be restricted to rectifiable sets, for which the two measures are equal.

We are given $B \subset \mathbb{R}^n$, $d \in (0, n)$, G, and a subgroup L of $\check{H}_{d-1}(B; G)$, and we set

$$\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L) = \left\{ \begin{array}{c} S \subset \mathbb{R}^n \mid S \text{ is a compact set that contains } B \\ \text{and spans } L \text{ in Čech homology} \end{array} \right\}.$$

For any $S \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ and any Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_B = \mathrm{id}_B$, we can easily get that $\varphi(S) \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$. Indeed, $i_{B,\varphi(S)} = \varphi \circ i_{B,S}$,

$$\check{H}_{d-1}(i_{B,\varphi(S)}) = \check{H}_{d-1}(\varphi) \circ \check{H}_{d-1}(i_{B,S}),$$

so, from this, we can see that

$$\ker\left(\check{H}_{d-1}(i_{B,S})\right) \subset \ker\left(\check{H}_{d-1}(i_{B,\varphi(S)})\right),$$

thus $\varphi(S)$ spans L in Čech homology.

We also set

$$m = m\left(\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L), F\right) = \inf\left\{J_F(S \setminus B)\,;\ S \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)\right\}.$$

As the reader may have guessed, we want to find $E \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ such that $J_F(S \setminus B) = m$. Our theorem is thus the following:

Theorem 1.3. *Let the compact set $B \subset \mathbb{R}^n$, an integrand $F \in \mathfrak{F}$, an Abelian group G, and a subgroup L of $\check{H}_{d-1}(B; G)$ be given. Suppose that*

$$m\left(\mathscr{C}_{\check{C}ech}(B, G, L), F\right) < +\infty.$$

Then there exists a compact set $E \in \mathscr{C}_{\check{C}ech}(B, G, L)$ such that

$$J_F(E \setminus B) = m\left(\mathscr{C}_{\check{C}ech}(B, G, L), F\right).$$

Notice that the statement is still true when $m = +\infty$, but it is not interesting.

As was mentioned before, this theorem was proved by Reifenberg in [20], under the additional assumption that G is compact and B is of Hausdorff dimension $d - 1$.

A slightly unfortunate feature of both statements is that they use the Čech homology groups. A similar statement with the singular homology groups would be very welcome, both because they are simpler and because connections with the theories of flat chains and currents would be much simpler. Unfortunately, singular homology does not pass to the limit as nicely as Čech homology.

Reifenberg was not the first person to give beautiful results on the Plateau Problem. Douglas [7] gave an essentially optimal existence result for the following parameterization problem: given a simple closed curve γ in $\mathbb{R}^n$, find a surface E, parameterized by the closed unit disk in the plane, so that the restriction of the parameterization to the unit circle parameterizes γ, and for which the area (computed with the Jacobian and counting multiplicity) is minimal.

But the most popular way to state and prove existence results for the Plateau problem has been through sets of finite perimeter (De Giorgi) and currents (Federer and Fleming). In particular, Federer and Fleming [11] gave a very general existence result for integral currents S whose mass is minimal under the boundary constraint $\partial S = T$, where T is a given integral current such that $\partial T = 0$. Mass-minimizing currents also have a very rich regularity theory; we refer to [16] for a nice overview.

In the author's view, Reifenberg's homological minimizers often give a better description of soap film than mass minimizers, and they are much closer to (the closed support of) size minimizing currents. Those are currents S that minimize the quantity $\mathrm{Size}(S)$ under a boundary constraint $\partial S = T$ as before, but where $\mathrm{Size}(S)$ is, roughly speaking, the $\mathcal{H}^d$-measure of the set where the multiplicity function that defines S as an integral current is nonzero. Thus the mass counts the multiplicity, while the size does not. We refer to [18] for precise definitions, and a more detailed account of the Plateau problem for size minimizing currents. We shall just mention two things here, in connection to the Reifenberg problem. Figure 1.1 depicts the support of a current which is size minimizing, but not mass minimizing (the multiplicity on the central disk is 2, so the mass is larger than the size).

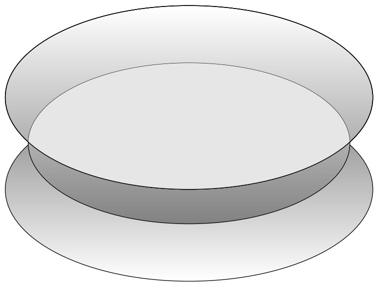

Figure 1.1. Size minimizing but not mass minimizing.

Even when the boundary current T is the current of integration on a smooth (but possibly linked) curve in $\mathbb{R}^3$, there is no general existence for a size minimizing current. However, Frank Morgan proved existence of a size minimizing current [16] when the boundary is a smooth submanifold contained in the boundary of a convex body and, in [19], Thierry de Pauw and Robert Hardt proved the existence of currents which minimize energies that lie somewhere between mass and size (typically, obtained by integration of some small power of the multiplicity).

The reason why the usual proof of existence for mass minimizers, using a compactness theorem, does not work for size minimizers is that the size of S does not give any control on the multiplicity, and so the limit of a minimizing sequence may well not have finite mass (or not even exist as a current). This issue is related to the reason why Reifenberg restricted to compact groups (so that multiplicities don't go to infinity).

In [1], F. Almgren proposed a scheme for proving Reifenberg's theorem, and even extending it to general groups and elliptic integrands. The scheme uses the then recently discovered varifolds, or flat chains, and a multiple layers argument to get rid of high multiplicities, but it is also very subtle and elliptic. Incidentally, Almgren uses Vietoris relative homology groups H_d^v instead of Čech homology groups. In his paper, a boundary B is a compact $(d-1)$-rectifiable subset of $\mathbb{R}^n$ with $\mathcal{H}^{d-1}(B) < +\infty$, a surface S is a compact d-rectifiable subset of $\mathbb{R}^n$. For any $\sigma \in H_d^v(\mathbb{R}^n, B; G)$, a surface S spans σ if $i_k(\sigma) = 0$, where we denote by $H_d^v(\mathbb{R}^n, B; G)$ the d-th Vietoris relative homology groups of $(\mathbb{R}^n, B)$, and

$$i_k : H_d^v(\mathbb{R}^n, B; G) \to H_d^v(\mathbb{R}^n, B \cup S; G)$$

is the homomorphism induced by the inclusion map $i : B \to B \cup S$. We should mention that Dowker, in [8, Theorem 2a], proved that Čech and Vietoris homology groups over an Abelian group G are isomorphic for arbitrary topological spaces.

There is some definite relation between Refenberg's homological Plateau problem and the size minimizing currents, and for instance T. De Pauw [18] shows that in the simple case when B is a nice curve, the infimums for the two problems are equal. In the same paper, T. De Pauw also extends Reifenberg's result (for curves in $\mathbb{R}^3$) to the group $G = \mathbb{Z}$. Unfortunately, even though the proof uses minimizations among currents, this does not yet give a size minimizer (one would need to construct an appropriate current on the minimizing set).

Let us look at an another related problem called free boundary Plateau problem, see for example in [17]. Given a compact set $B \subset \mathbb{R}^n$, an Abelian group G, and a subgroup L of $\check{H}_{d-1}(B; G)$, we call a compact set $X \subset \mathbb{R}^n$ a surface with free boundary including L, if

$$L \subset \check{H}_{d-1}(i_{X\cap B,B}) \left(\ker \check{H}_{d-1}(i_{X\cap B,X})\right).$$

We set

$$\mathscr{C}_{\text{free}}(B, G, L) = \left\{ E \subset \mathbb{R}^n \mid \begin{array}{c} E \text{ is a compact set with} \\ \text{free boundary including } L \end{array} \right\}.$$

Theorem 1.4. *Let B, G, L and F ba as in Theorem 1.3. We suppose that*

$$m\left(\mathscr{C}_{\text{free}}(B, G, L)\right) < +\infty.$$

Then we can find $E \in \mathscr{C}_{\text{free}}(B, G, L)$ such that

$$J_F(E \setminus B) = m\left(\mathscr{C}_{\text{free}}(B, G, L)\right).$$

We will give the proof in the end of the paper.

The proofs of Theorem 1.3 and Theorem 1.4 follow from Theorem 1.1, and the proof of Theorem 1.1 is more in the spirit of the initial proof of Reifenberg, but will rely on two more recent developments that make it work more smoothly and ignore multiplicity issues.

The first development is a lemma introduced by Dal Maso, Morel, and Solimini [14] in the context of the Mumford-Shah functional, and which gives a sufficient condition, on a sequence of sets E_k that converges to a limit E in Hausdorff distance, for the lower semicontinuity inequality

$$\mathcal{H}^d(E) \le \liminf_{k \to +\infty} \mathcal{H}^d(E_k).$$

This is very convenient here because we want to work with sets and we do not want to use weak limits of currents. Since we want to deal with integrands, we will show the following lower semicontinuity inequality,

$$J_F(E) \le \liminf_{k \to +\infty} J_F(E_k).$$

But our main tool will be a recent result of V. Feuvrier [13], where he uses a construction of polyhedral networks adapted to a given set (think about the usual dyadic grids, but where you manage to have faces that are very often parallel to the given set) to construct a minimizing sequence for our problem, but which has the extra feature that it is composed of locally uniformly quasiminimal sets, to which we can apply Dal Maso, Morel, and Solimini's lemma.

Such a construction was used by Xiangyu Liang, to prove existence results for sets that minimize Hausdorff measure under some homological generalization of a separation constraint (in codimension larger than 1).

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2. Existence of minimizers under Reifenberg homological conditions

In this section we prove an existence theorem for sets in $\mathbb{R}^n$ that minimize the Hausdorff measure under Reifenberg homological conditions.

Definition 2.1. A polyhedral complex $\mathcal{S}$ is a finite set of closed convex polytopes in $\mathbb{R}^n$, such that two conditions are satisfied:

(1) If $Q \in \mathcal{S}$, and F is a face of Q, then $F \in \mathcal{S}$;
(2) If $Q_1, Q_2 \in \mathcal{S}$, then $Q_1 \cap Q_2$ is a face of both Q_1 and Q_2 or $Q_1 \cap Q_2 = \emptyset$.

The subset $|\mathcal{S}| := \cup_{Q \in \mathcal{S}} Q$ of $\mathbb{R}^n$ equipped with the induced topology is called the underlying space of $\mathcal{S}$. The d-skeleton of $\mathcal{S}$ is the union of the faces whose dimension is at most d.

A dyadic complex is a polyhedral complex consisting of closed dyadic cubes. Let's refer to [12,13] for the precise definition of dyadic complex.

Let $\Omega \subset \mathbb{R}^n$ be an open subset, $1 \le M < +\infty$ and $0 < \delta \le +\infty$, and $\ell \in \mathbb{N}$ with $0 \le \ell \le n$. Let $f : \Omega \to \Omega$ be a Lipschitz map; we set

$$W_f = \{x \in \Omega \mid f(x) \ne x\}.$$

Definition 2.2. Let E be a relatively closed set in Ω. We say that E is an (Ω, M, δ)-quasiminimal set of dimension ℓ if $\mathcal{H}^\ell(E \cap B) < +\infty$, for every closed ball $B \subset \Omega$, and

$$\mathcal{H}^\ell(E \cap W_f) \leq M\mathcal{H}^\ell(f(E \cap W_f)),$$

for every Lipschitz map $f : \Omega \to \Omega$ such that $W_f \cup f(W_f)$ is relatively compact in Ω and $\operatorname{diam}(W \cup f(W_f)) < \delta$.

We denote by $\mathbf{QM}(\Omega, M, \delta, \mathcal{H}^\ell)$ the collection of all (Ω, M, δ)-quasiminimal sets of dimension ℓ.

We note that, for any open set $\Omega' \subset \Omega$, any positive numbers $\delta' \leq \delta$, and any $M' \geq M$, if $E \in \mathbf{QM}(\Omega, M, \delta, \mathcal{H}^\ell)$, then $E \cap \Omega' \in \mathbf{QM}(\Omega', M', \delta', \mathcal{H}^\ell)$.

Definition 2.3. Let Ω be an open subset of $\mathbb{R}^n$. A relatively closed set $E \subset \Omega$ is said to be locally Ahlfors-regular of dimension d if there is a constant $C > 0$ and $r_0 > 0$ such that

$$C^{-1}r^d \leq \mathcal{H}^d(E \cap B(x, r)) \leq Cr^d$$

for all $0 < r < r_0$ with $B(x, 2r) \subset \Omega$.

Lemma 2.4. *Let E be a d-rectifiable subset of $\mathbb{R}^n$. If E is a locally Ahlfors-regular and $\mathcal{H}^d(E) < +\infty$, then, for $\mathcal{H}^d$-a.e. $x \in E$, the set E has a true tangent plane at x,* i.e. *there exists a d-plane π such that, for any $\varepsilon > 0$, there is a $r_\varepsilon > 0$ such that*

$$E \cap B(x, r) \subset \mathcal{C}(x, \pi, r, \varepsilon) \text{ for } 0 < r < r_\varepsilon,$$

where

$$\mathcal{C}(x, \pi, r, \varepsilon) = \{y \in B(x, r) \mid \operatorname{dist}(y, \pi) \leq \varepsilon\, |y - x|\}.$$

Proof. Since E is rectifiable, by Theorem 15.11 in [15], for $\mathcal{H}^d$-a.e. $x \in E$, the set E has an approximate tangent plane π at x, *i.e.*

$$\limsup_{\rho \to 0} \frac{\mathcal{H}^d(E \cap B(x, \rho))}{\rho^d} > 0,$$

and for all $\varepsilon > 0$,

$$\lim_{\rho \to 0} \rho^{-d}\mathcal{H}^d(E \cap B(x, \rho) \setminus \mathcal{C}(x, \pi, \rho, \varepsilon)) = 0. \tag{2.1}$$

We will show that π is a true tangent plane. Suppose it is not, that is, there exists an $\varepsilon > 0$ such that $E \cap B(x, \rho) \setminus \mathcal{C}(x, \pi, \rho, \varepsilon) \neq \emptyset$ for all $\rho > 0$. We take a sequence of points $y_n \in E \setminus \mathcal{C}(x, \pi, \rho, \varepsilon)$ with $|y_n - x| \to 0$, we put $\rho_n = 2\,|y_n - x|$, then

$$B(x, \rho_n) \setminus \mathcal{C}\left(x, \pi, \rho_n, \frac{\varepsilon}{2}\right) \supset B\left(y_n, \frac{\varepsilon\rho_n}{4}\right)$$

and

$$\rho_n^{-d}\mathcal{H}^d\left(E\cap B(x,\rho_n)\setminus\mathcal{C}\left(x,\pi,\rho_n,\frac{\varepsilon}{2}\right)\right)\geq\rho_n^{-d}\mathcal{H}^d\left(E\cap B\left(y_n,\frac{\varepsilon\rho_n}{4}\right)\right)$$
$$\geq C^{-1}\left(\frac{\varepsilon}{4}\right)^d,$$

this is in contradiction with (2.1), so we proved the lemma. □

Let $\{E_k\}$ be a sequence of closed sets in Ω, and let E be a closed set in Ω. We say that E_k converges to E if

$$\lim_{k\to\infty} d_K(E,E_k)=0 \text{ for every compact set } K\subset\Omega,$$

where

$$d_K(E,E_k)=\sup\{\operatorname{dist}(x,E_k)\mid x\in E\cap K\}+\sup\{\operatorname{dist}(x,E)\mid x\in E_k\cap K\}.$$

For any set $E\subset\mathbb{R}^n$, we set

$$E^*=\{x\in E\mid \mathcal{H}^d(E\cap B(x,r))>0,\ \forall r>0\};$$

we call E^* the core of E. We will prove the following lower semicontinuity property:

Theorem 2.5. *Let $\Omega\subset\mathbb{R}^n$ be an open set. Let $\{E_k\}_{k\geq 1}$ be a sequence of quasiminimal sets in $\mathbf{QM}(\Omega,M,\delta,\mathcal{H}^d)$ such that $E_k=E_k^*$ and E_k converges to E. Then, for any $F\in\mathfrak{F}$,*

$$J_F(E)\leq\liminf_{k\to+\infty}J_F(E_k).$$

Proof. We may suppose that

$$\liminf_{k\to+\infty}\mathcal{H}^d(E_k)<+\infty,$$

otherwise we have nothing to prove. Let $C_1,C_2>0$ be two number such that $C_1\leq F(x,\pi)\leq C_2$ for all $x\in\mathbb{R}^n$, and all $\pi\in G(n,d)$.

By Theorem 3.4 in [4], we have that

$$\mathcal{H}^d(E)\leq\liminf_{k\to+\infty}\mathcal{H}^d(E_k)<+\infty.$$

We take $0<\varepsilon<\frac{1}{2}$, $\varepsilon'>0$ and $\rho\in(0,1)$ such that $M^2 3^d\varepsilon<1$, $\varepsilon'<\frac{\varepsilon}{8}$ and $1-(1-\rho)^d<\frac{\varepsilon}{2}$.

Applying Theorem 4.1 in [4], we get that $E\in\mathbf{QM}(\Omega,M,\delta,\mathcal{H}^d)$, hence it is rectifiable (see [3]), then by [15, Theorem 17.6], for $\mathcal{H}^d$-a.e. $x\in E$,

$$\lim_{r\to 0}\frac{\mathcal{H}^d(E\cap B(x,r))}{\omega_d r^d}=1,$$

where ω_d denotes the Hausdorff measure of d-dimensional unit ball. So we can find a set $E' \subset E$ with $\mathcal{H}^d(E \setminus E') = 0$ such that for any $x \in E'$ there exists $r'(\varepsilon', x) > 0$ for which

$$(1-\varepsilon')\omega_d r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq (1+\varepsilon')\omega_d r^d,$$

for all $0 < r < r'(\varepsilon', x)$. Then

$$\begin{aligned}\mathcal{H}^d(E \cap B(x,r)\backslash B(x,(1-\rho)r)) &\leq (1+\varepsilon')\omega_d r^d - (1-\varepsilon')\omega_d(1-\rho)^d r^d\\ &= \frac{(1+\varepsilon')-(1-\varepsilon')(1-\rho)^d}{1-\varepsilon'}(1-\varepsilon')\omega_d r^d\\ &\leq \left(\frac{2\varepsilon'}{1-\varepsilon'} + \Big(1-(1-\rho)^d\Big)\right)\mathcal{H}^d(E \cap B(x,r))\\ &\leq \varepsilon\mathcal{H}^d(E \cap B(x,r)).\end{aligned}$$

Since E is quasiminimal, by Proposition 4.1 in [6], we know that E is locally Ahlfors-regular; since E is rectifiable and $\mathcal{H}^d(E) < +\infty$, by Lemma 2.4, we have that, for $\mathcal{H}^d$-a.e. $x \in E$, the set E has a tangent plane $T_x E$ at x, so we can find $E'' \subset E'$ with $\mathcal{H}^d(E' \setminus E'') = 0$ such that for all $\varepsilon'' > 0$ and for all $x \in E''$ there exists $r''(\varepsilon'', x) > 0$ such that, for all $0 < r < r''(\varepsilon'', x)$,

$$E \cap B(x,r) \subset \mathcal{C}(x,r,\varepsilon''),$$

where

$$\mathcal{C}(x,r,\varepsilon'') = \left\{y \in \overline{B(x,r)} \mid \operatorname{dist}(y, T_x E) \leq \varepsilon'' |x-y|\right\}.$$

We consider the function $\psi_{\rho,r} : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_{\rho,r}(t) = \begin{cases} 0 & t \leq (1-\rho)r\\ \frac{3}{\rho r}(t-(1-\rho)r) & (1-\rho)r < t \leq (1-\frac{2\rho}{3})r\\ 1 & (1-\frac{2\rho}{3})r < t \leq (1-\frac{\rho}{3})r\\ -\frac{3}{\rho r}(t-r) & (1-\frac{\rho}{3})r < t \leq r\\ 0 & t > r.\end{cases}$$

It is easy to see that $\psi_{\rho,r}$ is a Lipschitz map with Lipschitz constant $\frac{3}{\rho r}$.

We take the Lipschitz map $\varphi_{x,\rho,r} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\varphi_{x,\rho,r}(y) = \psi_{\rho,r}(|y-x|)\Pi(y) + (1-\psi_{\rho,r}(|y-x|))y,$$

where we denote by $\Pi : \mathbb{R}^n \to T_x E$ the orthogonal projection. It is easy to check that $\varphi_{x,\rho,r}$ coincides with Π on the annulus $\overline{B(x,r')} \setminus B(x,r'')$, and

$$\varphi_{x,\rho,r}|_{B(x,(1-\rho)r)} = \mathrm{id}_{B(x,(1-\rho)r)},$$
$$\varphi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c}.$$

Let ε'' and h be such that $\varepsilon'' < \frac{\rho}{3}$ and $0 < \varepsilon'' < h < \frac{\rho}{3}$, and put

$$A_h = \left\{ y \in \overline{B(x,r)} \mid \operatorname{dist}(y, T_x E) \leq hr \right\},$$

then $\mathcal{C}(x, r; \varepsilon'') \subset A_h$. We will show that

$$\operatorname{Lip}\left(\varphi_{x,\rho,r}|_{A_h}\right) \leq 2 + \frac{3h}{\rho}.$$

We set

$$\Pi^{\perp}(y) = y - \Pi(y), \text{ for } y \in \mathbb{R}^n;$$

then

$$\left|\Pi^{\perp}(y)\right| \leq hr, \ \forall y \in A_h.$$

For any $y_1, y_2 \in A_h$,

$$\begin{aligned}
\varphi_{x,\rho,r}(y_1) - \varphi_{x,\rho,r}(y_2) &= y_1 - y_2 - \psi_{\rho,r}(|y_1 - x|)\Pi^{\perp}(y_1) \\
&\quad + \psi_{\rho,r}(|y_2 - x|)\Pi^{\perp}(y_2) \\
&= (y_1 - y_2) - \psi_{\rho,r}(|y_1 - x|)\left(\Pi^{\perp}(y_1) - \Pi^{\perp}(y_2)\right) \\
&\quad - \left(\psi_{\rho,r}(|y_1 - x|) - \psi_{\rho,r}(|y_2 - x|)\right)\Pi^{\perp}(y_2),
\end{aligned}$$

thus

$$\begin{aligned}
\left|\varphi_{x,\rho,r}(y_1) - \varphi_{x,\rho,r}(y_2)\right| &\leq |y_1 - y_2| + |y_1 - y_2| + \frac{3}{\rho r}\left||y_1| - |y_2|\right| rh \\
&\leq \left(2 + \frac{3h}{\rho}\right)|y_1 - y_2|,
\end{aligned}$$

and we get that

$$\operatorname{Lip}\left(\varphi_{x,\rho,r}|_{A_h}\right) \leq 2 + \frac{3h}{\rho}.$$

Since $E_k \to E$ in Ω, $\overline{B(x,r)} \subset \Omega$, $E \cap B(x,r) \subset \mathcal{C}(x, r, \varepsilon'')$ and $\overline{\mathcal{C}(x, r, \varepsilon'')} \subset A_{h'}$ for $h' \in (\varepsilon'', h)$, there exists a number k_h such that for $k \geq k_h$,

$$E_k \cap \overline{B(x,r)} \subset A_h.$$

Since

$$\varphi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c}$$

and

$$\varphi_{x,\rho,r}(B(x,r)) \subset B(x,r),$$

we have that

$$\varphi_{x,\rho,r}(E_k \cap B(x,r)) = \varphi_{x,\rho,r}(E_k) \cap B(x,r).$$

We put $r' = \left(1 - \frac{\rho}{3}\right) r$, $r'' = \left(1 - \frac{2\rho}{3}\right) r$, $r''' = (1-\rho)r$, $\pi = T_x E$. Note that

$$\partial B\left(x, r'\right) \cap \pi \subset \varphi_{x,\rho,r}(E_k)$$

and

$$\varphi_{x,\rho,r}(E_k) \cap B\left(x, r'\right) \subset B\left(x, r''\right) \cup \left(\left(B(x, r') \setminus B(x, r'')\right) \cap \pi\right).$$

We put

$$D_{\pi,r''} = \overline{B\left(x, r''\right)} \cap \pi$$

and

$$S_{k,r''} = \varphi_{x,\rho,r}(E_k) \cap \overline{B\left(x, r''\right)}.$$

We will show that no Lipschitz mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which is the identity on $\partial D_{\pi,r''}$ can map $S_{k,r''}$ into $\partial D_{\pi,r''}$. Suppose this is not the case, that is, there is a Lipschitz map $\varphi_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi_k|_{\partial D_{\pi,r''}} = \mathrm{id}_{\partial D_{\pi,r''}}$$

and

$$\varphi_k(S_{k,r''}) \subset \partial D_{\pi,r''}.$$

We consider the map

$$\tilde{\phi}_k : B(x,\eta)^c \cup \left[\left(B(x,\eta) \setminus B(x,r'')\right) \cap \pi\right] \cup B(x,r'') \to \mathbb{R}^n$$

defined by

$$\tilde{\phi}_k(x) = \begin{cases} x & x \in B(x,\eta)^c \cup \left[\left(B(x,\eta) \setminus B(x,r'')\right) \cap \pi\right] \\ \varphi_k(x) & x \in B(x,r''), \end{cases}$$

where η is any number such that $r'' < \eta < r'$. It is easy check that $\tilde{\phi}_k$ is a Lipschitz map, by Kirszbraun's theorem, see for example [10, 2.10.43 Kirszbraun's theorem], we can get a Lipschitz map $\phi_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\phi_k|_{B(x,r'')} = \varphi_k|_{B(x,r'')}$$

and

$$\phi_k|_{B(x,\eta)^c} = \mathrm{id}_{B(x,\eta)^c},$$

and

$$\phi_k|_{(B(x,\eta)\setminus B(x,r''))\cap\pi} = \mathrm{id}_{(B(x,\eta)\setminus B(x,r''))\cap\pi}.$$

By the construction of ϕ_k, we have that

$$\phi_k(S_{k,r''}) = \tilde{\phi}_k(S_{k,r''}) \subset \partial D_{\pi,r''}.$$

Recalling the construction of $\psi_{\rho,r}$, we have that

$$0 \le \psi_{\rho,r} \le 1,\ \psi_{\rho,r}|_{[0,r''']} = 0,\ \psi_{\rho,r}|_{[r'',r']} = 1,\ \psi_{\rho,r}|_{[r,+\infty)} = 0;$$

and then recalling the construction of $\varphi_{x,\rho,r}$, we have that

$$\varphi_{x,\rho,r}|_{B(x,r)^c} = \mathrm{id}_{B(x,r)^c},\ \varphi_{x,\rho,r}(B(x,r)) \subset B(x,r),\ \varphi_{x,\rho,r}(B(x,r'')) \subset B(x,r'')$$

and

$$\varphi_{x,\rho,r}|_{B(x,r')\setminus B(x,r'')} = \Pi_{B(x,r')\setminus B(x,r'')},\ \varphi_{x,\rho,r}|_{B(x,r''')} = \mathrm{id}_{B(x,r''')},$$

where Π is the orthogonal projection onto the plane π. Thus $\varphi_{x,\rho,r}$ coincides with the orthogonal projection Π on the annulus $\overline{B(x,r')} \setminus B(x,r'')$. Note that $\varepsilon'' < \frac{\rho}{3}$, we get that

$$1 - \frac{2\rho}{3} < \left(1 - \frac{\rho}{3}\right)\sqrt{1-\varepsilon''^2}.$$

We now let the number η be such that $r'' < \eta < (\sqrt{1-\varepsilon''^2})r'$. Since $E_k \cap B(x,r) \subset \mathcal{C}(x,r,\varepsilon'')$, we have that

$$\varphi_{x,\rho,r}(E_k \cap B(x,r'')) \subset B(x,r''),\ \varphi_{x,\rho,r}(E_k \cap (B(x,r') \setminus B(x,r'')) \subset \pi,$$

and

$$\varphi_{x,\rho,r}(E_k \cap (B(x,r) \setminus B(x,r'))) \subset B(x,r) \setminus B(x,\eta),$$

thus

$$\varphi_{x,\rho,r}(E_k) \cap \big(B(x,\eta) \setminus B(x,r'')\big) \subset \big(B(x,\eta) \setminus B(x,r'')\big) \cap \pi,$$

and

$$\phi_k\left(\varphi_{x,\rho,r}(E_k) \cap B(x,\eta) \setminus B(x,r'')\right) \subset \big(B(x,\eta) \setminus B(x,r'')\big) \cap \pi.$$

Thus we get that

$$\begin{aligned}\phi_k(\varphi_{x,\rho,r}(E_k \cap B(x,r))) &= \phi_k(\varphi_{x,\rho,r}(E_k) \cap B(x,r)) \\ &\subset \varphi_{x,\rho,r}(E_k) \cap \big(B(x,r) \setminus B\left(x,r''\right)\big) \\ &\subset \varphi_{x,\rho,r}\left(E_k \cap B(x,r) \setminus B(x,r'')\right).\end{aligned}$$

Since $\mathcal{H}^d(E) < \infty$, we have that $\mathcal{H}^d(E \cap \partial B(x,r)) = 0$ for almost every $r \in (0, r''(\varepsilon'',x))$, if we take any $r \in (0, r''(\varepsilon'',x))$ with $\mathcal{H}^d(E \cap \partial B(x,r)) = 0$ and $r < \delta$, by using Lemma 3.12 in [5], we get that

$$\limsup_{k\to+\infty} \mathcal{H}^d\left(E_k \cap \overline{B(x,r)} \setminus B(x,r''')\right) \le M\mathcal{H}^d\left(E \cap \overline{B(x,r)} \setminus B(x,r''')\right).$$

Then we have the following inequalities:

$$
\begin{aligned}
\mathcal{H}^d\left(E\cap\overline{B(x,r)}\right) &= \mathcal{H}^d(E\cap B(x,r))\\
&\le \liminf_{k\to+\infty}\mathcal{H}^d(E_k\cap B(x,r))\\
&\le \liminf_{k\to+\infty} M\mathcal{H}^d\left(\phi_k\circ\varphi_{x,\rho,r}(E_k\cap B(x,r))\right)\\
&\le \liminf_{k\to+\infty} M\mathcal{H}^d\left(\varphi_{x,\rho,r}(E_k\cap B(x,r)\setminus B(x,r''')))\right)\\
&\le \liminf_{k\to+\infty} M\mathcal{H}^d\left(\varphi_{x,\rho,r}(E_k\cap \overline{B(x,r)}\setminus B(x,r''')))\right)\\
&\le \liminf_{k\to+\infty} M\left(2+\frac{3h}{\rho}\right)^d\mathcal{H}^d\left(E_k\cap \overline{B(x,r)}\setminus B(x,r''')\right)\\
&\le M\left(2+\frac{3h}{\rho}\right)^d\limsup_{k\to+\infty}\mathcal{H}^d\left(E_k\cap \overline{B(x,r)}\setminus B(x,r''')\right)\\
&\le M\left(2+\frac{3h}{\rho}\right)^d\cdot M\mathcal{H}^d\left(E\cap \overline{B(x,r)}\setminus B(x,r''')\right)\\
&\le M^2\left(2+\frac{3h}{\rho}\right)^d\varepsilon\mathcal{H}^d\left(E\cap \overline{B(x,r)}\right)\\
&\le M^2 3^d\varepsilon\mathcal{H}^d\left(E\cap \overline{B(x,r)}\right).
\end{aligned}
$$

This is a contradiction since $M^2 3^d\varepsilon < 1$ and $\mathcal{H}^d\left(E\cap\overline{B(x,r)}\right) > 0$.

Since $F\in\mathfrak{F}$, by definition, we have that

$$J_F(D_{\pi,r''})\le J_F(S_{k,r''})+h(r'')(r'')^d,$$

where $h:\mathbb{R}\to[0,+\infty]$ is a nondecreasing function with $\lim_{r\to 0}h(r)=0$.

Since E is a d-rectifiable set and $\mathcal{H}^d(E)<+\infty$, the function $f:E\to G(n,d)$ defined by $f(x)=T_xE$ is $\mathcal{H}^d$-measurable. By Lusin's theorem, see for example [10, 2.3.5. Lusin's theorem], we can find a closed set $N\subset E$ with $\mathcal{H}^d(E\setminus N)<\varepsilon$ such that f restricted to N is continuous. We put $E'''=(E''\cap N)$, then $E'''\subset E$ and

$$\mathcal{H}^d(E\setminus E''')<\varepsilon,$$

by Lemma 15.18 in [15], we have that for $\mathcal{H}^d$-a.e. $x\in E'''$,

$$T_xE'''=T_xN=T_xE.$$

The map $\tilde f:E'''\to\mathbb{R}^n\times G(n,d)$ given by $\tilde f(x)=(x,T_xE)$ is continuous. Since F is continuous, and the function $F\circ\tilde f:E'''\to\mathbb{R}$ is continuous, for any $x\in E'''$, we can find $r(\varepsilon,x)>0$ such that

$$(1-\varepsilon)F(x,T_xE)\le F(y,T_yE)\le(1+\varepsilon)F(x,T_xE),$$

for any $y \in E''' \cap B(x, r(\varepsilon, x))$. Thus, for all $0 < r < r(\varepsilon, x)$,

$$(1-\varepsilon) J_F(T_x E \cap B(x,r)) \leq J_F(E''' \cap B(x,r)) \leq (1+\varepsilon) J_F(T_x E \cap B(x,r)).$$

For any $x \in \mathbb{R}^n$, there exists $r'''(\varepsilon, x) > 0$ such that $h(r) < \varepsilon$ for all $0 < r < r'''(\varepsilon, x)$. We put

$$r(x) = \min(r(\varepsilon, x), r'(\varepsilon', x), r''(\varepsilon'', x), r'''(\varepsilon, x), \delta), \text{ for } x \in E''',$$

then

$$\left\{ B(x,r) \mid x \in E''', 0 < r < r(x), \mathcal{H}^d(E \cap \partial B(x,r)) = 0 \right\}$$

is a Vitali covering of E''', so we can find a countable family of balls $(B_i)_{i \in J}$ such that

$$\mathcal{H}^d \left(E''' \setminus \bigcup_{i \in J} B_i \right) = 0.$$

We choose a finite set $I \subset J$ such that

$$\mathcal{H}^d \left(E''' \setminus \bigcup_{i \in I} B_i \right) < \varepsilon.$$

We assume that $B_i = B(x_i, r_i)$. We put

$$\varphi = \prod_{i \in I} \varphi_{x_i, \rho, r_i}.$$

Since $\varphi|_{B_i} = \varphi_{x_i,\rho,r_i}|_{B_i}$, we have that $\varphi|_{B(x_i, r_i''')} = \mathrm{id}_{B(x_i, r_i''')}$,

$$\varphi(E_k) \cap B(x_i, r_i'') \setminus B(x_i, r_i''') \subset \varphi(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i'''))$$

and

$$\pi_i \cap B = \pi_i \cap \left(\left(B(x_i, r_i) \setminus B(x_i, r_i'') \right) \cup \left(B(x_i, r_i'') \setminus B(x_i, r_i''') \right) \cup \left(B(x_i, r_i''') \right) \right),$$

so we get that

$$J_F(E''') = \sum_{i \in J} J_F(E''' \cap B_i) \leq \sum_{i \in I} J_F(E''' \cap B_i) + C_2 \varepsilon.$$

For any $i \in I$,

$$
\begin{aligned}
J_F(E''' \cap B_i) &\le (1+\varepsilon) J_F(\pi_i \cap B_i) \\
&\le (1+\varepsilon)\Big(J_F\left(\pi_i \cap B(x_i, r_i) \setminus B(x_i, r_i'')\right) + J_F(\pi_i \cap B(x_i, r_i''))\Big) \\
&\le (1+\varepsilon)\Big(C_2 (r_i^d - (r_i'')^d) + J_F(S_{k,r''}) + h(r_i'')(r_i'')^d \Big) \\
&\le (1+\varepsilon) J_F(S_{k,r''}) + 2\varepsilon (r_i'')^d + 2C_2 \left((r_i^d - (r_i'')^d) \right) \\
&\le (1+\varepsilon) J_F(E_k \cap B(x_i, r_i''')) \\
&\quad + (1+\varepsilon) J_F\left(\varphi\left(E_k \cap B(x_i, r_i) \setminus B(x_i, r_i''')\right)\right) \\
&\quad + \left(2\varepsilon + 2C_2 \left(1 - \left(1 - \frac{2\rho}{3} \right)^d \right) \right) r_i^d \\
&\le (1+\varepsilon) J_F\left(E_k \cap B\left(x_i, r_i'''\right)\right) \\
&\quad + 2C_2 \left(1 + \frac{3h}{\rho} \right)^d \mathcal{H}^d\left(E_k \cap B\left(x_i, r_i\right) \setminus B(x_i, r_i''')\right) \\
&\quad + \left(2\varepsilon + 2C_2 \cdot \frac{\varepsilon}{2} \right) \frac{2}{\omega_d} \mathcal{H}^d\left(E \cap B(x_i, r_i)\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_F(E''') &\le \sum_{i \in I} J_F(E''' \cap B_i) + C_2 \varepsilon \\
&\le (1+\varepsilon) J_F(E_k) + (2\varepsilon + C_2 \varepsilon) \frac{2}{\omega_d} \sum_{i \in I} \mathcal{H}^d(E \cap B(x_i, r_i)) + C_2 \varepsilon \\
&\quad + 2C_2 \left(2 + \frac{3h}{\rho} \right)^d \sum_{i \in I} \mathcal{H}^d \left(E_k \cap \overline{B(x_i, r_i)} \setminus B(x_i, r_i''') \right),
\end{aligned}
$$

thus

$$
\begin{aligned}
J_F(E''') &\le \liminf_{k \to +\infty} (1+\varepsilon) J_F(E_k) + (2\varepsilon + C_2 \varepsilon) \frac{2}{\omega_d} \mathcal{H}^d(E) + C_2 \varepsilon \\
&\quad + 2C_2 \left(2 + \frac{3h}{\rho} \right)^d \liminf_{k \to +\infty} \sum_{i \in I} \mathcal{H}^d \left(E_k \cap \overline{B(x_i, r_i)} \setminus B(x_i, r_i''') \right) \\
&\le \liminf_{k \to +\infty} (1+\varepsilon) J_F(E_k) + (2\varepsilon + C_2 \varepsilon) \frac{2}{\omega_d} \mathcal{H}^d(E) + C_2 \varepsilon \\
&\quad + 2C_2 \left(2 + \frac{3h}{\rho} \right)^d M \sum_{i \in I} \mathcal{H}^d \left(E \cap \overline{B(x_i, r_i)} \setminus B(x_i, r_i''') \right) \\
&\le \liminf_{k \to +\infty} (1+\varepsilon) J_F(E_k) + (2\varepsilon + C_2 \varepsilon) \frac{2}{\omega_d} \mathcal{H}^d(E) + C_2 \varepsilon \\
&\quad + 2C_2 \left(2 + \frac{3h}{\rho} \right)^d M \varepsilon \mathcal{H}^d(E),
\end{aligned}
$$

and

$$\begin{aligned} J_F(E) &= J_F(E''') + J_F(E \setminus E''') \\ &\le J_F(E''') + C_2\mathcal{H}^d(E \setminus E''') \\ &\le (1+\varepsilon) \liminf_{k\to+\infty} J_F(E_k) \\ &\quad + \left(\frac{4+2C_2}{\omega_d} + 2C_2 \left(2 + \frac{3h}{\rho}\right)^d M \right) \mathcal{H}^d(E)\varepsilon + 2C_2\varepsilon. \end{aligned}$$

We can let ε tend to 0, obtaining

$$J_F(E) \le \liminf_{k\to+\infty} J_F(E_k). \qquad \square$$

Proposition 2.6. *Suppose that* $0 < d < n$ *and that* F *is an integrand. Then there is a positive constant* $M > 0$ *such that for any open bounded domain* $U \subset \mathbb{R}^n$*, for any closed* d*-dimensional set* $E \subset U$ *and for any* $\varepsilon > 0$*, we can build a* n*-dimensional complex* $\mathcal{S}$ *and a Lipschitz map* $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the following properties:*

(1) $\phi|_{\mathbb{R}^n\setminus U} = \mathrm{id}_{\mathbb{R}^n\setminus U}$ *and* $|\phi(x) - x| \le \varepsilon$ *for all* $x \in U$*;*
(2) $\phi(E)$ *is contained in the union of* d*-skeleton of* $\mathcal{S}$*, and* $|\mathcal{S}| \subset U$*;*
(3) $J_F(\phi(E)) \le (1+\varepsilon)J_F(E)$.

This is only a small improvement of Theorem 4.3.17 in [12] and Theorem 3 in [13], but the proof is almost same as that of V. Feuvrier in [12,13].

The inequality $J_F(\phi(E)) \le (1+\varepsilon)J_F(E)$ can be replaced by $J_{F,f}(\phi(E)) \le (1+\varepsilon)J_{F,f}(E)$ with any function $f : \mathbb{R}^n \to \mathbb{R}^+$, but since the proof will be more complicated, we do not want to give it here.

Proof. We decompose E as $E = E_{\mathrm{rec}} \sqcup E_{\mathrm{irr}}$, where E_{rec} is d-rectifiable and E_{irr} is purely d-unrectifiable. For any $\varepsilon' > 0$, by Lemma 4 in [13], for $\mathcal{H}^d$ almost every $x \in E_{\mathrm{rec}}$, we can find $r_{\max}(x) > 0$, $\rho \in (0,1)$, $u > 0$ and a d-plane H such that for all $r \in (0, r_{\max}(x))$,

$$\mathcal{H}^d(\Pi_{H,\rho r,\mathcal{C}(x,r,u)}(E \cap B(x, r+r\rho) \setminus \mathcal{C}(x,r,u))) \le \varepsilon'\mathcal{H}^d(E \cap B(x, r+r\rho)); \tag{2.2}$$

indeed, H is the approximate tangent plane T_xE_{rec} of E_{rec} at point x.

Since E_{rec} is a d-rectifiable set and $\mathcal{H}^d(E_{\mathrm{rec}}) < +\infty$, the function $g : E_{\mathrm{rec}} \to G(n,d)$ defined by $g(x) = T_xE_{\mathrm{rec}}$ is $\mathcal{H}^d$-measurable. By Lusin's theorem, see for example [10, 2.3.5. Lusin's theorem], we can find a closed set $E' \subset E_{\mathrm{rec}}$ with $\mathcal{H}^d(E_{\mathrm{rec}} \setminus E') < \varepsilon'\mathcal{H}^d(E)$ such that g restricted to E' is continuous. Thus the function $E' \to E' \times G(n,d)$ defined by $x \mapsto (x, T_xE_{\mathrm{rec}})$ is continuous. Hence for any $x \in E'$, we can find $r'_{\max}(x) > 0$ such that for all $y \in E' \cap B(x, r'_{\max}(x))$,

$$(1-\varepsilon')F(x, T_xE_{\mathrm{rec}}) \le F(y, T_yE_{\mathrm{rec}}) \le (1+\varepsilon')F(x, T_xE_{\mathrm{rec}}). \tag{2.3}$$

We consider the function $\tilde{F} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{F}(x) = \sup_{\pi \in G(n,d)} F(x, \pi).$$

$\tilde{F}$ is continuous, thus for any $x \in \mathbb{R}^n$ there exists $r''_{\max}(x) > 0$ such that

$$(1 - \varepsilon')\tilde{F}(x) \le \tilde{F}(y) \le (1 + \varepsilon')\tilde{F}(x),\ \forall y \in B(x, r''_{\max}(x)).$$

Since F is a continuous function and $G(n, d)$ is compact, for any $x \in \mathbb{R}^n$ we can find $r'''_{\max}(x) > 0$ such that for all $y \in B(x, r'''_{\max}(x))$ and $\pi \in G(n, d)$,

$$(1 - \varepsilon')F(x, \pi) \le F(y, \pi) \le (1 + \varepsilon')F(x, \pi).$$

Let $\mathcal{B}$ be the collection of closed balls with center $x \in E'$ and radius at most

$$\min\left(\frac{r_{\max}(x)}{1+\rho}, r'_{\max}(x), r''_{\max}(x), r'''_{\max}(x), \frac{\varepsilon}{2}\right).$$

By a Vitali covering lemma, we can find a countable many pairwise disjoint balls $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ such that

$$\mathcal{H}^d\left(E' \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

We take an integer N such that

$$\mathcal{H}^d\left(E' \setminus \bigcup_{i=1}^{N} B_i\right) \le \varepsilon' \mathcal{H}^d(E).$$

We suppose that $B_i = \overline{B(x_i, r'_i)}$ with $x_i \in E'$ and $r'_i \le r_{\max}(x_i)$. We put $r_i = \frac{r'_i}{1+\rho_i}$, $K_i = \mathcal{C}(x_i, r_i, u_i)$ and $H_i = T_{x_i}E_{\mathrm{rec}}$, where ρ_i and u_i are the numbers ρ and u in (2.2).

We consider the map

$$\psi_0 = \prod_{i=1}^{N} \Pi_{H_i, r_i\rho_i, K_i},$$

which is γ-Lipschitz with

$$\gamma \le \left(2 + \max_{1 \le i \le N} \frac{u_i}{\rho_i}\right);$$

by the construction, we know that ψ_0 is identity on $(\cup_{i=1}^{N} B_i)^c$. Therefore

$$J_F\left(\psi_0\left(E \setminus \bigcup_{i=1}^{N} B_i\right)\right) = J_F\left(E \setminus \bigcup_{i=1}^{N} B_i\right).$$

Let $b = \sup_{(x,\pi)} F(x,\pi)$ and $a = \inf_{(x,\pi)} F(x,\pi)$. Since ψ_0 is γ-Lipschitz and

$$\mathcal{H}^d(E \cap B_i \setminus K_i) \le \varepsilon' \mathcal{H}^d(E \cap B_i),$$

we get that

$$J_F(\psi_0(E \cap B_i \setminus K_i)) \le \gamma^d b \varepsilon' \mathcal{H}^d(E \cap B_i) \le \frac{\gamma^d b \varepsilon'}{a} J_F(E \cap B_i).$$

and

$$\begin{aligned} J_F(\psi_0((E_{\mathrm{rec}} \setminus E') \cap K_i)) &\le \gamma^d b \mathcal{H}^d((E_{\mathrm{rec}} \setminus E') \cap K_i) \\ &\le \gamma^d b \mathcal{H}^d((E_{\mathrm{rec}} \setminus E') \cap B_i). \end{aligned}$$

By (2.3), we get that

$$(1-\varepsilon')F(x_i, H_i)\mathcal{H}^d(E' \cap K_i) \le J_F(E' \cap K_i) \le (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(E' \cap K_i),$$

and

$$\begin{aligned} J_F(\psi_0(E' \cap K_i)) &\le (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(\psi_0(E' \cap K_i)) \\ &\le (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(E' \cap K_i) \\ &\le \frac{1+\varepsilon'}{1-\varepsilon'} J_F(E' \cap K_i) \\ &\le \frac{1+\varepsilon'}{1-\varepsilon'} J_F(E' \cap B_i). \end{aligned}$$

Since ψ_0 is the orthogonal projection to H_i in a neighborhood of K_i,

$$\psi_0(E_{\mathrm{irr}} \cap K_i) \subset H_i \cap B_i,$$

thus $\psi_0(E_{\mathrm{irr}} \cap K_i)$ is rectifiable, and

$$\begin{aligned} J_F(\psi_0(E_{\mathrm{irr}} \cap K_i)) &\le (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(\psi_0(E_{\mathrm{irr}} \cap K_i)) \\ &\le (1+\varepsilon')F(x_i, H_i)\mathcal{H}^d(E_{\mathrm{irr}} \cap K_i) \\ &\le (1+\varepsilon')^2 J_F(E_{\mathrm{irr}} \cap K_i). \end{aligned}$$

Put $S = \cup_{i=1}^N B_i$. Note that

$$E \cap K_i = ((E_{\mathrm{rec}} \setminus E') \cap K_i) \cup (E' \cap K_i) \cup (E_{\mathrm{irr}} \cap K_i),$$

we have that

$$\begin{aligned} J_F(\psi_0(E)) &\le J_F(\psi_0(E \setminus S)) + J_F(\psi_0(E \cap S)) \\ &\le J_F(E \setminus S) + J_F(\psi_0(E \cap S)) \\ &\le J_F(E \setminus S) + \gamma^d b \mathcal{H}^d(E_{\mathrm{rec}} \setminus E' \cap S) \\ &\quad + \frac{1+\varepsilon'}{1-\varepsilon'} J_F(E' \cap S) + (1+\varepsilon')^2 J_F(E_{\mathrm{irr}} \cap S) \\ &\le \left(1 + \frac{2\varepsilon'}{1-\varepsilon'} + 2\varepsilon' + \varepsilon'^2 + \frac{\gamma^d b \varepsilon'}{a}\right) J_F(E). \end{aligned} \tag{2.4}$$

We put $E_1 = \psi_0(E)$, $D_i = H_i \cap B_i$. Since ψ_0 is the identity on S^c, we know that $\psi_0(E_{\mathrm{irr}} \setminus S) = E_{\mathrm{irr}} \setminus S$ is purely d-unrectifiable. Since $\psi_0(E \cap K_i) \subset D_i$, we know that $\psi_0(E \cap K_i)$ is rectifiable. Thus

$$(E_1)_{\mathrm{rec}} \setminus \bigcup_{i=1}^{N} D_i \subset \psi_0(E_{\mathrm{rec}} \setminus S) \cup \bigcup_{i=1}^{N} \psi_0(E \cap B_i \setminus K_i),$$

so

$$\begin{aligned}\mathcal{H}^d\left((E_1)_{\mathrm{rec}} \setminus \bigcup_{i=1}^{N} D_i\right) &\le \mathcal{H}^d\left(\psi_0(E_{\mathrm{rec}} \setminus S)\right) + \sum_{i=1}^{N} \mathcal{H}^d\left(\psi_0(E \cap B_i \setminus K_i)\right)\\ &\le 2\varepsilon' \mathcal{H}^d(E).\end{aligned}$$

Let $\alpha > 0$ be small number such that

$$\alpha < \min_{1 \le i \le N} \left(\frac{1}{2} r_i \rho_i\right).$$

For $1 \le i \le N$, we take a dyadic complex $\mathcal{S}_i$ of stride α in an orthonormal basis centered at x_i with d vectors parallel to H_i such that

$$K_i \subset |\mathcal{S}_i| \subset B\left(x_i, \left(1 + \frac{1}{2}\rho_i\right) r_i\right),$$

then for any $1 \le i < j \le N$,

$$\operatorname{dist}\left(|\mathcal{S}_i|, \left|\mathcal{S}_j\right|\right) \ge \min_{1 \le k \le N} \left(\frac{1}{2} r_k \rho_k\right),$$

we can apply [13, Theorem 1]: there is a complex $\mathcal{S}$ such that $|\mathcal{S}| \subset U$, and $\mathcal{S}_i$, for $1 \le i \le N$, are subcomplexes of $\mathcal{S}$.

Now, we do a Federer-Fleming projection which maps E_1 into the d-skeleton as in [13, page 33]. We get a Lipschitz map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi|_{U^c} = \mathrm{id}_{U^c}$ such that $\psi(E_1)$ is contained in the union of d-skeleton of $\mathcal{S}$, $\psi((E_1)_{\mathrm{irr}})$ is still purely d-unrectifiable, and

$$\psi\left(\bigcup_{i=1}^{N} D_i\right) \subset \bigcup_{i=1}^{N} D_i;$$

this is because that $\cup_{i=1}^{N} D_i$ is contained in the union of the d-skeleton of $\mathcal{S}$. Thus

$$\mathcal{H}^d(\psi((E_1)_{\mathrm{irr}})) = 0$$

and there exists a constant C, depending on n and d only, such that

$$\mathcal{H}^d\left(\psi\left((E_1)_{\mathrm{rec}} \setminus \bigcup_{i=1}^{N} D_i\right)\right) \le C\mathcal{H}^d\left((E_1)_{\mathrm{rec}} \setminus \bigcup_{i=1}^{N} D_i\right) \le 2C\varepsilon' \mathcal{H}^d(E).$$

We take $\phi = \psi \circ \psi_0$, then

$$J_F(\psi(E_1)) \leq \left(1 + \frac{bC\varepsilon'}{a}\right) J_F(E_1),$$

thus

$$J_F(\phi(E)) \leq \left(1 + \frac{bC\varepsilon'}{a}\right)\left(1 + \frac{2\varepsilon'}{1-\varepsilon'} + 2\varepsilon' + \varepsilon'^2 + \frac{\gamma^d b\varepsilon'}{a}\right) J_F(E).$$

If we take ε' small enough, we can get that

$$J_F(\phi(E)) \leq (1+\varepsilon) J_F(E).$$

As in [13], we can show that $|\phi(x) - x| \leq \varepsilon$ for any x. $\square$

Using this theorem, we can prove the following lemma:

Lemma 2.7. *Suppose that* $0 < d < n$ *and that* $U \subset \mathbb{R}^n$ *is open. Suppose that* F *is an integrand. Then there is a positive constant* $M' > 0$ *depending only on* d *and* n *such that for any relatively closed* d*-dimensional set* $E \subset U$*, for any relatively compact subset* $V \subset U$*,* i.e. *the closure of* V *in* $\mathbb{R}^n$ *is contained in* U*, and for all* $\epsilon > 0$*, we can find a* n*-dimensional complex* $\mathcal{S}$ *and a subset* $E'' \subset U$ *satisfying the following properties:*

(1) E'' *is a* $\operatorname{diam}(U)$*-deformation of* E *over* U *and by putting* $W = |\mathring{\mathcal{S}}|$ *we have* $V \subset W \subset \overline{W} \subset U$ *and there is a* d*-dimensional skeleton* $\mathcal{S}'$ *of* $\mathcal{S}$ *such that* $E'' \cap \overline{W} = |\mathcal{S}'|$*;*
(2) $J_F(E'') \leq (1+\epsilon) J_F(E)$*;*
(3) *there are* $d+1$ *complexes* $\mathcal{S}^0, \dots, \mathcal{S}^d$ *such that* $\mathcal{S}^\ell$ *is contained in the* ℓ*-skeleton of* $\mathcal{S}$ *and there is a decomposition*

$$E'' \cap W = E^d \sqcup E^{d-1} \sqcup \ldots \sqcup E^0,$$

where for each $0 \leq \ell \leq d$*,*

$$E^\ell \in \mathbf{QM}(W^\ell, M', \operatorname{diam}(W^\ell), \mathcal{H}^\ell),$$

where

$$\begin{cases} W^d = W \\ W^{\ell-1} = W^\ell \setminus E^\ell \end{cases} \qquad \begin{cases} E^d = |\mathcal{S}^d| \cap W^d \\ E^\ell = |\mathcal{S}^\ell| \cap W^\ell. \end{cases}$$

The proof of this lemma is the same as the proof of [12, Lemma 5.2.6] or [13, Lemma 9], therefore we omit it.

We now turn to prove the results stated at the beginning of the paper.

Proof of Theorem 1.1. We claim that we can find a ball $B(0, R)$ and a sequence of compact sets $\{E_k\}_{k\geq 1} \subset \mathscr{C}$ such that $B \subset B(0, R)$, $E_k \subset B(0, R)$ and

$$J_F(E_k \setminus B) \to m(\mathscr{C}, F).$$

We take any sequence of compact sets $\{E'_k\}_{k\geq 1}$ in $\mathscr{C}$ such that

$$J_F(E'_k \setminus B) \to m(\mathscr{C}, F).$$

We take

$$U'_k = \{x \in B(0, R_k) \mid \operatorname{dist}(x, B) > 2^{-k}\},$$

where

$$R_k > \max\{k, R_{k-1} + 1, \operatorname{dist}(0, E'_k) + \operatorname{diam}(E'_k) + 1\}.$$

By Lemma 2.7, we can find a Lipschitz map $\phi'_k : \mathbb{R}^n \to \mathbb{R}^n$ and a complex $\mathcal{S}_k$ such that

$$\phi'_k|_{U'^c_k} = \mathrm{id}_{U'^c_k},\ U'_{k-1} \subset |\mathcal{S}_k| \subset U'_k,\ E'_k \subset |\mathcal{S}_k|\,,$$

and

$$\phi'_k(E'_k) \cap W'_k = F_k \sqcup F'_k,$$

where $W'_k = |\mathring{\mathcal{S}}_k|$ the interior of $|\mathcal{S}_k|$,

$$F_k \in \mathbf{QM}(W'_k, M, \operatorname{diam}(W'_k), \mathcal{H}^d),$$

and F'_k is contained in the union of the $(d-1)$-dimensional skeleton of $\mathcal{S}_k$.

We now prove that $\{F_k\}_{k\geq 1}$ is bounded, *i.e.* we can find a large ball $B(0, r)$ such that $B \cup (\cup_k F_k) \subset B(0, r)$. Suppose there is none, that is, suppose that for any large number $r > R_1$ there exists $k > 4r$ such that $F_k \setminus B(0, 2r) \neq \emptyset$. If $x \in F_k \setminus B(0, 2r)$, we take a cube Q centered at x with $\operatorname{diam}(Q) = r$, then by using Proposition 4.1 in [6], we have that

$$\mathcal{H}^d(F_k \cap Q) \geq C^{-1}\operatorname{diam}(Q)^d,$$

where C only depends on n and M. If we take r large enough, for example

$$r^d > \frac{2C}{\inf F}(m(\mathscr{C}, F) + 1),$$

and we take k large enough such that $J_F(E'_k) < m(\mathscr{C}, F) + 1$, then

$$\begin{aligned} C^{-1}r^d &\leq \mathcal{H}^d(F_k \cap Q) \\ &\leq \frac{1}{\inf F} J_F(\phi'_k(E'_k)) \\ &\leq \frac{(1 + 2^{-k})}{\inf F} J_F(E'_k) \\ &< \frac{2}{\inf F}(m(\mathscr{C}, F) + 1), \end{aligned}$$

which is a contradiction. Thus $\cup_k F_k$ is bounded. It is easy to see that $\cup_k(\phi'_k(E'_k) \cap W'^c_k)$ is bounded, so we can assume that both $B \cup (\cup_k F_k)$ and $\cup_k(\phi'_k(E'_k) \cap W'^c_k)$ are contained in a large ball $B(0, R)$. We take the map $\rho : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\rho(x) = \begin{cases} x & x \in B(0, R) \\ \frac{R}{|x|} x & x \in B(0, R)^c, \end{cases}$$

which is 1-Lipschitz map. We put $E_k = \rho \circ \phi'_k(E'_k)$, then $E_k \in \mathscr{C}$, and

$$E_k = (\phi'_k(E'_k) \cap W'^c_k) \cup F_k \cup \rho(F'_k).$$

Since $\mathcal{H}^d(F'_k) = 0$, we have that

$$J_F(E_k \setminus B) = J_F(\phi'_k(E'_k) \setminus B) \le (1 + 2^{-k}) J_F(E'_k \setminus B),$$

therefore

$$J_F(E_k \setminus B) \to m(\mathscr{C}, F),$$

and $(E_k)_{k \ge 1}$ is a sequence as we desire. We proved our claim.

If $J_F(E_k \setminus B) = 0$ for some $k \ge 1$, then $m(\mathscr{C}, F) = 0$ and E_k is a minimizer: we have nothing to prove. We now suppose that $0 < J_F(E_k \setminus B) < +\infty$, for all $k \ge 1$. Thus $0 < \mathcal{H}^d(E_k \setminus B) < +\infty$.

We put

$$U = B(0, R+1) \setminus B, \ V_k = \left\{ x \in B(0, R + 1 - 2^{-k}) \mid \operatorname{dist}(x, B) > 2^{-k} \right\}.$$

By Lemma 2.7, we can find polyhedral complexes $\mathcal{S}_k$, Lipschitz maps $\phi_k : \mathbb{R}^n \to \mathbb{R}^n$ and a constant $M' = M'(n, d)$ such that:

(1) $V_k \subset |\mathcal{S}_k| \subset V_{k+1}, \phi_k|_{V^c_{k+1}} = \mathrm{id}_{V^c_{k+1}}$, and there exists a d-dimensional skeleton $\mathcal{S}'_k$ of $\mathcal{S}_k$ such that $E''_k \cap W_k = \left|\mathcal{S}'_k\right|$, where $E''_k = \phi_k(E_k)$ and $W_k = |\mathcal{S}_k|$;
(2) $J_F(E''_k \setminus B) \le (1 + 2^{-k}) J_F(E_k \setminus B)$;
(3) there exist complexes $\mathcal{S}^0_k, \dots, \mathcal{S}^d_k$ such that $\mathcal{S}^\ell_k$ is contained in the ℓ-skeleton of $\mathcal{S}_k$ and there is a disjoint decomposition

$$E''_k \cap \mathring{W}_k = E^d_k \sqcup E^{d-1}_k \sqcup \cdots \sqcup E^0_k,$$

where for each $0 \le \ell \le d$,

$$E^\ell_k \in \mathbf{QM}(W^\ell_k, M', \operatorname{diam}(W^\ell_k), \mathcal{H}^\ell),$$

where

$$\begin{cases} W^d_k = \mathring{W}_k \\ W^{\ell-1}_k = W^\ell_k \setminus E^\ell_k \end{cases} \qquad \begin{cases} E^d = \left|\mathcal{S}^d_k\right| \cap W^d_k \\ E^\ell = \left|\mathcal{S}^\ell_k\right| \cap W^\ell_k, \end{cases}$$

and $\mathring{W}_k$ is the interior of W_k.

We note that E_k'' and W_k are two compact subsets of $\mathbb{R}^n$ for each k, thus $E_k''\cap W_k$ is a compact subset of $\mathbb{R}^n$. We may suppose that $E_k'' \cap W_k \to E'$ in Hausdorff distance, passing to a subsequence if necessary. We put $E = E' \cup B$. We will show that E is a minimizer.

First of all, we show that E is in the class $\mathscr{C}$, *i.e.* $E \in \mathscr{C}$. Since ϕ_k is Lipschitz map and $\phi_k|_{V_{k+1}^c} = \mathrm{id}_{V_{k+1}^c}$, in particular, $\phi_k|_B = \mathrm{id}_B$, thus $E_k'' = \phi_k(E_k) \in \mathscr{C}$. Since $E_k \subset B(0, R)$ and $B \subset B(0, R)$, and by our construction of ϕ_k, we get that $\phi_k(E_k) \subset B(0, R + \frac{1}{2})$. Since $V_k \subset W_k \subset V_{k+1}$, we have that

$$B \subset \phi_k(E_k) \setminus W_k \subset B\left(0, R + \frac{1}{2}\right) \setminus V_k \subset B(2^{-k}),$$

where we denote by $B(\epsilon)$ the ϵ-neighborhood of B. Thus $E_k'' \setminus W_k \to B$ in Hausdorff distance, so

$$E_k'' = (E_k'' \cap W_k) \cup (E_k'' \setminus W_k) \to E' \cup B = E,$$

we have that $E \in \mathscr{C}$.

Next, we will show that $J_F^d(E \setminus B) = m(\mathscr{C}, F)$.

Passing to a subsequence if necessary, we may assume that

$$E_k^\ell \to E^\ell \text{ in } U, \text{ for } 0 \le \ell \le d,$$

For any $0 \le \ell \le d$, we put

$$U^\ell = U \setminus \bigcup_{\ell<\ell'\le d} E^{\ell'},$$

then

$$E \setminus B = \bigcup_{0\le\ell\le d} E^\ell.$$

Since

$$E_k^d \in \mathbf{QM}(W_k^d, M', \mathrm{diam}(W_k^d), \mathcal{H}^d),$$

we can apply the Theorem 2.5, and get that

$$J_F(E^d \cap W_k^d) \le \liminf_{m\to\infty} J_F(E_m^d \cap W_k^d) \le \liminf_{m\to\infty} J_F(E_m^d).$$

Since $V_k \subset W_k \subset V_{k+1}$ and $W_k^d = \mathring{W}_k$, we have

$$\bigcup_k W_k^d = \bigcup_k V_k = U,$$

thus

$$J_F(E^d) \le \liminf_{m\to\infty} J_F(E_m^d).$$

For any $0 \le \ell \le d$, for any $\varepsilon > 0$, we put $U^d_\varepsilon = B(0, R+1-\varepsilon) \cap U^d$ and

$$U^\ell_\varepsilon = \left\{ x \in B(0, R+1-\varepsilon) \mid \operatorname{dist}\left(x, \bigcup_{\ell < \ell' \le d} E^{\ell'} \right) > \varepsilon \right\}.$$

Then $U^\ell_{\varepsilon_1} \subset U^\ell_{\varepsilon_2}$ for any $0 < \varepsilon_2 < \varepsilon_1$, and

$$\bigcup_{\varepsilon > 0} U^\ell_\varepsilon = U^\ell.$$

Since $E^\ell_k \to E^\ell$ in U, we have that $E^\ell_k \cap U^\ell_\varepsilon \to E^\ell \cap U^\ell_\varepsilon$ in U^ℓ_ε. We will show that for any $\varepsilon > 0$, there exists k_ε such that for $k \ge k_\varepsilon$,

$$E^\ell_k \cap U^\ell_\varepsilon \in \mathbf{QM}(U^\ell_\varepsilon, M', \operatorname{diam}(U^\ell_\varepsilon), \mathcal{H}^\ell).$$

Indeed, for any $\varepsilon > 0$, we can find k_ε such that $U^\ell_\varepsilon \subset W^\ell_k$. We prove this by induction on ℓ.

First, we take a positive integer k_ε such that $2^{-k_\varepsilon} < \varepsilon$, then $U^d_\varepsilon \subset W^d_k$ for any $k \ge k_\varepsilon$.

Next, we suppose that there is an integer k_ε such that $U^\ell_\varepsilon \subset W^\ell_k$ for $k \ge k_\varepsilon$. Since $E^\ell_k \to E^\ell$ in U^ℓ and

$$W^{\ell-1}_k = W^\ell_k \setminus E^\ell_k,\ U^{\ell-1}_\varepsilon = \left\{ x \in U^\ell_\varepsilon \mid \operatorname{dist}\left(x, E^\ell \right) > \varepsilon \right\},$$

we can find k'_ε such that $U^{\ell-1}_\varepsilon \subset W^{\ell-1}_k$ for $k \ge k'_\varepsilon$. Thus we finished the induction proofs.

Since $U^\ell_\varepsilon \subset W^\ell_k$ and

$$E^\ell_k \in \mathbf{QM}(W^\ell_k, M', \operatorname{diam}(W^\ell_k), \mathcal{H}^\ell),$$

we get that

$$E^\ell_k \cap U^\ell_\varepsilon \in \mathbf{QM}(U^\ell_\varepsilon, M', \operatorname{diam}(U^\ell_\varepsilon), \mathcal{H}^\ell).$$

For any $\delta > 0$ fixed, we put $\Omega_\delta = \{x \in U_\varepsilon \mid \operatorname{dist}(x, U^c_\varepsilon) \ge 10\delta\}$. $E^\ell_k \cap \Omega_\delta$ is a bounded subset, indeed it is contained in $B(0, R+1)$. Since $E^\ell_k \cap \Omega_\delta$ is bounded, and $\{B(x, \delta) \mid x \in E^\ell_k \cap \Omega_\delta\}$ is a covering of $E^\ell_k \cap \Omega_\delta$, and each ball $B(x, \delta)$ has a fixed radius δ, we can find a collection of finitely many balls $\{B(x_i, \delta)\}_{i \in I}$ which is a covering of $E^\ell_k \cap \Omega_\delta$; by the 5-covering lemma, see for example [15, Theorem 2.1], we can find a subset $J \subset I$ such that $B(x_{j_1}, \delta) \cap B(x_{j_2}, \delta) = \emptyset$ for $j_1, j_2 \in J$ with $j_1 \ne j_2$, and

$$\bigcup_{i \in I} B(x_i, \delta) \subset \bigcup_{j \in J} B(x_j, 5\delta).$$

Since $B(x_{j_1},\delta)\cap B(x_{j_2},\delta)=\emptyset$ for $j_i,\ j_2\in J$, we have that

$$\mathcal{L}^n(U_\varepsilon)\geq\sum_{j\in J}\mathcal{L}^n(B(x_j,\delta)),$$

thus

$$\#J\leq\frac{\mathcal{L}^n(U_\varepsilon)}{\omega_n\delta^n}.$$

By [6, Proposition 4.1], we have that

$$C^{-1}(5\delta)^\ell\leq\mathcal{H}^\ell(E_k^\ell\cap B(x_j,5\delta))\leq C(5\delta)^\ell,$$

so

$$\mathcal{H}^\ell(E_k^\ell\cap\Omega_\delta)\leq\sum_{j\in J}\mathcal{H}^\ell(E_k^\ell\cap B(x_j,5\delta))\leq\sum_{j\in J}C(5\delta)^\ell\leq\omega_n^{-1}\mathcal{L}^n(U_\varepsilon)5^d\delta^{\ell-n}C.$$

Applying [4, Theorem 3.4], we get that

$$\mathcal{H}^\ell(E^\ell\cap\Omega_\delta)\leq\liminf_{k\to\infty}\mathcal{H}^\ell(E_k^\ell\cap\Omega_\delta)\leq\omega_n^{-1}\mathcal{L}^n(U_\varepsilon)5^d\delta^{\ell-n}C,$$

and $\dim_{\mathcal{H}}E^\ell\cap\Omega_\delta\leq\ell$, hence $\dim_{\mathcal{H}}E^\ell\leq\ell$, thus $\mathcal{H}^d(E^\ell)=0$.

We get that

$$\begin{aligned}J_F(E\setminus B)&=J_F(E^d)\\&\leq\liminf_{k\to\infty}J_F(E_k^d)\\&\leq\liminf_{k\to\infty}J_F(E_k''\setminus B)\\&\leq\liminf_{k\to\infty}(1+2^{-k})J_F(E_k\setminus B)\\&=\liminf_{k\to\infty}J_F(E_k\setminus B)\\&=m(\mathscr{C},F).\end{aligned}$$

Since $E\in\mathcal{F}$, we have that

$$J_F(E\setminus B)\geq m(\mathscr{C},F),$$

therefore

$$J_F(E\setminus B)=m(\mathscr{C},F).$$ □

The following proposition is taken from [18, Proposition 3.1].

Proposition 2.8. *Let $B\subset\mathbb{R}^n$ be a compact subset. For $j=1,2,\dots$, suppose that $S_j\subset\mathbb{R}^n$ is a compact set with $B\subset S_j$, and that S_j converges in Hausdorff distance to a compact set $S\subset\mathbb{R}^n$. Let $L\subset\check{H}_{k-1}(B;G)$ be a subgroup such that $L\subset\ker\check{H}_{k-1}(i_{B,S_j})$. Then $L\subset\ker\check{H}_{k-1}(i_{B,S})$.*

The proof of the proposition is essentially the same as the proof of [18, Proposition 3.1], so we omit it.

Proof of Theorem 1.3. We only need to prove that $\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ satisfies the conditions in Theorem 1.1.

For any Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi|_B = \mathrm{id}_B$, and any compact set $E \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$, we have $i_{B,\varphi(E)} = \varphi \circ i_{B,E}$, thus

$$\check{H}_{d-1}(i_{B,\varphi(E)})(L) = \check{H}_{d-1}(\varphi) \circ \check{H}_{d-1}(i_{B,E})(L) = 0,$$

this implies that

$$L \in \ker\left(\check{H}_{d-1}(i_{B,\varphi(E)})\right),$$

so

$$\varphi(E) \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L).$$

For any sequence

$$\{E_k\}_{k=1}^{\infty} \subset \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L),$$

if E_k converges to E in Hausdorff distance for some compact set $E \subset \mathbb{R}^n$, then by Proposition 2.8, we get that

$$E \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L).$$

The class $\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ satisfies the two conditions in Theorem 1.1, thus we can find a compact set $E \in \mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L)$ such that

$$J_F(E \setminus B) = m(\mathscr{C}_{\check{\mathrm{C}}\mathrm{ech}}(B, G, L), F). \qquad \square$$

Proof of Theorem 1.4. We will show that $\mathscr{C}_{\mathrm{free}}(B, G, L)$ satisfies the two conditions in Theorem 1.1. It is fairly easy to verify the first condition, so we omit the details of this point.

Let $\{E_n\}_{n=1}^{\infty} \subset \mathscr{C}_{\mathrm{free}}(B, G, L)$ be a sequence of compact sets such that $E_n \to E$ in Hausdorff distance for some compact sets $E \subset \mathbb{R}^n$. We put

$$X_n = \left(\bigcup_{k \geq n} E_k\right) \bigcup E.$$

By [17, Lemma 3], we know that $X_n \in \mathscr{C}_{\mathrm{free}}(B, G, L)$, then we apply [17, Lemma 4], and we get that $E = \bigcap_{n=1}^{\infty} X_n \in \mathscr{C}_{\mathrm{free}}(B, G, L)$. $\square$

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