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Bilinkage in codimension 3 and canonical surfaces of degree 18 in $\mathbb{P}^5$

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Abstract. We study the behavior of the bilinkage process in codimension 3. In particular, we construct a smooth canonically embedded and linearly normal surface of general type of degree 18 in $\mathbb{P}^5$; this is probably the highest degree such a surface may have. Next, we apply our construction to find a geometric description of Tonoli Calabi-Yau threefolds in $\mathbb{P}^6$.

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1. Introduction

Let S be a minimal surface of general type defined over the field of complex numbers. Then, by the inequality of Noether and Bogomolov-Miyaoka-Yau, we have

$$2\chi(\mathcal{O}_S) - 6 \leq K_S^2 \leq 9\chi(\mathcal{O}_S).$$

On the other hand, if we assume that the canonical system of S gives a birational map, then by the Castelnuovo inequality we deduce $3\chi(\mathcal{O}_S) - 10 \leq K_S^2$. Note that we know from [2] that $5K_S$ always gives a birational morphism for surfaces of general type. In this context, it is a natural problem (*cf.* [1,4]) to construct surfaces of general type with birational canonical map in the range $3\chi(\mathcal{O}_S) - 10 \leq K_S^2 \leq 9\chi(\mathcal{O}_S)$. Many works are related to this problem [1,9,20]; however, the part with $\chi(\mathcal{O}_S) \leq 7$ seems out of reach with those methods. The general surface of general type with $\chi(\mathcal{O}_S) = 7$ and $h^1(\mathcal{O}_S) = 0$ should admit a birational canonical map to $\mathbb{P}^5$. The image of such a map is a subcanonical surface of codimension 3 in $\mathbb{P}^5$.

On the other hand, it was proven in [23] that submanifolds $X \subset \mathbb{P}^N$ of codimension 3 in projective spaces with $N-3$ not divisible by 4 that are subcanonical are Pfaffian, *i.e.*, their ideal sheaf admits a Pfaffian resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-2s-t) \to E^*(-s-t) \xrightarrow{\varphi} E(-s) \xrightarrow{\psi} \mathcal{I}_X \to 0 \qquad (1.1)$$

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where E is a vector bundle of odd rank and $s, t \in \mathbb{Z}$. The study of codimension 3 manifolds is reduced in this way to the study of the Hartshorne-Rao modules of submanifolds. However, complicated algebraic problems appear when we want to classify such modules (see [14]). Catanese [4] applied the Pfaffian construction in order to construct canonically embedded surfaces in $\mathbb{P}^5$ and found constructions of surfaces with $K_S^2 \leq 17$. Later, in his thesis [22], Tonoli constructed Calabi-Yau threefolds in $\mathbb{P}^6$, but only found examples of degree ≤ 17. Since the Pfaffian construction becomes more and more complicated when the degree increases, it is natural to ask whether there are any canonical surfaces of degree ≥ 18 in $\mathbb{P}^5$. Our main result is the following:

Theorem 1.1. *There exists a surface of general type with $K^2 = 18$, $p_g = 6$, $q = 0$ whose canonical map is an isomorphism onto its image.*

We describe the construction of such surfaces in Section 3 concluding with Theorem 3.5. We expect, by [14], that this is the highest degree of such a canonically embedded surface in $\mathbb{P}^5$. In fact, in Theorem 3.5, we find an explicit description of a 20-dimensional subfamily of the at least 36-dimensional family of degree 18 canonical surfaces in $\mathbb{P}^5$. Having our existence result, it is a natural problem (see [14]) to find a Pfaffian resolution for a general canonical surface of degree 18 in $\mathbb{P}^5$.

The idea of the proof of Theorem 1.1 is to construct a special bilinkage. Recall that the relation of *linking* (or equivalently *liaison*) was introduced in [18]. Two closed subschemes V, W of $\mathbb{P}^N$ are algebraically linked by a subscheme X if they are equidimensional without embedded components and X is a complete intersection containing them such that $\mathcal{I}_{W|\mathbb{P}^N}/\mathcal{I}_{X|\mathbb{P}^N} = \mathrm{Hom}(\mathcal{O}_V, \mathcal{O}_X)$ and $\mathcal{I}_{V|\mathbb{P}^N}/\mathcal{I}_{X|\mathbb{P}^N} = \mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_X)$. When additionally W and V do not have common components (this is the situation we are interested in) then they are linked if $V \cup W = X$. We say that two irreducible varieties of the same dimension are *bilinked* if they are linked in two steps, *i.e.*, there exists a scheme T such that T is linked with both W and V. We shall also need another point of view on bilinkage: the notion of *generalized divisors* introduced by Hartshorne. With the notation as above with V and W bilinked we see that V and W are of codimension one in X and thus can be seen as "divisors" in X. Hartshorne [6, Section 2] generalizes the usual notion of Cartier divisor and linear equivalence of divisors "$\simeq$" (in our case X is very singular so V, W are not Cartier divisors) in order to obtain a relation $V \simeq W + nH$ where $n \in \mathbb{Z}$ and H is a hyperplane section of $X \subset \mathbb{P}^N$.

Let us now describe our construction of the surface in Theorem 1.1. We first take a special central projection to $\mathbb{P}^5$ of the image V_9 of the third Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$ and perform a bilinkage. More precisely, we find a special $\mathbb{P}^3 \subset \mathbb{P}^9$ such that the image $D_9 \subset \mathbb{P}^5$ of $V_9 \subset \mathbb{P}^9$ by the projection centered in this $\mathbb{P}^3$ is smooth and contained in the complete intersection of two cubics. Note here that this image is contained in a single cubic for a generic projection (*cf.* [11, Remark 5.5]). Then we perform a bilinkage of D_9 through the intersection of these two cubics, obtaining a special smooth canonically embedded general type surface of degree 18 (*cf.* [14]). In Proposition 3.8, we show that all the known examples of canonical surfaces in $\mathbb{P}^5$ from [4] can be obtained via that bilinkage construction. We believe that our construction can be applied in a more general classification problems concerning submanifolds of codimension 3.

Since the method of bilinkage works better than the Pfaffian construction in the case of surfaces, we apply it in Section 4 to study Tonoli Calabi-Yau threefolds in $\mathbb{P}^6$. Those threefolds were constructed in [22] (*cf.* [14,21]) by using the Pfaffian resolution (1.1). Our first result, Proposition 4.2, says that starting from Del Pezzo threefolds of degree $d \leq 7$ in $\mathbb{P}^6$ we obtain families of Tonoli Calabi-Yau threefolds of degree $d + 9$ by performing the bilinkage construction through the intersection of two cubics (*cf.* [8]). In the remaining degree 17, there are three families of Calabi-Yau threefolds that we call after Tonoli of type $k = 8$, $k = 9$, and $k = 11$. The corresponding degree 8 Del Pezzo threefold is the double Veronese embedding of $\mathbb{P}^3$ projected to $\mathbb{P}^6$. As before, we can find a special center of projection such that the image of the Del Pezzo threefold of degree 8 is smooth and contained in a three-dimensional space of cubics. Note that it is contained in no cubic for a general projection. So we can perform a bilinkage and its result is a natural degeneration of the degree 17 Tonoli family of type $k = 9$. Note that the examples of type $k = 8$ and $k = 11$ cannot be constructed by bilinkage. This shows that the construction that we propose in [14, Theorem 1.3], by operations on vector bundles from the Pfaffian resolution, is, in this context, a strict generalization of the one using bilinkages. We close Section 4 with the construction of a singular degree 18 threefold in $\mathbb{P}^6$ birational to a Calabi-Yau threefold.

In Subsection 2.1 we study the relation between the Pfaffian resolutions 1.1 of two bilinked subvarieties of codimension 3. Finally, in Section 5, we discuss relations between constructions by bilinkage and by unprojection, finding that the former are more general in our situation. This confirms the general Reid philosophy about the relation between these constructions. As a result, we analyze an example of non-Gorenstein unprojection that should be of independent interest.

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2. Preliminaries

We shall apply the following construction to relate a given Del Pezzo surface $F \subset \mathbb{P}^5$ (respectively Del Pezzo threefold) to a surface $X \subset \mathbb{P}^5$ of general type (respectively Calabi-Yau threefold).

Construction 2.1. We write

$$\mathbb{P}^n \supset F \rightrightarrows X' \rightsquigarrow X \subset \mathbb{P}^n,$$

where $F \rightrightarrows X'$ means that F and X' are bilinked and $X' \rightsquigarrow X$ means that X' is a degeneration of X, *i.e.* there is a proper flat family over a disc such that X' is its special element and X a general one. Then we say that X is constructed from F by the bilinkage construction.

Before we study the possible applications of this construction, let us consider bilinkages of Pfaffian varieties in general.

2.1. Bilinkages of Pfaffians

Let us make some useful remarks on the construction of bilinkages between Pfaffian varieties by the relating vector bundles defining them. More precisely, we aim at proving that, under some assumptions, if two bundles E and F of odd rank differ by a sum of line bundles then the Pfaffian varieties associated to general sections of their twisted wedge squares are in the same complete intersection biliaison class.

Let $X \subset \mathbb{P}^N$ be a Pfaffian variety defined by a section $\varphi \in H^0\big(\bigwedge^2 E \otimes \mathcal{O}_{\mathbb{P}^N}(t)\big)$ for some vector bundle E of rank $2r+1$ for some $r \in \mathbb{N}$ and $t \in \mathbb{Z}$. Denote $s = c_1(E) + rt$.

The map φ in the Pfaffian resolution (1.1) is identified with the section

$$\varphi \in H^0\left(\mathbb{P}^N, \textstyle\bigwedge^2 E \otimes \mathcal{O}_{\mathbb{P}^N}(t)\right),$$

and ψ is the map

$$E(-s) \to \mathcal{I}_X = \operatorname{Im}(\psi) \subset \textstyle\bigwedge^{2r+1} E \otimes \mathcal{O}_{\mathbb{P}^N}(rt - s) = \mathcal{O}_{\mathbb{P}^N}$$

defined as the wedge product with the r-th divided power of φ:

$$\frac{1}{r!}(\varphi \wedge \varphi \wedge \cdots \wedge \varphi) \in H^0(\mathbb{P}^N, \textstyle\bigwedge^{2r} E \otimes \mathcal{O}_{\mathbb{P}^N}(rt)).$$

Assumption 2.2. $H^1(E^*(l)) = 0$ for $l \in \mathbb{Z}$.

Observe that Assumption 2.2 is satisfied when E is obtained as the kernel of a surjective map between decomposable bundles.

Under Assumption 2.2 on E we claim that every hypersurface of degree d containing X is defined as a Pfaffian hypersurface given by a section of the bundle $\bigwedge^2(E \oplus \mathcal{O}_{\mathbb{P}^N}(d-s-t)) \otimes \mathcal{O}_{\mathbb{P}^N}(t)$ of even rank $2r+2$. Indeed, we can split the Pfaffian sequence into two short exact sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-2s-t) \to E^*(-s-t) \to F \to 0,$$

$$0 \to F \to E(-s) \xrightarrow{\psi} \mathcal{I}_X \to 0,$$

for some sheaf F. Taking the cohomology of the second, we obtain an exact sequence

$$H^0(E(d-s)) \to H^0(\mathcal{I}_X(d)) \to H^1(F(d)).$$

On the other hand, by the first exact sequence we have

$$H^1(E^*(d-s-t)) \to H^1(F(d)) \to H^2(\mathcal{O}_{\mathbb{P}^N}(d-2s-t)),$$

and it follows from the assumption on E and the fact that $N \geq 3$ that $H^1(F(d)) = 0$. We hence have a surjection

$$H^0(E(d-s)) \to H^0(\mathcal{I}_X(d))$$

induced by ψ. Thus every hypersurface of degree d in the ideal of X is identified with a section

$$\mathbf{s} \wedge \varphi^{(r)} \in H^0(\mathbb{P}^N, \bigwedge^{2r+1} E \otimes \mathcal{O}_{\mathbb{P}^N}(d-s+rt)),$$

for some section $\mathbf{s} \in H^0(E(d-s))$. It is now enough to observe that

$$\mathcal{O}_{\mathbb{P}^N}(d) = \bigwedge^{2r+1} E \otimes \mathcal{O}_{\mathbb{P}^N}(d-s+rt) = \bigwedge^{2r+2}(E \oplus \mathcal{O}_{\mathbb{P}^N}(d-s-t)) \otimes \mathcal{O}_{\mathbb{P}^N}((r+1)t)$$

and the section $\mathbf{s} \wedge \varphi^{(r)}$ corresponds to the Pfaffian of the section

$$\begin{aligned}(\varphi, \mathbf{s}) &\in H^0(\bigwedge^2(E \oplus \mathcal{O}_{\mathbb{P}^N}(d-s-t)) \otimes \mathcal{O}_{\mathbb{P}^N}(t)) \\ &= H^0(\bigwedge^2 E \otimes \mathcal{O}_{\mathbb{P}^N}(t)) \oplus H^0(E \otimes \mathcal{O}_{\mathbb{P}^N}(d-s))\end{aligned}$$

under the above identification.

Lemma 2.3. *Let X, E, r, s, φ be as above and let E satisfy Assumption 2.2. Assume that X is contained in two hypersurfaces H_{d_1} and H_{d_2} of degree d_1 and d_2 respectively. Let $\mathbf{s}_i$ be the section of $H^0(E(d_i - s))$ corresponding to H_{d_i} for $i = 1, 2$. Then $H_{d_1} \cap H_{d_2}$ is a codimension 2 complete intersection if and only if the section $(\varphi, \mathbf{s}_1, \mathbf{s}_2, \mathbf{l})$ in the decomposition*

$$\begin{aligned}&H^0(\bigwedge^2(E \oplus \mathcal{O}_{\mathbb{P}^6}(d_1-s-t) \oplus \mathcal{O}_{\mathbb{P}^6}(d_2-s-t))(t)) \\ &= H^0(\bigwedge^2 E(t)) \oplus H^0(E(d_1-s)) \oplus H^0(E(d_2-s)) \oplus H^0\left(\mathcal{O}_{\mathbb{P}^6}(d_1+d_2-2s-t)\right)\end{aligned}$$

defines a codimension 3 Pfaffian variety for general $\mathbf{l} \in H^0\left(\mathcal{O}_{\mathbb{P}^6}(d_1+d_2-2s-t)\right)$. Moreover, if Y is a Pfaffian variety defined by a section $(\varphi, \mathbf{s}_1, \mathbf{s}_2, \mathbf{l})$ then Y is bilinked to X via the intersection of the two hypersurfaces H_{d_1} and H_{d_2}.

Proof. Assume that $T = H_{d_1} \cap H_{d_2}$ is a codimension 2 complete intersection. Let us choose $\mathbf{l} \in H^0\left(\mathcal{O}_{\mathbb{P}^6}(d_1+d_2-2s-t)\right)$ such that $T \cap \{\mathbf{l} = 0\}$ is of codimension 3. We can now easily check that at $\lambda = 0$ the degeneracy loci of the sections $(\varphi, \mathbf{s}_1, \lambda\mathbf{s}_2, \mathbf{l})$ degenerate to a subvariety of $X \cup (Y \cap \{\mathbf{l} = 0\})$, hence the general element of the family has codimension 3 as expected. By simple base change we find that the map associated to $\left(\varphi, \mathbf{s}_1, \mathbf{s}_2, \frac{1}{\lambda}\mathbf{l}\right)$ degenerates along a codimension 3 Pfaffian variety for general λ.

Conversely, assume that a section $(\varphi, \mathbf{s}_1, \mathbf{s}_2, \mathbf{l})$ defines a codimension 3 variety. Consider two sections

$$g_1 = \mathbf{s}_1 \wedge \mathbf{s}_2 \wedge \underbrace{\varphi \wedge \cdots \wedge \varphi}_{r-1} \\ \in H^0\left(\bigwedge^{2r+3}(E \oplus \mathcal{O}_{\mathbb{P}^6}(d_1 - s - t) \oplus \mathcal{O}_{\mathbb{P}^6}(d_2 - s - t))((r+1)t)\right)$$

and

$$g_2 = \mathbf{l} \otimes (t \wedge \underbrace{\varphi \wedge \cdots \wedge \varphi}_{r}) \\ \in H^0\left(\bigwedge^{2r+3}(E \oplus \mathcal{O}_{\mathbb{P}^6}(d_1 - s - t) \oplus \mathcal{O}_{\mathbb{P}^6}(d_2 - s - t))((r+1)t)\right).$$

It is a simple exercise in linear algebra to check on the fibers of this bundle that these two sections are proportional on $H_{d_1} \cap H_{d_2}$, *i.e.*, $\mathbf{s}_i \wedge \underbrace{\varphi \wedge \cdots \wedge \varphi}_{r}$ vanishes for $i = 1, 2$. The ratio between these two sections defines a rational function g on $H_{d_1} \cap H_{d_2}$. Observe that the function $g + 1$ vanishes only along the Pfaffian variety defined by $(\varphi, \mathbf{s}_1, \mathbf{s}_2, \mathbf{l})$, which is of codimension 3 by assumption. It follows that $H_{d_1} \cap H_{d_2}$ is of codimension 2.

To prove the last statement of the lemma, let Y be the Pfaffian variety defined by the section $(\varphi, \mathbf{s}_1, \mathbf{s}_2, \mathbf{l})$. In particular, Y is of codimension 3 and $T = H_{d_1} \cap H_{d_2}$ is a complete intersection of two hypersurfaces. Now, Y defines a generalized divisor in the sense of [6]. We claim that Y is linearly equivalent as a generalized divisor to $X + H$, where H is the restriction of the hyperplane section to T. Indeed, $g - 1$ is a rational function on $H_{d_1} \cap H_{d_2}$ which defines $Y - X - H$. By the definition of biliaison, it follows that Y is related to X by a biliaison of height 1. Finally, by [6, Prop. 4.4], this means that Y is bilinked to X. □

3. Degree 9 Del Pezzo surface and degree 18 canonically embedded surfaces

The analogy discussed in [14] suggests that one might try to construct a canonically embedded surface of general type of degree 18 in $\mathbb{P}^5$ if one finds an appropriate description of a Del Pezzo surface of degree 9 in $\mathbb{P}^5$. In this section we collect information on such Del Pezzo surfaces and next present a construction of canonically embedded surfaces of general type of degree 18.

3.1. Del Pezzo surface of degree 9

Recall that a Del Pezzo surface of degree 9 is just $\mathbb{P}^2$ and its anticanonical embedding is the image V_9 of the triple Veronese embedding. We shall denote by

$D_9^\Lambda \subset \mathbb{P}^5 =: \mathbb{P}(W)$ the surface obtained as the image of the projection of the image of this embedding from a general 3-dimensional linear subspace $\Lambda \subset \mathbb{P}^9$. Let $D = D_9^\Lambda$ for some general 3-dimensional linear subspace $\Lambda \subset \mathbb{P}^9$. Our aim is to understand the module $M := \bigoplus_{k=0}^{\infty} H^1(\mathcal{I}_D(k+2))$, the shifted Hartshorne-Rao module of D. From [14, Lemma 4.1], we know that the Hilbert function of the Hartshorne-Rao module of D has values $(0, 4, 7, 0, \dots)$ starting from grade 0. We can thus write $M = M_{-1} \oplus M_0 = H^1(\mathcal{I}_D(1)) \oplus H^1(\mathcal{I}_D(2))$.

By working out a random example in Macaulay2, we can prove that M is generated in degree -1. Moreover, the minimal resolution of M is

$$0 \leftarrow M \leftarrow 4R(1) \leftarrow 17R \nwarrow \begin{matrix} 18R(-1) \\ \oplus \\ 29R(-2) \end{matrix} \longleftarrow \begin{matrix} 4R(-2) \\ \oplus \\ 80R(-3) \end{matrix} \nwarrow 81R(-4) \leftarrow 38R(-5) \leftarrow 7R(-6) \leftarrow 0$$

Trying to extend [14, Theorem 1.3] to the case of Del Pezzo surfaces of degree 9, we should look for Pfaffian varieties associated to bundles in $\operatorname{Ext}^1\big(2\mathcal{O}_{\mathbb{P}^5}, Syz^1(M)\big)$. However, for modules corresponding to general Del Pezzo surfaces D_9^Λ it is hard to find any such bundle for which a Pfaffian variety would exist. In particular, even the bundle $Syz^1(M) \oplus 2\mathcal{O}_{\mathbb{P}^5}$ has no twisted skew self-map defining a Pfaffian variety. For this reason, instead of trying to find a special element in $\operatorname{Ext}^1\big(2\mathcal{O}_{\mathbb{P}^5}, Syz^1(M)\big)$ for M being the shifted Hartshorne-Rao module of a general Del Pezzo surface of degree 9 in $\mathbb{P}^5$, we look for a special smooth projected Del Pezzo surface of degree 9 for which the Betti table of the resolution of its Hartshorne-Rao module has different shape from the generic one.

Proposition 3.1. There exists a 3-dimensional linear subspace $\Lambda \subset \mathbb{P}^9$ such that the projected Del Pezzo surface $D_9^\Lambda \subset \mathbb{P}^5$ is smooth and is contained in a complete intersection Y of two smooth cubic hypersurfaces and such that the singular locus of Y consists of 60 isolated singularities.

Proof. Let V_9 be the Del Pezzo surface of degree 9 obtained as the image of the following Veronese embedding:

$$\mathbb{P}^2 \ni (x:y:z) \mapsto (x^3:y^3:z^3:3x^2y:3xy^2:3x^2z:3xz^2:3y^2z:3yz^2:6xyz) \in \mathbb{P}^9.$$

Let us denote the corresponding coordinates in $\mathbb{P}^9$ by $(a_0, \dots, a_9)$.

Recall that the ideal of the Del Pezzo surface of degree 9 in $\mathbb{P}^9$ is defined by the 2×2 minors of the Catalecticant matrix

$$A = \begin{bmatrix} 3a_0 & a_4 & a_6 & 2a_3 & 2a_5 & a_9 \\ a_3 & 3a_1 & a_8 & 2a_4 & a_9 & 2a_7 \\ a_5 & a_7 & 3a_2 & a_9 & 2a_6 & 2a_8 \end{bmatrix}.$$

It is also known that the 3×3 minors of A define the secant locus (*i.e.*, the closure of the union of all secant lines) of the Del Pezzo surface. Moreover, the secant locus is of codimension 4 in $\mathbb{P}^9$.

By the probabilistic method, one can easily construct in positive characteristic an example of a three-dimensional projective space Λ_0 disjoint from the secant locus such that the projection is contained in a pencil of cubics. An example of a matrix defining a projection from such a Λ_0 in characteristic 17 is the following:

$$N_0 = \begin{bmatrix} 0 & 0 & -1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We check using Macaulay2 that the projection of V_9 from Λ_0 is contained in a complete intersection Y of two smooth cubics C_1 and C_2 such that Y has 60 distinct singular points.

We shall show that Λ_0 lifts to characteristic 0 to some Λ such that the projection of V_9 from Λ is contained in two cubics that specialize to C_1 and C_2. First observe that a projection from Λ corresponds to a linear map $\mathbb{C}^{10} \to \mathbb{C}^6$, hence a 10×6 matrix N with complex entries. This projection composed with the Veronese embedding is a map

$$\varphi : \mathbb{P}^2 \to \mathbb{P}^6$$

defined by a base point free linear system of cubics, described by N. If we further compose φ with the triple Veronese embedding $\psi : \mathbb{P}^6 \to \mathbb{P}^{55}$, then the dimension of the space of cubics containing $\Pi_\Lambda(V_9)$ is equal to the codimension of the span of the image $\psi \circ \varphi(\mathbb{P}^2)$ in $\mathbb{P}^{55}$. On the other hand, it is clear that $\psi \circ \varphi$ factors through a 9-tuple Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^{54}$ and a linear map $L_N : \mathbb{P}^{54} \to \mathbb{P}^{55}$. It follows that the image of the projection $\Pi_\Lambda(V_9)$ is always contained in a cubic hypersurface. Moreover, the projection is contained in a two-dimensional space if and only if L_N has non-maximal rank. Observe that, following the above description, L_N can be written explicitly as a 55×56 matrix depending on the entries of N. If we now consider a 60-dimensional vector space V parametrizing matrices N, then we obtain a 55×56 matrix L with entries being cubic polynomials on V. Denote by Γ the degeneracy locus of L. It is a subvariety of V of codimension ≤ 2. We can now proceed to describe an explicit lifting to characteristic 0 of the constructed case over $\mathbb{F}_{17}$.

Consider any lift N_0' of N_0 to $\mathbb{Z}$ and a random line l in V passing through N_0. More precisely, we choose l by choosing a parametrization $\mathcal{N} : \mathbb{C} \ni \lambda \mapsto$

$N_0' + \lambda N_1 \in l$ with random N_1, for example

$$\mathcal{N} := \begin{bmatrix} 0 & \lambda & -2\lambda-1 & -2 & 0 & 0 \\ 1 & 2\lambda & -2 & -\lambda & 0 & 0 \\ 0 & 0 & -1 & 0 & 2\lambda & -1 \\ \lambda+1 & 2\lambda & 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & -\lambda & 0 & 0 \\ 0 & -\lambda & 0 & -2\lambda & 2 & 2\lambda \\ -2\lambda-1 & 2\lambda & 1 & 1 & 1 & 1 \\ 2\lambda+1 & -\lambda & 0 & 0 & 0 & 0 \\ \lambda+1 & -2\lambda & -\lambda & -2\lambda & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We can now easily compute in Macaulay2 the Smith normal form L_{SNF} of the matrix $L_{\mathcal{N}}$ restricted to the line l (*i.e.*, over $\mathbb{Q}[\lambda]$). It is a matrix with polynomial entries in λ. More precisely, in our specific case, one entry is a polynomial p of degree 150 with integer coefficients, whereas the remaining diagonal entries are 1. By definition, $L_{\mathcal{N}}$ has non-maximal rank if and only if L_{SNF} has non-maximal rank. The degeneracy locus of L_{SNF} is clearly defined by the vanishing of p. We also check that the reduction mod 17 of this polynomial is also of degree 150. It follows by the Valuative Criterion for Properness that there exists a number field K and a prime $\mathfrak{p}$ in its ring of integers O_K with $O_K/\mathfrak{p} \cong \mathbb{F}_{17}$ such that N_0 is the specialization of an $O_{K,\mathfrak{p}}$ valued point of Γ. In our case, this can be shown explicitly. Indeed, the polynomial of degree 150 decomposes into two irreducible (over $\mathbb{Q}[\lambda]$) polynomials P_1 and P_2 with integer coefficients and of degrees 60 and 90 respectively. We check that P_1 reduced mod 17 has a root $\lambda = 0$. We can hence consider the number field $K = \mathbb{Z}[\lambda]/(P_1)$ and the prime ideal generated by $(17, \lambda)$ in O_K. It is clear that the projection Π defined by the matrix $L_{\mathcal{N}}$ with λ being any root of P_1 maps V_9 to a variety contained in two cubics.

Observe now that the two cubics containing $\Pi(V_9)$ are computed universally (with parameter λ) when computing the Smith normal form. More precisely, from the algorithm, we obtain matrices $\mathcal{S}_1, \mathcal{S}_2$ invertible over $\mathbb{Q}[\lambda]$ such that $\mathcal{S}_1 L_{\mathcal{N}} \mathcal{S}_2 = L_{SNF}$. In particular, the columns of $\mathcal{S}_2$ corresponding to the vanishing columns of L_{SNF} define cubics containing the image $\Pi(V_9)$. We check easily that the ideal generated by these two cubics specializes via our specialization map to the ideal generated by the two cubics, computed over $\mathbb{F}_{17}$. Moreover, since the kernel of the projection over $\mathbb{F}_{17}$ is disjoint from the secant locus, this is also the case for the lifted projection. It follows that there exists a lift of the projection found over $\mathbb{F}_{17}$ such that the image of V_9 is smooth and contained in a complete intersection Y of two smooth cubics and such that Y has singularities not worse than 60 isolated singular points. $\square$

Remark 3.2. Note that the computation of $c_2(\mathcal{I}_{D_9}/\mathcal{I}^2_{D_9}(3)) = 60$ gives us the expected number of singular points. We indeed check using Macaulay 2 that the two cubics containing the projection of V_9 constructed above are smooth and intersect in a variety having 60 isolated singularities. However, the singularities of Y are not nodes but isolated triple points with tangent cone being the cone over the projection of the cubic scroll to $\mathbb{P}^3$.

Remark 3.3. One can describe the set of matrices defining those projections of V_9 that are contained in two cubics in terms of maximal minors of L in V, hence, a variety of expected codimension 2. On the other hand, from the proof of Proposition 3.1, for a random choice of line l, the degeneracy locus restricted to this line has two components over $\mathbb{Q}[\lambda]$. One component of degree 60 passes through our lift, and the other of degree 90 corresponds to the locus of Λ's which intersect the secant locus of V_9 (it is a simple exercise to prove that for any such Λ the projection is indeed contained in two cubics). This suggests that the locus of projections satisfying the assertion of Proposition 3.1 is an open subset of a hypersurface of degree 60 in $\mathbb{P}(V)$. Additional evidence follows from the Jacobi formula for the derivative of the determinant. Indeed, using this formula we can compute the tangent space to the variety Γ at any constructed point, even if writing down the equations of Γ is out of reach for our computer. In our case, we get a codimension 1 tangent space. It is an interesting problem to find a geometric interpretation as in [8] for the centers of projection contained in Γ.

Another interesting problem is the geometric description of the cubics containing the projected variety. For instance, for a generic choice of the center of projection Λ, the surface $D_9 \subset \mathbb{P}^5$ is contained in a unique cubic singular along a non-degenerate curve of degree 6. From [16] such a cubic has to be determinantal.

Remark 3.4. If the center of projection Λ intersects the secant locus of the surface V_9 in one point, then D_9^Λ is also contained in a pencil of cubics. However, in this case, we have $h^1\big(\mathcal{I}_{D_9^\Lambda}\big) = 3$. Moreover, if Λ meets the secant locus in three (respectively four) points then D_9^Λ is contained in a four- (respectively five-) dimensional space of cubics. And if the intersection is a line, then there is a seven-dimensional space of cubics in the ideal.

3.2. Surfaces of general type of degree 18 in $\mathbb{P}^5$

Let us consider a special Λ such that $D_9^\Lambda \subset \mathbb{P}^5$ (see Proposition 3.1) is contained in a complete intersection threefold of degree 9 with 60 isolated singularities as above.

Theorem 3.5. *The surface D_9^Λ can be bilinked through the complete intersection of two cubics to a smooth surface of general type S_0 of degree* 18.

Proof. Let us denote by H the class of the hyperplane on $\mathbb{P}^5$. Denote by F_k the surface residual to D_9 through a complete intersection of type $(3, 3, k)$ defined as the intersection of the cubics containing D_9^Λ with a general hypersurface of degree k for some $k \geq 4$. From the exact sequence

$$0 \to \mathcal{I}_{D_9 \cup F_k} \to \mathcal{I}_{F_k} \to \omega_{D_9}(-k) \to 0,$$

using the fact that $D_9 \cup F_k$ is ACM and $\omega_{D_9} = \mathcal{O}_{D_9}(-1)$, we infer that $h^0(\mathcal{I}_{D_9 \cup F_k}(k+1)) + 1 = h^0(\mathcal{I}_{F_k}(k+1))$. The surface F_k is thus linked through two cubics and a surface of degree $k+1$ to a surface S_0. Let Y be the complete intersection of our cubics. Using Macaulay 2 in characteristic 17, we proved that the singular locus of Y is a smooth zero-dimensional scheme of degree 60. Moreover, in characteristic 17 this scheme is also the intersection scheme of D_9 and S_0. It follows that also in characteristic 0 the surfaces D_9 and S_0 intersect in isolated points in a transversal way, *i.e.*, their tangent spaces intersect transversely at each intersection point. Since S_0 and D_9 are contained in a smooth cubic, it follows that S_0 is smooth at each intersection point. To prove that S_0 is smooth everywhere, we use the Bertini theorem outside the singular locus of Y. Indeed, observe first that by [6, Proposition 4.4] the variety S_0 is an almost Cartier generalized divisor linearly equivalent to the almost Cartier generalized divisor $D_9 + H$. It follows that on $Y \setminus \mathrm{Sing}(Y)$ the surface S_0 is a general element of the system $|D_9 + H|$ which is base point free on $Y \setminus \mathrm{Sing}(Y)$ because it is so outside D_0 and because S_0 does not meet D_9 in $Y \setminus \mathrm{Sing}(Y)$. Furthermore, by the adjunction formula, since S_0 is a smooth divisor, Cartier in codimension 2, we have $K_{S_0} = (K_Y + D + H)|_{S_0} = H_{S_0}$. This means that S_0 is a surface of general type, canonically embedded in $\mathbb{P}^5$. It is clearly linearly normal, since by the basic properties of liaison its Hartshorne-Rao module is the Hartshorne-Rao module of the Del Pezzo surface with gradation lifted by 1. Finally, the degree of S_0 is 18 by construction. □

Remark 3.6. Observe that if E is the bundle defining the Del Pezzo surface D_9 through the Pfaffian construction, then S_0 is defined by the bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^5}$. Indeed, let us first point out that since we are dealing with almost Cartier divisors, we can perform all computations on $Y \setminus \mathrm{Sing}(Y)$ and then extend the result to Y. In particular, by Lemma 2.3 we have a 6-dimensional subsystem of the linear system $|D_9 + H|$, consisting of varieties obtained as sections of $E \oplus 2\mathcal{O}_{\mathbb{P}^5}$. To prove that this gives the complete linear system $|D_9 + H|$, we make a simple dimension count basing on the exact sequence

$$0 \to \mathcal{O}_Y(H) \to \mathcal{O}_Y(D+H) \to \mathcal{O}_D(D+H) \to 0,$$

the fact that the singularities of Y are normal of codimension 3, and the equalities $\mathcal{O}_D(D+H) = \mathcal{O}_D$ and $h^1(Y, \mathcal{O}_Y(H)) = 0$. We obtain $h^0(Y, \mathcal{O}_Y(D+H)) = h^0(Y, \mathcal{O}_Y(H)) + h^0(D, \mathcal{O}_D) = 7$, hence the system $|D_9 + H|$ is of dimension 6.

Remark 3.7. Observe that the dimension of the family of surfaces obtained in Theorem 3.5 is at most 20. Indeed, since we know that the general choice of a center of projection does not lead to a variety contained in the complete intersection of two cubics, it follows that the dimension of the space of Λ's up to linear automorphisms is at most $\dim G(4, 10) - 1 - 8 = 15$. We also find that for a given $D_9 \subset \mathbb{P}^5$ contained in the intersection of the cubics there is a 5-dimensional family $(= h^0(D+H) - 1)$ of bilinked surfaces. Altogether this gives a space of dimension at most 20. Moreover, by Remark 3.3 and the Betti table of the constructed surface, we expect this dimension to be exactly 20.

We know that the dimension of the Kuranishi space $\mathcal{K}$ of a surface of general type S is not smaller than $h^1(T_S) - h^2(T_S)$. On the other hand, by Noether's formula we find $c_2(S) = 66$. So, by the Riemann-Roch theorem applied to T_S, we infer that $h^1(T_S) - h^2(T_S) = 36 + h^0(T_S) \geq 36$. Since $h^1(T_{\mathbb{P}^5}|S) = 0$, we also have $H^0(N_{S|\mathbb{P}^5}) \twoheadrightarrow H^1(T_S)$. Thus there should be an at least 36-dimensional family of canonically embedded surfaces of general type of degree 18 in $\mathbb{P}^5$ with special element S_0. The general element of this family should have a simpler Hartshorne-Rao module. It is a natural problem to construct such a general Hartshorne-Rao module.

3.3. Bilinkages of surfaces of general type

Let us now study how Construction 2.1 works in the case of canonical surfaces in $\mathbb{P}^5$ constructed in [4].

Proposition 3.8. Any smooth linearly normal canonical surface $S \subset \mathbb{P}^5$ of degree $d_S \leq 17$ satisfying the maximal rank assumption is obtained by Construction 2.1 from a Del Pezzo surface D of degree $d_D = d_S - 9$ with bilinkage performed in a complete intersection of two cubics.

Proof. We compare the description, by [4], of a canonical surface S_d of degree $12 \leq d + 9 \leq 17$ in $\mathbb{P}^5$ satisfying the maximal rank assumption, with the description of a projected Del Pezzo surface D_d of degree d contained in [14]. Since all surfaces S_d satisfying the assumptions above are deformation equivalent, it is enough to obtain one such surface in each degree using Construction 2.1 as in the assertion. We observe that for $d \leq 6$ the bundle F_d constructed by Catanese is related to the corresponding bundle E_d from [14] by $E_d \oplus 2\mathcal{O}_{\mathbb{P}^6} = F_d$. Since for $d \leq 7$ the Del Pezzo surface D_d is contained in a complete intersection of two cubic hypersurfaces, by Lemma 2.3 the Del Pezzo surface D_d is bilinked to a Gorenstein surface of general type defined by the bundle $E_d \oplus 2\mathcal{O}_{\mathbb{P}^6}$ through the Pfaffian construction. Let us denote such general surface by $\tilde{S_d}$. Since, for $d \leq 6$, we have $E_d \oplus 2\mathcal{O}_{\mathbb{P}^6} = F_d$, it follows that $\tilde{S}_d$ is a smooth canonical surface in $\mathbb{P}^5$ of degree $d + 9$.

For $d = 7$, both $E_7 \oplus 2\mathcal{O}_{\mathbb{P}^5}$ and F_7 appear as kernels of some surjective maps $13\mathcal{O}_{\mathbb{P}^5} \to 2\mathcal{O}_{\mathbb{P}^5}$. Moreover F_7 is the kernel of a generic such map. It is now enough to take a one-parameter family parametrized by $\lambda \in \mathbb{C}$ of maps as above such that for $\lambda \neq 0$ the kernel is isomorphic to F_7, whereas for $\lambda = 0$ the kernel is $E_7 \oplus 2\mathcal{O}_{\mathbb{P}^5}$. It follows that there is a bundle $\mathfrak{E}$ on $\mathbb{P}^6 \times \mathbb{C}$ whose restriction to $\mathbb{P}^6 \times \{0\}$ is the bundle $\mathcal{F} \oplus 2\mathcal{O}_{\mathbb{P}^6}$ and the restriction to a fiber $\mathbb{P}^6 \times \{\lambda\}$ for $\lambda \neq 0$ is isomorphic to E. Moreover, since $h^0\big(\bigwedge^2 \big((E_7 \oplus 2\mathcal{O}_P^5)(1)\big)\big) = h^0\big(\bigwedge^2 F_7(1)\big)$, we infer that each section of $\bigwedge^2 \big(E_7 \oplus 2\mathcal{O}_{\mathbb{P}^6}\big)(1)$ is extendable to a section of $\bigwedge^2 \mathfrak{E}(1)$. It follows from [14, Lemma 3.4] that $\tilde{S}_7$ is a degeneration of a family of canonical surfaces of degree 16 satisfying the maximal rank assumption; since all such surfaces are deformation equivalent, the assertion follows in the case $d = 7$.

For $d = 8$ the situation is similar. More precisely, let D_8 be a Del Pezzo surface of degree 8 (either of type D_8^1 or D_8^2) defined by some bundle E_8. Then D is contained in a variety that is the complete intersection of two cubics, and $E_8 \oplus 2\mathcal{O}_{\mathbb{P}^5}$ is the kernel of some special surjective map $16\mathcal{O}_{\mathbb{P}^5} \to 3\mathcal{O}_{\mathbb{P}^5}$. The bundle F_8 which is the kernel of a generic such map defines by [4] a canonical surface of general type satisfying the maximal rank assumption. We then construct $\mathfrak{E}$ in the same way as above and conclude the proof by applying [14, Lemma 3.4] and the equality $h^0\big(\bigwedge^2\big((E_8 \oplus 2\mathcal{O}_P^5)(1)\big)\big) = h^0\big(\bigwedge^2 F_8(1)\big)$. □

4. Del Pezzo threefolds and Tonoli Calabi-Yau threefolds

In order to obtain a Calabi-Yau threefold by Construction 2.1, we consider Del Pezzo threefolds T, *i.e.* $K_T = -2H$, where H is ample embedded in $\mathbb{P}^6$ by a subsystem of the half-anticanonical class. The first examples of such threefolds we consider are the Del Pezzo threefolds of degree $3 \le d \le 5$ embedded by the complete linear system of the half-anticanonical class into a linear subspace of $\mathbb{P}^6$. Note that all the Del Pezzo threefolds are enumerated in [19, Theorem 3.3.1].

Recall that, for a threefold in $Y \subset \mathbb{P}^n$ with $n \ge 7$ which is not contained in a hyperplane, there exists a smooth projection of Y into $\mathbb{P}^6$ if and only if Y is defective, *i.e.*, the dimension of the secant locus of $Y \subset \mathbb{P}^n$ is ≤ 6. It follows that if we want to consider only smooth threefolds $F \subset \mathbb{P}^6$ in the case $d \ge 6$, we are restricted to the consideration of Del Pezzo threefolds defective in their half-anticanonical embedding. There are only a few examples of such. Let $V_t \subset \mathbb{P}^{t+1}$ with $t = 7, 8$ be the image of $\mathbb{P}^3$ by the map defined by all quadrics passing through $8 - t$ point in $\mathbb{P}^3$.

Lemma 4.1. *A smooth Del Pezzo threefold of degree $d \ge 6$ embedded in $\mathbb{P}^6$ by a subsystem of the half-anticanonical embedding is defective if and only if it is isomorphic to one of the following:*

1. *$T_6 \subset \mathbb{P}^6$ is the generic central projection of the hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
2. *$T_7 \subset \mathbb{P}^6$ is the projection of $V_7 \subset \mathbb{P}^8$ from a linear space disjoint from the secant locus $Sec(V_7) \subset \mathbb{P}^8$;*
3. *$T_8 \subset \mathbb{P}^6$ is the projection of the image V_8 of the double Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$ from a linear space disjoint from the secant locus $Sec(V_8) \subset \mathbb{P}^9$.*

Proof. This follows by comparing the classical classification of defective threefolds due to Scorza (see [5] for a modern approach) and the classification of Del Pezzo threefolds due to Iskovskikh (see [10]). □

We aim at proving the following.

Proposition 4.2. For $6 \le d \le 7$ every smooth Del Pezzo threefold T_d of degree d in $\mathbb{P}^6$ is related by Construction 2.1 to a smooth Calabi-Yau threefold of degree $d + 9$ in $\mathbb{P}^6$.

Proof. Recall that Del Pezzo threefolds are defined through the Pfaffian construction by the bundles E'_d from [14, Remark 4.6]. Since for $d \leq 7$ the Del Pezzo threefold T_d is contained in a complete intersection of two cubic hypersurfaces, by Lemma 2.3, the Del Pezzo threefold of degree d is bilinked to a Gorenstein Calabi-Yau threefold (a Gorenstein threefold with $\omega_X = 0$ and $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$) $\tilde{X}_d$ defined by the bundle $E'_d \oplus 2\mathcal{O}_{\mathbb{P}^6}$ through the Pfaffian construction. Now for $d \leq 6$ we have $E'_d \oplus 2\mathcal{O}_{\mathbb{P}^6} = F_d$ and hence $\tilde{X}_d$ is a smooth Tonoli Calabi-Yau threefold of degree $d+9$. For $d = 7$ it is enough to observe that there is a bundle $\mathfrak{E}$ on $\mathbb{P}^6 \times \mathbb{C}$ whose restriction to $\mathbb{P}^6 \times \{0\}$ is $E'_7 \oplus 2\mathcal{O}_{\mathbb{P}^6}$ and the restriction to any fiber $\mathbb{P}^6 \times \{\lambda\}$ for $\lambda \neq 0$ is isomorphic to F_7. We also compute using Macaulay 2 that the dimension of the space of sections of $\bigwedge^2 \left(E'_7 \oplus 2\mathcal{O}_{\mathbb{P}^6}\right)(1)$ is equal to the dimension of the space of sections of $\bigwedge^2 F_7(1)$. We infer that $\bigwedge^2 \left(E'_d \oplus 2\mathcal{O}_{\mathbb{P}^6}\right)(1)$ is extendable to a section of $\bigwedge^2 \mathfrak{E}(1)$. It follows from [14, Lemma 3.4] that $\tilde{X}_d$ is a degeneration of a family of Tonoli Calabi-Yau threefolds. □

Remark 4.3. Note that the above proof works for each Del Pezzo threefold of degree $3 \leq d \leq 5$ such that the half-anticanonical divisor gives an embedding into a linear subspace of $\mathbb{P}^6$. In this way we obtain all smooth ACM Calabi-Yau threefolds.

Let now study the most interesting case: let $T_8 \subset \mathbb{P}^6$ be a Del Pezzo threefold of degree 8 in $\mathbb{P}^6$ which is the projection of $V_8 \subset \mathbb{P}^9$ as above. Using the methods from [7] we deduce that the ideal of $T_8 \subset \mathbb{P}^6$ is generated by 45 quartics and is not contained in any cubic.

However, in order to perform a bilinkage we can find a special center of projection $L \subset \mathbb{P}^9$ also disjoint from the secant locus $Sec(V_8) \subset \mathbb{P}^9$, such that the image of the projection $T_8^L \subset \mathbb{P}^6$ is contained in a 3-dimensional system of cubics.

Proposition 4.4. There exists a center of projection $L \subset \mathbb{P}^9$ such that $T_8^L \subset \mathbb{P}^6$ can be bilinked to a Gorenstein Calabi-Yau threefold (not necessarily normal) $X' \subset \mathbb{P}^6$ of degree 17. Moreover, one can choose the bilinkage in such a way that $X' \subset \mathbb{P}^6$ admits a smoothing by the family of Tonoli Calabi-Yau threefolds of degree 17 with $k=9$.

Proof. Recall that the 2×2 minors of the matrix

$$A = \begin{bmatrix} a & x & y & z \\ x & b & t & u \\ y & t & c & v \\ z & u & v & w \end{bmatrix}$$

define the second Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^9(a, b, c, x, y, z, t, u, v, w)$. Let us consider a special $\Lambda = \mathbb{P}^2$, the center of projection defined by the following equations:

$$\begin{cases} -2a + b + c - 2y - z + 2t + 2u + 2v - w = 0 \\ 2a + 2b + c + x + y - z - t + 2u - 2v - w = 0 \\ -a - 2b - 2c - 2x + y + t + v + w = 0 \\ 2a + b - 2c + 2y - 2z + 2u - v + w = 0 \\ a + b - 2z + 2t + u + 2v - w = 0 \\ -2c - 2x + y - z - t + 2u + w = 0 \\ -c + 2x - y - 2u - v + 2w = 0\,. \end{cases}$$

Although it is hard to check that by hand, it is straightforward to verify using Macaulay 2 that the image of the projection $T_8^L \subset \mathbb{P}^6$ is contained in three independent cubics. Then the residual to T_8^L of the intersection of these cubics is a threefold G of degree 19 that is contained in a quartic that does not contain T_8^L (*cf.* [11, Lemma 3.1]). The residual to G in the intersection of the cubics with the quartic containing G is a threefold X' of degree 17 (we say that X' is bilinked with T_8^L through two cubics with height 1).

As in Propositions 3.8 and 4.2, by Lemma 2.3, we infer that there is a Pfaffian variety X associated to the vector bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^6}$ which is a Gorenstein Calabi-Yau threefold of degree 17.

For the second part, note that, by the general properties of bilinkage, the threefold X has the same Hartshorne-Rao module as T_8^L but shifted by one. It follows by [14, proof of Theorem 1.3] that this Hartshorne-Rao module is determined by some special $\mathbb{P}^{13} \subset \langle \mathbb{P}^2 \times \mathbb{P}^6 \rangle$ containing a linear space $\mathcal{P}$ spanned by the graph of some double Veronese embedding (composed with a linear embedding) of $\mathbb{P}^2$ to $\mathbb{P}^6$. Observe moreover that, by [14, Theorem 1.1], the Hartshorne-Rao module of a Tonoli Calabi-Yau threefold of degree 17 with $k = 9$ corresponds to a $\mathbb{P}^{15} \subset \langle \mathbb{P}^2 \times \mathbb{P}^6 \rangle$ containing such a linear space $\mathcal{P}$. We now claim that the bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^6}$ appears as a flat deformation of a family of bundles associated to such Calabi-Yau threefolds. Indeed, the bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^6}$ is obtained as the kernel of a map $16\mathcal{O}_{\mathbb{P}^6} \to 3\mathcal{O}_{\mathbb{P}^6}(1)$ defined by a matrix whose columns span the $\mathbb{P}^{13}$, whereas the chosen Tonoli Calabi-Yau threefolds appear as a Pfaffian variety associated to a bundle obtained as the kernel of a similar map, but with columns spanning a $\mathbb{P}^{15}$ containing our $\mathbb{P}^{13}$. It is easy to see that by degenerating two columns of the map to zero (for example by multiplying them by the parameter λ) one obtains the desired flat deformation.

Observe now that there exists a subspace V of dimension 9 of the space of sections of $\bigwedge^2(E \oplus 2\mathcal{O})(1)$, consisting of all sections which admit extensions to our deformation family. By [14, Lemma 3.4] the varieties given by these sections admit smoothings to Tonoli Calabi-Yau threefolds of degree 17 with $k = 9$. □

Remark 4.5. Observe that, in the proof of Proposition 4.4, not every section of $\bigwedge^2 \left(E \oplus 2\mathcal{O}_{\mathbb{P}^6}\right)(1)$ extends to the deformation family. It follows that taking a general section of $\bigwedge^2 \left(E \oplus 2\mathcal{O}_{\mathbb{P}^6}\right)(1)$ in the proof of Proposition 4.4 one obtains a Gorenstein Calabi-Yau threefold representing a different component of the Hilbert scheme of Calabi-Yau threefolds consisting possibly of only singular threefolds.

5. Unprojections

Recall that *unprojection* is the inverse process to projection (see [17] for a general discussion). In this section we discuss the relations between the constructions by unprojection and by bilinkage in the context of submanifolds of codimension 3.

For the construction of Calabi-Yau threefolds using bilinkages with Del Pezzo threefolds we are not restricted to starting from smooth Fano threefolds. A natural choice for singular Del Pezzo threefolds are cones over Del Pezzo surfaces. These are always contained in many cubics and a bilinkage can be performed. This construction enables one to directly relate the Del Pezzo surface to the Calabi-Yau threefolds constructed. When the cone is Gorenstein, it is related to so-called Kustin-Miller unprojections. This construction was studied in [11, 12, 15]. In particular, a straightforward generalization of [12, Proposition 4.1] (*cf.* [3]) shows that the unprojection of a codimension 3 variety defined by Pfaffians of a decomposable bundle E on $\mathbb{P}^n$ in a codimension 2 complete intersection can be seen as some special Pfaffian variety associated to the bundle $E' \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^{n+1}}(a_2)$ where E' denotes the trivial extension of the decomposable bundle E to $\mathbb{P}^{n+1}$, and a_1 and a_2 are appropriate numbers depending on the degrees of the generators of the complete intersection and the degrees in the decomposition of E.

In the case of a complete intersection of two cubics containing a projectively Gorenstein Del Pezzo surface in $\mathbb{P}^5$, the result of the unprojection is a special Pfaffian variety associated to $F = E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$. More precisely, it is given as the degeneracy locus of a skew-symmetric map $\rho : F^* \to F \otimes \mathcal{O}_{\mathbb{P}^6}(1)$ corresponding to a section of the form

$$\begin{aligned}(\varphi, c_1, c_2, x_6) \in H^0 &\left(\textstyle\bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}^6}(1)\right)\\ &= H^0\left(\textstyle\bigwedge^2 E' \otimes \mathcal{O}_{\mathbb{P}^6}(1)\right) \oplus 2H^0(E'(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^6}(1)),\end{aligned}$$

where φ defines the cone over the Del Pezzo surface, c_1, c_2 are sections which correspond via the Pfaffian resolution to two cubics containing the Del Pezzo surface, and x_6 is the new variable of $\mathbb{P}^6$. It follows:

Corollary 5.1. *Every Tonoli Calabi-Yau threefold of degree ≤ 14 is a smoothing of a Gorenstein Calabi-Yau threefold obtained as the unprojection of a Del Pezzo surface of degree $d \leq 5$ in a complete intersection of two cubics.*

In the case $d \geq 6$ a standard Kustin-Miller unprojection cannot be performed because the Del Pezzo surface is not projectively Gorenstein. This is the first case in which the cone over the Del Pezzo surface is not Gorenstein at its vertex and, as such, it cannot be written in terms of the Pfaffian construction applied to a vector bundle. We can, however, somehow ignore this fact and propose a non-Gorenstein unprojection instead of the standard construction due to Kustin and Miller. More precisely, by a non-Gorenstein unprojection we mean that having a variety $D \subset Y \subset \mathbb{P}^N$ with D not projectively Gorenstein we construct a variety $X \subset \mathbb{P}^{N+1}$ singular at some point p such that the projection of X from p is Y and the exceptional locus is D.

Proposition 5.2. A Tonoli Calabi-Yau threefold of degree 15 can be obtained as a smoothing of a singular variety obtained as a non-Gorenstein unprojection of a Del Pezzo surface of degree $d = 6$ in a complete intersection of two cubics.

Proof. Observe that, although the cone over the Del Pezzo surface D_6 is not Gorenstein, we have its description in terms of some similar Pfaffian construction applied to the sheaf E', trivially extending E to $\mathbb{P}^6$. In this case the special Pfaffian variety associated to the sheaf $F = E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ obtained by copying the unprojection procedure above in the context of sheaves is a non-Gorenstein variety X. We shall prove that it admits a smoothing to a Calabi-Yau threefold of degree 15. More precisely, we proceed in the following way. We start with a Del Pezzo surface D_6. It is obtained as a Pfaffian variety associated to the bundle $E = \Omega^1_{\mathbb{P}^5}(1) \oplus 2\mathcal{O}_{\mathbb{P}^5}$, *i.e.*, defined as the degeneracy locus of a general skew-symmetric map $\phi : E^*(-1) \to E$. We consider two cubics in the ideal of the Del Pezzo surface. From the Pfaffian sequence they correspond to two sections of $E(1)$ giving a map $\psi : 2\mathcal{O}_{\mathbb{P}^5}(-1) \to E$. We can now extend the bundle E to a sheaf E' on $\mathbb{P}^6$ defined as the kernel of the map $8\mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{\mathbb{P}^6}(1)$ given by the matrix $[x_0, \dots, x_5, 0, 0]$. Then we consider the skew-symmetric map $\rho : (E' \oplus 2\mathcal{O}_{\mathbb{P}^6})^\vee(-1) \to E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ defined by ϕ, ψ and multiplication by the new variable x_6. The degeneracy locus of ρ is a codimension 3 variety X' which is singular at the point $(x_0, \dots, x_6) = (0, \dots, 0, 1)$, the tangent cone being the cone over the projected Del Pezzo surface D_6. The latter singularity is not Gorenstein. Hence our variety cannot be described as a Pfaffian variety associated to a vector bundle (we have its description as a kind of Pfaffian variety associated to the sheaf E'). It is however straightforward to check that the projection from the point $(0, \dots, 0, 1) \in \mathbb{P}^6$ maps X' to the complete intersection of the two cubics containing the del Pezzo surface, and the exceptional locus is D_6.

Having the description of X' in terms of Pfaffians (associated to a sheaf), we perform a similar reasoning as in [13, Proposition 7.2] and prove that X', though not Gorenstein and not normal, can nonetheless be smoothed to a Tonoli Calabi-Yau threefold of degree 15. More precisely, following [4] we can consider ρ as a 10×10 skew-symmetric matrix A of linear forms satisfying the the following

$$[x_0, \dots, x_5, 0, 0, 0, 0] \cdot A = 0.$$

The degeneracy locus of ρ is given by 8×8 Pfaffians of A. Observe that by the shape of unprojection and the assumption on A we can write A in the form

$$\left(\begin{array}{c|c|c} B & \begin{matrix} a_0 \\ \vdots \\ a_5 \end{matrix} & D^T \\ \hline -a_0 \ \dots \ -a_5 & 0 & a_7 \ a_8 \ a_9 \\ \hline D & \begin{matrix} -a_7 \\ -a_8 \\ -a_9 \end{matrix} & K \end{array}\right),$$

where the variable x_6 appears only in the matrix K (more precisely, in a 2×2 skew-symmetric submatrix of K). Since $[a_0, \dots, a_5]$ satisfies a Koszul relation, there exists a skew-symmetric 5×5 matrix B' with complex entries such that

$$\begin{bmatrix} a_0 \\ \vdots \\ a_5 \end{bmatrix} = B' \cdot \begin{bmatrix} x_0 \\ \vdots \\ x_5 \end{bmatrix}.$$

Moreover since a_i do not depend on x_6, there is clearly a unique 3×6 matrix D' with complex entries such that

$$\begin{bmatrix} a_7 \\ a_8 \\ a_9 \end{bmatrix} = D' \cdot \begin{bmatrix} x_0 \\ \vdots \\ x_5 \end{bmatrix}.$$

Consider now the family of skew-symmetric matrices

$$A_\lambda = \left(\begin{array}{c|c|c} B + \lambda x_6 B' & \begin{matrix} a_0 \\ \vdots \\ a_5 \end{matrix} & D^T + (\lambda x_6 D')^T \\ \hline -a_0 \ \dots \ -a_5 & 0 & a_7 \ a_8 \ a_9 \\ \hline D + \lambda x_6 D' & \begin{matrix} -a_7 \\ -a_8 \\ -a_9 \end{matrix} & K \end{array}\right),$$

parametrized by $\lambda \in \mathbb{C}$. Observe that in this case $[x_0, \dots, x_5, \lambda x_6, 0, 0, 0] \cdot A_\lambda = 0$. Hence the matrices A_λ induce sections of $\bigwedge^2 E_\lambda(1)$, with E_λ isomorphic to $\Omega^1_{\mathbb{P}^6}(1) \oplus 3\mathcal{O}_{\mathbb{P}^6}$, and the ideals generated by their 8×8 Pfaffians correspond to Pfaffian varieties associated to E_λ. To finish the proof, it is enough to observe that the above family is flat around $\lambda = 0$. □

Remark 5.3. Observe that we have used the special form of the section defining the unprojected variety. In particular the construction could not be performed if we were unable to find a matrix A with all the a_i for $i \in \{7, 8, 9\}$ independent of x_6. This suggests that the Hilbert scheme of Calabi-Yau threefolds of degree 15 has at least two components: one giving the Tonoli family of degree 15; the other parametrizing a family of non-Gorenstein threefolds probably birational to the degree 15 threefolds in $\mathbb{P}(1, 1, 1, 1, 1, 1, 1, 2)$ constructed in [11]. If that is indeed the case, the unprojected threefolds X' above would correspond to some points in the intersection of these two components.

One can try to extend the construction from the case of $d = 6$ to higher degree Del Pezzo surfaces. For instance, as in the proof of Proposition 4.4, the Hartshorne-Rao modules of the cones over D_8^1 and D_8^2 are degenerations of Hartshorne-Rao modules associated to Tonoli Calabi-Yau threefolds of degree 17 and $k = 9, 11$ respectively. The sheafified first syzygy modules of their Hartshorne-Rao modules are not vector bundles, but more general sheaves. However, one can still hope that, as in the case of degree $d = 6$, these non-Gorenstein threefolds admit smoothing to Calabi-Yau threefolds. Proceeding further, we compute the dimension of the space of sections of the twisted second wedge power corresponding to the unprojection and in each case we obtain a bigger space than the space of sections of the second wedge power of the bundle defining the appropriate families of Tonoli Calabi-Yau threefolds. Thus again (*cf.* Remarks 4.5, 5.3) we obtain distinct components of the Hilbert scheme of Calabi-Yau threefolds of degree 17 in $\mathbb{P}^6$. The smoothing might possibly be performed only for very special unprojections. It is also not clear whether the varieties representing the general points of any of these components are smooth Calabi-Yau threefolds.

5.1. Calabi-Yau threefolds of degree 18 via unprojection

Using the method of unprojection, we can also construct a non-Gorenstein projective threefold with one singular point with singularity locally isomorphic to the cone over a projected Del Pezzo surface of degree 9. More precisely, let us start with a Del Pezzo surface D_9^Λ from Proposition 3.1. It is contained in a complete intersection Y of two cubic hypersurfaces. Let E be the vector bundle on $\mathbb{P}^5$ defining D_9^Λ. Consider the non-Gorenstein unprojection of D_9^Λ in Y, *i.e.*, a threefold X defined as the degeneracy locus of a special skew-symmetric map between the sheaf $E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ and its twisted dual, as in the case of degree $d = 6$. Here, E' is the sheaf on $\mathbb{P}^6$ obtained as the trivial extension of E. In this case X is a threefold with one singular point such that the projection from this point is Y and the exceptional locus is D_9^Λ. Moreover, X has degree 18 and is birational to a Calabi-Yau threefold. Unfortunately, X has no smoothing.

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