

Curvature bounds via an isoperimetric comparison for Ricci flow on surfaces

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Abstract. A comparison theorem for the isoperimetric profile on the universal cover of surfaces evolving by normalised Ricci flow is proven. For any initial metric, a model comparison is constructed that initially lies below the profile of the initial metric and which converges to the profile of the constant curvature metric. The comparison theorem implies that the evolving metric is bounded below by the model comparison for all time and hence converges to the constant curvature profile. This yields a curvature bound and a bound on the isoperimetric constant, leading to a direct proof that the metric converges to the constant curvature metric.

Mathematics Subject Classification (2010): 53C44 (primary); 35K55, 58J35 (secondary).

1. Introduction

The Ricci flow is the nonlinear geometric parabolic evolution equation

$$\begin{cases} \frac{\partial}{\partial t} g &= -2 \operatorname{Ric}(t) \\ g(0) &= g_0 \end{cases} \quad (1.1)$$

for a smooth family of Riemannian metrics $g(t)$ on a smooth manifold M with Ricci curvature $\operatorname{Ric}(t)$ and an arbitrary smooth initial metric g_0 . Here we are interested in the case of closed surfaces, that is, 2-dimensional, compact manifolds M without boundary. The results here pertain to the *normalised* flow, preserving the 2-dimensional volume of M . After rescaling the initial metric to have volume 4π and applying the Gauss-Bonnet formula, the normalised flow on surfaces takes the form

$$\frac{\partial}{\partial t} g = -2(K - \bar{K})g \quad (1.2)$$

Received April 23, 2014; accepted in revised form March 31, 2015.

Published online September 2016.

where K is the Gaussian curvature and $\bar{K} = \frac{1}{4\pi} \int_M K \mu_g$ is the average Gauss curvature on M [1]. An important consequence of writing the equation in this way is that it may be lifted to the universal cover $\tilde{M} \rightarrow M$. That is, the pullback metric $\tilde{g}(t) = \pi^* g(t)$ also evolves according to equation (1.2).

The main theorem of this paper is a comparison theorem for the isoperimetric profile, I of a surface with metric evolving by the normalised Ricci flow, generalising the comparison theory in [1] for $M = \mathbb{S}^2$ to arbitrary closed surfaces. Recall, the isoperimetric profile is the least boundary area enclosing a given volume (see Section 2 for a precise definition).

Theorem 1.1 (Main Theorem 3.4). *Let $(M, g(t))$ be a Ricci flow of a closed surface of genus λ , and $(\tilde{M}, \tilde{g}(t))$ be the lift to the universal cover. Let $\phi : (0, |\tilde{M}|) \times [0, T) \rightarrow \mathbb{R}$ be a smooth, strictly positive, strictly concave function satisfying*

$$\frac{\partial \phi}{\partial t} \leq \phi'' \phi^2 - (\phi')^2 \phi + \phi' (4\pi - 2(1 - \lambda)a_0) + (1 - \lambda)\phi.$$

along with the asymptotic behaviour

$$\limsup_{a \rightarrow 0} \frac{\phi(a, t)}{\sqrt{4\pi a}} \leq 1$$

and

$$\limsup_{a \rightarrow \infty} (I(a, t) - \phi(a, t)) \geq 0.$$

If the initial inequality, $\phi(a, 0) < I_{\tilde{g}(0)}(a)$ for all $a \in (0, |\tilde{M}|)$ holds, then $\phi(a, t) \leq I_{\tilde{g}(t)}(a)$ for all a, t with strict inequality if the inequality in (3.7) is strict.

As an application, by a standard bootstrapping argument, a new, direct proof of the Hamilton-Chow Theorem [7, 14] (Theorem 1.2 below) is obtained immediately as a corollary of the comparison Theorem 3.4. This is achieved by suitable choices of comparison functions, leading to explicit curvature bounds and bounds on the isoperimetric constant as described in Sections 4 and 5. The proof, obtaining direct explicit curvature bounds by isoperimetric comparison is of a completely different character than the original proof which employed a Harnack inequality and entropy estimate. Also note here that we must rely on the Uniformisation Theorem for surfaces in the hyperbolic case $\lambda > 1$ in order to obtain the large scale asymptotics of the isoperimetric profile. In the case $\lambda = 0$, we do not require uniformisation since M is compact. In the case $\lambda = 1$, the result of [3] implies that

$$\limsup_{a \rightarrow \infty} \frac{I(a)}{\sqrt{a}} \leq C$$

with $0 < C \leq 4\pi$ and $C = 4\pi$ if and only if M is flat. This is precisely the required asymptotics in the theorem for $\lambda = 1$ surfaces obtained without requiring the use of the Uniformisation Theorem.

Theorem 1.2 ([7,14]). *Given any initial metric g_0 on a complete, two dimensional manifold M of genus λ , there exists a unique solution to the normalised Ricci flow existing for all time $t \in [0, \infty)$ and such that as $t \rightarrow \infty$, $g(t)$ converges smoothly to g_λ , the metric of constant curvature $K = 1 - \lambda$.*

Working in the other direction, the Hamilton-Chow Theorem in fact provides a proof of the Uniformisation Theorem since the Ricci flow preserves the conformal class of any initial metric g_0 . Thus for $\lambda = 0, 1$, the results here imply the Uniformisation Theorem, and for $\lambda > 1$, the missing piece is obtaining an analogous result to [3] controlling the large scale asymptotics of the isoperimetric profile for $\lambda > 1$ surfaces. The original proof of the Hamilton-Chow Theorem also relied on the Uniformisation Theorem. Some time later this requirement was removed in [10], which also contains further discussion of and references for the connection between Ricci flow and uniformisation.

Let us finish up this section with a brief outline of the structure of this paper. The use of the isoperimetric profile here is an extension from the 2-sphere to arbitrary surfaces of the results in [1], which in turn are based on the isoperimetric estimates in [15]. We begin in Section 2 with a treatment of the isoperimetric profile of surfaces, deriving a viscosity equation via variational techniques which forms the heart of the Comparison Theorem. The Comparison Theorem is proven in Section 3 by coupling the time-variation of the isoperimetric profile under the normalised Ricci flow with the spatial viscosity equation, yielding the parabolic version of the viscosity equation. Section 4 is devoted to constructing suitable model comparisons. The construction on the 2-sphere was given in [1] and is briefly described. Curiously, the most difficult case to deal with is for surfaces of genus $\lambda > 1$, which historically was perhaps the easiest case by applying the maximum principle and introducing a potential function [14]. For initially negatively curved surfaces however, the model given here is quite appealing. Finally in Section 5, the bootstrapping convergence argument is briefly described.

ACKNOWLEDGEMENTS. I would like to thank Professor Ben Andrews for his generous guidance and support supervising my Ph.D. thesis on which this paper is based. I would also like to thank Professor Gang Tian and BICMR for sponsoring a very enjoyable stay in Beijing where my thoughts on variational techniques were greatly clarified on the banks of the Weiming lake. Finally, I must thank the anonymous referees for providing helpful comments regarding the exposition. This paper was completed during my time as SEW Assistant Professor at UCSD.

2. The isoperimetric profile

2.1. Definition and basic properties

Definition 2.1. The *isoperimetric profile*, $I_M : (0, |M|) \rightarrow \mathbb{R}_+$ of M is defined by

$$I_M(a) = \inf \{ |\partial\Omega| : |\Omega| = a \}$$

where the infimum is taken over all relatively compact open sets Ω with smooth boundary. Such Ω are said to be *admissible regions*. If Ω is an admissible region such that $I_M(|\Omega|) = |\partial\Omega|$, we will call Ω an *isoperimetric region*.

A basic theorem we will assume here is that for every $a \in (0, |M|)$, there exists a corresponding isoperimetric region (with smooth boundary apart from a set of Hausdorff dimension at most $n - 7$ on an n dimensional manifold) provided M is either compact or co-compact. In particular, smooth isoperimetric regions exist on a closed surface and its universal cover equipped with the pull-back metric. The proof of this fact is a standard result of geometric measure theory [22, pages 128-129]. A simplified proof in the case of surfaces is given in [21] using regularity techniques developed in [17]. The more hardy reader may consult the original source, [11, Subsections 4.4.2, 5.1.6].

It will be important for us to understand the behaviour of the isoperimetric profile near the end points $\{0, |M|\}$. In the situation where M is compact, then the complement of an isoperimetric region is again an isoperimetric region, so the isoperimetric profile is symmetric about $|M|/2$ and it suffices to consider only the behaviour near 0. In the non-compact case, the behaviour near 0 is the same as for the compact case, so let us begin with the behaviour near 0.

Theorem 2.2. *Let M be a smooth Riemannian surface without boundary and such that $\sup_M K < \infty$. Then the isoperimetric profile satisfies*

$$I(a) = \sqrt{4\pi a} - \frac{\sup_M K}{4\sqrt{\pi}} a^{3/2} + O(a^{5/2}) \quad \text{as } a \rightarrow 0.$$

Proof. Small geodesic balls about any point p are admissible regions. The result of [12, Theorem 3.1] gives $|B_r(p)| = \pi r^2(1 - \frac{K(p)}{12}r^2 + O(r^4))$ and $|\partial B_r(p)| = 2\pi r(1 - \frac{K(p)}{6}r^2 + O(r^4))$. The upper bound follows since $|\partial B_r(p)| \geq I(|B_r(p)|)$.

To prove the lower bound, first choose a_0 sufficiently small to ensure that $I(a_0)$ is much smaller than the injectivity radius of M . Then an isoperimetric region Ω_0 corresponding to a_0 lies inside a geodesic ball about some point p (width is bounded above by perimeter for surfaces). Since geodesic balls are simply connected and $K \leq K_0 = \sup_M K$, the Bol-Fiala inequality (see [23]) then gives

$$I(a_0) \geq \sqrt{4\pi a_0 - K_0 a_0^2} = \sqrt{4\pi a_0} - \frac{K_0}{4\sqrt{\pi}} a_0^{3/2} + O(a_0^2). \quad \square$$

Next, we have the asymptotics of the isoperimetric profile near ∞ for non-compact \tilde{M} .

Theorem 2.3. *Let M be a closed, genus $\lambda \geq 1$ surface with metric g , normalised to have $|M| = 4\pi$ and let $\pi : \tilde{M} \rightarrow M$ be the universal cover of M equipped with the pull-back metric $\tilde{g} = \pi^* g$. Then*

$$\limsup_{a \rightarrow \infty} \frac{I_{\tilde{g}}(a)}{\sqrt{4\pi a - (1 - \lambda)a^2}} \leq C$$

for some $C > 0$ depending on M and g_0 .

Proof. By the Uniformisation Theorem (see, e.g., [18, Chapter 3]), \tilde{g} is conformal to a metric of constant curvature so that

$$\tilde{g} = \phi g_{1-\lambda}$$

with $g_{1-\lambda}$ the metric of constant curvature $1 - \lambda$ and ϕ a positive function $\phi : \tilde{M} \rightarrow \mathbb{R}$ invariant under the deck transformation group of \tilde{M} . In particular, ϕ is uniformly bounded above and below.

The isoperimetric inequality for simply connected Riemannian surfaces of constant curvature $1 - \lambda$ implies that the isoperimetric profile $I_{1-\lambda}$ of the constant curvature metric $g_{1-\lambda}$ is given by

$$I_{1-\lambda}(a) = \sqrt{4\pi a - (1 - \lambda)a^2}.$$

Since \tilde{g} is conformal to $g_{1-\lambda}$ with conformal factor ϕ uniformly bounded, we have

$$\begin{aligned} \frac{1}{C_1} |\partial\Omega|_{g_{1-\lambda}} &\leq |\partial\Omega|_{\tilde{g}} \leq C_1 |\partial\Omega|_{g_{1-\lambda}} \\ \frac{1}{C_2} |\Omega|_{g_{1-\lambda}} &\leq |\Omega|_{\tilde{g}} \leq C_2 |\Omega|_{g_{1-\lambda}} \end{aligned}$$

for constants $C_1, C_2 > 0$ depending on M and g_0 . This gives the result with C dependent only on C_1, C_2 . □

2.2. Variational formulae and consequences

Our techniques are based on applying the standard variational formula for isoperimetric regions, with a slight change in the second variation, obtained by applying the Gauss-Bonnet Theorem. Let us briefly recall the applicable variational formulae and describe the approach used here in obtaining the second variation.

Let Ω_0 be an isoperimetric region and Ω_ϵ a smooth normal variation with variational vector field $\eta\vec{n}$ for a smooth function $\eta : \partial\Omega_0 \rightarrow \mathbb{R}$. The first variation formulae are

$$\frac{\partial}{\partial\epsilon} |\partial\Omega_\epsilon| = \int_{\partial\Omega_\epsilon} \eta\kappa \tag{2.1}$$

and

$$\frac{\partial}{\partial\epsilon} |\Omega_\epsilon| = \int_{\partial\Omega_\epsilon} \eta \tag{2.2}$$

where κ is the geodesic curvature of $\partial\Omega_\epsilon$. In particular, the vanishing of the first variation for all functions η such that $\int_{\partial\Omega_\epsilon} \eta = 0$ (area preserving variations) implies that κ is constant.

For the second variation, we need only to consider unit-speed variations ($\eta \equiv 1$) and so immediately conclude the second variation for area,

$$\frac{\partial^2}{\partial \epsilon^2} |\Omega_\epsilon| = \int_{\partial \Omega_\epsilon} \kappa. \tag{2.3}$$

For the second variation of boundary length, it suits our purposes to first apply the Gauss-Bonnet formula and then differentiate equation (2.1). Thus

$$\begin{aligned} \frac{\partial^2}{\partial \epsilon^2} |\partial \Omega_\epsilon| &= \frac{\partial}{\partial \epsilon} \int_{\partial \Omega_\epsilon} \kappa \\ &= \frac{\partial}{\partial \epsilon} \left(2\pi \chi_{\Omega(\epsilon)} - \int_{\Omega(\epsilon)} K_M \right) \end{aligned}$$

where K_M is the Gauss curvature of M , and χ_{Ω_ϵ} is the Euler characteristic of Ω_ϵ which is independent of ϵ since by the definition of smooth variations, each Ω_ϵ is diffeomorphic to Ω_0 . As K_M has no explicit dependence on ϵ , the Reynold's Transport Theorem (or differentiating under the integral sign) yields

$$\frac{\partial^2}{\partial \epsilon^2} |\partial \Omega_\epsilon| = - \int_{\partial \Omega_\epsilon} K_M. \tag{2.4}$$

Our approach is based on weak differential inequalities for the isoperimetric profile arising from the variational formulae.

Definition 2.4. A function $f : (a, b) \rightarrow \mathbb{R}$ has weak derivatives satisfying

$$\frac{\partial f^-}{\partial x} \leq C_1 \leq \frac{\partial f^+}{\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} \leq C_2$$

in the *support* (or sometimes *Calabi*) sense at x_0 if f supports a smooth function ϕ at x_0 ($f(x_0) = \phi(x_0)$ and $f(x) \leq \phi(x)$ for x near x_0) such that

$$\frac{\partial \phi}{\partial x}(x_0) = C_1 \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2}(x_0) = C_2.$$

Proposition 2.5 ([4] (see also [6, pages 249–251])). *For each $a_0 \in (0, |M|)$, let Ω_0 be a corresponding isoperimetric region with constant curvature $\kappa(a_0)$ along the boundary. Then the isoperimetric profile satisfies*

$$\frac{\partial I^-}{\partial a} \leq \kappa(a_0) \leq \frac{\partial I^+}{\partial a} \quad \text{and} \quad \frac{\partial^2 I}{\partial a^2} \leq -\frac{1}{I^2} \left(\kappa(a_0)^2 I + \int_{\partial \Omega_0} K_M \right)$$

in the *support* sense. Moreover, if $K_M \geq K_0$, the function

$$a \mapsto I(a)^2 + K_0 a^2$$

is concave, hence I^2 is locally Lipschitz and in particular I is continuous.

Corollary 2.6. *With the notation of the proposition, if $K_0 \geq 0$ then I is concave and so is I^2 too. If the inequality is strict, then I and I^2 are strictly concave.*

The last results of this section concern the topology of isoperimetric regions. We generally do not have a priori control over the topology of isoperimetric regions and so we do not know the precise form of the differential inequality for I because of the integral over the unknown regions Ω_0 . However, there is a useful sufficient condition for obtaining control of the topology of isoperimetric regions. The idea comes from [25].

Lemma 2.7. *Let $a_0 \in (0, |M|)$ and Ω_0 a corresponding isoperimetric region. If there exists a strictly positive, strictly concave function $\phi : (0, |M|) \rightarrow \mathbb{R}$ supporting I at a_0 ($\phi(a_0) = I(a_0)$ and $\phi(a) \leq I(a)$ for all $a \in (0, |M|)$), then Ω_0 is connected. If M is compact then Ω_0 has connected complement.*

Remark 2.8. It is worth pointing out that while the conclusion of the lemma is local, pertaining to a particular value of a_0 and corresponding isoperimetric region, the hypotheses are global in nature in that we need a *globally defined* supporting function ϕ (not just in a neighbourhood of a_0).

Proof. First note that since $\phi > 0$ on $(0, |M|)$, $\phi \leq I$, and $I(0) = 0$, we have $\phi(0) = 0$. Thus since ϕ is strictly concave, ϕ is strictly subadditive.

Now suppose Ω_0 is not connected. Then we can write $\Omega_0 = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$. Since $\partial\Omega_0$ is smooth we must have $\partial\Omega_0 = \partial\Omega_1 \cup \partial\Omega_2$ and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$. Thus we have $|\Omega_0| = |\Omega_1| + |\Omega_2|$ and $|\partial\Omega_0| = |\partial\Omega_1| + |\partial\Omega_2|$, and all of these are non-zero. But then we get

$$\begin{aligned} \phi(|\Omega_1|) + \phi(|\Omega_2|) &\leq |\partial\Omega_1| + |\partial\Omega_2| \\ &= |\partial\Omega_0| \\ &= \phi(|\Omega_0|) \\ &= \phi(|\Omega_1| + |\Omega_2|) \\ &< \phi(|\Omega_1|) + \phi(|\Omega_2|). \end{aligned}$$

This is a contradiction, so Ω_0 is connected.

If M is compact, then $M \setminus \Omega_0$ is also an isoperimetric region with $|M \setminus \Omega_0| = |M| - |\Omega_0| = |M| - a_0$ and $I(|M| - a_0) = |\partial M \setminus \Omega_0| = |\partial\Omega_0| = I(a_0)$. Reflecting ϕ about $a = |M|/2$ gives a function satisfying the hypothesis of the proposition at $|M| - a_0$ hence $M \setminus \Omega_0$ is also connected. □

Corollary 2.9. *With the hypothesis of Lemma 2.7, if M is diffeomorphic to \mathbb{S}^2 then Ω_0 is simply connected.*

Proof. Follows from the Jordan Curve Theorem for \mathbb{S}^2 . □

Corollary 2.10. *With the hypothesis of Lemma 2.7, if M is diffeomorphic to \mathbb{R}^2 then Ω_0 is simply connected.*

Proof. Since \mathbb{R}^2 is not compact, we cannot immediately conclude that Ω_0 has connected complement as before. To achieve this result, first note that if M is \mathbb{R}^2 , then ϕ is a (strictly) positive concave function on $(0, \infty)$ and hence is strictly increasing. Since Ω_0 is connected, topologically it is a disc with finitely many discs removed. Let Ω_1 denote the interior of the external boundary of Ω_0 , *i.e.*, Ω_1 is equal to Ω_0 with the “holes” filled in. Then Ω_1 has strictly larger area than Ω_0 and strictly smaller boundary length. But then

$$\phi(|\Omega_0|) = I(|\Omega_0|) = |\partial\Omega_0| > |\partial\Omega_1| \geq I(|\Omega_1|) \geq \phi(|\Omega_1|)$$

contradicting that ϕ is increasing. Therefore $\mathbb{R}^2 \setminus \Omega_0$ is connected and now the Jordan Curve Theorem implies Ω_0 is simply connected. \square

Let us finish by noting that in positive curvature, we have complete knowledge of the topology of isoperimetric regions.

Corollary 2.11. *If M is diffeomorphic to either \mathbb{S}^2 or \mathbb{R}^2 (for instance if M is the universal cover of a closed surface) equipped with any metric (not necessarily the pull-back from a compact surface) and $K_0 > 0$, then all isoperimetric regions are simply connected.*

Proof. By Corollary 2.6, I is strictly concave so the hypotheses of Corollaries 2.9 and 2.10 are satisfied at any $a_0 \in (0, |M|)$ by choosing $\phi = I$ itself. \square

2.3. A viscosity equation for the isoperimetric profile

The results in this section formalise some of the ideas used in [1]. We obtain a differential inequality in the viscosity sense for the isoperimetric profile of a surface. This is somewhat dual to the results in the previous section and those in [2] in that we assert conditions which *lower* supporting functions must satisfy as opposed to the aforementioned results which assert the existence of an *upper* supporting function with bounds on the derivatives. The methods, however, are essentially the same and the support inequality implies the viscosity inequality. As the isoperimetric profile is defined as an extrema, viscosity equations turn out to be well suited to this situation. Indeed, viscosity equations were introduced in [9] to study Hamilton-Jacobi equations, also arising from optimisation problems. A central feature of viscosity equations, that forms the basis of the Comparison Theorem 3.4, is that they enjoy a maximum principle. See [5] for more on viscosity equations.

Definition 2.12. A lower semi-continuous function $f : (a, b) \rightarrow \mathbb{R}$ is a *viscosity super-solution* of the second order differential equation

$$F(x, f, f', f'') = 0$$

for F and elliptic operator, if for every $x_0 \in (a, b)$ and every C^2 function ϕ such that $\phi(x_0) = f(x_0)$ and $\phi(x) \leq f(x)$ in a neighbourhood of x_0 , we have $F(x_0, \phi(x_0), \phi'(x_0), \phi''(x_0)) \geq 0$. An upper semi-continuous function is a *viscosity sub-solution* if the same statements hold with all the inequalities reversed.

For f a viscosity super(sub)-solution of $F(x, f, f', f'') = 0$, we will abuse notation slightly and write $F(x, f, f', f'') \geq 0(\leq 0)$ (in the viscosity sense).

Theorem 2.13. *The isoperimetric profile is a viscosity super-solution of*

$$-\left(I'' I^2 + (I')^2 I + \int_{\partial\Omega_0} K_M\right) = 0 \tag{2.5}$$

where Ω_0 is any isoperimetric region corresponding to $a_0 - |\Omega_0| = a_0$ and $I(a_0) = |\partial\Omega_0| -$ and K_M is the Gauss curvature of M .

In particular, if $K_M \geq K_0$ is bounded below on M , then

$$-\left(I'' I^2 + (I')^2 I + K_0 I\right) \geq 0$$

in the viscosity sense.

Remark 2.14. The integral term in the first equation is difficult to deal with; even though the Gauss curvature K is a given function on the ambient space M , we do not have any a-priori knowledge of Ω_0 . Nevertheless, the first form will be the most useful to us when considering the Ricci flow, since the integral term will also appear in the time variation of isoperimetric regions under the Ricci flow allowing us to connect the spatial variational formulae with the time variational formulae.

Proof. The isoperimetric profile is continuous by Proposition 2.5 and the remark following it.

Let ϕ be a smooth function defined on a neighbourhood of $a_0 \in (0, |M|)$ such that $\phi \leq I$ and $\phi(a_0) = I(a_0)$. Let Ω_0 be an isoperimetric region corresponding to a_0 . Choose a unit speed normal variation of $\partial\Omega_0$ and define

$$f(\epsilon) = |\partial\Omega_\epsilon| - \phi(|\Omega_\epsilon|).$$

Then we have

$$f(\epsilon) \geq I(|\Omega_\epsilon|) - \phi(|\Omega_\epsilon|) \geq 0$$

and

$$f(0) = |\partial\Omega_0| - \phi(|\Omega_0|) = I(|\Omega_0|) - \phi(|\Omega_0|) = 0.$$

Thus 0 is a minima of f so that $\partial f / \partial \epsilon(0) = 0$ and $\partial^2 f / \partial \epsilon^2(0) \geq 0$. Now we use the first variation formula to compute

$$\frac{\partial f}{\partial \epsilon} = \int_{\partial\Omega_\epsilon} \kappa - \phi' |\partial\Omega_\epsilon|$$

which at $\epsilon = 0$ gives

$$0 = \int_{\partial\Omega_0} \kappa - \phi'(a_0) |\partial\Omega_0| = (\kappa - \phi'(a_0)) |\partial\Omega_0|$$

since κ is constant along $\partial\Omega_0$. Thus $\kappa = \phi'(a_0)$ along $\partial\Omega_0$.

The second variation gives

$$\begin{aligned} 0 &\leq \frac{\partial^2 f}{\partial \epsilon^2} = \frac{\partial^2}{\partial \epsilon^2} |\partial \Omega_\epsilon| - \phi'' \left(\frac{\partial}{\partial \epsilon} |\Omega_\epsilon| \right)^2 - \phi' \frac{\partial^2}{\partial \epsilon^2} |\Omega_\epsilon| \\ &= - \int_{\partial \Omega_\epsilon} K - \phi'' (|\partial \Omega_\epsilon|)^2 - \phi' \int_{\partial \Omega_\epsilon} \kappa \\ &= - \int_{\partial \Omega_0} K - \phi''(a_0) \phi^2(a_0) - (\phi')^2(a_0) \phi(a_0). \end{aligned}$$

Recalling that $\phi(a_0) = |\Omega_0|$ and using that $\kappa = \phi'(a_0)$ along $\partial \Omega_0$ as just obtained from the first variation. □

3. A comparison theorem

3.1. Comparison equation under the Ricci flow

Let us now couple the spatial viscosity equation with the Ricci flow. For this we need to know the time-variation of isoperimetric regions under the Ricci flow. It is quite remarkable that this is possible at all and heavily relies on the fact that M is 2-dimensional. It would be interesting to see if similar results hold in higher dimensions, though this seems unlikely unless some topological and/or curvature restrictions are imposed.

We first need the parabolic version of viscosity equations.

Definition 3.1. A lower semi-continuous function $f : (a, b) \times [0, T) \rightarrow \mathbb{R}$ is a *viscosity super-solution* of the second order parabolic equation

$$\frac{\partial f}{\partial t} + A(x, t, f, f', f'') = 0$$

if for every $(x_0, t_0) \in (a, b) \times [0, T)$ and every C^2 function ϕ such that $\phi(x_0, t_0) = f(x_0, t_0)$ and $\phi(x, t) \leq f(x, t)$ for x in a neighbourhood of x_0 and $t \leq t_0$ near t_0 , we have $\frac{\partial \phi}{\partial t}(x_0, t_0) + A(x_0, t_0, \phi, \phi', \phi'') \geq 0$. An upper semi-continuous function is a *viscosity sub-solution* if the same statements hold with the inequalities reversed.

Theorem 3.2. Let M be a closed surface of genus λ , $g(t)$ a solution of the normalised Ricci flow on M and $\tilde{g}(t) = \pi^* g(t)$ the corresponding solution on the universal cover $\pi : \tilde{M} \rightarrow M$. For any a_0 , let χ_0 be the Euler characteristic of Ω_0 an isoperimetric region corresponding to a_0 . Then the isoperimetric profile, $I_{\tilde{g}(t)}$ satisfies

$$\frac{\partial}{\partial t} I - \left[I'' I^2 + (I')^2 I + (4\pi \chi_0 - 2(1 - \lambda)a) I' + (1 - \lambda) I \right] \geq 0 \tag{3.1}$$

in the viscosity sense.

Proof. For convenience sake, let us write $|\cdot|_t = |\cdot|_{\tilde{g}(t)}$ and $I_t = I_{\tilde{g}(t)}$. Let ϕ be a C^2 function such that $\phi(a_0, t_0) = I_{t_0}(a_0)$ and $\phi \leq I$ for a near a_0 and $t \leq t_0$ near t_0 . We need to show that ϕ satisfies the differential inequality (3.1).

We compute the time variation of isoperimetric regions. Given a_0 , let $\Omega_0 \subset \tilde{M}$ be an isoperimetric region in \tilde{M} with respect to the metric $\tilde{g}(t_0)$. That is $|\Omega_0|_{t_0} = a_0$ and $|\partial\Omega_0|_{t_0} = \phi(a_0, t_0)$. Since $I_t(a) \geq \phi(a, t)$ for $t \leq t_0$ and a near a_0 , we have

$$|\partial\Omega_0|_t \geq \phi(|\Omega_0|_t, t)$$

for $t \leq t_0$, and equality holds when $t = t_0$. Since both sides of this equation are differentiable in t , it follows that under the normalised Ricci flow,

$$\frac{\partial}{\partial t} \Big|_{t=t_0} |\partial\Omega_0|_t \leq \frac{\partial\phi}{\partial t}(a_0, t_0) + \phi'(a_0, t_0) \frac{\partial}{\partial t} \Big|_{t=t_0} |\Omega_0|_t. \tag{3.2}$$

The time derivative on the left can be computed as follows: parametrise $\partial\Omega_0$ by $\gamma : u \in \mathbb{S}^1 \mapsto M$ and write $\gamma_u = \gamma_* \frac{\partial}{\partial u}$. Then recalling that the metric evolves by the normalised Ricci flow, $\frac{\partial}{\partial t} \tilde{g} = -2(K - (1 - \lambda))\tilde{g}$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=t_0} |\partial\Omega_0| &= \frac{\partial}{\partial t} \int_{\partial\Omega_0} \sqrt{\tilde{g}_t(\gamma_u, \gamma_u)} du = - \int_{\partial\Omega_0} (K_M - (1 - \lambda)) ds \\ &= - \int_{\partial\Omega_0} K_M ds + (1 - \lambda)\phi(a_0, t_0), \end{aligned}$$

where ds is the arc-length element along $\partial\Omega_0$.

For the right-hand side, by differentiating the determinant and using the normalised Ricci flow equation again, we have $\frac{\partial}{\partial t} \mu_{\tilde{g}} = -2(K_M - (1 - \lambda))\mu_{\tilde{g}}$ where $\mu_{\tilde{g}}$ is the measure on \tilde{M} induced by the metric \tilde{g} . Thus,

$$\frac{\partial}{\partial t} \Big|_{t=t_0} |\Omega_0|_t = -2 \int_{\Omega_0} (K_M - (1 - \lambda)) d\mu_{\tilde{g}(t_0)}.$$

Writing $\chi_0 = \chi(\Omega_0)$ the Euler characteristic of Ω_0 and applying the Gauss-Bonnet Theorem yields

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=t_0} |\Omega_0|_t &= 2(1 - \lambda) |\Omega_0| - 2 \left(2\pi \chi_0 - \int_{\partial\Omega_0} \kappa ds \right) \\ &= 2(1 - \lambda)a_0 - 4\pi \chi_0 + 2 \int_{\partial\Omega_0} \kappa ds, \end{aligned}$$

where κ is the geodesic curvature of the curve $\partial\Omega_0$. Thus the inequality (3.2) becomes

$$- \int_{\partial\Omega_0} K_M ds + (1 - \lambda)\phi \leq \frac{\partial}{\partial t} \phi + \phi' \left(2(1 - \lambda)a_0 - 4\pi \chi_0 + 2 \int_{\partial\Omega_0} \kappa ds \right). \tag{3.3}$$

Now recall that Theorem 2.13 states that for each time t , the isoperimetric profile I_t satisfies

$$-\left(I'' I^2 + (I')^2 I + \int_{\partial\Omega_0} K_M\right) \geq 0$$

in the viscosity sense. Since at a_0 , $\phi(-, t_0)$ is a supporting function for $I_{t_0}(-)$ we also have

$$\phi'' \phi^2 + (\phi')^2 \phi \leq - \int_{\partial\Omega_0} K_M. \tag{3.4}$$

Also, the vanishing of the first spatial variation gives $\kappa = \phi'(a_0)$ is constant along $\partial\Omega_0$ and so

$$\int_{\partial\Omega_0} \kappa ds = \phi(a_0)\phi'(a_0). \tag{3.5}$$

Putting together the inequalities (3.3) and (3.4) and using (3.5) we obtain

$$\begin{aligned} \frac{\partial\phi}{\partial t} &\geq - \int_{\partial\Omega_0} K_M ds + (1 - \lambda)\phi - \phi' (2(1 - \lambda)a_0 - 4\pi \chi_0 + 2\phi\phi') \\ &\geq \phi'' \phi^2 + (\phi')^2 \phi + (1 - \lambda)\phi + \phi' (4\pi \chi_0 - 2(1 - \lambda)a_0) - 2\phi(\phi')^2 \\ &= \phi'' \phi^2 - (\phi')^2 \phi + (1 - \lambda)\phi + \phi' (4\pi \chi_0 - 2(1 - \lambda)a_0) \end{aligned} \tag{3.6}$$

which is the required inequality. □

Remark 3.3. The viscosity equation includes the χ_0 term which, without any topological knowledge of isoperimetric regions, is essentially unknown and could, a priori, take on any possible value. By Corollary 2.11, in the particular case that $K_M > 0$, we may conclude that $\chi_0 = 1$ for all a_0 . In general however, we need not to expect any particular bound on Euler characteristic from a curvature bound alone.

Even though the topological uncertainty is a real problem, for our purposes we may avoid it entirely by appealing to the underlying concavity of the isoperimetric profile. This is exploited in the next theorem, the Comparison Theorem, which is the central result of this paper.

Theorem 3.4. *Let $(M, g(t))$, $(\tilde{M}, \tilde{g}(t))$ be as in the previous theorem. Let $\phi : (0, |\tilde{M}|) \times [0, T) \rightarrow \mathbb{R}$ be a smooth, strictly positive, strictly concave function satisfying*

$$\frac{\partial\phi}{\partial t} \leq \phi'' \phi^2 - (\phi')^2 \phi + \phi' (4\pi - 2(1 - \lambda)a_0) + (1 - \lambda)\phi \tag{3.7}$$

along with the asymptotic behaviour

$$\limsup_{a \rightarrow 0} \frac{\phi(a, t)}{\sqrt{4\pi a}} \leq 1$$

and

$$\limsup_{a \rightarrow \infty} (I(a, t) - \phi(a, t)) \geq 0.$$

Then if the initial inequality, $\phi(a, 0) < I_{\tilde{g}(0)}(a)$ for all $a \in (0, |\tilde{M}|)$ holds, the inequality $\phi(a, t) \leq I_{\tilde{g}(t)}(a)$ holds for all a, t with strict inequality if the inequality in (3.7) is strict.

Remark 3.5. The large scale asymptotic requirements are rather imprecise because we do not have a priori control over the constant C in Theorem 2.3. However, this will not prove problematic for us by Proposition 3.10 below.

Proof. First suppose that we have strict inequality in the differential inequality and in the asymptotic inequalities. We argue by contradiction. The conditions $\phi(a, 0) < I_{\tilde{g}(0)}(a)$ and $\phi(a, t) < I_{\tilde{g}(t)}(a)$ for a sufficiently close to $\{0, |\tilde{M}|\}$ imply that, if the theorem is false, there is a first time $t_0 > 0$ and an $a_0 \in (0, |\tilde{M}|)$ such that $\phi(a_0, t_0) = I_{t_0}(a_0)$. Thus $\phi(a, t) \leq I_t(a)$ for $t \leq t_0$ with equality at (a_0, t_0) . Since ϕ is strictly concave, the hypotheses of Lemma 2.7 are satisfied, so Ω_0 is simply connected and $\chi_0 = 1$.

But now observe that ϕ is a lower supporting function for I_t at a_0 and by Theorem 3.2,

$$\frac{\partial \phi}{\partial t} \geq \phi'' \phi^2 - (\phi')^2 \phi + \phi' (4\pi - 2(1 - \lambda)a_0) + (1 - \lambda)\phi$$

a contradiction, hence the theorem is true when the inequalities are strict.

If any of the inequalities is not strict, define

$$\phi_\epsilon = (1 - \epsilon)\phi$$

for any ϵ with $0 < \epsilon < 1$. Then we have $\phi_\epsilon < \phi$ giving strict inequality for the asymptotics. We also have

$$\begin{aligned} & \frac{\partial \phi_\epsilon}{\partial t} - (\phi''_\epsilon \phi_\epsilon^2 - (\phi'_\epsilon)^2 \phi_\epsilon) - \phi'_\epsilon (4\pi - 2(1 - \lambda)a_0) - (1 - \lambda)\phi_\epsilon \\ &= (1 - \epsilon) \left(\frac{\partial \phi}{\partial t} - (1 - \epsilon)^2 (\phi'' \phi^2 - (\phi')^2 \phi) - \phi' (4\pi - 2(1 - \lambda)a_0) - (1 - \lambda)\phi \right) \\ &\leq \epsilon(1 - \epsilon)(2 - \epsilon)(\phi^2 \phi'' - (\phi')^2 \phi) \\ &< 0 \end{aligned}$$

since $\phi'' < 0$.

Thus $\phi_\epsilon(a, t) < I(a, t)$ by the result for strict inequalities and the result follows by letting $\epsilon \rightarrow 0$. □

Remark 3.6. It is not entirely clear whether strict concavity may be relaxed to mere concavity. A strictly concave approximation to ϕ may increase ϕ violating the inequality $\phi \leq I$.

Using the theorem, and the asymptotics of I from Theorem 2.2,

$$I(a) = \sqrt{4\pi a} \left(1 - \frac{\sup_M K}{8\pi} a + O(a^2) \right)$$

as $a \rightarrow 0$, we may now obtain a curvature bound for $\tilde{g}(t)$ and hence for $g(t)$.

Corollary 3.7. *With the notation of the previous theorem, ϕ satisfying the hypothesis of the theorem and such that*

$$\phi(a, t) = \sqrt{4\pi a} \left(1 - \frac{K_0(t)}{8\pi} a + O(a^2) \right),$$

we have

$$\sup_M K_M(t) \leq K_0(t). \quad \square$$

The isoperimetric constant of a non-compact surface is defined to be

$$\mathcal{I} = \inf \left\{ \frac{|\partial\Omega|^2}{|\Omega|} : \Omega \text{ admissible} \right\} = \inf \left\{ \frac{I(a)^2}{a} : 0 < a < \infty \right\}.$$

For a compact surface, the (modified) isoperimetric constant is defined by

$$\mathcal{I} = \inf \left\{ \frac{|\partial\Omega|^2}{\min(|\Omega|, |M \setminus \Omega|)} : \Omega \text{ admissible} \right\} = \inf \left\{ \frac{I(a)^2}{a} : 0 < a < \frac{|M|}{2} \right\}.$$

Corollary 3.8. *With the notation of the previous theorem and, ϕ satisfying the hypothesis of the theorem we have*

$$\mathcal{I}_{\tilde{M}}(t) \geq \inf \left\{ \frac{\phi(a, t)^2}{a} : 0 < a < \frac{|\tilde{M}|}{2} \right\}.$$

Remark 3.9. Note that area (2 dimensional volume) on \tilde{M} equipped with the pull-back metric $\pi^* g$ grows like the growth of the fundamental group, but boundary length cannot be controlled so easily. For instance, the torus with arbitrarily small ratio of principal radii may be equipped with the flat metric, giving control of the isoperimetric constant on \tilde{M} , but with arbitrarily small isoperimetric constant on M . For the matter at hand, when $\lambda > 0$ (so that \tilde{M} is not compact), we cannot immediately transfer control of the isoperimetric constant on \tilde{M} to control of the isoperimetric constant on M .

Let us finish this section by recording a useful result for surfaces of genus $\lambda \geq 1$ that shows the large scale asymptotics of ϕ are superfluous.

Proposition 3.10. *Let M be a closed surface of genus ≥ 1 (so that \tilde{M} is not compact). Let ϕ be a strictly positive, strictly concave function satisfying the differential inequality (3.7) and the small scale asymptotics from the Comparison Theorem 3.4. Then if $\phi(a, 0) < I_{\tilde{g}(0)}(a)$ for all $a \in (0, \infty)$, then $\phi(a, t) \leq I_{\tilde{g}(t)}(a)$ for all a, t .*

Proof. The only thing missing from Theorem 3.4 is the large scale asymptotics. It is convenient to work with the function $v = \phi^2$. This satisfies

$$\frac{\partial v}{\partial t} \leq v^2 \Delta \ln v + (4\pi - 2(1 - \lambda)a)v' + 2(1 - \lambda)v. \tag{3.8}$$

For any $C > 0$, define

$$u_C(a, t) = Ce^{2(1-\lambda)t}.$$

Then u_C satisfies equality in equation (3.8). Since u_C is constant for each fixed t and I grows at least linearly as $a \rightarrow \infty$ by Theorem 2.3, we have also have $u_C(a, t) < I_{\tilde{g}(t)}(a)$ for all a large enough. Now take the harmonic mean,

$$H(a, t) = \left(\frac{1}{v(a, t)} + \frac{1}{u_C(a, t)} \right)^{-1}.$$

This has the property that for any (a, t) we have

$$v(a, t) = \lim_{C \rightarrow \infty} \frac{v(a, t)u_C(a, t)}{v(a, t) + u_C(a, t)} = \lim_{C \rightarrow \infty} H(a, t).$$

Therefore to prove the result, we need to show H satisfies the hypotheses of Theorem 3.4 since this will give the inequality for H for every $C > 0$ and so also for v being the limit $C \rightarrow \infty$ of H .

First, since $0 < H \leq v, u_C$, the initial inequality $H < I_0$ is satisfied along with the necessary small and large scale asymptotics.

For strict concavity of H , we use

$$H = \frac{u_C v}{u_C + v}$$

and $(u_C)' = 0$ to compute

$$H' = \frac{u_C v'}{u_C + v} - \frac{u_C v v'}{(u_C + v)^2} = \frac{u_C^2 v'}{(u_C + v)^2}$$

and so

$$H'' = \frac{u_C^2 v''}{(u_C + v)^2} - \frac{2u_C^2 (v')^2}{(u_C + v)^3} < 0$$

by strict concavity of v and positivity of v, u_C . Thus H is strictly concave.

Now let us consider the differential inequality. Define

$$L_{\pm} = (4\pi - 2(1 - \lambda)a) \frac{\partial}{\partial a} \pm 2(1 - \lambda).$$

The differential inequality (3.8) then reads

$$\left(\frac{\partial}{\partial t} - L_{-} \right) v \leq v^2 \Delta \ln v.$$

For any function f we have

$$\left(\frac{\partial}{\partial t} - L_{\pm} \right) \frac{1}{f} = -\frac{1}{f^2} \left(\frac{\partial}{\partial t} - L_{\mp} \right) f. \quad (3.9)$$

Applying equation (3.9) to $H = 1/f$ with $f = v^{-1} + u_C^{-1}$ gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} - L_{-} \right) H &= -H^2 \left(\left(\frac{\partial}{\partial t} - L_{+} \right) \frac{1}{v} + \left(\frac{\partial}{\partial t} - L_{+} \right) \frac{1}{u_C} \right) \\ &= -H^2 \left(\left(\frac{\partial}{\partial t} - L_{+} \right) \frac{1}{v} - 2(1 - \lambda) \frac{1}{u_C} \right) \end{aligned}$$

since $L_{\pm}(u_C) = 0$ and u_C satisfies equality in (3.8).

Next applying equation (3.9) to v^{-1} we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} - L_{-} \right) H &= \frac{H^2}{v^2} \left(\frac{\partial}{\partial t} - L_{-} \right) v + 2H^2(1 - \lambda) \frac{1}{u_C} \\ &\leq \frac{H^2}{v^2} v^2 \Delta \ln v \\ &= H^2 \Delta \ln v, \end{aligned}$$

since $(1 - \lambda) \leq 0$. Here the inequality is strict if v (or equivalently u) satisfies strict inequality in the differential inequality. We want to show the right-hand side is less than or equal to $H^2 \Delta \ln H$. We compute

$$\begin{aligned} H^2 \Delta \ln H &= H H'' - (H')^2 \\ &= \frac{v u_C}{v + u_C} \left[\left(\frac{u_C}{v + u_C} \right)^2 v'' - \frac{2u_C^2}{(v + u_C)^3} (v')^2 \right] - \left(\frac{u_C}{v + u_C} \right)^4 (v')^2 \\ &= \left(\frac{v u_C}{v + u_C} \right)^2 \left[\left(\frac{u_C}{v + u_C} \right) \frac{v''}{v} - \frac{2u_C v}{(v + u_C)^2} \frac{(v')^2}{v^2} - \frac{u_C^2}{(v + u_C)^2} \frac{(v')^2}{v^2} \right] \\ &= H^2 \left[\left(\frac{u_C}{v + u_C} \right) \frac{v''}{v} - \left(\frac{(v + u_C)^2 - v^2}{(v + u_C)^2} \right) \frac{(v')^2}{v^2} \right] \\ &\geq H^2 \left[\frac{v''}{v} - \frac{(v')^2}{v^2} \right] = H^2 \Delta \ln v \end{aligned}$$

where the inequality follows from the concavity of v and the positivity of v and u_C . □

3.2. A connection with logarithmic porous media

For positive functions ϕ , the differential inequality

$$\frac{\partial \phi}{\partial t} < \phi^2 \phi'' - \phi(\phi')^2 + (4\pi - 2(1 - \lambda)a)\phi' + (1 - \lambda)\phi$$

is equivalent to the logarithmic porous media inequality

$$\frac{\partial u}{\partial t} > \Delta \ln u.$$

To see this, observe that

$$\phi^3 \Delta \ln \phi = \phi^2 \phi'' - \phi(\phi')^2.$$

Letting $u = \phi^{-2}$ we have $\Delta \ln u = -2\Delta \ln \phi$ and so

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{-2}{\phi^3} \frac{\partial \phi}{\partial t} > \frac{-2}{\phi^3} \left[\phi^3 \Delta \ln \phi + (4\pi - 2(1 - \lambda)a)\phi' + (1 - \lambda)\phi \right] \\ &= \Delta \ln u + (4\pi - 2(1 - \lambda)a)u' - 2(1 - \lambda)u. \end{aligned}$$

A change of the independent variables (a, t) can now be made to get rid of the lower order terms. This point of view may prove useful since the logarithmic porous media equation has been extensively studied, but we do not use it here. For more details on the logarithmic porous media equation, see [19,20,24,28]. Of particular interest is the connection between the Ricci flow and the porous media equation (namely the conformal factor evolves by logarithmic porous media) [26,27]. An interesting question would be to study how the conformal factor relates to the isoperimetric profile given both are solutions of the same equation.

4. Model solutions

This section is devoted to exhibiting suitable comparison functions ϕ and hence curvature and isoperimetric bounds for metrics evolving by the normalised Ricci flow via Corollaries 3.7–3.8. We will need to treat the cases $\lambda = 0, \lambda = 1, \lambda > 1$ separately. The next and final section briefly outlines how such bounds lead, via standard arguments, to the convergence results described in Theorem 1.2.

4.1. Genus 0

In [1], we showed that the isoperimetric profile of the Rosenau solution provided a suitable comparison solution. Let us briefly recall the result. The Rosenau solution is an explicit axially symmetric solution of the normalized Ricci flow on the two-sphere. The metric is given by $\bar{g}(t) = u(x, t)(dx^2 + dy^2)$, where $(x, y) \in \mathbb{R} \times [0, 4\pi]$, and

$$u(x, t) = \frac{\sinh(e^{-2t})}{2e^{-2t} (\cosh(x) + \cosh(e^{-2t}))}.$$

This extends to a smooth metric on the two-sphere at each time with area 4π , and which evolves according to the normalized Ricci flow equation (1.2). A direct computation gives the isoperimetric profile,

$$\varphi(a, t) = \sqrt{4\pi} \sqrt{\frac{\sinh(ae^{-2t}) \sinh((1-a)e^{-2t})}{\sinh(e^{-2t})e^{-2t}}}. \tag{4.1}$$

By translating $t \mapsto t - t_0$ with t_0 chosen so that initial inequality of the isoperimetric profile holds, the Comparison Theorem leads to the following bounds for solutions of the normalised Ricci flow on the 2-sphere:

Theorem 4.1. *Let g be a solution of the normalised Ricci flow on \mathbb{S}^2 . Then there exist constants $A, C > 0$ depending only on the metric at the initial time such that*

$$\sup_{\mathbb{S}^2} K(t) \leq 1 + Ce^{-At}.$$

There also exists a constant $\mathcal{I}_0 > 0$, depending only on the initial metric g_0 , such that

$$\mathcal{I}(t) > \mathcal{I}_0$$

where $\mathcal{I}(t)$ is the isoperimetric constant of $(\mathbb{S}^2, g(t))$.

4.2. Genus 1

Next, let us describe a comparison solution for the universal cover of surfaces of genus $\lambda = 1$, *i.e.* for \mathbb{R}^2 .

Recall, we need to find a function satisfying the differential inequality

$$\phi_t \geq \phi^2 \phi'' - \phi(\phi')^2 + 4\pi \phi'.$$

We look for solutions with equality. First, to simplify matters, let $v = \phi^2$ which satisfies the equation

$$v_t = vv'' - (v')^2 + 4\pi v' = v^2 \left(\frac{v'}{v} - \frac{4\pi}{v} \right)'.$$

Taking the Ansatz $v(a, t) = tV(a/t)$, and letting $z = a/t$, we obtain the integrable equation,

$$V^2 \left(\frac{V'}{V} - \frac{4\pi + z}{V} \right)' = 0.$$

Adding in the limiting behaviour $V(0) = 0$, this has the family of solutions

$$V_C(z) = \frac{1}{C} \left(4\pi - \frac{1}{C} \right) \left(1 - e^{-Cz} \right) + \frac{z}{C}.$$

That is, we have

$$v_C(a, t) = \frac{a}{C} + \frac{t}{C} \left(4\pi - \frac{1}{C} \right) \left(1 - e^{-\frac{Ca}{t}} \right). \tag{4.2}$$

We can now use v_C as a comparison for $\lambda = 1$ surfaces, as in the following:

Theorem 4.2. *Let $g(t)$ be any solution of the normalised Ricci flow on M a closed, genus 1 surface and let $\tilde{g} = \pi^* g$ be the pull back metric to the universal cover $\tilde{M} = \mathbb{R}^2$. Then there exists a $C > 0$ such that the function $\phi = \sqrt{v_C}$ where v_C is defined by (4.2) satisfies $\phi(a, t) < I_{\tilde{g}}(a, t)$ for all $a \in (0, \infty)$ and $t \in [0, T)$. Therefore the Gauss curvature K of M satisfies the bound*

$$\sup_M K \leq \frac{A}{t}$$

for a constant $A > 0$ depending only on the initial metric g_0 .

Proof. We know that v_C satisfies the differential inequality and it's easy to see that v_C is strictly concave, so we need to show that v_C meets the other requirements for the Comparison Theorem in the form of Proposition 3.10. At $t = 0$, we have $v_C(a, 0) = \frac{a}{C}$ so by choosing C large enough, we have the initial comparison since $I \simeq \sqrt{C_1 a + C_2 a^2}$ as $a \rightarrow \infty$.

On the small scale we have

$$\phi(a, t) = \sqrt{4\pi a} \left(1 - \left(\frac{C}{4} - \frac{1}{16\pi} \right) \frac{1}{t} a + O(a^2) \right)$$

as required for the small scale asymptotics and also providing the stated curvature bound with $A = 2\pi C - 1/2$ (which is positive for $C > 1/4\pi$) by Corollary 3.7. \square

4.3. Genus > 1

In this subsection, we construct the model comparison solution for the final case, $\lambda > 1$. When $\sup_{M_0} K > 0$, the construction is a little involved.

4.3.1. $K < 0$ case

First let us consider the case where $\sup_{M_0} K \leq 0$, since it admits a simple, appealing comparison solution.

For any $A, C > 0$, let

$$v(a, t) = 4\pi a + B(t)a^2 \tag{4.3}$$

with

$$B(t) = -(1 - \lambda) - \frac{C}{1 + Ae^{(\lambda-1)t}}.$$

Direct computation shows that v_C is a solution of the differential equation

$$v_t = vv'' - (v')^2 + (4\pi - (1 - \lambda)a)v' + 2(1 - \lambda)v,$$

which is the required equation for $v = \phi^2$ as in the genus 1 case above.

Theorem 4.3. *Let $g(t)$ be any solution of the normalised Ricci flow on M a closed, genus > 1 surface with $\sup_{M_0} K \leq 0$ and let $\tilde{g} = \pi^* g$ the pull back to \mathbb{H}^2 with $\pi : \mathbb{H}^2 \rightarrow M$ the universal cover. Then for $\phi = \sqrt{v}$ where v is defined by (4.3), there exist $A, C > 0$ such that $\phi(a, t) < I_{\tilde{g}}(a, t)$ for all $a \in (0, \infty)$. Therefore, the Gauss curvature K_M satisfies the bound*

$$\sup_{M_t} K \leq 1 - \lambda + C_1 e^{-C_2 t}$$

for positive constants C_1, C_2 .

Proof. Since the comparison function is a quadratic with zero constant term and linear coefficient equal to 4π , the small scale asymptotics are satisfied providing that $B(t) \leq -\text{const} \sup_M K$, by the asymptotics of the isoperimetric profile given in Theorem 2.2. Since we require $B(t) \geq 0$, this can only be achieved in the case $\sup_{M_t} K \leq 0$ which is true by the maximum principle under the assumption $\sup_{M_0} K \leq 0$. In this case, we choose A, C large enough so that the initial comparison holds. Concavity is easily checked. Proposition 3.10 completes the proof that $\phi(a, t) < I_{\tilde{g}}(a, t)$ for all $a \in (0, \infty)$. The curvature bound now follows directly from Corollary 3.7. □

4.3.2. General case

Stationary solution. Recall we have the equation,

$$\frac{\partial}{\partial t} v - \left\{ vv'' - (v')^2 + [4\pi - 2(1 - \lambda)a]v' + 2(1 - \lambda)v \right\} \leq 0.$$

We can write this as

$$\frac{\partial}{\partial t} v \leq v^2 \left(\frac{v'}{v} - \frac{4\pi - 2(1 - \lambda)a}{v} \right)'. \tag{4.4}$$

Stationary solutions (with equality) to this equation that satisfy the conditions $v(0) = 0$ and $\limsup_{x \rightarrow \infty} \frac{v(x)}{x^2} < \infty$ are given by

$$\begin{aligned}
 v_C(x) &= \frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \left[1 - e^{-Cx} \right] - \frac{2(1-\lambda)}{C^2} (Cx) \\
 &= 4\pi x + \frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \left[1 - Cx - e^{-Cx} \right] \\
 &= 4\pi x - \frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \frac{(Cx)^2}{2} \\
 &\quad + \frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \left[1 - Cx + \frac{(Cx)^2}{2} - e^{-Cx} \right]
 \end{aligned}
 \tag{4.5}$$

for any $C \geq 0$. The last line is obtained from the Taylor expansion for e^{-Cx} . Each of the three expressions illustrates different properties of v_C . For instance, the first line shows that v_C grows at most linearly. The second and third lines give the first and second order Taylor expansions with explicit remainders.

For later use, the first and second derivatives of v_C are

$$v'_C = 4\pi + \left[4\pi + \frac{2(1-\lambda)}{C} \right] \left[-1 + e^{-Cx} \right]
 \tag{4.6}$$

$$v''_C = -C \left[4\pi + \frac{2(1-\lambda)}{C} \right] e^{-Cx}.
 \tag{4.7}$$

In particular, provided that

$$C \geq C_{\text{crit}} = -\frac{1-\lambda}{2\pi}$$

we have $\left[4\pi + \frac{2(1-\lambda)}{C} \right] \geq 0$ and so V_C is concave, strictly so when $C > C_{\text{crit}}$.

Such functions prove useful, but are not quite sufficient for our purposes. The comparison is constructed from the function,

$$f(x, t) = \sqrt{v_C(x) + bx^2}
 \tag{4.8}$$

with $b \geq 0$.

Lemma 4.4. *Let f be defined as in equation (4.8) with $C \geq C_{\text{crit}}$. Then f is concave, if and only if*

$$b \leq b_{\text{crit}} = \frac{(1-\lambda)^2}{\frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C^2} \right]},$$

with strict concavity corresponding to strict inequality.

Proof. We have

$$f'' = \frac{1}{2f^3} \left[v_C v_C'' - \frac{1}{2}(v_C')^2 + b(2v_C - 2xv_C' + x^2v_C'') \right]$$

so that f is concave if and only if

$$b \leq \frac{\frac{1}{2}(v_C')^2 - v_C v_C''}{(2v_C - 2xv_C' + x^2v_C'')}$$

since b is non-negative.

First, consider the numerator. Since $v_C \geq 0$ and $v_C'' \leq 0$ we have

$$\frac{1}{2}(v_C')^2 - v_C v_C'' \geq \frac{1}{2}(v_C')^2.$$

Again using $v_C'' \leq 0$ we have

$$v_C'(x) \geq \lim_{x \rightarrow \infty} v_C' = \frac{-2(1-\lambda)}{C}.$$

Also $\lim_{x \rightarrow \infty} v_C v_C'' = 0$ so that in fact,

$$\inf_x \left(\frac{1}{2} \left[(v_C')^2 - v_C v_C'' \right] \right) = \frac{2(1-\lambda)^2}{C^2}.$$

For the denominator, we have

$$\begin{aligned} 2v_C - 2xv_C' + x^2v_C'' &= \frac{2}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \left[1 - e^{-Cx} (1 + Cx + (Cx)^2/2) \right] \\ &\leq \frac{2}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \end{aligned}$$

with equality at $x = 0$ so that

$$\sup_x \left(2v_C - 2xv_C' + x^2v_C'' \right) = \frac{2}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right].$$

Therefore f is concave if and only if

$$b \leq \frac{\frac{2(1-\lambda)^2}{C^2}}{\frac{2}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right]} = \frac{(1-\lambda)^2}{\frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right]}.$$

□

Remark 4.5. Observe that the denominator in b_{crit} is zero for $C = C_{\text{crit}}$, is positive for $C > C_{\text{crit}}$ and, approaches 0 as $C \rightarrow \infty$. Thus for $C_0 \geq C_{\text{crit}}$, $b_{\text{crit}}([C_{\text{crit}}, C_0])$ is bounded below away from 0. This will prove useful later.

Let us also record the small and large scale asymptotics of f in a lemma for later reference.

Lemma 4.6. *The function v_C satisfies the asymptotic behaviour*

$$v_C(x) = 4\pi x - \frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \frac{(Cx)^2}{2} + \mathcal{O}((Cx)^3)$$

as $Cx \rightarrow 0$. Moreover,

$$\limsup_{x \rightarrow \infty} v_C(x) = -\frac{2(1-\lambda)}{C^2}(Cx).$$

Therefore, f satisfies the asymptotic behaviour

$$f^2(x) = 4\pi x + \left(\frac{2b}{C^2} - \frac{1}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \right) \frac{(Cx)^2}{2} + \mathcal{O}((Cx)^3)$$

as $Cx \rightarrow 0$. Moreover

$$\limsup_{x \rightarrow \infty} f^2 = bx^2. \quad \square$$

Remark 4.7. In particular notice that the coefficient of $x^2/2$ (rather than $(Cx)^2/2$) from the small-scale asymptotics of f is

$$2b - 4\pi C - 2(1 - \lambda)$$

and this can be made arbitrarily large negative by choosing $0 \leq b \ll 1 - \lambda$ and $C \gg C_{\text{crit}}$.

Construction of the comparison function. The comparison is built from the function f defined in equation (4.8) by letting $C = C(t)$, $b = b(t)$. If $C(t) \searrow C_{\text{crit}}$ and $b(t) \nearrow -(1 - \lambda)$ as $t \rightarrow \infty$, with $b(t) \leq b_{\text{crit}}(C(t))$ (which choice is possible by Remark 4.5), then

$$f(x, t) = \sqrt{v_{C(t)} + b(t)x^2} \tag{4.9}$$

is a concave function with

$$\lim_{t \rightarrow \infty} f(x, t) = \sqrt{4\pi x + (1 - \lambda)x^2}$$

the isoperimetric profile of the metric of constant curvature $1 - \lambda$ (which is the curvature of the metric lifted from the constant curvature surface with area 4π and genus λ by the Gauss-Bonnet Theorem).

By choosing $C(0) > C_{\text{crit}}$ sufficiently large and $0 \leq b(0) < \max\{b_{\text{crit}}, 1 - \lambda\}$, sufficiently small, Lemma 4.6 and Remark 4.7 imply that for any initial metric, initial inequality is satisfied along with the asymptotic behaviour required by Comparison Theorem in the form of Proposition 3.10.

Thus for f to be a suitable comparison function, we need to choose $C(t)$ and $b(t)$ so that the differential inequality is satisfied. As before, it is more convenient to work with $v = f^2 = v_c + bx^2$.

Lemma 4.8. *Let*

$$b(t) = \left[\left(\frac{1}{b_0} + \frac{1}{1 - \lambda} \right) e^{4(1-\lambda)t} - \frac{1}{1 - \lambda} \right]^{-1}$$

$$C(t) = (C_0 - C_{\text{crit}}) \sqrt{b_0} e^{2(1-\lambda)t} \left[\left(\frac{1}{b_0} + \frac{1}{1 - \lambda} \right) e^{4(1-\lambda)t} - \frac{1}{1 - \lambda} \right]^{-1/2} + C_{\text{crit}}.$$

Then $v = f^2$ satisfies the differential inequality (4.4) with f defined by (4.9).

Proof. First, for the time derivative we have

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{dC}{dt} \frac{\text{partial} v_C}{\partial C} + \frac{db}{dt} x^2 \\ &= -\frac{\partial C}{\partial t} \frac{1}{C^2} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \frac{(Cx)^2}{2} \\ &\quad - \frac{dC}{dt} \left(\frac{1}{C^2} \left[4\pi + \frac{4(1-\lambda)}{C} \right] \right) \left(1 - Cx + \frac{(Cx)^2}{2} - e^{-Cx} \right) \\ &\quad - \frac{dC}{dt} \left(\frac{1}{C^2} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \right) (Cx) \left(1 - Cx - e^{-Cx} \right) \\ &\quad + \frac{2}{C^2} \frac{db}{dt} \frac{(Cx)^2}{2} \tag{4.10} \\ &< \frac{1}{C^2} \left(2 \frac{db}{dt} - \frac{dC}{dt} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \right) \frac{(Cx)^2}{2} \\ &\quad - \frac{dC}{dt} \left(\frac{1}{C^2} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \right) \left(1 - Cx + \frac{(Cx)^2}{2} - e^{-Cx} \right) \\ &\quad - \frac{dC}{dt} \left(\frac{1}{C^2} \left[4\pi + \frac{2(1-\lambda)}{C} \right] \right) (Cx) \left(1 - Cx - e^{-Cx} \right). \end{aligned}$$

The inequality occurs in the second line after the inequality by replacing the 4 with a 2 using the fact that C is decreasing and $(1 - \lambda) < 0$.

For the spatial part, we first use the fact that for two functions g, h we have

$$\begin{aligned} (g + h)^2 \Delta \ln(g + h) &= (g + h)(g + h)'' - (g + h)' \\ &= g^2 \Delta \ln g + h^2 \Delta \ln h + gh'' + hg'' - 2g'h'. \end{aligned}$$

Thus with $g = v_C$ and $h = bx^2$ we get

$$v^2 \Delta \ln v = v_C^2 \Delta \ln v_C - 2b^2x^2 + b(2v_C - 4xv'_C + x^2v''_C)$$

so that

$$\begin{aligned} v^2 \Delta \ln v + L[v] &= v_C^2 \Delta \ln v_C + L[v_C] + 8\pi bx - 2(b^2 + (1 - \lambda)b)x^2 \\ &\quad + b(2v_C - 4xv'_C + x^2v''_C) \\ &= 8\pi bx - 2(b^2 + (1 - \lambda)b)x^2 + b(2v_C - 4xv'_C + x^2v''_C) \end{aligned}$$

since v_C satisfies $v_C^2 \Delta \ln v_C + L[v_C] = 0$. Expand the last term in parenthesis in a Taylor series using equations (4.5), (4.6) and (4.7) to get

$$\begin{aligned} v^2 \Delta \ln v + L[v] &= \left(\frac{4b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] - \frac{4(b^2 + (1 - \lambda)b)}{C^2} \right) \frac{(Cx)^2}{2} \\ &\quad + \frac{2b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] \left(1 - Cx + \frac{(Cx)^2}{2} - e^{-Cx} \right) \\ &\quad + \frac{4b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] (Cx) (1 - Cx - e^{-Cx}) \\ &\quad + \frac{2b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] \frac{(Cx)^2}{2} (1 - e^{-Cx}). \end{aligned} \tag{4.11}$$

Now we compare the terms from equation (4.10) with those of equation (4.11) to obtain the following necessary inequalities:

- $(Cx)^2/2$:

$$\begin{aligned} \frac{1}{C^2} \left(2 \frac{db}{dt} - \frac{dC}{dt} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] \right) \\ < \left(\frac{4b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] - \frac{4(b^2 + (1 - \lambda)b)}{C^2} \right); \end{aligned}$$

- $1 - Cx + \frac{(Cx)^2}{2} - e^{-Cx}$:

$$-\frac{dC}{dt} \left(\frac{1}{C^2} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] \right) < \frac{2b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right];$$

- $Cx(1 - Cx - e^{-Cx})$:

$$-\frac{dC}{dt} \left(\frac{1}{C^2} \left[4\pi + \frac{2(1 - \lambda)}{C} \right] \right) < \frac{4b}{C} \left[4\pi + \frac{2(1 - \lambda)}{C} \right];$$

$$\bullet \frac{(Cx)^2}{2} (1 - e^{-Cx});$$

$$0 < \frac{2b}{C} \left[4\pi + \frac{2(1-\lambda)}{C} \right].$$

All the above inequalities are satisfied if

$$\frac{db}{dt} < -4 \left(b^2 + (1-\lambda)b \right) \tag{4.12}$$

$$\frac{d}{dt} \ln C > -2b \tag{4.13}$$

and $C \geq C_{\text{crit}}$ which ensures that $4\pi + \frac{2(1-\lambda)}{C} \geq 0$.

It is now a simple matter, left to the reader, to check that $b(t)$ as given in the statement of the lemma satisfies equality in the Riccati equation (4.12) and that equality in (4.13) is satisfied by

$$\tilde{C} = (C_0 - C_{\text{crit}}) \sqrt{b_0} e^{2(1-\lambda)t} \left[\left(\frac{1}{b_0} + \frac{1}{1-\lambda} \right) e^{4(1-\lambda)t} - \frac{1}{1-\lambda} \right]^{-1/2}.$$

Since $C(t) = \tilde{C} + C_{\text{crit}} > \tilde{C}$ and $\frac{d}{dt} \tilde{C} < 0$, we then have

$$\frac{d}{dt} \ln C = \frac{\frac{d}{dt} \tilde{C}}{\tilde{C} + C_{\text{crit}}} > \frac{\frac{d}{dt} \tilde{C}}{\tilde{C}} = -2b$$

completing the proof. □

Remark 4.9. Observe that with $b(t), C(t)$ as given in Lemma 4.8, $b(t)$ monotonically increases from b_0 to $-(1-\lambda)$ and $C(t)$ monotonically decreases from C_0 to C_{crit} so that $f = \sqrt{v}$ converges to the constant curvature $1-\lambda$ isoperimetric profile.

Finally, applying Corollary 3.7 we obtain

Theorem 4.10. *Let $g(t)$ be any solution of the normalised Ricci flow on M a closed, genus > 1 surface and let $\tilde{g} = \pi^* g$ the pull back to \mathbb{H}^2 with $\pi : \mathbb{H}^2 \rightarrow M$ the universal cover. Then for $\phi = \sqrt{v_{C(t)} + b(t)a^2}$ where $C(t), b(t)$ are defined as in Lemma 4.8 there exist $C_0, b_0 > 0$ such that $\phi(a, t) < I_{\tilde{g}}(a, t)$ for all $a \in (0, \infty)$. Therefore, the Gauss curvature K_M satisfies the bound*

$$\sup_M K_M(t) \leq \left(\frac{C(t)}{2} \left[4\pi + \frac{2(1-\lambda)}{C(t)} \right] - b(t) \right)$$

which decays exponentially fast to $(1-\lambda)$ as $t \rightarrow \infty$.

Remark 4.11. The exponential decay in the theorem follows from the fact that $b(t) \rightarrow -(1-\lambda)$ exponentially fast, $C(t) \rightarrow C_{\text{crit}}$ exponentially fast and hence $\left[4\pi + \frac{2(1-\lambda)}{C(t)} \right] \rightarrow 0$ exponentially fast.

5. Convergence

In this last section, let us briefly discuss the proof of Theorem 1.2. The argument is very standard, following from bootstrapping the curvature bounds to higher derivative bounds. Here we will only outline the steps, indicating how the results here may be applied.

Proof of Theorem 1.2. First, observe that by Theorems 4.1–4.3 and 4.10 and the fact that $\pi : \tilde{M} \rightarrow M$ is a local isometry, we have uniform upper bounds $K_M \leq K_0(t) + (1 - \lambda)$ with $K_0(t)$ uniformly bounded and such that $\lim_{t \rightarrow \infty} K_0(t) = 0$. Since the Gauss curvature evolves according to $\frac{\partial}{\partial t} K = \Delta K + K(K - (\lambda - 1))$, by an ODE comparison we also have uniform lower bounds converging to 0 as $t \rightarrow \infty$. Thus $|K(t)|$ is uniformly bounded for all $t > 0$ and hence the solution exists for all time.

L^1 convergence of the curvature to $1 - \lambda$ now follows easily. By Gauss-Bonnet, $K \leq K_0(t) + (1 - \lambda)$ and, by the fact that $|M| = 4\pi$,

$$0 = \int_M K - (1 - \lambda) d\mu \leq - \int_{K \leq (1-\lambda)} |K - (1 - \lambda)| d\mu + 4\pi K_0(t),$$

which rearranges to give $\int_{K \leq (1-\lambda)} |K - (1 - \lambda)| d\mu \leq 4\pi K_0(t)$. Therefore we get

$$\int_M |K - (1 - \lambda)| d\mu \leq 8\pi K_0(t)$$

which converges to 0 as $t \rightarrow \infty$.

Next we bound the higher derivatives of K . By the bootstrapping argument described in [16, Section 7], and from the uniform curvature bounds we obtain

$$\left| \nabla^{(j)} K \right|^2 \leq C_j ((1 - \lambda) + t^{-j})$$

for constants $C_j > 0$.

In the genus $\lambda = 0$ case, the lower bound on the isoperimetric constant affords very strong analytic control allowing us to apply Gagliardo-Nirenberg inequalities to deduce that $K \rightarrow_{C^\infty} (1 - \lambda)$ uniformly as $t \rightarrow \infty$. See [1].

For higher genus surfaces, we do not have such control as noted in Remark 3.9. Thus instead, for $\lambda > 0$ surfaces, with a little more work, another bootstrapping argument gives $K \rightarrow_{C^\infty} (1 - \lambda)$ uniformly as $t \rightarrow \infty$ [16, Section 7].

Finally, in the cases $\lambda \neq 1$, we have $K_0(t) = Ce^{-at}$ which gives for any non-zero $v \in TM$,

$$\left| \frac{\partial}{\partial t} \ln g(t)(v, v) \right| = 2 |K - (1 - \lambda)| \leq C^{-at}$$

which is integrable in t on $[0, \infty)$. Smooth convergence of the metric now follows by the argument in [13, Section 17].

For the case $\lambda = 0$ we only have $K_0(t) = C/t$ which is not integrable. However, we may use the fact that $|\nabla K_0| \leq C/t^{3/2}$ which is integrable to again deduce smooth convergence [8]. \square

Remark 5.1. Note that we have control of the isoperimetric constant on \tilde{M} and a curvature bound. We cannot however use these to obtain the simpler convergence proof using Gagliardo-Nirenberg inequalities, since these rely on L^1 convergence of the curvature. But this is invalid for $\lambda > 1$ since \tilde{M} is not compact. Perhaps one might deduce L^1_{loc} convergence, but note that the above L^1 convergence argument uses Gauss-Bonnet. In the L^1_{loc} case, we would need to deal with boundary terms arising from Gauss-Bonnet and I do not know how to control these. This is perhaps related to transferring isoperimetric control from \tilde{M} to M .

References

- [1] B. ANDREWS and P. BRYAN, *Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere*, Calc. Var. Partial Differential Equations **39** (2010), 419–428.
- [2] V. BAYLE, *A differential inequality for the isoperimetric profile*, Int. Math. Res. Not. **7** (2004), 311–342.
- [3] D. BURAGO and S. IVANOV, *On asymptotic volume of tori*, Geom. Funct. Anal. **5** (1995), 800–808.
- [4] C. BAVARD and P. PANSU, *Sur le volume minimal de R^2* , Ann. Sci. École Norm. Sup. (4) **19** (1986), 479–490.
- [5] L. A. CAFFARELLI and X. CABRÉ, “Fully Nonlinear Elliptic Equations”, American Mathematical Society Colloquium Publications, Vol. 43, American Mathematical Society, Providence, RI, 1995.
- [6] I. CHAVEL, “Riemannian Geometry: A Modern Introduction”, Second edition, Cambridge Studies in Advanced Mathematics, Vol. 98., Cambridge University Press, Cambridge, 2006.
- [7] B. CHOW, *The Ricci flow on the 2-sphere*, J. Differential Geom. **33** (1991), 325–334.
- [8] B. CHOW and D. KNOPF, “The Ricci Flow: an Introduction”, Mathematical Surveys and Monographs, Vol. 110, American Mathematical Society, Providence, RI, 2004.
- [9] M. G. CRANDALL and P.-L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.
- [10] X. CHEN, P. LU and G. TIAN, *A note on uniformization of Riemann surfaces by Ricci flow*, Proc. Amer. Math. Soc. **134** (2006), 3391–3393 (electronic).
- [11] H. FEDERER, “Geometric Measure Theory”, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
- [12] A. GRAY, *The volume of a small geodesic ball of a Riemannian manifold*, Michigan Math. J. **20** (1973), 329–344.
- [13] R. S. HAMILTON, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.
- [14] R. S. HAMILTON, *The Ricci flow on surfaces*, In: “Mathematics and General Relativity (Santa Cruz, CA, 1986)”, J. A. Isenberg (ed.), Contemporary Mathematics, Vol. 71, American Mathematical Society, Providence, RI, 1988, 237–262.
- [15] R. S. HAMILTON, *An isoperimetric estimate for the Ricci flow on the two-sphere*, In: “Modern Methods in Complex Analysis (Princeton, NJ, 1992)”, T. Bloom, D. W. Catlin, J. P. D’Angelo and Y.-T. Siu (eds.), Annals of Mathematical Studies, Vol. 137, Princeton University Press, Princeton, NJ, 1995, 191–200.

- [16] R. S. HAMILTON, *The formation of singularities in the Ricci flow*, In: “Surveys in Differential Geometry, Vol. II (Cambridge, MA, 1993)”, International Press, Cambridge, MA, 1995, 7–136.
- [17] J. HASS and F. MORGAN, *Geodesics and soap bubbles in surfaces*, Math. Z. **223** (1996), 185–196.
- [18] J. JOST, “Two-dimensional Geometric Variational Problems”, Pure and Applied Mathematics, John Wiley & Sons, Ltd., Chichester, 1991.
- [19] J. R. KING, *Exact similarity solutions to some nonlinear diffusion equations*, J. Phys. A **23** (1990), 3681.
- [20] J. R. KING, *Some non-local transformations between nonlinear diffusion equations*, J. Phys. A **23** (1990), 5441.
- [21] F. MORGAN, M. HUTCHINGS and H. HOWARDS, *The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature*, Trans. Amer. Math. Soc. **352** (2000), 4889–4909.
- [22] F. MORGAN, “Geometric Measure Theory. A Beginner’s Guide”, Fourth edition, Elsevier/Academic Press, Amsterdam, 2009.
- [23] R. OSSERMAN, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.
- [24] P. ROSENAU, *Fast and superfast diffusion processes*, Phys. Rev. Lett. **74** (1995), 1056–1059.
- [25] P. STERNBERG and K. ZUMBRUN, *On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint*, Comm. Anal. Geom. **7** (1999), 199–220.
- [26] J. L. VÁZQUEZ, J. R. ESTEBAN and A. RODRÍGUEZ, *The fast diffusion equation with logarithmic nonlinearity and the evolution of conformal metrics in the plane*, Adv. Differential Equations **1** (1996), 21–50.
- [27] L.-F. WU, *A new result for the porous medium equation derived from the Ricci flow*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), 90–94.
- [28] I. T. PEDRON, R. S. MENDES, T. J. BURATTA, L. C. MALACARNE and E. K. LENZI, *Logarithmic diffusion and porous media equations: A unified description*, Phys. Rev. E **72** (2005), 031106. Eprint: cond-mat/0508343.

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