

Regularity of roots of polynomials

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Abstract. We show that smooth curves of monic complex polynomials $P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j}$, $a_j : I \rightarrow \mathbb{C}$ with $I \subset \mathbb{R}$ a compact interval, have absolutely continuous roots in a uniform way. More precisely, there exist a positive integer k and a rational number $p > 1$, both depending only on the degree n , such that if $a_j \in C^k$ then any continuous choice of roots of P_a is absolutely continuous with derivatives in L^q for all $1 \leq q < p$, in a uniform way with respect to $\max_j \|a_j\|_{C^k}$. The uniformity allows us to deduce also a multiparameter version of this result. The proof is based on formulas for the roots of the universal polynomial P_a in terms of its coefficients a_j which we derive using resolution of singularities. For cubic polynomials we compute the formulas as well as bounds for k and p explicitly.

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Introduction

This paper is dedicated to the solution of a basic problem in perturbation theory and differential analysis. Given a monic polynomial with smooth coefficients (or a matrix with smooth entries) it is natural to ask for the optimal regularity of the roots (or of the eigenvalues). For instance, this question arises in finding local solutions of partial differential equations with multiple characteristics.

In connection with the study of a class of pseudodifferential systems, Spagnolo [26] asked whether a smooth (C^∞) curve of monic complex polynomials admits a locally absolutely continuous parameterization of its roots. And if so, whether it is possible to choose the absolutely continuous roots uniformly with respect to the coefficients on compact subintervals. We answer these questions affirmatively and prove the following stronger result; see also Theorem 3.5. Our proof builds on the recent result of Ghisi and Gobbino [7] who found the optimal regularity of radicals of functions that we combine with the resolution of singularities.

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Main Theorem. *For every $n \in \mathbb{N}_{>0}$ there are $k = k(n) \in \mathbb{N}_{>0}$ and $p = p(n) > 1$ such that the following holds. Let $I \subset \mathbb{R}$ be a compact interval and let*

$$P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t)Z^{n-j} \in C^k(I)[Z]$$

be a monic polynomial with coefficients $a_j \in C^k(I)$, for $j = 1, \dots, n$.

- (1) *Let $\lambda_j \in C^0(I)$, for $j = 1, \dots, n$, be a continuous parameterization of the roots of P_a on I . Then the distributional derivative of each λ_j in I is a measurable function $\lambda'_j \in L^q(I)$ for every $q \in [1, p)$. In particular, each $\lambda_j \in W^{1,q}(I)$ for every $q \in [1, p)$.*
- (2) *This regularity of the roots is uniform. Let $\{P_{a_\nu}; \nu \in \mathcal{N}\}$, where*

$$P_{a_\nu(t)}(Z) = Z^n + \sum_{j=1}^n a_{\nu,j}(t)Z^{n-j} \in C^k(I)[Z], \quad \text{for } \nu \in \mathcal{N},$$

be a family of curves of polynomials, indexed by ν in some set \mathcal{N} , so that the set of coefficients $\{a_{\nu,j}; \nu \in \mathcal{N}, j = 1, \dots, n\}$ is bounded in $C^k(I)$. Then the set

$$\{\lambda_\nu \in C^0(I); P_{a_\nu}(\lambda_\nu) = 0 \text{ on } I, \nu \in \mathcal{N}\}$$

is bounded in $W^{1,q}(I)$ for every $q \in [1, p)$.

L^q denotes the Lebesgue space and $W^{1,q}$ the Sobolev space with respect to the Lebesgue measure. We want to stress the fact that a continuous curve of monic complex polynomials $P_{a(t)}$, for $t \in \mathbb{R}$, allows for a continuous parameterization of its roots. This is no longer true if the parameter space has more than one dimension due to monodromy. For multiparameter families of polynomials we obtain the following result; see also Theorem 4.1.

Multiparameter Theorem. *Let $k = k(n) \in \mathbb{N}_{>0}$ and $p = p(n) > 1$ be as above. Let $U \subset \mathbb{R}^m$ be open and let $P_{a(x)}(Z) \in C^k(U)[Z]$ be a monic polynomial with coefficients $a_j \in C^k(U)$, for $j = 1, \dots, n$.*

- (1) *Let $\lambda \in C^0(V)$ represent a root of P_a , i.e., $P_a(\lambda) = 0$, on a relatively compact open subset $V \Subset U$. Then the distributional gradient of λ in V is a measurable function $\nabla \lambda \in [L^q(V)]^m$ for every $q \in [1, p)$. In particular, $\lambda \in W^{1,q}(V)$ for every $q \in [1, p)$.*
- (2) *The regularity of the roots is uniform. Let $\{P_{a_\nu}; \nu \in \mathcal{N}\}$ be a family of polynomials, indexed by ν in some set \mathcal{N} , so that the set of coefficients $\{a_{\nu,j}; \nu \in \mathcal{N}, j = 1, \dots, n\}$ is bounded in $C^k(U)$. Let $V \Subset U$. Then the set*

$$\{\lambda_\nu \in C^0(V); P_{a_\nu}(\lambda_\nu) = 0 \text{ on } V, \nu \in \mathcal{N}\}$$

is bounded in $W^{1,q}(V)$ for every $q \in [1, p)$.

In [26] Spagnolo proved that the pseudodifferential $n \times n$ system

$$u_t + iA(t, D_x)u + B(t, D_x)u = f(t, x), \quad \text{for } (t, x) \in I \times U,$$

where $A(t, \xi), B(t, \xi)$ are matrix symbols of order 1 and 0, respectively, and $A(t, \xi)$ is homogeneous of degree 1 in ξ for $|\xi| \geq 1$, is locally solvable in the Gevrey class G^s for $1 \leq s \leq n/(n - 1)$ and semi-globally solvable in G^s for $1 < s < n/(n - 1)$ under the following assumptions: the eigenvalues of $A(t, \xi)$ admit a parameterization $\tau_1(t, \xi), \dots, \tau_n(t, \xi)$ such that each $\tau_j(t, \xi)$ is absolutely continuous in t , uniformly with respect to ξ , *i.e.*,

$$|\partial_t \tau_j(t, \xi)| \leq \mu(t, \xi) \left(1 + |\xi|^2\right)^{\frac{1}{2}}, \quad \text{with } \mu(\cdot, \xi) \text{ equi-integrable on } I, \quad (\mathcal{A}_1)$$

and for each ξ the imaginary parts of the $\tau_j(t, \xi)$ do not change sign for varying t and j , *i.e.*,

$$\forall \xi \quad \text{either } \operatorname{Im} \tau_j(t, \xi) \geq 0, \quad \forall t, j, \quad \text{or } \operatorname{Im} \tau_j(t, \xi) \leq 0, \quad \forall t, j. \quad (\mathcal{A}_2)$$

Our Main Theorem implies that the Assumption (\mathcal{A}_1) is automatically satisfied. Indeed, this follows by applying the Main Theorem to the characteristic polynomial of the matrix $(1 + |\xi|^2)^{-\frac{1}{2}} A(t, \xi)$ and noting that the entries of $(1 + |\xi|^2)^{-\frac{1}{2}} A(t, \xi)$ and its iterated partial derivatives with respect to t are globally bounded in ξ , since $A(t, \xi)$ is a symbol of order 1.

Spagnolo formulated the removal of Assumption (\mathcal{A}_1) as an open problem in [26, page 1122], and he successfully tackled the case of quadratic and cubic polynomials in [25]. Note that the problems of smoothly choosing roots of polynomials on one hand and eigenvalues of *arbitrary* quadratic matrices on the other hand are equivalent; whereas the perturbation theory for normal matrices is easier and allows for stronger results, *cf.* Rainer [24] and references therein.

We would like to remark that our result represents a complex analogue of Bronshtein’s Theorem on the regularity of the roots of hyperbolic polynomials. A monic polynomial is called hyperbolic if all its roots are real. Bronshtein’s Theorem, first proved in Bronshtein [4] and generalizing the classical Glaeser inequality [8], states that any continuous parameterization of the roots of a hyperbolic polynomial of degree n with $C^{n-1,1}$ coefficients is locally Lipschitz. It plays a crucial role for weakly hyperbolic Cauchy problems. Different proofs appeared in Wakabayashi [28] and in Parusiński and Rainer [20].

In the absence of hyperbolicity the roots cannot fulfill a Lipschitz condition and in a certain sense absolute continuity is the best one can hope for; in fact the degree of summability p tends to 1 as n goes to ∞ . The first result towards absolute continuity of the roots is probably Lemma 1 in Colombini, Jannelli, and Spagnolo [5] which states that for a real-valued non-negative function f of class $C^{k,\alpha}$ on a compact interval I , with $k \in \mathbb{N}_{\geq 1}$ and $0 \leq \alpha \leq 1$, the radical $f^{1/(k+\alpha)}$ is absolutely continuous on I and satisfies

$$\left\| \left(f^{\frac{1}{k+\alpha}} \right)' \right\|_{L^1(I)}^{k+\alpha} \leq C(k, \alpha, I) \|f\|_{C^{k,\alpha}(I)}.$$

Tarama [27] extended this lemma to real-valued functions (not necessarily non-negative). A better summability for the weak partial derivatives of $f^{1/(k+1)}$ was obtained by Colombini and Lerner [6] for non-negative C^{k+1} functions f of several real variables.

The case of radicals of functions was completely settled recently by Ghisi and Gobbino [7] by finding their optimal regularity. They showed that, if f is a real-valued continuous function and there exists $g \in C^{k,\alpha}(I)$ so that $|f|^{k+\alpha} = |g|$ on I , then $f' \in L^p_w(I)$, where $\frac{1}{p} + \frac{1}{k+\alpha} = 1$, and

$$\|f'\|_{p,w,I} \leq C(k) \max \left\{ \left[\text{Höld}_{\alpha,I}(g^{(k)}) \right]^{\frac{1}{k+\alpha}} |I|^{\frac{1}{p}}, \|g'\|_{L^\infty(I)}^{\frac{1}{k+\alpha}} \right\};$$

in particular, $f \in W^{1,q}(I)$ for every $q \in [1, p)$. Here $L^p_w(I)$ denotes the weak Lebesgue space equipped with the quasi-norm $\|f\|_{p,w,I} := \sup_{r>0} \{r \cdot \mathcal{L}^1(\{t \in I; |f(t)| > r\})^{\frac{1}{p}}\}$, where \mathcal{L}^1 is the one dimensional Lebesgue measure. By $\text{Höld}_{\alpha,I}(g^{(k)})$ we mean the α -Hölder constant of $g^{(k)}$ on I , and $|I| = \mathcal{L}^1(I)$ is the length of the interval I . Ghisi and Gobbino also provided examples that show that the assumptions as well as the conclusion in their theorem are the best possible. We use this result in a substantial way.

The mentioned results all treat special cases, where the algebraic structure of the polynomials is very simple: the roots are either given by radicals or can be expressed by radicals (by Cardano’s formulas). A different approach was pursued in Rainer [22], where no restrictions on the algebraic structure of the polynomial were imposed. Under the assumption that no two roots meet with infinite order of contact it was shown that the roots of a C^∞ curve of monic polynomials are locally absolutely continuous. We also mention Rainer [23], where it was proved that the roots of a monic polynomial whose coefficients are functions in several variables that belong to any quasianalytic class satisfying some stability properties admit a parameterization by (special) functions of bounded variation.

Our proof of the Main Theorem is based on formulas for the roots of the universal monic polynomial P_a in terms of its coefficients $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. The derivation of these formulas represents the third major result of this paper; see Theorem 1.6. Using Hironaka’s resolution of singularities [11], we construct a tower of smooth principalizations

$$M_1 = \mathbb{C}^n \xleftarrow{\sigma_2} M_2 \xleftarrow{\sigma_{3,2}} M_3 \xleftarrow{\sigma_{4,3}} \dots \xleftarrow{\sigma_{n,n-1}} M_n$$

which successively principalize the generalized discriminant ideals $\mathcal{D}_m \subset \mathbb{C}[a]$, for $m = 2, \dots, n$, that encode the stratification of the space of polynomials by root multiplicity. In fact, the zero set of \mathcal{D}_m is exactly the set of those $a \in \mathbb{C}^n$ for which P_a has at most $m - 1$ distinct roots. We show that, locally on M_n , the roots of the pulled back polynomial $P_{\sigma_n^*(a)}$ are given by rational linear combinations $\sum_{m=1}^n A_m \varphi_m \circ \sigma_{n,m}$ where

$$\varphi_m = f_m^{\alpha_m} \psi_m \left(y_{m,1}^{1/q_m}, \dots, y_{m,r_m}^{1/q_m}, y_{m,r_m+1}, \dots, y_{m,n} \right).$$

Here $\sigma_m = \sigma_2 \circ \sigma_{3,2} \circ \dots \circ \sigma_{m,m-1}$, $\sigma_{n,m} = \sigma_{m+1,m} \circ \dots \circ \sigma_{n,n-1}$, $f_m \in \mathcal{D}_m$ is a local generator of $\sigma_m^*(\mathcal{D}_m)$, ψ_m is a convergent power series, $q_m \in \mathbb{N}_{\geq 1}$, $\alpha_m \in \frac{1}{q_m}\mathbb{N}_{\geq 1}$, and $(y_{m,i})$ is a privileged system of local coordinates such that $f_m^{-1}(0)$ is given by $y_{m,1} \cdots y_{m,r_m} = 0$ (cf. Subsection 1.3 and Definition 1.4). These formulas are obtained in a natural way by a consecutive factorization procedure of the pull-backs $P_{\sigma_i^*(a)}$, for $i = 2, \dots, n$, so that each step contributes exactly to one summand. Thanks to these formulas we are able to reduce the problem to radicals of functions and use the result of Ghisi and Gobbino (cf. Lemma 3.4).

The paper is divided into three parts. The first part presents the three main results of this paper. Section 1 is devoted to the formulation of the result on the formulas for the roots; see Theorem 1.6. In Section 2 we mainly fix notation on function spaces. The Main Theorem 3.5 is proved in Section 3, assuming validity of Theorem 1.6, and the Multiparameter Theorem 4.1 is deduced in Section 4.

The second part of the paper is dedicated to the proof of Theorem 1.6. The strategy of the proof is briefly outlined in Section 5. In Section 6 we find a convenient criterion of principality of the ideals \mathcal{D}_m . In Sections 7 and 8 we further develop the necessary tools utilized in the proof of Theorem 1.6 which is finally carried out in Section 9.

In the third part we illustrate our method of proof by discussing the case of cubic polynomials in detail. Here the resolution is explicit, and we can specify more precisely the degree of differentiability of the coefficients and the degree of summability of the derivative of the roots, namely $k(3) = 6$ and $p(3) = 6/5$.

Notation and terminology. By a *normal crossing* we mean a function that is locally equivalent to a monomial, i.e., equals a monomial times an analytic unit. The zero set of an ideal \mathcal{I} will be denoted by $V(\mathcal{I})$. For two real-valued functions φ and ψ we write $\varphi \sim \psi$ if there exists $C > 0$ such that $\varphi \leq C\psi$ and $\psi \leq C\varphi$. By $\lceil x \rceil$ we denote the ceiling function, that is the least integer bigger than or equal to x .

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Part 1. Absolute continuity of roots

1. Formulas for the roots

1.1. Generalized discriminant ideals

Let

$$P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j} \tag{1.1}$$

be a unitary polynomial with coefficients $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. We denote by $\xi(a) = \{\xi_1(a), \dots, \xi_n(a)\}$ the unordered set of roots of P_a and assign to a_i the weight i so that homogeneous permutation invariant polynomials in ξ are precisely the weighted homogeneous polynomials in a .

Let $N \in \mathbb{N}$ be a large constant fixed throughout the paper ($N \geq \max_{1 \leq s \leq n} \binom{n}{s}$). For $2 \leq m \leq n$ we denote by $\mathcal{D}_{N,m}$, or simply by \mathcal{D}_m , the ideal of $\mathbb{C}[a]$ generated by all $f^{N!/s} \in \mathbb{C}[a]$, where f runs over all homogeneous polynomials in $\prod_{i \neq j \in I} (\xi_i - \xi_j)$, for $I \subset \{1, \dots, n\}$ with $|I| = m$, of degree $s \leq N$ in ξ , that are invariant under the permutations of ξ_i 's.

Example 1.1. $\mathcal{D}_{N,n}$ is the principal ideal generated by the $N!/n(n - 1)$ -th power of the discriminant of P_a .

By replacing Z by $Z - a_1/n$ we define a new polynomial

$$P_{\hat{a}}(Z) = Z^n + \hat{a}_2 Z^{n-2} + \dots + \hat{a}_n := P_a(Z - a_1/n).$$

Each \hat{a}_j is a weighted homogeneous polynomial in the a_i 's of the weighted degree j . This transformation $a \mapsto \hat{a}$, called *Tschirnhausen transformation*, shifts the roots of P_a by a_1/n . Since the polynomials in $\prod_{i \neq j \in I} (\xi_i - \xi_j)$ are invariant by shifts of the roots, the Tschirnhausen transformation does not change the ideal \mathcal{D}_m . Therefore, in what follows, we may suppose without loss of generality that P_a is in *Tschirnhausen form*

$$P_a(Z) = Z^n + a_2 Z^{n-2} + \dots + a_n. \tag{1.2}$$

Proposition 1.2. *Suppose that P_a is in Tschirnhausen form (1.2). Then $\mathcal{D}_{N,2}$ is the ideal of $\mathbb{C}[a_2, \dots, a_n]$ generated by the weighted homogeneous polynomials in a_2, \dots, a_n of weighted degree $N!$.*

Proof. If P_a is in Tschirnhausen form then $\sum_{i=1}^n \xi_i = 0$ and hence $\xi_i = \frac{1}{n} \sum_j (\xi_i - \xi_j)$. Therefore any polynomial in the ξ_i 's is a polynomial in the $(\xi_i - \xi_j)$'s. \square

Thus the zero set $V(\mathcal{D}_2)$ of \mathcal{D}_2 is exactly the set of those a for which P_a has precisely one root, (i.e., $a_2 = \dots = a_n = 0$ if P_a is in Tschirnhausen form (1.2) and then this root is zero). In general, the zero set of \mathcal{D}_m consists of those $a \in \mathbb{C}^n$ for which P_a has at most $m - 1$ distinct roots, cf. Corollary 6.4 below.

1.2. Smooth principalization of an ideal

Let $\mathcal{I} = (f_1, \dots, f_s)$ be an ideal of $\mathbb{C}[x]$, $x = (x_1, \dots, x_m)$. We shall assume $\mathcal{I} \neq (0)$. The blowing-up of \mathcal{I} , $\sigma : M = Bl_{\mathcal{I}} \mathbb{C}^m \rightarrow \mathbb{C}^m$ can then be realized as follows, see for instance [10, Example 7.17.2]. The variety M is the irreducible component of

$$\left\{ (x, y) \in \mathbb{C}^m \times \mathbb{P}^{s-1}; f_i(x)y_j = f_j(x)y_i, \quad \text{for } i, j = 1, \dots, s \right\}$$

that projects surjectively onto \mathbb{C}^m and σ is the projection on the first factor. Then M is the union of finitely many standard affine charts \mathcal{V}_i , where

$$\mathcal{V}_i = M \cap \left\{ (x, y_1, \dots, \widehat{y}_i, \dots, y_s) \in \mathbb{C}^m \times \mathbb{C}^{s-1}; f_i(x)y_j = f_j(x), \right. \\ \left. \text{for } j = 1, \dots, \hat{i}, \dots, s \right\}$$

is a subvariety of the affine space $\mathbb{C}^m \times \mathbb{C}^{s-1} = \{(x, y) \in \mathbb{C}^m \times \mathbb{P}^{s-1}; y_i = 1\}$ with the coordinates x_j , for $j = 1, \dots, m$, and $y_j = f_j/f_i$, for $j = 1, \dots, \hat{i}, \dots, s$. The pullback of \mathcal{I} on \mathcal{V}_i is generated by f_i , since $f_j = y_j f_i$ for $j \neq i$, and hence is an invertible ideal, *i.e.*, principal and generated by a non-zero divisor. The pullback of \mathcal{I} on M , denoted by $\sigma^*(\mathcal{I})$, is an invertible sheaf of ideals.

In general, for arbitrary \mathcal{I} , the blow-up space $Bl_{\mathcal{I}}\mathbb{C}^m$ is singular. Using Hironaka’s resolution of singularities [11], it is possible to give an algorithm of principalization of any ideal by composing a sequence of smooth blow-ups (that is blowing-ups with smooth nowhere dense centers). Then the blow-up space is nonsingular. Such a principalization map is not unique. We will use the following theorem that is a special case of [13, Principalization III], see also [3, Theorem 1.10].

Theorem 1.3. *Let $\mathcal{I} \neq (0)$ be an ideal of $\mathbb{C}[x]$. Then there exists a composition of smooth blowing-ups, $\sigma : M \rightarrow \mathbb{C}^m$, such that:*

- (1) $\sigma^*(\mathcal{I})$ is invertible and its zero set $V(\sigma^*(\mathcal{I}))$ is a simple normal crossing divisor.
- (2) The restriction $\sigma|_{M \setminus V(\sigma^*(\mathcal{I}))} : M \setminus V(\sigma^*(\mathcal{I})) \rightarrow \mathbb{C}^m \setminus V(\mathcal{I})$ is an isomorphism.

Recall that a simple normal crossing divisor $E \subset M$ is the union $E = \cup_i E_i$ of nonsingular hypersurfaces E_i intersecting transversally.

1.3. Privileged coordinate system

The map σ of Theorem 1.3 is itself a blowing-up of an ideal $\mathcal{K} \subset \mathbb{C}[x]$ such that $V(\mathcal{K}) = V(\mathcal{I})$. (Because σ is birational and projective, see [10, Chapter II Theorem. 7.17 and Exercise 7.11 (c)].) Thus $M = Bl_{\mathcal{K}}\mathbb{C}^m$. Denote $V(\sigma^*(\mathcal{I}))$ by E and let $E = \cup_i E_i$ be the decomposition into irreducible components.

Let $\mathcal{I} = (f_1, \dots, f_s)$ and $\mathcal{K} = (h_1, \dots, h_l)$ and let $\mathcal{V} \subset Bl_{\mathcal{K}}\mathbb{C}^m$ be a standard affine chart of $Bl_{\mathcal{K}}\mathbb{C}^m$. Then on \mathcal{V} , $\sigma^*(\mathcal{I})$ is a principal ideal that is generated by some f_i , say, f_1 , and $\sigma^*(\mathcal{K})$ by some h_i , say, h_1 . \mathcal{V} is a subvariety of $\mathbb{C}^m \times \mathbb{C}^{l-1}$ with the coordinates x_i , for $i = 1, \dots, m$, and h_j/h_1 , for $j = 2, \dots, l$. Each component $E_i \cap \mathcal{V}$ is the zero set of a finite number of functions regular on \mathcal{V} , that is functions of the form P/h_1^s , for $s \in \mathbb{N}$, where P is a polynomial in x .

Let $p \in \mathcal{V}$. There is a neighborhood \mathcal{U} of p in \mathcal{V} , and a coordinate system y_1, \dots, y_m on \mathcal{U} , such that $y_i = P_i/h_1^s$, $P_i \in \mathbb{C}[x]$, $s \in \mathbb{N}$, and $E \cap \mathcal{U}$ is given by

$y_1 \cdots y_r = 0$. (By taking the maximum we may choose s independent of i .) Then

$$f_1 = \text{unit} \cdot \prod_{i=1}^r y_i^{n_i}, \quad h_1 = \text{unit} \cdot \prod_{i=1}^r y_i^{m_i}, \tag{1.3}$$

with $n_i > 0$ and $m_i > 0$. Here by a unit we mean an analytic, nowhere-vanishing function on \mathcal{U} .

1.4. Tower of smooth principalizations

For the ideals \mathcal{D}_m , for $m = 2, \dots, n$, we construct a tower of smooth principalizations

$$M_1 = \mathbb{C}^n \xleftarrow{\sigma_2} M_2 \xleftarrow{\sigma_{3,2}} M_3 \xleftarrow{\sigma_{4,3}} \cdots \xleftarrow{\sigma_{n,n-1}} M_n. \tag{1.4}$$

We take as σ_2 a smooth principalization of \mathcal{D}_2 satisfying the conclusion of Theorem 1.3. Given $\sigma_2, \sigma_{3,2}, \dots, \sigma_{m,m-1}$, we denote by $\sigma_m : M_m \rightarrow \mathbb{C}^n$ the composition $\sigma_m = \sigma_2 \circ \sigma_{3,2} \circ \cdots \circ \sigma_{m,m-1}$, and take as $\sigma_{m+1,m}$ a smooth principalization of $\sigma_m^*(\mathcal{D}_{m+1})$. Then $\sigma_{m+1} = \sigma_m \circ \sigma_{m+1,m}$ is a smooth principalization of \mathcal{D}_{m+1} . We denote $\sigma_{n,m} = \sigma_{m+1,m} \circ \cdots \circ \sigma_{n,n-1}$.

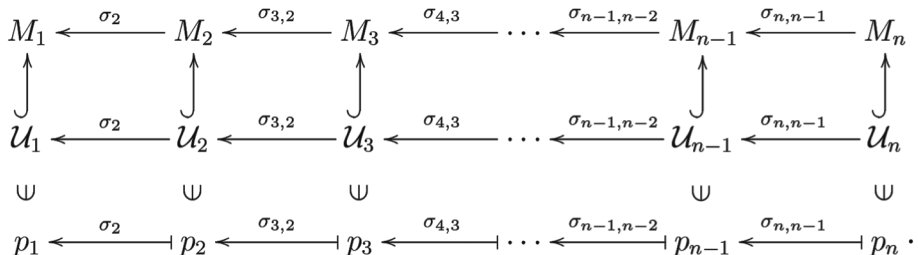
By Subsection 1.3, σ_m is the blowing-up of an ideal $\mathcal{K}_m \subset \mathbb{C}[a]$.

Definition 1.4. By local data (f, h, P_i, s, r) for $p \in M_m$ we mean the following. A polynomial $f \in \mathcal{D}_m$ that generates $\sigma_m^*(\mathcal{D}_m)$ at p , a polynomial $h \in \mathcal{K}_m$ that generates $\sigma_m^*(\mathcal{K}_m)$, a positive integer s and polynomials P_i such that $y_i = P_i/h^s$, for $i = 1, \dots, n$, is a privileged system of coordinates in a neighborhood \mathcal{U} of p , and r such that $f^{-1}(0)$ is given by $y_1 \cdots y_r = 0$.

We fix such local data for every $p \in M_m$ (but allow to replace the neighborhood \mathcal{U} by a smaller one if necessary).

Definition 1.5. By a chain $\mathbf{C} = (p_m, f_m, h_m, P_{m,i}, s_m, r_m)$ for $p_n \in M_n$ we mean the points $p_m := \sigma_{n,m}(p_n)$, for $m = 1, \dots, n$, and the local data $(f_m, h_m, P_{m,i}, s_m, r_m)$ for p_m . We complete this data for $m = 1$ by putting $f_1 = h_1 = 1, P_{1,i} = a_i$, and $s_1 = r_1 = 0$.

When we specify the neighborhoods $\mathcal{U}_m \subset M_m$ of p_m on which these local data are defined we always assume that $\sigma_{m,m-1}(\mathcal{U}_m) \subset \mathcal{U}_{m-1}$:



We pull back the polynomial P_a onto M_m via σ_m ,

$$P_{\sigma_m^*(a)}(Z) = Z^n + \sum_{i=1}^n (a_i \circ \sigma_m) Z^{n-i}.$$

The roots of $P_{\sigma_n^*(a)}$ are the pull-backs of the roots of P_a .

1.5. Formulas for the roots

Theorem 1.6 (Formulas for the roots). *Given a tower (1.4), we may associate with every $p_m \in M_m$ a convergent power series ψ_m , an integer $q_m \geq 1$, and a positive exponent $\alpha_m \in \frac{1}{q_m}\mathbb{N}_{>0}$, such that the following holds. For any chain $\mathbf{C} = (p_m, f_m, h_m, P_{m,i}, s_m, r_m)$ the roots of $P_{\sigma_n^*(a)}$ in a neighborhood of p_n are given by*

$$\sum_{m=1}^n A_m \varphi_m \circ \sigma_{n,m}, \tag{1.5}$$

where $A_m \in \mathbb{Q}$ and

$$\varphi_m = f_m^{\alpha_m} \psi_m \left(y_{m,1}^{1/q_m}, \dots, y_{m,r_m}^{1/q_m}, y_{m,r_m+1}, \dots, y_{m,n} \right). \tag{1.6}$$

Theorem 1.6 will be proved in Section 9.

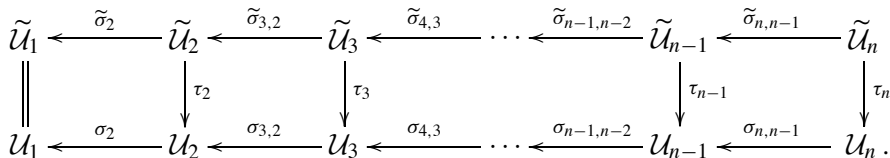
Remark 1.7. Because f_m is a normal crossing in $y_{m,i}$, cf. (1.3),

$$f_m^{\alpha_m} \in \mathbb{C} \left\{ \left(y_{m,1}^{1/q_m}, \dots, y_{m,r_m}^{1/q_m}, y_{m,r_m+1}, \dots, y_{m,n} \right) \right\}.$$

Hence φ_m of (1.6) is a fractional power series. It can be interpreted geometrically as follows. Set

$$y_{m,i} = \begin{cases} t_i^{q_m} & \text{if } i \leq r_m \\ t_i & \text{if } i > r_m + 1. \end{cases} \tag{1.7}$$

Then φ_m is a convergent power series in $t = (t_1, \dots, t_n)$. There are neighborhoods \mathcal{U}_m of p_m , $\sigma_{m,m-1}(\mathcal{U}_m) \subset \mathcal{U}_{m-1}$, and their branched covers $\tau_m : \tilde{\mathcal{U}}_m \rightarrow \mathcal{U}_m$, given by the formulas (1.7), such that $\psi_m \circ \tau_m$ and $\varphi_m \circ \tau_m$ are analytic on $\tilde{\mathcal{U}}_m$. Since $\sigma_{m+1,m}^{-1}(f_m^{-1}(0)) \subset f_{m+1}^{-1}(0)$, $y_{m,i} \circ \sigma_{m+1,m}$, for $i \leq r_m$, is a normal crossing in $y_{m+1,1}, \dots, y_{m+1,r_{m+1}}$ and therefore, we may suppose that $\sigma_{m+1,m} \circ \tau_{m+1}$ factors through τ_m , changing q_{m+1} if necessary. Thus we obtain a sequence of branched covers τ_i making the following diagram commutative:



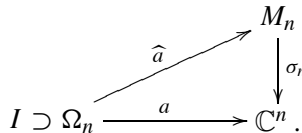
Then Theorem 1.6 says that the roots of $P_{\tilde{\sigma}_n^*(a)}$ are combinations of analytic functions on $\tilde{\mathcal{U}}_n$ that are pull-backs of analytic functions on the $\tilde{\mathcal{U}}_m$'s.

Definition 1.8. By an *extended chain* $E = (p_m, f_m, h_m, P_{m,i}, s_m, r_m, \mathcal{U}_m)$ for $p_n \in M_n$ we mean a chain $C = (p_m, f_m, h_m, P_{m,i}, s_m, r_m)$ and a system of neighborhoods \mathcal{U}_m of p_m as in Remark 1.7. By Theorem 1.6 for every $p_n \in M_n$ there is an extended chain.

We filter the coefficient space \mathbb{C}^n by the zero sets of discriminant ideals $\Sigma_m := V(\mathcal{D}_m)$,

$$\mathbb{C}^n \supset \Sigma_n \supset \dots \supset \Sigma_2.$$

By Corollary 6.4, $a \in \Sigma_m$ if and only if $P_a(Z)$ has at most $m - 1$ distinct roots. If $a(t) : \mathbb{R} \supset I \rightarrow \mathbb{C}^n$ is continuous, then $\Omega_m := I \setminus a^{-1}(\Sigma_m)$ defines a filtration by open subsets $I \supset \Omega_2 \supset \dots \supset \Omega_n$. Because σ_m is an isomorphism over $\mathbb{C}^n \setminus \Sigma_m$, $a|_{\Omega_m}$ has a lift \widehat{a}_m to M_m . For $m = n$ we write $\widehat{a} := \widehat{a}_n$.



Lemma 1.9 (Addendum to Theorem 1.6). Let $E = (p_m, f_m, h_m, P_{m,i}, s_m, r_m, \mathcal{U}_m)$ be an extended chain for $p_n \in M_n$ and let J be a connected component of $\widehat{a}^{-1}(\mathcal{U}_n)$. Let $\lambda(t)$ be a continuous root of $P_{a(t)}(Z)$ on J . Then there are continuous choices of radicals $f_m^{\alpha_m}(a(t))$ and $y_{m,1}^{1/q_m}(\widehat{a}_m(t)), \dots, y_{m,r_m}^{1/q_m}(\widehat{a}_m(t))$, such that

$$\lambda(t) = \sum_{m=1}^n A_m \varphi_m(t), \tag{1.8}$$

where

$$\varphi_m(t) = f_m^{\alpha_m}(a(t)) \psi_m \left(y_{m,1}^{1/q_m}(\widehat{a}_m(t)), \dots, y_{m,r_m}^{1/q_m}(\widehat{a}_m(t)), y_{m,r_m+1}(\widehat{a}_m(t)), \dots, y_{m,n}(\widehat{a}_m(t)) \right). \tag{1.9}$$

Lemma 1.9 will be proved in Subsection 9.1.

2. Function spaces

In this section we fix notation for function spaces and present an extension lemma.

2.1. Function spaces

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We denote by $C^0(\Omega)$ the space of continuous complex-valued functions on Ω . For $k \in \mathbb{N} \cup \{\infty\}$ we set

$$\begin{aligned}
 C^k(\Omega) &= \{f \in C^\Omega; \partial^\alpha f \in C^0(\Omega), \text{ for } 0 \leq |\alpha| \leq k\}, \\
 C^k(\overline{\Omega}) &= \{f \in C^k(\Omega); \partial^\alpha f \text{ has a continuous extension to } \overline{\Omega}, \text{ for } 0 \leq |\alpha| \leq k\},
 \end{aligned}$$

where $\overline{\Omega}$ denotes the closure of Ω .

Note that $C^k(\overline{\Omega})$ is a Banach space when equipped with the norm

$$\|f\|_{C^k(\overline{\Omega})} := \sup_{\substack{|\alpha| \leq k \\ x \in \Omega}} |\partial^\alpha f(x)|.$$

For $k \in \mathbb{N}$ and $p \geq 1$ we consider the Sobolev space

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); \partial^\alpha f \in L^p(\Omega), \text{ for } 0 \leq |\alpha| \leq k\},$$

where $\partial^\alpha f$ denote distributional derivatives. On bounded intervals $I \subset \mathbb{R}$ the Sobolev space $W^{1,1}(I)$ coincides with the space $AC(I)$ of absolutely continuous functions on I if we identify each $W^{1,1}$ -function with its unique continuous representative. Recall that a function $f : \Omega \rightarrow \mathbb{R}$ on an open subset $\Omega \subset \mathbb{R}$ is absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ so that $\sum_{i=1}^n |a_i - b_i| < \delta$ implies $\sum_{i=1}^n |f(a_i) - f(b_i)| < \epsilon$ whenever $[a_i, b_i]$, for $i = 1, \dots, n$, are non-overlapping intervals contained in Ω .

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $1 \leq p < \infty$. A measurable function $f : \Omega \rightarrow \mathbb{C}$ belongs to the weak L^p -space $L_w^p(\Omega)$ if

$$\|f\|_{p,w,\Omega} := \sup_{r>0} \left\{ r \cdot \mathcal{L}^n(\{x \in \Omega; |f(x)| > r\})^{\frac{1}{p}} \right\} < \infty,$$

where \mathcal{L}^n denotes the n -dimensional Lebesgue measure. For $1 \leq q < p < \infty$ we have (cf. [9, Example 1.1.11])

$$\|f\|_{q,w,\Omega} \leq \|f\|_{L^q(\Omega)} \leq \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \mathcal{L}^n(\Omega)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{p,w,\Omega} \tag{2.1}$$

and hence $L^p(\Omega) \subset L_w^p(\Omega) \subset L^q(\Omega) \subset L_w^q(\Omega)$ with strict inclusions. Note that $\|\cdot\|_{p,w,\Omega}$ is only a quasinorm; more precisely, for $f_j \in L_w^p(\Omega)$ we have

$$\left\| \sum_{j=1}^m f_j \right\|_{p,w,\Omega} \leq m \sum_{j=1}^m \|f_j\|_{p,w,\Omega}. \tag{2.2}$$

If Ω_i is a finite or countable family of open sets whose union is Ω then (cf. [7, Lemma 3.1])

$$\|f\|_{p,w,\Omega} \leq \sum_i \|f\|_{p,w,\Omega_i}, \quad \forall f \in L_w^p(\Omega). \tag{2.3}$$

If $p > 1$ then there exists a norm equivalent to $\|\cdot\|_{p,w,\Omega}$ which makes $L_w^p(\Omega)$ into a Banach space.

Analogously we may consider $L_w^p(K)$ for compact sets $K \subset \mathbb{R}^n$. We shall also use $W_{\text{loc}}^{k,p}, AC_{\text{loc}}$, as well as $L^p(\Omega, \mathbb{C}^m) = [L^p(\Omega)]^m, W^{k,p}(\Omega, \mathbb{C}^m) = [W^{k,p}(\Omega)]^m$, etc., with the obvious meaning.

2.2. Extension lemma

The following lemma is a generalization of [7, Lemma 3.2] with essentially the same proof.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}$ be open and bounded, let $f : \Omega \rightarrow \mathbb{C}$ be continuous, and set $\Omega_0 := \{t \in \Omega; f(t) \neq 0\}$. Assume that $f|_{\Omega_0} \in AC_{loc}(\Omega_0)$ and that $f'|_{\Omega_0} \in L^p_w(\Omega_0)$ for some $p > 1$ (note that f is differentiable a.e. in Ω_0). Then the distributional derivative of f in Ω is a measurable function $f' \in L^p_w(\Omega)$ and*

$$\|f'\|_{p,w,\Omega} = \|f'|_{\Omega_0}\|_{p,w,\Omega_0}. \tag{2.4}$$

Proof. The function $\psi : \Omega \rightarrow \mathbb{C}$ defined by

$$\psi(t) := \begin{cases} f'(t) & \text{if } t \in \Omega_0 \\ 0 & \text{if } t \in \Omega \setminus \Omega_0 \end{cases}$$

clearly belongs to $L^p_w(\Omega)$. We show that ψ is the distributional derivative of f in Ω . Let $\phi \in C^\infty_c(\Omega)$ be a test function with compact support in Ω and let \mathcal{C} denote the (at most countable) set of connected components of Ω_0 . Then, using integration by parts for the Lebesgue integral (see, e.g., [15, Corollary 3.37])

$$\begin{aligned} \int_{\Omega} f \phi' dt &= \int_{\Omega_0} f \phi' dt = \sum_{J \in \mathcal{C}} \int_J f \phi' dt = - \sum_{J \in \mathcal{C}} \int_J f' \phi dt \\ &= - \int_{\Omega_0} f' \phi dt = - \int_{\Omega} \psi \phi dt. \end{aligned}$$

(If $J = (a, b)$ then $\int_a^b f \phi' dt = \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\epsilon} f \phi' dt = - \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\epsilon} f' \phi dt = - \int_a^b f' \phi dt$, by the dominated convergence theorem, continuity of f , and (2.1).) Moreover, we have $\|f'\|_{p,w,\Omega} = \|\psi\|_{p,w,\Omega} = \|\psi\|_{p,w,\Omega_0} = \|f'|_{\Omega_0}\|_{p,w,\Omega_0}$. \square

3. Absolute continuity of roots

3.1. Optimal regularity of radicals of differentiable functions

We need the following variant of [7, Theorem 2.2].

Proposition 3.1. *Let $I \subset \mathbb{R}$ be a bounded interval and let $k \in \mathbb{N}_{>0}$. For each $f \in C^k(\bar{I}, \mathbb{C})$ we have*

$$|f'(t)| \leq \Lambda_k(t) |f(t)|^{1-\frac{1}{k}} \quad \text{a.e. in } I \tag{3.1}$$

for some $\Lambda_k = \Lambda_{k,f} \in L^p_w(I, \mathbb{R}_{\geq 0})$, where $\frac{1}{p} + \frac{1}{k} = 1$, and such that

$$\|\Lambda_k\|_{p,w,I} \leq C(k) \max \left\{ \|f^{(k)}\|_{L^\infty(I)}^{\frac{1}{k}} |I|^{\frac{1}{p}}, \|f'\|_{L^\infty(I)}^{\frac{1}{k}} \right\}. \tag{3.2}$$

Proof. If the real and imaginary part of f satisfy (3.1), then so does f . Hence it suffices to consider the case that f is real-valued.

Set $h = |f|^{\frac{1}{k}}$. Then [7, Theorem 2.2] implies that $h' \in L^p_w(I, \mathbb{R})$ and

$$\|h'\|_{p,w,I} \leq C(k) \max \left\{ \|f^{(k)}\|_{L^\infty(I)}^{\frac{1}{k}} |I|^{\frac{1}{p}}, \|f'\|_{L^\infty(I)}^{\frac{1}{k}} \right\}.$$

In particular, $h \in W^{1,q}(I, \mathbb{R})$ for each $q \in [1, p)$. By differentiating $h^k = |f|$ we find

$$|f'(t)| = k|h'(t)||f(t)|^{1-\frac{1}{k}} \quad \text{for all } t \in I \text{ with } f(t) \neq 0.$$

The derivative f' vanishes at the accumulation points of $f^{-1}(0)$, and the isolated points of $f^{-1}(0)$ form an at most countable set. So we conclude that (3.1) holds with $\Lambda_k := k|h'|$. □

Remark 3.2. Proposition 3.1 is optimal in the following sense. Λ_k cannot, in general, be chosen in L^p . Indeed, for $f : (-1, 1) \rightarrow \mathbb{R}, f(t) = t$, we have

$$\left(\frac{|f'|}{|f|^{1-\frac{1}{k}}} \right)^p = \left(|t|^{\frac{1}{k}-1} \right)^p = |t|^{-1},$$

which is not integrable near 0. See [7, Example 4.3].

Remark 3.3. In Proposition 3.1 it is actually enough to require that $f \in C^{k-1,1}(\bar{I}, \mathbb{C})$; cf. [7, Theorem 2.2].

3.2. Set-valued functions and curves of polynomials

In the following we shall be dealing with multi-valued functions arising from complex radicals, their composition with single-valued functions, and their addition and multiplication.

The (usual) composition $G \circ F : X \rightsquigarrow Z$ of two set-valued functions $F : X \rightsquigarrow Y$ and $G : Y \rightsquigarrow Z$ is given by $(G \circ F)(x) = \cup_{y \in F(x)} G(y)$. The addition $F + G$ and multiplication FG of $F : X \rightsquigarrow \mathbb{C}$ and $G : X \rightsquigarrow \mathbb{C}$ are then well-defined.

A selection of a set-valued function $F : X \rightsquigarrow Y$ is a single-valued function $f : X \rightarrow Y$ satisfying $f(x) \in F(x)$ for all $x \in X$. A parameterization of a set-valued function $F : X \rightsquigarrow Y$ is a pair (f, Z) , where $f : X \times Z \rightarrow Y$ is a single-valued function so that $F(x) = \{f(x, z); z \in Z\}$ for every $x \in X$. We shall only be concerned with multi-valued functions F so that the cardinality $|F(x)|$ is finite and bounded, i.e., $\max_{x \in X} |F(x)| =: N < \infty$. Then a parameterization of F is an N -tuple of single-valued functions (with multiplicities at points x where $|F(x)| < N$).

If the coefficients of the polynomial P_a in (1.1) are complex-valued continuous functions $a_j \in C^0(I)$ defined in an interval $I \subset \mathbb{R}$, we say that $P_a(t) = P_{a(t)}$, for $t \in I$, is a curve of polynomials. The roots of a curve of polynomials form a

multi-valued function $\lambda : I \rightsquigarrow \mathbb{C}$ which admits a continuous parameterization; see [12, Chapter II Theorem 5.2]. (This is no longer true if the parameter space is higher-dimensional due to monodromy.) Moreover, any continuous selection of $\lambda : I \rightsquigarrow \mathbb{C}$ can be completed to a continuous parameterization $\lambda_1, \dots, \lambda_n$ such that $P_{a(t)}(Z) = \prod_i (Z - \lambda_i(t))$; see [24, Lemma 6.17].

Lemma 3.4. *Let $\alpha \in \mathbb{Q}_{>0}$, and $q, s \in \mathbb{N}_{>0}$. Suppose that $k \geq \lceil \max \{ \frac{s}{\alpha}, q \} \rceil$, and $p = \frac{k}{k-1}$. Let $I \subset \mathbb{R}$ be a bounded interval and let $\mathcal{U} \subset \mathbb{C}^n$ be open and bounded. Let $\psi \in C^1(\overline{\mathcal{U}})$, $h, P_j \in C^k(\overline{I})$, and let $\Omega \subset I$ be an open subset of I so that $y^{1/q}(\Omega) \subset \mathcal{U}$, where we put $y = (y_1, \dots, y_n) = (P_1/h^s, \dots, P_n/h^s)$ and $y^{1/q} = (y_1^{1/q}, \dots, y_n^{1/q})$. Consider the multi-valued function*

$$\varphi = h^\alpha \psi \left(y^{\frac{1}{q}} \right) = h^\alpha \psi \left(y_1^{\frac{1}{q}}, \dots, y_n^{\frac{1}{q}} \right).$$

Then φ admits a continuous parameterization on Ω and for any such parameterization ϕ the distributional derivative of ϕ in Ω is a measurable function $\phi' \in L^p_w(\Omega)$ and

$$\|\phi'\|_{p,w,\Omega} \leq C_1(\alpha, s, q, \mathcal{U}) \|\psi\|_{C^1(\overline{\mathcal{U}})} N_{\alpha,k,s,I}(h) \max_j \{ H_{k,I}(h), H_{k,I}(P_j) \} \quad (3.3)$$

for a positive constant $C_1(\alpha, s, q, \mathcal{U})$, where for any function $g \in C^k(\overline{I})$ we set

$$N_{\alpha,k,s,I}(g) := \max \left\{ \|g\|_{L^\infty(I)}^{\alpha - \frac{s}{k}}, \|g\|_{L^\infty(I)}^{\alpha - \frac{1}{k}} \right\},$$

$$H_{k,I}(g) := \max \left\{ \|g^{(k)}\|_{L^\infty(I)}^{\frac{1}{k}} |I|^{\frac{1}{p}}, \|g'\|_{L^\infty(I)}^{\frac{1}{k}} \right\}.$$

Proof. First we show that φ admits a continuous parameterization on Ω . Consider the open subsets $\Omega_1 \subset \Omega_0 \subset \Omega \subset I$ given by

$$\Omega_0 = \{t \in \Omega; h(t) \neq 0\} \quad \text{and} \quad \Omega_1 = \{t \in \Omega_0; \forall j P_j \neq 0\}.$$

Each multi-valued function $y_j^{1/q} = (P_j/h^s)^{1/q}$ has a continuous parameterization on Ω_0 and thus so does $\psi(y^{1/q})$. The multi-valued function h^α has a continuous parameterization on I , which vanishes on the zero set of h . Since $\psi(y^{1/q})$ is bounded on Ω , we may conclude that φ admits a continuous parameterization on Ω .

Let ϕ be any continuous parameterization of φ on Ω . By a slight abuse of notation, we denote by ϕ also any single component in the parameterization ϕ .

Then ϕ is C^1 on Ω_1 and its derivative satisfies

$$\begin{aligned} |\phi'| &\leq \alpha \left| \psi \left(y^{\frac{1}{q}} \right) \right| \frac{|h'|}{|h|^{1-\alpha}} + \frac{1}{q} \sum_{j=1}^n \left| \partial_j \psi \left(y^{\frac{1}{q}} \right) \right| \left(\frac{|P'_j|}{|P_j|^{1-\frac{1}{q}}} |h|^{\alpha-\frac{s}{q}} + s \left| \frac{P_j}{h^s} \right|^{\frac{1}{q}} \frac{|h'|}{|h|^{1-\alpha}} \right) \\ &\leq \alpha \left| \psi \left(y^{\frac{1}{q}} \right) \right| \frac{|h'|}{|h|^{1-\frac{1}{k}}} |h|^{\alpha-\frac{1}{k}} \\ &\quad + \frac{1}{q} \sum_{j=1}^n \left| \partial_j \psi \left(y^{\frac{1}{q}} \right) \right| \left(\frac{|P'_j|}{|P_j|^{1-\frac{1}{k}}} \left| \frac{P_j}{h^s} \right|^{\frac{1}{q}-\frac{1}{k}} |h^s|^{\frac{\alpha}{s}-\frac{1}{k}+s} \left| \frac{P_j}{h^s} \right|^{\frac{1}{q}} \frac{|h'|}{|h|^{1-\frac{1}{k}}} |h|^{\alpha-\frac{1}{k}} \right) \end{aligned} \quad (3.4)$$

Next we claim that ϕ is locally absolutely continuous on Ω_0 . Indeed, every continuous parameterization of h^α , respectively $P_j^{1/q}$, is AC on I by Proposition 3.1, and consequently every continuous parameterization of $y_j^{1/q} = (P_j/h^s)^{1/q}$ is AC_{loc} on Ω_0 ; note that on each compact subinterval of Ω_0 any continuous parameterization of $1/h^{s/q}$ is C^1 . Since ψ is C^1 , we may infer from [17, Lemma 2.1] that each continuous parameterization of $\psi(y^{1/q})$, and thus of $h^\alpha \psi(y^{1/q})$, is locally absolutely continuous on Ω_0 . This shows the claim.

In particular, ϕ is differentiable almost everywhere in Ω_0 . We argue that (3.4) holds almost everywhere in Ω_0 , if we define

$$\frac{|P'_j|}{|P_j|^{1-\frac{1}{k}}} := 0 \quad \text{on accumulation points of } P_j^{-1}(0).$$

Indeed, by [17, Lemma 2.1] the chain rule holds almost everywhere and the derivative of any continuous, and hence absolutely continuous, parameterization of $P_j^{1/q}$ exists almost everywhere and vanishes on accumulation points of $P_j^{-1}(0)$. The isolated points of $P_j^{-1}(0)$ form an at most countable set.

Applying Proposition 3.1 we may conclude that

$$|\phi'| \leq \Lambda_k \Psi \quad \text{a.e. in } \Omega_0 \text{ for some } \Psi \in L^\infty(\Omega, \mathbb{R}) \text{ and } \Lambda_k \in L_w^p(I, \mathbb{R}). \quad (3.5)$$

Here we set $\Lambda_k = \max\{\Lambda_{k,h}, \Lambda_{k,P_j}\}$.

Extending Ψ by 1 on $I \setminus \Omega$ and using $L_w^p \cdot L^\infty \subset L_w^p$ we obtain

$$|\phi'| \leq \tilde{\Lambda}_k \quad \text{a.e. in } \Omega_0 \text{ for some } \tilde{\Lambda}_k \in L_w^p(I, \mathbb{R}).$$

Using Lemma 2.1 we may conclude that the distributional derivative of ϕ in Ω is a measurable function $\phi' \in L_w^p(\Omega)$.

The estimate (3.3) follows from (3.5), (3.2), and (2.4). \square

3.3. Main Theorem

Theorem 3.5. *For every $n \in \mathbb{N}_{>0}$ there are $k = k(n) \in \mathbb{N}_{>0}$ and $p = p(n) > 1$ such that the following holds. Let $I \subset \mathbb{R}$ be a compact interval and let*

$$P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t)Z^{n-j} \in C^k(I)[Z]$$

be a monic polynomial with coefficients $a_j \in C^k(I)$, for $j = 1, \dots, n$.

- (1) *Let $\lambda_j \in C^0(I)$, for $j = 1, \dots, n$, be a continuous parameterization of the roots of P_a on I . Then the distributional derivative of each λ_j in I is a measurable function $\lambda'_j \in L^q(I)$ for every $q \in [1, p)$. In particular, each $\lambda_j \in W^{1,q}(I)$ for every $q \in [1, p)$.*
- (2) *This regularity of the roots is uniform. Let $\{P_{a_v}; v \in \mathcal{N}\}$,*

$$P_{a_v(t)}(Z) = Z^n + \sum_{j=1}^n a_{v,j}(t)Z^{n-j} \in C^k(I)[Z], \quad \text{for } v \in \mathcal{N},$$

be a family of curves of polynomials, indexed by v in some set \mathcal{N} , so that the set of coefficients $\{a_{v,j}; v \in \mathcal{N}, j = 1, \dots, n\}$ is bounded in $C^k(I)$. Then the set

$$\{\lambda_v \in C^0(I); P_{a_v}(\lambda_v) = 0 \text{ on } I, \quad \text{for } v \in \mathcal{N}\}$$

is bounded in $W^{1,q}(I)$ for every $q \in [1, p)$.

The rest of this section will be devoted to the proof of Theorem 3.5.

3.4. Definition of $k(n)$ and $p(n)$

Let $E = (p_m, f_m, h_m, P_{m,i}, s_m, r_m, \mathcal{U}_m)$ be an extended chain. By (1.3), we may express φ_m of (1.6) as follows

$$\varphi_m = h_m^{\tilde{\alpha}_m} \tilde{\psi}_m \left(y_{m,1}^{1/\tilde{q}_m}, \dots, y_{m,r_m}^{1/\tilde{q}_m}, y_{m,r_m+1}, \dots, y_{m,n} \right), \tag{3.6}$$

where $\tilde{\alpha}_k \in \frac{1}{\tilde{q}_m} \mathbb{N}_{>0}$ and \tilde{q}_m is a positive integer possibly much bigger than q_m . Then we define

$$k_E := \max_m \left[\max \left\{ \frac{s_m}{\tilde{\alpha}_m}, \tilde{q}_m \right\} \right].$$

We fix an open bounded neighborhood \mathcal{B} of the origin in \mathbb{C}^n and a finite family of extended chains

$$\mathcal{CV} = \{E_j\} = \{(p_{j,m}, f_{j,m}, h_{j,m}, P_{j,m,i}, s_{j,m}, r_{j,m}, \mathcal{U}_{j,m})\} \tag{3.7}$$

such that

$$\sigma_n^{-1}(\overline{\mathcal{B}}) \subset \bigcup_j \mathcal{U}_{j,n}.$$

Then we set

$$k = k(n) := \max_j k_{E_j}, \quad \text{and} \quad p = p(n) := \frac{k}{k-1} \in \mathbb{Q}_{>1} \cup \{\infty\}.$$

3.5. Real analytic case

We begin the proof of Theorem 3.5 with the following special case. We suppose that $a(t)$ is real analytic and that $a(t) \in \mathcal{B}$ for all $t \in I$. We suppose moreover that the discriminant of $P_{a(t)}$ is not identically equal to zero. Under these assumptions we show that

$$\begin{aligned} & \|\lambda'_j\|_{p,w,I} \\ & \leq C(\mathcal{CV}) \sum_{j,m} N_{\tilde{\alpha}_{j,m,k,s_{j,m},I}}(h_{j,m}(t)) \max_i \{H_{k,I}(h_{j,m}(t)), H_{k,I}(P_{j,m,i}(t))\}, \end{aligned} \tag{3.8}$$

where the constant $C(\mathcal{CV})$ depends only on the family \mathcal{CV} .

Recall that \widehat{a} denotes the lift of $a|_{\Omega_n}$ over σ_n , cf. Subsection 1.5. We remark that actually a has a unique real analytic lift to M_n on the whole interval I , by the universal property of blowing-ups, see [10, Proposition 7.14]; but we will not use this fact.

All the roots of $P_{a(t)}$ on Ω_n are distinct and hence, by the implicit function theorem, depend analytically on t . Thus Lemmas 3.4 and 1.9 give (3.8) with I replaced by $\widehat{a}^{-1}(\mathcal{U}_n) \cap \Omega_n$. We set $I_i := \widehat{a}^{-1}(\mathcal{U}_{i,n})$ and $\Omega_{i,n} := I_i \cap \Omega_n$. Then, by (2.3),

$$\|\lambda'_j\|_{p,w,\Omega_n} \leq \sum_i \|\lambda'_j\|_{p,w,\Omega_{i,n}}.$$

Since $a(t)$ is real analytic, $a^{-1}(\Sigma_n) = I \setminus \Omega_n$ is finite, and hence the derivative λ'_j of λ_j exists almost everywhere in I and belongs to $L^1(I)$ by (2.1). It coincides with the distributional derivative of λ_j in I , and $\|\lambda'_j\|_{p,w,I} = \|\lambda'_j\|_{p,w,\Omega_n}$. (If $\phi \in C_c^\infty(I)$ and (a, b) is a connected component of Ω_n , then $\int_a^b \lambda_j \phi' dt = \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\epsilon} \lambda_j \phi' dt = \lambda_j(b)\phi(b) - \lambda_j(a)\phi(a) - \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^{b-\epsilon} \lambda'_j \phi dt = \lambda_j(b)\phi(b) - \lambda_j(a)\phi(a) - \int_a^b \lambda'_j \phi dt$, by the dominated convergence theorem. Since $I \setminus \Omega_n$ is finite, all boundary terms cancel and thus $\int_I \lambda_j \phi' dt = - \int_I \lambda'_j \phi dt$.) This implies (3.8).

3.6. Weighted homogeneity

Let $a(t)$ be real analytic and suppose that the discriminant of $P_{a(t)}$ is not identically equal to zero. We do not assume any longer that $a(t) \in \mathcal{B}$ for all $t \in I$. We extend the bound of the previous subsection to such curves using the weighted homogeneity.

For $\eta > 0$ and $a \in \mathbb{C}^n$ we define $\eta * a \in \mathbb{C}^n$ by $(\eta * a)_i = \eta^i a_i$. Then λ is a root of P_a if and only if $\eta\lambda$ is a root of $P_{\eta * a}$.

Fix $\rho \geq \max\{1, \sup_{t \in I} \|a(t)\|\}$. Then $\|\rho^{-1} * a\| \leq 1$. For a polynomial $g \in \mathbb{C}[a]$, set $\tilde{g}(t) := g(\rho^{-1} * a(t))$. Then by (3.8)

$$\|\lambda'_j\|_{p,w,I} \leq \rho C(\mathcal{CV}) \sum_{j,m} N_{\tilde{\alpha}_{j,m,k,s_{j,m},I}}(\tilde{h}_{j,m}) \max_i \{H_{k,I}(\tilde{h}_{j,m}), H_{k,I}(\tilde{P}_{j,m,i})\}. \tag{3.9}$$

3.7. General Case

Let $a \in C^k(I, \mathbb{C}^n)$. By the classical Weierstrass theorem there is a sequence of polynomial curves $(a_\nu) \subset C^\omega(I, \mathbb{C}^n)$, such that

$$a_\nu \longrightarrow a \quad \text{in } C^k(I, \mathbb{C}^n) \quad \text{as } \nu \rightarrow \infty.$$

By replacing a_ν by $a_\nu + (0, \dots, 0, \varepsilon_\nu)$, with $\varepsilon_\nu > 0$ sufficiently small, we may suppose moreover that the discriminant of each P_{a_ν} is not identically zero.

For each ν choose a continuous parameterization $\lambda_\nu = (\lambda_{\nu,1}, \dots, \lambda_{\nu,n}) \in C^0(I, \mathbb{C}^n)$ of the roots of $P_{a_\nu}(t)$, $t \in I$. Since (a_ν) is bounded in $C^k(I, \mathbb{C}^n)$, we may infer from (3.8) that the set of distributional derivatives $\{\lambda'_\nu; \nu\}$ is bounded in $L^q(I, \mathbb{C}^n)$ for every $q \in [1, p)$.

Fix $q \in (1, p)$. By the Arzelá-Ascoli theorem, as (λ_ν) is equi-Hölder, or alternatively by the Rellich-Kondrachov compactness theorem, there is a subsequence $(\lambda_{\nu(\ell)})$ that converges in $C^0(I, \mathbb{C}^n)$ to some λ .

Since $L^q(I)$ is reflexive, we also have (after possibly passing to a subsequence again) that $(\lambda'_{\nu(\ell)})$ converges to some λ' weakly in $L^q(I, \mathbb{C}^n)$. Then λ' is the distributional derivative of λ and thus $\lambda \in W^{1,q}(I, \mathbb{C}^n)$. It is clear that λ forms a parameterization of the roots of P_a on I .

Lemma 3.6. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be two parameterizations of the roots of $P_a(t)(Z) \in C^k(I)[Z]$. If $\lambda \in W^{1,q}(I, \mathbb{C}^n)$, for $q \in [1, p)$, and $\mu \in C^0(I, \mathbb{C}^n)$, then also $\mu \in W^{1,q}(I, \mathbb{C}^n)$ and*

$$\sum_{i=1}^n \|\mu'_i\|_{L^q(I)}^q = \sum_{i=1}^n \|\lambda'_i\|_{L^q(I)}^q. \tag{3.10}$$

Proof. For each j we have

$$\text{length}(\mu_j) \leq \sum_{i=1}^n \text{length}(\lambda_i) < \infty$$

and so μ_j is of bounded variation. Moreover, for any subset $E \subset I$

$$\mu_j(E) \subset \bigcup_{i=1}^n \lambda_i(E)$$

and hence μ_j has the Luzin (N) property. We may conclude that each μ_j is absolutely continuous on I and hence the derivative μ'_j of μ_j exists almost everywhere in I and coincides with the distributional derivative of μ_j in I .

At points t , where each μ_j and each λ_i is differentiable, the sets $\{\mu'_j(t)\}$ and $\{\lambda'_i(t)\}$ coincide together with the multiplicities of its elements. These points form a subset of I of full measure and therefore $\mu'_j \in L^q(I)$ and satisfies (3.10). \square

Uniformity can be seen by repeating the proof with an additional parameter v . The weak limits in the reasoning above are weakly bounded and thus bounded in $L^q(I)$.

4. Multiparameter families of polynomials

Theorem 4.1. *Let $k = k(n) \in \mathbb{N}_{>0}$ and $p = p(n) > 1$ be as in Subsection 3.4. Let $U \subset \mathbb{R}^m$ be open and let*

$$P_{a(x)}(Z) = Z^n + \sum_{j=1}^n a_j(x)Z^{n-j} \in C^k(U)[Z]$$

be a monic polynomial with coefficients $a_j \in C^k(U)$, for $j = 1, \dots, n$.

- (1) *Let $\lambda \in C^0(V)$ represent a root of P_a , i.e., $P_a(\lambda) = 0$, on a relatively compact open subset $V \Subset U$. Then the distributional gradient of λ in V is a measurable function $\nabla\lambda \in [L^q(V)]^m$ for every $q \in [1, p)$. In particular, $\lambda \in W^{1,q}(V)$ for every $q \in [1, p)$.*
- (2) *The regularity of the roots is uniform. Let $\{P_{a_v}; v \in \mathcal{N}\}$,*

$$P_{a_v(x)}(Z) = Z^n + \sum_{j=1}^n a_{v,j}(x)Z^{n-j} \in C^k(U)[Z], \quad \text{for } v \in \mathcal{N},$$

be a family of polynomials, indexed by v in some set \mathcal{N} , so that the set of coefficients $\{a_{v,j}; v \in \mathcal{N}, \text{ for } j = 1, \dots, n\}$ is bounded in $C^k(U)$. Let $V \Subset U$. Then the set

$$\{\lambda_v \in C^0(V); P_{a_v}(\lambda_v) = 0 \text{ on } V, \quad \text{for } v \in \mathcal{N}\}$$

is bounded in $W^{1,q}(V)$ for every $q \in [1, p)$.

Remark 4.2. The roots of a polynomial depending on at least two parameters, in general, do not admit a continuous parameterization due to monodromy. For instance, the radical $\mathbb{C} \ni (x + iy) \mapsto (x + iy)^{1/n}$ does not admit continuous parameterizations on \mathbb{C} .

Proof of Theorem 4.1. By Theorem 3.5, λ is absolutely continuous along each affine line parallel to the coordinate axes. So λ possesses the partial derivatives $\partial_i \lambda$, for $i = 1, \dots, m$, which are defined almost everywhere and are measurable. It clearly suffices to show that all partial derivatives $\partial_j \lambda$ belong to $L^q(V)$, for every $q \in [1, p)$.

Set $x = (t, y)$, where $t = x_1, y = (x_2, \dots, x_m)$, and let V_1 be the orthogonal projection of V on the hyperplane $\{x_1 = 0\}$. For each $y \in V_1$ we denote by $V^y := \{t \in \mathbb{R}; (t, y) \in V\}$ the corresponding section of V ; note that V^y is open in \mathbb{R} . Then by Fubini's theorem,

$$\int_V |\partial_1 \lambda(x)|^q dx = \int_{V_1} \int_{V^y} |\partial_1 \lambda(t, y)|^q dt dy. \tag{4.1}$$

We may cover V by finitely many open boxes $K = I_1 \times \dots \times I_m$ contained in U . Let K be fixed and set $L = I_2 \times \dots \times I_m$. Fix $y \in V_1 \cap L$ and let λ_j^y , for $j = 1, \dots, n$, be a continuous parameterization of the roots of $P_a(\cdot, y)$ on $\Omega^y := V^y \cap I_1$ such that $\lambda(\cdot, y) = \lambda_1^y$; it exists since $\lambda(\cdot, y)$ can be completed to a continuous parameterization of the roots of $P_a(\cdot, y)$ on each connected component of Ω^y by [24, Lemma 6.17]. Our goal is to bound

$$\|\partial_t \lambda(\cdot, y)\|_{L^q(\Omega^y)} = \left\| (\lambda_1^y)' \right\|_{L^q(\Omega^y)}$$

uniformly with respect to $y \in V_1 \cap L$.

To this end let \mathcal{C}^y denote the set of connected components J of the open set Ω^y . For each $J \in \mathcal{C}^y$ extend the parameterization $\lambda_j^y|_J$, for $j = 1, \dots, n$, continuously to I_1 , i.e., choose

$$\lambda_j^{y,J} \in C^0(I_1), \quad \text{for } j = 1, \dots, n, \quad \text{so that}$$

$$\forall j \quad \lambda_j^{y,J}|_J = \lambda_j^y|_J \quad \text{and} \quad P_a(t, y)(Z) = \prod_{j=1}^n (Z - \lambda_j^{y,J}(t)), \quad \text{for } t \in I_1.$$

This is possible since $\lambda_j^y|_J$ has a continuous extension to the endpoints of the (bounded) interval J , by [14, Lemma 4.3], and can then be extended on the left and on the right of J by a continuous parameterization of the roots of $P_a(\cdot, y)$ on I_1 after suitable permutations.

By Theorem 3.5, for each $y \in V_1 \cap L$, each $J \in \mathcal{C}^y$, and each $j = 1, \dots, n$, $\lambda_j^{y,J}$ is absolutely continuous on I_1 and $(\lambda_j^{y,J})' \in L^q(I_1)$ with

$$\sup_{y, J, j} \left\| (\lambda_j^{y,J})' \right\|_{L^q(I_1)} < \infty. \tag{4.2}$$

Let $J, J_0 \in \mathcal{C}^y$ be arbitrary. By Lemma 3.6, $(\lambda_j^y)'$ as well as $(\lambda_j^{y, J_0})'$ belong to $L^q(J)$ and we have

$$\sum_{j=1}^n \left\| (\lambda_j^y)' \right\|_{L^q(J)}^q = \sum_{j=1}^n \left\| (\lambda_j^{y, J})' \right\|_{L^q(J)}^q = \sum_{j=1}^n \left\| (\lambda_j^{y, J_0})' \right\|_{L^q(J)}^q .$$

Thus, for arbitrary fixed $J_0 \in \mathcal{C}^y$,

$$\begin{aligned} \sum_{j=1}^n \left\| (\lambda_j^y)' \right\|_{L^q(\Omega^y)}^q &= \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \left\| (\lambda_j^y)' \right\|_{L^q(J)}^q \\ &= \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \left\| (\lambda_j^{y, J_0})' \right\|_{L^q(J)}^q \\ &= \sum_{j=1}^n \left\| (\lambda_j^{y, J_0})' \right\|_{L^q(\Omega^y)}^q \\ &\leq \sum_{j=1}^n \left\| (\lambda_j^{y, J_0})' \right\|_{L^q(I_1)}^q . \end{aligned}$$

In view of (4.2) we may conclude that $\sup_{y \in V_1 \cap L} \|(\lambda_1^y)'\|_{L^q(\Omega^y)} < \infty$. By (4.1) and since the number of boxes K is finite, this implies that $\partial_1 \lambda \in L^q(V)$. The other partial derivatives can be treated analogously. This shows (1).

In order to see (2) it suffices to repeat the proof of (1) paying attention to the additional dependence on ν . □

Part 2. Formulas for the roots. Proof of Theorem 1.6

5. Strategy of the proof

The main ideas of the proof of Theorem 1.6 are the following. Let $x = (x_1, \dots, x_r)$ be local coordinates at $0 \in \mathbb{C}^r$. Suppose that $a_i \in \mathbb{C}\{x\}$ and let

$$P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j} .$$

Thus we may consider \mathcal{D}_m , defined in Section 1, as an ideal of $\mathbb{C}\{x\}$. If $a_1 = 0$ and \mathcal{D}_2 is principal and generated by a monomial, then we may split P_a , that is factor it $P_a = P_b P_c$; see Step 2 of the proof, Section 9. This requires introducing fractional powers. If we can continue this process by splitting P_b, P_c , and then their factors, etc., then we eventually arrive at linear factors, (i.e., of degree 1) whose coefficients are the roots. As we show in the next three sections this can be guaranteed by the principalization of the higher order discriminant ideals \mathcal{D}_m .

A subtle point of Theorem 1.6 and hence of its proof is to obtain the exponent α_m in (1.6) strictly positive. This forces us to blow-up the ideals \mathcal{D}_m one by one. Then we put each factor in Tschirnhausen form, which amounts to subtracting a fraction of its first coefficient from the roots. The remaining part of the roots vanishes on $V(\mathcal{D}_{m+1})$ and hence we may continue.

This consecutive splitting process can be compared to the proof of the Abhyankar-Jung Theorem of [19], that gives a formula for the roots in one shot by making the discriminant normal crossing, but without the property $\alpha_m > 0$ which is crucial for us. In the splitting process of [19], at each stage, the coefficients of the factors, say, defined on M_{m+1} , are expressed in terms of their product, which is well-defined on M_m . The complexity of our proof comes from the fact that, in the formula (1.6), we need each f_m to be the pull-back of a polynomial in the coefficients a_i , that is, of a polynomial defined on \mathbb{C}^n . Similarly, each $y_{m,i}$ has to be the pull-back of a rational function on \mathbb{C}^n .

6. A characterization of principality

Let $x = (x_1, \dots, x_r)$ be local coordinates at $0 \in \mathbb{C}^r$. Suppose that $a_i \in \mathbb{C}\{x\}$, for $i = 1, \dots, n$. We denote by $\xi(x) = \{\xi_1(x), \dots, \xi_n(x)\}$ the unordered set of roots of P_a . Let

$$S_m(x) = \max_{|I|=m} \prod_{i \neq j \in I} |\xi_i(x) - \xi_j(x)|.$$

By definition S_m is the germ at the origin of a non-negative real-valued function of $x \in \mathbb{C}^r$.

Proposition 6.1.

- (1) *There is a finite family $g_1, \dots, g_p \in \mathcal{D}_m$ such that $S_m^{N!/m(m-1)} \sim \max_j |g_j|$. Moreover, we may take as g_1, \dots, g_p any system of generators of \mathcal{D}_m .*
- (2) *$g \in \mathcal{D}_m$ generates \mathcal{D}_m if and only if $S_m^{N!/m(m-1)} \sim |g|$.*

Proof. (1) We can choose as g_1, \dots, g_p the powers $\sigma_i^{N!/im(m-1)}$ of the elementary symmetric functions $\sigma_i, i = 1, \dots, \binom{n}{m}$, in $\lambda_I = \prod_{i \neq j \in I} (\xi_i(x) - \xi_j(x))$, for $I \subset \{1, \dots, n\}$, with $|I| = m$. Indeed

$$\begin{aligned} \left| \sigma_s(\lambda_I; |I| = m) \right|^{\frac{N!}{sm(m-1)}} &\leq \binom{n}{m}^{\frac{N!}{sm(m-1)}} \max_I |\lambda_I|^{\frac{N!}{m(m-1)}} \\ &= \binom{n}{m}^{\frac{N!}{sm(m-1)}} S_m(x)^{\frac{N!}{m(m-1)}}. \end{aligned}$$

The converse estimate follows from Lemma 6.2 below.

(2) By (1) if g generates \mathcal{D}_m then $|g| \sim S_m^{N!/m(m-1)}$. Suppose now that $|g| \sim S_m^{N!/m(m-1)}$. Then for any $f \in \mathcal{D}_m$ the quotient f/g is bounded and hence holomorphic. \square

Lemma 6.2 ([16, page 56] , or [21, Theorem 1.1.4.]). *If $a_1, \dots, a_n, z \in \mathbb{C}$ satisfy the equation $z^n + \sum_{j=1}^n a_j z^{n-j} = 0$, then $|z| \leq 2 \max_j |a_j|^{1/j}$.*

Corollary 6.3. *If \mathcal{D}_m is principal then $\mathcal{D}_l \subset \mathcal{D}_m$ for $l \geq m$.*

Corollary 6.4. *The zero set of \mathcal{D}_m equals $\{x; |\{\xi_1(x), \dots, \xi_n(x)\}| < m\}$.*

Corollary 6.5. *Suppose that $a_1 = 0$. Then \mathcal{D}_2 is principal if and only if so is $(a_2^{N!/2}, \dots, a_n^{N!/n})$, and then both ideals coincide.*

Thus, if $\mathcal{D}_2 = (g)$ then g divides each $a_i^{N!/i}$ and there is i_0 such that $|g| \sim |a_{i_0}^{N!/i_0}|$.

Proof. Shortly speaking, this corollary follows from the fact that, by Proposition 1.2, the ideals \mathcal{D}_2 and $(a_2^{N!/2}, \dots, a_n^{N!/n})$ have the same integral closure.

More precisely, in the Tschirnhausen case $a_1 = 0$,

$$\begin{aligned} \max_j |\xi_j| &= \max_j \left| \frac{1}{n} \sum_k (\xi_j - \xi_k) \right| \leq \max_j \frac{1}{n} \sum_k |\xi_j - \xi_k| \leq \max_{k \neq j} |\xi_j - \xi_k|, \\ \max_{k \neq j} |\xi_j - \xi_k| &\leq \max_{k \neq j} (|\xi_j| + |\xi_k|) \leq 2 \max_j |\xi_j|, \end{aligned}$$

and hence, by Lemma 6.2,

$$S_2 \sim \max_i |\xi_i|^2 \sim \max_j |a_j|^{2/j}. \tag{6.1}$$

Thus if $a_i^{N!/i}$ generates $(a_2^{N!/2}, \dots, a_n^{N!/n})$ then for any $f \in \mathcal{D}_2$, $f/a_i^{N!/i}$ is bounded and hence holomorphic. Therefore $a_i^{N!/i}$ generates \mathcal{D}_2 .

Conversely, if g generates \mathcal{D}_2 then by (6.1) one of the $a_i^{N!/i}/g$ does not vanish at the origin and hence $g \in (a_2^{N!/2}, \dots, a_n^{N!/n})$. \square

7. Convexity

For a power series in one variable t , $\lambda \in \mathbb{C}\{t\}$, we define its order $\text{ord}_0 \lambda$ as the leading exponent $\lambda(t) = a_0 t^{\text{ord}_0 \lambda} + \dots$ and set $\text{ord}_0 0 := \infty$. Given the power series $\lambda_i \in \mathbb{C}\{t\}$, for $i = 1, \dots, n$, we define, for $2 \leq m \leq n$,

$$\alpha(m) := \min_{|I|=m} \text{ord}_0 \prod_{i \neq j \in I} (\lambda_i - \lambda_j).$$

Proposition 7.1. For $3 \leq m \leq n - 1$,

$$2\alpha(m) + \alpha(2) \leq \alpha(m - 1) + \alpha(m + 1). \tag{7.1}$$

Example 7.2. If all $\lambda_i - \lambda_j$ have the same order, say, equal to 1, then $\alpha(m) = 2\binom{m}{2}$ and we have equality.

For the proof we first make some reduction. By shifting all λ_i 's by $\frac{1}{n} \sum_i \lambda_i$ we may assume that $\sum_i \lambda_i = 0$. Then $\alpha(2) = 2 \min_i \text{ord}_0 \lambda_i$. Indeed,

$$\begin{aligned} \alpha(2)/2 &= \min_{i \neq j} \text{ord}_0(\lambda_i - \lambda_j) \geq \min_{i \neq j} \min\{\text{ord}_0 \lambda_i, \text{ord}_0 \lambda_j\} = \min_i \text{ord}_0 \lambda_i \\ &= \min_i \text{ord}_0 \frac{1}{n} \sum_k (\lambda_i - \lambda_k) \geq \min_i \min_{i \neq k} \text{ord}_0(\lambda_i - \lambda_k) = \alpha(2)/2. \end{aligned}$$

Thus, by dividing all λ_i by $t^{\alpha(2)/2}$, we may suppose that $\alpha(2) = 0$. Then (7.1) becomes a genuine convexity relation.

Let us divide the set of λ_i into the union of two disjoint non-empty subsets $\{\lambda_i; i \in I\}$ and $\{\lambda_j; j \in J\}$, for $|I| = n_1 < n, |J| = n_2 < n$, and $n_1 + n_2 = n$, so that for each $i \in I, j \in J$, one has $\text{ord}_0(\lambda_i - \lambda_j) = 0$. We shall call such a partition a *splitting*.

The corresponding orders for these two families we denote by $\beta(m)$, for $2 \leq m \leq n_1$, and $\gamma(m)$, for $2 \leq m \leq n_2$. Then for each $2 \leq m \leq n$ there is $0 \leq m_1 \leq n_1$ such that

$$\alpha(m) = \beta(m_1) + \gamma(m - m_1). \tag{7.2}$$

It is possible that m_1 or $m - m_1$ is equal to 0 or 1. In this case we put $\beta(0) = \beta(1) = \gamma(0) = \gamma(1) = 0$.

For a couple of integers $a \leq b$ we denote by $[a .. b] := \{c \in \mathbb{Z}; a \leq c \leq b\}$ the set of all integers between a and b and call it an *interval*.

Proposition 7.3. For each m the set of m_1 such that (7.2) holds is an interval. If we denote this interval by $[\underline{K}_1(m) .. \overline{K}_1(m)]$ then, for $2 \leq m \leq n - 1$, one has $\underline{K}_1(m) \leq \underline{K}_1(m + 1) \leq \underline{K}_1(m) + 1$ and $\overline{K}_1(m) \leq \overline{K}_1(m + 1) \leq \overline{K}_1(m) + 1$.

Proof of Propositions 7.1 and 7.3. First we show Proposition 7.3 assuming Proposition 7.1 for β and γ . Let

$$\varphi_m(m_1) := \beta(m_1) + \gamma(m - m_1).$$

The set of m_1 such that (7.2) holds is the set of m_1 at which $\varphi_m(m_1)$ is minimal. It is an interval since, by assumption, $\varphi_m(m_1)$ is convex. Moreover, by convexity, φ_m is decreasing on $[2 .. \underline{K}_1(m)]$ and increasing on $[\overline{K}_1(m) .. n_1]$.

We show that it is not possible that $\varphi_m(m_1) \leq \varphi_m(m_1 + 1)$ and $\varphi_{m+1}(m_1 + 1) \geq \varphi_{m+1}(m_1 + 2)$ with one of these inequalities strict. Indeed, if this were the case then

$$\begin{aligned} \varphi_m(m_1) + \varphi_{m+1}(m_1 + 2) &= \beta(m_1) + \gamma(m - m_1) + \beta(m_1 + 2) + \gamma(m - m_1 - 1) \\ &< \varphi_m(m_1 + 1) + \varphi_{m+1}(m_1 + 1) = 2\beta(m_1 + 1) + \gamma(m - m_1 - 1) + \gamma(m - m_1) \end{aligned}$$

which contradicts the convexity of β . This implies that $\overline{K}_1(m+1) \leq \overline{K}_1(m)+1$ and $\underline{K}_1(m+1) \leq \underline{K}_1(m)+1$. By interchanging I, β and J, γ we obtain $\overline{K}_2(m+1) \leq \overline{K}_2(m)+1$ and $\underline{K}_2(m+1) \leq \underline{K}_2(m)+1$ which are the desired inequalities in view of $\overline{K}_2(m) = m - \underline{K}_1(m)$ and $\underline{K}_2(m) = m - \overline{K}_1(m)$, where $\underline{K}_2, \overline{K}_2$ play the roles of $\underline{K}_1, \overline{K}_1$ for γ .

Now we show how Proposition 7.3 implies Proposition 7.1 for α . Fix m such that $3 \leq m \leq n-1$. By Proposition 7.3 we may assume $[\underline{K}_1(m-1) .. \overline{K}_1(m-1)] \cap [\underline{K}_1(m) .. \overline{K}_1(m)] \neq \emptyset$. (If this is not the case then $\underline{K}_1(m) = \overline{K}_1(m) = \underline{K}_1(m-1) + 1 = \overline{K}_1(m-1) + 1$ and thus $\underline{K}_2(m-1) = \overline{K}_2(m-1) = \underline{K}_2(m) = \overline{K}_2(m)$.)

Fix $m_1 \in [\underline{K}_1(m-1) .. \overline{K}_1(m-1)] \cap [\underline{K}_1(m) .. \overline{K}_1(m)]$. Then, by Proposition 7.3, either m_1 or $m_1 + 1$ belongs to $[\underline{K}_1(m+1) .. \overline{K}_1(m+1)]$. In the former case

$$\begin{aligned} \alpha(m-1) + \alpha(m+1) &= \beta(m_1) + \gamma(m - m_1 - 1) + \beta(m_1) + \gamma(m - m_1 + 1) \\ &\geq 2(\beta(m_1) + \gamma(m - m_1)) = 2\alpha(m). \end{aligned}$$

In the latter case

$$\begin{aligned} \alpha(m-1) + \alpha(m+1) &= \beta(m_1) + \gamma(m - m_1 - 1) + \beta(m_1 + 1) + \gamma(m - m_1) \\ &\geq 2(\beta(m_1) + \gamma(m - m_1)) = 2\alpha(m), \end{aligned}$$

since $\beta(m_1+1)+\gamma(m-m_1-1) \geq \beta(m_1)+\gamma(m-m_1)$ as $m_1 \in [\underline{K}_1(m) .. \overline{K}_1(m)]$. \square

8. Splitting

We suppose that $a_i \in \mathbb{C}\{x\}, x = (x_1, \dots, x_r)$, and that P_a factors

$$P_a(Z) = P_b(Z)P_c(Z),$$

where $P_b(Z) = Z^{n_1} + b_1Z^{n_1-1} + \dots + b_{n_1}$, and $P_c(Z) = Z^{n_2} + c_1Z^{n_2-1} + \dots + c_{n_2}$, for $b_i, c_j \in \mathbb{C}\{x\}$, and for $n = n_1 + n_2$, with $n_1, n_2 > 0$.

Assumption. We shall assume that the resultant of P_b and P_c does not vanish at 0, that is $P_{b(0)}$ and $P_{c(0)}$ do not have common roots.

In order to distinguish the ideals \mathcal{D}_m for polynomials P_a, P_b , and P_c we shall denote them by $\mathcal{D}_{a,m}, \mathcal{D}_{b,m}$, and $\mathcal{D}_{c,m}$ respectively; likewise for the size functions $S_{a,m}, S_{b,m}, S_{c,m}$.

Proposition 8.1.

- (1) Suppose that $\mathcal{D}_{a,m}$ is a principal ideal. Then there are $m_1 \geq 0$, and $m_2 \geq 0$, with $m_1 + m_2 = m$, such that

$$S_{a,m} \sim S_{b,m_1} S_{c,m_2}, \tag{8.1}$$

and for any such m_1, m_2 , the ideals \mathcal{D}_{b,m_1} and \mathcal{D}_{c,m_2} are principal. (We put $S_0 = S_1 = 1$.) Moreover, the set of those m_1 for which (8.1) holds, with $m_2 = m - m_1$, is an interval. We denote this interval by $[\underline{K}_1(m) .. \overline{K}_1(m)]$.

- (2) If the ideals $\mathcal{D}_{a,m}$ and $\mathcal{D}_{a,m+1}$ are principal then $\underline{K}_1(m) \leq \underline{K}_1(m+1) \leq \underline{K}_1(m) + 1$ and $\overline{K}_1(m) \leq \overline{K}_1(m+1) \leq \overline{K}_1(m) + 1$.
- (3) Suppose that the ideals $\mathcal{D}_{a,i}$ are principal for $2 \leq i \leq m$ and that (8.1) holds. Then $\mathcal{D}_{b,i}$ and $\mathcal{D}_{c,j}$ are principal for all $i \leq m_1$ and $j \leq m_2$.

Proof. For fixed x we may order the roots of $P_{a(x)}$. Given $I \subset \{1, \dots, n\}$, with $|I| = m$, we divide $I = I' \cup I''$, so that I' , with $|I'| = m_1$, labels the roots of $P_{b(x)}$ and I'' , with $|I''| = m_2$, the roots of $P_{c(x)}$. Then

$$\prod_{i,j \in I, i \neq j} (\xi_i(x) - \xi_j(x)) = \prod_{i \neq j \in I'} (\xi_i(x) - \xi_j(x)) \cdot \prod_{i \neq j \in I''} (\xi_i(x) - \xi_j(x)) \cdot \varphi(x),$$

and $\varphi(x) = \prod_{i \in I', j \in I''} (\xi_i(x) - \xi_j(x))$ is non-zero and close to $\prod_{i \in I', j \in I''} (\xi_i(0) - \xi_j(0))$. Therefore, as functions of x ,

$$S_{a,m} \sim \max_{m_1+m_2=m} S_{b,m_1} S_{c,m_2}. \tag{8.2}$$

Suppose that $\mathcal{D}_{a,m}$ is generated by f . By (8.2) there exist $m_1 + m_2 = m$ and $h \in \mathcal{D}_{b,m_1}, g \in \mathcal{D}_{c,m_2}$ such that $(gh/f)(0) \neq 0$. Then (8.1) holds and, by Proposition 6.1, g generates \mathcal{D}_{b,m_1} and h generates \mathcal{D}_{c,m_2} .

Thus if $\mathcal{D}_{a,m}$ is principal then the set of m_1 for which (8.1) holds, which we denote by \mathcal{M} , is non-empty. By the curve selection lemma, see for instance [18], (8.1) holds if and only if it holds on every real analytic curve $x(t) : (\mathbb{R}, 0) \rightarrow (\mathbb{C}^r, 0)$. Therefore, by (8.1) and Proposition 7.3, \mathcal{M} , as an intersection of intervals, is an interval. Denote it by $[\underline{K}_1(m) .. \overline{K}_1(m)]$. Moreover, if $\mathcal{D}_{a,m}$ and $\mathcal{D}_{a,m+1}$ are principal then $\underline{K}_1(m), \overline{K}_1(m)$ satisfy the inequalities of Proposition 7.3. Indeed, suppose for instance that $\overline{K}_1(m) \leq \overline{K}_1(m+1)$ fails. Then there exists a real analytic curve on which $S_{a,m+1} \sim S_{b,m_1} S_{c,m+1-m_1}$ with $m_1 = \overline{K}_1(m)$ fails but is satisfied for some $m_1 < \overline{K}_1(m)$. But this contradicts Proposition 7.3 since (8.1) holds on this curve. This shows (1) and (2).

Finally (3) follows from (1) and (2). □

Proposition 8.2. *Suppose that $\mathcal{D}_{a,i}$ are principal for all $2 \leq i \leq n$. Let $m_1(m)$, and $m_2(m) = m - m_1(m)$ be two non-decreasing integer-valued functions defined for $0 \leq m \leq n$, such that $0 \leq m_1(m) \leq n_1$, and $0 \leq m_2(m) \leq n_2$, for every $0 \leq m \leq n$, and such that (8.1) holds for each triple $(m, m_1(m), m_2(m))$, for $0 \leq m \leq n$. Let $\mathcal{D}_{a,m} = (f_m), \mathcal{D}_{b,m_1} = (g_{m_1}), \mathcal{D}_{c,m_2} = (h_{m_2})$.*

Then, for every $1 \leq m \leq n$ exactly one of the following two cases happens

- (1) $m_1(m) = m_1(m-1) + 1, m_2(m) = m_2(m-1)$;
- (2) $m_2(m) = m_2(m-1) + 1, m_1(m) = m_1(m-1)$.

Moreover, if (1) holds then $g_{m_1} | f_m | g_{m_1}^m$, and symmetrically, if (2) holds then $h_{m_2} | f_m | h_{m_2}^m$.

Proof. Two non-decreasing non-negative functions m_1, m_2 such that $m_1(m) + m_2(m) = m$ must satisfy either (1) or (2). Thus it suffices to show that if (1) is satisfied then $g_{m_1} | f_m | g_{m_1}^m$. The first condition $g_{m_1} | f_m$ follows easily from the proof of Proposition 8.1. If (8.1) holds then $g_{m_1} h_{m_2}$ generates $\mathcal{D}_{a,m}$ and we may suppose that

$$f_m = g_{m_1} h_{m_2}. \tag{8.3}$$

By Corollary 6.3, $1 = f_1 | f_2 | \dots | f_n$, and similarly $1 = g_1 | g_2 | \dots | g_{n_1}$ and $1 = h_1 | h_2 | \dots | h_{n_2}$.

Let r be given by $m_1(m) = m_1(m - 1) + 1 = m_1(m - 2) + 1 = \dots = m_1(m - r) + 1 = m_1(m - r - 1) + 2$; we have Case (1) for m and $m - r$ and Case (2) in between. We write $m_1 = m_1(m)$, $m_2 = m_2(m)$ for short. Then

$$\begin{aligned} f_{m-1} &= g_{m_1-1} h_{m_2} | g_{m_1} h_{m_2-1} \\ &\dots \\ f_{m-r+1} &= g_{m_1-1} h_{m_2-r+2} | g_{m_1} h_{m_2-r+1} \\ f_{m-r} &= g_{m_1-1} h_{m_2-r+1}. \end{aligned}$$

Hence

$$h_{m_2} \left| (g_{m_1}/g_{m_1-1}) h_{m_2-1} \right| \dots \left| (g_{m_1}/g_{m_1-1})^{r-1} h_{m_2-r+1} \right|.$$

Consequently

$$f_m = g_{m_1} h_{m_2} \left| (g_{m_1}/g_{m_1-1})^r g_{m_1-1} h_{m_2-r+1} \right| = (g_{m_1}/g_{m_1-1})^r f_{m-r}.$$

Since $m_1(m - r) + 1 = m_1(m - r - 1) + 2$, by induction on m_1 , $f_{m-r} | g_{m_1-1}^{m-r}$ (we may start the induction by formally putting $f_1 = g_1 = h_1 = 1$), which shows $f_m | g_{m_1}^m$ as $g_{m_1-1} | g_{m_1}$. □

Remark 8.3. By Proposition 8.1, both $m_1(m) = \underline{K}_1(m)$ and $m_1(m) = \overline{K}_1(m)$ satisfy the assumptions of Proposition 8.2 if we complete them by putting $\underline{K}_1(1) = 0$, $\overline{K}_1(1) = 1$. In particular, there exists a function $m_1(m)$ satisfying the assumptions of Proposition 8.2, such that $m_1(2) = m_2(2) = 1$. This follows from the assumption that $P_{b(0)}$ and $P_{c(0)}$ do not have common roots; in particular, not all roots of $P_{a(0)}$ coincide and thus $S_{a,2} \sim 1$.

Fix such a function $m_1(m)$. For $2 \leq m_1 \leq n_1$ define $m_b(m_1)$ as the smallest m such that $m_1 = m_1(m)$, i.e., $m_b(m_1)$ as a function is the lowest inverse of $m_1(m)$. Similarly we define $m_c(m_2)$ for $2 \leq m_2 \leq n_2$. The functions m_b and m_c are strictly increasing and each $3 \leq m \leq n$ is in the image of precisely one of them. See the example in Figure 8.1.

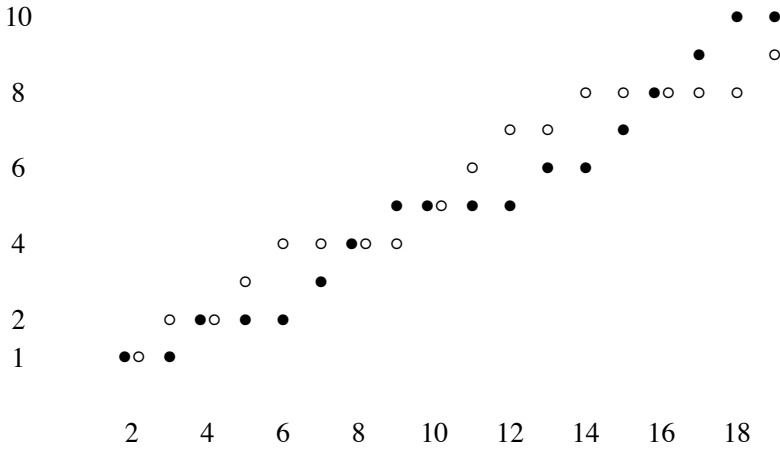


Figure 8.1. An example for the functions $m_1(m)$, bullets \bullet , and $m_2(m)$, circles \circ . In this example $(m_b(2), m_b(3), \dots) = (4, 7, 8, 9, 13, 15, 16, 17, 18)$ and $(m_c(2), m_c(3), \dots) = (3, 5, 6, 10, 11, 12, 14, 19)$; these are the points where the respective sequence increases by 1. The two sets form a partition of the integers between 3 and 19.

9. Proof of Theorem 1.6

Proof. We first put P_a into Tschirnhausen form (1.2), then split it into two factors after blowing up \mathcal{D}_2 . These factors are defined locally on the blow-up space M_2 . Then we put each of these factors into Tschirnhausen form and continue the process by splitting the subsequent factors on M_3, M_4 and so on, putting first every new factor into Tschirnhausen form. At each stage the Tschirnhausen transformation shifts the roots exactly by the term $A_m\varphi_m$ of (1.5).

In order to simplify the notation we use the same letters for the functions and their pull-backs to the blow-up spaces, for instance we write a_k for $a_k \circ \sigma_m$.

Step 1. First we perform the Tschirnhausen transformation by replacing Z by $Z - a_1/n$ and hence shifting the roots by a_1/n . Thus we put

$$\varphi_1 := a_1 \quad \text{and} \quad A_1 := \frac{1}{n}.$$

(Recall that the Tschirnhausen transformation does not change the ideals \mathcal{D}_m .) After this transformation we may assume that $P_a(Z)$ is in the Tschirnhausen form (1.2).

Step 2. Fix $p_2 \in M_2$ and a privileged system of coordinates $y_{2,1}, \dots, y_{2,n}$ at p_2 . We shall split P_a at p_2

$$P_a = P_b P_c, \tag{9.1}$$

where $P_b(Z) = Z^{n_1} + \sum_{i=1}^{n_1} b_i Z^{n_1-i}$, and $P_c(Z) = Z^{n_2} + \sum_{j=1}^{n_2} c_j Z^{n_2-j}$, with $n_1 > 0, n_2 > 0$, with $b_i, c_j \in \mathbb{C}\{y_{2,1}^{1/q_2}, \dots, y_{2,n}^{1/q_2}\}$, and $q_2 = N!$, as follows. Let f_2 generate \mathcal{D}_2 at p_2 . By Corollary 6.5, f_2 divides each $a_i^{N!/i}$ and there is i_0 such that f_2 equals $a_{i_0}^{N!/i_0}$ times a unit. Consider an auxiliary polynomial

$$Q_{\bar{a}}(\tilde{Z}) := f_2^{-n/q_2} P_a \left(f_2^{1/q_2} \tilde{Z} \right) = \tilde{Z}^n + \bar{a}_2 \tilde{Z}^{n-2} + \dots + \bar{a}_n, \tag{9.2}$$

where $\bar{a}_i = f_2^{-i/q_2} a_i$, and $\bar{a}_{i_0}(p_2) \neq 0$. Because f_2 is a normal crossing in $y_{2,i}$, cf. (1.3),

$$f_2^{1/q_2} \in \mathbb{C} \left\{ \left(y_{2,1}^{1/q_2}, \dots, y_{2,r_2}^{1/q_2}, y_{2,r_2+1}, \dots, y_{2,n} \right) \right\}.$$

We first split $Q_{\bar{a}}(\tilde{Z}) = Q_{\bar{b}}(\tilde{Z})Q_{\bar{c}}(\tilde{Z})$ using the following lemma, see, e.g., [1] or [2].

Lemma 9.1. *Let $Q_{a'} = Q_{b'}Q_{c'}$, for $a' = (a'_1, \dots, a'_n) \in \mathbb{C}^n, b' = (b'_1, \dots, b'_{n_1}) \in \mathbb{C}^{n_1}$, and $c' = (c'_1, \dots, c'_{n_2}) \in \mathbb{C}^{n_2}$, be monic complex polynomials. Suppose that $Q_{b'}$ and $Q_{c'}$ have no common root. Then there are complex analytic mappings $b(a), c(a)$, defined in a neighborhood of a' in \mathbb{C}^n , such that*

$$Q_a = Q_{b(a)}Q_{c(a)},$$

and $b' = b(a'), c' = c(a')$.

Proof. If we write $Q_a = Q_bQ_c$ and compute a as a function of b and c , denoted by $a(b, c)$, then the Jacobian determinant of $a(b, c)$ equals the resultant of Q_b and Q_c which is nonzero by assumption. Thus the lemma follows from the inverse function theorem. □

Since $\bar{a}_{i_0}(p_2) \neq 0$ and $Q_{\bar{a}}(\tilde{Z})$ is in Tschirnhausen form, $Q_{\bar{a}(p_2)}(\tilde{Z}) = \tilde{Z}^n + \bar{a}_2(p_2)\tilde{Z}^{d-2} + \dots + \bar{a}_n(p_2)$ has at least two distinct complex roots and thus can be written as the product of two factors with no common roots. Then Lemma 9.1 allows us to extend this splitting to a neighborhood of p_2

$$Q_{\bar{a}}(\tilde{Z}) = Q_{\bar{b}}(\tilde{Z})Q_{\bar{c}}(\tilde{Z}). \tag{9.3}$$

This splitting induces a splitting (9.1) of P_a by setting $P_b(Z) := f_2^{n_1/q_2} Q_{\bar{b}}(f_2^{-1/q_2} Z)$, where $n_1 = \deg Q_{\bar{b}}$, that is

$$b_i := f_2^{i/q_2} \eta_i \left(y_{2,1}^{1/q_2}, \dots, y_{2,r_2}^{1/q_2}, y_{2,r_2+1}, \dots, y_{2,n} \right), \tag{9.4}$$

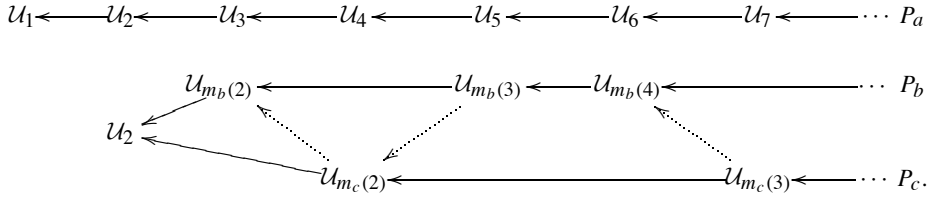
for $i = 1, \dots, n_1 = \deg P_b$,

where η_i is a convergent power series. Similar formulas hold for P_c . The next step involves putting both P_b and P_c in Tschirnhausen form. Thus if we put

$$\varphi_2 := b_1 = -c_1$$

then by the Tschirnhausen transformation the roots of P_b are shifted by $-\frac{\varphi_2}{n_1}$ and those of P_c by $\frac{\varphi_2}{n_2}$.

Step 3. Let $\mathbf{C} = (p_m, f_m, h_m, P_{m,i}, s_m, r_m)$ be a chain for $p_n \in M_n$ with neighborhoods \mathcal{U}_m of p_m . From \mathbf{C} we extract the chains for P_b and P_c as follows. Choose the functions $m_1(m), m_2(m), m_b(m_1), m_c(m_2)$ as in Remark 8.3 and split the chain



We claim that by Propositions 8.1 and 8.2

$$\mathcal{U}_2 \longleftarrow \mathcal{U}_{m_b(2)} \longleftarrow \dots \longleftarrow \mathcal{U}_{m_b(n_1)} \tag{9.5}$$

defines a chain for p_n associated with P_b . Indeed, the pull-back of \mathcal{D}_{b,m_1} on $(\mathcal{U}_m, p_m), m = m_b(m_1)$, is invertible and generated by a (fractional) normal crossing in $y_{m,1}, \dots, y_{m,r_m}$, this generator is denoted g_{m_1} in Proposition 8.2. The rest of the data defining the chain $(h_m, P_{m,i}, s_m, r_m)$ is the same.

Then, if $m = m_b(m_1) \geq 2$, we set $\varphi_m := \varphi_{b,m_1}$. By Proposition 8.2

$$\varphi_m = g_{m_1}^{\alpha_{b,m_1}} \psi_{b,m_1} = f_m^{\alpha_{b,m_1}/m} \psi_m,$$

(this equation defines ψ_m). By the inductive assumption every root of P_b is a combination of the $\varphi_{b,i}$. Since each root of P_a is either a root of P_b or of P_c , the proof is complete. □

9.1. Proof of Lemma 1.9

We prove Lemma 1.9 closely following the steps of the proof of Theorem 1.6.

Step 1. Clearly $\varphi_1(t) = a_1(t)$. Thus after a shift of $\lambda(t)$ by $\frac{1}{n}\varphi_1(t)$ we may assume that $P_{a(t)}$ is in Tschirnhausen form.

Step 2. The crucial observation is that, for all $m \geq 2$, $f_m(a(t)), y_{m,1}(\widehat{a}_m(t)), \dots, y_{m,r_m}(\widehat{a}_m(t))$ do not vanish on J . Hence we may choose their q_m -th radicals continuously, and even of the same regularity (C^k , real analytic, etc.) as the coefficients $a_i(t)$.

Thus a root $\lambda(t)$ of $P_{a(t)}$ induces a root $f_2^{-\alpha_2}(t)\lambda(t)$ of $Q_{\bar{a}(t)}$, whose coefficients are now well-defined as functions of $t \in J$. Since the roots of $Q_{\bar{b}(t)}, Q_{\bar{c}(t)}$, are distinct, $f_2^{-\alpha_2}(t)\lambda(t)$ is a root of precisely one of them. Thus we may consider $\lambda(t)$ as a root of $P_{b(t)}$ for instance.

Step 3. Then, on $M_m, m = m_b(2)$, perform the Tschirnhausen transformation of $P_{b(t)}$, split it and by choosing the radical $g_{m_1}^{\alpha_{b,m_1}}(t)$ identify $\lambda(t)$ (shifted by the

Tschirnhausen transformation) with a root of one of these factors. We continue these procedure until the last factor is of degree 1.

We note that on J we have n everywhere distinct continuous roots of $P_{a(t)}$. They separate in the above process; any two of them, shifted first by common Tschirnhausen transformations, are roots of different factors at some stage.

Part 3 – Example. Roots of cubic polynomials

10. Statement of the result

We give a detailed presentation of the degree 3 case as an example. In this case the resolution is explicit and the result can be made more precise.

Theorem 10.1. *Let $I \subset \mathbb{R}$ be a bounded open interval. Consider a monic polynomial*

$$P(t)(Z) = Z^3 + a_1(t)Z^2 + a_2(t)Z + a_3(t), \quad \text{for } t \in I, \quad (10.1)$$

with coefficients $a_j \in C^6(\bar{I})$, for $j = 1, 2, 3$. Then:

- (1) If $\lambda_j : I \rightarrow \mathbb{C}$, for $j = 1, 2, 3$, denotes a continuous parameterization of the roots of P , then each $\lambda'_j \in L_w^{6/5}(I)$, in particular, each $\lambda_j \in W^{1,q}(I)$, for $q \in [1, 6/5)$.
- (2) Let $\{P_v; v \in \mathcal{N}\}$ be a family of curves of polynomials (10.1) so that the set of coefficients $\{a_{v,j}; v \in \mathcal{N}, j = 1, 2, 3\}$ is bounded in $C^6(\bar{I})$. Then the set $\{\lambda'_v; \lambda_v \in C^0(I)$ with $P_v(\lambda_v) = 0$ on I , for $v \in \mathcal{N}\}$ is bounded in $L_w^{6/5}(I)$.

Remark 10.2. In Theorem 10.1 it is actually enough to require that $f \in C^{5,1}(\bar{I}, \mathbb{C})$; cf. [7].

We sketch below the proof of Theorem 10.1. Thus consider P after Tschirnhausen transformation

$$P(Z) = Z^3 + pZ + q. \quad (10.2)$$

The discriminant of P equals

$$\Delta = -27q^2 - 4p^3.$$

We assume that $p, q : I \rightarrow \mathbb{C}$ belong to $C^6(\bar{I})$. By Proposition 3.1, for each $\delta \in [1/6, 1)$,

$$\begin{aligned} |p'(t)| &\leq \Lambda_{\delta-1}(t) |p(t)|^{1-\delta} \\ |q'(t)| &\leq \Lambda_{\delta-1}(t) |q(t)|^{1-\delta} \\ |\Delta'(t)| &\leq \Lambda_{\delta-1}(t) |\Delta(t)|^{1-\delta} \end{aligned} \quad (10.3)$$

a.e. in I for some $\Lambda_{\delta-1} \in L_w^p(I, \mathbb{R})$, where $p = \frac{1}{1-\delta}$.

Here we set $\Lambda_{\delta-1} := \max\{\Lambda_{\delta-1,p}, \Lambda_{\delta-1,q}, \Lambda_{\delta-1,\Delta}\}$. Note that each formula of (10.3) holds for every t outside the zero set of p, q , or Δ , respectively.

11. Resolution of the discriminant

Consider the embedded resolution of the discriminant given by a sequence of three point blowing-ups. We denote it by $\sigma : M \rightarrow \mathbb{C}^2$. Note that σ resolves also the ideal $\mathcal{I} = (p^3, q^2)$, that is, makes it locally principal and generated by a monomial. Thus in the notation of Section 1, σ is a smooth principalization of \mathcal{D}_2 and \mathcal{D}_3 at the same time. Moreover, in this case, the formulas are explicit and we do not have to use the ideal \mathcal{K} of Subsection 1.3.

We describe the pull-back of P by σ in the affine charts and in each chart we give a formula for the roots of P . These formulas give the bounds on the derivative of the roots with respect to t .

11.1. Chart 1

$p = XY, q = Y$. This is one of two standard charts of the blowing-up of the origin. Then $\sigma^*\mathcal{I} = (Y^2)$ and

$$\sigma^*P(Z) = Y(\tilde{Z}^3 + XY^{1/3}\tilde{Z} + 1) = Y\tilde{P}(\tilde{Z}),$$

where $Z = Y^{1/3}\tilde{Z}$. The polynomial \tilde{P} has distinct roots near the exceptional divisor $Y = 0$. Therefore by the implicit function theorem (IFT) near the exceptional divisor the roots are of the form

$$Z = Y^{1/3}\Phi\left(XY^{1/3}\right) = q^{1/3}\Phi\left(p/q^{2/3}\right), \tag{11.1}$$

where Φ is an analytic function (given locally by a convergent power series). Hence by (10.3)

$$|Z'(t)| \leq \frac{|q'|}{|q|^{2/3}}|\Phi| + \frac{|p'|}{|q|^{1/3}}|\Phi'| + \frac{|pq'|}{|q|^{4/3}}|\Phi'| \leq \Lambda_3(t) (|\Phi| + C|\Phi'|) \tag{11.2}$$

taking into account that $p/q = X$ is bounded.

11.2. Chart 2

We take the other standard chart of the blowing-up of the origin $p = x, q = xy$. The pull-back of the discriminant is not normal crossing in this chart and we have to blow up the origin again.

11.3. Chart 2a

$p = X, q = X^2Y$. This is one of the standard charts of the blowing-up of the origin of Chart 2: $x = X, y = XY$. Then $\sigma^*\mathcal{I} = (X^3)$ and

$$\sigma^*P(Z) = X^{3/2}\left(\tilde{Z}^3 + \tilde{Z} + X^{1/2}Y\right) = X^{3/2}\tilde{P}(\tilde{Z}),$$

where $Z = X^{1/2}\tilde{Z}$. The discriminant of \tilde{P} equals $-(4 + 27XY^2)$ and is non-zero near the exceptional divisor. Therefore by the IFT the roots of \tilde{P} are convergent powers series in $X^{1/2}Y$ and hence the roots of P are of the form

$$Z = X^{1/2}\Phi\left(X^{1/2}Y\right) = p^{1/2}\Phi\left(q/p^{3/2}\right). \quad (11.3)$$

Hence by (10.3)

$$|Z'(t)| \leq \frac{|p'|}{2|p|^{1/2}}|\Phi| + \frac{|q'|}{|p|}|\Phi'| + \frac{|qp'|}{|p|^2}|\Phi'| \leq \frac{1}{2}\Lambda_2(t)(|\Phi| + C|\Phi'|) \quad (11.4)$$

taking into account that $q/p^2 = Y$ is bounded.

11.4. Chart 2b

$p = XY, q = XY^2$. The Chart 2b is the other standard affine chart obtained by blowing up the origin in Chart 2: $x = x'y', y = y'$. Again the discriminant is not normal crossing in this chart so we have to blow-up again.

11.5. Chart 2b(i)

$p = X^2Y, q = X^3Y^2$. This is one of the standard charts of blowing up the origin on Chart 2b, the other one is 2b(ii). On Chart 2b(i), $\sigma^*\mathcal{S} = (X^6Y^3)$ and $\Delta \circ \sigma = X^6Y^3(-4 - 27Y)$. Then

$$\sigma^*P(Z) = X^3Y^{3/2}\left(\tilde{Z}^3 + \tilde{Z} + Y^{1/2}\right) = X^3Y^{3/2}\tilde{P}(\tilde{Z}),$$

where $Z = XY^{1/2}\tilde{Z}$. On the set where the discriminant of $\tilde{P} = -(4 + 27Y)$ is non-zero we may again use the IFT. Then the roots of P are of the form

$$Z = XY^{1/2}\Phi\left(Y^{1/2}\right) = p^{1/2}\Phi\left(q/p^{3/2}\right),$$

where Φ is a convergent power series, as on Chart 2a. Now

$$\begin{aligned} |Z'(t)| &\leq \frac{1}{2}\frac{|p'|}{|p|^{1/2}}|\Phi| + \frac{|q^{2/3}|}{|p|}\frac{|q'|}{|q^{2/3}|}|\Phi'| + \frac{3}{2}\frac{|q|}{|p^{3/2}|}\frac{|p'|}{|p|^{1/2}}|\Phi'| \\ &\leq \frac{1}{2}\Lambda_2(t)|\Phi| + C\left(\Lambda_3(t) + \frac{3}{2}\Lambda_2(t)\right)|\Phi'|. \end{aligned} \quad (11.5)$$

Near $Y = -\frac{4}{27}$ we introduce a new system of coordinates $\tilde{X} = X, \tilde{Y} = Y + \frac{4}{27} = \frac{q^2}{p^3} + \frac{4}{27} = \frac{27q^2 + 4p^3}{27p^3}$. Then $Y^{1/2}$ is a convergent power series in \tilde{Y} . At $\tilde{X} = \tilde{Y} = 0$

the polynomial \tilde{P} has one single and one double root. Therefore, by Lemma 9.1 we can factorize locally

$$\tilde{P}(\tilde{Z}) = (\tilde{Z}^2 + b_1(\tilde{Y})\tilde{Z} + b_2(\tilde{Y}))(\tilde{Z} + c(\tilde{Y})), \tag{11.6}$$

where b_1, b_2 and $c = -b_1$ are convergent power series in \tilde{Y} . Thus one root of P equals $-XY^{1/2}c(\tilde{Y})$ and hence it can be written in the form

$$Z = X\Phi(\tilde{Y}) = \frac{p^2}{q}\Phi\left(\frac{q^2}{p^3}\right) = p^{1/2}\frac{p^{3/2}}{q}\Phi\left(\frac{q^2}{p^3}\right) = q^{1/3}\frac{p^2}{q^{4/3}}\Phi\left(\frac{q^2}{p^3}\right), \tag{11.7}$$

where Φ is a convergent power series. Then, taking into account that $p^3 \sim q^2$ near $\tilde{X} = \tilde{Y} = 0$

$$\begin{aligned} |Z'(t)| &\leq \frac{2|pp'|}{|q|}|\Phi| + \frac{|p^2q'|}{|q^2|}|\Phi| + 3\frac{|q|}{|p^{3/2}|}\frac{|p'|}{|p^{1/2}|}|\Phi'| + 2\frac{|q^{2/3}|}{|p|}\frac{|q'|}{|q^{2/3}|}|\Phi'| \\ &\leq C(\Lambda_2(t) + \Lambda_3(t))|\Phi| + C(3\Lambda_2(t) + 2\Lambda_3(t))|\Phi'|. \end{aligned} \tag{11.8}$$

Denote the factors of (11.6) by P_b and P_c . The discriminant of P_b is the product of the discriminant of \tilde{P} that is $-27\tilde{Y}$, and an invertible convergent power series in \tilde{Y} . Indeed, this follows from the fact that the discriminant of \tilde{P} equals the discriminant of P_b times the square of the resultant of P_b and P_c which is invertible.

Let $\tilde{P}_b = \tilde{Z}^2 + \tilde{b}_2$ be the Tschirnhausen transformation of P_b . The roots of \tilde{P}_b equal, up to a constant, the square root of the discriminant of P_b . This gives the following form for the remaining two roots of P

$$Z = Z_0 \pm Z_1, \tag{11.9}$$

where $Z_0 = -\frac{1}{2}X\Phi(\tilde{Y})$ is coming from (11.7), and Z_1 is of the form

$$Z_1 = X\tilde{Y}^{1/2}\Phi_1(\tilde{Y}) = \frac{p^{1/2}\Delta^{1/2}}{q}\Phi_1\left(\frac{q^2}{p^3}\right). \tag{11.10}$$

It remains to give a bound for $Z'_1(t)$,

$$\begin{aligned} |Z'_1(t)| &\leq \left(\frac{|p'|}{|p^{1/2}|}\frac{|\Delta^{1/2}|}{|q|} + \frac{|p^{1/2}|}{|q^{1/3}|}\frac{|\Delta^{1/3}|}{|q^{2/3}|}\frac{|\Delta'|}{|\Delta^{5/6}|} + \frac{|q'|}{|q^{2/3}|}\frac{|p^{1/2}|}{|q^{1/3}|}\frac{|\Delta^{1/2}|}{|q|}\right)|\Phi_1| \\ &\quad + \frac{|\Delta^{1/2}|}{|q|}\left(2\frac{|q^{2/3}|}{|p|}\frac{|q'|}{|q^{2/3}|} + 3\frac{|q^2|}{|p^3|}\frac{|p'|}{|p^{1/2}|}\right)|\Phi'_1| \\ &\leq C(\Lambda_2(t) + \Lambda_6(t) + \Lambda_3(t))|\Phi_1| + C(2\Lambda_3(t) + 3\Lambda_2(t))|\Phi'_1|. \end{aligned} \tag{11.11}$$

We have used that $p^3 \sim q^2$ and that Δ/q^2 is bounded near $\tilde{X} = \tilde{Y} = 0$.

11.6. Chart 2b(ii)

$p = XY^2, q = XY^3$. Then $\sigma^* \mathcal{S} = (X^2Y^6)$ and $\Delta \circ \sigma = X^2Y^6(-4X - 27)$. We only consider the points near the origin. The other points on this chart, including the strict transform of the discriminant, are also on Chart 2b(i) and were considered before. On Chart 2b(ii)

$$\sigma^* P(Z) = XY^3(\tilde{Z}^3 + X^{1/3}\tilde{Z} + 1) = XY^3\tilde{P}(\tilde{Z}),$$

where $Z = X^{1/3}Y\tilde{Z}$. Since \tilde{P} has distinct roots near $X = Y = 0$, by the IFT, the roots are of the form

$$Z = X^{1/3}Y\Phi(X^{1/3}) = q^{1/3}\Phi(p/q^{2/3}),$$

as on Chart 1. Then

$$|Z'(t)| \leq \frac{|q'|}{|q|^{2/3}}|\Phi| + \frac{|p'|}{|q|^{1/3}}|\Phi'| + \frac{|pq'|}{|q|^{4/3}}|\Phi'| \leq \Lambda_3(t)(|\Phi| + C|\Phi'|) \quad (11.12)$$

taking into account that $p/q^{2/3} = X$ is bounded.

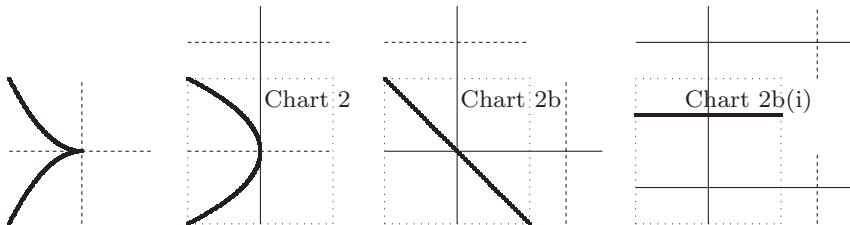


Figure 11.1. Bold curves represent the discriminant set and its strict transforms, thin (continuous) lines the exceptional divisors.

11.7. Proof of Theorem 10.1

The proof follows the reasoning of Subsection 3.5.

Let $p(t), q(t) \in C^6(\bar{I})$, let $\lambda(t)$ be a continuous root of (10.2), and let $\Omega = \{t \in I; (p(t), q(t)) \neq (0, 0)\}$. Then $(p(t), q(t))|_{\Omega}$ lifts to M and Subsection 3.5 gives a bound on $\|\lambda'\|_{6/5, w, \Omega}$. Since $\lambda \equiv 0$ on $I \setminus \Omega$, Theorem 10.1 follows from Lemma 2.1.

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