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Pointwise estimates and existence of solutions of porous medium and p-Laplace evolution equations with absorption and measure data

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Abstract. Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 2)$. We obtain a necessary and a sufficient condition, expressed in terms of capacities, for the existence of a solution to the porous medium equation with absorption

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = \sigma \end{cases}$$

where σ and μ are bounded Radon measures, $q > \max(m, 1)$, and $m > \frac{N-2}{N}$. We also obtain a sufficient condition for the existence of a solution to the p-Laplace evolution equation

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = \sigma \end{cases}$$

where $q > p - 1$ and $p > 2$.

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1. Introduction and main results

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$, and $\Omega_T = \Omega \times (0, T)$. In this paper we study the existence of solutions to the following two types of evolution problems: the porous medium problem with absorption

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = \sigma \end{cases} \tag{1.1}$$

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where $m > \frac{N-2}{N}$ and $q > \max(1, m)$, and the p-Laplace evolution problem with absorption

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = \sigma \end{cases} \tag{1.2}$$

where $q > p - 1 > 1$, and μ and σ are bounded Radon measures respectively on Ω_T and Ω. In the sequel, for any bounded domain O of $\mathbb{R}^l (l \geq 1)$, we denote by $\mathcal{M}_b(O)$ the set of bounded Radon measures in O, and by $\mathcal{M}_b^+(O)$ its positive cone. For any $\nu \in \mathcal{M}_b(O)$, we denote by ν^+ and ν^- respectively its positive and negative part.

When $m = 1$, $p = 2$ and $q > 1$ the problem has been studied by Brezis and Friedman [8] with $\mu = 0$. It is shown that in the subcritical case $q < 1 + 2/N$, the problem can be solved for any $\sigma \in \mathcal{M}_b(\Omega)$, and it has no solution when $q \geq 1 + 2/N$ and σ is a Dirac mass. The general case has been solved by Baras and Pierre [2] and their results are expressed in terms of capacities. For $s > 1, \alpha > 0$, the capacity $\mathrm{Cap}_{\mathbf{G}_\alpha,s}$ of a Borel set $E \subset \mathbb{R}^N$ is defined by

$$\mathrm{Cap}_{\mathbf{G}_\alpha,s}(E) = \inf\left\{ \|g\|^s_{L^s(\mathbb{R}^N)} : g \in L^s_+\left(\mathbb{R}^N\right), \mathbf{G}_\alpha * g \geq 1 \text{ on } E \right\},$$

where G_α is the Bessel kernel of order α and the capacity $\mathrm{Cap}_{2,1,s}$ of a compact set $K \subset \mathbb{R}^{N+1}$ is defined by

$$\mathrm{Cap}_{2,1,s}(K) = \inf\left\{ \|\varphi\|^s_{W^{2,1}_s(\mathbb{R}^{N+1})} : \varphi \in S\left(\mathbb{R}^{N+1}\right), \varphi \geq 1 \text{ in a neighborhood of } K \right\},$$

where

$$\|\varphi\|_{W^{2,1}_s(\mathbb{R}^{N+1})} = \|\varphi\|_{L^s(\mathbb{R}^{N+1})} + \|\varphi_t\|_{L^s(\mathbb{R}^{N+1})} + \| \, |\nabla\varphi| \, \|_{L^s(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} \|\varphi_{x_i x_j}\|_{L^s(\mathbb{R}^{N+1})}.$$

The capacity $\mathrm{Cap}_{2,1,s}$ is extended to Borel sets by the usual method. Note the relation between the two capacities:

$$C^{-1}\,\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}(E) \leq \mathrm{Cap}_{2,1,s}(E \times \{0\}) \leq C\,\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}(E)$$

for any Borel set $E \subset \mathbb{R}^N$, see [19, Corollary 4.21]. In particular, for any $\omega \in \mathcal{M}_b(\mathbb{R}^N)$ and $a \in \mathbb{R}$, the measure $\omega \otimes \delta_{\{t=a\}}$ in $\mathbb{R}^{N+1}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,s}$ (in $\mathbb{R}^{N+1}$) if and only if ω is absolutely continuous with respect to the capacity $\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}$ (in $\mathbb{R}^N$). We recall that a measure μ is absolutely continuous with respect to the capacity Cap if, for any Borel set E,

$$\mathrm{Cap}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

From [2], the problem

$$\begin{cases} u_t - \Delta u + |u|^{q-1}u = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = \sigma \end{cases}$$

has a solution if and only if the measures μ and σ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω respectively, where $q' = \frac{q}{q-1}$.

In Section 2 we study problem (1.1).

For $m > 1$, Chasseigne [10] has extended the results of [8] for $\mu = 0$ in the new subcritical range $m < q < m + \frac{2}{N}$. The supercritical case $q \geq m + \frac{2}{N}$ where $\mu = 0$ and σ is positive is studied in [9]. He has essentially proved that if problem (1.1) has a solution, then $\sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}$, defined for any compact set $K \subset \mathbb{R}^{N+1}$ by

$$\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}(K) = \inf\left\{ \|\varphi\|^{\frac{q}{q-m}}_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} : \varphi \in S\left(\mathbb{R}^N\right), \varphi \geq 1 \text{ in a neighborhood of } K \right\},$$

where

$$\|\varphi\|_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} = \|\varphi\|_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})} + \|\varphi_t\|_{L^{q'}(\mathbb{R}^{N+1})} + \|\,|\nabla\varphi|\,\|_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} \|\varphi_{x_i x_j}\|_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})}.$$

In this section we first give necessary conditions on the measures μ and σ for existence, which cover the results mentioned above.

Theorem 1.1. *Let $q > \max(1,m)$ and $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$. If problem (1.1) has a very weak solution, then μ and $\sigma \otimes \delta_{\{t=0\}}$ are absolutely continuous with respect to* $\mathrm{Cap}_{2,1,\frac{q}{q-m},\frac{q}{q-1}}$.

Remark 1.2. The capacity $\mathrm{Cap}_{2,1,\frac{q}{q-m},\frac{q}{q-1}}$ is absolutely continuous with respect to $\mathrm{Cap}_{2,1,\frac{q}{q-\max\{m,1\}}}$, since

$$\|\varphi\|_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} \leq C(|\operatorname{supp}(\varphi)|)\|\varphi\|_{W^{2,1}_{\frac{q}{q-\max\{m,1\}}}(\mathbb{R}^{N+1})}, \quad \forall \varphi \in C_c^\infty\left(\mathbb{R}^{N+1}\right).$$

Therefore μ and $\sigma \otimes \delta_{\{t=0\}}$ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,\frac{q}{q-\max\{m,1\}}}$. In particular σ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_{\frac{2\max\{m,1\}}{q}},\frac{q}{q-\max\{m,1\}}}$.

The main result of this section is the following sufficient condition for existence, where we use the notion of R-truncated Riesz parabolic potential $\mathbb{I}_2$ on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathcal{M}_b^+(\Omega_T)$, defined by

$$\mathbb{I}_2^R[\mu](x,t) = \int_0^R \frac{\mu\big(\tilde{Q}_\rho(x,t)\big)}{\rho^N}\frac{d\rho}{\rho} \quad \text{for any } \ (x,t) \in \mathbb{R}^{N+1},$$

with $R \in (0, \infty]$, and $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t-\rho^2, t+\rho^2)$.

Theorem 1.3. *Let $m > \frac{N-2}{N}$, $q > \max(1,m)$, $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$.*

i. *If $m > 1$ and μ and σ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω, then there exists a very weak solution u of* (1.1)*, satisfying for a.e.$(x,t) \in \Omega_T$*

$$|u(x,t)| \le C\Bigg(\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\sigma|(\Omega)+|\mu|(\Omega_T)+1 + \mathbb{I}_2^{2d}\left[|\sigma|\otimes\delta_{\{t=0\}}+|\mu|\right](x,t)\Bigg), \tag{1.3}$$

where $C = C(N,m) > 0$ and

$$m_1 = \frac{(N+2)(2mN+1)}{m(mN+2)(1+2N)}, \qquad d = \mathrm{diam}(\Omega) + T^{1/2}.$$

ii. *If $\frac{N-2}{N} < m \le 1$, and μ and σ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,\frac{2q}{2(q-1)+N(1-m)}}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2-N(1-m)}{q}},\frac{2q}{2(q-1)+N(1-m)}}$ in Ω, there exists a very weak solution u of* (1.1)*, such that for a.e.$(x,t) \in \Omega_T$*

$$|u(x,t)| \le C\Bigg(\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d}\left[|\sigma|\otimes\delta_{\{t=0\}}+|\mu|\right](x,t)\right)^{\frac{2}{2-N(1-m)}}\Bigg), \tag{1.4}$$

where $C = C(N,m) > 0$ and

$$m_2 = \frac{2N(N+2)(m+1)}{(2+Nm)(2-N(1-m))(2+N(1+m))}.$$

Moreover we give existence results in the subcritical case, for any $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$, see Theorem 2.9.

We also give other types of sufficient conditions for measures which are good in time, that means

$$\begin{aligned} &\sigma \in L^1(\Omega) \quad \text{ and } |\mu| \le f + \omega \otimes F, \\ &\text{where } f \in L^1_+(\Omega_T),\ F \in L^1_+((0,T)),\ \omega \in \mathcal{M}_b^+(\Omega), \end{aligned} \tag{1.5}$$

see Theorem 2.10. The proof is based on estimates for the stationary problem in terms of elliptic Riesz potential.

In Section 3, we consider problem (1.2). Let us recall some former results about it.

For $q > p-1 > 0$, Pettitta, Ponce and Porretta [21] have proved that it admits a (unique renormalized) solution provided $\sigma \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega_T)$ is a diffuse measure, *i.e.*, absolutely continuous with respect to the C_p-capacity in Ω_T, defined on a compact set $K \subset \Omega_T$ by

$$C_p(K, \Omega_T) = \inf\left\{ \|\varphi\|_W : \varphi \in C_c^\infty(\Omega_T),\ \varphi \ge 1 \text{ on } K \right\}, \tag{1.6}$$

where

$$W = \Big\{ z \in L^p\left((0,T); W_0^{1,p}(\Omega) \cap L^2(\Omega)\right) : \\ z_t \in L^{p'}\left((0,T); W^{-1,p'}(\Omega) + L^2(\Omega)\right) \Big\}.$$

In the recent work [4], we have proved a stability result for the p-Laplace parabolic equation, see Theorem 3.5 below, for $p > \frac{2N+1}{N+1}$. As a first consequence, in the new subcritical range

$$q < p - 1 + \frac{p}{N},$$

problem (1.2) admits a renormalized solution for any measures $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in L^1(\Omega)$. Moreover, we have obtained sufficient conditions for existence, for measures that have a good behavior in time, of the form (1.5). It is shown that (1.2) has a renormalized solution if $\omega \in \mathcal{M}_b^+(\Omega)$ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}$. The proof is based on estimates of [5] for the stationary problem which involve Wolff potentials.

Here we give new sufficient conditions when $p > 2$. Our second main result is the following:

Theorem 1.4. *Let $q > p - 1 > 1$ and $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$. If μ and σ are absolutely continuous with respect to* $\mathrm{Cap}_{2,1,q'}$ *in Ω_T and* $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}}, q'}$ *in Ω, then there exists a distribution solution of problem* (1.2) *which satisfies the pointwise estimate*

$$|u(x,t)| \le C\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|\right](x,t)\right) \tag{1.7}$$

for a.e $(x,t) \in \Omega_T$ *with* $C = C(N,p)$ *and*

$$\begin{aligned} m_3 &= \frac{(N+p)(\lambda+1)(p-1)}{((p-1)N+p)(1+\lambda(p-1))}, \qquad \lambda = \min\{1/(p-1), 1/N\}, \\ D &= \operatorname{diam}(\Omega) + T^{1/p}. \end{aligned} \tag{1.8}$$

Moreover, if $\sigma \in L^1(\Omega)$*, u is a renormalized solution.*

2. Porous medium equation

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\}, -k\}$.

2.1. Weak solutions

The solutions of (1.1) are considered in a weak sense:

Definition 2.1. Let $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$ and $g \in C(\mathbb{R})$.

i. A function u is a weak solution of problem

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + g(u) = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = \sigma & \text{in } \Omega \end{cases} \tag{2.1}$$

if $u \in C([0,T]; L^2(\Omega))$, $|u|^m \in L^2((0,T); H_0^1(\Omega))$ and $g(u) \in L^1(\Omega_T)$, and for any $\varphi \in C_c^{2,1}(\Omega \times [0,T))$,

$$\begin{aligned} &-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} \nabla(|u|^{m-1}u) \cdot \nabla\varphi dxdt \\ &+ \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_\Omega \varphi(0) d\sigma. \end{aligned}$$

ii. A function u is a very weak solution of (2.1) if $u \in L^{\max\{m,1\}}(\Omega_T)$ and $g(u) \in L^1(\Omega_T)$, and for any $\varphi \in C_c^{2,1}(\Omega \times [0,T))$,

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_{\Omega_T} |u|^{m-1}u\Delta\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_\Omega \varphi(0) d\sigma.$$

2.2. Necessary conditions for existence

Next we show the necessary conditions stated in Theorem 1.1.

Proof of Theorem 1.1. As in [2, Proof of Proposition 3.1], it is enough to claim that, for any compact $K \subset \Omega \times [0,T)$ such that $\mu^-(K) = 0$ and $(\sigma^- \otimes \delta_{\{t=0\}})(K) = 0$ and $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}(K) = 0$, there holds $\mu^+(K) = 0$ and $(\sigma^+ \otimes \delta_{\{t=0\}})(K) = 0$. For $\varepsilon > 0$ we choose an open set O such that $(|\mu| + |\sigma| \otimes \delta_{\{t=0\}})(O \backslash K) < \varepsilon$ and $K \subset O \subset \Omega \times (-T,T)$. One can find a sequence $\{\varphi_n\} \subset C_c^\infty(O)$ which satisfies $0 \le \varphi_n \le 1$, $\varphi_n|_K = 1$ and $\varphi_n \to 0$ in $W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})$ and almost everywhere in O (see [2, Proposition 2.2]). We get

$$\begin{aligned}
&\int_{\Omega_T} \varphi_n d\mu + \int_\Omega \varphi_n(0) d\sigma \\
&\quad = -\int_{\Omega_T} u(\varphi_n)_t dxdt - \int_{\Omega_T} |u|^{m-1} u \Delta\varphi_n dxdt + \int_{\Omega_T} |u|^{q-1} u\varphi_n dxdt \\
&\quad \le \left(\|u\|_{L^q(\Omega_T)} + \|u\|^m_{L^q(\Omega_T)} \right) \|\varphi_n\|_{W^{2,1}_{\frac{q}{q-m},\frac{q}{q-1}}(\mathbb{R}^{N+1})} + \int_{\Omega_T} |u|^q \varphi_n dxdt.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\Omega_T} \varphi_n d\mu + \int_\Omega \varphi_n(0) d\sigma &\ge \mu^+(K) + \left(\sigma^+ \otimes \delta_{\{t=0\}}\right)(K) - \left(|\mu| + |\sigma| \otimes \delta_{\{t=0\}}\right)(O \backslash K) \\
&\ge \mu^+(K) + \left(\sigma^+ \otimes \delta_{\{t=0\}}\right)(K) - \varepsilon.
\end{aligned}$$

This implies

$$\begin{aligned}
&\mu^+(K) + \left(\sigma^+ \otimes \delta_{\{t=0\}}\right)(K) \\
&\quad \le \left(\|u\|_{L^q(\Omega_T)} + \|u\|^m_{L^q(\Omega_T)} \right) \|\varphi_n\|_{W^{2,1}_{\frac{q}{q-m},\frac{q}{q-1}}(\mathbb{R}^{N+1})} + \int_{\Omega_T} |u|^q \varphi_n dxdt + \varepsilon.
\end{aligned}$$

As $n \to \infty$, we get $\mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) \le \varepsilon$. Therefore, $\mu^+(K) = (\sigma^+ \otimes \delta_{\{t=0\}})(K) = 0$. □

2.3. Estimates on the porous media equation without absorption

The proof of existence results for problem 1.1 is highly dependent on estimates for the equation of porous media without absorption. We begin by simple a priori estimates:

Proposition 2.2. *Let $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2((0,T); H_0^1(\Omega))$ be a weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = \sigma & \text{in } \Omega \end{cases} \tag{2.2}$$

with $\sigma \in C_b(\Omega)$ *and* $\mu \in C_b(\Omega_T)$. *Then,*

$$\|u\|_{L^\infty((0,T);L^1(\Omega))} \le |\sigma|(\Omega) + |\mu|(\Omega_T), \tag{2.3}$$

$$\|u\|_{L^{m+2/N,\infty}(\Omega_T)} \le C_1\big(|\sigma|(\Omega) + |\mu|(\Omega_T)\big)^{\frac{N+2}{mN+2}}, \tag{2.4}$$

$$\big\||\nabla(|u|^{m-1}u)|\big\|_{L^{\frac{mN+2}{mN+1},\infty}(\Omega_T)} \le C_2\big(|\sigma|(\Omega) + |\mu|(\Omega_T)\big)^{\frac{m(N+1)+1}{mN+2}}, \tag{2.5}$$

where $C_1 = C_1(N,m)$, $C_2 = C_2(N,m)$.

Proof of Proposition 2.2. By using Steklov averages, we can take $T_k(|u|^{m-1}u)$, $k > 0$ as a test function. Setting $H_k(a) = \int_0^a T_k(|y|^{m-1}y)dy$, we find for any $\tau \in (0,T)$

$$\int_{\Omega_\tau} (H_k(u))_t dxdt + \int_{\Omega_\tau} |\nabla T_k(|u|^{m-1}u)|^2 dxdt = \int_{\Omega_\tau} T_k(|u|^{m-1}u)d\mu(x,t).$$

This leads to

$$\int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt \le k(|\sigma|(\Omega) + |\mu|(\Omega_T)) \quad \text{and} \tag{2.6}$$
$$\int_\Omega (H_k(u))(\tau)dx \le k(|\sigma|(\Omega) + |\mu|(\Omega_T)), \ \forall \tau \in (0,T).$$

Since $H_k(a) \ge k\left(|a| - k^{\frac{1}{m}}\right)$ for any a and $k > 0$, we find

$$\int_\Omega \left(|u|(\tau) - k^{\frac{1}{m}}\right) dx \le |\sigma|(\Omega) + |\mu|(\Omega_T), \ \forall \tau \in (0,T).$$

Letting $k \to 0$, we get (2.3).

Next we prove (2.4). By the Gagliardo-Nirenberg embedding theorem, there holds

$$\begin{aligned}
&\int_{\Omega_T} \left|T_k\left(|u|^{m-1}u\right)\right|^{\frac{2(N+1)}{N}} dxdt \\
&\le C_1 \left\|T_k\left(|u|^{m-1}u\right)\right\|^{2/N}_{L^\infty((0,T);L^1(\Omega))} \int_{\Omega_T} \left|\nabla T_k\left(|u|^{m-1}u\right)\right|^2 dxdt \\
&\le C_1 k^{\frac{2(m-1)}{mN}} \|u\|^{2/N}_{L^\infty((0,T);L^1(\Omega))} \int_{\Omega_T} \left|\nabla T_k\left(|u|^{m-1}u\right)\right|^2 dxdt.
\end{aligned}$$

Thus, from (2.6) and (2.3) we get

$$\begin{aligned}
k^{\frac{2(N+1)}{N}} \left|\left\{|u|^m > k\right\}\right| &\le \int_{\Omega_T} \left|T_k\left(|u|^{m-1}u\right)\right|^{\frac{2(N+1)}{N}} dxdt \\
&\le c_1 k^{\frac{2(m-1)}{mN}+1} \big(|\sigma|(\Omega) + |\mu|(\Omega_T)\big)^{\frac{N+2}{N}},
\end{aligned}$$

which implies (2.4). Finally, we prove (2.5). Thanks to (2.6) and (2.4) we have for $k, k_0 > 0$

$$\begin{aligned}
&\left|\left\{\left|\nabla\left(|u|^{m-1}u\right)\right| > k\right\}\right| \le \frac{1}{k^2}\int_0^{k^2} |\{|\nabla(|u|^{m-1}u)| > \ell\}|d\ell \\
&\le \left|\left\{|u|^m > k_0\right\}\right| + \frac{1}{k^2}\int_{\Omega_T} \left|\nabla T_{k_0}\left(|u|^{m-1}u\right)\right|^2 dxdt \\
&\le C_1 k_0^{-\frac{2}{mN}-1}\left(|\sigma|(\Omega)+|\mu|(\Omega_T)\right)^{\frac{N+2}{N}} + k_0 k^{-2}(|\sigma|(\Omega)+|\mu|(\Omega_T)).
\end{aligned}$$

Choosing $k_0 = k^{\frac{Nm}{Nm+1}}(|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{m}{Nm+1}}$, we get (2.5). $\square$

The crucial result used to establish Theorem 1.3 is the following a priori estimates, due to of Liskevich and Skrypnik [17] for $m \ge 1$ and Bogelein, Duzaar and Gianazza [7] for $m \le 1$.

Theorem 2.3. *Let $m > \frac{N-2}{N}$ and $\mu \in (C_b(\Omega_T))^+$. Let $u \in L^\infty_+(\Omega_T)$ with $u^m \in L^2((0,T); H^1_{\mathrm{loc}}(\Omega))$ be a weak solution to equation*

$$u_t - \Delta(u^m) = \mu \;\; in \; \Omega_T.$$

Then there exists $C = C(N,m)$ such that, for almost all $(y,\tau) \in \Omega_T$ and any cylinder $\tilde{Q}_r(y,\tau) \subset\subset \Omega_T$, there holds:

i. *If $m > 1$*

$$u(y,\tau) \le C\Bigg(\left(\frac{1}{r^{N+2}}\int_{\tilde{Q}_r(y,\tau)} |u|^{m+\frac{1}{2N}} dxdt\right)^{\frac{2N}{1+2N}} + \|u\|_{L^\infty((\tau-r^2,\tau+r^2);L^1(B_r(y)))} + 1 + \mathbb{I}_2^{2r}[\mu](y,\tau)\Bigg);$$

ii. *If $m \le 1$,*

$$u(y,\tau) \le C\Bigg(\left(\frac{1}{r^{N+2}}\int_{\tilde{Q}_r(y,s)} |u|^{\frac{2(1+mN)}{N(1+m)}} dxdt\right)^{\frac{2N(m+1)}{(2-N(1-m))(2+N(1+m))}} + 1 + \left(\mathbb{I}_2^{2r}[\mu](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\Bigg).$$

As a consequence we get a new a priori estimate for the porous medium equation:

Corollary 2.4. *Let $m > \frac{N-2}{N}$ and $\mu \in C_b(\Omega_T)$. Let $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2((0,T); H^1_0(\Omega))$ be the weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = \mu & in \; \Omega_T \\ u = 0 & on \; \partial\Omega \times (0,T) \\ u(0) = 0 & in \;\; \Omega. \end{cases}$$

Then there exists $C = C(N, m)$ *such that, for a.e.* $(y, \tau) \in \Omega_T$*:*

i. *If* $m > 1$,

$$|u(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\mu|(\Omega_T) + 1 + \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right); \qquad (2.7)$$

ii. *If* $m \le 1$,

$$|u(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right), \qquad (2.8)$$

where m_1, m_2 *and* d *are defined in Theorem* 1.3.

Proof. Let $x_0 \in \Omega$, and $Q = B_{2d}(x_0) \times (-(2d)^2, (2d)^2)$. Consider the function $U \in (C_b(Q))^+$, with $U^m \in L^p((-(2d)^2, (2d)^2); H_0^1(B_{2d}(x_0)))$ such that U is weak solution of

$$\begin{cases} U_t - \Delta(U^m) = \chi_{\Omega_T}|\mu| & \text{in } B_{2d}(x_0) \times \big(-(2d)^2, (2d)^2\big) \\ U = 0 & \text{on } \partial B_{2d}(x_0) \times \big(-(2d)^2, (2d)^2\big) \\ U(-(2d)^2) = 0 & \text{in } B_{2d}(x_0). \end{cases} \qquad (2.9)$$

From Theorem 2.3, we get, for a.e. $(y, \tau) \in \Omega_T$,

$$U(y,\tau) \le c_1\Bigg(\left(\frac{1}{d^{N+2}}\int_{\tilde{Q}_d(y,\tau)} |U|^{m+\frac{1}{2N}}dxdt\right)^{\frac{2N}{1+2N}} + \|U\|_{L^\infty((\tau-d^2,\tau+d^2);L^1(B_d(y)))} + 1 + \mathbb{I}_2^{2d}[|\mu|](y,\tau)\Bigg)$$

if $m > 1$; and

$$U(y,\tau) \le C\Bigg(\left(\frac{1}{d^{N+2}}\int_{\tilde{Q}_d(y,s)} |u|^{\frac{2(1+mN)}{N(1+m)}}dxdt\right)^{\frac{2N(m+1)}{(2-N(1-m))(2+N(1+m))}} + 1 + \left(\mathbb{I}_2^{2r}[\mu](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\Bigg)$$

if $m \le 1$. By Proposition 2.2, we have

$$\begin{aligned} &\|U\|_{L^\infty((\tau-d^2,\tau+d^2);L^1(B_d(y)))} \le |\mu|(\Omega_T), \\ &|\{|U| > \ell\}| \le c_2(|\mu|(\Omega_T))^{\frac{2+N}{N}}\ell^{-\frac{2}{N}-m} \qquad \forall \ell > 0. \end{aligned}$$

Thus, for any $\ell_0 > 0$,

$$\begin{aligned}\int_Q U^{m+\frac{1}{2N}}dxdt &= \left(m+\frac{1}{2N}\right)\int_0^\infty \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell\\ &= \left(m+\frac{1}{2N}\right)\int_0^{\ell_0} \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell\\ &\quad+\left(m+\frac{1}{2N}\right)\int_{\ell_0}^\infty \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell\\ &\le c_3 d^{N+2}\ell_0^{m+\frac{1}{2N}}+c_4\ell_0^{\frac{1}{2N}-\frac{2}{N}}(|\mu|(\Omega_T))^{\frac{2+N}{N}}.\end{aligned}$$

Choosing $\ell_0 = \left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{\frac{N+2}{mN+2}}$, we get

$$\int_Q U^{(\lambda+1)(p-1)}dxdt \le c_5 d^{N+2}\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{\frac{(N+2)(2mN+1)}{2mN(mN+2)}}.$$

Thus, for a.e. $(y,\tau)\in\Omega_T$,

$$U(y,\tau)\le c_6\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1}+|\mu|(\Omega_T)+1+\mathbb{I}_2^{2d}[|\mu|](y,\tau)\right)$$

if $m>1$. Similarly, we also obtain for a.e. $(y,\tau)\in\Omega_T$,

$$U(y,\tau)\le c_7\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2}+1+\left(\mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right),$$

if $m\le 1$. By the comparison principle we get $|u|\le U$ in Ω_T, and (2.7)-(2.8) follow. □

2.4. Sufficient conditions for existence

In this section we prove Theorem 1.3 by following several steps of approximation.

2.4.1. *Case of bounded nonlinearity and zero initial data*

First, we show that the existence of solution to equations

$$\begin{cases} u_t-\Delta(|u|^{m-1}u)+g(u)=\mu & \text{in } \Omega_T\\ u=0 & \text{on } \partial\Omega\times(0,T)\\ u(0)=0 & \text{in } \Omega\end{cases}\tag{2.10}$$

when $g:\mathbb{R}\to\mathbb{R}$ is a nondecreasing continuous and *bounded* function, such that $g(0)=0$, and $\mu\in\mathcal{M}_b(\Omega_T)$. We first consider the case where μ is continuous and bounded.

Lemma 2.5. *Let $g \in C_b(\mathbb{R})$ be nondecreasing with $g(0) = 0$, and $\mu \in C_b(\Omega_T)$. There exists a weak solution $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2((0,T); H_0^1(\Omega))$ of problem* (2.10).

Moreover, the comparison principle holds for these solutions: if u_1, u_2 are weak solutions of (2.10) *when (μ, g) is replaced by (μ_1, g_1) and (μ_2, g_2), where $\mu_1, \mu_2 \in C_b(\Omega_T)$ with $\mu_1 \geq \mu_2$ and g_1, g_2 have the same properties as g with $g_1 \leq g_2$ in $\mathbb{R}$ then $u_1 \geq u_2$ in Ω_T.*

As a consequence, if $\mu \geq 0$ then $u \geq 0$.

Proof of Lemma 2.5. Set

$$a_n(s) = \begin{cases} m|s|^{m-1} & \text{if } 1/n \leq |s| \leq n \\ m|n|^{m-1} & \text{if } |s| \geq n \\ m(1/n)^{m-1} & \text{if } |s| \leq 1/n \end{cases}$$

and $A_n(\tau) = \int_0^\tau a_n(s)ds$. Then one can find u_n being a weak solution of the following problem:

$$\begin{cases} (u_n)_t - \operatorname{div}(a_n(u_n)\nabla u_n) + g(u_n) = \mu & \text{in } \Omega_T \\ u_n = 0 & \text{on } \partial\Omega \times (0,T) \\ u_n(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.11}$$

It is easy to see that $|u_n(x,t)| \leq t\|\mu\|_{L^\infty(\Omega_T)}$ for all $(x,t) \in \Omega_T$. Thus, choosing $A_n(u_n)$ as a test function, we obtain

$$\int_{\Omega_T} |\nabla A_n(u_n)|^2 dxdt \leq C_1\big(T, \|\mu\|_{L^\infty(\Omega_T)}\big). \tag{2.12}$$

Now set $\Phi_n(\tau) = \int_0^\tau |A_n(s)|ds$. Choosing $|A_n(u_n)|\varphi$ as a test function in (2.11), where $\varphi \in C_c^{2,1}(\Omega_T)$, we get the relation

$$\begin{aligned}(\Phi_n(u_n))_t - \operatorname{div}(|A_n(u_n)|\nabla A_n(u_n)) + \nabla A_n(u_n).\nabla|A_n(u_n)| + |A_n(u_n)|g(u_n) \\ = |A_n(u_n)|\mu\end{aligned}$$

in $\mathcal{D}'(\Omega_T)$. Hence,

$$\begin{aligned}&\|(\Phi_n(u_n))_t\|_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))} \\ &\leq \big\||A_n(u_n)\nabla A_n(u_n)|\big\|_{L^2(\Omega_T)} + \big\||\nabla A_n(u_n)|\big\|^2_{L^2(\Omega_T)} \\ &\quad + \|A_n(u_n)g(u_n)\|_{L^1(\Omega_T)} + \|A_n(u_n)\mu\|_{L^1(\Omega_T)}.\end{aligned}$$

Combining this with (2.12) and the estimate $|A_n(u_n)| \leq C_2(T, \|\mu\|_{L^\infty(\Omega)})$, we deduce that

$$\sup_n \|(\Phi_n(u_n))_t\|_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))} < \infty.$$

On the other hand, since $|A_n(u_n)| \le |u_n| a_n(u_n) \le T\|\mu\|_{L^\infty(\Omega)} a_n(u_n)$, there holds

$$\begin{aligned}\int_{\Omega_T} |\nabla \Phi_n(u_n)|^2 dxdt &= \int_{\Omega_T} |A_n(u_n)|^2 |\nabla u_n|^2 dxdt \\ &\le T\|\mu\|_{L^\infty(\Omega)} \int_{\Omega_T} |a_n(u_n)|^2 |\nabla u_n|^2 dxdt \\ &\le T\|\mu\|_{L^\infty(\Omega)} \int_{\Omega_T} |\nabla A_n(u_n)|^2 dxdt \le C_3\big(T, \|\mu\|_{L^\infty(\Omega)}\big).\end{aligned}$$

Therefore, $\Phi_n(u_n)$ is relatively compact in $L^1(\Omega_T)$. Note that

$$\Phi_n(s) = \begin{cases} \frac{m}{2}\left(\frac{1}{n}\right)^m |s|^2 \mathrm{sign}(s) & \text{if } |s| \le \frac{1}{n} \\ (m-1)\left(\frac{1}{n}\right)^m \left(|s| - \frac{1}{n}\right) \mathrm{sign}(s) + \frac{1}{m+1}\left(|s|^{m+1} - \left(\frac{1}{n}\right)^{m+1}\right) \mathrm{sign}(s) & \\ & \text{if } \frac{1}{n} \le |s| \le n. \end{cases}$$

So, for every $n_1, n_2 \ge n$ and $|s_1|, |s_2| \le T\|\mu\|_{L^\infty(\Omega)}$,

$$\frac{1}{m+1}\big||s_1|^m s_1 - |s_2|^m s_2\big| \le C_4\left(m, T|\mu|_{L^\infty(\Omega)}\right)\left(\frac{1}{n}\right)^m + \big|\Phi_{n_1}(s_1) - \Phi_{n_2}(s_2)\big|.$$

Hence, for any $\varepsilon > 0$,

$$\left|\left\{\frac{1}{m+1}\big||u_{n_1}|^m u_{n_1} - |u_{n_2}|^m u_{n_2}\big| > 2\varepsilon\right\}\right| \le \big|\{|\Phi_{n_1}(u_{n_1}) - \Phi_{n_2}(u_{n_2})| > \varepsilon\}\big|,$$

for all $n_1, n_2 \ge \left(C_4(m, T\|\mu\|_{L^\infty(\Omega)})/\varepsilon\right)^{1/m}$. Thus, up to a subsequence $\{u_n\}$ converges a.e. in Ω_T to a function u. From (2.11) we can write

$$-\int_{\Omega_T} u_n \varphi_t dxdt - \int_{\Omega_T} A_n(u_n)\Delta\varphi dxdt + \int_{\Omega_T} g(u_n)\varphi dxdt = \int_{\Omega_T} \varphi d\mu,$$

for any $\varphi \in C_c^{2,1}(\Omega_T)$. Thanks to the dominated convergence Theorem we deduce that

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_{\Omega_T} |u|^{m-1} u \Delta\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu.$$

By the Fatou Lemma and (2.12) we also get $|u|^m \in L^2((0,T); H_0^1(\Omega))$.

Furthermore, from the classical maximum principle, see [15, Theorem 9.7], if $\{\tilde{u}_n\}$ is a sequence of solutions to equations (2.11) where (g, μ) is replaced by (h, ν) such that $\nu \in C_b(\Omega_T)$ with $\nu \ge \mu$ and h has the same properties as g, satisfying $h \le g$ in $\mathbb{R}$, then $u_n \le \tilde{u}_n$. As $n \to \infty$, we get $u \le \tilde{u}$. This achieves the proof. □

Next we come to the general case where μ is a bounded measure:

Lemma 2.6. *Let* $m > \frac{N-2}{N}$ *and* $g \in C_b(\mathbb{R})$, *such that* g *is nondecreasing and* $g(0) = 0$, *and let* $\mu \in \mathcal{M}_b(\Omega_T)$.

There exists a very weak solution u *of equation* (2.10) *which satisfies* (2.7)-(2.8) *and*

$$\int_{\Omega_T} |g(u)|dxdt \le |\mu|(\Omega_T), \quad \left\|u\right\|_{L^{m+2/N,\infty}(\Omega_T)} \le C(|\mu|(\Omega_T))^{\frac{N+2}{mN+2}}. \tag{2.13}$$

where $C = C(m, N) > 0$.

Moreover, the comparison principle holds for these solutions: if u_1, u_2 *are very weak solutions of* (2.10) *when* (μ, g) *is replaced by* (μ_1, g_1) *and* (μ_2, g_2), *where* $\mu_1, \mu_2 \in \mathcal{M}_b(\Omega_T)$ *with* $\mu_1 \ge \mu_2$ *and* g_1, g_2 *have the same properties as* g *with* $g_1 \le g_2$ *in* $\mathbb{R}$ *then* $u_1 \ge u_2$ *in* Ω_T.

Proof. Let $\{\mu_n\}$ be a sequence in $C_c^\infty(\Omega_T)$ converging to μ in $\mathcal{M}_b(\Omega_T)$, such that $|\mu_n| \le \varphi_n * |\mu|$ and $|\mu_n|(\Omega_T) \le |\mu|(\Omega_T)$ for any $n \in \mathbb{N}$, where $\{\varphi_n\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$. By Lemma 2.5 there exists a very weak solution u_n of problem

$$\begin{cases} (u_n)_t - \Delta(|u_n|^{m-1}u_n) + g(u_n) = \mu_n & \text{in } \Omega_T \\ u_n = 0 & \text{on } \partial\Omega \times (0, T) \\ u_n(0) = 0 & \text{in } \Omega \end{cases}$$

which satisfies for a.e. $(y, \tau) \in \Omega_T$,

$$|u_n(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\mu|(\Omega_T) + 1 + \varphi_n * \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right) \text{ if } m > 1,$$
$$|u_n(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\varphi_n * \mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right) \text{ if } m \le 1,$$

and

$$\int_{\Omega_T} \left|\nabla T_k\left(|u_n|^{m-1}u_n\right)\right|^2 dxdt \le k|\mu|(\Omega_T), \qquad \forall k > 0, \tag{2.14}$$
$$|\{|u_n| > \ell\}| \le C_1 \ell^{-\frac{2}{N}-m}(|\mu|(\Omega_T))^{\frac{N+2}{N}}, \qquad \forall \ell > 0, \tag{2.15}$$
$$\int_{\Omega_T} |g(u_n)|dxdt \le |\mu|(\Omega_T).$$

For $l > 0$, we consider $S_l \in C_c^2(\mathbb{R})$ such that

$$S_l(a) = |a|^m a, \quad \text{for } |a| \le l, \quad \text{and} \quad S_l(a) = (2l)^{m+1}\text{sign}(a), \quad \text{for } |a| \ge 2l.$$

Then we find the relation

$$(S_l(u_n))_t - \operatorname{div}\left(S_l'(u_n)\nabla\left(|u_n|^{m-1}u_n\right)\right) + m|u_n|^{m-1}|\nabla u_n|^2 S_l''(u_n) + g(u_n)S_l'(u_n)$$
$$= S_l'(u_n)\mu_n$$

in $\mathcal{D}'(\Omega_T)$. It leads to

$$\begin{aligned}
&\left\|(S_l(u_n))_t\right\|_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))}\\
&\le \left\|S_l'(u_n)|\nabla(|u_n|^{m-1}u_n)|\right\|_{L^2(\Omega_T)} + m\left\||u_n|^{m-1}|\nabla u_n|^2 S_l''(u_n)\right\|_{L^1(\Omega_T)}\\
&\quad + \left\|g(u_n)S_l'(u_n)\right\|_{L^1(\Omega_T)} + \left\|S_l'(u_n)\mu_n\right\|_{L^1(\Omega_T)}.
\end{aligned}$$

Since $|S_l'(u_n)| \le C_2\chi_{[-2l,2l]}(u_n)$ and $|S_l''(u_n)| \le C_3|u_n|^{m-1}\chi_{[-2l,2l]}(u_n)$, we obtain

$$\begin{aligned}
&\left\|(S_l(u_n))_t\right\|_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))}\\
&\le C_4\left(\left\|\left|\nabla T_{(2l)^m}\left(|u_n|^{m-1}u_n\right)\right|\right\|_{L^2(\Omega_T)} + \|g\|_{L^\infty(\mathbb{R})}|\Omega_T| + |\mu_n|(\Omega_T)\right).
\end{aligned}$$

From (2.14) we deduce that $\{(S_l(u_n))_t\}$ is bounded in $L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))$ and for any $n \in \mathbb{N}$,

$$\begin{aligned}
&\left\|(S_l(u_n))_t\right\|_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))}\\
&\le C_4\left((2l)^{m/2}(|\mu|(\Omega_T))^{1/2} + \|g\|_{L^\infty(\mathbb{R})}|\Omega_T| + |\mu|(\Omega_T)\right).
\end{aligned}$$

Moreover, $\{S_l(u_n)\}$ is bounded in $L^2((0,T);H_0^1(\Omega))$. Hence, $\{S_l(u_n)\}$ is relatively compact in $L^1(\Omega_T)$ for any $l > 0$. Thanks to (2.15) we find

$$\begin{aligned}
&\left|\left\{\left||u_{n_1}|^m u_{n_1} - |u_{n_1}|^m u_{n_1}\right| > \ell\right\}\right|\\
&\le \left|\{|u_{n_1}| > l\}| + |\{|u_{n_2}| > l\}\right| + \left|\{|S_l(u_{n_1}) - S_l(u_{n_2})| > \ell\}\right|\\
&\le 2C_2 l^{-\frac{2}{N}-m}|\mu|(\Omega_T)^{\frac{N+2}{N}} + \left|\{|S_l(u_{n_1}) - S_l(u_{n_2})| > \ell\}\right|.
\end{aligned}$$

Thus, up to a subsequence $\{u_n\}$ converges a.e. in Ω_T to a function u. Consequently, u is a very weak solution of equation (2.10) and satisfies (2.13) and (2.7)-(2.8). The other conclusions follow in the same way. □

Remark 2.7. If $\operatorname{supp}(\mu) \subset \overline{\Omega} \times [a,T]$ for some $a > 0$, then the solution u in Lemma 2.6 satisfies $u = 0$ in $\Omega \times [0,a)$.

2.4.2. *Proof of Theorem* 1.3

Now we recall the important approximation property of Radon measures which was proved in [3] and [19].

Proposition 2.8. *Let $s > 1$ and $\mu \in \mathcal{M}_b^+(\Omega_T)$. If μ is absolutely continuous with respect to* $\mathrm{Cap}_{2,1,s'}$ *in Ω_T, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathcal{M}_b^+(\Omega_T)$, with compact support in Ω_T which converges to μ weakly in $\mathcal{M}_b(\Omega_T)$ and satisfies $\mathbb{I}_2^R[\mu_n] \in L^s_{\mathrm{loc}}(\mathbb{R}^{N+1})$ for any $R > 0$.*

We are now ready to prove Theorem 1.3. We reduce to the case of zero initial data by considering the problem on $(-T, T)$ with the measure $\sigma^+ \otimes \delta_{\{t=0\}} + \mu$ in $\Omega \times (-T, T)$.

Proof of Theorem 1.3. First suppose $m > 1$. Assume that μ, σ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. Then $\sigma^+ \otimes \delta_{\{t=0\}} + \mu^+, \sigma^- \otimes \delta_{\{t=0\}} + \mu^-$ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,q'}$ in $\Omega \times (-T, T)$. Applying Proposition 2.8 to $\sigma^+ \otimes \delta_{\{t=0\}} + \mu^+, \sigma^- \otimes \delta_{\{t=0\}} + \mu^-$, there exist two nondecreasing sequences $\{\upsilon_{1,n}\}$ and $\{\upsilon_{2,n}\}$ of positive bounded measures with compact support in $\Omega \times (-T, T)$ which converge respectively to $\sigma^+ \otimes \delta_{\{t=0\}} + \mu^+$ and $\sigma^- \otimes \delta_{\{t=0\}} + \mu^-$ in $\mathcal{M}_b(\Omega \times (-T, T))$ and such that $\mathbb{I}_2^{2d_1}[\upsilon_{1,n}], \mathbb{I}_2^{2d_1}[\upsilon_{2,n}] \in L^q(\Omega \times (-T, T))$ for all $n \in \mathbb{N}$.

Step 1. For any $n_1, n_2 \in \mathbb{N}$, we show that there exists a very weak solution $u^{n_1,n_2} := u$ of

$$\begin{cases} u_t - \Delta\left(|u|^{m-1}u\right) + |u|^{q-1}u = \upsilon_{1,n_1} - \upsilon_{2,n_2} & \text{in } \Omega \times (-T, T) \\ u = 0 & \text{on } \partial\Omega \times (-T, T) \\ u(-T) = 0 & \text{in } \Omega. \end{cases} \tag{2.16}$$

By Lemma 2.6, for $k_1, k_2 > 0$ there exists a weak solution u_{k_1,k_2} of the problem

$$\begin{cases} \left(u_{k_1,k_2}\right)_t - \Delta\left(\left|u_{k_1,k_2}\right|^{m-1}u_{k_1,k_2}\right) + T_{k_1}\left(\left(u^+_{k_1,k_2}\right)^q\right) & \\ \quad -T_{k_2}\left(\left(u^-_{k_1,k_2}\right)^q\right) = \upsilon_{1,n_1} - \upsilon_{2,n_2} & \text{in } \Omega \times (-T, T) \\ u_{k_1,k_2} = 0 & \text{on } \partial\Omega \times (-T, T) \\ u_{k_1,k_2}(-T) = 0 & \text{in } \Omega \end{cases}$$

which satisfies

$$\begin{aligned} |u_{k_1,k_2}| \le C\Bigg(&\left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{d^N}\right)^{m_1} \\ &+ |\sigma|(\Omega) + |\mu|(\Omega_T) + 1 + \mathbb{I}_2^{2d}\left[\upsilon_{1,n_1} + \upsilon_{2,n_2}\right]\Bigg), \end{aligned} \tag{2.17}$$

and

$$\int_{\Omega_T} T_{k_1}\left(\left(u^+_{k_1,k_2}\right)^q\right) dxdt + \int_{\Omega_T} T_{k_2}\left(\left(u^-_{k_1,k_2}\right)^q\right) dxdt \le |\mu|(\Omega_T).$$

Moreover, for any $n_1 \in \mathbb{N}, k_2 > 0$, $\{u_{k_1,k_2}\}_{k_1}$ is non-increasing and for any $n_2 \in \mathbb{N}$, $k_1 > 0$, $\{u_{k_1,k_2}\}_{k_2}$ is non-decreasing. Therefore, thanks to the fact that $\mathbb{I}_2^{2d_1}[\upsilon_{1,n}]$, $\mathbb{I}_2^{2d_1}[\upsilon_{2,n}] \in L^q(\Omega \times (-T,T))$ and from (2.17) and the dominated convergence theorem, $u = \lim_{k_1\to\infty} \lim_{k_2\to\infty} u_{k_1,k_2}$ is a very weak solution of (2.16).

Step 2. We show that $u = \lim_{n_2\to\infty} \lim_{n_1\to\infty} u^{n_1,n_2}$ is a very weak solution of (1.1). By Lemma 2.6, $\{u^{n_1,n_2}\}_{n_1}$ is non-increasing, $\{u^{n_1,n_2}\}_{n_2}$ is non-decreasing and (2.17) is true when u_{k_1,k_2} is replaced by u^{n_1,n_2}, and

$$\int_{\Omega_T} |u^{n_1,n_2}|^q dxdt \le |\mu|(\Omega_T) \ \ \forall\, n_1, n_2 \in \mathbb{N}.$$

From the monotone convergence theorem we obtain that $u = \lim_{n_2\to\infty} \lim_{n_1\to\infty} u_{n_1,n_2}$ is a very weak solution of

$$\begin{cases} u_t - \Delta\left(|u|^{m-1}u\right) + |u|^{q-1}u = \sigma \otimes \delta_{\{t=0\}} + \chi_{\Omega_T}\mu & \text{in } \Omega \times (-T,T) \\ u = 0 & \text{on } \partial\Omega \times (-T,T) \\ u(-T) = 0 & \text{in } \ \Omega \end{cases}$$

with $u = 0$ in $\Omega \times (-T, 0)$, and u satisfies (1.3). Clearly, u is a very weak solution of equation (1.1).

Next suppose $m \le 1$. The proof is similar, with the new capacitary assumptions, and (1.3) is replaced by (1.4). □

2.4.3. *The subcritical case*

We also obtain the description of the subcritical case.

Theorem 2.9. *Let* $m > \frac{N-2}{N}$ *and* $0 < q < m + \frac{2}{N}$. *Then problem* (1.1) *has a very weak solution for any* $\mu \in \mathcal{M}_b(\Omega_T)$ *and* $\sigma \in \mathcal{M}_b(\Omega)$.

Proof. As the proof of Theorem 1.3, we can reduce to the case $\sigma = 0$. By Lemma 2.6, there exists a very weak solution u_{k_1,k_2} of

$$\begin{cases} \left(u_{k_1,k_2}\right)_t - \Delta\left(\left|u_{k_1,k_2}\right|^{m-1} u_{k_1,k_2}\right) + T_{k_1}\left(\left(u^+_{k_1,k_2}\right)^q\right) & \\ \quad -T_{k_2}\left(\left(u^-_{k_1,k_2}\right)^q\right) = \mu & \text{in } \Omega_T \\ u_n = 0 & \text{on } \partial\Omega \times (0,T) \\ u_n(0) = 0 & \text{in } \ \Omega \end{cases}$$

such that $\{u_{k_1,k_2}\}_{k_1}$ and $\{u_{k_1,k_2}\}_{k_2}$ are monotone sequences and

$$\left\|u_{k_1,k_2}\right\|_{L^{m+2/N,\infty}(\Omega_T)} \le C(|\mu|(\Omega_T))^{\frac{N+2}{mN+2}}.$$

In particular, $\{u_{k_1,k_2}\}$ is a uniformly bounded in $L^s(\Omega_T)$ for any $0 < s < m + \frac{2}{N}$.

Therefore, we get that $u = \lim_{k_2\to\infty} \lim_{k_1\to\infty} u_{k_1,k_2}$ is a very weak solution of (1.1). This completes the proof. □

2.4.4. *Existence for good measures in time*

Next, from an idea of [4, Theorem 2.3], we obtain an existence result for measures which present a good behaviour in time:

Theorem 2.10. *Let* $m > \frac{N-2}{N}$, $q > \max(1, m)$ *and* $f \in L^1(\Omega_T)$, $\mu \in \mathcal{M}_b(\Omega_T)$, *such that*

$$|\mu| \leq \omega \otimes F \quad \textit{for some } \omega \in \mathcal{M}_b^+(\Omega) \textit{ and } F \in L^1_+((0,T)).$$

If ω *is absolutely continuous with respect to* $\mathrm{Cap}_{\mathbf{G}_2, \frac{q}{q-m}}$ *in* Ω, *then there exists a very weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = f + \mu & \textit{in } \Omega_T \\ u = 0 & \textit{on } \partial\Omega \times (0,T) \\ u(0) = 0. \end{cases} \tag{2.18}$$

Proof. For $R \in (0, \infty]$, we define the R-truncated Riesz elliptic potential of a measure $\nu \in \mathcal{M}_b^+(\Omega)$ by

$$\mathbf{I}_2^R[\nu](x) = \int_0^R \frac{\nu(B_\rho(x))}{\rho^{N-2}} \frac{d\rho}{\rho} \quad \forall x \in \Omega.$$

By [5, Theorem 2.6], there exists sequence $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ω in $\mathcal{M}_b(\Omega)$ and such that $\mathbf{I}_2^{2\mathrm{diam}(\Omega)}[\omega_n] \in L^{q/m}(\Omega)$ for any $n \in \mathbb{N}$. We can write

$$f + \mu = \mu_1 - \mu_2, \qquad \mu_1 = f^+ + \mu^+, \qquad \mu_2 = f^- + \mu^-,$$

and $\mu^+, \mu^- \leqq \omega \otimes F$. We set

$$\mu_{1,n} = T_n(f^+) + \inf\{\mu^+, \omega_n \otimes T_n(F)\}, \quad \mu_{2,n} = T_n(f^-) + \inf\{\mu^-, \omega_n \otimes T_n(F)\}.$$

Then $\{\mu_{1,n}\}$, $\{\mu_{2,n}\}$ are nondecreasing sequences converging to μ_1, μ_2 respectively in $\mathcal{M}_b(\Omega_T)$ and $\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}$, with $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$ and $\mathbf{I}_2^{2\mathrm{diam}(\Omega)}[\tilde{\omega}_n] \in L^{q/m}(\Omega)$. As in the proof of Theorem 1.3, there exists a sequence of weak solution $\{u_{n_1,n_2,k_1,k_2}\}$ of equations

$$\begin{cases} \left(u_{n_1,n_2,k_1,k_2}\right)_t - \Delta\left(\left|u_{n_1,n_2,k_1,k_2}\right|^{m-1} u_{n_1,n_2,k_1,k_2}\right) + T_{k_1}\left(\left(u^+_{n_1,n_2,k_1,k_2}\right)^q\right) & \\ \quad - T_{k_2}\left(\left(u^-_{n_1,n_2,k_1,k_2}\right)^q\right) = \mu_{1,n_1} - \mu_{2,n_2} & \text{in } \Omega_T \\ u_{n_1,n_2,k_1,k_2} = 0 & \text{on } \partial\Omega \times (0,T) \\ u_{n_1,n_2,k_1,k_2}(0) = 0 & \text{in } \Omega. \end{cases} \tag{2.19}$$

Using the comparison principle as in [4], we can assume that

$$-v_{n_2} \le |u_{n_1,n_2,k_1,k_2}|^{m-1} u_{n_1,n_2,k_1,k_2} \le v_{n_1},$$

where for any $n \in \mathbb{N}$, v_n is a nonnegative weak solution of

$$\begin{cases} -\Delta v_n = \tilde{\omega}_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$v_n \le c_1 \mathbf{I}_2^{2\mathrm{diam}(\Omega)}[\tilde{\omega}_n] \ \ \forall\, n \in \mathbb{N}.$$

Hence, utilizing the arguments in the proof of Theorem 1.3, it is easy to obtain the result as desired. □

3. $p-$Laplacian evolution equation

Here we consider solutions in the weak sense of distributions, or in the renormalized sense.

3.1. Distribution and renormalized solutions

We first consider weak solutions in the sense of distributions:

Definition 3.1. Let $\mu \in \mathcal{M}_b(\Omega_T)$, $\sigma \in \mathcal{M}_b(\Omega)$ and $B \in C(\mathbb{R})$. A measurable function u is a *distribution* solution of problem

$$\begin{cases} u_t - \Delta_p u + B(u) = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \ \partial\Omega \times (0,T) \\ u(0) = \sigma & \text{in } \ \Omega \end{cases} \tag{3.1}$$

if $u \in L^s((0,T); W_0^{1,s}(\Omega))$ for any $s \in \left[1, p - \frac{N}{N+1}\right)$, and $B(u) \in L^1(\Omega_T)$, such that

$$-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} |\nabla u|^{p-2}\nabla u.\nabla\varphi dxdt + \int_{\Omega_T} B(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0)d\sigma$$

for every $\varphi \in C_c^1(\Omega \times [0,T))$.

Remark 3.2. Let $\sigma' \in \mathcal{M}_b(\Omega)$ and $a' \in (0,T)$, set $\omega = \mu + \sigma' \otimes \delta_{\{t=a'\}}$. Let u be a distribution solution of problem (3.1) with data ω and $\sigma = 0$, such that

$$\mathrm{supp}(\mu) \subset \overline{\Omega} \times [a',T], \qquad \text{and } u = 0,\, B(u) = 0 \quad \text{in } \Omega \times (0,a').$$

Then $\tilde{u} := u|_{\Omega\times[a',T)}$ is a distribution solution of problem (3.1) in $\Omega \times (a',T)$ with data μ and σ'.

As it is well known, when $p \neq 2$, this notion is not well adapted to the quasilinear problem. The notion of renormalized solution is stronger. It was first introduced by Blanchard and Murat [6] to obtain uniqueness results for the p-Laplace evolution problem for L^1 data μ and σ, and developed by Petitta [20] for measure data μ. It requires a decomposition of the measure μ, that we recall now.

Let $\mathcal{M}_0(\Omega_T)$ be the space of Radon measures in Ω_T which are absolutely continuous with respect to the C_p-capacity, defined at (1.6), and $\mathcal{M}_s(\Omega_T)$ be the space of measures in Ω_T with support on a set of zero C_p-capacity. Classically, any $\mu \in \mathcal{M}_b(\Omega_T)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathcal{M}_0(\Omega_T) \cap \mathcal{M}_b(\Omega_T)$ and $\mu_s \in \mathcal{M}_s(\Omega_T)$. In turn μ_0 can be decomposed under the form

$$\mu_0 = f - \operatorname{div} g + h_t,$$

where $f \in L^1(\Omega_T)$, $g \in (L^{p'}(\Omega_T))^N$ and $h \in L^p((0,T); W_0^{1,p}(\Omega))$, see [12]; and we say that (f, g, h) is a decomposition of μ_0. We say that a sequence of $\{\mu_n\}$ in $\mathcal{M}_b(\Omega_T)$ converges to $\mu \in \mathcal{M}_b(\Omega_T)$ in the narrow topology of measures if

$$\lim_{n\to\infty} \int_{\Omega_T} \varphi d\mu_n = \int_{\Omega_T} \varphi d\mu \quad \forall \varphi \in C(\Omega_T) \cap L^\infty(\Omega_T).$$

We recall that if u is a measurable function defined and finite a.e. in Ω_T, such that $T_k(u) \in L^p((0,T); W_0^{1,p}(\Omega))$ for any $k > 0$, there exists a measurable function $v : \Omega_T \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\leq k} v$ a.e. in Ω_T and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$.

Definition 3.3. Let $p > \frac{2N+1}{N+1}$ and $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega_T)$, $\sigma \in L^1(\Omega)$ and $B \in C(\mathbb{R})$. A measurable function u is a *renormalized* solution of (3.1) if there exists a decomposition (f, g, h) of μ_0 such that

$$\begin{aligned} &v = u - h \in L^s\Big((0,T); W_0^{1,s}(\Omega)\Big) \cap L^\infty\Big((0,T); L^1(\Omega)\Big),\ \forall s \in \Big[1, p - \frac{N}{N+1}\Big),\\ &T_k(v) \in L^p\Big((0,T); W_0^{1,p}(\Omega)\Big) \quad \forall k > 0, \quad B(u) \in L^1(\Omega_T), \end{aligned} \tag{3.2}$$

and:

(i) For any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$\begin{aligned} &-\int_\Omega S(\sigma)\varphi(0)dx - \int_{\Omega_T} \varphi_t S(v)dxdt + \int_{\Omega_T} S'(v)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dxdt\\ &+ \int_{\Omega_T} S''(v)\varphi|\nabla u|^{p-2}\nabla u \cdot \nabla v dxdt + \int_{\Omega_T} S'(v)\varphi B(u)dxdt\\ &= \int_{\Omega_T} (fS'(v)\varphi + g.\nabla(S'(v)\varphi)dxdt \end{aligned} \tag{3.3}$$

for any $\varphi \in L^p((0,T); W_0^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ such that $\varphi_t \in L^{p'}((0,T); W^{-1,p'}(\Omega)) + L^1(\Omega_T)$ and $\varphi(.,T) = 0$;

(ii) For any $\phi \in C(\overline{\Omega_T})$,

$$\lim_{m\to\infty} \frac{1}{m} \int_{\{m\le v<2m\}} \phi|\nabla u|^{p-2}\nabla u\cdot\nabla v dxdt = \int_{\Omega_T} \phi d\mu_s^+ \quad \text{and} \tag{3.4}$$

$$\lim_{m\to\infty} \frac{1}{m} \int_{\{-m\ge v>-2m\}} \phi|\nabla u|^{p-2}\nabla u\cdot\nabla v dxdt = \int_{\Omega_T} \phi d\mu_s^-. \tag{3.5}$$

We first mention a convergence result of [4].

Proposition 3.4. *Let $\{\mu_n\}$ be bounded in $\mathcal{M}_b(\Omega_T)$ and $\{\sigma_n\}$ be bounded in $L^1(\Omega)$, and $B \equiv 0$. Let u_n be a renormalized solution of* (3.1) *with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$ and initial data σ_n.*

If $\{f_n\}$ is bounded in $L^1(\Omega_T)$, $\{g_n\}$ bounded in $(L^{p'}(\Omega_T))^N$ and $\{h_n\}$ convergent in $L^p((0,T); W_0^{1,p}(\Omega))$, then, up to a subsequence, $\{u_n\}$ converges to a function u in $L^1(\Omega_T)$. Moreover, if $\{\mu_n\}$ is bounded in $L^1(\Omega_T)$, then $\{u_n\}$ is convergent in $L^s((0,T); W_0^{1,s}(\Omega))$ for any $s \in \left[1, p - \frac{N}{N+1}\right)$.

Next we recall the fundamental stability result of [4].

Theorem 3.5. *Suppose that $p > \frac{2N+1}{N+1}$ and $B \equiv 0$. Let $\sigma \in L^1(\Omega)$ and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(\Omega_T),$$

with $f \in L^1(\Omega_T)$, $g \in (L^{p'}(\Omega_T))^N$, $h \in L^p((0,T); W_0^{1,p}(\Omega))$ and $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(\Omega_T)$. Let $\sigma_n \in L^1(\Omega)$ and

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(\Omega_T),$$

with $f_n \in L^1(\Omega_T)$, $g_n \in (L^{p'}(\Omega_T))^N$, $h_n \in L^p((0,T); W_0^{1,p}(\Omega))$, and $\rho_n, \eta_n \in \mathcal{M}_b^+(\Omega_T)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div}\rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div}\eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(\Omega_T)$, $\rho_n^2, \eta_n^2 \in (L^{p'}(\Omega_T))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(\Omega_T)$.

Assume that $\{\mu_n\}$ is bounded in $\mathcal{M}_b(\Omega_T)$, $\{\sigma_n\}, \{f_n\}, \{g_n\}, \{h_n\}$ converge to σ, f, g, h in $L^1(\Omega)$, weakly in $L^1(\Omega_T)$, in $(L^{p'}(\Omega_T))^N$, in $L^p((0,T); W_0^{1,p}(\Omega))$ respectively; and $\{\rho_n\}, \{\eta_n\}$ converge to μ_s^+, μ_s^- in the narrow topology of measures; and $\left\{\rho_n^1\right\}, \left\{\eta_n^1\right\}$ are bounded in $L^1(\Omega_T)$, and $\left\{\rho_n^2\right\}, \left\{\eta_n^2\right\}$ bounded in $(L^{p'}(\Omega_T))^N$.

Let $\{u_n\}$ be a sequence of renormalized solutions of

$$\begin{cases} (u_n)_t - \Delta_p u_n = \mu_n & \text{in } \Omega_T \\ u_n = 0 & \text{on } \partial\Omega\times(0,T) \\ u_n(0) = \sigma_n & \text{in } \Omega \end{cases} \tag{3.6}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ *of* $\mu_{n,0}$. *Let* $v_n = u_n - h_n$.

Then up to a subsequence, $\{u_n\}$ *converges a.e. in* Ω_T *to a renormalized solution* u *of* (3.1)*, and* $\{v_n\}$ *converges a.e. in* Ω_T *to* $v = u - h$. *Moreover,* $\{\nabla v_n\}$ *converge to* ∇v *a.e. in* Ω_T*, and* $\{T_k(v_n)\}$ *converges to* $T_k(v)$ *strongly in* $L^p((0,T); W_0^{1,p}(\Omega))$ *for any* $k > 0$.

In order to apply this result, we need some the following properties concerning approximate measures of $\mu \in \mathcal{M}_b^+(\Omega_T)$, see also [4].

Proposition 3.6. *Let* $\mu = \mu_0 + \mu_s \in \mathcal{M}_b^+(\Omega_T)$*,* $\mu_0 \in \mathcal{M}_0(\Omega_T) \cap \mathcal{M}_b^+(\Omega_T)$ *and* $\mu_s \in \mathcal{M}_s(\Omega_T)$. *Let* $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ *be sequences of mollifiers in* $\mathbb{R}^N$*,* $\mathbb{R}$ *respectively.*

There exist sequences of measures $\mu_{n,0} = (f_n, g_n, h_n)$*, and* $\mu_{n,s}$*, such that* $f_n, g_n, h_n, \mu_{n,s} \in C_c^\infty(\Omega_T)$ *and strongly converge to* f, g, h *in* $L^1(\Omega_T)$*,* $(L^{p'}(\Omega_T))^N$ *and* $L^p((0,T); W_0^{1,p}(\Omega))$ *respectively,* $\mu_{n,s}$ *converges to* $\mu_s \in \mathcal{M}_s^+(\Omega_T)$*, and* $\mu_n = \mu_{n,0} + \mu_{n,s}$ *converges to* μ*, in the narrow topology, and satisfying* $0 \leq \mu_n \leq (\varphi_{1,n}\varphi_{2,n}) * \mu$*, and*

$$\|f_n\|_{L^1(\Omega_T)} + \|g_n\|_{(L^{p'}(\Omega_T))^N} + \|h_n\|_{L^p((0,T);W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega_T) \\ \leq 2\mu(\Omega_T) \text{ for any } n \in \mathbb{N}.$$

Proposition 3.7. *Let* $\mu = \mu_0 + \mu_s,\ \mu_n = \mu_{n,0} + \mu_{n,s} \in \mathcal{M}_b^+(\Omega_T)$ *with* $\mu_0, \mu_{n,0} \in \mathcal{M}_0(\Omega_T) \cap \mathcal{M}_b^+(\Omega_T)$ *and* $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(\Omega_T)$ *such that* $\{\mu_n\}$ *is nondecreasing and converges to* μ *in* $\mathcal{M}_b(\Omega_T)$.

Then, $\{\mu_{n,s}\}$ *is nondecreasing and converging to* μ_s *in* $\mathcal{M}_b(\Omega_T)$*; and there exist decompositions* (f, g, h) *of* μ_0*,* (f_n, g_n, h_n) *of* $\mu_{n,0}$ *such that* $\{f_n\}, \{g_n\}, \{h_n\}$ *strongly converge to* f, g, h *in* $L^1(\Omega_T)$*,* $(L^{p'}(\Omega_T))^N$ *and* $L^p((0,T); W_0^{1,p}(\Omega))$ *respectively, satisfying*

$$\|f_n\|_{L^1(\Omega_T)} + \|g_n\|_{(L^{p'}(\Omega_T))^N} + \|h_n\|_{L^p((0,T);W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega_T) \\ \leq 2\mu(\Omega_T) \text{ for any } n \in \mathbb{N}.$$

3.2. Estimates on the p-Laplace equation without absorption

Here the crucial point for proving existence results for problem (1.2) is a result of Liskevich, Skrypnik and Sobol [16] for the p-Laplace evolution problem without absorption:

Theorem 3.8. *Let* $p > 2$*, and* $\mu \in \mathcal{M}_b(\Omega_T)$. *Let* $u \in C([0,T]; L^2_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}((0,T); W^{1,p}_{\text{loc}}(\Omega))$ *be a distribution solution to equation*

$$u_t - \Delta_p u = \mu \text{ in } \Omega_T. \tag{3.7}$$

Then there exists $C = C(N, p)$ such that, for every Lebesgue point $(x, t) \in \Omega_T$ of u, and any $\rho > 0$ such that $Q_{\rho,\rho^p}(x, t) := B_\rho(x) \times (t - \rho^p, t + \rho^p) \subset \Omega_T$, there holds

$$|u(x,t)| \leq C\left(1+\left(\frac{1}{\rho^{N+p}}\int_{Q_{\rho,\rho^p}(x,t)}|u|^{(\lambda+1)(p-1)}\right)^{\frac{1}{1+\lambda(p-1)}}+\mathbf{P}_p^\rho[\mu](x,t)\right) \quad (3.8)$$

where $\lambda = \min\{1/(p-1), 1/N\}$ and

$$\mathbf{P}_p^\rho[\mu](x,t) = \sum_{i=0}^{\infty} D_p(\rho_i)(x,t),$$

$$D_p(\rho_i)(x,t) = \inf_{\tau>0}\left\{(p-2)\tau^{-\frac{1}{p-2}} + \frac{1}{2(p-1)^{p-1}}\frac{|\mu|(Q_{\rho_i,\tau\rho_i^p}(x,t))}{\rho_i^N}\right\}$$

with $\rho_i = 2^{-i}\rho$, $Q_{\rho,\tau\rho^p}(x,t) = B_\rho(x) \times (t - \tau\rho^p, t + \tau\rho^p)$.

As a consequence, we deduce the following estimate:

Proposition 3.9. *Let u be a distribution solution of the problem*

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(0) = 0 & \text{in } \Omega \end{cases}$$

with data $\mu \in C_b(\Omega_T)$. Then there exists $C = C(N, p)$ such that for a.e. $(x, t) \in \Omega_T$,

$$|u(x,t)| \leq C\left(1 + D + \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}[|\mu|](x,t)\right), \quad (3.9)$$

where m_3 and D are defined at (1.8).

Proof. Let $x_0 \in \Omega$ and $Q = B_{2D}(x_0) \times (-(2D)^p, (2D)^p)$. Let

$$U \in L^p\big((-(2D)^p, (2D)^p); W_0^{1,p}(B_{2D}(x_0))\big)$$

with $U \in C(Q)$ be the distribution solution of

$$\begin{cases} U_t - \Delta_p U = \chi_{\Omega_T}|\mu| & \text{in } Q \\ u = 0 & \text{on } \partial B_{2D}(x_0) \times (-(2D)^p, (2D)^p) \\ u(-(2D)^p) = 0 & \text{in } B_{2D}(x_0) \end{cases} \quad (3.10)$$

for $x_0 \in \Omega$. Thus, by Theorem 3.8 we find, for any $(x, t) \in \Omega_T$,

$$U(x,t) \leq c_1\left(1+\left(\frac{1}{D^{N+p}}\int_{Q_{D,D^p}(x,t)}|U|^{(\lambda+1)(p-1)}\right)^{\frac{1}{1+\lambda(p-1)}}+\mathbf{P}_p^D[\mu](x,t)\right), \quad (3.11)$$

where $Q_{D,D^p}(x,t) = B_D(x) \times (t - D^p, t + D^p)$.

According to Proposition 4.8 and [4, Remark 4.9], there exists a constant $C_2 > 0$ such that

$$|\{|U| > \ell\}| \le c_2(|\mu|(\Omega_T))^{\frac{p+N}{N}} \ell^{-p+1-\frac{p}{N}} \qquad \forall \ell > 0.$$

Thus, for any $\ell_0 > 0$,

$$\begin{aligned}\int_Q |U|^{(\lambda+1)(p-1)} dxdt &= (\lambda+1)(p-1)\int_0^\infty \ell^{(\lambda+1)(p-1)-1}|\{|U| > \ell\}|d\ell \\ &= (\lambda+1)(p-1)\bigg(\int_0^{\ell_0} \ell^{(\lambda+1)(p-1)-1}|\{|U| > \ell\}|d\ell \\ &\qquad + \int_{\ell_0}^\infty \ell^{(\lambda+1)(p-1)-1}|\{|U| > \ell\}|d\ell\bigg) \\ &\le c_3 D^{N+p}\ell_0^{(\lambda+1)(p-1)} + c_4 \ell_0^{(\lambda+1)(p-1)-p+1-\frac{p}{N}}(|\mu|(\Omega_T))^{\frac{p+N}{N}}.\end{aligned}$$

Choosing $\ell_0 = \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{\frac{N+p}{(p-1)N+p}}$ we get

$$\int_Q |U|^{(\lambda+1)(p-1)} dxdt \le c_5 D^{N+p}\left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{\frac{(N+p)(\lambda+1)(p-1)}{(p-1)N+p}}. \tag{3.12}$$

Next we show that

$$\mathbf{P}_p^{d_2}[\mu](x,t) \le (p-2)D + c_6 \mathbb{I}_2^{2D}[|\mu|](x,t). \tag{3.13}$$

Indeed, we have

$$D_p(\rho_i)(x,t) \le (p-2)\rho_i + \frac{1}{2(p-1)^{p-1}}\frac{|\mu|(\tilde{Q}_{\rho_i}(x,t))}{\rho_i^N},$$

where $\rho_i = 2^{-i}D$. Thus,

$$\begin{aligned}\mathbf{P}_p^D[\mu](x,t) &\le (p-2)D + \frac{1}{2(p-1)^{p-1}}\sum_{i=0}^\infty \frac{|\mu|(\tilde{Q}_{\rho_i}(x,t))}{\rho_i^N} \\ &\le (p-2)D + C_5\int_0^{2D}\frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho}.\end{aligned}$$

So from (3.12), (3.13) and (3.11) we get, for any $(x,t) \in \Omega_T$,

$$|U(x,t)| \le C\left(1 + D + \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}[|\mu|](x,t)\right).$$

By the comparison principle we get $|u| \le U$ in Ω_T, thus (3.9) follows. ☐

As a consequence we obtain a new existence result for equation (3.7):

Proposition 3.10. *Let $p > 2$, and $\mu \in \mathcal{M}_b(\Omega_T)$, $\sigma \in \mathcal{M}_b(\Omega)$. There exists a distribution solution u of problem*

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = \sigma \end{cases} \tag{3.14}$$

which satisfies for any $(x,t) \in \Omega_T$

$$|u(x,t)| \leq C\left(1+D+\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|\right](x,t)\right), \tag{3.15}$$

where $C = C(N,p)$. Moreover, if $\sigma \in L^1(\Omega)$, u is a renormalized solution.

Proof. Let $\{\varphi_{1,n}\}$, $\{\varphi_{2,n}\}$ be sequences of standard mollifiers in $\mathbb{R}^N$ and $\mathbb{R}$. Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega_T)$, with $\mu_0 \in \mathcal{M}_0(\Omega_T)$, $\mu_s \in \mathcal{M}_s(\Omega_T)$.

By Lemma 3.6, there exist sequences of nonnegative measures $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$ and $\mu_{n,s,i}$ such that $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(\Omega_T)$ and strongly converge to some f_i, g_i, h_i respectively in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p((0,T);W_0^{1,p}(\Omega))$, and $\mu_{n,1}, \mu_{n,2}, \mu_{n,s,1}, \mu_{n,s,2} \in C_c^\infty(\Omega_T)$ converge to $\mu^+, \mu^-, \mu_s^+, \mu_s^-$ in the narrow topology, with $\mu_{n,i} = \mu_{n,0,i} + \mu_{n,s,i}$, for $i = 1, 2$, and satisfying

$$\mu_0^+ = (f_1, g_1, h_1),\ \mu_0^- = (f_2, g_2, h_2) \text{ and}$$
$$0 \leq \mu_{n,1} \leq (\varphi_{1,n}\varphi_{2,n}) * \mu^+, 0 \leq \mu_{n,2} \leq (\varphi_{1,n}\varphi_{2,n}) * \mu^-.$$

Let $\sigma_{1,n}, \sigma_{2,n} \in C_c^\infty(\Omega)$, converging to σ^+ and σ^- in the narrow topology, and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$, such that

$$0 \leq \sigma_{1,n} \leq \varphi_{1,n} * \sigma^+, 0 \leq \sigma_{2,n} \leq \varphi_{1,n} * \sigma^-.$$

Set $\mu_n = \mu_{n,1} - \mu_{n,2}$ and $\sigma_n = \sigma_{1,n} - \sigma_{2,n}$.

Let u_n be solution of the approximate problem

$$\begin{cases} (u_n)_t - \Delta_p u_n = \mu_n & \text{in } \Omega_T \\ u_n = 0 & \text{on } \partial\Omega \times (0,T) \\ u_n(0) = \sigma_n & \text{on } \Omega. \end{cases} \tag{3.16}$$

We set $g_{n,m}(x,t) = \sigma_n(x) \int_{-T}^t \varphi_{2,m}(s)ds$. By Theorem 3.5, we can see that there exists a sequence $\{u_{n,m}\}_m$ of solutions of the problem

$$\begin{cases} (u_{n,m})_t - \Delta_p u_{n,1,m} = \left(g_{n,m}\right)_t + \chi_{\Omega_T}\mu_n & \text{in } \Omega \times (-T,T) \\ u_{n,1,m} = 0 & \text{on } \partial\Omega \times (-T,T) \\ u_{n,m}(-T) = 0 & \text{on } \Omega \end{cases} \tag{3.17}$$

which converges to u_n in $\Omega \times (0,T)$. By Proposition 3.9, there holds, for any $(x,t) \in \Omega_T$,

$$|u_{n,m}(x,t)| \le C\Bigg(1 + D + \left(\frac{|\mu_n|(\Omega_T) + (|\sigma_n| \otimes \varphi_{2,m})(\Omega \times (-T,T))}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}[|\mu_n| + |\sigma_n| \otimes \varphi_{2,m}](x,t)\Bigg).$$

Therefore

$$|u_{n,m}(x,t)| \le C\left(1 + D + \left(\frac{|\mu_n|(\Omega_T) + (|\sigma_n| \otimes \varphi_{2,m})(\Omega \times (-T,T))}{D^N}\right)^{m_3}\right) + C(\varphi_{1,n}\varphi_{2,m}) * \mathbb{I}_2^{2D}\big[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}\big](x,t).$$

Letting $m \to \infty$, we deduce that

$$|u_n(x,t)| \le C\left(1 + D + \left(\frac{|\mu_n|(\Omega_T) + |\sigma_n|(\Omega)}{D^N}\right)^{m_3}\right) + c_1(\varphi_{1,n}) * \left(\mathbb{I}_2^{2D}\big[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}\big](\cdot,t)\right)(x).$$

Therefore, by Proposition 3.4 and Theorem 3.5, up to a subsequence, $\{u_n\}$ converges to a distribution solution u of (3.14) (a renormalized solution if $\sigma \in L^1(\Omega)$), and satisfying (3.15). □

3.3. Sufficient conditions for existence

In this part we prove Theorem 1.4.

Proof of Theorem 1.4.

Step 1. First, assume that $\sigma \in L^1(\Omega)$. Since μ is absolutely continuous with respect to $\text{Cap}_{2,1,q'}$, the same happens for μ^+ and μ^-. Applying Proposition 2.8 to μ^+, μ^-, there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω_T which converge to μ^+ and μ^- in $\mathcal{M}_b(\Omega_T)$ respectively and such that $\mathbb{I}_2^{2D}[\mu_{1,n}], \mathbb{I}_2^{2D}[\mu_{2,n}] \in L^q(\Omega_T)$ for all $n \in \mathbb{N}$.

For $i = 1, 2$, set $\tilde{\mu}_{i,1} = \mu_{i,1}$ and $\tilde{\mu}_{i,j} = \mu_{i,j} - \mu_{i,j-1} \ge 0$, so $\mu_{i,n} = \sum_{j=1}^{n} \tilde{\mu}_{i,j}$. We write

$$\mu_{i,n} = \mu_{i,n,0} + \mu_{i,n,s}, \tilde{\mu}_{i,j} = \tilde{\mu}_{i,j,0} + \tilde{\mu}_{i,j,s},$$
$$\text{with } \mu_{i,n,0}, \tilde{\mu}_{i,n,0} \in \mathcal{M}_0(\Omega_T), \mu_{i,n,s}, \tilde{\mu}_{i,n,s} \in \mathcal{M}_s(\Omega_T).$$

Let $\{\varphi_m\}$ be a sequence of mollifiers in $\mathbb{R}^{N+1}$. As in the proof of Proposition 3.10, for any $j \in \mathbb{N}$ and $i = 1, 2$, there exist sequences of nonnegative measures $\tilde{\mu}_{m,i,j,0} = (f_{m,i,j}, g_{m,i,j}, h_{m,i,j})$ and $\tilde{\mu}_{m,i,j,s}$ such that $f_{m,i,j}, g_{m,i,j}, h_{m,i,j} \in$

$C_c^\infty(\Omega_T)$ and strongly converge to some $f_{i,j}, g_{i,j}, h_{i,j}$ in $L^1(\Omega_T)$, $(L^{p'}(\Omega_T))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively; and $\tilde{\mu}_{m,i,j}, \tilde{\mu}_{m,i,j,s} \in C_c^\infty(\Omega_T)$ converge to $\tilde{\mu}_{i,j}, \tilde{\mu}_{i,j,s}$ in the narrow topology, with $\tilde{\mu}_{m,i,j} = \tilde{\mu}_{m,i,j,0} + \tilde{\mu}_{m,i,j,s}$, which satisfy $\tilde{\mu}_{i,j,0} = (f_{i,j}, g_{i,j}, h_{i,j})$, and

$$\begin{aligned} &0 \le \tilde{\mu}_{m,i,j} \le \varphi_m * \tilde{\mu}_{i,j},\ \tilde{\mu}_{m,i,j}(\Omega_T) \le \tilde{\mu}_{i,j}(\Omega_T), \\ &\left\| f_{m,i,j} \right\|_{L^1(\Omega_T)} + \left\| g_{m,i,j} \right\|_{(L^{p'}(\Omega_T))^N} + \left\| h_{m,i,j} \right\|_{L^p((0,T);W_0^{1,p}(\Omega))} \\ &\quad + \mu_{m,i,j,s}(\Omega_T) \le 2\tilde{\mu}_{i,j}(\Omega_T). \end{aligned} \tag{3.18}$$

Note that, for any $n, m \in \mathbb{N}$,

$$\sum_{j=1}^{n} \left(\tilde{\mu}_{m,1,j} + \tilde{\mu}_{m,2,j}\right) \le \varphi_m * \left(\mu_{1,n} + \mu_{2,n}\right)$$
$$\text{and } \sum_{j=1}^{n} \left(\tilde{\mu}_{m,1,j}(\Omega_T) + \tilde{\mu}_{m,2,j}(\Omega_T)\right) \le |\mu|(\Omega_T).$$

Step 1.a For any $n, k \in \mathbb{N}$, we show that there exist a renormalized solution $u_{n,k} := u$ to

$$\begin{cases} u_t - \Delta_p u + T_k\left(|u|^{q-1}u\right) = \mu_{1,n} - \mu_{2,n} & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = T_n(\sigma^+) - T_n(\sigma^-) & \text{on } \Omega \end{cases} \tag{3.19}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j} - \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} - \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} - \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0} - \mu_{2,n,0}$ and a renormalized solution $v_{n,k} := v$ to

$$\begin{cases} v_t - \Delta_p v + T_k(v^q) = \mu_{1,n} + \mu_{2,n} & \text{in } \Omega_T \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v(0) = T_n(|\sigma|) & \text{on } \Omega, \end{cases} \tag{3.20}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j} + \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} + \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} + \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0} + \mu_{2,n,0}$, such that

$$\begin{aligned} |u| \le v \le C &\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[T_n(|\sigma|) \otimes \delta_{\{t=0\}}\right]\right) \\ &+ C\mathbb{I}_2^{2D}\left[\mu_{1,n} + \mu_{2,n}\right], \end{aligned} \tag{3.21}$$

and

$$\int_{\Omega_T} T_k(v^q)dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega). \tag{3.22}$$

Indeed, for any $m \in \mathbb{N}$, let $u_{n,k,m} := u_m, v_{n,k,m} := v_m \in W$ be solutions of problems

$$\begin{cases} (u_m)_t - \Delta_p u_m + T_k\left(|u_m|^{q-1}u_m\right) = \sum_{j=1}^{n} \left(\tilde{\mu}_{m,1,j} - \tilde{\mu}_{m,2,j}\right) & \text{in } \Omega_T \\ u_m = 0 & \text{on } \partial\Omega \times (0,T) \\ u_m(0) = T_n(\sigma^+) - T_n(\sigma^-) & \text{on } \Omega \end{cases}$$

and

$$\begin{cases} (v_m)_t - \Delta_p v_m + T_k\left(v_m^q\right) = \sum_{j=1}^{n} \left(\tilde{\mu}_{m,1,j} + \tilde{\mu}_{m,2,j}\right) & \text{in } \Omega_T \\ v_m = 0 & \text{on } \partial\Omega \times (0,T) \\ v_m(0) = T_n(|\sigma|) & \text{on } \Omega. \end{cases}$$

By the comparison principle and Proposition 3.9 we have

$$|u_m| \le v_m \le c_1\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[T_n(|\sigma|)\otimes\delta_{\{t=0\}}\right]\right) + c_1\varphi_m * \mathbb{I}_2^{2D}\left[\mu_{1,n} + \mu_{2,n}\right].$$

Moreover,

$$\int_{\Omega_T} T_k(v_m^q)dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

From Proposition 3.4, up to subsequences, $\{u_m\}_m, \{v_m\}_m$ converge to some u, v a.e. in Ω_T. Then, applying Theorem 3.5 to data $(\sum_{j=1}^{n}(\tilde{\mu}_{m,1,j} - \tilde{\mu}_{m,2,j}) - T_k(|u_m|^{q-1}u_m), T_n(\sigma^+) - T_n(\sigma^-))$ and $(\sum_{j=1}^{n}(\tilde{\mu}_{m,1,j} + \tilde{\mu}_{m,2,j}) - T_k(v_m^q), T_n(|\sigma|))$, up to subsequences, $\{u_m\}_m$ converges to a renormalized solution u of problem (3.19) and $\{v_m\}_m$ converges to a solution v of (3.20). Clearly, u and v satisfy (3.21) and (3.22).

Step 1.b For any $n \in N$, we show that there exist renormalized solutions $u^n := u, v^n := v$ to

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu_{1,n} - \mu_{2,n} & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = T_n(\sigma^+) - T_n(\sigma^-) & \text{on } \Omega \end{cases} \tag{3.23}$$

relative to the decomposition $(\sum_{j=1}^{n} f_{1,j} - \sum_{j=1}^{n} f_{2,j}, \sum_{j=1}^{n} g_{1,j} - \sum_{j=1}^{n} g_{2,j}, \sum_{j=1}^{n} h_{1,j} - \sum_{j=1}^{n} h_{2,j})$ of $\mu_{1,n,0} - \mu_{2,n,0}$ and

$$\begin{cases} v_t - \Delta_p v + v^q = \mu_{1,n} + \mu_{2,n} & \text{in } \Omega_T \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v(0) = T_n(|\sigma|) & \text{on } \Omega \end{cases} \tag{3.24}$$

relative to the decomposition ($\sum_{j=1}^{n} f_{1,j} + \sum_{j=1}^{n} f_{2,j}, \sum_{j=1}^{n} g_{1,j} + \sum_{j=1}^{n} g_{2,j}$, $\sum_{j=1}^{n} h_{1,j} + \sum_{j=1}^{n} h_{2,j}$) of $\mu_{1,n,0} + \mu_{2,n,0}$, respectively and u, v satisfies (3.21) and

$$\int_{\Omega_T} v^q dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega). \tag{3.25}$$

Indeed, for any $k \in \mathbb{N}$, by Step 1.a, there exist renormalized solutions $u_{n,k}, v_{n,k}$ of equations (3.19) and (3.20), respectively, which satisfy (3.21) and (3.22) with $u = u_{n,k}, v = v_{n,k}$.

Thanks to Proposition 3.4, up to subsequences, $\{u_{n,k}\}_k, \{v_{n,k}\}_k$ converge to some u^n, v^n a.e. in Ω_T. Then, $\{T_k(|u_{n,k}|^{q-1}u_{n,k})\}_k$, $\{T_k(v_{n,k}^q)\}_k$ converge to some $|u^n|^{q-1}u^n$, $(v^n)^q$ in $L^1(\Omega_T)$, respectively, from (3.21) and the dominated convergence Theorem, since $\mathbb{I}_2^{2D}[\mu_{1,n} + \mu_{2,n}] \in L^q(\Omega_T)$ for any $n \in \mathbb{N}$. Thus, by Theorem 3.5, up to a subsequence, $\{u_{n,k}\}_k$ $\{v_{n,k}\}_k$ converge to renormalized solutions u^n, v^n of problems (3.23) and (3.24) which still satisfy (3.21) with $u = u^n, v = v^n$ and (3.25).

Moreover, we can see that the sequence $\{v^n\}_n$ is increasing. Note that from (3.18) we have

$$\|f_{i,j}\|_{L^1(\Omega_T)} + \|g_{i,j}\|_{(L^{p'}(\Omega_T))^N} + \|h_{i,j}\|_{L^p((0,T);W_0^{1,p}(\Omega))} \le 2\tilde{\mu}_{i,j}(\Omega_T),$$

which implies

$$\left\|\sum_{j=1}^{n} f_{i,j}\right\|_{L^1(\Omega_T)} + \left\|\sum_{j=1}^{n} g_{i,j}\right\|_{(L^{p'}(\Omega_T))^N} + \left\|\sum_{j=1}^{n} h_{i,j}\right\|_{L^p((0,T);W_0^{1,p}(\Omega))} \le 2\tilde{\mu}_{i,n}(\Omega_T) \le 2|\mu|(\Omega_T). \tag{3.26}$$

Step 1.c We show that, up to subsequence, $\{u^n\}_n$ converges to a renormalized solution u of problem

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = \sigma & \text{in } \Omega \end{cases} \tag{3.27}$$

relative to the decomposition ($\sum_{j=1}^{\infty} f_{1,j} - \sum_{j=1}^{\infty} f_{2,j}, \sum_{j=1}^{\infty} g_{1,j} - \sum_{j=1}^{\infty} g_{2,j}$, $\sum_{j=1}^{\infty} h_{1,j} - \sum_{j=1}^{\infty} h_{2,j}$) of μ_0 , and $\{v^n\}_n$ converges to a renormalized solution v of problem

$$\begin{cases} v_t - \Delta_p v + v^q = |\mu| & \text{in } \Omega_T \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v(0) = |\sigma| & \text{in } \Omega \end{cases} \tag{3.28}$$

relative to the decomposition ($\sum_{j=1}^{\infty} f_{1,j} + \sum_{j=1}^{\infty} f_{2,j}, \sum_{j=1}^{\infty} g_{1,j} + \sum_{j=1}^{\infty} g_{2,j}$, $\sum_{j=1}^{\infty} h_{1,j} + \sum_{j=1}^{\infty} h_{2,j}$) of $|\mu_0|$ and

$$|u| \le v \le C\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|\right]\right). \tag{3.29}$$

Indeed, by Proposition 3.4, up to subsequences, $\{u^n\}_n$, $\{v^n\}_n$ converges to some u, v a.e. in Ω_T. Then, thanks to (3.25) with $v = v^n$, the fact that $\{v^n\}_n$ is increasing and the monotone convergence Theorem, we deduce that u^n, v^n converge to u, v in $L^q(\Omega_T)$.

Therefore, from (3.26), we can apply Theorem 3.5 to obtain that, up to subsequences, $\{u^n\}_n$, $\{v^n\}_n$ converge to renormalized solutions u, v of problems (3.27) and (3.28) which satisfy (3.29).

Note that, if $\sigma \equiv 0$ and $\mathrm{supp}(\mu) \subset \overline{\Omega} \times [a, T]$, $a > 0$, then $u = v = 0$ in $\Omega \times (0, a)$, since $u_{n,k} = v_{n,k} = 0$ in $\Omega \times (0, a)$.

Step 2. We consider any $\sigma \in \mathcal{M}_b(\Omega)$ such that σ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. Then $\mu + \sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to $\mathrm{Cap}_{2,1,q'}$ in $\Omega \times (-T, T)$. As above, we verify that there exists a renormalized solution u of

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \chi_{\Omega_T}\mu + \sigma \otimes \delta_{\{t=0\}} & \text{in } \Omega \times (-T, T) \\ u = 0 & \text{on } \partial\Omega \times (-T, T) \\ u(-T) = 0 & \text{on } \Omega \end{cases}$$

satisfying $u = 0$ in $\Omega \times (-T, 0)$ and (1.7). Finally, we get the result from Remark 3.2, achieving the proof. □

References

[1] D. R. ADAMS and L. I. HEDBERG, "Function Spaces and Potential Theory", Grundlehren der Mathematischen Wissenschaften, Vol. 31, Springer-Verlag, 1999.

[2] P. BARAS and M. PIERRE, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18** (1984), 111–149.

[3] P. BARAS and M. PIERRE, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **3** (1985), 185–212.

[4] M. F. BIDAUT-VÉRON and Q. H. NGUYEN, *Stability properties for quasilinear parabolic equations with measure data and applications*, J. Eur. Math. Soc. (JEMS) **17** (2015), 2103–2135.

[5] M. F. BIDAUT-VÉRON, Q. H NGUYEN and L. VÉRON, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. **102** (2014), 315–337.

[6] D. BLANCHARD and F. MURAT, *Renormalized solutions of nonlinear parabolic equation with L^1 data: existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 1153–1179.

[7] V. BOGELEIN, F. DUZAAR and U. GIANAZZA, *Very weak solutions of singular porous medium equations with measure data*, Inst. Mittag-Leffler, Commun. Pure Appl. Math. Anal. **14** (2015), 23–49.

[8] H. BREZIS and A. FRIEDMAN, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. **62** (1983), 73–97.

[9] E. CHASSEIGNE, "Contribution à la théorie des traces pour les équations quasilinéaires paraboliques", Thesis, Univ. Tours, France, 2000.

[10] E. CHASSEIGNE, *Initial trace for a porous medium equation: I. The strong absorption case*, Ann. Mat. Pura Appl. (4) **179** (2001), 413–458.
[11] A. DALL'AGLIO and L. ORSINA, *Existence results for some nonlinear parabolic equations with nonregular data*, Differential Integral Equations **5** (1992), 1335–1354.
[12] J. DRONIOU, A. PORRETTA and A. PRIGNET, *Parabolic capacity and soft measures for nonlinear equations*, Potential Anal. **19** (2003), 99–161.
[13] D. FEYEL and A. DE LA PRADELLE, *Topologies fines et compactifications associées à certains espaces de Dirichlet*, Ann. Inst. Fourier Grenoble **27** (1977), 121–146.
[14] J. HEINONEN, T. KILPELAINEN and O. MARTIO, "Nonlinear Potential Theory of Degenerate Elliptic Equations", Unabridged rep. of the 1993 original, Dover Publications, Inc., Mineola, N.Y., 2006.
[15] G. M. LIEBERMAN, "Second Order Parabolic Differential Equations", World Scientific press, River Edge, 1996.
[16] V. LISKEVICH, I. SKRYPNIK and Z. SOBOL, *Estimates of solutions for the parabolic p-Laplacian equation with measure via parabolic nonlinear potentials*, Commun. Pure Appl. Anal. **12** (2013), 1731–1744.
[17] V. LISKEVICH and I. SKRYPNIK, *Pointwise estimates for solutions to the porous medium equation with measure as a forcing term*, Isreal J. Math. **194** (2013), 259–275.
[18] Q. H. NGUYEN and L. VÉRON, *Quasilinear and Hessian type equations with exponential reaction and measure data*, preprint, Arch. Ration. Mech. Anal. **214** (2014), 235–267.
[19] Q. H. NGUYEN, *Potential estimates and quasilinear equations with measure data*, preprint, arXiv:1405.2587v1.
[20] F. PETITTA, *Renormalized solutions of nonlinear parabolic equations with general measure data*, Ann. Mat. Pura Appl. (4) **187** (2008), 563–604.
[21] F. PETITTA, A. PONCE and A. PORRETTA, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equ. **11** (2011), 861–905.

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