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Boundary Trace of Positive Solutions of Nonlinear Elliptic Inequalities

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Abstract. We develop a new method for proving the existence of a boundary trace, in the class of Borel measures, of nonnegative solutions of $-\Delta u+g(x,u)\geq 0$ in a smooth domain Ω under very general assumptions on g. This new definition which extends the previous notions of boundary trace is based upon a sweeping technique by solutions of Dirichlet problems with measure boundary data. We also prove a boundary pointwise blow-up estimate of any solution of such inequalities in terms of the Poisson kernel. If the nonlinearity is very degenerate near the boundary, for example if $g(x,u)\approx\exp(-\rho^{-1}_{\partial\Omega}(x))u^q$, we exhibit a new full boundary blow-up phenomenon.

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Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$ with a C^2 boundary $\partial\Omega$. This paper is concerned with the study of the generalized boundary value problem for the equation

$$-\Delta u+g(x,u)=0 \quad \text{in} \quad \Omega, \tag{0.1}$$

where $(x,r)\mapsto g(x,r)$ is a continuous function defined on $\Omega\times\mathbb{R}$, nondecreasing in the r variable, and nonnegative if $r\geq 0$. When $g(r)=r^q$ this problem has been thoroughly investigated with a probabilistic approach by Le Gall [19], [20] in the case $N=2=q$, then by Marcus and Véron [21], [22], [23] in the general case $q>1$, $N>1$ by analytic tools. Related studies were carried on by Dynkin and Kuznetsov [10], [11] with a mixing of probabilistic and analytic methods. In [16] the same problem is investigated with $g(r)=\exp(r)$. In all those cases, the boundary trace dichotomy argument is settled upon duality techniques which were first introduced by Baras and Pierre [1], but in the case of general nonlinearity, this method fails.

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In [27] an new approach of the boundary trace is developed for positive solutions of (0.1). This approach is settled upon two ingredients:

I - The coerciveness, which asserts that the set of nonnegative solutions of (0.1) is bounded in the local uniform topology upon $C(\Omega)$.

II - The strong-barrier property which is the property that for any boundary point z and for any $r > 0$, small enough, there exist supersolutions φ of (0.1) in $\Omega \cap B_r(z)$ with infinite value on $\partial B_r(z) \cap \Omega$, and zero value on $\partial\Omega \cap B_r(z)$.

When g depends only of r, those two notions coincide thanks to the Osserman-Keller condition,

$$\int_a^\infty \frac{ds}{\sqrt{G(s)}} < \infty, \qquad \text{for any } a > 0, \tag{0.2}$$

where $G(s) = \displaystyle\int_0^s g(t)dt$. The same equivalence holds if $\inf_{x\in\Omega} g(x,r) = g(r)$, and g satisfies (0.2). Moreover, in these cases, the local uniform upper bound of any positive solution of (0.1) achieves the following form

$$u(x) \leq \psi_g(\rho_{\partial\Omega}(x)), \quad \forall x \in \Omega, \tag{0.3}$$

where $\psi_g(t) = \displaystyle\int_t^\infty \frac{ds}{\sqrt{2G(s)}}$, and $\rho_{\partial\Omega}(x)) = \text{dist}\,(x, \partial\Omega)$.

If the strong barrier property is uniform with respect to $z \in \partial\Omega$, it implies the coerciveness, but when $\lim_{\rho_{\partial\Omega}(x))\to 0} g(x,r) = 0$ for any $r > 0$, (we say that the nonlinearity degenerates near the boundary), the reverse implication may not hold. However, if

$$g(x,r) \geq \rho^\alpha_{\partial\Omega}(x) r^q \quad \forall (x,r) \in \Omega \times \mathbb{R}_+,$$

for some $\alpha > 0$ and $q > 1$, it is proved in [27] that the equivalence still holds.

We adopt here a different point of view in connecting the existence of a boundary trace and the question of solving a Dirichlet problem with measure data. If $\mu \in \mathfrak{M}(\partial\Omega)$, the set of Radon measures on $\partial\Omega$, and $(x,r) \mapsto g(x,r)$ is a continuous function defined on $\Omega \times \mathbb{R}$, a function u defined in Ω is a solution of

$$\begin{aligned} -\Delta u + g(x,u) = 0 \;\text{ in }\; \Omega, \\ u = \mu \;\text{ on }\; \partial\Omega, \end{aligned} \tag{0.4}$$

if $u \in L^1(\Omega)$, $g(.,u) \in L^1(\Omega, \rho_{\partial\Omega} dx)$ and

$$\int_\Omega (-u\Delta\zeta + g(x,u)\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu(y), \tag{0.5}$$

for any $\zeta \in C_c^{1,1}(\bar{\Omega})$, the subspace of $C^1(\bar{\Omega})$ functions with Lipschitz continuous gradient and zero value on $\partial\Omega$. If $g(., x)$ is nondecreasing, this solution is unique whenever it exists and we denote it by $u = u_\mu$, since

$$(0.6)\quad \|u_\mu - u_{\mu^*}\|_{L^1(\Omega)} + \|\rho_{\partial\Omega}\left(g(u_\mu, .) - g(u_{\mu^*}, .)\right)\|_{L^1(\Omega)} \leq C\|\mu - \mu^*\|_{\mathfrak{M}(\partial\Omega)}.$$

The mapping $\mu \mapsto u_\mu$ is nondecreasing, moreover if $g(x, 0) = 0$, $\mu \geq 0 \Longrightarrow u_\mu \geq 0$. Conditions for existence are various.

Let $\mathcal{G}_0$ be the set of continuous functions g defined in $\Omega \times \mathbb{R}$ such that $g(x, 0) = 0$ and $r \mapsto g(r, x)$ is nondecreasing for any $x \in \Omega$, and $(x, y) \mapsto P(x, y)$ be the Poisson kernel in $\Omega \times \partial\Omega$. If $\mu \in \mathfrak{M}(\partial\Omega)$, we denote by $\mathbb{P}_\mu$ its Poisson's potential. If $g \in \mathcal{G}_0$ we say that μ is *g-admissible* if

$$(0.7)\qquad \int_\Omega g(x, \mathbb{P}_{|\mu|}(x))\rho_{\partial\Omega}(x) < \infty.$$

It is proved in [27] that problem (0.4) is uniquely solvable if μ is g-admissible.

However to check this condition on every measure might be far out of reach and a more tractable condition is introduced. We denote $\mathcal{HG}_0$ the subset of $g \in \mathcal{G}_0$ such that there exist two continuous, nondecreasing and nonnegative functions h and f defined on $\mathbb{R}_+$, such that

$$(0.8)\qquad \begin{aligned} &0 \leq |g(x, r)| \leq h(\rho_{\partial\Omega}(x)) f(|r|), &&\forall (x, r) \in \Omega \times \mathbb{R},\\ &\int_0^1 h(s) f(\sigma s^{1-N}) s^N ds < \infty, &&\forall \sigma \geq 0,\\ &\text{either } h(s) = s^\alpha, \text{ for some } \alpha \geq 0, \text{ or } f \text{ is convex}. \end{aligned}$$

In the first section of this article we prove the following.

If $g \in \mathcal{HG}_0$, then for any $\mu \in \mathfrak{M}(\partial\Omega)$, problem (0.4) *admits a unique solution u_μ. Moreover the problem is stable, in the sense that if $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges to μ in the weak sense of measures on $\partial\Omega$, the corresponding solutions $\{u_{\mu_n}\}$ converge to u_μ, locally uniformly in Ω.*

In the second section we introduce a new definition of the boundary trace for nonnegative solutions of elliptic inequalities.

$$(0.9)\qquad -\Delta u + g(x, u) \geq 0 \quad \text{in} \quad \Omega,$$

which extends the previous results concerning equations. A key observation for defining this notion is a supremum technique introduced by Richard and Véron [30] in the study of isolated singularities of elliptic inequalities. Following [27] we say that a function $g \in \mathcal{G}_0$ is *positively subcritical* if for any $\mu \in \mathfrak{M}_+(\partial\Omega)$, problem (0.4) admits a solution u_μ (unique and nonnegative). If $u \in C(\Omega)$ such that $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$ is a nonnegative solution of (0.9), then $w_\mu = \min\{u, u_\mu\}$ satisfies (0.9), and it admits a boundary trace $\gamma_u(\mu) \in \mathfrak{M}_+(\partial\Omega)$. Furthermore

$\mu \mapsto \gamma_u(\mu)$ is nondecreasing, concave if $r \mapsto g(x,r)$ is convex. Therefore the formula

$$\nu = \sup_{\mu \in \mathfrak{M}_+(\partial\Omega)} \gamma_u(\mu) \tag{0.10}$$

defines a *Borel measure* $\nu = \mathrm{Tr}^e_{\partial\Omega}(u)$ on $\partial\Omega$, that we call *the extended boundary trace of* u. This measure may not be an outer regular one except in some particular cases. A particularly important case deals with the choice $\mu = \lambda\delta_a$ for $\lambda > 0$, $a \in \partial\Omega$. The corresponding solution $u_{\lambda\delta_a}$ is called a *fundamental solution*. In such a case the boundary trace of $w_{\lambda\delta_a}$ is a measure concentrated at a, that we denote $\tilde{\gamma}_u(a,\lambda)\delta_a$. The mapping $\lambda \mapsto \tilde{\gamma}_u(a,\lambda)$ is a nondecreasing on $\mathbb{R}_+$, and satisfies

$$0 \leq \tilde{\gamma}_u(a,\lambda) \leq \lambda, \quad \forall\lambda \geq 0,\ \forall a \in \partial\Omega.$$

We define

$$\tilde{\gamma}_u(a) = \lim_{\lambda\to\infty} \tilde{\gamma}_u(a,\lambda),$$

and denote by $\mathcal{A}(u)$ the set of atoms of u,

$$\mathcal{A}(u) = \{a \in \partial\Omega : \ \gamma_u(a) > 0\}.$$

The *regular set* $\mathcal{R}(u)$ of u is the relatively open subset of the boundary points a with the property that there exists a relatively neighborhood of a, $\mathcal{O} \subset \partial\Omega$ such that

$$\sum_{\omega\in\mathcal{O}} \tilde{\gamma}_u(\omega) < \infty.$$

The *singular set* $\mathcal{S}(u)$ of u is the closed subset of the boundary points a with the property that for any relatively open neighborhood $\mathcal{O} \subset \partial\Omega$ of a, there holds

$$\sum_{\omega\in\mathcal{O}} \tilde{\gamma}_u(\omega) = \infty.$$

Those two definitions extend the classical notions of regular or singular sets of the boundary trace of the solution of an equation (see [22], [24]).
We prove in particular

$$\nu(a) = \tilde{\gamma}_u(a), \quad \forall a \in \partial\Omega,$$

and the equivalence between

(i) $\nu(\mathcal{O}) = \infty$, for any relatively open neighborhood $\mathcal{O} \subset \partial\Omega$ of a,

and

(ii) $u_{\infty,a} \leq u$,

under a general stability condition which holds in particular if $g \in \mathcal{HG}_0$.

As for $u_{\infty,a}$, different features may occur, in particular,

- $u_{\infty,a} \equiv +\infty$, *the full blow-up case.*
- $u_{\infty,a}(x) < \infty$ for any $x \in \Omega$, but $\lim_{\rho_{\partial\Omega}(x)\to 0} u_{\infty,a}(x) = \infty$, *the uniform boundary blow-up case.*
- $u_{\infty,a}(x) < \infty$ for any $x \in \bar{\Omega}\setminus\{a\}$, and $\lim_{x\to a} u_{\infty,a}(x) = \infty$ (non-tangential limit), *the strong isolated singularity case.*

Using precise pointwise estimates of positive super-harmonic functions near the boundary and the sweeping of any positive solution u of (0.9) by the solutions of (0.4) with Dirac masses as boundary data, we prove that for any $a \in \partial\Omega$,

$$x \mapsto |x-a|^{N-1} u(x)$$

converges in measure on the set $\{\sigma = (y-a)/|y-a| : y \in \Omega\}$ to $C(N)\tilde{\gamma}_u(a)$ as $x \to a$, where $C(N)$ is some positive constant depending only on the dimension.

If $g(x,r)$ satisfies

$$g(x,r) \geq \tilde{h}(x)\tilde{g}(r), \ \forall (x,r) \in \Omega \times \mathbb{R}_+, \tag{0.11}$$

where $\tilde{h} \in C(\Omega)$ takes positive values, and $\tilde{g}$ is nondecreasing and satisfies (0.2), there exists a maximal solution U_M to (0.1) in Ω (actually the global positivity of $\tilde{h}$ can be weakened, since the positivity near $\partial\Omega$ is sufficient for the existence of U_M). In that case (ii) implies

$$u_{\infty,a}(x) \leq u(x) \leq U_M(x),$$

which rules out the full blow-up case. The nature of $u_{\infty,a}$ depends strongly on $\tilde{h}$ and $\tilde{g}$. For example if it is assumed that $\tilde{h}$ is a positive constant, it follows from the method of construction of maximal solutions that the uniform boundary blow-up case does not hold, and we are left with the strong isolated singularity case. However, this situation also holds even if $\tilde{h}$ depends truly of x. It is proved in [27] that if

$$h(x) = \rho^{\alpha}_{\partial\Omega}(x), \ \text{with } \alpha > -2 \ \text{and } 1 < q < (N+\alpha+1)/(N-1) = q_c(\alpha), \tag{0.12}$$

the strong isolated singularity case occurs. We prove here that if

$$g(x,r) = \exp(-1/\rho_{\partial\Omega}(x))r^q, \quad \text{with } q > 1, \tag{0.13}$$

the uniform boundary blow-up case occurs for any $a \in \partial\Omega$. In such a case, either the boundary trace is a bounded Borel measure, or $u \equiv U_M$.

A parabolic version of this phenomenon has been observed in [28].

When the nonlinearity is not degenerate in the sense that the function $g \in \mathcal{HG}_0$ satisfies

$$(0.14)\quad 0 \le |g(x,r)| \le f(|r|),\ \forall (x,r) \in \Omega \times \mathbb{R},\ \text{ and } \int_0^1 f(s^{1-N}) s^N ds < \infty,$$

where f is a continuous nondecreasing function defined on $\mathbb{R}_+$, we recover the classical definition of the boundary trace in the class of outer regular Borel measures. More precisely, if u is a nonnegative solution of (0.9) with extended boundary trace ν, then for any point $a \in \partial\Omega$ the following dichotomy occurs: either

(i) $a \in \mathcal{S}(u)$ and for any $\mathcal{O} \in \mathcal{N}_a$ (the set of its relatively open neighborhoods $\mathcal{O} \subset \partial\Omega$), $\nu(\mathcal{O}) = \infty$. This is equivalent to

$$\lim_{t \to 0} \int_{\mathcal{O}_t} u(y) dS_t = \infty,\ \forall \mathcal{O} \in \mathcal{N}_a,$$

where $\mathcal{O}_t$ is the subset of points in Ω at distance $t > 0$ from $\partial\Omega$, with projection in $\mathcal{O}$ and dS_t the induced $(N-1)$-dimensional Hausdorff measure, or

(ii) $a \in \mathcal{R}(u)$, there exists $\mathcal{O} \in \mathcal{N}_a$ such that $\nu(\mathcal{O}) < \infty$ and for any $\mathcal{O}' \subset \bar{\mathcal{O}}' \subset \mathcal{O}$

$$\sup_{t \in (0,\beta_0]} \int_{\mathcal{O}_t} u(y) dS < \infty.$$

Furthermore, for any $\phi \in C_c(\mathcal{R}(u))$

$$\lim_{t \to 0} \int_{\mathcal{O}_t} u(y)\phi dS_t = \int_{\mathcal{R}(u)} \phi d\nu\,.$$

Our paper is organised as follows: In Section 1 we study the boundary value problem with Radon measures. In Section 2 we define and study the extended boundary trace of nonnegative solutions of inequalities. In Section 3 we give a boundary pointwise estimate for solutions of inequalities. In Section 4 we give properties of the boundary trace when the nonlinearity is not degenerate at the boundary. In Section 5 we study different examples of limit of a fundamental solution when the mass goes to infinity.

1. – Measure boundary data

Throughout this section, Ω is a bounded domain with a C^2 boundary $\partial\Omega$ and $\rho_{\partial\Omega}(x) = \operatorname{dist}(x, \partial\Omega)$. We put

$$(1.1)\quad \mathcal{G}_0 = \{g \in C(\Omega \times \mathbb{R}) \text{ s.t. } g(x,0) = 0 \text{ and } r \mapsto g(x,r) \text{ nondecreasing},\ \forall x \in \Omega\}.$$

We denote by $C_c^{1,1}(\bar{\Omega})$, the subspace of $C^1(\bar{\Omega})$-functions with Lipschitz continuous gradient and zero value on $\partial\Omega$, $\mathfrak{M}(\partial\Omega)$ the space of Radon measures on $\partial\Omega$, and $\mathfrak{M}_+(\partial\Omega)$ its positive cone. If $P(x,y)$ is the Poisson kernel in $\Omega\times\partial\Omega$, the Poisson potential of μ denoted by $\mathbb{P}_\mu$ is defined by

$$\mathbb{P}_\mu(x) = \int_{\partial\Omega} P(x,y)d\mu(y), \quad \forall x \in \Omega. \tag{1.2}$$

The next variant of Herglotz' theorem is due to Brezis [5] (see [32] for a proof).

LEMMA 1.1. *Let $f \in L^1(\Omega; \rho_{\partial\Omega}dx)$ and $\varphi \in L^1(\partial\Omega)$. Then there exists a unique $u \in L^1(\Omega)$ such that*

$$-\int_\Omega u\Delta\zeta = \int_\Omega f\zeta dx - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\varphi dS \tag{1.3}$$

for any $\zeta \in C_c^{1,1}(\bar{\Omega})$. Moreover there exists $C = C(\Omega) > 0$ such that

$$\|u\|_{L^1(\Omega)} \leq C\left(\left\|\rho_{\partial\Omega} f\right\|_{L^1(\Omega)} + \|\varphi\|_{L^1(\partial\Omega)}\right). \tag{1.4}$$

Finally u satisfies

$$-\int_\Omega |u|\,\Delta\zeta + \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\,|\varphi|\,dS \leq \int_\Omega f\zeta sgn(u)dx, \tag{1.5}$$

and

$$-\int_\Omega u^+\Delta\zeta + \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\varphi^+ dS \leq \int_\Omega f\zeta sgn^+(u)dx, \tag{1.6}$$

for any $\zeta \in C_c^{1,1}(\bar{\Omega})$, $\zeta \geq 0$.

DEFINITION 1.2. Let $\mu \in \mathfrak{M}(\partial\Omega)$. A function $u \in L^1(\Omega)$ is a solution of

$$\begin{aligned} -\Delta u + g(x,u) &= 0 \quad \text{in } \Omega, \\ u &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

if $g(.,u) \in L^1(\Omega; \rho_{\partial\Omega}dx)$ and

$$\int_\Omega (-u\Delta\zeta + g(x,u)\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu(y), \tag{1.8}$$

for any $\zeta \in C_c^{1,1}(\bar{\Omega})$.

Uniqueness is a straightforward consequence of (1.5). In the case where $g(x,r) = g(r)$ and $\mu \in L^1(\partial\Omega)$, existence of a solution to (1.7) is due to Brezis [5]. If g is continuous in $\bar{\Omega}\times\mathbb{R}$, the proof of Brezis result goes through without any difficulty. If $x \mapsto g(x,r)$ is merely continuous in Ω and unbounded near $\partial\Omega$, problem (1.7) may not have any solution, even with very regular data μ.

DEFINITION 1.2. A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is g-admissible if

$$g(.,\mathbb{P}_{|\mu|}) \in L^1(\Omega; \rho_{\partial\Omega}dx). \tag{1.9}$$

The two next results can be found in [27]

PROPOSITION 1.1. *Let $g \in \mathcal{G}_0$ and $\mu \in \mathfrak{M}(\partial\Omega)$ be g-admissible. Then problem* (1.7) *possesses a unique solution u_μ. Moreover the mapping $\mu \mapsto u_\mu$ is increasing and continuous from $\mathfrak{M}(\partial\Omega)$ endowed with the total variation norm into $C(\Omega)$ with the local uniform topology.*

PROPOSITION 1.2. *Let $g \in \mathcal{G}_0$ satisfy*

$$g(.,c) \in L^1(\Omega; \rho_{\partial\Omega} dx), \quad \forall c \in \mathbb{R}. \tag{1.10}$$

Then for any $\mu \in L^1(\partial\Omega)$, problem (0.4) *admits a unique solution.*

Let dH_{N-1} be the (N-1)-dimensional Hausdorff measure. If $\mu \in \mathfrak{M}(\partial\Omega)$ we denote by μ_R (resp. μ_s) its regular (resp. singular) part in the Lebesgue decomposition

$$\mu = \mu_R + \mu_s,$$

where $\mu_R \prec dH_{N-1}$) and $\mu_s \perp \mu_R$. A variant of the next result can be found in [25].

PROPOSITION 1.3. *Assume $g \in \mathcal{G}_0$ satisfies* (1.10) *and*

$$|g(x,2r)| \leq K(|g(x,r)| + \ell(x), \ \forall (x,r) \in \Omega \times \mathbb{R}, \tag{1.11}$$

for some fixed $K > 0$ and $\ell \in L^1(\Omega; \rho_{\partial\Omega} dx)$. If $\mu \in \mathfrak{M}(\partial\Omega)$, $\mu = \mu_R + \mu_s$ and μ_s is g-admissible, then the conclusions of Proposition 1.1 *still hold.*

PROOF. First notice that relation (1.11), called the Δ_2-condition, implies

$$\left|g(x,r+r')\right| \leq K(|g(x,r)| + \left|g(x,r')\right|) + \ell(x), \ \forall (x,r,r') \in \Omega \times \mathbb{R}^2. \tag{1.12}$$

STEP 1. Assume that μ is nonnegative, and so are μ_R and μ_s. Let $\{\mu_{R,n}\}$ be a sequence of smooth functions on $\partial\Omega$ converging to μ_R in $L^1(\partial\Omega)$. Since $\mathbb{P}_{\mu_{R,n}}$ is bounded, $g(x,.)$ is nondecreasing and (1.10) is satisfied, it follows from (1.12) that $\mu_n = \mu_{R,n} + \mu_S$ is g-admissible. Let u_n and v_n be the solutions of (0.4) with respective measure boundary data μ_n and $\mu_{R,n}$. Applying (1.5) with $u = v_n - v_p$, $f = -g(.,v_n) + g(.,v_p)$ and $\zeta = \mathbb{P}_1$, we obtain

$$\left\|v_n - v_p\right\|_{L^1(\Omega)} + \left\|\rho_{\partial\Omega}(g(.,v_n) - g(.,v_p))\right\|_{L^1(\Omega)} \leq C \left\|\mu_{R,n} - \mu_{R,p}\right\|_{L^1(\partial\Omega)}. \tag{1.13}$$

Thus $\{v_n\}$ and $\{g(.,v_n\}$ converge to v and $g(.,v)$, respectively in $L^1(\Omega)$ and $L^1(\Omega; \rho_{\partial\Omega} dx)$. Furthermore v is the solution of (0.4) with measure boundary data μ_R. Because $v_n + \mathbb{P}_{\mu_S}$ is a supersolution of (0.1) with boundary data μ_n, there also holds

$$0 \leq u_n \leq v_n + \mathbb{P}_{\mu_S}, \tag{1.14}$$

thus $\{u_n\}$ is uniformly integrable in $L^1(\Omega)$, and also locally compact in the $C^1_{\text{loc}}(\Omega)$ topology, by the elliptic equations regularity theory. Since inequality (1.12) leads to

$$0 \le g(x, u_n) \le K(g(x, \mathbb{P}_{\mu_S}) + g(x, v_n)) + \ell(x), \tag{1.15}$$

it follows from the assumption on μ_S that the sequence $\{g(., u_n)\}$ is uniformly integrable in $L^1(\Omega; \rho_{\partial\Omega}dx)$. By the Vitali theorem $u_{n_k} \to u$ and $g(., u_{n_k}) \to g(., u)$ respectively in $L^1(\Omega)$ and $L^1(\Omega; \rho_{\partial\Omega}dx)$ and u is the solution of (0.4).

STEP 2. Let $\tilde{\mu}_{R,n}$ and $\bar{\mu}_{R,n}$ be smooth L^1-approximations of μ_R^+ and μ_R^-, and denote by u_n, $\tilde{v}_n$ and $\bar{v}_n$ the solutions of (0.4) with respective measure boundary data $\mu_n = \tilde{\mu}_{R,n} + \bar{\mu}_{R,n} + \mu_S$, $\tilde{\mu}_{R,n}$ and $-\bar{\mu}_{R,n}$. By monotonicity and (1.12) there holds

$$\bar{v}_n - \mathbb{P}_{\mu_S^-} \le u_n \le \tilde{v}_n + \mathbb{P}_{\mu_S^+},$$

and

$$K(g(x, -\mathbb{P}_{\mu_S^-}) + g(x, \bar{v}_n)) - \ell(x) \le g(x, u_n) \le K(g(x, \mathbb{P}_{\mu_S^+}) + g(x, \tilde{v}_n)) + \ell(x).$$

Since $\bar{v}_n$, $\tilde{v}_n$, $g(., \tilde{v}_n)$ and $g(., \bar{v}_n)$ inherit the uniform integrability properties of Step 1, we conclude again by the Vitali theorem. □

Let u_μ denote the solution of (0.4) with boundary data μ. The g admissibility condition on μ does not imply the weak continuity of the mapping $\mu \mapsto u_\mu$, thus a more uniform assumption is needed.

DEFINITION 1.3. We denote by $\mathcal{HG}_0$ the subset of the $g \in \mathcal{G}_0$ such that there exist two continuous, nondecreasing and nonnegative functions h and f defined on $\mathbb{R}_+$, with the property

$$0 \le |g(x, r)| \le h(\rho_{\partial\Omega}(x)) f(|r|), \quad \forall (x, r) \in \Omega \times \mathbb{R}, \tag{1.16}$$

$$\int_0^1 h(s) f(\sigma s^{1-N}) s^N ds < \infty, \quad \forall \sigma \ge 0 \tag{1.17}$$

$$\text{either } h(s) = s^\alpha, \text{ for some } \alpha \ge 0, \text{ or } f \text{ is convex .} \tag{1.18}$$

The main result of this section is an existence and stability theorem which extends a previous one due to Gmira and Véron [15]. The technique involved is based upon the use of Marcinkiewicz spaces first introduced by Benilan and Brezis [3], [6] for solving semilinear equations with right-hand side measure.

THEOREM 1.1. *Let $g \in \mathcal{HG}_0$. Then any measure μ on $\partial\Omega$ is g-admissible. Moreover, if $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ converges to μ in the weak sense of measures, the corresponding solutions u_{μ_n} of (0.4) with boundary data μ_n converge to u_μ locally uniformly in Ω and $g(., u_{\mu_n}) \to g(., u_\mu)$ in $L^1(\Omega; \rho_{\partial\Omega}dx)$.*

PROOF. STEP 1. The Marcinkiewicz space framework. For any nonnegative locally bounded Borel measure β in Ω and real number $p > 1$, we denote

$$M^p(\Omega; d\beta) = \{v \in L^1_{\mathrm{loc}}(\Omega; d\beta) : \|v\|_{M^p(\Omega;d\beta)}\} < \infty\}, \tag{1.19}$$

where

$$\|v\|_{M^p(\Omega;d\beta)} = \inf\left\{c \in [0,\infty] \text{ s.t. } \int_K |v|\, d\beta \le c \left(\int_K d\beta\right)^{1-1/p} \forall K \subset \Omega,\ K \text{ Borel}\right\}. \tag{1.20}$$

Besides the classical imbedding of $M^p(\Omega; d\beta)$ into $L^{\tilde{p}}_{\mathrm{loc}}(\Omega; d\beta)$ for any $1 \le \tilde{p} < p$, the next inequality plays an important role

$$C(p)\|v\|_{M^p(\Omega;d\beta)}\} \le \sup_{\lambda>0}\left\{\lambda^p \int_{\{|u|>\lambda\}} d\beta\right\} \le \|v\|_{M^p(\Omega;d\beta)}. \tag{1.21}$$

Moreover the following estimates are proved in [16]: there exists $K = K(\Omega) > 0$ such that for any $\nu \in \mathfrak{M}(\partial\Omega)$,

$$\|\mathbb{P}_\nu\|_{M^{(N+1)/(N-1)}(\Omega;\rho_{\partial\Omega}dx)} \le K\|\nu\|_{L^1(\partial\Omega)}, \tag{1.22}$$

$$\|\mathbb{P}_\nu\|_{M^{N/(N-1)}(\Omega)} \le K\|\nu\|_{L^1(\partial\Omega)}, \tag{1.23}$$

$$\|\mathbb{P}_\nu\|_{L^\infty(\Omega^c_r)} \le Kr^{1-N}\|\nu\|_{L^1(\partial\Omega)}, \tag{1.24}$$

where $\Omega_r = \{x \in \Omega : \rho_{\partial\Omega}(x) \le r\}$, and $\Omega^c_r = \Omega \setminus \bar{\Omega}_r = \{x \in \Omega : \rho_{\partial\Omega}(x) > r\}$.

STEP 2. We claim that there exist two positive constants $C_1 = C_1(\Omega)$ and $C_2 = C_2(N)$ such that for any $a \in \partial\Omega$ and $\lambda > 0$

$$\beta_a(\lambda) = \int_{\Gamma_a(\lambda)} h(\rho(x))\rho(x)dx \le C_2 \int_0^{(C_1/\lambda)^{1/(N-1)}} h(s)s^N ds, \tag{1.25}$$

where

$$\Gamma_a(\lambda) = \{x \in \Omega : P(x,a) > \lambda\}.$$

Since

$$C_1^{-1}\rho_{\partial\Omega}(x)|x-a|^{-N} \le P(x,a) \le C_1\rho_{\partial\Omega}(x)|x-a|^{-N}, \tag{1.26}$$

for some $C_1 > 0$ independent of $(x,a) \in \Omega \times \partial\Omega$,

$$\Gamma_a(\lambda) \subset \left\{x \in \Omega : \rho_{\partial\Omega}(x)|x-a|^{-N} > \lambda/C_1\right\} \subset \Omega \cap B_{r_\lambda}(a),$$

with $r_\lambda = (C_1/\lambda)^{1/(N-1)}$. Since h is nondecreasing,

$$\int_{\Gamma_a(\lambda)} h(\rho(x))\rho(x)dx \le \int_{B_{r_\lambda}(a)} h(|x|)\rho(|x|)dx = \left|S^{N-1}\right| \int_0^{r_\lambda} h(s)s^N ds,$$

which implies (1.25).

STEP 3. Let $G \subset \Omega$ be a Borel subset, then for any $m > 0$, $\lambda > 0$, and $a \in \partial\Omega$ there holds

$$\begin{aligned}\int_G h(\rho_{\partial\Omega}) f(mP(.,a)\rho_{\partial\Omega} dx \le f(\lambda) \int_G h(\rho_{\partial\Omega})\rho_{\partial\Omega} dx \\ + C_3 m^{(N+1)/(N-1)} \int_\lambda^\infty f(s) h((mC_1/s)^{1/(N-1)}) s^{-2N/(N-1)} ds,\end{aligned} \tag{1.27}$$

with $C_3 = C_3(N) > 0$. Actually,

$$\begin{aligned}\int_G h(\rho_{\partial\Omega}(x)) f(mP(x,a)\rho_{\partial\Omega}(x)dx = \int_{G\cap\{P(x,a)\le\lambda/m\}} h(\rho_{\partial\Omega}(x)) f(mP(x,a)\rho_{\partial\Omega}(x)dx \\ + \int_{G\cap\Gamma_a(\lambda/m)} h(\rho_{\partial\Omega}(x)) f(mP(x,a)\rho_{\partial\Omega}(x)dx.\end{aligned}$$

Since f is nondecreasing,

$$\int_{G\cap\{P(x,a)\le\lambda/m\}} f(mP(x,a)h(\rho_{\partial\Omega}(x))\rho_{\partial\Omega}(x)dx \le f(\lambda) \int_G h(\rho_{\partial\Omega}(x))\rho_{\partial\Omega}(x)dx.$$

Moreover

$$\int_{G\cap\Gamma_a(\lambda/m)} h(\rho_{\partial\Omega}(x)) f(mP(x,a)\rho_{\partial\Omega}(x)dx \le -\int_{\lambda/m}^\infty f(ms)d\beta_a(s).$$

But

$$-\int_{\lambda/m}^\infty f(ms)d\beta_a(s) = f(\lambda)\beta_a(\lambda) + \int_{\lambda/m}^\infty \beta_a(s)df(ms).$$

Using (1.25) in Step 2 infers

$$-\int_{\lambda/m}^\infty f(ms)d\beta_a(s) \le f(\lambda)\beta_a(\lambda) + C_2 \int_{\lambda/m}^\infty \int_0^{(C_1/s)^{1/(N-1)}} h(\tau)\tau^N d\tau df(ms).$$

Since

$$\begin{aligned}\int_{\lambda/m}^\infty \int_0^{(C_1/s)^{1/(N-1)}} h(\tau)\tau^N d\tau df(ms) = -f(\lambda) \int_0^{(mC_1/\lambda)^{1/(N-1)}} h(s)s^N ds \\ + \frac{C_1^{(N+1)/(N-1)}}{N-1} \int_{\lambda/m}^\infty h((C_1/s)^{1/(N-1)}) s^{-2N/(N-1)} f(ms)ds,\end{aligned}$$

(1.27) follows by change of variable, with $C_3 = C_2 C_1^{(N+1)/(N-1)}$. It is important to notice that this integral is convergent because of (1.16).

STEP 4. Suppose f is convex, then for any $\mu \in \mathfrak{M}^+(\partial\Omega)$ with total mass m and any Borel subset $G \subset \Omega$, there holds

$$\begin{aligned}\int_G f(\mathbb{P}_\mu)h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx &\leq f(\lambda)\int_G h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx \\ &+ C_3 m^{(N+1)/(N-1)}\int_\lambda^\infty f(s)h((mC_1/s)^{1/(N-1)})s^{-2N/(N-1)}ds.\end{aligned} \tag{1.28}$$

First, let us assume that

$$\mu = m\sum_{i=1}^k \theta_i \delta_{a_i}$$

for some $a_i \in \partial\Omega$ and $\theta_i > 0$ with $\sum_{i=1}^k \theta_i = 1$. Then

$$\mathbb{P}_\mu(x) = m\sum_{i=1}^k \theta_i P(x, a_i).$$

Since

$$f(\mathbb{P}_\mu(x)) = f(m\sum_{i=1}^k \theta_i P(x, a_i)) \leq \sum_{i=1}^k \theta_i f(mP(x, a_i))$$

$$\int_G f(\mathbb{P}_\mu)h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx \leq \sum_{i=1}^k \theta_i \int_G f(mP(x, a_i)h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx.$$

Therefore (1.28) follows from (1.27).

For a general nonnegative measure μ with total mass m, there exists a sequence of finite combinations of positive Dirac measures μ_n with same total mass converging to μ in the weak sense of measures. Then $\mathbb{P}_{\mu_n}$ converges to $\mathbb{P}_\mu$ locally uniformly in Ω and in $L^p(\Omega)$ for any $1 \leq p < N/(N-1)$. Thus (1.28) follows by the Fatou's lemma.

STEP 5. Suppose $h(s) = s^\alpha$, then for any $\mu \in \mathfrak{M}^+(\partial\Omega)$ with total mass m and any Borel subset $G \subset \Omega$, there holds

$$\begin{aligned}&\int_G f(\mathbb{P}_\mu)h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx \\ &\quad\leq f(\lambda)\int_G \rho_{\partial\Omega}^{1+\alpha}dx + C_7 m^{(N+1+\alpha)/(N-1)}\int_\lambda^\infty s^{-(2N+\alpha)/(N-1)}f(s)ds\,.\end{aligned} \tag{1.29}$$

By Step 2 there exists $C_4 = C_4(\Omega, \alpha) > 0$ such that

$$\beta_a(\lambda) \leq C_4 \lambda^{(N+1+\alpha)/(N-1)}$$

for any $\lambda > 0$. Thus we derive an estimate of $P(.,a)$ in $M^{(N+1+\alpha)/(N-1)}$,

$$\int_G P(x,a)\rho_{\partial\Omega}^{1+\alpha}dx \leq C_5\left(\int_G \rho_{\partial\Omega}^{1+\alpha}dx\right)^{(2+\alpha)/(N+1+\alpha)}. \tag{1.30}$$

Therefore

$$\begin{aligned}\int_G \mathbb{P}_\mu(x))\rho_{\partial\Omega}^{1+\alpha}dx &= \int_{\partial\Omega} d\mu(a)\int_G P(x,a)\rho_{\partial\Omega}^{1+\alpha}dx,\\ &\leq \left(\int_{\partial\Omega} d\mu(a)\right)\max_{a\in\partial\Omega}\int_G P(x,a)\rho_{\partial\Omega}^{1+\alpha}dx.\end{aligned}$$

From this estimate follows

$$\int_G \mathbb{P}_\mu(x))\rho_{\partial\Omega}^{1+\alpha}dx \leq C_5\|\mu\|_{L^1(\partial\Omega)}\left(\int_G \rho_{\partial\Omega}^{1+\alpha}dx\right)^{(2+\alpha)/(N+1+\alpha)}. \tag{1.31}$$

Now

$$\begin{aligned}\int_G f(\mathbb{P}_\mu)h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx &= \int_G f(\mathbb{P}_\mu)\rho_{\partial\Omega}^{1+\alpha}dx\\ &\leq f(\lambda)\int_G \rho_{\partial\Omega}^{1+\alpha}dx + \int_{\{\mathbb{P}_\mu>\lambda\}} f(\mathbb{P}_\mu)\rho_{\partial\Omega}^{1+\alpha}dx.\end{aligned} \tag{1.32}$$

But

$$\int_{\{\mathbb{P}_\mu>\lambda\}} f(\mathbb{P}_\mu)\rho_{\partial\Omega}^{1+\alpha}dx = -\int_\lambda^\infty f(s)d\beta^\mu(s),$$

where

$$\beta^\mu(s) = \int_{\Gamma^\mu(s)} \rho_{\partial\Omega}^{1+\alpha}dx \quad \text{with} \quad \Gamma^\mu(s) = \{x\in\Omega : \mathbb{P}_\mu(x) > s\}.$$

Moreover

$$\beta^\mu(s) \leq C_6 m^{(N+1+\alpha)/(N-1)}\lambda^{-(N+1+\alpha)/(N-1)}$$

by (1.30), with $G = \Gamma^\mu(\lambda)$ and $C_6 = C_5^{(N+1+\alpha)/(N-1)}$. Therefore

$$\begin{aligned}-\int_\lambda^\infty f(s)d\beta^\mu(s) &= f(\lambda)\beta^\mu(\lambda) + \int_\lambda^\infty \beta^\mu(s)df(s),\\ &\leq C_6 m^{(N+1+\alpha)/(N-1)}\lambda^{-(N+1+\alpha)/(N-1)}f(\lambda)\\ &\quad + C_6 m^{(N+1+\alpha)/(N-1)}\int_\lambda^\infty s^{-(N+1+\alpha)/(N-1)}df(s),\\ &\leq C_7 m^{(N+1+\alpha)/(N-1)}\int_\lambda^\infty s^{-(2N+\alpha)/(N-1)}f(s)ds,\end{aligned} \tag{1.33}$$

where $C_7 = (N + 1 + \alpha)C_6 m^{(N+1+\alpha)/(N-1)}$. Combining (1.32) and (1.33) yields (1.29).

If we take $G = \Omega$ in (1.27) and (1.29) we derive that (1.9) holds with μ and we conclude by Proposition 1.1. However those two estimates are much more powerfull since they leads to uniform-integrability properties.

STEP 6. Put $v_n = \mathbb{P}_{|\mu_n|}$ and $v = \mathbb{P}_{|\mu|}$. Then

$$0 \leq |u_n| \leq v_n \text{ and } 0 \leq |u| \leq v. \tag{1.34}$$

The fact that u_n is locally bounded in Ω independently of n follows from (1.24). Since g is continuous, $g(., u_n)$ remains also locally bounded in Ω. By the elliptic equations regularity theory there exists a subsequence $\{u_{n_k}\}$ and a $C^1(\Omega)$-function u such that $u_{n_k} \to u$ in the $C^1_{\text{loc}}(\Omega)$-topology. This clearly implies that u solves (0.1) in Ω.

STEP 7. We claim that u is a solution of (0.4) with μ as boundary data. By the definition of the Marcinkiewicz norm,

$$\begin{aligned}\int_\Omega |u_{n_k} - u|\,dx &= \int_{\Omega_r^c} |u_{n_k} - u|\,dx + \int_{\Omega_r} |u_{n_k} - u|\,dx,\\ &\leq \int_{\Omega_r^c} |u_{n_k} - u|\,dx + 2\|u_{n_k} - u\|_{M^{N/(N-1)}(\Omega)} (meas.\Omega_r)^{1/N}.\end{aligned}$$

But $\|u_{n_k} - u\|_{M^{N/(N-1)}(\Omega)}$ remains bounded independently of n_k by (1.22). Thus $u_{n_k} \to u$ in $L^1(\Omega)$. In order to prove that $g(., u_{n_k}) \to g(., u)$ in $L^1(\Omega, \rho_{\partial\Omega}dx)$ put $m_n = \int_{\partial\Omega} d\,|\mu_n|$ and let $G \subset \Omega$ be a Borel set. Because of (1.16) and (1.34) there holds

$$|g(., u_n)| \leq f(v_n)h(\rho_{\partial\Omega}).$$

If f is convex (Step 4) it follows

$$\begin{aligned}\int_G |g(., u_n)|\,\rho_{\partial\Omega}dx &\leq f(\lambda)\int_G h(\rho_{\partial\Omega})\rho_{\partial\Omega}dx\\ &\quad + C_3 m_n^{(N+1)/(N-1)} \int_\lambda^\infty f(s)h((m_n C_1/s)^{1/(N-1)})s^{-2N/(N-1)}ds.\end{aligned} \tag{1.35}$$

If $h(s) = s^\alpha$ (Step 5), then

$$\begin{aligned}&\int_G |g(., u_n)|\,\rho_{\partial\Omega}dx\\ &\qquad \leq f(\lambda)\int_G \rho_{\partial\Omega}^{1+\alpha}dx + C_7 m_n^{(N+1+\alpha)/(N-1)} \int_\lambda^\infty s^{-(2N+\alpha)/(N-1)} f(s)ds.\end{aligned} \tag{1.36}$$

Because $\{m_n\}$ is bounded, for any $\varepsilon > 0$, we first choose $\lambda > 0$ large enough so that, for any $n \in \mathbb{N}$,

$$C_3 m_n^{(N+1)/(N-1)} \int_\lambda^\infty f(s)h((m_n C_1/s)^{1/(N-1)}) s^{-2N/(N-1)} ds \leq \varepsilon/2$$

in the case f is convex, or

$$C_7 m_n^{(N+1+\alpha)/(N-1)} \int_\lambda^\infty s^{-(2N+\alpha)/(N-1)} f(s) ds \leq \varepsilon/2$$

in the case $h(s) = s^\alpha$. Then we take $meas.G$ small enough so that

$$f(\lambda) \int_G h(\rho_{\partial\Omega})\rho_{\partial\Omega} dx, \leq \varepsilon/2$$

and we conclude that

$$\int_G |g(., u_n)|\, \rho_{\partial\Omega} dx \leq \varepsilon$$

independently of n. Therefore $\{g(., u_n\}$ is uniformly integrable for the measure $\rho_{\partial\Omega} dx$. Since g is continuous and $u_n \to u$ in Ω, it follows that $g(., u_n) \to g(., u)$ in $L^1(\Omega, \rho_{\partial\Omega} dx)$. Letting $n \to \infty$ in the integral formulation of u_n for (0.4) implies that u solves (0.4) with μ as boundary data. □

The next stability result is a straightforward extension of the previous result

Proposition 1.4. *Let $(x, r) \mapsto g_n(x, r)$ be a sequence of functions in $\in C(\Omega \times \mathbb{R})$, nondecreasing with respect to r, vanishing at $r = 0$ for any $x \in \Omega$, and satisfying* (1.16)-(1.18) *uniformly with respect to n. If there exists $g \in C(\Omega \times \mathbb{R})$ such that $g_n(x, r) \to g(x, r)$ pointwise in $\Omega \times \mathbb{R}$, then g satisfies* (1.16). *Moreover, if $\mu_n \in \mathfrak{M}(\partial\Omega)$ converges to μ in the weak sense of measures, the sequence of solutions u_{μ_n, g_n} of*

$$-\Delta u_n + g_n(x, u_n) = 0 \quad in \ \ \Omega,$$
$$u_n = \mu_n \ on \ \ \partial\Omega,$$

converges locally uniformly in Ω to the solution u_μ of (0.4).

Proof. By assumption

$$0 \leq g_n(x, r) \leq h(\rho_{\partial\Omega}(x)) f(r), \ \forall n \in \mathbb{N}^*, \ \forall (x, r) \in \bar{\Omega} \times \mathbb{R},$$

and (1.16) holds. Then g satisfies the same upper bound. Since $0 \leq |u_n| \leq \mathbb{P}_{|\mu_n|}$, the inequalities (1.35)-(1.36) hold with g replaced by g_n which infers the uniform integrability of $\{g_n(., u_n)\}$ in $L^1(\Omega; \rho_{\partial\Omega} dx)$. The rest of the proof is similar to the one of Proposition 1.1. □

2. – The extended boundary trace

If Ω is a C^3 bounded domain in $\mathbb{R}^N$ and $x \in \partial\Omega$, let $\mathbf{n}_x$ denote the normal pointing outward unit vector at x. Let us recall some notations and definitions from [25]. The mapping Π from $\partial\Omega \times (0, \infty)$ into $\mathbb{R}^N$ is defined by

$$\Pi(x, t) = x - t\mathbf{n}_x \quad \forall (x, t) \in \partial\Omega \times (0, 1). \tag{2.1}$$

It is known that there exists $0 < \beta_0$ such that Π is a diffeomorphism from $\partial\Omega \times [0, \beta_0)$ onto

$$\Omega_{\beta_0} = \{x \in \Omega : \rho_{\partial\Omega}(x) < \beta_0\}. \tag{2.2}$$

In particular, for any $t \in [0, \beta_0)$ the set

$$\Sigma_t = \{y = x - t\mathbf{n}_x : \ x \in \partial\Omega\} \tag{2.3}$$

is diffeomorphic to $\partial\Omega = \Sigma_0$ (for the sake of simplicity, we shall denote $\Sigma_0 = \Sigma$), and for any $y \in \Sigma_t$, $\rho_{\partial\Omega}(y) = t$. If $U \subset \partial\Omega$, we denote

$$U_t = \{y = x - t\mathbf{n}_x : \ x \in U\},$$

and if ζ is a function defined in U_t,

$$\zeta_t(y) = \zeta(x) \quad \text{for any} \quad y = x - t\mathbf{n}_x \in U_t.$$

We denote by $\mathfrak{H}_t$ the mapping from Σ_t to Σ defined by $\mathfrak{H}_t(x) = \sigma(x)$ for $x \in \Sigma_t$. Thus $\mathfrak{H}_t^{-1}(x) = \Pi^{-1}(., t)$.

Given $t \in [0, \beta_0)$, a Borel measure μ and a function f on Σ_t, we define a corresponding measure μ^t and function f^t on Σ by

$$\begin{cases} \mu^t(E) = \mu(\mathfrak{H}_t^{-1}(E)), & \forall E \subset \Sigma, \ E \text{ Borel}, \\ f^t(\sigma) = f(\sigma - t\mathbf{n}_\sigma), & \forall \sigma \in \Sigma. \end{cases} \tag{2.4}$$

Then

$$\mu \in \mathfrak{M}(\Sigma_t), \ f \in L^1(\Sigma_t, |\mu|) \Longrightarrow \begin{cases} f^t \in L^1(\Sigma, |\mu^t|), \\ \displaystyle\int_{\Sigma_t} f d\mu = \int_\Sigma f^t d\mu^t. \end{cases} \tag{2.5}$$

This section is devoted to the definition and properties of the notion of extended boundary trace for nonnegative solutions of

$$-\Delta u + g(x, u) \geq 0 \quad \text{in } \Omega, \tag{2.6}$$

and of the associated equation

$$-\Delta u + g(x, u) = 0 \quad \text{in } \Omega. \tag{2.7}$$

DEFINITION 2.1. I- We say that a function $g \in \mathcal{G}_0$ is *positively subcritical* if for any $\mu \in \mathfrak{M}_+(\partial\Omega)$, problem (0.4) admits a solution u_μ.

II- The function g is said *positively subcritical and stable* if $\mu_n \to \mu$ weakly in $\mathfrak{M}_+(\partial\Omega)$ implies

$$u_{\mu_n} \to u_\mu \text{ locally uniformly in } \Omega \text{ and } g(., u_{\mu_n}) \to g(., u_\mu) \text{ in } L^1(\Omega; \rho_{\partial\Omega} dx).$$

REMARK 2.1. I- The convergence of μ_n to μ in the weak sense of measures implies that $\mathbb{P}_{\mu_n}$ converges to $\mathbb{P}_\mu$ in $L^s(\Omega)$ for any $1 \leq s < N/(N-1)$, by classical potential analysis. Because $0 \leq u_{\mu_n} \leq \mathbb{P}_{\mu_n}$, $u_{\mu_n} \to u_\mu$ in $L^s(\Omega)$ (by Vitali's theorem) and in the local uniform topology of $C(\Omega)$). If $\zeta \in C_c^2(\bar{\Omega})$, $\zeta \geq 0$, then

$$\int_\Omega \left(-u_{\mu_n} \Delta\zeta + g(x, u_{\mu_n})\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu_n.$$

Since $u_{\mu_{n_k}} \to \tilde{u}$ and $g(., u_{\mu_{n_k}}) \to g(., \tilde{u})$ locally uniformly in Ω, Fatou's lemma infers

$$\int_\Omega (-\tilde{u}\Delta\zeta + g(x, \tilde{u})\zeta)\, dx \leq -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu,$$

and there always holds $\tilde{u} \leq u_\mu$. If it is assumed that $\tilde{u} = u_\mu$, then

$$\lim_{n\to\infty} \int_\Omega g(x, u_{\mu_n})\zeta dx = \int_\Omega g(x, u_\mu)\zeta dx,$$

and this convergence holds for any $\zeta \in C_c(\bar{\Omega})$ such that $\zeta/\rho_{\partial\Omega}$ is bounded. However it does not imply weak convergence in $L^1(\Omega; \rho_{\partial\Omega} dx)$. Notice that in such a case, weak convergence in $L^1(\Omega; \rho_{\partial\Omega} dx)$ implies strong convergence by Dunford-Pettis and Vitali's theorems.

II- It follows by Theorem 1.1 that any $g \in \mathcal{H}\mathcal{G}_0$ is positively subcritical and stable.

DEFINITION 2.2. Let U be a relatively open subset of Σ and $\mu \in \mathfrak{M}(U)$. We say that a function $v \in C(\Omega)$ admits μ for trace on $\mathcal{O}$, and we denote it by $\mathrm{Tr}_U(v)$, if

$$\lim_{t\to 0} \int_{U_t} v(x)\phi(\sigma(x)) dS = \int_U \phi d\mu, \quad \forall\phi \in C_c(U). \tag{2.8}$$

It is proved in [27, Corollary 1.3] that the solution u_μ of (0.4) admits μ for trace on $\partial\Omega$. The role of nonnegative super-solutions is enlighted by the next result.

Proposition 2.1. *Let* $g \in \mathcal{G}_0$ *and* $u \in C(\Omega)$ *be a nonnegative solution of* (2.6) *in* Ω *such that* $g(.,u) \in L^1(\Omega, \rho_{\partial\Omega} dx)$. *Then* u *admits a boundary trace* $\mu \in \mathfrak{M}_+(\partial\Omega)$. *Moreover, if*

$$u^* = \sup\{v \in C(\Omega) : 0 \le v \le u,\ v \text{ solution of } (2.7)\}, \tag{2.9}$$

then u^* *is a solution of* (2.7) *and*

$$\mathrm{Tr}_{\partial\Omega}(u) = \mathrm{Tr}_{\partial\Omega}(u^*).$$

Proof. Let $\psi = \mathbb{G}_{g(.,u)}$ be the Green potential of the function $g(.,u)$. Then $\psi + u$ is nonnegative and super-harmonic. Therefore $\psi + u$ admits a boundary trace belonging to $\mathfrak{M}_+(\partial\Omega)$. Thus the same holds for u since ψ vanishes on $\partial\Omega$ and the boundary trace of u is a nonnegative Radon measure. The construction of u^* is performed in considering the sequence of smooth domains Ω_n $(n \ge 1)$, defined by

$$\Omega_n = \{x \in \Omega : \rho_{\partial\Omega}(x) > \beta_n\},$$

where $0 < \beta_{n+1} < \beta_n < \beta_0$ and $\beta_n \to 0$ as $n \to \infty$. For each n we denote by u_n the solution of

$$\begin{aligned} -\Delta u_n + g(x,u_n) &= 0 \text{ in } \Omega_n, \\ u_n &= u \text{ on } \partial\Omega_n. \end{aligned}$$

Since u is a super solution, $u \ge u_n \ge 0$ and consequently the sequence $\{u_n\}$ is decreasing. Therefore u^* exists as the decreasing limit of the u_n. By the regularity theory of elliptic equations, the convergence holds in $C^1_{\mathrm{loc}}(\Omega)$ and u^* satisfies (2.7). If $\tilde{u}$ is any nonnegative solution of (2.7) dominated by u, then $\tilde{u} \le u^* = u_n$ on $\partial\Omega_n$, thus $\tilde{u} \le u_n$ in Ω_n. Letting $n \to \infty$ yields $\tilde{u} \le u^*$. Clearly the correspondence $u \mapsto u^*$ which associates to a solution u of (2.6) the largest solution of (2.7) dominated by u inherits the following properties:

$$u_1 \le u_2 \Longrightarrow u_1^* \le u_2^*, \tag{2.10}$$

and

$$\left(u^*\right)^* = u^*. \tag{2.11}$$

In the above construction, the boundary trace of u plays no role. In order to prove that $\mathrm{Tr}_{\partial\Omega}(u^*) = \mu$, let $\zeta \in C_c^{1,1}(\bar{\Omega})$ and ζ_n be the solution of

$$\begin{aligned} -\Delta\zeta_n &= -\Delta\zeta \quad \text{in } \Omega_n, \\ \zeta_n &= 0 \qquad \text{on } \partial\Omega_n. \end{aligned}$$

Although $\zeta_n \notin C_c^{1,1}(\bar{\Omega}_n)$, ζ_n remains uniformly bounded in $C_c^{1,\alpha}(\bar{\Omega}_n)$ for any $\alpha \in (0,1)$, $\Delta\zeta_n$ is bounded and

$$\int_{\Omega_n} (-u_n \Delta\zeta_n + g(x,u_n)\zeta_n)\, dx = -\int_{\Sigma_{\beta_n}} \frac{\partial \zeta_n}{\partial \mathbf{n}} u(y) dS,$$

by approximation. We extend u_n by putting the zero value outside $\bar{\Omega}_n$ and call $\tilde{u}_n$ this extension. For ζ_n we perform an extension by reflexion following the normal direction and define $\tilde{\zeta}_n$ thanks to the following formula

$$\forall x \in \bar{\Omega} \setminus \Omega_n \text{ with } \rho_{\partial\Omega}(x) = t_x \text{ and } x = \sigma(x) - t_x \mathbf{n}_{\sigma(\mathbf{x})},$$
$$\tilde{\zeta}_n(x) = -\zeta_n(\sigma(x) - (2\beta_n - t_x)\mathbf{n}_{\sigma(\mathbf{x})}).$$

Notice that we have to assume $2\beta_n \leq \beta_0$. It follows by the elliptic equations regularity theory that there exist a subsequence ζ_{n_k} and some $\tilde{\zeta} \in C_c^{1,\alpha}(\bar{\Omega})$ such that $\zeta_{n_k} \to \tilde{\zeta}$ in $C^1(\bar{\Omega})$. By the uniqueness of the solution of the Dirichlet problem $\tilde{\zeta} = \zeta$, and the wole sequence $\{\zeta_n\}$ is convergent. Moreover

$$0 \leq \tilde{u}_n(x) \leq u(x) \Longrightarrow 0 \leq g(x, \tilde{u}_n) \leq g(x, u), \quad \forall x \in \Omega.$$

Thus the sequence $\{g(., \tilde{u}_n)\rho_{\partial\Omega}\}$ is uniformly integrable in Ω. By Vitali's theorem

$$\int_\Omega \left(-u^*\Delta\zeta + g(x, u^*)\zeta\right) dx = -\lim_{n_k\to 0} \int_{\Sigma_{\beta_{n_k}}} \frac{\partial \zeta_{n_k}}{\partial \mathbf{n}} u(y)dS = \int_{\partial\Omega} \frac{\partial\zeta}{\partial \mathbf{n}} d\mu.$$

This indicates that u^* is a solution of (0.4) with boundary μ and therefore $\mathrm{Tr}_{\partial\Omega}(u^*) = \mu$. □

The key observation on which is based the definition of the boundary trace is the following

Proposition 2.2. *Let $g \in \mathcal{G}_0$ is positively subcritical and u a nonnegative solution of* (2.6) *in Ω. If $\mu \in \mathfrak{M}_+(\partial\Omega)$ set $w_\mu = \min(u, u_\mu)$. Then w_μ satifies*

$$-\Delta w_\mu + g(., w_\mu) \geq 0 \quad \text{in } \Omega, \tag{2.12}$$

and there exists $\gamma_u(\mu) \in \mathfrak{M}_+(\partial\Omega)$ such that $\mathrm{Tr}_{\partial\Omega}(w_\mu) = \gamma_u(\mu)$. The mapping $\mu \mapsto \gamma_u(\mu)$ is nondecreasing and $0 \leq \gamma_u(\mu) \leq \mu$. Moreover, if for any $x \in \Omega$ the function $r \mapsto g(x, r)$ is convex on $\mathbb{R}_+$, the mapping $\mu \mapsto \gamma_u(\mu)$ is concave on $\mathfrak{M}_+(\partial\Omega)$.

Proof. Let $\delta > 0$ and p be the $C^{1,1}$ even convex function defined on $\mathbb{R}$ by

$$p(t) = \begin{cases} |t| - \delta/2 & \text{for } |t| \geq \delta, \\ t^2/2\delta & \text{for } |t| \leq \delta\,. \end{cases}$$

Then $\omega_\delta = \dfrac{1}{2}(u + u_\mu - p(u - u_\mu))$ satisfies

$$\Delta\omega_\delta = \frac{1}{2}\left(\Delta u + \Delta u_\mu - p'(u + u_\mu)\Delta(u - u_\mu) - p''(u - u_\mu)\left|D(u - u_\mu)\right|^2\right),$$
$$\leq \frac{1}{2}\left(\Delta u + \Delta u_\mu - p'(u + u_\mu)\Delta(u - u_\mu)\right) = F.$$

Put

$$
\begin{aligned}
G_1 &= \{x \in \Omega : \ (u - u_\mu)(x) > \delta\},\\
G_2 &= \{x \in \Omega : \ (u - u_\mu)(x) < -\delta\},\\
G_3 &= \{x \in \Omega : \ \left|u - u_\mu\right|(x) \leq \delta\}.
\end{aligned}
\tag{2.13}
$$

On G_1, $p'(u - u_\mu) = 1$ and

$$F = \Delta u_\mu = g(., u_\mu) = g(., \omega_\delta - \delta/4).$$

On G_2, $p'(u - u_\mu) = -1$ and

$$F = \Delta u \leq g(., u_\mu) = g(., \omega_\delta - \delta/4).$$

On G_3, $p'(u - u_\mu) = \delta^{-1}(u - u_\mu)$ and

$$
\begin{aligned}
F &= \frac{1}{2}\left(1 - \frac{u - u_\mu}{\delta}\right)\Delta u + \frac{1}{2}\left(1 + \frac{u - u_\mu}{\delta}\right)\Delta u_\mu\\
&\leq \left(1 - \frac{u - u_\mu}{\delta}\right) g(., u) + \frac{1}{2}\left(1 + \frac{u - u_\mu}{\delta}\right) g(., u_\mu).
\end{aligned}
$$

By continuity of $r \mapsto g(x, r)$ there exists $\theta = \theta(x) \in [0, 1]$ such that

$$F \leq g(., \theta u + (1 - \theta)u_{\mu)}) \leq g(., \omega_\delta + 3\delta/4) \leq g(., v + \delta).$$

Combining those inequalities infers

$$\Delta\omega_\delta \leq g(., \omega_\delta + 3\delta/4) \leq g(., u_\mu + \delta). \tag{2.14}$$

If we let $\delta \to 0$, $\omega_\delta \to w_\mu = \min(u, u_\mu)$ and (2.12) holds in the sense of distributions in Ω. Since $0 \leq g(., w_\mu) \leq g(., u_\mu)$, $g(., w_\mu) \in L^1(\Omega; \rho_{\partial\Omega})dx$. For the last assertion let $\mu_i \in \mathfrak{M}_+(\partial\Omega)$ $(i = 1, 2)$, $\theta \in [0, 1]$, $\mu_\theta = \theta\mu_1 + (1-\theta)\mu_2$ and $u_\theta = \theta u_{\mu_1} + (1 - \theta)u_{\mu_2}$. Since

$$g(x, u_\theta) \leq \theta g(x, u_{\mu_1}) + (1 - \theta)g(x, u_{\mu_2}),$$

there holds

$$-\Delta u_\theta + g(x, u_\theta) \leq 0,$$

and $u_\theta \leq u_{\mu_\theta}$ by the comparison principle between solutions of (0.4). Moreover,

$$
\begin{aligned}
u + u_\theta - |u - u_\theta| &= \theta(u + u_{\mu_1}) + (1-\theta)(u + u_{\mu_2}) - \left|\theta(u - u_{\mu_1}) + (1-\theta)(u - u_{\mu_2})\right|\\
&\geq \theta\left(u + u_{\mu_1} - \left|u - u_{\mu_1}\right|\right) + (1 - \theta)\left(u + u_{\mu_2} - \left|u - u_{\mu_2}\right|\right)\\
&= \theta\min\{u, u_{\mu_1}\} + (1 - \theta)\min\{u, u_{\mu_2}\}.
\end{aligned}
$$

Thus

$$\min\{u, u_{\mu_\theta}\} \geq \min\{u, u_\theta\} \geq \theta \min\{u, u_{\mu_1}\} + (1-\theta) \min\{u, u_{\mu_2}\},$$

which implies

$$\gamma_u(\theta\mu_1 + (1-\theta)\mu_2) \geq \theta\gamma_u(\mu_1) + (1-\theta)\gamma_u(\mu_2). \qquad \square$$

REMARK 2.2. It follows also from [8], [9] that $\Delta w_\mu \in L^1(\Omega; \rho_{\partial\Omega} dx)$. Moreover

$$\int_\Omega \big(-w_\mu \Delta\zeta + g(x, w_\mu)\big)\, dx = \int_\Omega \Phi\zeta\, dx - \int_{\partial\Omega} \frac{\partial\zeta}{\partial \mathbf{n}} d\gamma_u(\mu), \ \forall\zeta \in C_c^{1,1}(\bar\Omega), \tag{2.15}$$

where $\Phi = [-\Delta w_\mu + g(x, w_\mu)]$.

PROPOSITION 2.3. *Under the assumptions of Proposition* 2.2 *set*

$$\nu = \sup\{\gamma_u(\mu) : \ \mu \in \mathfrak{M}_+(\partial\Omega)\}. \tag{2.16}$$

Then ν *is a Borel measure on* $\partial\Omega$.

PROOF. It is clear that ν is an outer measure in the sense that

$$\nu(\emptyset) = 0, \quad \text{and } \nu(A) \leq \sum_{k=1}^{\infty} \nu(A_k), \quad \text{whenever } A \subset \bigcup_{k=1}^{\infty} A_k. \tag{2.17}$$

Let A and $B \subset \partial\Omega$ be disjoint Borel subsets. In order to prove that

$$\nu(A \cup B) = \nu(A) + \nu(B), \tag{2.18}$$

we first notice that the relation holds if $\max\{\nu(A), \nu(B)\} = \infty$. Therefore we assume that $\nu(A)$ and $\nu(B)$ are finite. For $\varepsilon > 0$ there exist two bounded positive measures μ_1 and μ_2 such that

$$\gamma_u(\mu_1)(A) \leq \nu(A) \leq \gamma_u(\mu_1)(A) + \varepsilon/2$$

and

$$\gamma_u(\mu_1)(B) \leq \nu(B) \leq \gamma_u(\mu_1)(B) + \varepsilon/2$$

Hence

$$\begin{aligned} \nu(A) + \nu(B) &\leq \gamma_u(\mu_1)(A) + \gamma_u(\mu_2)(B) + \varepsilon \\ &\leq \gamma_u(\mu_1 + \mu_2)(A) + \gamma_u(\mu_1 + \mu_2)(B) + \varepsilon \\ &= \gamma_u(\mu_1 + \mu_2)(A \cup B) + \varepsilon \\ &\leq \nu(A \cup B) + \varepsilon. \end{aligned}$$

Therefore ν is a finitely additive measure. If $\{A_k\}$ $(k \geq 0)$ is a sequence of of disjoint Borel sets and $A = \cup A_k$, then

$$\nu(A) \geq \nu\left(\bigcup_{1\leq k\leq n} A_k\right) = \sum_{k=1}^{n} \nu(A_k) \Longrightarrow \nu(A) \geq \sum_{k=1}^{\infty} \nu(A_k).$$

By (2.17), it implies that ν is a countably additive measure. □

REMARK 2.3. The measure ν may not be regular. If $\nu(B) = \infty$ then $\nu(\mathcal{O}) = \infty$ for any relatively open subset $\mathcal{O}$ containing B. On the other hand, if $\nu(B) < \infty$, there exists a sequence of positive Radon measures μ_n such that

$$\gamma_u(\mu_n)(B) \uparrow \nu(B) \quad \text{as } n \to \infty.$$

Even if for each $n \in \mathbb{N}_*$ and $\epsilon > 0$ there exists a relatively open subset $\mathcal{O}_{n,\epsilon}$ containing B such that

$$\gamma_u(\mu_n)(\mathcal{O}_{n,\epsilon}) \leq \gamma_u(\mu_n)(B) + \epsilon,$$

there is no reason that there exists some open subset containing B such that $\gamma_u(\mu_n)(\mathcal{O})$ would remain bounded independently of n.

DEFINITION 2.3. The outer Borel measure ν defined by the above process is called the *extended boundary trace of u* and denoted by $\mathrm{Tr}^e_{\partial\Omega}(u)$.

The next result shows that in the study of the extended boundary trace, it is always possible to replace the inequation by an equation.

PROPOSITION 2.4. *Let $g \in \mathcal{G}_0$ be positively subcritical. If u is a nonnegative solution of* (2.6) *and u^* is the largest solution of* (0.1) *dominated by u, then*

$$\mathrm{Tr}^e_{\partial\Omega}(u) = \mathrm{Tr}^e_{\partial\Omega}(u^*). \tag{2.19}$$

If, in addition, there exist an open domain $\mathcal{O} \subset \mathbb{R}^N$ and a nonnegative Radon measure $\tilde{\mu}$ on $\mathcal{O} \cap \partial\Omega$ such that

$$\lim_{\beta\to 0} \int_{\Sigma_\beta \cap \mathcal{O}} u(x)\phi(\sigma(x))dS = \int_{\mathcal{O}\cap\partial\Omega} \phi d\tilde{\mu}. \tag{2.20}$$

for any $\phi \in C(\mathcal{O} \cap \partial\Omega)$, then

$$\mathrm{Tr}^e_{\partial\Omega}(u)|_{\partial\Omega\cap\mathcal{O}} = \tilde{\mu}. \tag{2.21}$$

PROOF. Because of the definition of the extended boundary trace, it is sufficient to prove that for any $\mu \in \mathfrak{M}_+(\partial\Omega)$,

$$\gamma_u(\mu) = \gamma_{u^*}(\mu). \tag{2.22}$$

Because $u^* \leq u$, then $\min\{u^*, u_\mu\} \leq \min\{u, u_\mu\}$ and $\gamma_{u^*}(\mu) \leq \gamma_u(\mu)$. Conversely

$$u \geq \min\{u, u_\mu\} \Longrightarrow u^* \geq [\min\{u, u_\mu\}]^*,$$

and

$$u_\mu \geq \min\{u, u_\mu\} \Longrightarrow u_\mu^* = u_\mu \geq [\min\{u, u_\mu\}]^*,$$

by (2.10). Therefore

$$\min\{u^*, u_\mu\} \geq [\min\{u, u_\mu\}]^*,$$

and

$$[\min\{u^*, u_\mu\}]^* \geq \left([\min\{u, u_\mu\}]^*\right)^* = [\min\{u, u_\mu\}]^*,$$

by (2.11). By Proposition 2.1, $\min\{u, u_\mu\}$ and $[\min\{u, u_\mu\}]^*$ have the same boundary trace, and the same holds $\min\{u^*, u_\mu\}$ and $[\min\{u^*, u_\mu\}]^*$. Therefore

$$\gamma_{u^*}(\mu) \geq \gamma_u(\mu),$$

which implies (2.22).

For the second assertion, we assume that u admits $\tilde{\mu}$ for boundary trace on $\mathcal{O} \cap \partial\Omega$. Let $\lambda \in \mathfrak{M}_+(\partial\Omega)$ and $\phi \in C_c(\partial\Omega \cap \mathcal{O})$, $\phi \geq 0$. Since

$$\int_{\Sigma_\beta \cap \mathcal{O}} u(x)\phi(\sigma(x))dS \geq \int_{\Sigma_\beta \cap \mathcal{O}} \min\{u(x), u_\lambda(x)\}\phi(\sigma(x))dS,$$

there holds, as $\beta \to 0$,

$$\int_{\partial\Omega \cap \mathcal{O}} \phi d\tilde{\mu} \geq \int_{\partial\Omega \cap \mathcal{O}} \phi d\gamma_u(\lambda),$$

thus

$$\tilde{\mu} \geq \mathrm{Tr}^e_{\partial\Omega}(u)|_{\partial\Omega \cap \mathcal{O}}. \tag{2.23}$$

Conversely, by reducing the set $\mathcal{O}$, we first suppose that $\tilde{\mu}$ is bounded and we extend it by 0 outside $\mathcal{O} \cap \partial\Omega$. We can also suppose that $\mathcal{O} \cap \Omega$ is C^2 and that $u \in L^1(\Omega \cap \partial\mathcal{O})$ by (2.20) and Fubini's theorem. Let $v = v_{\tilde{\mu}}^{\mathcal{O}}$ be the solution of

$$\begin{aligned} -\Delta v + g(x, v) &= 0 \quad \text{in } \mathcal{O} \cap \Omega, \\ v &= 0 \quad \text{in } \Omega \cap \partial\mathcal{O}, \\ v &= \tilde{\mu} \quad \text{in } \partial\Omega \cap \mathcal{O}. \end{aligned} \tag{2.24}$$

Since $u|_{\mathcal{O}\cap\Omega}$ satisfies the same equation, with the exception of the data on $\Omega \cap \partial\mathcal{O}$ which is an integrable nonnegative function, $g(., u)|_{\mathcal{O}\cap\Omega} \in L^1(\mathcal{O}\cap\Omega; \rho_{\partial(\mathcal{O}\cap\Omega)} dx)$ and

$$v_{\tilde{\mu}}^{\mathcal{O}} \leq u \quad \text{in } \mathcal{O} \cap \Omega,$$

by the maximum principle and [5]. If $\tilde{v} = \tilde{v}^{\mathcal{O}}_{\tilde{\mu}}$, is the extension of v by zero in $\Omega \setminus (\mathcal{O} \cap \Omega)$, then

$$\tilde{v} \leq u_{\tilde{\mu}} \leq \min\{u, u_{\tilde{\mu}}\}.$$

Therefore

$$\int_{\mathcal{O}} \phi d\tilde{\mu} \leq \int_{\mathcal{O}} \phi d\gamma_u(\tilde{\mu}),$$

for any $\phi \in C_c(\partial\Omega \cap \mathcal{O})$. Clearly the same relation holds even if we no longer assume that $\tilde{\mu}$ is bounded. Thus

$$\tilde{\mu}(E) \leq \gamma_u(\tilde{\mu})(E), \quad \forall E \subset \partial\Omega \cap \mathcal{O},\ E \text{ Borel},$$

and consequently

$$\tilde{\mu} \leq \mathrm{Tr}^e_{\partial\Omega}(u)|_{\partial\Omega\cap\mathcal{O}}. \tag{2.25}$$ □

REMARK 2.4. The relation (2.20) means that u admits a boundary trace in the usual sense on $\mathcal{O} \cap \partial\Omega$ which is precisely $\tilde{\mu}$. The reverse implication "(2.21) $\Longrightarrow$ (2.20)" holds under an additional strong stability assumption which will be developped in Section 4. However we can give a weaker form of this implication if u^* is dominated by the minimal large solution of (2.7), whenever it exists.

Let us denote by λ_n $(n \geq 0)$ the measure $n\chi_{\partial\Omega} dS$ and

$$u_m = \sup\{u_{\lambda_n} \ :\ n \in \mathbb{N}\}. \tag{2.26}$$

If u_m is locally bounded in Ω, it is a solution of (2.7) which blows up on the boundary. In such a case it is called *the minimal large solution*. Depending upon the nonlinearity, u_m may also be infinite in whole Ω or in part of Ω. Moreover, by the maximum principle, it dominates any solution u of the same equation which is obtained by approximation by solutions with finite values as boundary data.

PROPOSITION 2.5. *Let $g \in \mathcal{H}_0$ be positively subcritical. If $\mathrm{Tr}^e_{\partial\Omega}(u)$ is a bounded Borel measure and $u^* \leq u_m$, then*

$$\mathrm{Tr}^e_{\partial\Omega}(u^*) = \mathrm{Tr}_{\partial\Omega}(u^*) \in \mathfrak{M}_+(\partial\Omega). \tag{2.27}$$

PROOF. By assumption $\nu = \mathrm{Tr}^e_{\partial\Omega}(u)$ is bounded. Thus there exists a sequence $\{\mu_n\} \subset \mathfrak{M}_+(\partial\Omega)$ such that

$$\gamma_u(\mu_n)(1) = \gamma_{u^*}(\mu_n)(1) = \mathrm{Tr}_{\partial\Omega}([\min\{u^*, u_{\mu_n}\}]^*)(1) \uparrow \nu(1),$$

since u and u^* have the same extended boundary trace. Because the extended boundary trace is defined by a supremum over all measures, it can also be assumed that the regular part of μ_n is a.e. bounded from below by n. Let $\zeta \in C_c^{1,1}(\bar{\Omega})$ be the solution of

$$\begin{aligned} -\Delta\zeta &= 1 \text{ in } \Omega, \\ \zeta &= 0 \text{ on } \partial\Omega. \end{aligned}$$

By the definition of the boundary trace in $\mathfrak{M}_+(\partial\Omega)$,

$$\begin{aligned} \int_\Omega \big([\min\{u^*, u_{\lambda_n}\}]^* + \zeta g(x, [\min\{u^*, u_{\lambda_n}\}]^*\big)\, dx &= -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\operatorname{Tr}_{\partial\Omega}([\min\{u^*, u_{\lambda_n}\}]^*), \\ &\leq C\nu(1). \end{aligned}$$

Since u_m dominates u^*, $\lim_{n\to\infty}\min\{u^*, u_{\lambda_n}\} = u^*$ and

$$u^* \in L^1(\Omega), \text{ and } \ g(., u^*) \in L^1(\Omega; \rho_{\partial\Omega} dx). \tag{2.28}$$

It follows from [27, Corollary 1.3] that u^* has a boundary trace, say μ^* in $\mathfrak{M}_+(\partial\Omega)$, and $u^* = u_{\mu^*}$. Consequently for any measure μ larger than μ^*, $\min\{u_\mu, u_{\mu^*}\} = u_{\mu^*}$ and

$$\operatorname{Tr}^e_{\partial\Omega}(u^*) = \operatorname{Tr}^e_{\partial\Omega}(u_{\mu^*}) = \sup_{\mu\geq\mu^*} \gamma_{u^*}(\mu) = \mu^* = \operatorname{Tr}_{\partial\Omega}(u^*). \qquad \square$$

REMARK 2.5. The previous result still holds if, in the domination assumption $u^* \leq u_m$, the function u_m is no longer the minimal large solution, but any σ-moderate solution in the sense of Dynkin and Kuznetsov [12], that is a solution of (2.7) which is an increasing limit of solutions u_{μ_n} for some $\mu_n \in \mathfrak{M}_+(\partial\Omega)$

PROPOSITION 2.6. *Let $g \in \mathcal{G}_0$ be positively subcritical and stable, and let u be a nonnegative solution of* (0.9). *If $\{\mu_n\} \subset \mathfrak{M}_+(\partial\Omega)$ converges weakly to μ, then* $\limsup_{n\to\infty} \gamma_u(\mu_n) \leq \gamma_u(\mu)$. *If we assume moreover that the sequence $\{\mu_n\}$ is nonincreasing,* $\lim_{n\to\infty} \gamma_u(\mu_n) = \gamma_u(\mu)$.

PROOF. Since $\mu_n \to \mu$ in the weak sense of measures on $\partial\Omega$, $u_{\mu_n} \to u_\mu$ locally uniformly in Ω by definition of the positive subcriticality and stability. Thus

$$w_{\mu_n} = \min\{u^*, u_{\mu_n}\} \to w_\mu = \min\{u^*, u_\mu\} \text{ in the } C_{\text{loc}}(\Omega)\text{-topology.}$$

Since $u_{\mu_n} \leq \mathbb{P}_{\mu_n}$ and $P_{\mu_n} \to P_\mu$ in $L^1(\Omega)$, the convergence of w_{μ_n} to w_μ holds also in $L^1(\Omega)$. Moreover

$$g(., w_{\mu_n}) \to g(., w_\mu)$$

in $C_{\text{loc}}(\Omega)$. Since

$$0 \leq g(., w_{\mu_n}) \leq g(., u_{\mu_n}),$$

and

$$g(., u_{\mu_n}) \to g(., u_\mu)$$

in $L^1(\Omega; \rho_{\partial\Omega} dx)$,

$$g(., w_{\mu_n}) \to g(., w_\mu) \tag{2.29}$$

also in $L^1(\Omega; \rho_{\partial\Omega} dx)$. Put $v_n^* = [w_{\mu_n}]^*$. By the elliptic equations regularity theory, $\{v_n^*\}$ remains bounded in $C^1_{\mathrm{loc}}(\Omega)$. Since $\gamma_u(\mu_n)$ is dominated by μ_n which is bounded let us consider a subsequence $\gamma_u(\mu_{n_k})$ weakly convergent to some nonnegative measure λ. Up to an extraction of a subsequence, it is always possible to assume that $v_{n_k}^*$ converges (in the $C_{\mathrm{loc}}(\Omega)$-topology) to $\bar{v}$. Clearly $\bar{v}$ is a solution of (0.1) in Ω and

$$v_n^* \le w_{\mu_n} \Longrightarrow \bar{v} \le \lim_{n\to\infty} w_{\mu_n} = w_\mu. \tag{2.30}$$

Therefore

$$\bar{v} = \bar{v}^* \le v^* = [w_\mu]^*. \tag{2.31}$$

Inasmuch

$$v_{n_k}^* \to \bar{v} \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad g(., v_{n_k}^*) \to g(., \bar{v}) \quad \text{in} \quad L^1(\Omega; \rho_{\partial\Omega} dx),$$

(for the second relation we use $0 \le g(., v_{n_k}^*) \le g(., w_{\mu_n})$, together with (2.29) and Vitali's theorem) and

$$v_{n_k} + \mathbb{G}_{g(.,v_{n_k}^*)} = \mathbb{P}_{\gamma_u(\mu_n)},$$

it follows

$$\bar{v} + \mathbb{G}_{g(.,\bar{v})} = \mathbb{P}_\lambda.$$

But

$$v^* + \mathbb{G}_{g(.,v^*)} = \mathbb{P}_{\gamma_u(\mu)}.$$

As $\bar{v} \le v^*$ and $g(., \bar{v}) \le g(., v^*)$,

$$\mathbb{P}_\lambda \le \mathbb{P}_{\gamma_u(\mu)} \Longrightarrow \lambda \le \gamma_u(\mu), \tag{2.32}$$

which is the first assertion.

If we assume that $\{\mu_n\}$ is nonincreasing, the same holds with $\{u_{\mu_n}\}$, $\{w_{\mu_n}\}$, $\{\gamma_u(\mu_n)\}$ and $\{v_n^*\}$. If $\mu_n \downarrow \mu$, any solution of (0.1) dominated by w_μ is dominated by w_{μ_n}. Thus $v^* \le v_n^*$ and $\bar{v} = \lim_{n\to\infty} v_n^* = v^*$ by (2.30) and (2.31). □

A particularly important case deals with the choice $\mu = \lambda\delta_a$, with $a \in \partial\Omega$ and $\lambda > 0$. Let $u = u_{\lambda\delta_a}$ be the solution of

$$\begin{aligned} -\Delta u + g(x, u) &= 0, && \text{in } \Omega, \\ u &= \lambda\delta_a, && \text{on } \partial\Omega. \end{aligned}$$

Since $g(x, .)$ is nondecreasing, $\lambda \mapsto u_{\lambda\delta_a}$ is increasing. Set

$$u_{\infty,a} = \lim_{\lambda\to\infty} u_{\lambda\delta_a}.$$

On any open subset of Ω where it is locally finite, $u_{\infty,a}$ is a solution of (0.1).

LEMMA 2.1. *Let* $a \in \partial\Omega$, $\lambda > 0$ *and* $w_{\lambda\delta_a} = \min\{u, u_{\lambda\delta_a}\}$. *Then*

$$\mathrm{Tr}_{\partial\Omega}(w_{\lambda\delta_a}) = \gamma_u(\lambda\delta_a) = \tilde{\gamma}_u(a,\lambda)\delta_a, \tag{2.34}$$

where $0 \leq \tilde{\gamma}_u(a,\lambda) \leq \lambda$. *Moreover the mapping* $\lambda \mapsto \tilde{\gamma}_u(a,\lambda)$ *is nondecreasing, there exists*

$$\tilde{\gamma}_u(a) = \lim_{\lambda\to\infty} \tilde{\gamma}_u(a,\lambda), \tag{2.35}$$

and

$$\min(u, u_{\infty,a}) \geq u_{\tilde{\gamma}_u(a)}. \tag{2.36}$$

PROOF. Because $\mathrm{Tr}_{\partial\Omega}(w_{\lambda\delta_a}) \leq \lambda\delta_a$, this trace is concentrated at a and achieves the form $\tilde{\gamma}_u(a,\lambda)\delta_a$. Moreover $\lambda \mapsto \tilde{\gamma}_u(a,\lambda)$ is nondecreasing as is the mapping $\lambda \mapsto \min(u, u_{\lambda\delta_a})$. Since

$$w_{\lambda\delta_a} \geq u_{\tilde{\gamma}_u(a,\lambda)\delta_a},$$

and $u_{\lambda\delta_a}$ admits a limit, finite or not, when $\lambda \to \infty$, assertions (i) or (ii) follow. □

The next result points out the role of $u_{\infty,a}$

PROPOSITION 2.7. *Let* $g \in \mathcal{G}_0$ *be positively subcritical. If* u *is a nonnegative solution of* (2.6) *with boundary trace* ν *and* $a \in \partial\Omega$, *then*

$$u \geq u_{\infty,a} \Longrightarrow \nu(a) = \infty. \tag{2.37}$$

If we assume moreover that g *is positively subcritical and stable, then*

$$\nu(a) = \infty \Longrightarrow u \geq u_{\infty,a}. \tag{2.38}$$

PROOF. Let $a \in \partial\Omega$ be such that

$$\nu(a) = \infty.$$

Then for any relatively open subset $\mathcal{O} \subset \partial\Omega$ containing a,

$$\nu(\mathcal{O}) = \infty,$$

which means that there exists a sequence of positive Radon measures μ_n such that

$$\lim_{n\to\infty} \gamma_u(\mu_n)(\mathcal{O}) = \infty.$$

Without any restriction we can suppose that the sequence of restricted measures $\gamma'_u(\mu_n) = \chi_{\mathcal{O}}\gamma_u(\mu_n)$ is increasing and

$$u \geq u_{\gamma'_u(\mu_n)},$$

because $u \geq u_{\gamma_u(\mu_n)}$ and $\gamma_u(\mu_n) \geq \gamma'_u(\mu_n)$. For any $k \in \mathbf{N}_*$, there exists $\epsilon_{n,k} > 0$ such that, if we take $\mathcal{O}_{n,k} = B_{\epsilon_{n,k}}(a) \cap \partial\Omega$, there holds

$$\gamma_u(\mu_n)(\mathcal{O}_{n,k}) = k.$$

Set $\mu_{n,k} = \chi_{\mathcal{O}_{n,k}} \gamma_u(\mu_n)$. Then

$$\gamma_u(\mu_n) \geq \mu_{n,k} \Longrightarrow u_{\gamma_u(\mu_n)} \geq u_{\mu_{n,k}}.$$

Since $\lim_{n\to\infty} \epsilon_{n,k} = 0$, $\lim_{n\to\infty} \mu_{n,k} = k\delta_a$. Consequently

$$u(x) \geq u_{k\delta_a}(x).$$

Letting $k \to \infty$ yields to the following implication

$$\nu(a) = \infty \Longrightarrow u \geq u_{\infty a}.$$

Conversely, assume $u \geq u_{\infty a}$, then for any $k > 0$, $u \geq u_{k\delta_a}$. On one hand the boundary trace of $w_{k\delta_a} = \min\{u, u_{k\delta_a}\}$ is the measure $\gamma_u(a,k)\delta_a$. But $\min\{u, u_{k\delta_a}\} = u_{k\delta_a}$ implies $\gamma_u(a,k) = k$ and therefore $\gamma_u(k\delta_a) = k\delta_a$. By the definition of ν,

$$\nu(a) \geq \gamma_u(k\delta_a)(a) = k.$$

Since this holds for any $k > 0$, $\nu(a) = \infty$. □

The characterisation of Borel subsets on which the boundary trace of u takes finite values is less complete, however there holds

PROPOSITION 2.8. *Assume the assumptions of Proposition* 2.2 *are fulfilled and u is a nonnegative solution of* (2.6) *with boundary trace ν. If $\mathcal{O} \subset \partial\Omega$ is a relatively open subset of $\partial\Omega$ such that*

$$\int_{\mathcal{O}_t} u(y)dS_t$$

remains bounded independently of $t \in (0, \beta_0]$, then $\nu(\mathcal{O})$ is finite.

PROOF. Let μ be a nonnegative measure. Since $u_{\gamma_u(\mu)} \leq u$ in Ω,

$$\int_{\mathcal{O}_t} u_{\gamma_u(\mu)}(y)dS_t \leq \int_{\mathcal{O}_t} u(y)dS_t.$$

Thus

$$\gamma_u(\mu)(\mathcal{O}) \leq \sup_{0<\beta\leq\beta_0} \int_{\mathcal{O}_t} u(y)dS_t = M.$$

Therefore

$$\nu(\mathcal{O})) = \sup_{\mu\in\mathcal{M}_+(\partial\Omega)} \gamma_u(\mu)(\mathcal{O}) \leq M.$$ □

REMARK 2.6. The reverse implication

$$\nu(\mathcal{O}) < \infty \Longrightarrow \int_{\mathcal{K}_t} u(y)dS_t \leq M, \quad \forall t \in (0, \beta_0],$$

for any compact subset $K \subset \mathcal{O}$, where $M = M(K) > 0$, may not hold in the case of general inequalities. However, $g \in \mathcal{HG}_0$ with $\alpha = 0$, that is g satisfies (0.14), or if

$$g(x, r) \leq \rho_{\partial\Omega}^{\alpha}(x) r^q = 0, \quad \text{in } \Omega \times \mathbb{R}_+,$$

with $\alpha > -2$, and $1 < q < (N + 1 + \alpha)/(N - 1)$, such a result is still valid. Under both assumptions the proof is much intricated : in the first one it is given in next section, and in the second one, in [27]. In both cases the proof is settled on the notion of stability from inside approximations of the Dirichlet problems (0.4) which means that if a sequence of measures $\lambda_n \in \mathfrak{M}_+(\Omega)$ converges weakly to a measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ the solutions v_n of the semilinear equation with forcing term

$$-\Delta v_n + g(x, v_n) = \lambda_n \quad \text{in } \Omega, \tag{2.39}$$

$$v_n = 0 \quad \text{on } \partial\Omega, \tag{2.40}$$

converges to u_μ locally uniformly in Ω.

The real number $\gamma_u(a)$ plays an important role in the study of the boundary behavior of u at a. If $a \in \partial\Omega$, we denote by $\mathcal{N}_a$ the set of relatively open neighborhoods of a in $\partial\Omega$.

DEFINITION 2.4. We define by $\mathcal{A}(u)$ the *set of atoms* of u,

$$\mathcal{A}(u) = \{a \in \partial\Omega : \ \gamma_u(a) > 0\},$$

by $\mathcal{S}(u)$ the *singular set* of u,

$$\mathcal{S}(u) = \{a \in \mathcal{A} : \ \forall\, N_a \in \mathcal{N}_a, \sum_{\omega \in N_a} \gamma_u(\omega) = \infty\},$$

the symbol $\sum$ being taken in the sense of summable family, and by $\mathcal{R}(u)$ the *regular set* of u,

$$\mathcal{R}(u) = \partial\Omega \setminus \mathcal{S}(u) = \{a \in \mathcal{A} : \ \exists\, N_a \in \mathcal{N}_a, \sum_{\omega \in N_a} \gamma_u(\omega) < \infty\}.$$

The set $\mathcal{S}(u)$ is closed and $\mathcal{R}(u)$ relatively open. Moreover, if $a \in \mathcal{R}(u)$, there exists a relatively open neighborhood $N_a \in \mathcal{N}_a$ such that $\mathcal{A}(u) \cap N_a$ is at most countable.

The next result complements Propositions 2.7 and 2.8

THEOREM 2.1. *Assume the assumptions of Proposition* 2.2 *are fulfilled, u is a nonnegative solution of* (2.6) *with boundary trace ν, and $\mathcal{O}$ is a relatively open subset of $\partial\Omega$ such that $\nu(\mathcal{O}) < \infty$. Then*

$$\sum_{a\in\mathcal{O}} \tilde{\gamma}_u(a) < \infty.$$

If we assume moreover that $g \in \mathcal{HG}_0$, then, for any $\omega \in \partial\Omega$, there holds

$$\nu(\omega) = \tilde{\gamma}_u(\omega),$$

and the measure $\chi_{\mathcal{O}}\nu - \sum_{\omega\in\mathcal{O}} \tilde{\gamma}_u(\omega)\delta_\omega$ has no atom.

PROOF. Let K be a finite subset of $\mathcal{R}(u) \cap \mathcal{O}$ and put $\mu_K = \sum_{a\in K} \delta_a$. Then for any $\lambda > 0$

$$\sum_{a\in K} \tilde{\gamma}_u(a,\lambda) \le \gamma_u(\lambda\mu_K)(\mathcal{O}) \le \nu(\mathcal{O}).$$

Therefore the following family $\{\tilde{\gamma}_u(a)\}_{a\in\mathcal{O}}$ is summable, and

$$\sum_{a\in\mathcal{O}} \tilde{\gamma}_u(a) \le \nu(\mathcal{O}).$$

For the next statement, for any $\lambda > 0$ and $\omega \in \partial\Omega$, there holds

$$\nu(\omega) \ge \tilde{\gamma}_u(\omega,\lambda) \Longrightarrow \nu(\omega) \ge \tilde{\gamma}_u(\omega).$$

Conversely, for any relatively open neighborhood of ω, $\mathcal{O}_\omega$, there exists a sequence of Radon measures $\mu_n \in \mathfrak{M}_+(\partial\Omega)$ such that

$$\int_{\mathcal{O}_\omega} d\mu_n \uparrow \nu(\mathcal{O}_\omega), \quad \text{as} \quad n \to \infty.$$

If we assume that $\nu(\omega) = \infty$, we know from Proposition 2.7 that $\tilde{\gamma}_u(\omega) = \infty$. Thus we assume $\nu(\omega) < \infty$. For $\epsilon > 0$, there exists $\mu_\epsilon \in \mathfrak{M}_+(\partial\Omega)$ such that

$$\gamma_u(\mu_\epsilon)(\omega) \le \nu(\omega) \le \gamma_u(\mu_\epsilon)(\omega) + \epsilon\,,$$

and there exists $\eta_0 > 0$ such that $0 < \eta \le \eta_0$ implies

$$\int_{\Gamma_\eta(\omega)} d\gamma_u(\mu_\epsilon) - \epsilon \le \nu(\omega) \le \int_{\Gamma_\eta(\omega)} d\gamma_u(\mu_\epsilon) + \epsilon,$$

where $\Gamma_\eta(\omega) = B_\eta(\omega) \cap \partial\Omega$, which yields to

$$\left| \int_{\Gamma_\eta(\omega)} d\gamma_u(\mu_{\epsilon,\eta}) - \nu(\omega) \right| \le 2\epsilon.$$

If we take $\epsilon = 1/n$, then $\eta_0 = \eta_0(n) \to 0$ and $\chi_{\Gamma_{\eta(\omega)}}\gamma_u(\mu_\epsilon) \to \nu(\omega)\delta_\omega$ as $n \to \infty$. But

$$u_{\chi_{\Gamma_{\eta(\omega)}}\gamma_u(\mu_\epsilon)} \leq u_{\gamma_u(\mu_\epsilon)} \leq w_{\gamma_u(\mu_\epsilon)} \leq u.$$

Letting $n \to \infty$ and using the fact that $u_{\chi_{\Gamma_{\eta(\omega)}}\gamma_u(\mu_\epsilon)} \to u_{\nu(\omega)\delta_\omega}$ implies

$$u_{\nu(\omega)\delta_\omega} \leq u.$$

Therefore

$$u_{\nu(\omega)\delta_\omega} = \min\{u, u_{\nu(\omega)\delta_\omega}\} = w_{\nu(\omega)\delta_\omega} = u_{\tilde{\gamma}_u(\omega,\nu(\omega))\delta_\omega} \leq u_{\tilde{\gamma}_u(\omega)\delta_\omega}.$$

This implies $\nu(\omega) \leq \gamma_u(\omega)$ and the equality follows. Consequently $\chi_{\mathcal{O}}\nu - \sum_{\omega\in\mathcal{O}} \tilde{\gamma}_u(\omega)\delta_\omega$ has no atom. □

3. – Pointwise boundary behaviour of solutions of general inequalities

In this section, we give a precise description of the behaviour of a solution u of (2.6) near an atom of its extended boundary trace. We say that *the coordinates are proper at* $a = (a_1, \ldots, a_N) \in \partial\Omega$ *relatively to* Ω if the plane $x_1 - a_1 = 0$ is tangent to $\partial\Omega$ at a, and that the inward pointing vector to $\partial\Omega$ is the direction $x_1 - a_1 > 0$.

Definition 3.1. Let (E, Σ, μ) be a measured space, where Σ is σ-algebra of subsets of E and μ a positive and σ-additive measure with finite mass. We recall that a set of μ-measurable functions $x \mapsto \psi_r(x)$ $(r > 0)$, defined over E converges in measure to ψ when $r \to 0$, if for any $\varepsilon > 0$ there holds

$$\lim_{r\to 0} \mu\{x \in E : |\psi_r(x) - \psi(x)| > \varepsilon\} = 0.$$

The functions ψ_r converges in measure to ∞, if for any $k > 0$,

$$\lim_{r\to 0} \mu\{x \in E : \psi_r(x) \leq k\} = 0.$$

The convergence is equivalent to the following statement: from any sequence $\{r_n\}$ converging to 0 one can extract a subsequence $\{r_{n_k}\}$ such that $\psi_{r_{n_k}}$ converges to ψ (or ∞), μ-a.e. in E.

THEOREM 3.1. *Assume $g \in \mathcal{G}_0$ is positively subcritical, u is a nonnegative solution of* (2.6) *and $a \in \partial\Omega$. If the coordinates are proper at a relatively to Ω, the following alternative holds. Either*

(i) *$\tilde{\gamma}_u(a)$ is finite and the following convergence holds*

$$\lim_{\substack{x\to a\\(x_1-a_1)/|x-a|\to\eta_1}} |x-a|^{N-1}u(x) - C(N)\tilde{\gamma}_u(a)\eta_1 = 0, \tag{3.1}$$

in measure on S_+^{N-1},

or

(ii) *$\tilde{\gamma}_u(a)$ is infinite and*

$$\lim_{r\to 0} |x-a|^{N-1}u(x) = \infty, \tag{3.2}$$

in measure on S_+^{N-1}.

For $s > 0$, put $\Omega \cap B_s(a) = \Omega_s(a)$, $\Omega_s^c(a) = \Omega \setminus \bar{\Omega}_s(a)$ and $\partial B_s(a) \cap \Omega = \Gamma_s^\Omega(a)$. The next series of results deals with the boundary behaviour of the Green potential of a weighted integrable function. In the flat case where $\Omega = \mathbb{R}_+^N = \{x = (x_1, \dots, x_N) : x_1 > 0\}$ the computation can be explicited

LEMMA 3.1. *Let $\Omega = \mathbb{R}_+^N$, $N \geq 2$, $\Phi \in L^1(\mathbb{R}_+^N;\, x_1 dx)$ and $v = \mathbb{G}_\Phi$. Then for any $a \in \partial\mathbb{R}_+^N$ there holds*

$$\lim_{x\to a} |x-a|^{-1} \int_{\Gamma_s^\Omega(a)} |v|\, x_1 dS = 0. \tag{3.3}$$

PROOF. We can assume that $a = 0$, $\Phi \geq 0$, and so is v. For $\varepsilon > 0$, let $s > 0$ such that

$$\int_{B_s(a)} \Phi\rho_{\partial\Omega} dx \leq \varepsilon.$$

Let $(r, \sigma) \in (0, +\infty) \times S^{N-1}$ be the spherical coordinates in $\mathbb{R}^N$, $S_+^{N-1} = S^{N-1} \cap \mathbb{R}_+^N$ and $v(x) = v(r, \sigma)$, then

$$-\partial_{rr}v - \frac{N-1}{r}\partial_r v - \frac{1}{r^2}\Delta_\sigma v = \Phi,$$

where Δ_σ is the Laplace Beltami operator on S^{N-1}. Since $N-1$ is the first eigenvalue of $-\Delta_\sigma$ in $W_0^{1,2}(S_+^{N-1})$ and $\phi_1(\sigma) = x_1|_{S_+^{N-1}}$, the first eigenfunction, there holds

$$-\bar{v}_{rr} - \frac{N-1}{r}\bar{v}_r + \frac{N-1}{r^2}\bar{v} = \bar{\Phi},$$

where

$$\bar{v}(r) = \int_{S_+^{N-1}} v(r,\sigma)\phi_1(\sigma)d\sigma \quad \text{and} \quad \bar{\Phi}(r) = \int_{S_+^{N-1}} \Phi(r,\sigma)\phi_1(\sigma)d\sigma.$$

Integrating the above differential equation yields to

$$\bar{v}(r) = \alpha r^{1-N} + \beta r - \frac{r}{N}\int_0^r \bar{\Phi}(s)ds + \frac{r^{1-N}}{N}\int_0^r \bar{\Phi}(s)s^N ds,$$

for some constants $\alpha \geq 0$ and β. But $\alpha = 0$ otherwhile v would be bounded from below by $\alpha C(N)P(x,0)$. This is impossible because v admits the zero measure for trace on the boundary. Thus

$$\limsup_{r\to 0} r^{N-1}\bar{v}(r) = 0,$$

since

$$\int_0^r \bar{\Phi}(s)s^N ds = \int_{B_r(0)} \Phi(x)x_1 dx,$$

and the result follows. □

This result is immediately extendable for any domain which can be deduced by a conformal transformation from a half space.

LEMMA 3.2. *Let Ω be a ball or the complementary of a ball, $\Phi \in L^1(\Omega; \rho_{\partial\Omega}dx)$ and $v = \mathbb{G}_\Phi$. Then for any $a \in \partial\Omega$ there holds*

$$\lim_{x\to a} |x-a|^{-1}\int_{\Gamma_s^\Omega(a)} |v|\,\rho_{\partial\Omega}dS = 0. \tag{3.4}$$

In the next lemma we prove that this result is actually always valid. Our proof involves Marcinkiewicz space estimates on the Green potential of a weighted integrable function. The following estimates, similar to (1.22), (1.23), can be found in [4, Theorem 2.6]

$$\|\mathbb{G}_\Phi\|_{M^{(N+1)/(N-1)}(\Omega;\rho_{\partial\Omega}dx)} \leq K\|\Phi\|_{L^1(\Omega;\rho_{\partial\Omega}dx)}, \tag{3.5}$$

$$\|\mathbb{G}_\Phi\|_{M^{N/(N-1)}(\Omega)} \leq K\|\Phi\|_{L^1(\Omega;\rho_{\partial\Omega}dx)}. \tag{3.6}$$

Actually (3.5) is obtained in [4] only in the case $N \geq 3$, but an easy adaptation of the proof fills the gap.

LEMMA 3.3. *Let $N \geq 2$, $\Phi \in L^1(\Omega; \rho_{\partial\Omega}dx)$ and $v = \mathbb{G}_\Phi$. If $a \in \partial\Omega$, there holds*

$$\lim_{r\to 0} r^{-1}\int_{\Gamma_s^\Omega(a)} |v|\,\rho_{\partial\Omega}dS = 0. \tag{3.7}$$

PROOF. We still assume $\Phi \geq 0$. For $\varepsilon > 0$ let $s > 0$ be such that

$$\int_{\Omega_s(a)} \Phi\rho_{\partial\Omega}dx \leq \varepsilon.$$

Set $\Phi_s = \chi_{\Omega_s^c(a)}$ and $v_s = \mathbb{G}_{\Phi_s}$. Since v_s is harmonic in $\Omega_s(a)$, with zero trace on $\partial\Omega \cap B_s(a)$, it is continuous in a neighborhood of a and

$$\lim_{r\to 0} r^{-1} \int_{\Gamma_s^\Omega(a)} |v|_s \, \rho_{\partial\Omega} dS = 0.$$

Thus there is no loss of generality in assuming that Φ has support in $\Omega_s(a)$ and

$$\|\Phi\|_{L^1(\Omega;\rho_{\partial\Omega}dx)} \leq \varepsilon.$$

For any $r > 0$ and any $\zeta \in C_c^{1,1}(\bar{\Omega}_r(a))$, there holds

$$-\int_{\Omega_r(a)} v\Delta\zeta dx + \int_{\Gamma_r^\Omega(a)} \frac{\partial\zeta}{\partial\mathbf{n}} v dS = \int_{\Omega_r(a)} \Phi\zeta dx. \tag{3.8}$$

This can be established in assuming first that $\Phi = \Phi_n$ is regular, and then by density in $L^1(\Omega;\rho_{\partial\Omega}dx)$. By translation it can be assumed that $a = 0$. We set $\zeta(x) = |x|\,\eta_r(x)$, where η_r is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega_r(0))$, and let λ_r be the corresponding eigenvalue. Notice that $r^2\lambda_r \approx \lambda_1$ where λ_1 is the first eigenvalue of the operator

$$\ell \mapsto -\ell'' - \frac{N-1}{s}\ell' + \frac{N-1}{s^2}\ell \quad \text{on} \quad (0,1), \tag{3.9}$$

subject to the limit conditions $\ell'(0) = 0$, $\ell(1) = 0$ (thus the corresponding eigenfuction for (3.9) is a Bessel function, say B_1, and $\eta_r(x) \equiv B_1(x/r)x_1$ as $r \to 0$). Then (3.8) becomes

$$\begin{aligned}\lambda_r \int_{\Omega_r(0)} v\,|x|\,\eta_r(x)dx &- \int_{\Omega_r(0)} v|x|^{-1}\langle x.\nabla\eta_r\rangle dx \\ &= (N-1)\int_{\Omega_r(0)} \eta_r u dx - \int_{\Gamma_r^\Omega(0)} (r + \langle x.\nabla\eta_r\rangle) v dS + \int_{\Omega_r(0)} \Phi\,|x|\,\eta_r dx.\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{r\to 0} \int_{\Gamma_r^\Omega(0)} \langle x.\nabla\eta_r\rangle v dS &\leq \limsup_{r\to 0} \lambda_r \int_{\Omega_r(0)} v\,|x|\,\eta_r(x)dx \\ &+ \limsup_{r\to 0} \int_{\Omega_r(0)} v|x|^{-1}\,|\langle x.\nabla\eta_r\rangle|\,dx.\end{aligned} \tag{3.10}$$

But

$$|x|\,\eta_r(x) \leq C\rho_{\partial\Omega}(x)\,|\langle x.\nabla\eta_r\rangle| \leq C\,|x|\,/r,$$

and more precisely,

$$\lim_{r\to 0}\langle x.\nabla\eta_r\rangle|_{|x|=r} = \phi_1(\sigma) = x_1/\,|x|\,.$$

Then

$$
\begin{aligned}
\int_{\Omega_r(0)} v\,|x|\,\eta_r(x)dx &\le C\int_{\Omega_r(0)} \rho_{\partial\Omega}(x)v dx, \\
&\le C\|v\|_{M^{(N+1)/(N-1)}(\Omega;\rho_{\partial\Omega}dx)}\left(\int_{\Omega_r(0)}\rho_{\partial\Omega}dx\right)^{2/(N+1)}, \\
&\le CC'r^2\|\Phi\|_{L^1(\Omega;\rho_{\partial\Omega}dx)},
\end{aligned} \tag{3.11}
$$

and

$$
\begin{aligned}
\int_{\Omega_r(0)} v|x|^{-1}\,|\langle x.\nabla\eta_r\rangle|\,dx &\le Cr^{-1}\int_{\Omega_r(0)} v dx, \\
&\le Cr^{-1}\|v\|_{M^{N/(N-1)}(\Omega)}\,|\Omega_r(0)|^{1/N}, \\
&\le CC'\|\Phi\|_{L^1(\Omega;\rho_{\partial\Omega}dx)}\,.
\end{aligned} \tag{3.12}
$$

Combining (3.10), (3.11) and (3.12) yields to

$$
\limsup_{r\to 0} r^{-1}\int_{\Gamma_r^{\Omega}(0)} v\rho_{\partial\Omega}dS \le C"\|\Phi\|_{L^1(\Omega;\rho_{\partial\Omega}dx)} \le C\varepsilon, \tag{3.13}
$$

which ends the proof since ε is arbitrary. □

LEMMA 3.4. *Assume the assumptions of Theorem* 3.1 *are fulfilled, u is a non-negative solution of* (2.6), *$\lambda > 0$ and $a \in \partial\Omega$. If the coordinates are proper at a relatively to Ω, then for any $q \in [1,\infty)$,*

$$
\lim_{r\to 0} r^{N-2}\left(\int_{\Gamma_r^{\Omega}(a)} \left|u_{\lambda\delta_a}(y) - \lambda P(y,a)\right|^q \rho_{\partial\Omega}(y)dS\right)^{1/q} = 0. \tag{3.14}
$$

PROOF. Recall that $g(.,u_{\lambda\delta_a}) \in L^1(\Omega;\rho_{\partial\Omega}dx)$, we put $v = \mathbb{G}_{g(.,u_{\lambda\delta_a})}$. Since $u_{\lambda\delta_a} = \lambda P(.,a) - v$, it follows from Lemma 3.2

$$
\lim_{r\to 0} r^{-1}\int_{\Gamma_r^{\Omega}(a)} \left|u_{\lambda\delta_a}(y) - \lambda P(y,a)\right| \rho_{\partial\Omega}(y)dS = 0. \tag{3.15}
$$

Since $0 \le u_{\lambda\delta_a} \le \lambda P(.,a)$,

$$
y \mapsto r^{N-2}\sup_{y\in\Gamma_r^{\Omega}(a)} \left|u_{\lambda\delta_a}(y) - \lambda P(y,a)\right| \rho_{\partial\Omega}(y)
$$

is bounded independently of r. Thus the result follows by Hölder's inequality. □

Under a pointwise growth estimate on g the convergence of $u_{\lambda\delta_a}$ is much more precise.

LEMMA 3.5. *Let the conditions of Theorem* 3.1 *be fulfilled. Assume also that there exists* $\varepsilon_0 > 0$ *such that the mapping* $(k, x) \mapsto k^{N+1}g(k(x-a)+a, k^{1-N})$ *remains bounded for* $(k, x) \in (0, \varepsilon_0] \times \{x \in a + k^{-1}(\Omega - a) : 1-\varepsilon_0 \le |x| \le 1+\varepsilon_0\}$. *Then for any* $\eta_1 > 0$,

$$\lim_{\substack{x\to a\\(x-a)/|x-a|\to\eta_1}} |x-a|^{N-1}u_{\lambda\delta_a}(x) = \lambda C(N)\eta_1, \tag{3.16}$$

for some constant $C(N) > 0$. *Moreover, for any* $\eta > 0$, *the convergence is uniform in the cone* $\eta_1 \ge \eta$.

PROOF. We can assume $a = 0$ and set $u_k(x) = k^{N-1}u(kx)$. Then

$$\Delta u_k(x) = k^{N+1}g(kx, u(kx)).$$

Since $0 \le u \le \lambda P(x, 0) \le \lambda C(N)|x|^{1-N}$, $u_k(x)$ and $(k, x) \mapsto k^{N+1}g(kx, u(kx))$ remains bounded for $(k, x) \in (0, \varepsilon_0] \times \{x \in k^{-1}\Omega : 1-\varepsilon_0 \le |x| \le 1+\varepsilon_0\}$. Thus $\{u_k\}$ is relatively compact in $\{x \in k^{-1}\Omega : 1-\varepsilon_0/2 \le |x| \le 1+\varepsilon_0/2\}$, and there exist a sequence $\{k_n\}$ and some function $\zeta \in C^1\left(\mathbb{R}^N_+ \cap \left(\bar{B}_{1+\varepsilon_0}(0) \setminus B_{1-\varepsilon_0}(0)\right)\right)$ such that $u_{k_n} \to \zeta$ and $\nabla u_{k_n} \to \nabla\zeta$ uniformly on $\mathbb{R}^N_+ \cap \left(\bar{B}_{1+\varepsilon_0/2}(0) \setminus B_{1-\varepsilon_0/2}(0)\right)$. Putting $|x| = 1$, it implies

$$\lim_{k_n\to 0} k_n^{N-1}u(k_n, \sigma) = \zeta(\sigma),$$

uniformly on any compact subset of S^{N-1}_+. Since $P(x, 0) = P(r, \sigma, 0) = C(N)r^{1-N}\phi_1(\sigma)$, with $\phi_1(\sigma) = x_1|_{S^{N-1}_+}$, the relation (3.15) yields to $\zeta(\sigma) = C(N)\lambda\phi_1(\sigma)$, and finally

$$\lim_{k\to 0} k_n^{N-1}u(k, .) = C(N)\lambda\phi_1(.). \qquad \square$$

When $u_{\lambda\delta_a}$ is replaced by $w_{\lambda\delta_a}$, the convergence is comparable to the one of Lemma 3.4.

LEMMA 3.6. *Let the assumption of Theorem* 3.1 *be fulfilled. If* $\lambda > 0$, $a \in \partial\Omega$ *and the coordinates are proper at a relatively to* Ω, *there holds*

$$\lim_{r\to 0} r^{N-2}\left(\int_{\Gamma_r^\Omega(a)} \left|w_{\lambda\delta_a(y)} - \gamma(a, \lambda)P(y, a)\right|^q \rho_{\partial\Omega}(y)dS\right)^{1/q} = 0, \tag{3.17}$$

for any $1 \le q < \infty$.

PROOF. Since $\Delta w_{\lambda\delta_a}$ and $g(., w_{\lambda\delta_a})$ belong to $L^1(\Omega; \rho_{\partial\Omega}dx)$, there exists $\Phi \in L^1(\Omega; \rho_{\partial\Omega}dx)$ such that

$$\begin{aligned} -\Delta w_{\lambda\delta_a} &= \Phi && \text{in } \Omega, \\ w_{\lambda\delta_a} &= \gamma(a,\lambda)\delta_0 && \text{on } \partial\Omega. \end{aligned}$$

Then $w_{\lambda\delta_a} = \mathbb{G}_\Phi + \gamma(a,\lambda)P(., a)$ and

$$\left|w_{\lambda\delta_a} - \gamma(a,\lambda)P(., a)\right| \leq \mathbb{G}_{|\Phi|}.$$

By Lemma 3.3

$$\lim_{r\to 0} r^{N-2} \int_{\Gamma_r^\Omega(a)} |\mathbb{G}_\Phi(y)|\, dS = 0,$$

thus

$$\lim_{r\to 0} r^{N-2} \int_{\Gamma_r^\Omega(a)} \left|w_{\lambda\delta_a}(y) - \gamma(a,\lambda)P(y, a)\right| \rho_{\partial\Omega}(y)dS = 0. \tag{3.18}$$

Since $0 \leq w_{\lambda\delta_a} \leq \lambda P(., a)$, and $r^{N-2}\rho_{\partial\Omega}P(., a)$ is bounded on $\Gamma_r^\Omega(a)$, (3.17) follows. □

PROOF OF THEOREM 3.1. Up to a translation, we can assume that $a = 0$. We can assume that $S_+^{N-1}$is the intersection of the unit sphere with the half space $\{x_1 > 0\}$ and ∂S_+^{N-1} the intersection of S^{N-1} with the hyperplane $\{x_1 = 0\}$. Thus ϕ_1, the first eigenvalue of the Laplace-Beltrami operator $-\Delta_\sigma$ in $W_0^{1,2}(S_+^{N-1})$ is the restriction to S^{N-1} of the coordinate function $x \mapsto x_1$, and the corresponding eigenvalue is $N - 1$. We normalize by $\max\phi_1 = 1$.

CASE 1. $\gamma(a) < \infty$. For $\lambda > \gamma(a)$ the following convergences hold in $L^q(S_+^{N-1})$ for $1 \leq q < \infty$:

$$\lim_{r\to 0} r^{N-1} w_{\lambda\delta_a}(r, .) = C(N)\gamma(a,\lambda)\phi_1$$

by Lemma 3.6, and

$$\lim_{r\to 0} r^{N-1} u_{\lambda\delta_a}(r, \sigma) = C(N)\lambda\phi_1(\sigma)$$

by Lemma 3.4. If $\{r_n\}$ is some sequence converging to 0, there exists a subsequence $\{r_{n_k}\}$ such that

$$\lim_{r_{n_k}\to 0} r_{n_k}^{N-1} w_{\lambda\delta_a}(r_{n_k}, \sigma) = C(N)\tilde{\gamma}_u(a,\lambda)\phi_1(\sigma),$$

and

$$\lim_{r_{n_k}\to 0} r_{n_k}^{N-1} u_{\lambda\delta_a}(r_{n_k}, \sigma) = C(N)\lambda\phi_1(\sigma),$$

for almost all $\sigma \in S_+^{N-1}$. Therefore for almost all $\sigma \in S_+^{N-1}$, there exists n_{k_0} such that for $n_k \geq n_{k_O}$, $w_{\lambda\delta_a}(r_{n_k}, \sigma) = u(r_{n_k}, \sigma)$. Consequently there holds

$$\lim_{r\to 0} r^{N-1} u(r_{n_k}, \sigma) = C(N)\tilde{\gamma}_u(a, \lambda)\phi_1(\sigma),$$

for almost all $\sigma \in S_+^{N-1}$. Let $\theta > \lambda$. It follows from Lemma 3.5 applied to the $w_{\theta\delta_a}(r_{n_k}, .)$ and the previous argument, that, up to some subsequence $r_{n_{k_\ell}}$,

$$\lim_{r\to 0} r^{N-1} u(r_{n_{k_\ell}}, \sigma) = C(N)\tilde{\gamma}_u(a, \theta)\phi_1(\sigma)$$

almost everywhere. Therefore $\tilde{\gamma}_u(a, \theta) = \tilde{\gamma}_u(a, \lambda) = \tilde{\gamma}_u(a)$. This infers (i).

CASE 2. $\tilde{\gamma}_u(a) = \infty$. From Lemma 2.1,

$$\min(u, u_{\infty,a}) \geq u_{\infty,a} \Longrightarrow u \geq u_{\infty,a} > u_{\lambda\delta_a}, \quad \forall\lambda > 0.$$

By Lemma 3.4, for any $\varepsilon > 0$,

$$\liminf_{\substack{x\to a \\ (x_1-a_1)/|x-a|\to\eta_1 \\ \eta_1\geq\varepsilon}} |x-a|^{N-1} u(x) \geq C(N)\lambda\varepsilon. \tag{3.19}$$

Since λ is arbitrary, (ii) holds. □

REMARK 3.1. In the core of the proof in Case 1 we have seen that $\tilde{\gamma}_u(\lambda, a) = \tilde{\gamma}_u(a)$ for any $\lambda > \gamma(a)$. Actually the same proof gives also $\tilde{\gamma}_u(\lambda, a) = \tilde{\gamma}_u(a)$ for $\lambda = \tilde{\gamma}_u(a)$.

REMARK 3.2. If it is supposed moreover that $(k, x) \mapsto k^{N+1} g(k(x-a) + a, k^{1-N})$ remains bounded for $(k, x) \in (0, \varepsilon_0] \times \{x \in a + k^{-1}(\Omega - a) : 1 - \varepsilon_0 \leq |x| \leq 1 + \varepsilon_0\}$, assertion (ii) can be replaced by:

or

(ii*) $\tilde{\gamma}_u(a)$ is infinite and

$$\lim_{\substack{x\to a \\ (x_1-a_1)/|x-a|\to\eta_1}} |x-a|^{N-1} u(x) = \infty, \tag{3.20}$$

uniformly for $\eta_1 > 0$.

4. – Boundary trace of solutions with uniform absorption

In this section Ω is a C^2 bounded domain in $\mathbb{R}^N$, $g \in \mathcal{G}_0$ satisfies a uniform condition with respect to x in the sense that

$$0 \leq |g(x,r)| \leq f(r),\ \forall (x,r) \in \Omega \times \mathbb{R}_+, \quad \text{with} \int_0^1 f(\sigma s^{1-N}) s^N ds < \infty,\ \forall \sigma > 0, \tag{4.1}$$

where f is a continuous nondecreasing function defined on $\mathbb{R}_+$. The next result provides a precise characterisation of the boundary trace of solutions of inequalities with a uniform absorption in terms of *outer regular Borel measures*, without introducing the notion of coercivity and the strong barrier property as in [27].

THEOREM 4.1. *Assume $g \in \mathcal{G}_0$ satisfies* (4.1) *and u is a nonnegative solution of* (2.6) *with boundary trace ν. For any $a \in \partial\Omega$ the following dichotomy occurs. Either,*

(i) $\nu(\mathcal{O}) = \infty$ *for any $\mathcal{O} \in \mathcal{N}_a$. In this case $a \in \mathcal{S}(u)$ and $u \geq u_{\infty,a}$. Consequently*

$$\lim_{t\to 0} \int_{\mathcal{O}_t} u(y) dS_t = \infty,\ \forall \mathcal{O} \in \mathcal{N}_a. \tag{4.2}$$

Or

(ii) *there exists $\mathcal{O} \in \mathcal{N}_a$ such that $\nu(\mathcal{O}) < \infty$. In this case $a \in \mathcal{R}(u)$ and*

$$\sup_{0<t\leq\beta_0} \int_{\mathcal{O}'_t} u(y) dS_t < \infty. \tag{4.3}$$

for relatively every open subset $\mathcal{O}' \subset \bar{\mathcal{O}}' \subset \mathcal{O}$. Furthermore

$$\lim_{t\to 0} \int_{\Sigma_t} u(y)\phi(\sigma(y))(y) dS_t = \int_{\mathcal{R}(u)} \phi(y) d\nu(y), \quad \forall \phi \in C_c(\mathcal{R}(u)) \tag{4.4}$$

A major point in the proof of the theorem is the following completion of Proposition 2.7 which gives a characterization of the regular part of the extended boundary trace of a solution u based upon a local L^1 bound.

PROPOSITION 4.1. *Assume $g \in \mathcal{G}_0$ satisfies* (4.1) *and u is a nonnegative solution of* (2.6) *with extended boundary trace ν. Let $\mathcal{O}$ be a relatively open subset of $\partial\Omega$. If $\nu(\mathcal{O}) < \infty$, then for any compact subset $K \subset \mathcal{O}$, $\int_{K_t} u(y) dS_t$ remains bounded independently of $t \in (0, \beta_0]$.*

We recall some notations introduced in Section 2 : for $0 < \beta \leq \beta_0$, we put $\Omega^c_\beta = \Omega \setminus \bar{\Omega}_\beta = \{x \in \Omega : \rho_{\partial\Omega}(x) > \beta\}$, and $\Sigma_\beta = \partial\Omega_\beta = \partial\Omega^c_\beta$. The next result which extends Theorem 1.1 deals with the stability of the boundary value problem when the boundary is approximated from inside by a sequence of smooth domains.

LEMMA 4.1. *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$, $\{\varepsilon_n\}$ be a sequence of positive numbers converging to 0, $\mu_n \in L^1_+(\Sigma_{\varepsilon_n})$, with corresponding pull-back $\mu_n^{\varepsilon_n} \in L^1_+(\partial\Omega)$. If $\mu_n^{\varepsilon_n} \to \mu$ in the weak sense of measures, as $n \to \infty$, then the sequence of solutions $u_n = u_{\mu_n,\varepsilon_n}$ of*

$$-\Delta u_n + g(x,u_n) = 0 \quad in \ \ \Omega^c_{\varepsilon_n}, \qquad u_n = \mu_n \ on \ \ \partial\Omega^c_{\varepsilon_n}, \tag{4.5}$$

converges locally uniformly in Ω to the solution u_μ of (0.4).

PROOF. STEP 1. Since g is continuous on $\bar\Omega^c_{\varepsilon_n} \times \mathbb{R}$, existence of a unique solution to (4.6) follows from Brezis' result. The shred of the proof of the convergence is parallel to Theorem 1.1 and Proposition 1.4. Set P^{ε_n} the Poisson kernel in $\Omega^c_{\varepsilon_n}$ and $\mathbb{P}^{\varepsilon_n}_\nu$ the Poisson potential of any given Radon measure ν on Σ_{ε_n}. Since the mapping Π is C^2 there exists $C_2 > 0$ independent ε_n such that for any $(x,a) \in \Omega^c_{\varepsilon_n} \times \Sigma_{\varepsilon_n}$,

$$C_2^{-1}\rho_{\partial\Omega^c_{\varepsilon_n}}(x)|x-a|^{-N} \leq P^{\varepsilon_n}(x,a) \leq C_2\rho_{\partial\Omega^c_{\varepsilon_n}}(x)|x-a|^{-N}, \tag{4.6}$$

provided $0 \leq \varepsilon_n \leq \beta_0$, and $\rho_{\partial\Omega^c_{\varepsilon_n}}(x) = \rho_{\partial\Omega}(x) - \varepsilon_n$. Estimates (1.22), (1.23), (1.24) are valid under the form

$$\left\|\mathbb{P}^{\varepsilon_n}_\nu\right\|_{M^{(N+1)/(N-1)}(\Omega^c_{\varepsilon_n};\rho_{\partial\Omega^c_{\varepsilon_n}}dx)} \leq K\|\nu\|_{L^1(\Sigma_{\varepsilon_n})}, \tag{4.7}$$

$$\left\|\mathbb{P}^{\varepsilon_n}_\nu\right\|_{M^{N/(N-1)}(\Omega^c_{\varepsilon_n})} \leq K\|\nu\|_{L^1(\Sigma_{\varepsilon_n})}, \tag{4.8}$$

$$\left\|\mathbb{P}^{\varepsilon_n}_\nu\right\|_{L^\infty(\Omega^c_{r+\varepsilon_n})} \leq Kr^{1-N}\|\nu\|_{L^1(\Sigma_{\varepsilon_n})}, \tag{4.9}$$

Since $0 \leq u_n \leq v_n = \mathbb{P}^{\varepsilon_n}_{\mu_n}$, estimates (4.9) and (1.16)-(1.18) and the classical regularity theory for elliptic equations imply that the set of u_n is relatively compact in the $C^1_{\mathrm{loc}}(\Omega)$-topology, and any cluster point of the sequence $\{u_n\}$ is a solution of (0.1) in Ω. If $\eta \in C^{1,1}_c(\bar\Omega^c_{\varepsilon_n})$ there holds

$$\int_{\Omega^c_{\varepsilon_n}} (-u_n\Delta\eta + g(.,u_n)\eta)dy = -\int_{\Sigma_n} \frac{\partial\eta}{\partial\mathbf{n}_y}\mu_n dS(y) \tag{4.10}$$

If $\zeta \in C^{1,1}_c(\bar\Omega)$, with support in $\bar\Omega_{\beta_0}$, we set

$$\zeta_n(y) = \zeta(y + \varepsilon_n\mathbf{n}_y), \quad \forall y \in \bar\Omega^c_{\varepsilon_n} \Longleftrightarrow \zeta(x) = \zeta_n(x - \varepsilon_n\mathbf{n}_x), \quad \forall x \in \bar\Omega.$$

In relation (4.10) we take $\eta = \zeta_n$ and get

$$\int_{\Omega^c_{\varepsilon_n}} (-u_n\Delta\zeta_n + g(.,u_n)\zeta_n)dy = -\int_{\Sigma_n} \frac{\partial\zeta_n}{\partial\mathbf{n}}\mu_n dS_{\varepsilon_n}. \tag{4.11}$$

But the pointing outward normal vector $\mathbf{n}_y$ at $y \in \Sigma_n$ is the same as the pointing outward normal vector at $y + \varepsilon_n \mathbf{n}_y y \in \partial\Omega$, therefore

$$\int_{\Sigma_n} \frac{\partial \zeta_n}{\partial \mathbf{n}} \mu_n dS_{\varepsilon_n} = \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} \mu_n^{\varepsilon_n} dS, \tag{4.12}$$

by (2.5). Moreover, if we perform the change of variable $x = y + \varepsilon_n \mathbf{n}_y$ in (4.11) and denote by $\mathbf{n}_y^j$ the coordinates of $\mathbf{n}_y$, we get

$$\frac{\partial \zeta_n}{\partial y_i}(y) = \sum_j \frac{\partial \zeta}{\partial x_j}(x) \left(\delta_{ij} + \varepsilon_n \frac{\partial \mathbf{n}_y^j}{\partial y_i} \right),$$

$$\begin{aligned}\frac{\partial^2 \zeta_n}{\partial y_i^2}(y) = &\sum_{k,j} \frac{\partial^2 \zeta}{\partial x_k \partial x_j}(x) \left(\delta_{ij} + \varepsilon_n \frac{\partial \mathbf{n}_y^j}{\partial y_i} \right) \left(\delta_{ik} + \varepsilon_n \frac{\partial \mathbf{n}_y^k}{\partial y_i} \right) \\ &+ \varepsilon_n \sum_j \frac{\partial \zeta}{\partial x_j}(x) \left(\frac{\partial^2 \mathbf{n}_y^j}{\partial y_i^2} \right),\end{aligned}$$

$$\begin{aligned}\Delta \zeta_n(y) = &\sum_i \frac{\partial^2 \zeta}{\partial x_i}(x) \left(1 + \varepsilon_n \frac{\partial \mathbf{n}_y^i}{\partial y_i} \right)^2 \\ &+ \sum_{\substack{i,j,k \\ k \neq i \ ou j \neq i}} \frac{\partial^2 \zeta}{\partial x_k \partial x_j}(x) \left(\delta_{ij} + \varepsilon_n \frac{\partial \mathbf{n}_y^j}{\partial y_i} \right) \left(\delta_{ik} + \varepsilon_n \frac{\partial \mathbf{n}_y^k}{\partial y_i} \right) \\ &+ \varepsilon_n \sum_j \frac{\partial \zeta}{\partial x_j}(x) \Delta \mathbf{n}_y^j.\end{aligned}$$

Then

$$\Delta \zeta_n(y) = \Delta \zeta(x) + \varepsilon_n \mathcal{L}(D\zeta(x), D^2\zeta(x)), \tag{4.13}$$

where $\mathcal{L}(D\zeta, D^2\zeta)$ is a linear second order operator with continuous coefficients. Plugging (4.12) and (4.13) into (4.11) yields

$$\int_\Omega \left(-\tilde{u}_n(\Delta\zeta + \varepsilon_n \mathcal{L}(D\zeta, D^2\zeta)) + g_n(., \tilde{u}_n)\zeta \right) J dx = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} \mu_n^{\varepsilon_n} dS, \tag{4.14}$$

where $\tilde{u}_n(x) = u_n(x - \varepsilon_n \mathbf{n}_x)$, $g_n(x, r) = g(x - \varepsilon_n \mathbf{n}_x, r)$ and $J(x) = \left|\det(I - \varepsilon_n D\mathbf{n}_y)\right|$.

STEP 2 From (1.26) and (4.6) there exists $C_3 > 0$ independent of ε_n and $(y, b) \in \Omega_{\varepsilon_n}^c \times \Sigma_n$ such that

$$C_3^{-1} P(y + \varepsilon_n \mathbf{n}_y, b + \varepsilon_n \mathbf{n}_y) \leq P^{\varepsilon_n}(y, b) \leq C_3 P(y + \varepsilon_n \mathbf{n}_y, b + \varepsilon_n \mathbf{n}_y), \tag{4.15}$$

provided $\varepsilon_n \le \beta_0$. Because $0 \le u_n(y) \le \mathbb{P}^{\varepsilon_n}_{\mu_n}$, the above inequality and the Poisson representation formula imply

$$0 \le \tilde{u}_n \le C_3 \mathbb{P}_{\mu_n^{\varepsilon_n}} \tag{4.16}$$

in $\bar{\Omega}_{\beta_0-\varepsilon_n}$. Jointly with (1.23) it implies that $\{\tilde{u}_n\}$ is uniformly integrable in $\Omega_{\beta_0-\varepsilon_n}$, and thus in Ω.

STEP 3 From (4.1), (4.16),

$$0 \le g_n(x, \tilde{u}_n)(x) \le f(C_3 \mathbb{P}_{\mu_n^{\varepsilon_n}}(x)), \tag{4.17}$$

for any $x \in \Omega_{\beta_0-\varepsilon_n}$. For $\lambda \ge 0$, put $\Gamma_\lambda = \{x \in \Omega : P(x,a) \ge \lambda\}$ and

$$\beta_{\varepsilon_n}(\lambda) = \int_{\Gamma_\lambda} \rho_{\partial\Omega} dx.$$

Then (see Step 2 in the proof of Theorem 1.1),

$$\beta_{\varepsilon_n}(\lambda) \le C_2 \int_0^{(C_1/\lambda)^{1/(N-1)}} s^N ds \le \frac{C_2}{N+1} \left(\frac{C_1}{\lambda}\right)^{(N+1)/(N-1)}. \tag{4.18}$$

It follows from (4.17) that for any Borel set $G \subset \Omega$,

$$\int_G g_n(., \tilde{u}_n) \rho_{\partial\Omega} dx \le \int_G f(C_3 \mathbb{P}_{\mu_n^{\varepsilon_n}}) \rho_{\partial\Omega} dx. \tag{4.19}$$

In order to estimate the right-hand side of (4.19), we follow the proof of Theorem 1.1. Let $m > 0$ and $a \in \partial\Omega$, then

$$\begin{aligned} &\int_G f(mP(., a)) \rho_{\partial\Omega} dx \\ &\qquad \le f(\lambda) \int_G \rho_{\partial\Omega} dx + C_4 m^{(N+1)/(N-1)} \int_\lambda^\infty f(s) s^{-2N/(N-1)} ds. \end{aligned} \tag{4.20}$$

We take $m = m_n = \int_{\partial\Omega} d\mu_n^{\varepsilon_n}$ and we deduce that the $\{g_n(., \tilde{u}_n)\}$ are uniformly integrable, as in the proof of Theorem 1.1-Step 7. If $\tilde{u}_{n_k}$ is a subsequence converging in the C^1_{loc} topology to some function u, we can pass to the limit in (4.14) and get

$$\int_\Omega (-u\Delta\zeta + g(., u)\zeta) dy = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \tag{4.21}$$

Because of uniqueness, the whole sequence $\tilde{u}_n$ converges to u. □

REMARK 4.1. This above stability result of solutions with respect to approximations from inside is no longer valid if the absorption term is truly degenerate, for example if

$$g(x,r) = \rho_{\partial\Omega}^{\alpha}(x)|u|^{q-1}u,$$

for some $\alpha > 0$ and $q > 1$. In that case Problem (0.4) is solvable in Ω for any Radon measure μ when $1 < q < (N+1+\alpha)/(N-1)$ and is not solvable when μ is a Dirac mass if $q \geq (N+1+\alpha)/(N-1)$ (see [27]). Therefore, even if the data μ_n are smooth functions on Σ_{ε_n}, if they concentrate too quickly to a Dirac mass on the boundary, the corresponding solutions u_n of (4.6) converge to 0.

PROOF OF PROPOSITION 4.1. We proceed by contradiction in assuming that there exist a compact $K \subset \mathcal{O}$ and a sequence $\{\varepsilon_n\}$ converging to 0 such that

$$\lim_{n\to\infty} \int_{K_{\varepsilon_n}} u(y)dS_{\varepsilon_n} = \infty. \tag{4.22}$$

Since K is compact, there exist a sequence $\{a_n\{\subset K$ converging to some $a \in K$ and a sequence $\{t_n\}$ converging to 0 such that

$$\lim_{n\to\infty} \int_{K_{\varepsilon_n}\cap B_{t_n}(a_n)} u(y)dS_{\varepsilon_n} = \infty. \tag{4.23}$$

For $k > 0$, there exists $\ell_k > 0$ such that

$$\lim_{n\to\infty} \int_{K_{\varepsilon_n}\cap B_{t_n}(a_n)} \min\{\ell_k, u(y)\}dS_{\varepsilon_n} = k, \tag{4.24}$$

and $\ell_k \to 0$ as $n \to \infty$. We set $\mu_n = \min\{\ell_k, u(y)\}\chi_{K_{\varepsilon_n}\cap B_{t_n}(a_n)}$ and denote by u_n the solution of (4.6) in $\Omega^c_{\varepsilon_n}$ with this specific boundary data. Then

$$u \geq u_n \quad \text{in} \quad \Omega^c_{\varepsilon_n}.$$

Since the corresponding measure $\gamma_u(\mu_n) = \mu_n^{\varepsilon_n}$ on $\partial\Omega$ converges to $k\delta_a$, and u_n converges to $u_{k\delta_a}$ by Lemma 4.1, it leads to $u \geq u_{k\delta_a}$ in Ω. Since k is arbitrary,

$$u \geq u_{\infty,a} \quad \text{in} \quad \Omega.$$

Therefore

$$\nu(\mathcal{O}) \geq \nu(a) = \tilde{\gamma}(a) = \infty,$$

by Proposition 2.7, a contradiction. □

For any $a \in \partial\Omega$, we recall that $\mathcal{N}_a$ is the set of relatively open neighborhoods of a in $\partial\Omega$.

PROOF OF THEOREM 4.1. Let $a \in \partial\Omega$. If (i) holds, inequality $u \geq u_{\infty,a}$ follows from Proposition 2.7, and

$$\lim_{t\to 0}\int_{\mathcal{O}_t} u(y)dS_t \geq \lim_{t\to 0}\int_{\mathcal{O}_t} u_{\infty,a}(y)dS_t = \infty,$$

which is equivalent to $\nu(\mathcal{O}) = \infty$.

Next we assume that (i) does not hold, and there exists $\mathcal{O} \in \mathcal{N}_a$ such that $\nu(\mathcal{O}) < \infty$. By Proposition 4.1, for any compact subset $\mathcal{K} \in \mathcal{N}_a$ such that $\bar{\mathcal{K}} \subset \mathcal{O}$, there exists a constant $M_K > 0$ such that

$$\int_{\mathcal{K}_t} u(y)dS_t \leq M_K, \quad \text{on } (0, \beta_0].$$

Let $\mathcal{O}$ be any relatively open subset with compact closure in $\mathcal{R}(u)$, $0 < \beta < \beta_0$ and

$$\mathfrak{O}_\beta = \{x = \sigma(x) - t\mathbf{n}_x : \sigma(x) \in \mathcal{O},\ \beta < t < \beta_0\},$$

then $u \in L^1(\mathfrak{O}_0)$. As in the proof of [27, Lemma 1.8], if $\varphi \in C_c^2(\mathcal{O})$, $\varphi > 0$, we define a test function which vanishes on Gs_β by

$$\zeta(x) = \varphi(\sigma(x))(\rho_{\partial\Omega}(x) - \beta) \quad \forall x \in \mathfrak{O}_\beta,$$

and derive that the largest solution u^* of (0.1) dominated by u satisfies

$$\begin{aligned}\int_{\mathfrak{O}_\beta} \left(-u^*\Delta\zeta + g(x,u^*)\zeta\right) dx = \int_{\mathcal{O}_\beta} u^*\varphi^\beta dS_\beta - \int_{\mathcal{O}_{\beta_0}} u^*\varphi^{\beta_0} dS_{\beta_0} \\ + \int_{\mathcal{O}_{\beta_0}} \frac{\partial u^*}{\partial \mathbf{n}}\zeta dS.\end{aligned} \tag{4.25}$$

Therefore $\int_{\mathfrak{O}_\beta} g(x,u^*)\zeta dx$ is bounded independently of β. Letting $\beta \to 0$ yields to

$$\int_{\mathfrak{O}_0} g(x,u^*)\varphi\rho_{\partial\Omega} dx < \infty. \tag{4.26}$$

Since $g(x,u^*) \in L^1(\mathfrak{O}_0; \rho_{\partial\Omega}dx)$, [27, Corollary 1.3] implies that there exists a nonnegative Radon measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$ such that

$$\lim_{t\to 0}\int_{\mathcal{O}_t} \varphi_t u^*(y)dS_t = \int_{\mathcal{O}} \varphi d\mu,$$

for any $\varphi \in C_c(\mathcal{O})$. The measure ν, which is equal to μ on $\mathcal{R}(u)$, is therefore a regular Borel measure.

Because of (4.3) from any sequence $\{\epsilon_n\}$ converging to 0 one can extract a subsequence, still denoted by $\{\epsilon_n\}$, such that $\{u(\epsilon_n,)\chi_{\mathcal{R}(u)}dS\}$ converges in the weak sense of measures to some $\eta \in \mathfrak{M}_+(\mathcal{R}(u))$. We claim that

$$\eta = \nu|_{\mathcal{R}(u)} = \chi_{\mathcal{R}(u)}\nu. \tag{4.27}$$

Since $u \geq u^*$, $\eta \geq \nu|_{\mathcal{R}(u)}$. If $\mathcal{O}$ is any relatively open subset of $\partial\Omega$ with compact closure in $\mathcal{R}(u)$, we put $\mu_n = u(\epsilon_n,)\chi_{\mathcal{R}(u)}dS$ and denote by u_n the solution of (4.6) in $\Omega^c_{\epsilon_n}$. By Lemma 4.1, $\{u_n\}$ converges locally uniformly in Ω to the solution $\tilde{u}$ of (0.4) with boundary data $\chi_{\mathcal{O}}\eta$. Since $u \geq u_n$, $u \geq \tilde{u}$ and thus $u^* \geq \tilde{u}$. Therefore $\chi_{\mathcal{O}}\eta \leq \nu_{\mathcal{O}}$. This implies (4.27). Finally, as η is independent of the sequence $\{\epsilon_n\}$, it follows that $u(t, .)\chi_{\mathcal{R}(u)}dS$ converges to $\chi_{\mathcal{R}(u)}\nu$ in the sense of measures, as $t \to 0$. This ends the proof. □

5. – Some examples

In this section Ω is a C^2 bounded domain and we consider absorption terms g which are split under the form

$$g(x, r) = \tilde{h}(x)\tilde{g}(r) \tag{5.1}$$

where both $\tilde{h}$ and $\tilde{g}$ are nonnegative continuous functions defined respectively on Ω and $\mathbb{R}_+$. We assume also that $\tilde{g}$ vanishes at 0 and is nondecreasing, and that $\tilde{h}(x) > 0$ in Ω. If the the Keller-Osserman condition is fulfilled, that is there exists some $c \geq 0$ such that

$$\int_c^\infty \frac{ds}{\sqrt{G(s)}} < \infty \tag{5.2}$$

where $G(s) = \displaystyle\int_0^s \tilde{g}(t)dt$, then for any compact subset $K \subset \Omega$ there exists a constant $C(K) > 0$ such that any nonnegative solution u of

$$-\Delta u + g(x, u) \leq 0, \quad \text{in } \Omega, \tag{5.3}$$

satisfies

$$u(x) \leq C(K), \quad \forall x \in K.$$

If the Keller-Osserman condition is not satisfied, and $\tilde{h}$ is a positive constant, no such a priori upper bound can exist [31]. If $\tilde{g}(r) = kr$ for some $k > 0$ and g is uniformly admissible, it is clear that

$$\lim_{\lambda\to\infty} u_{\lambda\delta_a}(x) = \infty, \quad \forall a \in \partial\Omega,\ \forall x \in \Omega, \tag{5.4}$$

but it appears difficult to find a general condition on $\tilde{g}$ which implies that (5.4) holds. However, it is proved in [13] that if

$$g(x, r) = r(\ln(r + 1))^{\gamma},$$

with $0 < \gamma \leq 2$, this property holds. As a consequence we have the following

COROLLARY 5.1. *Let u be a nonnegative solution of*

$$-\Delta u + u(\ln(u + 1))^{\gamma} = 0, \quad \text{in } \Omega. \tag{5.5}$$

Then the boundary trace of u is a Radon measure μ*,* $u(\ln(u+1))^{\gamma} \in L^1(\Omega; \rho_{\partial\omega} dx)$ *and*

$$u = \mathbb{P}_{\mu} - \mathbb{G}(u(\ln(u + 1))^{\gamma}).$$

PROOF. It is clear that $g(u) = u(\ln(u + 1))^{\gamma}$ is uniformly admissible. By Theorem 4.1, $\mathrm{Tr}^e_{\partial\Omega}(u)$ is an outer regular Borel measure, which admits no singular part by [13], therefore it is a Radon measure. Thus the nonlinearity is integrable for the measure $\rho_{\partial\omega} dx$, and the representation follows. □

REMARK 5.1. If

$$g(x, r) = r(\ln(r + 1))^{\gamma}$$

with $\gamma > 2$, or if

$$g(x, r) = \rho^{\alpha}_{\partial\Omega}(x)\, |r|^{q-1}\, r,$$

with $\alpha > -2$ and $1 < q < (N + 1 + \alpha)/(N - 1)$, it is proved respectively in [13] and [27] that for any $a \in \partial\Omega$, $u_{\infty,a}$ is a solution of (0.1) in Ω vanishing on $\partial\Omega \setminus \{a\}$, with a strong singularity at a. In those two cases the boundary trace of a nonnegative solution of (0.1) can be any outer regular Borel measure on $\partial\Omega$.

Another interesting type of problems deals with the situation in which the absorption term is strongly degenerate at the boundary. The model example is

$$g(x, r) = \exp(-\kappa/\rho_{\partial\omega}(x))u^{q},$$

with $q > 1$ and $\kappa > 0$. In this case the function g belongs to $\mathcal{H}\mathcal{G}_0$ for any $q > 1$. Therefore fundamental solutions always exist, but a new phenomenon appears which is to be compared with what is called instantaneous or complete blow-up for parabolic equations ([2]) or elliptic equations ([7]), linear or nonlinear, with an inverse square potential.

PROPOSITION 5.1. *For any* $q > 1$*,* $\kappa > 0$ *and* $a \in \partial\Omega$

$$u_{\infty,a} = u_m$$

where u_m is the minimal solution of

$$(5.6)\qquad \begin{gathered} -\Delta u + \exp(-\kappa/\rho_{\partial\omega}(x))u^q = 0 \quad \text{in } \Omega, \\ \lim_{\rho_{\partial\Omega}(x)\to 0} u(x) = \infty. \end{gathered}$$

PROOF. Without any loss of generality we assume $a = O$. Let $\{x = (x_1, x')\}$ be the coordinates in $\mathbb{R}^N$. We assume that the hyperplane $H_0 = \{x = (0, x')\}$ is tangent to $\partial\Omega$ at O and $S_+^{N-1} = B_1(O) \cap \{x = (x_1, x') : x_1 > 0\}$. We write

$$\exp(-\kappa/\rho_{\partial\omega}(x)) = h(\rho_{\partial\omega}(x)).$$

STEP 1 The case $1 < q < (N+1)/(N-1)$. For $0 < \varepsilon \le \beta_0$, the function $u_{\infty,O}$ is minorized in $\Omega_\varepsilon^m = \{x \in \Omega : |x| < m,\ 0 < \rho_{\partial\omega}(x) < \varepsilon\}$ ($m > 0$ small enough) by the function

$$\ell(\varepsilon)U_\varepsilon$$

where $\ell(\varepsilon) = h^{-1/(q-1)}(\varepsilon)$ and U_ε is the unique solution of

$$(5.7)\qquad \begin{aligned} -\Delta v &= v^q \quad \text{in } \Omega_\varepsilon^m, \\ v &= \infty\delta_O \quad \text{on } \partial\Omega_\varepsilon^m. \end{aligned}$$

Moreover there holds (see [15])

$$(5.8)\qquad \lim_{\substack{x\to O\\ x\in\Omega_\varepsilon}} |x|^{2/(q-1)} U_\varepsilon(x) = \omega(x/|x|),$$

where ω is the unique positive solution of

$$(5.9)\qquad \begin{aligned} -\Delta_\sigma \omega - \left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right)\omega + \omega^q &= 0 \quad \text{in } S_+^{N-1}, \\ \omega &= 0 \quad \text{on } \partial S_+^{N-1}. \end{aligned}$$

If we write

$$U_\varepsilon(x) = \varepsilon^{-2/(q-1)} U_{1,\varepsilon}(x/\varepsilon) = \varepsilon^{-2/(q-1)} U_{1,\varepsilon}(y), \quad y = x/\varepsilon,$$

the function $U_{1,\varepsilon}$ satisfies

$$\Delta U_{1,\varepsilon} = U_{1,\varepsilon}^q \quad \text{in } \mathfrak{D}_\varepsilon^m = \varepsilon^{-1}\Omega_\varepsilon^m.$$

When $\varepsilon \to 0$, D_ε^m converges to $\mathfrak{D}_0 = (0,1) \times \mathbb{R}^{N-1}$ in the sense of sets. Thus, for any $0 < \theta_1 < \theta_2 < 1$, there exists ε_0 such that if $0 < \varepsilon \le \varepsilon_0$, the following inclusion holds

$$\mathfrak{G}_\theta^m = \{y = (y_1, y') : \theta_1 < y_1 < \theta_2,\ 1 < |y'| \le m/2\varepsilon\} \subset \mathfrak{D}_\varepsilon^m.$$

Because

$$U_{1,\varepsilon}(y) \leq C|y|^{-2/(q-1)},$$

$U_{1,\varepsilon}$ converges uniformly, as $\varepsilon \to 0$, on any compact subset of $\mathfrak{D}_0 \setminus \{O\}$ to the unique solution U_1 of

$$\Delta U_1 = U_1^q \quad \text{in} \quad \mathfrak{D}_0,$$

which vanishes on $\partial\mathfrak{D}_0 \setminus \{O\}$ and satisfies

$$\lim_{\substack{y \to O \\ y \in \mathfrak{D}_0}} |y|^{2/(q-1)} U_1(y) = \omega(y/\,|y|). \tag{5.10}$$

Therefore there exists some $\eta \in (0, 1)$ such that

$$U_{1,\varepsilon}(y_1, y') \geq \eta \sin\left(\frac{\pi(y_1 - \theta_1)}{\theta_2 - \theta_1}\right), \quad \text{for } y_1 \in [\theta_1, \theta_2] \text{ and } \left|y'\right| = 1. \tag{5.11}$$

Notice that the function $y_1 \mapsto \psi_\theta(y_1) = \sin(\pi(y_1 - \theta_1)/(\theta_2 - \theta_1))$ vanishes for $y_1 = \theta_1$ and for $y_1 = \theta_2$.

We first suppose $N = 2$. Then for any $\beta > 0$ the function

$$y' \mapsto \varphi_\beta(y') = \frac{\sinh\left(\beta\left(\dfrac{m}{2\varepsilon} - \left|y'\right|\right)\right)}{\sinh\left(\beta\left(\dfrac{m}{2\varepsilon} - 1\right)\right)}, \tag{5.12}$$

is nonnegative takes the value 1 for $\left|y'\right| = 1$, and vanishes for $\left|y'\right| = m/(2\varepsilon)$. If we set

$$\zeta_{\theta,\beta}(y_1, y') = \eta\psi_\theta(y_1)\varphi_\beta(y'),$$

there holds

$$\Delta\zeta_{\theta,\beta} = \left(\beta^2 - \frac{\pi^2}{(\theta_2 - \theta_1)^2}\right)\zeta_{\theta,\beta}. \tag{5.13}$$

Since $\zeta_{\theta,\beta} \leq \eta$, it follows

$$\Delta\zeta_{\theta,\beta} \geq \left(\beta^2 - \frac{\pi^2}{(\theta_2 - \theta_1)^2}\right)\eta^{1-q}\zeta_{\theta,\beta}^q \qquad \text{in} \quad \mathfrak{G}_\theta^m, \tag{5.14}$$

furthermore

$$\begin{aligned}
\zeta_{\theta,\beta}(y_1, y') &= 0 \quad \text{for} \quad y_1 = \theta_i,\ i = 1,\ 2,\\
\zeta_{\theta,\beta}(y_1, y') &= 0 \quad \text{for} \quad \left|y'\right| = m/2\varepsilon,\\
\zeta_{\theta,\beta}(y_1, y') &= 0 \quad \text{for} \quad \left|y'\right| = 1.
\end{aligned}$$

We chose β such that $\dfrac{\pi^2}{(\theta_2-\theta_1)^2}\eta^{1-q}=1$. By (5.11) and the maximum principle one obtains

$$U_{1,\varepsilon}(y_1,y')\geq \zeta_{\theta,\beta}(y_1,y') \quad \text{in} \quad \mathfrak{G}^m_\theta. \tag{5.15}$$

Therefore

$$\begin{aligned}u(x_1,x')&\geq \ell(\varepsilon)\varepsilon^{-2/(q-1)}U_{1,\varepsilon}(x_1/\varepsilon,x'/\varepsilon)\\&\geq \ell(\varepsilon)\varepsilon^{-2/(q-1)}\zeta_{\theta,\beta}(x_1/\varepsilon,x'/\varepsilon),\end{aligned}$$

for $\theta_1\varepsilon\leq x_1\leq\theta_1\varepsilon$ and $\varepsilon\leq|x'|\leq m/2$. Take $x_1=\varepsilon(\theta_1+\theta_2)/2=\theta\varepsilon$, then

$$u(\theta\varepsilon,x')\geq \eta\ell(\varepsilon)\varepsilon^{-2/(q-1)}\frac{\sinh\left(\beta\left(\dfrac{m-2|x'|}{2\varepsilon}\right)\right)}{\sinh\left(\beta\left(\dfrac{m}{2\varepsilon}-1\right)\right)}.$$

If $|x|'\leq m/4$,

$$\frac{\sinh\left(\beta\left(\dfrac{m-2|x'|}{2\varepsilon}\right)\right)}{\sinh\left(\beta\left(\dfrac{m}{2\varepsilon}-1\right)\right)}=e^{\beta(1-|x|'/\varepsilon)}(1+\circ(1)) \quad \text{as} \quad \varepsilon\to 0,$$

Thus

$$u(\theta\varepsilon,x')\geq \eta\ell(\varepsilon)\varepsilon^{-2/(q-1)}e^{\beta}e^{-\beta|x'|/\varepsilon}(1+\circ(1)),$$

and

$$\liminf_{\varepsilon\to 0}u(\theta\varepsilon,x')\geq \eta e^{\beta}\liminf_{\varepsilon\to 0}\ell(\varepsilon)\varepsilon^{-2/(q-1)}e^{-\beta|x'|/\varepsilon}.$$

Since $\ell(\varepsilon)=e^{\kappa/((q-1)\varepsilon}$,

$$\ell(\varepsilon)\varepsilon^{-2/(q-1)}e^{\beta(1-|x|'/\varepsilon)}\geq \varepsilon^{-2/(q-1)}\exp\left[\varepsilon^{-1}\left(\kappa/(q-1)-\beta|x|'\right)\right]$$

If we fix $|x|'<\beta\kappa/(q-1)$, then

$$\liminf_{\varepsilon\to 0}u(\theta\varepsilon,x')=\lim_{\rho_{\partial\Omega}(x)\to 0}u(x)=\infty,$$

and this limit is uniform on any compact subset of $\{x' : |x|'<\beta\kappa/(q-1)\}$, which is equivalent to any compact of $\{x : |\sigma(x)|<\beta\kappa/(q-1)\}$. Put $\tau=\beta\kappa/(2(q-1))$. Because this blow-up holds in a fixed neighborhood $\partial\Omega\cap\bar{B}_\tau(O)$ of O, we can replace O by any point P in $\partial\Omega\cap\bar{B}_\tau(O)$ and conclude that

$$\lim_{\rho_{\partial\Omega}(x)\to 0}u(x)=\infty,$$

uniformly if $|\sigma(x) - P| \leq \tau$. Iterating this process infers that

$$\lim_{\rho_{\partial\Omega}(x)\to 0} u(x) = \infty.$$

Next we assume $N \geq 3$. Let $\beta > 0$ to be fixed and

$$\Gamma_\varepsilon = \{y' \in \mathbb{R}^{N-1} \,:\, 1 < |y'| < m/2\varepsilon\}$$

and let $B_\beta(y')$ be the solution of

$$\begin{aligned} &\Delta_{y'} B_\beta = \beta^2 B_\beta \quad \text{in } \Gamma_\varepsilon, \\ &B_\beta(y') = 1 \quad \text{if } |y'| = 1, \\ &B_\beta(y') = 0 \quad \text{if } |y'| = m/2\varepsilon. \end{aligned} \tag{5.16}$$

The function $\zeta_{\theta,\beta}(y_1, y') = \eta\psi_\theta(y_1)B_\beta(y')$ satisfies also (5.13) in $\mathfrak{G}^m_\theta$. Therefore, if we chose β as in the case $N = 2$, (5.15) is still valid. Since B_β is a Bessel function, its behaviour at infinity is classical and there holds, for $|x|' \leq m/4$,

$$B_\beta(y) = C_\beta \left(\frac{|x|'}{\varepsilon}\right)^{1-N/2} e^{-\beta|x|'/\varepsilon}(1 + \circ(1)), \quad \text{as } \varepsilon \to 0.$$

We conclude as in the case $N = 2$.

STEP 2. The general case. If $q \geq (N+1)/(N-1)$ let $\alpha > 0$ such that

$$q < (N + 1 + \alpha)/(N - 1).$$

We write

$$h(\rho_{\partial\Omega}(x)) = \rho^\alpha_{\partial\Omega}(x)\tilde{h}(\rho_{\partial\Omega}(x)),$$

with

$$\tilde{h}(\rho_{\partial\Omega}(x)) = \rho^{-\alpha}_{\partial\Omega}(x)h(\rho_{\partial\Omega}(x)).$$

We can assume that $r \mapsto \tilde{h}(r)$ is nondecreasing near $r = 0$ and we extend it by continuity at $r = 0$ by putting $\tilde{h}(0) = 0$. Thus there holds

$$\Delta u \leq \tilde{h}(\varepsilon)\rho^\alpha_{\partial\Omega}(x)u^q \quad \text{in } \Omega_\varepsilon.$$

The equation

$$-\Delta U + \rho^\alpha_{\partial\Omega}(x)U^q = 0,$$

admits weak and strong isolated singularities on the boundary and any positive solution with a strong singularity at $x = O$ satisfies

$$\lim_{\substack{x\to O \\ x\in\Omega_\varepsilon}} |x|^{(2+\alpha)/(q-1)} U_\varepsilon(x) = \omega_\alpha(x/|x|), \tag{5.17}$$

where ω_α is the unique positive solution of

$$(5.18)\qquad -\Delta_\sigma \omega_\alpha - \left(\frac{2+\alpha}{q-1}\right)\left(\frac{2q+\alpha}{q-1} - N\right)\omega_\alpha + \omega_\alpha^q = 0 \quad \text{in } S_+^{N-1}, \qquad \omega = 0 \quad \text{on } \partial S_+^{N-1},$$

we proceed as in Step 1, with some minor changes of coefficients.

STEP 3. End of the proof. The minimal solution u_m of (5.6) is constructed by considering the increasing sequence u_k of solutions of

$$(5.19)\qquad -\Delta u_k + \exp(-\kappa/\rho_{\partial\omega}(x))u_k^q = 0 \quad \text{in } \Omega, \qquad u_k(x) = k \quad \text{on } \partial\Omega.$$

When $k \to \infty$, $u_k \to u_m$, thus $u_{\infty,a} \geq u_m$. On the other hand, $u_{\lambda\delta_a}$ is constructed by approximating the Dirac mass on the boundary by bounded functions g_λ. Thus the corresponding solutions u_{g_λ} of (0.4) are all dominated by u_m. Therefore

$$u_{\lambda\delta_a} \leq u_m \Longrightarrow u_{\infty,a} \leq u_m. \qquad \square$$

REMARK 5.2. If the domain Ω is starshaped with respect to some point, say O, the Iscoe uniqueness method (see [17]) of scaling applies straightforwardly to prove the uniqueness of the solution of (5.6). We recall this method. Let $\ell > 0$ and $u_\ell(x) = \ell^{2/(q-1)}u(\ell x)$, then u_ℓ satisfies

$$-\Delta u_\ell + e^{-\kappa/\rho_{\partial\Omega}(\ell x)}u_\ell^q = 0 \quad \text{in } \Omega_\ell = \frac{1}{\ell}\Omega, \qquad u_\ell = \infty \quad \text{on } \partial\Omega_\ell.$$

But $\Omega_\ell \subset \Omega$ and $e^{-\kappa/\rho_{\partial\Omega}(\ell x)} \leq e^{-\kappa/\rho_{\partial\Omega}(x)}$ if $\ell > 1$. Therefore u_ℓ satisfies

$$-\Delta u_\ell + e^{-\kappa/\rho_{\partial\Omega}(x)}u_\ell^q \geq 0, \quad \text{in } \Omega_\ell.$$

If $\hat{u}$ is another of (5.6) in Ω, then $u_\ell \geq \hat{u}$. Letting $\ell \to 1$ infers $u \geq \hat{u}$. In the same way $\hat{u} \geq u$. In a much more elaborated manner, if Ω is locally a continuous graph, the method of local translations developped by the authors in [21] can be adapted and once again uniqueness of the solution of (5.6) holds.

Combining Proposition 5.1, the previous remark and Theorem 4.1 (in the case $1 < q < (N+1)/(N-1)$) we derive,

COROLLARY 5.2. *Let $q > 1$ and u be a nonnegative solution of*

$$(5.20)\qquad -\Delta u + \exp(-1/\rho_{\partial\omega}(x))u^q = 0 \quad \textit{in } \Omega.$$

Then either

(i) $\mathcal{S}(u) = \partial\Omega$, $\mathrm{Tr}^e_{\partial\Omega}(u)$ *the Borel measure indentically equal to* ∞, *and*

$$u_{\infty,a} = u_m,$$

or

(ii) $\mathcal{R}(u) = \partial\Omega$ *and* $\mathrm{Tr}^e_{\partial\Omega}(u) = \nu$ *is a bounded Borel measure. Moreover, if* $1 < q < (N+1)/(N-1)$, ν *is a Radon measure and* $u = u_\nu$.

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