Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVI (2016), 581-601

Schrödinger-type operators with unbounded diffusion and potential terms

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Abstract. We prove that the realization A_p in $L^p(\mathbb{R}^N)$, for $1 < p < \infty$, of the Schrödinger-type operator $A = (1 + |x|^\alpha)\Delta - |x|^\beta$ with domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}$ generates a strongly continuous analytic semigroup provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Moreover this semigroup is consistent, irreducible, immediately compact and ultracontractive.

Mathematics Subject Classification (2010): 47D07 (primary); 47D08, 35J10, 35K20 (secondary).

1. Introduction

In this paper we study the generation of analytic semigroups in L^p-spaces of Schrödinger-type operators of the form

$$Au(x) = a(x)\Delta u(x) - V(x)u(x), \quad \text{for} \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $a(x) = 1 + |x|^\alpha$ and $V(x) = |x|^\beta$ with $\alpha > 2$ and $\beta > \alpha - 2$. We also investigate spectral properties of such semigroups. In the case where $\alpha \in [0, 2]$ and $\beta \geq 0$, generation results of analytic semigroups for suitable realizations A_p of the operator A in $L^p(\mathbb{R}^N)$ have been proved in [4].

For $\beta = 0$ and $\alpha > 2$, the generation results depend upon N as it is proved in [8]. More specifically, if $N = 1, 2$ no realization of A in $L^p(\mathbb{R}^N)$ generates a strongly continuous (resp. analytic) semigroup. The same happens if $N \geq 3$ and $p \leq N/(N-2)$. On the other hand, if $N \geq 3$ and $p > N/(N-2)$, then the maximal realization A_p of the operator A in $L^p(\mathbb{R}^N)$ generates a positive analytic semigroup, which is also contractive if $\alpha \geq (p-1)(N-2)$.

This work has been supported by the M.I.U.R. research project Prin 2010MXMAJR and INdAM-GNAMPA 2014.

Received September 9, 2014; accepted in revised form February 26, 2015.

Published online June 2016.

Generation results concerning the case where $\beta = 0$ and with drift terms of the form $|x|^{\alpha-2}x$ were obtained recently in [9]. The operator with a more general diffusion term was also investigated in [10] and [14].

We also quote the recent paper [5]. Here the authors studied the generation of C_0 and analytic semigroups in $L^p(\mathbb{R}^N)$, for $1 < p < \infty$, of operators of the form $\mathcal{A} = |x|^\alpha \Delta + c|x|^{\alpha-2}x \cdot \nabla - b|x|^{\alpha-2}$. They prove for $\alpha \neq 2$, in particular for $c = 0$ and $b = 1$, that a suitable L^p-realization of $\mathcal{A}$ generates a bounded analytic semigroup in $L^p(\mathbb{R}^N)$ if and only if $N/p < (N-2)/2 + \sqrt{1 + (N-2)^2/4}$, see [5, Theorem 1.2]. We note here that $\beta = \alpha - 2$ corresponds to a critical case. The methods used in [5] are completely different from ours and lead to results which are not comparable with our case ($\beta > \alpha - 2$).

Here we consider the case where $\alpha > 2$ and assume that $N > 2$. Let us denote by A_p the realization of A in $L^p(\mathbb{R}^N)$ endowed with its maximal domain

$$D_{p,\max}(A) = \left\{u \in L^p\left(\mathbb{R}^N\right) \cap W^{2,p}_{\mathrm{loc}}\left(\mathbb{R}^N\right) \,:\, Au \in L^p\left(\mathbb{R}^N\right)\right\}. \tag{1.2}$$

After proving a priori estimates, we deduce that $D_{p,\max}(A)$ coincides with

$$D_p(A) := \left\{u \in W^{2,p}\left(\mathbb{R}^N\right) : Vu, \left(1 + |x|^{\alpha-1}\right)|\nabla u|, \left(1 + |x|^\alpha\right)|D^2u| \in L^p\left(\mathbb{R}^N\right)\right\}.$$

So we show in the main result of this paper that, for any $1 < p < \infty$, the realization A_p of A in $L^p(\mathbb{R}^N)$, with domain $D_p(A)$, generates a positive strongly continuous and analytic semigroup $(T_p(t))_{t\geq 0}$ for any $\beta > \alpha - 2$. This semigroup is also consistent, irreducible, immediately compact and ultracontractive.

The paper is structured as follows. In Section 2 we study the invariance of $C_0(\mathbb{R}^N)$ under the semigroup generated by A in $C_b(\mathbb{R}^N)$ and show its compactness. In Section 3 we use reverse Hölder classes and some results in [13] to study the solvability of the elliptic problem in $L^p(\mathbb{R}^N)$. Finally, in Section 4 we prove the generation results.

Notation. For any $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C_c^k(\mathbb{R}^N)$ the set of all functions $f : \mathbb{R}^N \to \mathbb{R}$ that are continuously differentiable in $\mathbb{R}^N$ up to k-th order and have compact support (denoted supp(f)). The space $C_b(\mathbb{R}^N)$ is the set of all bounded and continuous functions $f : \mathbb{R}^N \to \mathbb{R}$, and we denote by $\|f\|_\infty$ its sup-norm, *i.e.*, $\|f\|_\infty = \sup_{x\in\mathbb{R}^N} |f(x)|$. We use also the space $C_0(\mathbb{R}^N) := \{f \in C_b(\mathbb{R}^N) : \lim_{|x|\to\infty} f(x) = 0\}$. If f is smooth enough we set

$$|\nabla f(x)|^2 = \sum_{i=1}^N |D_i f(x)|^2, \qquad |D^2 f(x)|^2 = \sum_{i,j=1}^N |D_{ij} f(x)|^2.$$

For any $x_0 \in \mathbb{R}^N$ and any $r > 0$ we denote by $B(x_0, r) \subset \mathbb{R}^N$ the open ball, centered at x_0 with radius r. We simply write $B(r)$ when $x_0 = 0$. The function χ_E denotes the characteristic function of the (measurable) set E, *i.e.*, $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ otherwise.

For any $p \in [1, \infty)$ we denote by $L^p(\mathbb{R}^N)$ the Banach space of all measurable and p-integrable functions in $\mathbb{R}^N$ with respect to the Lebesgue measure endowed with its usual norm $\|\cdot\|_p$. Finally, by $x \cdot y$ we denote the Euclidean scalar product of the vectors $x, y \in \mathbb{R}^N$.

ACKNOWLEDGEMENTS. We are grateful to the referee for his many helpful remarks and suggestions.

2. Generation of semigroups in $C_0(\mathbb{R}^N)$

In this section we recall some properties of the elliptic and parabolic problems associated with A in $C_b(\mathbb{R}^N)$. We prove the existence of a Lyapunov function for A in the case where $\alpha > 2$ and $\beta > \alpha - 2$. This implies the uniqueness of the solution semigroup $(T(t))_{t\geq 0}$ to the associated parabolic problem. Using a domination argument, we show that $T(t)$ is compact and $T(t)C_0(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$.

First, we endow A with its maximal domain in $C_b(\mathbb{R}^N)$

$$D_{\max}(A) = \left\{u \in C_b\left(\mathbb{R}^N\right) \cap W^{2,p}_{\text{loc}}\left(\mathbb{R}^N\right), \ \text{ for } 1 \leq p < \infty : \ Au \in C_b\left(\mathbb{R}^N\right)\right\}.$$

Then, we consider for any $\lambda > 0$ and $f \in C_b(\mathbb{R}^N)$ the elliptic equation

$$\lambda u - Au = f. \tag{2.1}$$

It is well-known that equation (2.1) admits at least one solution in $D_{\max}(A)$ (see [3, Theorem 2.1.1]). A solution is obtained as follows.

Take the unique solution to the Dirichlet problem associated with $\lambda - A$ into the balls $B(0, n)$ for $n \in \mathbb{N}$. Using Schauder interior estimates one can prove that the sequence of solutions so obtained converges to a solution u of (2.1). It is also known that a solution to (2.1) is in general not unique. The solution u, which we obtained by approximation, is nonnegative whenever $f \geq 0$.

As regards the parabolic problem

$$\begin{cases} u_t(t, x) = Au(t, x) & \text{for } x \in \mathbb{R}^N \text{ and } t > 0 \\ u(0, x) = f(x) & \text{for } x \in \mathbb{R}^N\ , \end{cases} \tag{2.2}$$

where $f \in C_b(\mathbb{R}^N)$, it is well-known that one can find a semigroup $(T(t))_{t\geq 0}$ of bounded operator in $C_b(\mathbb{R}^N)$ such that $u(t, x) = T(t)f(x)$ is a solution of (2.2) in the following sense:

$$u \in C\left([0, +\infty) \times \mathbb{R}^N\right) \cap C^{1+\frac{\sigma}{2}, 2+\sigma}_{\text{loc}}\left((0, +\infty) \times \mathbb{R}^N\right)$$

and u solves (2.2) for any $f \in C_b(\mathbb{R}^N)$ and some $\sigma \in (0, 1)$. Uniqueness of solutions to (2.2) in general is not guaranteed. Moreover the semigroup $(T(t))_{t\geq 0}$

is not strongly continuous in $C_b(\mathbb{R}^N)$ and does not preserve in general the space $C_0(\mathbb{R}^N)$. We note here that the obtained solution u is the minimal solution among all positive solutions of (2.2). For this reason the semigroup $T(t)$ will be called the minimal semigroup. For more details we refer to [3, Chapter 2, Section 2].

Uniqueness is obtained if there exists a positive function $\varphi(x) \in C^2(\mathbb{R}^N)$, called *Lyapunov function*, such that $\lim_{|x|\to\infty} \varphi(x) = +\infty$ and $A\varphi - \lambda\varphi \le 0$ for some $\lambda > 0$.

Proposition 2.1. *Let $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Let $\varphi = 1 + |x|^\gamma$ where $\gamma > 2$. Then there exists a constant $C > 0$ such that*

$$A\varphi \le C\varphi.$$

Proof. An easy computation gives

$$A\varphi = \gamma(N+\gamma-2)(1+|x|^\alpha)|x|^{\gamma-2} - (1+|x|^\gamma)|x|^\beta .$$

Then, since $\beta > \alpha - 2$, there exists a $C > 0$ such that

$$\gamma(N+\gamma-2)(1+|x|^\alpha)|x|^{\gamma-2} \le (1+|x|^\gamma)|x|^\beta + C(1+|x|^\gamma). \qquad \square$$

Then we can assert that problem (2.2) admits a unique solution in $C([0,\infty)\times\mathbb{R}^N) \cap C^{1,2}((0,\infty)\times\mathbb{R}^N)$ and problem (2.1) admits a unique solution in $D_{\max}(A)$.

In order to investigate the compactness of the semigroup and the invariance of $C_0(\mathbb{R}^N)$ we check the behaviour of $T(t)\mathbf{1}$. We use the following result (see [3, Theorem 5.1.11]):

Theorem 2.2. *Let us fix $t > 0$. Then $T(t)\mathbf{1} \in C_0(\mathbb{R}^N)$ if and only if $T(t)$ is compact and $C_0(\mathbb{R}^N)$ is invariant under $T(t)$.*

Let A_0 be the operator defined by $A_0 := a(x)\Delta$. By [6, Example 7.3] or [8, Proposition 2.2 (iii)], we have that the minimal semigroup $(S(t))$ is generated by $(A_0, D_{\max}(A_0)\cap C_0(\mathbb{R}^N))$. Moreover the resolvent and the semigroup map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and are compact.

Set $v(t,x) = S(t)f(x)$ and $u(t,x) = T(t)f(x)$ for $t > 0$, $x \in \mathbb{R}^N$ and $0 \le f \in C_b(\mathbb{R}^N)$. Then the function $w(t,x) = v(t,x) - u(t,x)$ solves

$$\begin{cases} w_t(t,x) = A_0 w(t,x) + V(x)u(t,x) & \text{for } t > 0 \\ w(0,x) = 0 & \text{for } x \in \mathbb{R}^N. \end{cases}$$

So, applying [3, Theorem 4.1.3], we have $w \ge 0$ and hence $T(t) \le S(t)$. Thus, $T(t)\mathbf{1} \in C_0(\mathbb{R}^N)$, since $S(t)\mathbf{1} \in C_0(\mathbb{R}^N)$ for any $t > 0$ (see [8, Proposition 2.2 (iii)]). Thus, $T(t)$ is compact and $C_0(\mathbb{R}^N)$ is invariant under $T(t)$ (*cf.* [3, Theorem 5.1.11]). Then we have proved the following proposition:

Proposition 2.3. *The semigroup $(T(t))$ is generated by $(A, D_{\max}(A) \cap C_0(\mathbb{R}^N))$, maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and is compact.*

3. Solvability of the elliptic problem in $L^p(\mathbb{R}^N)$

In this section we study the existence and uniqueness of solutions of the elliptic problem $\lambda u - A_p u = f$ for a given $f \in L^p(\mathbb{R}^N)$, where $1 < p < \infty$ and $\lambda \geq 0$. Let us consider first the case $\lambda = 0$.

We note that the equation $(1+|x|^\alpha)\Delta u - Vu = f$ is equivalent to the equation

$$\Delta u - \frac{V}{1+|x|^\alpha} u = \frac{f}{1+|x|^\alpha} =: \tilde{f}.$$

Therefore we focus our attention to the L^p-realization $\tilde{A}_p$ of the Schrödinger operator

$$\tilde{A} = \Delta - \frac{V}{1+|x|^\alpha} = \Delta - \tilde{V} .$$

Let us denote by G the Green function (or the fundamental solution) for $\tilde{A}$, *i.e.*,

$$u(x) = \int_{\mathbb{R}^N} G(x,y)\tilde{f}(y)dy. \tag{3.1}$$

Thus, $u(x) = \int_{\mathbb{R}^N} G(x,y)\frac{f(y)}{1+|y|^\alpha}dy$ solves $Au = f$ for every $f \in L^p(\mathbb{R}^N)$. So we have to study the operator

$$u(x) = Lf(x) := \int_{\mathbb{R}^N} G(x,y)\frac{f(y)}{1+|y|^\alpha}dy. \tag{3.2}$$

To this purpose, we use the bounds of $G(x,y)$ obtained in [13] when the potential of $\tilde{A}_p$ belongs to the reverse Hölder class B_q for some $q \geq N/2$.

We recall that a nonnegative locally L^q-integrable function V on $\mathbb{R}^N$ is said to be in B_q, for $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right)$$

holds for every ball B in $\mathbb{R}^N$. A nonnegative function $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ is in B_∞ if

$$\|V\|_{L^\infty(B)} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right)$$

for any ball B in $\mathbb{R}^N$.

One can verify that

$$\tilde{V} \in \begin{cases} B_\infty & \text{if } \beta - \alpha \geq 0 \\ B_q & \text{if } \beta - \alpha > -\frac{N}{q} \\ B_{\frac{N}{2}} & \text{if } \beta - \alpha > -2 \\ B_N & \text{if } \beta - \alpha > -1 \end{cases} \tag{3.3}$$

for some $q > 1$. So, it follows from [13, Theorem 2.7] that, if $\beta - \alpha > -2$, then for any $k > 0$ there is some constant $C_k > 0$ such that, for any $x,\ y \in \mathbb{R}^N$,

$$|G(x,y)| \le \frac{C_k}{(1+m(x)|x-y|)^k} \cdot \frac{1}{|x-y|^{N-2}}, \tag{3.4}$$

where the function m is defined by

$$\frac{1}{m(x)} := \sup_{r>0} \left\{ r \ :\ \frac{1}{r^{N-2}} \int_{B(x,r)} \tilde{V}(y)dy \le 1 \right\}, \quad \text{for } x \in R^N. \tag{3.5}$$

Due to the importance of the auxiliary function m, we establish for it a lower bound:

Lemma 3.1. *Let $\alpha - 2 < \beta < \alpha$. There exists $C = C(\alpha, \beta, N)$ such that*

$$m(x) \ge C\,(1+|x|)^{\frac{\beta-\alpha}{2}}. \tag{3.6}$$

Proof. Fix $x \in \mathbb{R}^N$, and set $f_x(r) = \frac{1}{r^{N-2}} \int_{B(x,r)} \tilde{V}(y)dy,\ r > 0$. Since $\tilde{V} \in B_{N/2}$ implies $V \in B_q$ for some $q > \frac{N}{2}$, by [13, Lemma 1.2], we have

$$\lim_{r\to 0} f_x(r) = 0 \ \text{ and } \ \lim_{r\to\infty} f_x(r) = \infty.$$

Thus, $0 < m(x) < \infty$.

In order to estimate $\frac{1}{m(x)}$ we need to find $r_0 = r_0(x)$ such that $r \in [r_0, \infty[$ implies $f_x(r) \ge 1$. In this case we will have $\frac{1}{m(x)} \le r_0$.

Since $\tilde{V} \in B_{N/2}$, there exists a constant C_1 depending only α, β, N such that

$$\left(\frac{1}{|B|}\int_B \tilde{V}^{N/2}(y)dy\right)^{2/N} \le C_1 \left(\frac{1}{|B|}\int_B \tilde{V}(y)\,dy\right)$$

for any ball B in $\mathbb{R}^N$. Then we have

$$\begin{aligned} f_x(r) &= N^{-1}\sigma_N r^2 \frac{1}{|B(x,r)|}\int_{B(x,r)} \tilde{V}(y)dy \\ &\ge \frac{N^{-1}\sigma_N r^2}{C_1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \tilde{V}(y)^{N/2}dy\right)^{2/N} \\ &= \frac{(N^{-1}\sigma_N)^{1-2/N}}{C_1}\left(\int_{B(x,r)} \tilde{V}(y)^{N/2}dy\right)^{2/N}, \end{aligned}$$

where σ_N is the $(N-1)$-dimensional measure of $\partial B(0,1)$. Hence, if

$$\int_{B(x,r)} \tilde{V}(y)^{N/2}dy - C_2 \ge 0, \tag{3.7}$$

then $f_x(r) \geq 1$, where $C_2 = C_2(\alpha, \beta, N) = \frac{C_1^{N/2}}{(N^{-1}\sigma_N)^{N/2-1}}$. Note that $\tilde{V} \geq \tilde{V}^*$ in $\mathbb{R}^N \setminus B(0,1)$ with $\tilde{V}^*(x) = \frac{1}{2}|x|^{\beta-\alpha}$. Hence,

$$\begin{aligned}
\int_{B(x,r)} \tilde{V}(y)^{N/2} dy &\geq \int_{B(x,r)\setminus B(0,1)} \tilde{V}(y)^{N/2} dy \geq \int_{B(x,r)\setminus B(0,1)} \tilde{V}^*(y)^{N/2} dy \\
&= \int_{B(x,r)} \tilde{V}^*(y)^{N/2} dy - \int_{B(x,r)\cap B(0,1)} \tilde{V}^*(y)^{N/2} dy \\
&\geq \int_{B(x,r)} \tilde{V}^*(y)^{N/2} dy - \int_{B(0,1)} \tilde{V}^*(y)^{N/2} dy \\
&= \int_{B(x,r)} \tilde{V}^*(y)^{N/2} dy - \frac{2^{1-N/2}\sigma_N}{N(2-\alpha+\beta)} \\
&\geq N^{-1}\sigma_N r^N \inf_{B(x,r)} (\tilde{V}^*)^{N/2} - C_3(\alpha, \beta, N) \qquad (3.8) \\
&= N^{-1}\sigma_N \frac{2^{-N/2} r^N}{(|x|+r)^{\frac{\alpha-\beta}{2}N}} - C_3(\alpha, \beta, N). \qquad (3.9)
\end{aligned}$$

Let $\eta = \frac{\alpha-\beta}{2} < 1$, let $\delta > 0$ be a parameter to be chosen later, and set

$$r_0 = \delta(1+|x|)^\eta .$$

By (3.8) condition (3.7) becomes

$$\begin{aligned}
\int_{B(x,r_0)} \tilde{V}(y)^{N/2} dy - C_2 &\geq N^{-1}\sigma_N \frac{2^{-N/2} r_0^N}{(|x|+r_0)^{\frac{\alpha-\beta}{2}N}} - C_2 - C_3 \\
&= N^{-1} 2^{-N/2} \sigma_N \frac{\delta^N (1+|x|)^{\eta N}}{(|x| + \delta(1+|x|)^\eta)^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&\geq N^{-1} 2^{-N/2} \sigma_N \frac{\delta^N (1+|x|)^{\eta N}}{(1+|x| + \delta(1+|x|)^\eta)^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&\geq N^{-1} 2^{-N/2} \sigma_N \frac{\delta^N (1+|x|)^{\eta N}}{((\delta+1)(1+|x|))^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&= N^{-1} 2^{-N/2} \sigma_N \left(\frac{\delta}{(1+\delta)^{\frac{\alpha-\beta}{2}}} \right)^N - C_4 .
\end{aligned}$$

Since $\frac{\alpha-\beta}{2} < 1$ we can choose $\delta > 0$ such that $N^{-1}2^{-N/2}\sigma_N \left(\frac{\delta}{(1+\delta)^{\frac{\alpha-\beta}{2}}} \right)^N - C_4 \geq 0$.

So, (3.7) is satisfied for $r = r_0$ and hence it is satisfied for any $r > r_0$. Thus, $f_x(r) \geq 1$ for $r > r_0$, and, hence, $\frac{1}{m(x)} \leq r_0 = \delta(1+|x|)^\eta$. □

The same lower bound holds in the case $\beta \geq \alpha$ as the following lemma shows:

Lemma 3.2. *Let $\beta \geq \alpha$. There exists $C = C(\alpha, \beta, N)$ such that*

$$m(x) \geq C\,(1+|x|)^{\frac{\beta-\alpha}{2}} . \tag{3.10}$$

Proof. From [13, Lemma 1.4 (c)], there exist $C_1 > 0$ and $0 < \eta_0 < 1$ such that, for $x, y \in \mathbb{R}^N$,

$$m(x) \geq \frac{C_1 m(y)}{(1+|x-y|m(y))^{\eta_0}} .$$

In particular,

$$m(x) \geq \frac{C_1 m(0)}{(1+|x|m(0))^{\eta_0}},$$

where $\frac{1}{m(0)} = \sup_{r>0}\{r \,:\, f_0(r) \leq 1\}$ with

$$f_0(r) = \frac{1}{r^{N-2}} \int_{B(0,r)} \frac{|z|^\beta}{1+|z|^\alpha} dz = \frac{\sigma_N}{r^{N-2}} \int_0^r \frac{\rho^{\beta+N-1}}{1+\rho^\alpha} d\rho .$$

We have $\frac{\sigma_N}{(\beta+N)(1+r^\alpha)} r^{\beta+2} \leq f_0(r) \leq \frac{\sigma_N}{\beta+N} r^{\beta+2}$. Since $\beta > 0$ and $\beta - \alpha + 2 > 0$ it follows that $\lim_{r\to 0} f_0(r) = 0$ and $\lim_{r\to\infty} f_0(r) = \infty$. Consequently,

$$0 < \sup_{r>0}\{r \,:\, f_0(r) \leq 1\} < \infty$$

and, hence, $m(0) = C_2$ for some constant $C_2 > 0$. Then

$$m(x) \geq \frac{C_1 C_2}{(1+C_2|x|)^{\eta_0}} \geq \frac{C_3}{(1+|x|)^{\eta_0}} \tag{3.11}$$

for some constant $C_3 > 0$.

On the other hand, since $\beta \geq \alpha$, we obtain by (3.3) that $\tilde{V} \in B_\infty$. Then, by [13, Remark 2.9], we have

$$m(x) \geq C_5 \tilde{V}^{1/2}(x) = C_5 |x|^{\frac{\beta}{2}} (1+|x|)^{-\frac{\alpha}{2}} . \tag{3.12}$$

The thesis follows taking into account (3.11) and (3.12). □

Applying the estimate (3.4) and the previous lemma we obtain the following upper bounds for the Green function G:

Lemma 3.3. *Let $G(x, y)$ denote the Green function of the Schrödinger operator $\Delta - \frac{|x|^\beta}{1+|x|^\alpha}$ and assume that $\beta > \alpha - 2$. Then,*

$$G(x, y) \leq C_k \frac{1}{1+|x-y|^k\,(1+|y|)^{\frac{\beta-\alpha}{2}k}} \frac{1}{|x-y|^{N-2}}, \quad \textit{for } x, y \in \mathbb{R}^N, \tag{3.13}$$

for any $k > 0$ and some constant $C_k > 0$ depending on k.

Using the above lemma we have the following estimate:

Lemma 3.4. *Assume that* $\alpha > 2$*,* $N > 2$ *and* $\beta > \alpha - 2$*. Then there exists a positive constant* C *such that for every* $0 \le \gamma \le \beta$ *and* $f \in L^p(\mathbb{R}^N)$

$$\||x|^\gamma Lf\|_p \le C\|f\|_p, \tag{3.14}$$

where L *is defined in* (3.2).

Proof. Let $\Gamma(x,y) = \frac{G(x,y)}{1+|y|^\alpha}$, $f \in L^p(\mathbb{R}^N)$ and

$$u(x) = \int_{\mathbb{R}^N} \Gamma(x,y) f(y) dy.$$

We have to show that

$$\||x|^\gamma u\|_p \le C\|f\|_p.$$

Let us consider the regions $E_1 := \{|x-y| \le (1+|y|)\}$ and $E_2 := \{|x-y| > (1+|y|)\}$ and write

$$u(x) = \int_{E_1} \Gamma(x,y) f(y) dy + \int_{E_2} \Gamma(x,y) f(y) dy =: u_1(x) + u_2(x)\ .$$

In E_1 we have

$$\frac{1+|x|}{1+|y|} \le \frac{1+|x-y|+|y|}{1+|y|} \le 2.$$

So, by Lemma 3.2

$$\begin{aligned} \big||x|^\gamma u_1(x)\big| &\le |x|^\gamma \int_{E_1} \Gamma(x,y)|f(y)|dy \le \frac{1+|x|^\beta}{1+|x|^\alpha} \int_{E_1} \frac{1+|x|^\alpha}{1+|y|^\alpha} G(x,y)|f(y)|dy \\ &\le C(1+|x|)^{\beta-\alpha} \int_{\mathbb{R}^N} G(x,y)|f(y)|dy \le Cm^2(x)\tilde{u}(x), \end{aligned}$$

where $\tilde{u}(x) = \int_{\mathbb{R}^N} G(x,y)|f(y)|dy$. By (3.3) we have $\tilde{V} \in B_{\frac{N}{2}}$. So, applying [13, Corollary 2.8], we obtain $\|m^2\tilde{u}\|_p \le C\|f\|_p$ and then $\||x|^\gamma u_1\|_p \le C\|f\|_p$.

In the region E_2, we have, by Hölder's inequality,

$$\begin{aligned} \big||x|^\gamma u_2(x)\big| &\le |x|^\gamma \int_{E_2} \Gamma(x,y)|f(y)|dy \\ &= \int_{E_2} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p'}} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p}} |f(y)|dy \\ &\le \left(\int_{E_2} |x|^\gamma \Gamma(x,y) dy\right)^{\frac{1}{p'}} \left(\int_{E_2} |x|^\gamma \Gamma(x,y)|f(y)|^p dy\right)^{\frac{1}{p}}\ . \end{aligned} \tag{3.15}$$

We propose to estimate first $\int_{E_2} |x|^\gamma \Gamma(x, y)dy$. In E_2 we have $1 + |x| \leq 1 + |y| + |x - y| \leq 2|x - y|$, then from (3.13) it follows that

$$\begin{aligned} |x|^\gamma \Gamma(x, y) &\leq |x|^\gamma G(x, y) \\ &\leq C \frac{1 + |x|^\beta}{|x - y|^k (1 + |y|)^{k\frac{\beta-\alpha}{2}}} \frac{1}{|x - y|^{N-2}} \\ &\leq C \frac{1}{|x - y|^{k-\beta+N-2}} \frac{1}{(1 + |y|)^{k\frac{\beta-\alpha}{2}}} \ . \end{aligned}$$

For every $k > \beta - N + 2$, taking into account that $\frac{1}{|x-y|} < \frac{1}{1+|y|}$, we get

$$|x|^\gamma \Gamma(x, y) \leq \frac{1}{(1 + |y|)^{k\frac{\beta-\alpha+2}{2}+N-2-\beta}} \ .$$

Since $\beta - \alpha + 2 > 0$ we can choose k such that $\frac{k}{2}(\beta - \alpha + 2) + N - 2 - \beta > N$, then

$$\int_{E_2} |x|^\gamma \Gamma(x, y)dy \leq \int_{E_2} |x|^\gamma G(x, y)dy \leq C \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{\frac{k}{2}(2+\beta-\alpha)+N-2-\beta}} dy < C \ .$$

Moreover by the symmetry of G we have

$$\begin{aligned} |x|^\gamma \Gamma(x, y) &\leq |x|^\gamma G(x, y) \\ &\leq C \frac{1 + |x|^\beta}{|x - y|^k (1 + |x|)^{k\frac{\beta-\alpha}{2}}} \frac{1}{|x - y|^{N-2}} \\ &\leq C \frac{1}{|x - y|^{k-\beta+N-2}} \frac{1}{(1 + |x|)^{k\frac{\beta-\alpha}{2}}} \ . \end{aligned}$$

Taking into account that $\frac{1}{|x-y|} \leq 2\frac{1}{1+|x|}$, arguing as above we obtain

$$\int_{E_2} |x|^\gamma \Gamma(x, y)dx \leq C. \tag{3.16}$$

Hence (3.15) implies

$$\left||x|^\gamma u_2(x)\right|^p \leq C \int_{E_2} |x|^\gamma \Gamma(x, y)|f(y)|^p dy. \tag{3.17}$$

Thus, by (3.17) and (3.16), we have

$$\begin{aligned} \||x|^\gamma u_2\|_p^p &\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x|^\gamma \Gamma(x, y) \chi_{\{|x-y|>1+|y|\}}(x, y)|f(y)|^p dydx \\ &= C \int_{\mathbb{R}^N} |f(y)|^p \left(\int_{E_2} |x|^\gamma \Gamma(x, y)dx \right) dy \leq C\|f\|_p^p \ . \end{aligned}$$

□

We are now ready to show the invertibility of A_p and $D_{p,\max}(A) \subset D(V)$:

Proposition 3.5. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the operator A_p is closed and invertible. Moreover there exists $C > 0$ such that, for every $0 \leq \gamma \leq \beta$, we have*

$$\| |\cdot|^\gamma u\|_p \leq C\|A_p u\|_p, \quad \forall u \in D_{p,\max}(A) \,. \tag{3.18}$$

Proof. Let us first prove the injectivity of A_p. Let $u \in D_{p,\max}(A)$ such that $A_p u = 0$, in particular $\tilde{A}_p u = 0$. It follows that $u \in D_{p,\max}(\tilde{A}) = D(\Delta) \cap D\left(\frac{|x|^\beta}{1+|x|^\alpha}\right)$, (see [11]). Then multiplying $A_p u$ by $u|u|^{p-2}$ and integrating over $\mathbb{R}^N$ we obtain, by [7],

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} u|u|^{p-2}\Delta u\, dx - \int_{\mathbb{R}^N} \frac{|x|^\beta}{1+|x|^\alpha}|u|^p dx \\ &= -(p-1)\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 dx - \int_{\mathbb{R}^N} \frac{|x|^\beta}{1+|x|^\alpha}|u|^p dx, \end{aligned}$$

from which we have $u \equiv 0$. On the other hand, we recall that the function given by (3.2) solves $Au = f$ for every $f \in L^p(\mathbb{R}^N)$. Applying Lemma 3.4 with $\gamma = 0$, we deduce that $u \in L^p(\mathbb{R}^N)$ and so by elliptic regularity we have $u \in D_{p,\max}(A)$. This, together with the injectivity of A_p gives the invertibility of A_p and $A_p^{-1} \in \mathcal{L}(L^p(\mathbb{R}^N))$. This implies in particular that A_p is closed. Finally, the estimate (3.18) follows from (3.14). □

The previous theorem gives in particular the A_p-boundedness of the potential V and the following regularity result:

Corollary 3.6. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then*:

(i) *there exists $C > 0$ such that for every $u \in D_{p,\max}(A)$*

$$\|(1+V)u\|_p \leq C\|A_p u\|_p;$$

(ii)

$$D_{p,\max}(A) = \left\{u \in W^{2,p}\left(\mathbb{R}^N\right) \mid Au \in L^p\left(\mathbb{R}^N\right)\right\}.$$

Proof. We have only to prove the inclusion $D_{p,\max}(A) \subset \{u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N)\}$. Let $u \in D_{p,\max}(A)$. Then, by (i), $Vu \in L^p(\mathbb{R}^N)$ and hence

$$\Delta u = \frac{Au + Vu}{1+|x|^\alpha} \in L^p\left(\mathbb{R}^N\right).$$

So, the thesis follows from the Calderon-Zygmund inequality. □

We can now state the main result of this section:

Theorem 3.7. *Assume that* $N > 2$, $\beta > \alpha - 2$ *and* $\alpha > 2$. *Then,* $[0, +\infty) \subset \rho(A_p)$ *and* $(\lambda - A_p)^{-1}$ *is a positive operator on* $L^p(\mathbb{R}^N)$ *for any* $\lambda \geq 0$. *Moreover, if* $f \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, *then* $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$.

Proof. Let us first prove that if $0 \leq \lambda \in \rho(A_p)$, then $(\lambda - A_p)^{-1}$ is a positive operator on $L^p(\mathbb{R}^N)$. To this purpose, take $0 \leq f \in L^p(\mathbb{R}^N)$ and set $u = (\lambda - A_p)^{-1} f$. Then, by Corollary 3.6, $u \in D(\tilde{A}_p)$ and

$$-\left(\tilde{A}_p - \lambda q\right) u = qf =: \tilde{f},$$

where $q(x) = \frac{1}{1+|x|^\alpha}$. Since $\tilde{A}_p$ generates an exponentially stable and positive C_0-semigroup $(\tilde{T}_p(t))_{t\geq 0}$ on $L^p(\mathbb{R}^N)$ (see [4, Theorem 2.5]), it follows that the semigroup $(e^{-t\lambda q}\tilde{T}_p(t))_{t\geq 0}$ generated by $\tilde{A}_p - \lambda q$ is positive and exponentially stable. Hence,

$$u = \left(\lambda q - \tilde{A}_p\right)^{-1} \tilde{f} \geq 0.$$

We show that $E = [0, +\infty) \cap \rho(A_p)$ is a non-empty open and closed set in $[0, +\infty)$. By Proposition 3.5 we have $0 \in \rho(A_p)$ and hence $E \neq \emptyset$. On the other hand, using the above positivity property and the resolvent equation we have $(\lambda - A_p)^{-1} \leq (-A_p)^{-1} = L$ for any $\lambda \in E$ and therefore

$$\left\|(\lambda - A_p)^{-1}\right\| \leq \|L\| \,. \tag{3.19}$$

It follows that the operator norm of $(\lambda - A_p)^{-1}$ is bounded in E and consequently E is closed. Finally, since $\rho(A_p)$ is an open set, it follows that E is open in $[0, +\infty)$. Thus, $E = [0, +\infty)$.

Now in order to show the last statement we may assume $f \in C_c^\infty$, the thesis will follow by density. Setting $u := (\lambda - A_p)^{-1} f$, we obtain, by local elliptic regularity (*cf.* [2, Theorem 9.19]), that $u \in C^{2+\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \sigma < 1$. On the other hand, $u \in W^{2,p}(\mathbb{R}^N)$, by Corollary 3.6. If $p \geq \frac{N}{2}$, then by the Sobolev's inequality, $u \in L^q(\mathbb{R}^N)$ for all $q \in [p, +\infty)$. In particular, $u \in L^q(\mathbb{R}^N)$ for some $q > \frac{N}{2}$ and hence $Au = -f + \lambda u \in L^q(\mathbb{R}^N)$. Moreover, since $u \in C^{2+\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$. So, $u \in D_{q,\max}(A) \subset W^{2,q}(\mathbb{R}^N) \subset C_b(\mathbb{R}^N)$, by Corollary 3.6 and Sobolev's embedding theorem, since $q > \frac{N}{2}$.

Let us now suppose that $p < \frac{N}{2}$. Take the sequence (r_n), defined by $r_n = 1/p - 2n/N$ for any $n \in \mathbb{N}$, and set $q_n = 1/r_n$ for any $n \in \mathbb{N}$. Let n_0 be the smallest integer such that $r_{n_0} \leq 2/N$ noting that $r_{n_0} > 0$. Then, $u \in D_{p,\max}(A) \subset L^{q_1}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, by Sobolev's embedding theorem. As above we obtain that $u \in D_{q_1,\max}(A) \subset L^{q_2}(\mathbb{R}^N)$. Iterating this argument, we deduce that $u \in D_{q_{n_0},\max}(A)$.

So we can conclude that $u \in C_b(\mathbb{R}^N)$ arguing as in the previous case. Thus, $Au = -f + \lambda u \in C_b(\mathbb{R}^N)$. Again, since $u \in C^{2+\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $q \in (1, +\infty)$. Hence, $u \in D_{\max}(A)$. So, by the uniqueness of the solution of the elliptic problem, we have $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$ for any $f \in C_c^\infty(\mathbb{R}^N)$. □

4. Generation of semigroups

In this section we show that A_p generates an analytic semigroup on $L^p(\mathbb{R}^N)$, for $1 < p < \infty$, provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$.

We start by giving the characterization of the domain of A. More precisely we prove that the maximal domain $D_{p,\max}(A)$ coincides with the weighted Sobolev space $D_p(A)$ defined by

$$D_p(A) := \left\{ u \in W^{2,p}\left(\mathbb{R}^N\right) : Vu, \left(1 + |x|^{\alpha-1}\right) \nabla u, \left(1 + |x|^\alpha\right) D^2 u \in L^p\left(\mathbb{R}^N\right) \right\}$$

endowed with its canonical norm.

To this purpose we need the following covering result, see [1, Proposition 6.1], to prove a weighted gradient estimate:

Proposition 4.1. *For every* $0 \le k < 1/2$ *there exists a natural number* $\zeta = \zeta(N, k)$ *with the following property: given* $\mathcal{F} = \{B(x, \rho(x))\}_{x \in \mathbb{R}^N}$, *where* $\rho : \mathbb{R}^N \to \mathbb{R}_+$ *is a Lipschitz continuous function with Lipschitz constant* k, *there exists a countable subcovering* $\{B(x_n, \rho(x_n))\}_{n \in \mathbb{N}}$ *of* $\mathbb{R}^N$ *such that at most* ζ *among the double balls* $\{B(x_n, 2\rho(x_n))\}_{n \in \mathbb{N}}$ *overlap.*

We need the following weighted gradient and second derivative estimate:

Lemma 4.2. *Assume that* $N > 2$, $\alpha > 2$ *and* $\beta > \alpha - 2$. *Then there exists a constant* $C > 0$ *such that for every* $u \in D_p(A)$ *we have*

$$\left\| \left(1 + |x|^{\alpha-1}\right) \nabla u \right\|_p \le C \|A_p u\|_p \,, \tag{4.1}$$

$$\left\| \left(1 + |x|^{\alpha}\right) D^2 u \right\|_p \le C \|A_p u\|_p \,. \tag{4.2}$$

Proof. Let $u \in D_p(A)$. We fix $x_0 \in \mathbb{R}^n$ and choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B(1)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B(2)$. Moreover, we set $\vartheta_\rho(x) = \vartheta\left(\frac{x - x_0}{\rho}\right)$, where $\rho = \frac{1}{4}(1 + |x_0|)$. We apply the well-known inequality

$$\begin{gathered} \|\nabla v\|_{L^p(B(R))} \le C \|v\|^{1/2}_{L^p(B(R))} \|\Delta v\|^{1/2}_{L^p(B(R))}, \\ \text{where } v \in W^{2,p}(B(R)) \cap W^{1,p}_0(B(R)) \quad \text{and } R > 0, \end{gathered} \tag{4.3}$$

to the function $\vartheta_\rho u$ and obtain, for every $\varepsilon > 0$,

$$\begin{aligned}
&\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \le \|(1+|x_0|)^{\alpha-1}\nabla(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}\\
&\le C\left\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\right\|^{\frac12}_{L^p(B(x_0,2\rho))}\left\|(1+|x_0|)^{\alpha-2}\vartheta_\rho u\right\|^{\frac12}_{L^p(B(x_0,2\rho))}\\
&\le C\left(\varepsilon\left\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\right\|_{L^p(B(x_0,2\rho))}+\frac{1}{4\varepsilon}\left\|(1+|x_0|)^{\alpha-2}\vartheta_\rho u\right\|_{L^p(B(x_0,2\rho))}\right)\\
&\le C\left(\varepsilon\left\|(1+|x_0|)^{\alpha}\Delta u\right\|_{L^p(B(x_0,2\rho))}+\frac{2M}{\rho}\varepsilon\left\|(1+|x_0|)^{\alpha}\nabla u\right\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{\varepsilon M}{\rho^2}\left\|(1+|x_0|)^{\alpha}u\right\|_{L^p(B(x_0,2\rho))}+\frac{1}{4\varepsilon}\left\|(1+|x_0|)^{\alpha-2}u\right\|_{L^p(B(x_0,2\rho))}\right)\\
&\le C\left(\varepsilon\left\|(1+|x_0|)^{\alpha}\Delta u\right\|_{L^p(B(x_0,2\rho))}+8M\varepsilon\left\|(1+|x_0|)^{\alpha-1}\nabla u\right\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\left(16\varepsilon M+\frac{1}{4\varepsilon}\right)\left\|(1+|x_0|)^{\alpha-2}u\right\|_{L^p(B(x_0,2\rho))}\right)\\
&\le C(M)\left(\varepsilon\left\|(1+|x_0|)^{\alpha}\Delta u\right\|_{L^p(B(x_0,2\rho))}+\varepsilon\left\|(1+|x_0|)^{\alpha-1}\nabla u\right\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{1}{\varepsilon}\left\|(1+|x_0|)^{\alpha-2}u\right\|_{L^p(B(x_0,2\rho))}\right),
\end{aligned}$$

where $M = \|\nabla\vartheta\|_\infty + \|\Delta\vartheta\|_\infty$. Since $2\rho = \frac12(1+|x_0|)$ we get

$$\frac12(1+|x_0|) \le 1+|x| \le \frac32(1+|x_0|), \quad \text{for } x \in B(x_0, 2\rho).$$

Thus,

$$\begin{aligned}
&\left\|(1+|x|)^{\alpha-1}\nabla u\right\|_{L^p(B(x_0,\rho))} \le \left(\frac32\right)^{\alpha-1}\left\|(1+|x_0|)^{\alpha-1}\nabla u\right\|_{L^p(B(x_0,\rho))}\\
&\le C\left(\varepsilon\left\|(1+|x_0|)^{\alpha}\Delta u\right\|_{L^p(B(x_0,2\rho))}+\varepsilon\left\|(1+|x_0|)^{\alpha-1}\nabla u\right\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{1}{\varepsilon}\left\|(1+|x_0|)^{\alpha-2}u\right\|_{L^p(B(x_0,2\rho))}\right) \qquad (4.4)\\
&\le C\left(2^\alpha\varepsilon\left\|(1+|x|)^{\alpha}\Delta u\right\|_{L^p(B(x_0,2\rho))}+2^{\alpha-1}\varepsilon\left\|(1+|x|)^{\alpha-1}\nabla u\right\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{2^{\alpha-2}}{\varepsilon}\left\|(1+|x|)^{\alpha-2}u\right\|_{L^p(B(x_0,2\rho))}\right).
\end{aligned}$$

Let $\{B(x_n,\rho(x_n))\}$ be a countable covering of $\mathbb{R}^N$ as in Proposition 4.1 such that at most ζ among the double balls $\{B(x_n, 2\rho(x_n))\}$ overlap.

We write (4.4) with x_0 replaced by x_n and sum over n. Taking into account the above covering result, we get

$$\left\|(1+|x|)^{\alpha-1}\nabla u\right\|_p \le C\Big(\varepsilon\left\|(1+|x|)^{\alpha}\Delta u\right\|_p + \varepsilon\left\|(1+|x|)^{\alpha-1}\nabla u\right\|_p + \frac{1}{\varepsilon}\left\|(1+|x|)^{\alpha-2}u\right\|_p\Big).$$

Choosing ε such that $\varepsilon C < 1/2$ we have

$$\frac{1}{2}\left\|(1+|x|)^{\alpha-1}\nabla u\right\|_p \le \frac{1}{2}\left\|(1+|x|)^{\alpha}\Delta u\right\|_p + \frac{C}{\varepsilon}\left\|(1+|x|)^{\alpha-2}u\right\|_p .$$

Furthermore $\||x|^{\alpha-2}u\|_p \le \|(1+|x|^{\beta})u\|_p \le C\|A_p u\|_p$ for any $u \in D_p(A) \subset D_{p,\max}(A)$ and some $C > 0$ by Corollary 3.6. Hence,

$$\left\|(1+|x|)^{\alpha-1}\nabla u\right\|_p \le C\left(\|A_p u\|_p + \|u\|_p\right).$$

As regards the second order derivatives we consider the classical Calderón-Zygmund inequality on $B(1)$

$$\left\|D^2 v\right\|_{L^p(B(1))} \le C\|\Delta v\|_{L^p(B(1))}, \quad v \in W^{2,p}(B(1)) \cap W_0^{1,p}(B(1)) ,$$

by rescaling and translating we get

$$\left\|D^2 v\right\|_{L^p(B(x_0,R))} \le C\|\Delta v\|_{L^p(B(x_0,R))} \tag{4.5}$$

for every $x_0 \in \mathbb{R}^N$, $R > 0$ and $v \in W^{2,p}(B(x_0,R)) \cap W_0^{1,p}(B(x_0,R))$. We observe that the constant C does not depend on R and x_0.

Then we fix $x_0 \in \mathbb{R}^n$ and choose ρ and $\vartheta_\rho \in C_c^\infty(\mathbb{R}^N)$ as above. Applying (4.5) to the function $\vartheta_\rho u$ in $B(x_0, 2\rho)$, we obtain

$$\begin{aligned}\left\|(1+|x_0|)^{\alpha}D^2 u\right\|_{L^p(B(x_0,\rho))} &\le \left\|(1+|x_0|)^{\alpha}D^2(\vartheta_\rho u)\right\|_{L^p(B(x_0,2\rho))}\\ &\le C\left\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\right\|_{L^p(B(x_0,2\rho))}.\end{aligned}$$

Reasoning as above we obtain

$$\begin{aligned}&\left\|(1+|x|)^{\alpha}D^2 u\right\|_p\\ &\le C\left(\left\|(1+|x|)^{\alpha}\Delta u\right\|_p + \left\|(1+|x|)^{\alpha-1}\nabla u\right\|_p + \left\|(1+|x|)^{\alpha-2}u\right\|_p\right).\end{aligned}$$

The lemma follows by Corollary 3.6 and by the gradient estimate (4.1). □

The following lemma shows that $C_c^\infty(\mathbb{R}^N)$ is a core for $(A, D_p(A))$.

Lemma 4.3. *Assume $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. The space $C_c^\infty(\mathbb{R}^N)$ is dense in $D_p(A)$ with respect to the graph norm.*

Proof. Let us first observe that $C_c^\infty(\mathbb{R}^N)$ is dense in $W_c^{2,p}(\mathbb{R}^N)$ with respect to the operator norm. Let $u \in W_c^{2,p}(\mathbb{R}^N)$ and consider $u_n = \rho_n * u$, where ρ_n are standard mollifiers. We have $u_n \in C_c^\infty(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$ and $D^2 u_n \to D^2 u$ in $L^p(\mathbb{R}^N)$. Moreover, $\mathrm{supp}\, u_n \subset \mathrm{supp}\, u + B(1) := K$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\left\| A_p u - A u_n \right\|_p = \left\| A_p u - A u_n \right\|_{L^p(K)} \\
&\le \left\| \left(1 + |x|^\alpha\right) \Delta(u - u_n) \right\|_{L^p(K)} + \left\| |x|^\beta (u - u_n) \right\|_{L^p(K)} \\
&\le \left\| \left(1 + |x|^\alpha\right) \right\|_{L^\infty(K)} \|\Delta(u - u_n)\|_{L^p(K)} \\
&\quad + \left\| |x|^\beta \right\|_{L^\infty(K)} \|(u - u_n)\|_{L^p(K)} \to 0 \text{ as } n \to \infty \,.
\end{aligned}$$

Now, let u in $D_{p,\max}(A)$ and let η be a smooth function such that $\eta = 1$ in $B(1)$, $\eta = 0$ in $\mathbb{R}^N \setminus B(2)$, $0 \le \eta \le 1$ and set $\eta_n(x) = \eta\left(\frac{x}{n}\right)$. Then consider $u_n = \eta_n u \in W_c^{2,p}(\mathbb{R}^N)$. First we have $u_n \to u$ in $L^p(\mathbb{R}^N)$ by dominated convergence. As regard $A_p u_n$ we have

$$\begin{aligned}
A_p u_n(x) &= \left(1 + |x|^\alpha\right)\Delta(\eta_n u)(x) - |x|^\beta \eta_n(x) u(x) \\
&= \eta_n(x) A_p u(x) + 2\left(1 + |x|^\alpha\right)\nabla\eta_n(x)\nabla u(x) + \left(1 + |x|^\alpha\right)\Delta\eta_n(x) u(x) \\
&= \eta_n(x) A_p u(x) + \frac{2}{n}\left(1+|x|^\alpha\right)\nabla\eta\left(\frac{x}{n}\right)\nabla u(x) + \frac{1}{n^2}\left(1+|x|^\alpha\right)\Delta\eta\left(\frac{x}{n}\right) u(x)
\end{aligned}$$

and

$$\eta_n A_p u \to A_p u \qquad \text{in} \qquad L^p\left(\mathbb{R}^N\right)$$

by dominated convergence. As regards the last terms we note that $\nabla\eta(x/n)$ and $\Delta\eta(x/n)$ can be different from zero only for $n \le |x| \le 2n$, then we have

$$\frac{1}{n}\left(1 + |x|^\alpha\right) \left|\nabla\eta\left(\frac{x}{n}\right)\right| |\nabla u| \le C\left(1 + |x|^{\alpha-1}\right)|\nabla u| \chi_{\{n \le |x| \le 2n\}}$$

and

$$\frac{1}{n^2}\left(1 + |x|^\alpha\right) \left|\Delta\eta\left(\frac{x}{n}\right)\right| |u| \le C\left(1 + |x|^{\alpha-2}\right)|u| \chi_{\{n \le |x| \le 2n\}} \,.$$

The right-hand sides tend to 0 as $n \to \infty$, since by Proposition 3.5 and Lemma 4.2 we have $\|(1 + |x|^{\alpha-2})u\|_p \le C\|A_p u\|_p$ and $\|(1 + |x|^{\alpha-1})\nabla u\|_p \le C\|A_p u\|_p$. So, applying again the dominated convergence theorem, we obtain $A_p u_n \to A_p u$ in $L^p(\mathbb{R}^N)$. This ends the proof of the lemma. □

We can give now the complete characterization of $D_{p,\max}(A)$.

Theorem 4.4. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then maximal domain $D_{p,\max}(A)$ coincides with $D_p(A)$.*

Proof. We have to prove only the inclusion $D_{p,\max}(A) \subset D_p(A)$.

Let $\tilde{u} \in D_{p,\max}(A)$ and set $f = A\tilde{u}$. The operator A in $B(\rho)$, for $\rho > 0$, is an elliptic operator with bounded coefficients, then the problem

$$\begin{cases} Au = f & \text{in } B(\rho) \\ u = 0 & \text{on } \partial B(\rho)\ , \end{cases} \tag{4.6}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$ (*cf.* [2, Theorem 9.15]). Now $u_\rho \in D_p(A)$ and by Lemma 4.2 and Corollary 3.6 (i)

$$\begin{aligned} &\left\| \left(1 + |x|^{\alpha-2}\right) u_\rho \right\|_{L^p(B(\rho))} + \left\| \left(1 + |x|^{\alpha-1}\right) \nabla u_\rho \right\|_{L^p(B(\rho))} \\ &+ \left\| \left(1 + |x|^{\alpha}\right) D^2 u_\rho \right\|_{L^p(B(\rho))} + \|V u_\rho\|_{L^p(B(\rho))} \le C \|A u_\rho\|_p \end{aligned}$$

with C independent of ρ. Using a standard weak compactness argument we can construct a sequence u_{ρ_n} which converges to a function u in $W^{2,p}_{\text{loc}}$ such that $Au = f$. Since the estimates above are independent of ρ, also $u \in D_p(A)$. Then we have $A\tilde{u} = Au$ and since $D_p(A) \subset D_{p,\max}(A)$ and A is invertible on $D_{p,\max}(A)$ by Proposition 3.5, we have $\tilde{u} = u$. □

Let us give now the main result of this section:

Theorem 4.5. *Assume $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the operator A_p with domain $D_{p,\max}(A)$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$.*

Proof. Let $f \in L^p$, and $\rho > 0$. Consider the operator $\widetilde{A_p} := A_p - \omega$ where ω is a constant which will be chosen later. It is known that the elliptic problem in $L^p(B(\rho))$

$$\begin{cases} \lambda u - \widetilde{A_p} u = f & \text{in } B(\rho) \\ u = 0 & \text{on } \partial B(\rho)\ , \end{cases} \tag{4.7}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$ for $\lambda > 0$, (*cf.* [2, Theorem 9.15]).

Let us prove that that $e^{\pm i\theta}\widetilde{A_p}$ is dissipative in $B(\rho)$ for $0 \le \theta \le \theta_\alpha$ with suitable $\theta_\alpha \in (0, \frac{\pi}{2}]$. To this purpose observe that

$$\widetilde{A_p} u_\rho = \text{div}\left((1 + |x|^\alpha)\nabla u_\rho\right) - \alpha |x|^{\alpha-1} \frac{x}{|x|} \nabla u_\rho - |x|^\beta u_\rho - \omega u_\rho\ .$$

Set $u^\star = \overline{u}_\rho |u_\rho|^{p-2}$ and recall that $a(x) = 1 + |x|^\alpha$. Multiplying $\widetilde{A_p} u_\rho$ by $u^\star$ and integrating over $B(\rho)$, we obtain

$$\begin{aligned}\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx = & -\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Re}(\overline{u}_\rho \nabla u_\rho)\right|^2 dx \\ & -\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Im}(\overline{u}_\rho \nabla u_\rho)\right|^2 dx \\ & -(p-2)\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \overline{u}_\rho \nabla u_\rho \mathrm{Re}(\overline{u}_\rho \nabla u_\rho) dx \\ & -\alpha \int_{B(\rho)} \overline{u}_\rho \left|u_\rho\right|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \nabla u_\rho\, dx - \int_{B(\rho)} \left(|x|^\beta + \omega\right) |u_\rho|^p dx.\end{aligned}$$

We note here that the integration by part in the singular case $1 < p < 2$ is allowed thanks to [7]. By taking the real and imaginary part of the left- and the right-hand side, we have

$$\begin{aligned}&\mathrm{Re}\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right) \\ &= -(p-1)\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Re}\big(\overline{u}_\rho \nabla u_\rho\big)\right|^2 dx - \int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Im}\big(\overline{u}_\rho \nabla u_\rho\big)\right|^2 dx \\ &\quad - \alpha \int_{B(\rho)} \left|u_\rho\right|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \mathrm{Re}\big(\overline{u}_\rho \nabla u_\rho\big)\, dx - \int_{B(\rho)} \left(|x|^\beta + \omega\right) \left|u_\rho\right|^p dx \\ &= -(p-1)\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Re}\big(\overline{u}_\rho \nabla u_\rho\big)\right|^2 dx - \int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Im}\big(\overline{u}_\rho \nabla u_\rho\big)\right|^2 dx \\ &\quad - \frac{\alpha}{p} \int_{B(\rho)} |x|^{\alpha-1} \frac{x}{|x|} \nabla\left(\left|u_\rho\right|^p\right) dx - \int_{B(\rho)} \left(|x|^\beta + \omega\right) |u_\rho|^p dx \\ &= -(p-1)\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Re}\big(\overline{u}_\rho \nabla u_\rho\big)\right|^2 dx - \int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \left|\mathrm{Im}\big(\overline{u}_\rho \nabla u_\rho\big)\right|^2 dx \\ &\quad + \int_{B(\rho)} \left(\frac{\alpha(N-2+\alpha)}{p} |x|^{\alpha-2} - |x|^\beta - \omega\right) \left|u_\rho\right|^p dx\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im}\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right) = & -(p-2)\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4} \mathrm{Im}\big(\overline{u}_\rho \nabla u_\rho\big) \mathrm{Re}\big(\overline{u}_\rho \nabla u_\rho\big)\, dx \\ & - \alpha \int_{B(\rho)} \left|u_\rho\right|^{p-2} |x|^{\alpha-1} \frac{x}{|x|} \mathrm{Im}\big(\overline{u}_\rho \nabla u_\rho\big)\, dx.\end{aligned}$$

We can choose $\tilde{c} > 0$ and $\omega > 0$ (depending on $\tilde{c}$) such that

$$\frac{\alpha(N-2+\alpha)}{p} |x|^{\alpha-2} - |x|^\beta - \omega \le -\tilde{c}|x|^{\alpha-2}.$$

So, we obtain

$$-\operatorname{Re}\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right) \geq (p-1)\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4}\left|\operatorname{Re}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx$$
$$+\int_{B(\rho)} a(x)\left|u_\rho\right|^{p-4}\left|\operatorname{Im}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx + \tilde{c}\int_{B(\rho)} \left|u_\rho\right|^p |x|^{\alpha-2} dx$$
$$= (p-1)B^2 + C^2 + \tilde{c}D^2.$$

Moreover,

$$\left|\operatorname{Im}\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right)\right|$$
$$\leq |p-2|\left(\int_{B(\rho)} \left|u_\rho\right|^{p-4} a(x)\left|\operatorname{Re}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx\right)^{\frac{1}{2}}$$
$$\cdot\left(\int_{B(\rho)} \left|u_\rho\right|^{p-4} a(x)\left|\operatorname{Im}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx\right)^{\frac{1}{2}}$$
$$+\alpha\left(\int_{B(\rho)} \left|u_\rho\right|^{p-4}|x|^{\alpha}\left|\operatorname{Im}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_{B(\rho)} \left|u_\rho\right|^{p}|x|^{\alpha-2} dx\right)^{\frac{1}{2}}$$
$$= |p-2|BC + \alpha CD,$$

where

$$B^2 = \int_{B(\rho)} \left|u_\rho\right|^{p-4} a(x)\left|\operatorname{Re}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx,$$
$$C^2 = \int_{B(\rho)} \left|u_\rho\right|^{p-4} a(x)\left|\operatorname{Im}\left(\overline{u}_\rho \nabla u_\rho\right)\right|^2 dx,$$
$$D^2 = \int_{B(\rho)} \left|u_\rho\right|^{p}|x|^{\alpha-2} dx.$$

Let us observe that, choosing $\delta^2 = \frac{|p-2|^2}{4(p-1)} + \frac{\alpha^2}{4\tilde{c}}$ (which is independent of ρ), we obtain

$$\left|\operatorname{Im}\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right)\right| \leq \delta\left\{-\operatorname{Re}\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right)\right\}.$$

If $\tan\theta_\alpha = \delta$, then $e^{\pm i\theta}\widetilde{A_p}$ is dissipative in $B(\rho)$ for $0 \leq \theta \leq \theta_\alpha$. From [12, Theorem I.3.9] follows that the problem (4.7) has a unique solution u_ρ for every $\lambda \in \Sigma_\theta$, $0 \leq \theta < \theta_\alpha$ where

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\operatorname{Arg}\lambda| < \pi/2 + \theta\}.$$

Moreover, there exists a constant C_θ which is independent of ρ, such that

$$\|u_\rho\|_{L^p(B(\rho))} \leq \frac{C_\theta}{|\lambda|}\|f\|_{L^p}, \quad \text{for } \lambda \in \Sigma_\theta. \tag{4.8}$$

Let us now fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$ and a radius $r > 0$. We apply the interior L^p estimates (*cf.* [2, Theorem 9.11]) to the functions u_ρ with $\rho > r + 1$. So, by (4.8), we have

$$\|u_\rho\|_{W^{2,p}(B(r))} \le C_1 \left(\|\lambda u_\rho - \widetilde{A}_p u_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))} \right) \le C_2 \|f\|_{L^p}.$$

Using a weak compactness and a diagonal argument, we can construct a sequence $(\rho_n) \to \infty$ such that the functions (u_{ρ_n}) converge weakly in $W^{2,p}_{\mathrm{loc}}$ to a function u which satisfies $\lambda u - \widetilde{A}_p u = f$ and

$$\|u\|_p \le \frac{C_\theta}{|\lambda|} \|f\|_p, \quad \text{for } \lambda \in \Sigma_\theta. \tag{4.9}$$

Moreover, $u \in D_{p,\max}(A_p)$. We have now only to show that $\lambda - \widetilde{A}_p$ is invertible on $D_{p,\max}(A_p)$ for $\lambda \in \Sigma_\theta$. Consider the set

$$E = \left\{ r > 0 : \Sigma_\theta \cap C(r) \subset \rho\big(\widetilde{A}_p\big) \right\},$$

where $C(r) := \{\lambda \in \mathbb{C} : |\lambda| < r\}$. Since, by Theorem 3.7, 0 is in the resolvent set of $\widetilde{A}_p$, then $R = \sup E > 0$. On the other hand, the norm of the resolvent is bounded by $C_\theta/|\lambda|$ in $C(R) \cap \Sigma_\theta$, consequently it cannot explode on the boundary of $C(R)$, then $R = \infty$ and this ends the proof of the theorem. □

Remark 4.6. Since A_p generates an analytic semigroup $T_p(\cdot)$ on $L^p(\mathbb{R}^N)$ and the semigroups $T_q(\cdot)$, for $q \in (1, \infty)$ are consistent, see Theorem 3.7, one can deduce (as in the proof of [4, Proposition 2.6]) using Corollary 3.6 that $T_p(t)L^p(\mathbb{R}^N) \subset C_b^{1+\nu}(\mathbb{R}^N)$ for any $t > 0$, $\nu \in (0, 1)$ and for any $p \in (1, \infty)$.

We end this section by studying the spectrum of A_p. We recall from Proposition 3.5 that

$$\left\| |x|^\beta u \right\|_p \le C \|A_p u\|_p, \quad \forall u \in D_{p,\max}(A).$$

So, arguing as in [4], we obtain the following results:

Proposition 4.7. *Assume $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then:*

(i) *The resolvent of A_p is compact in L^p;*
(ii) *The spectrum of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$. Moreover, $\sigma(A_p)$ is independent of p;*
(iii) *The semigroup $T_p((\cdot)$ is irreducible, the eigenspace corresponding to the largest eigenvalue λ_0 of A is one-dimensional and is spanned by a strictly positive function ψ, which is radial, belongs to $C_b^{1+\nu}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ for any $\nu \in (0, 1)$ and tends to 0 when $|x| \to \infty$.*

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