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Hölder regularity of generic manifolds

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Dedicated to Professor Józef Siciak for his 85^{th} *birthday*

Abstract. In this paper we study Hölder continuity of the pluricomplex Green function with logarithmic growth at infinity of a smooth generic submanifold of $\mathbb{C}^n$. In particular we prove that the pluricomplex Green function of any C^2-smooth generic compact submanifold of $\mathbb{C}^n$ (without boundary) is Lipschitz continuous in $\mathbb{C}^n$.

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1. Introduction and statement of the main result

Real $m-$planes $\Pi \subset \mathbb{C}^n$, $\dim_{\mathbb{R}} \Pi = m$, $m \in \mathbb{N}^+$, which are not contained in any proper complex subspace of $\mathbb{C}^n$ are important in complex analysis and pluripotential theory. The $\mathbb{C}-$hull of such plane Π is equal to all $\mathbb{C}^n$, *i.e.*, $\Pi + J\Pi = \mathbb{C}^n$ (J is the standard complex structure on $\mathbb{C}^n$) and any non-empty open subset of Π is non-pluripolar in $\mathbb{C}^n$. Such planes are called *generic* (real) subspaces of $\mathbb{C}^n$. Correspondingly, a real smooth submanifold $M \subset \mathbb{C}^n$ is said to be *generic* if for each $z \in M$, its real tangent space $T_z M$ is a generic subspace of $\mathbb{C}^n$, *i.e.*, $T_z M + J T_z M = \mathbb{C}^n$. Such submanifold has real dimension $m \geq n$. The case of minimal dimension $\dim M = n$ is the most relevant for our concern. In this case for each $z \in M$, the tangent space $T_z M$ does not contain any complex line, *i.e.*, $T_z M \cap J T_z M = \{0\}$ and M is said to be *totally real*.

Observe that any smooth Jordan curve in $\mathbb{C}$ is totally real, hence any product of n smooth Jordan curves in $\mathbb{C}$ is a smooth compact totally real submanifold of dimension n in $\mathbb{C}^n$. Moreover the class of smooth compact totally real submanifolds of dimension n in $\mathbb{C}^n$ is stable under small C^2-perturbations.

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Generic submanifolds of $\mathbb{C}^n$ play an important role in Complex Analysis and Pluripotential Theory (see [3,5,8,13,14,17]).

In our previous paper [17], we used the method of attached analytic discs to investigate non plurithinness of generic submanifolds of $\mathbb{C}^n$. We proved in [17], that subsets of full measure in a generic C^2-smooth submanifold are non-plurithin at any point.

Here we continue our investigations concerning the *pluripotential* properties (*pluripolarity, pluriregularity*) of generic submanifolds in $\mathbb{C}^n$ by studying Hölder continuity of their pluricomplex Green functions.

All these properties can be expressed in terms of the pluricomplex Green function defined as follows.

Given a (bounded) subset $E \Subset \mathbb{C}^n$, we define its pluricomplex Green function as follows:

$$V_E(z) := \sup \left\{ u(z) : u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0 \right\},$$

where $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of psh functions u in $\mathbb{C}^n$ with logarithmic growth at infinity, *i.e.*, $\sup\{u(z) - \log^+(z) : z \in \mathbb{C}^n\} < +\infty$ (see [9,18,19,22]).

Our main result is the following.

Main Theorem. *Let $M \subset \mathbb{C}^n$ be a C^2-smooth generic compact submanifold without boundary. Then its pluricomplex Green function V_M is Lipschitz continuous in $\mathbb{C}^n$.*

This theorem is concerned with compact submanifolds without boundary. In Section 3, we will consider the more general case of a C^2-smooth generic submanifold and prove that its extremal function is Lipschitz near each of its compact subsets (see Theorem 5.1). In the last section we consider the case of a compact C^2-smooth generic submanifold with boundary and discuss the Hölder continuity property of its pluricomplex Green function.

From Lipschitz continuity, or more generally the Hölder continuity of the pluricomplex Green function $V_E(z)$ of a compact set $E \subset \mathbb{C}^n$, it follows that the compact set E satisfies the following Markov's inequality: there exists positive constants $A, r > 0$ such that

$$\|\nabla P(z)\|_E \leq A d^r \|P(z)\|_E, \ z \in \mathbb{C}^n$$

for any polynomial P of degree d.

This inequality plays an important role in approximation theory, gives sharp inequalities for polynomials and is useful for constructing continuous extension operators for smooth functions from subsets of $\mathbb{R}^n$ to $\mathbb{C}^n$ (see [12,25]). On the other hand, Complex Dynamic gives a lot of examples of compact subsets for which the pluricomplex Green function $V_E(z)$ is Hölder continuous (see [6]).

More recently an important result of C. T. Dinh, V. A. Nguyen and N. Sibony shows that the Monge-Ampère measure of a Hölder continuous plurisubharmonic function is a *moderate measure* (see [4]). In particular the equilibrium Monge-Ampère measure $\mu_E := (dd^c V_E)^n$ of a compact subset whose pluricomplex Green function $V_E(z)$ is Hölder continuous is a moderate measure, which means that it satisfies the following uniform version of Skoda's integrability theorem: for any compact family $\mathcal{U}$ of *psh* functions in a neighborhood of a given ball $\mathbb{B} \subset \mathbb{C}^n$, there exists $\varepsilon > 0$ and a constant $C > 0$ such that

$$\int_{\mathbb{B}} e^{-\varepsilon u} d\mu_E \leq C, \forall u \in \mathcal{U}.$$

From this property it follows that the equilibrium measure μ_E is "well dominated" by the Monge-Ampère capacity (see [26]), in the sense that for any given ball $\mathbb{B} \subset \mathbb{C}^n$, there is a constant $A > 0$ such that for any Borel set $S \subset \mathbb{B}$,

$$\mu_E(S) \leq A \exp\left(-A \mathrm{cap}_{\mathbb{B}}(S)^{-1/n}\right),$$

where $\mathrm{cap}_{\mathbb{B}}(S)$ is the Monge-Ampère capacity [2].

This property turns out to play an important role in the theory of complex Monge-Ampère equations as was discovered by S. Kolodziej (see [7,10]).

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2. Definitions and preliminaries

Let us recall the following definitions:

Definition 2.1. 1. We say that a subset $P \subset \mathbb{C}^n$ is *pluripolar* if there is a plurisubharmonic (psh) function $u : u \not\equiv -\infty$ but $u|_P \equiv -\infty$.

2. We say that E is *pluriregular* if its pluricomplex Green function satisfies $V_E^*|E = 0$, *i.e.*, $V_E = V_E^*$ on $\mathbb{C}^n$.

Observe that any pluriregular set is non-pluripolar. It is well-known that if E in non-pluripolar then $V_E^* \in \mathcal{L}(\mathbb{C}^n)$. Moreover if E is pluriregular compact set then $V_E = V_E^*$ is continuous in $\mathbb{C}^n$ (see [19]).

On the other hand, we know from ([2]) that if E is non-pluripolar then the locally bounded psh function V_E^* satisfies the following complex Monge-Ampère equation

$$(dd^c V_E^*)^n = 0, \text{ on } \mathbb{C}^n \setminus \overline{E},$$

which means that the equilibrium measure of E defined as

$$\mu_E := \left(dd^c V_E^*\right)^n$$

is a Borel measure supported in the closed set $\overline{E}$.

Here we will introduce the following important notion.

Definition 2.2. We say that a set E is Λ_α-pluriregular, $\alpha > 0$, if for every compact $K \subset E$ there exist a constant $A = A_K > 0$ and a neighborhood $O = O_K$ of K such that

$$V_E(z) \leq A d^\alpha(z, K), \ \forall z \in O, \tag{2.1}$$

where d is the Euclidean distance in $\mathbb{C}^n$.

Roughly speaking, this definition means that the pluricomplex Green function V_E of the set E is Hölder continuous near any compact subset $K \subset E$. The following observation, which is essentially due to Z. Błocki, shows that if the set E is compact itself, the definition means that its pluricomplex Green function is Hölder continuous (see [20]).

Lemma 2.3. *If $E \subset \mathbb{C}^n$ is a Λ_α-pluriregular compact set then its pluricomplex Green function V_E is Hölder continous of order α globally in $\mathbb{C}^n$,* i.e.*, for any $z, w \in \mathbb{C}^n$, we have*

$$|V_E(z) - V_E(w)| \leq A|z - w|^\alpha.$$

Proof. Observe that V_E^* has a logarithmic growth at infinity. Therefore if E is a Λ_α-pluriregular compact set then its pluricomplex Green function V_E satisfies (2.1) for all $z \in \mathbb{C}^n$, *i.e.*, for some constant $A > 0$ we have

$$V_E(z) \leq A d^\alpha(z, E), \ \forall z \in \mathbb{C}^n. \tag{2.2}$$

To prove that V_E is Hölder continous of order α globally in $\mathbb{C}^n$, fix $h \in \mathbb{C}^n$ such that $|h| < \delta$ and observe, that for any $z \in E$, $d(z + h, E) \leq \delta^\alpha$, which implies by the Hölder condition (2.2) that for any $z \in E$, $V_E(z + h) \leq A\delta^\alpha$. Therefore the function defined by $u(z) := V_E(z + h) - A\delta^\alpha$ is a plurisubharmonic function such that $u \in \mathcal{L}(\mathbb{C}^n)$ and $u \leq 0$ on E. By the definition of V_E, we conclude that $u \leq V_E(z)$ for any $z \in \mathbb{C}^n$, which implies that V_E is Hölder continuous. □

3. Analytic discs attached to generic manifolds

3.1. Construction of attached analytic discs

Let $\mathbb{U} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disc and $\mathbb{T} := \partial\mathbb{U}$ the unit circle. An analytic disc of $\mathbb{C}^n$ is a continuous function $f : \overline{\mathbb{U}} \longrightarrow \mathbb{C}^n$, which is holomorphic on $\mathbb{U}$. Let $M \subset \mathbb{C}^n$ be a given subset of $\mathbb{C}^n$ and $\gamma \subset \overline{U}$ a given connected subset of the closed disc $\overline{\mathbb{U}}$. We say that the analytic disc f is attached to M along γ if $f(\gamma) \subset M$.

If $f : \overline{\mathbb{U}} \longrightarrow \mathbb{C}^n$ is an analytic disc and F is a holomorphic function on a neighboorhood D of $f(\overline{\mathbb{U}})$, then $F \circ f$ is a holomorphic function on the unit disc $\mathbb{U}$. If u is a plurisubharmonic function on D, then $u \circ f$ is a subharmonic function on $\mathbb{U}$. Therefore analytic discs enable us to reduce multidimentional complex problems to corresponding one dimensional complex problems.

In the proof of our theorem, we need a smooth family of analytic discs. We will use Bishop's equation for construct such a family (see [1, 13]). Let M be a totally real submanifold of dimension n given locally by the following equation

$$M := \left\{z = x + iy \in B \times \mathbb{R}^n : y = h(x)\right\},$$

where $B \subset \mathbb{R}^n$ is a ball of center 0 and $h : B \to \mathbb{R}^n$ a smooth map, such that

$$h(0) = 0 \text{ and } Dh(0) = 0.$$

Let $v(\tau) : \mathbb{T} \to \mathbb{R}^+$ a C^∞ function on the unit circle $\mathbb{T}$ such that

$$v|_{\{e^{i\theta} :\ \theta\in(0,\pi)\}} = 0 \text{ and } v|_{\{e^{i\theta} :\ \theta\in(\pi,2\pi)\}} > 0.$$

Assume that there exists a continuous mapping $X : \mathbb{T} \to \mathbb{R}^n$ which is a solution of the following Bishop's equation

$$X(\tau) = c - \Im(h \circ X + tv)(\tau), \quad \tau \in \mathbb{T}, \tag{3.1}$$

where $(c, t) \in Q = Q_c \times Q_t \subset \mathbb{R}^n \times \mathbb{R}^n$ is a fixed parameter and $\Im$ is the harmonic conjugate operator defined by the Schwarz integral formula

$$\Im(X)(\zeta) = \frac{1}{2\pi} \int_{\mathbb{T}} X(\tau) \operatorname{Im} \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\tau\ , \quad \zeta = re^{i\theta}, \tag{3.2}$$

normalized by the condition

$$\Im X(0) = 0.$$

We will consider the unique harmonic extension $X(\zeta)$ of the mapping X to the unit disc $\mathbb{U}$. Then the following mapping

$$\begin{aligned} \Phi(c, t, \zeta) : &= X(c, t, \zeta) + i\left[h^*(c, t, \zeta) + tv(\zeta)\right] \\ &= c + i\left\{h^*(c, t, \zeta) + tv(\zeta) + i\Im\left[h^*(c, t, \zeta) + tv(\zeta)\right]\right\} \end{aligned} \tag{3.3}$$

provides a family of analytic discs $\Phi(c, t, \zeta) : \overline{\mathbb{U}} \to \mathbb{C}^n$ such that

$$\forall (c, t) \in Q\ ,\ \forall \tau \in \gamma\ ,\ \Phi(c, t, \tau) \in M. \tag{3.4}$$

Here $X(c, t, \zeta)$, $h^*(c, t, \zeta)$ and $v(\zeta)$ are harmonic extensions of $X(c, t, \tau)$, $h \circ X(c, t, \tau)$ and $v(\tau)$ to the unit disc $\mathbb{U}$ respectively.

We need a smooth family of discs $\Phi(c, t, \zeta)$. Many constructions of analytic discs attached to generic manifolds along a part of the circle have been given by many different authors, depending on the smoothness properties of the manifold (see [3, 13, 14]). The most general and sharp result was proved by B. Coupet [3]:

Theorem 3.1 ([3]). *Let $p > 2n+1, q \geq 1$ be integers and $h \in C^q(B)$. Then there exist a constant $\delta_0 > 0$, independent of h and p, such that for arbitrary C^q-smooth mapping $k(c,t,\tau) : \mathbb{R}^{2n+1} \to \mathbb{R}^n$, with compact support and $\|k\|_{W^{q,p}} \leq \delta_0$, the equation*

$$u = -\Im(h \circ u) + k \tag{3.5}$$

has a unique solution $u \in W^{q,p}(\mathbb{T} \times \mathbb{R}^{2n})$.

Moreover, the harmonic extensions of u and $h \circ u$ to the unit disc $\mathbb{U}$ belong to $C^q(\mathbb{U} \times \mathbb{R}^{2n})$.

Let now $h \in C^q(\mathbb{B})$. Observe that Bishop's equation (3.1) is a particular case of the equation (3.5). Therefore, from the theorem of Coupet and Sobolev's embedding theorem $W^{q,p} \subset C^{q-1}$, it follows that for a small enough neighborhood $Q \ni 0$, $(c,t) \in Q$, the Bishop equation (3.1) has unique solution $X(\tau,c,t)$: $X, h \circ X \in C^{q-1}(\overline{U} \times Q) \cap C^q(U \times Q)$. Note that the operator $\Im : W^{q,p} \to W^{q,p}$ is continuous.

Therefore, for a C^2-smooth generic submanifold $M \subset \mathbb{C}^n$, we obtain a smooth family of discs (3.3), attached to M, such that

$$\|\Im X\|_1 \leq A \|X\|_1, \|\Im h \circ X\|_1 \leq A \|h \circ X\|_1,$$

where A is a constant and $\|\cdot\|_1$ is the C^1-norm in $\tau \in T$.

3.2. Harmonic measure of boundary set of the unit disc

For arbitrary $\gamma \subset \mathbb{T}$ we put $\aleph(\gamma, \mathbb{U})$– class of functions

$$\left\{u(\zeta) : u \in \text{sh}(\mathbb{U}) \cap C(\overline{\mathbb{U}}),\ u|_{\mathbb{U}} < 0,\ u|_\gamma \leq -1\right\},$$

and set

$$\omega(\zeta, \gamma, \mathbb{U}) = \sup\{u(z) : u \in \aleph\},\ \zeta \in \mathbb{U}.$$

Then (negative of) the upper semi-continuous regularization $\omega^*(\zeta,\gamma,\mathbb{U})$ is called the *harmonic measure* of γ with respect to $\mathbb{U}$ at the point ζ. The function ω^* is the unique solution of the Dirichlet problem:

$$\Delta\omega^* = 0,\ \omega^*|_{\mathbb{T}} = -\chi_\gamma,$$

where χ_γ is the characteristic function of γ. By Poisson formula

$$\omega^*(\zeta,\gamma,\mathbb{U}) = -\frac{1}{2\pi}\int_{\mathbb{T}} \chi_\gamma(\tau)\,\text{Re}\,\frac{e^{i\tau}+\zeta}{e^{i\tau}-\zeta}d\tau\ ,\quad \zeta = re^{i\theta}.$$

For $\gamma = \{e^{i\varphi} : 0 \leq \varphi \leq \pi\}$ the harmonic measure ω^* can be expressed as follows.

$$\omega^*(\zeta,\gamma,\mathbb{U}) = \frac{1}{\pi}\arg i\,\frac{1-\zeta}{1+\zeta} - 1. \tag{3.6}$$

Let us define the sector at the point $1 = e^{i\cdot 0} \in \overline{\mathbb{U}}$ as follows

$$\Omega_{0,\alpha} = \cup\big\{l \cap \overline{\mathbb{U}} :\ l \ni 1,\ \pi/2 \le \arg l \le \pi/2 + \alpha\big\},$$

where l stands for a real line passing through the point 1 and $0 \le \alpha \le \pi/2$ is fixed. The sector $\Omega_{a,\alpha}$ at the point $e^{ia} \in \overline{\mathbb{U}}$ can define in the same way. From (3.6) it follows clearly that

$$\omega^*(\zeta, \gamma, \mathbb{U}) \le -1 + \alpha/\pi, \quad \forall \zeta \in \Omega_{0,\alpha}. \tag{3.7}$$

A sector $\Omega_{a,\alpha}$ at the point e^{ia} is said to be *admissible* if $\Omega_{a,\alpha} \cap \partial\mathbb{U} \subset \gamma$. From the last fact, we deduce the following statement.

Lemma 3.2. *Let $\gamma = \operatorname{arc}[e^{ia}, e^{ib}] \subset \mathbb{T}$, $0 \le a < b \le 2\pi$ be an arbitrary arc on $\mathbb{T}$, and let $\Omega_{a,\alpha}$ be an admissible sector at the point e^{ia}. Then $\omega^*(\zeta, \gamma, \mathbb{U})$ is Λ_1-continuous in $\mathbb{U} \cup \mathbb{T} \setminus \{e^{ia}, e^{ib}\}$, $\omega^*|_{\gamma^\circ} \equiv -1$, $\omega^*|_{\mathbb{T}\setminus\gamma} \equiv 0$ and ω^* satisfies* (3.7) *in $\Omega_{a,\alpha}$.*

Here γ° denote the interior of the arc γ. We note that if $\gamma_0 \Subset \gamma$ is an arc with non-empty interior, then there exist $\alpha = \alpha(\gamma_0, \gamma) > 0$ such that $\Omega_{\tau,\alpha}$ is admissible for every $\tau \in \gamma_0$.

4. Transversality of attached discs to a generic manifold

It is clear that the family of analytic discs constructed above

$$\Phi(c, t, \zeta) = X(c, t, \zeta) + i(h^*(c, t, \zeta) + tv(\zeta)),$$

for $(c, t) \in Q = Q_c \times Q_t$, $\zeta \in \bar{\mathbb{U}}$, satisfies the following properties:

$$X(c, t, \tau) = c - \Im\left(h \circ X(c, t, \tau) + tv(\tau)\right),\ (c, t) \in Q,\ \tau \in \partial\mathbb{U}. \tag{4.1}$$

$$h^*(c, t, \tau) = h \circ X(c, t, \tau), (c, t) \in Q, \tau \in \partial\mathbb{U} \tag{4.2}$$

$$\begin{gathered} X(c, 0, \zeta) \equiv c, h^*(c, 0, \zeta) \equiv h(c) \text{ so that} \\ \Phi(c, 0, \zeta) \equiv c + ih(c) \in M, c \in Q_c \end{gathered} \tag{4.3}$$

$$X(c, t, 0) = \frac{1}{2\pi} \int_T X(c, t, \tau) d\tau \equiv c,\ (c, t) \in Q. \tag{4.4}$$

$$\|X\| \le O(\|c\| + \|t\|), \|D_\tau X\| \le O(\|t\|). \tag{4.5}$$

Here and below $\|\cdot\|$ is the Euclidean norm.

The following geometric transversality property will be crucial for the proof of our main theorem.

Lemma 4.1. *Let $\gamma_0 \Subset \gamma$ be an arc with non-empty interior. Then for small enough Q the attached discs $\Phi(c, t, \zeta)$, $t \neq 0$, for $\zeta \to \tau \in \gamma_0$ meet M transversally.*

Proof. For the normal derivative $D_{\vec{n}}$ at the points $\tau \in \gamma_0$ we have

$$\mathrm{Im} D_{\vec{n}} \Phi(c,t,\tau) = D_{\vec{n}} h^*(c,t,\tau) + t D_{\vec{n}} v(\tau)$$

and

$$\left|\mathrm{Im} D_{\vec{n}} \Phi(c,t,\tau)\right| \geq \|t\| b - O(\varepsilon)\|t\| = \|t\|(b - O(\varepsilon)),$$

where

$$b := \inf_{\gamma_0} \left| D_{\vec{n}} v(\tau) \right| > 0 \text{ and } \varepsilon = \sup\{\|c\| + \|t\| : c \in Q_c, \ t \in Q_t\}.$$

It follows, that for $O(\varepsilon) < \frac{b}{2}$

$$\left|\mathrm{Im} D_{\vec{n}} \Phi(c,t,\tau)\right| \geq \|t\| b/2 \quad \forall \tau \in \gamma_0, \tag{4.6}$$

i.e., the discs $\Phi(c,t,\zeta)$ meet M for $\zeta \to \tau \in \gamma_0$ transversally. □

Corollary 4.2. *Let $Q' = \{\|t\| = \sigma\} \subset Q_t$, where $\sigma > 0$. Then there exist a neighborhood $\Omega' \supset \gamma_0$ and a constant $C > 0$ such that*

$$\begin{aligned} d_{\mathbb{C}}(\zeta, \gamma_0) &\leq C d_{\mathbb{C}^n}[\Phi(c,t,\zeta), M] \\ d_{\mathbb{C}^n}[\Phi(c,t,\zeta), \Phi(c,t,\gamma_0)] &\leq C d_{\mathbb{C}^n}[\Phi(c,t,\zeta), M], \end{aligned} \tag{4.7}$$

$\forall\ \zeta \in \Omega = \overline{\mathbb{U} \cap \Omega'}$, $t \in Q'$, $c \in \overline{Q}_c$. Here $d_{\mathbb{C}}$ and $d_{\mathbb{C}^n}$ are Euclidean distances on $\mathbb{C}$ and $\mathbb{C}^n$, respectively.

Proof. The statement clearly follows from (4.6), because for every fixed $t^0 \in Q'$, $c^0 \in \overline{Q}_c$ we can write (4.7), which then will be true in some neighborhoods $B_c \ni c^0$, $B_t \ni t^0$. □

Lemma 4.3. *For every $\Omega' \supset \gamma_0$ and for every $Q'_t = \{\|t\| = \sigma\} \subset Q_t$, $\sigma > 0$ small enough the closed set $W = \{\Phi(c,t,\zeta) \in \mathbb{C}^n : c \in \overline{Q}_c,\ t \in Q'_t, \zeta \in \Omega = \overline{\mathbb{U} \cap \Omega'}\}$ contains the point $0 \in M$ in its interior in $\mathbb{C}^n$*, i.e., $0 \in \dot{W}$.

Proof. By (4.3) $X(c,t,\zeta) \equiv c$ if $t = 0$. Since X is smooth, then for small enough fixed t^0 and for arbitrary fixed $\tau^0 \in \gamma_0$ the image $X(c, t^0, \tau^0) : c \in Q_c$ contains $0 \in \mathbb{R}^n$. It follows, that $0 \in W$. Moreover, $\dot{W} \neq \emptyset$ and if for some $\|t^0\| \leq \sigma$, $\zeta^0 \in \overline{U}$

$$x\big(c, t^0, \zeta^0\big) \in \frac{1}{2} Q_c, \text{ then } c \in Q_c \tag{4.8}$$

Now we assume by contradiction that $0 \in \partial W$. Then $\mathbb{C}^n \setminus W$ is open and contains 0 on its boundary. It is clear that near 0 there exists a point $p^0 = (x^0, y^0) \in \partial \dot{W} \setminus M$ such that $x^0 \in \frac{1}{2} Q_c$ and $p^0 = \Phi(c^0, t^0, \zeta^0)$ for some $c^0 \in \overline{Q}_c$, $\|t^0\| = \sigma$, $\zeta^0 \in \Omega' \cap U$.

For simplicity we may assume that $t^0 = (0, \dots, 0, \sigma)$ and set $'c = (c_1, \dots, c_{n-1})$, $'t = (t_1, \dots, t_{n-1})$. From (4.8) it follows also that $c \in Q_c$.

We consider the transformation

$$S\left('c, 't, \zeta\right) = \Phi\left('c, c_n^0, 't, t_n^0, \zeta\right) : \ 'Q \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}^n, \tag{4.9}$$

where $'Q := \{z \in Q : \ c_n = c_n^0, t_n = t_n^0\} \subset \mathbb{R}^{2n-2}$.

Then $S\left('c^0, 't^0, \zeta^0\right) = p^0$ and its Jacobian is given by

$$J('c, 't, \zeta) = \hat{J}(c, t, \zeta)|_{c_n = c_n^0, t_n = t_n^0},$$

where

$$\hat{J}(c, t, \zeta) = \left|\begin{array}{ccc|ccc|cc}
\frac{\partial X_1}{\partial c_1} & \cdots & \frac{\partial X_{n-1}}{\partial c_1} & \frac{\partial Y_1}{\partial c_1} & \cdots & \frac{\partial Y_{n-1}}{\partial c_1} & \frac{\partial X_n}{\partial c_1} & \frac{\partial Y_n}{\partial c_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{\partial X_1}{\partial c_{n-1}} & \cdots & \frac{\partial X_{n-1}}{\partial c_{n-1}} & \frac{\partial Y_1}{\partial c_{n-1}} & \cdots & \frac{\partial Y_{n-1}}{\partial c_{n-1}} & \frac{\partial X_n}{\partial c_{n-1}} & \frac{\partial Y_n}{\partial c_{n-1}} \\
\hline
\frac{\partial X_1}{\partial t_1} & \cdots & \frac{\partial X_{n-1}}{\partial t_1} & \frac{\partial Y_1}{\partial t_1} & \cdots & \frac{\partial Y_{n-1}}{\partial t_1} & \frac{\partial X_n}{\partial t_1} & \frac{\partial Y_n}{\partial t_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{\partial X_1}{\partial t_{n-1}} & \cdots & \frac{\partial X_{n-1}}{\partial t_{n-1}} & \frac{\partial Y_1}{\partial t_{n-1}} & \cdots & \frac{\partial Y_{n-1}}{\partial t_{n-1}} & \frac{\partial X_n}{\partial t_{n-1}} & \frac{\partial Y_n}{\partial t_{n-1}} \\
\hline
\frac{\partial X_1}{\partial \zeta'} & \cdots & \frac{\partial X_{n-1}}{\partial \zeta'} & \frac{\partial Y_1}{\partial \zeta'} & \cdots & \frac{\partial Y_{n-1}}{\partial \zeta'} & \frac{\partial X_n}{\partial \zeta'} & \frac{\partial Y_n}{\partial \zeta'} \\
\frac{\partial X_1}{\partial \zeta''} & \cdots & \frac{\partial X_{n-1}}{\partial \zeta''} & \frac{\partial Y_1}{\partial \zeta''} & \cdots & \frac{\partial Y_{n-1}}{\partial \zeta''} & \frac{\partial X_n}{\partial \zeta''} & \frac{\partial Y_n}{\partial \zeta''}
\end{array}\right|.$$

Here $\zeta = \zeta' + i\zeta''$ and $Y_k(c, t, \zeta) = h_k^* \circ X(c, t, \zeta) + t_k v(\zeta)$, $k = 1, \dots, n$.

The determinant J, is composed by 9 block matrices D_{ij}, $i, j = 1, 2, 3$.

We will show that $J('c^0, 't^0, \zeta^0) \neq 0$, which will imply that the operator S is a local diffeomorphism in a neighborhood of the point $\left('c^0, 't^0, \zeta^0\right)$.

Indeed, by (4.3) $X(c, 0, \zeta) \equiv c$, $h^*(c, 0, \zeta) \equiv h(c)$ and then

$$\left|\begin{matrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{matrix}\right|_{(c,0,\zeta)} = D_{11} \cdot D_{22} = v^{n-1}(\zeta)$$

and

$$\left|\begin{matrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{matrix}\right|_{(c,t,\zeta)} = v^{n-1}(\zeta) + O(\varepsilon), \tag{4.10}$$

where we recall that $\varepsilon = \sup\{\|c\| + \|t\| : \ c \in Q_c, \ t \in Q_t\}$. Note also that

$$D_{33} = \begin{vmatrix} \frac{\partial X_n}{\partial \zeta'} & \frac{\partial Y_n}{\partial \zeta'} \\ \frac{\partial X_n}{\partial \zeta''} & \frac{\partial Y_n}{\partial \zeta''} \end{vmatrix} = \left| \frac{d}{d\zeta}(X_n + iY_n) \right|^2 .$$

Now consider the right hand side near the arc γ. It is clear that for every $s > 0$, there is an open set $\tilde{\Omega} \supset \gamma_0$ such that

$$\left| \frac{d}{d\zeta}(X_n + iY_n)(c,t,\zeta) \right|^2 \geq |D_\tau X_n(c,t,\tau)|^2 - s, \forall \zeta \in U \cap \tilde{\Omega}, \tau \in \gamma_0. \quad (4.11)$$

We calculate $D_\tau X\,(c,t,\tau)$, for $(c,t) \in Q$, $\tau \in T$,

$$D_\tau X\,(c,t,\tau) = -D_\tau \Im h \circ X\,(c,t,\tau) - t D_\tau \Im v\,(\tau)\,. \quad (4.12)$$

Since, $D_\tau \Im v\,(\tau) = D_{\vec{n}} v\,(\tau)$, where $D_{\vec{n}}$ is the normal derivative $\vec{n}$, then (4.12) implies

$$D_\tau X\,(c,t,\tau) + t D_{\vec{n}} v\,(\tau) = \Im D_\tau h \circ X\,(c,t,\tau)\,.$$

For k-coordinate of vector $X\,(\tau) = X\,(c,t,\tau)$ we have

$$\begin{aligned} \left\| D_\tau X_k\,(c,t,\tau) + t_k D_{\vec{n}} v\,(\tau) \right\| &= \|\Im D_\tau h_k \circ X\,(c,t,\tau)\| \\ &\leq \text{const}\,\|D_\tau h_k \circ X\,(c,t,\tau)\| \\ &\leq O\,(\varepsilon)\,\|D_\tau X\,(c,t,\tau)\|\,. \end{aligned} \quad (4.13)$$

Therefore,

$$\begin{aligned} \left| t_k D_{\vec{n}} v\,(\tau) \right| - O\,(\varepsilon)\,\|t\| &\leq |D_\tau X_k\,(c,t,\tau)| \\ &\leq \left| t_k D_{\vec{n}} v\,(\tau) \right| + O\,(\varepsilon)\,\|t\|\,,\ 1 \leq k \leq n,\ \tau \in T. \end{aligned} \quad (4.14)$$

The second part of (4.14) implies

$$\|D_\tau X\,(c,t,\tau)\| \leq C\,\|t\|\,,\ \ (c,t,\tau) \in Q \times T,\ \ C - \text{constant}. \quad (4.15)$$

As in Lemma 4.1 if $b = \inf_{\gamma_0} \left| D_{\vec{n}} v\,(\tau) \right| > 0$ and $O(\varepsilon) < \frac{b}{2}$, then the first part of (4.14) implies

$$|D_\tau X_k(c,t,\tau)| \geq |t_k| b - \|t\| b/2, \quad (4.16)$$

for $\tau \in \gamma$, $1 \leq k \leq n$.

By (4.10) and (4.11) it follows that

$$\begin{aligned} \left| J('c,'t,\zeta) \right| &= |D_{11}| \cdot |D_{22}| \cdot \left| \frac{d}{d\zeta}(x_n + iy_n)('c,'t,\zeta) \right|^2 + O(\varepsilon) \\ &\geq \left[v^{n-1}(\zeta) + O(\varepsilon) \right] \cdot \left[|t_n b/2|^2 - s \right] + O(\varepsilon), \end{aligned}$$

for all $('c,'t,\zeta) \in {}'Q \times \left[\mathbb{U} \cap \Omega' \right]$, because $\|t^0\| = |t_n^0|$.

We can take $\tilde{\Omega} \cap \Omega'$ instead of Ω and observe that all functions $O(\cdot)$ do not depend on ζ. Therefore if we take ε, s small enough, then $\left|J('c^0,' t^0, \zeta^0)\right| > 0$.

Since the plane $\{t_n = t_n^0\}$ is tangent to the sphere $\|t\| = \sigma$ at the point t^0, the Jacobian of the restriction $\breve{S} = \Phi\left('c, c_n^0, \sqrt{|t_1|^2 + ... + |t_{n-1}|^2}, \zeta\right)$ also is not zero at the point $('c^0,' t^0, \zeta^0)$. In particular, the operator

$$\breve{S}: \; U_1 \times U_2 \times U_3 \to U(p)$$

is a homeomorphism, where $U_1 \subset \mathbb{R}^{n-1}-$ a neighborhood of the point $'c^0$, $U_2 \subset Q_t'-$ a neighborhood of $t^0 \in Q_t'$ and $U_3 = \{|\zeta - \zeta_0| < \sigma'\} \subset \Omega, \sigma' > 0$, is a neighborhood of ζ_0. It follows that the open set $U(p) \subset W$, that is contradiction to $p \in \partial W$. □

5. Proof of the main theorem

First we observe that from the results of Edigarian-Wiegerinck [5] and the authors [17], it follows that $M \subset \mathbb{C}^n$ is a pluriregular set. Indeed, it was proved in [17] that a set of full measure in a generic manifold M is not thin. Since the set $P = \{z \in M : V_M^*(z) > 0\}$, where $V_M^*(z)$ is Green function, is pluripolar by Bedford and Taylor [2], it has zero-measure (see [3,14]) and then the set $M \setminus P$ is not thin. Therefore $V_M^* \equiv 0$ on M, *i.e.*, M is pluriregular. Note that in [5] non-thinness of $M \setminus P$ was proved for C^1-smooth manifold M and for a pluripolar set $P \subset M$, which implies that an arbitrary C^1-smooth generic manifold is pluriregular.

Our main theorem will be a consequence of the following result, thanks to Lemma 2.3.

Theorem 5.1. *Any C^2-smooth generic submanifold $M \subset \mathbb{C}^n$ is Λ_1-pluriregular.*

Proof. We first reduce to the case of a totally real submanifold. Fix a point, say $z^0 = 0 \in M$. Changing holomorphic coordinates in $\mathbb{C}^n$, we can assume that the tangent space T_0M, which by definition does not contain any complex hyperplane, can be written as

$$T_0M = \{z = x + iy \in \mathbb{C}^n : y_1 = \cdots = y_{2n-m} = 0\}.$$

Hence for a small neighborhood $G = G_1 \times G_2$ of the origin with

$$G_1 = \{(x, y'') = (x, y_{2n-m+1}, ..., y_n) \in \mathbb{R}^n \times \mathbb{R}^{m-n} : |x| \leq \delta, |y''| < \delta\},$$

$$G_2 = \{y' = (y_1, \cdots, y_{2n-m}) \in \mathbb{R}^{2n-m} : |y'| < \delta\},$$

we can represent M as a graph

$$M \cap G = \{z \in G : y' = h(x, y'')\},$$

where h is C^2-smooth mapping from G_1 into G_2.

Observe that for each small enough y_0'' the intersection $M \cap \Pi\{y_0''\}$ of M with the plane $\Pi\{y_0''\} := \{z \in \mathbb{C}^n : y'' = y_0''\}$ is an $n-$dimensional generic manifold. Moreover, since the Green function is monotonic, *i.e.*, $V(z, E_1) \geq V(z, E_2)$ for $E_1 \subset E_2$, it is enough to prove the theorem in the case when M is generic of dimension n, hence totally real of dimension n.

In this case, we show the local Hölder pluriregularity of M, using previous results from Section 4. Fix a point $p \in M$, a ball $B(p) = B_x \times B_y \subset \mathbb{C}^n$ centered at the point p such that $M_p := M \cap B(p)$ is the graph of a C^2-smooth function. Then by Corollary 4.2, for arbitrary fixed small $\sigma > 0$ there exist a neighborhood $\Omega' \supset \gamma_0$ and a constant $C > 0$, depending on the point p, such that the inequalities (4.7) hold. By Lemma 4.3 $O(p) \ni p$, where $O(p) = W^0$ is the interior of the set $W = W(p, \Omega', \sigma)$, constructed in Lemma 4.3.

Fix a point $z^0 \in O(p) \setminus M$ and a disc $\Phi(c, t, \zeta) : \Phi(c^0, t^0, \zeta^0) = z^0$, with $c^0 \in \overline{Q}_c$, $t^0 \in Q_t'$, $\zeta^0 \in \mathbb{U} \cap \Omega'$. Then the function $V_{M_p} \circ \Phi(c^0, t^0, \zeta) \in sh(\mathbb{U})$ and $V_{M_p} \circ \Phi|_\gamma \equiv 0$. Let $C'' = \max_{B(p)} V_{M_p}(z) < \infty$. By the theorem of two constants we have

$$V_{M_p} \circ \Phi(c^0, t^0, \zeta) \leq C''[\omega^*(\zeta, \gamma, \mathbb{U}) + 1], \ \zeta \in \mathbb{U}. \tag{5.1}$$

Therefore the first part of Lemma 3.2 and (4.7) yields the inequality

$$\begin{aligned} V_{M_p}(z^0) = V_{M_p} \circ \Phi(c^0, t^0, \zeta^0) &\leq C'' \left[\omega^*(\zeta^0, \gamma, \mathbb{U}) + 1)\right] \\ &\leq C'C'' d_{\mathbb{C}}(\zeta^0, \gamma_0) \leq C_p d_{\mathbb{C}^n}(z^0, M_p), \end{aligned} \tag{5.2}$$

for all $z^0 \in O(p)$, where $C_p := CC'C''$ depends on the fixed point $p \in M$ and on the corresponding family of analytic discs, attached to M locally, in a neighborhood of p.

Now given a compact set $K \subset M$ we can apply the previous estimate to each point of K. Then by compactness we can find a finite number of points $p_1, \dots, p_k$ of K, a finite number of balls balls $B(p_1), \dots, B(p_k)$ and a finite numbers of open sets $O(p_1), \dots, O(p_k)$ such that

$$V_{M_p}(z) \leq C_p d_{\mathbb{C}^n}(z, M_p),$$

for any $z \in O(p)$ and $p = p_1, \dots, p_k$. Now observe that $O = \cup_{1 \leq i \leq k} O(p_i)$ is a neighborhood of K and shrinking a little bit the open sets O_p we can assume that for any $p = p_i$ and $z \in O_p$, $d_{\mathbb{C}^n}(z, M_p) \leq d(z, M)$. Since $V_M \leq V_{M_p}$, it follows that $V_M(z) \leq A d_{\mathbb{C}^n}(z, K)$, for any $z \in O$. □

6. Open problems

Let $D \subset M$ a domain with C^1-smooth boundary ∂D. Lemma 4.2 states that a neighborhood of the generic manifold locally consists in the interior $\dot{W}$ of the set $W = \{\Phi(c, t, \zeta) : c \in \overline{Q}_c, t \in Q_t', \zeta \in \Omega = \overline{\mathbb{U} \cap \Omega'}\}$. It seems clear, at least intuitively, that if here, instead of Ω, we take its part $\Omega_{a,\alpha}$, $\alpha > 0$ (see Lemma 3.2), then we should see that $\dot{W}$ contains some wedge

$$\left\{z \in \mathbb{C}^n : \ d_{\mathbb{C}^n}(z, M) < C_\alpha \cdot d_{\mathbb{C}^n}(z, \partial D)\right\},$$

where $C_\alpha > 0$ is a constant. If this is true then we could prove: *arbitrary close* C^1*-domain* $\overline{D}$ *in* C^2*-smooth generic manifold is pluriregular,* i.e.*, the Green function* $V^*(z, \overline{D})$ *is continuous in* $\mathbb{C}^n$. The proof easily follows by the well-known criteria of pluriregularity (see [Sa80]) and by the following lemma.

Lemma 6.1. *If* $f(\lambda)$ *is a* C^1*-smooth function on* $[0, 1] \subset \mathbb{R}$*, then for every* $\varepsilon > 0$ *there exist a polynomial* $p(\lambda)$ *such that*

$$|p(\lambda) - f(\lambda)| \le \varepsilon\lambda, \ \ \lambda \in [0, 1].$$

The authors do not know any proof of the following:

Conjecture 6.2. Let $D \subset M$ be a bounded domain in M with smooth boundary. Then $\overline{D} \subset \mathbb{C}^n$ is $\Lambda_{1/2}$-pluriregular, *i.e.*, its pluricomplex Green function $V(z, \overline{D})$ is Hölder continuous of order $1/2$ in $\mathbb{C}^n$.

We note that if M is real analytic generic manifold then the conjecture is true.

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