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Almost-periodic solutions to an initial boundary value problem for model equations of resistive drift wave turbulence

SHINTARO KONDO AND ATUSI TANI

Abstract. In this paper we are concerned with the drift wave turbulence in a strong magnetic field. The existence and the uniqueness of a strong Stepanov-almost-periodic solution to the initial boundary value problems are established both for the model equations of the resistive drift wave turbulence and for the three-dimensional Hasegawa–Wakatani equations when the initial data are Stepanov-almost-periodic in the magnetic field direction.

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1. Introduction

Tokamak is the most advanced magnetic confinement device, in which an axisymmetric plasma is confined by a strong magnetic field (toroidal magnetic field); see, for example, [34, 46]. In tokamak, plasma is heated at very high temperature until the thermonuclear fusion of the nuclei occurs. In general, if the electric fields are set up in plasma by charge separation, both positive and negative charged particles obtain the drift velocities. It has been well known that the spatial gradients in plasma lead to the drift waves, whose turbulence is a natural cause of anomalous transport bringing on the dramatic reduction in confinement time ([1,14,24]). Experimentally it was found that low frequency fluctuations in tokamak turbulence plasmas are in the frequency domain of drift waves ([50, 51]). Besides, a vast variety of plasma wave phenomena are found in the planet's magnetosphere, where the anomalous transport occurs ([4,19,42]). Thereby the analysis of such drift wave turbulences is important from various point of view.

In order to study the resistive drift wave turbulence in tokamak, Hasegawa and Wakatani ([22]) in 1983 proposed the following equations for the perturbations of

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the electrostatic potential ϕ and the plasma density n:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2\Delta^2\phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) (n + \log n^*) = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) \end{cases} \tag{1.1}$$

(Hasegawa-Wakatani equations) from the two fluids model in a homogeneous strong magnetic field $\mathbf{B} = B_0\vec{e}$ and an inhomogeneous plasma equilibrium density $n^* = n^*(|x'|)$ $(x = (x_1, x_2, x_3) = (x', x_3))$ (see, [21,23,32]). Here B_0 is the strength of a magnetic field (constant), $\vec{e} = (0,0,1)$, $c_1 = T_e/(e^2\eta\omega_{ci})$, $c_2 = \mu/(\rho_s^2\omega_{ci})$, T_e is the electron temperature, e is the elementary charge, μ is the kinematic ion-viscosity coefficient, η is the resistivity, $\omega_{ci} = eB_0/m_i$ is the cyclotoron frequency, $\rho_s = \sqrt{T_e}/(\omega_{ci}\sqrt{m_i})$ is the ion Larmor radius and m_i is the ion mass. For simplicity we assume that c_1 and c_2 are positive constants.

Later in 2005 Das, Sen, Kaw, Benkadda and Beyer ([17]) studied the magnetic-curvature-driven Rayleigh–Taylor instability for the plasma density, the electrostatic potential and the vector potential for electromagnetic perturbations, and derived the model equations for it. By neglecting the effects of electromagnetic perturbations and gravitational drift, the model equations become

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2\Delta^2\phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) (n+\log n^*) = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) + D\Delta(n+\log n^*). \end{cases} \tag{1.2}$$

Here $D = m_eT_e\nu_e/(eB_0)^2$ is the diffusion coefficient, m_e is the electron mass and ν_e is the collision frequency of the electron (see [39,40]). We also assume that D is a non-negative constant for simplicity.

Concerning the mathematical issues for (1.2) and for (1.1) we have a few results [25–27]. In [25] we established the existence and uniqueness of a strong global in time solution to the initial boundary value problem first for (1.2) and second for (1.1) in the framework of periodic functions to the magnetic field direction. In [26] and [27] we proved that the solution of Hasegawa–Wakatani equations converges strongly to that of the model equations of drift wave turbulence with zero resistivity as the resistivity tends to zero.

In this paper, we consider the initial boundary value problems for (1.2) first and for (1.1) second in $\Omega \times (0, \infty)$ under the initial and boundary conditions

$$\begin{cases} \phi(x,0) = \phi_0(x), \quad n(x,0) = n_0(x) & \text{for } x \in \Omega, \\ \phi(x,t) = \Delta\phi(x,t) = n(x,t) = 0 & \text{for } x \in \Gamma,\ t > 0, \end{cases} \tag{1.3}$$

when the initial data are Stepanov-almost-periodic in the direction $\vec{e}$. Here $\Omega = \omega \times \mathbb{R}$, $\omega = \{x' \in \mathbb{R}^2 \mid |x'| < R\}$, $\partial\omega = \{x' \in \mathbb{R}^2 \mid |x'| = R\}$, $\Gamma = \partial\omega \times \mathbb{R}$, and R is a positive real number.

We use the same letters $n(x,t)$ and $n_0(x)$ in place of $n(x,t) + \log n^*(|x'|) - \log n^*(R)$ and $n_0(x) + \log n^*(|x'|) - \log n^*(R)$, respectively. Then equations (1.1) and (1.2) become

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2 \Delta^2 \phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) n = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) \quad \text{for } x \in \Omega,\ t > 0 \end{cases} \tag{1.4}$$

and

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2 \Delta^2 \phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) n = -\dfrac{c_1}{n^*} \dfrac{\partial^2}{\partial x_3^2}(\phi - n) + D\Delta n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \Omega,\ t > 0, \end{cases} \tag{1.5}$$

respectively, but (1.3) is unchanged.

The theory of almost-periodic functions was constructed by Bohr ([11]). His theory is restricted to the class of uniformly continuous functions, and so its generalizations have been until now. Among them Stepanov generalized the definition of almost-periodic functions to the class of functions in L^p_{loc} $(1 \le p < \infty)$ ([43,44]). Certainly there are some structural affinities between (Stepanov-)almost-periodic functions and purely periodic functions. Since (Stepanov-)almost-periodic functions are effective in various applications, they are used for ordinary and partial differential equations ([2, 16, 30, 38]). For some other generalizations of almost-periodic functions to partial differential equations see, for example, [5,48].

Concerning Stepanov-almost-periodic solutions of Navier–Stokes equations, we have had some results. When the external force fields are sufficiently small and Stepanov-almost-periodic in time variable, the existence and uniqueness of such solutions of the initial boundary value problem for incompressible Navier–Stokes equations were proved by Foias ([20]) in 1962 in three-dimensional case and by Prouse ([36]) in 1963 in two-dimensional case. For compressible Navier-Stokes equations, similar result was obtained by Marcati and Valli ([31]) in 1985 in three-dimensional case. The basic scheme of the proof essentially consists of the following steps ([37]): i) global existence on $[0, +\infty)$ with zero initial data; ii) global existence on $(-\infty, +\infty)$; iii) Stepanov-almost-periodicity by contradiction.

The aim of the present paper is to solve problems (1.5), (1.3) and (1.4), (1.3) with Stepanov-almost-periodic initial data to the magnetic field direction in Sobolev-Slobodetskiĭ spaces. In the proof we apply the theory of Bohr–Fourier series of

Stepanov-almost-periodic functions. However, the Riesz–Fischer theorem does not hold true for Stepanov-almost-periodic functions ([6]), so that we should pay special attention to this point. We solve the linear problem whit the Galerkin method without relying on the Riesz–Fischer theorem and then the nonlinear problem by the method of successive approximations. There have been no results to overcome the difficulty caused by the inapplicability of the Riesz–Fischer theorem as far as the present authors know.

Before describing the main theorem we introduce the function spaces and the almost periodic functions that we use in the sequel ([2,3,7,9–11,16]).

Let Ω be a domain in $\mathbb{R}^m$ for $m = 1, 2, 3, \ldots$. By $W_2^l(\Omega)$ for $l \in \mathbb{R}$, and $l \geq 0$ we denote the space of functions $u(x),\ x \in \Omega$, equipped with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{|\alpha|<l} \left\|\mathrm{D}_x^\alpha u\right\|^2_{L^2(\Omega)} + \|u\|^2_{\dot{W}_2^l(\Omega)},$$

where

$$\|u\|^2_{\dot{W}_2^l(\Omega)} = \begin{cases} \displaystyle\sum_{|\alpha|=l} \left\|\mathrm{D}_x^\alpha u\right\|^2_{L^2(\Omega)} & \text{for } l \in \mathbb{Z} \\ \displaystyle\sum_{|\alpha|=[l]} \int_\Omega\int_\Omega \frac{\left|\mathrm{D}_x^\alpha u(x) - \mathrm{D}_y^\alpha u(y)\right|^2}{|x-y|^{m+2(l-[l])}}\,\mathrm{d}x\,\mathrm{d}y & \text{for } l \notin \mathbb{Z}. \end{cases}$$

Here $[l]$ is the integral part of l, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a multi-index, and $\mathrm{D}_x^\alpha u = \partial^{|\alpha|}u/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_m^{\alpha_m}$ is the generalized derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_m$. For $1 \leq p < \infty$, we denote by $\|\cdot\|_{L^p(\Omega)}$ the norm of the Lebesgue space $L^p(\Omega)$.

The anisotropic Sobolev-Slobodetskiĭ space $W_2^{l,l/2}(Q_T)$ $(Q_T \equiv \Omega \times (0,T))$ is defined as $L^2(0,T;\, W_2^l(\Omega)) \cap L^2(\Omega;\, W_2^{l/2}(0,T))$, equipped with the norm

$$\begin{aligned} \|u\|^2_{W_2^{l,l/2}(Q_T)} &= \|u\|^2_{W_2^{l,0}(Q_T)} + \|u\|^2_{W_2^{0,l/2}(Q_T)} \\ &\equiv \int_0^T \|u(t)\|^2_{W_2^l(\Omega)}\,\mathrm{d}t + \int_\Omega \|u(x)\|^2_{W_2^{l/2}(0,T)}\,\mathrm{d}x. \end{aligned}$$

Let X be a Banach space with the norm $\|\cdot\|_X$. By $S^p(\mathbb{R}; X)$ for $1 \leq p < \infty$ we denote the subspace of $L^p_{\mathrm{loc}}(\mathbb{R}; X)$ equipped with the finite norm

$$\|u\|^p_{S^p(\mathbb{R};X)} \equiv \sup_{s\in\mathbb{R}} \int_s^{s+1} \|u(x)\|^p_X\,\mathrm{d}x.$$

The function $f(x) \in L^p_{\mathrm{loc}}(\mathbb{R}; X)$ is called Stepanov-almost-periodic (S^p-a.p.) ([43, 44,47]) if for any $\varepsilon > 0$ the set

$$E_\varepsilon(f) \equiv \left\{\sigma \in \mathbb{R} \;\middle|\; \sup_{s\in\mathbb{R}} \left(\int_s^{s+1} \|f(x+\sigma) - f(x)\|^p_X\,\mathrm{d}x\right)^{1/p} \leq \varepsilon\right\}$$

is relatively dense in $\mathbb{R}$, that is, there exists $L = L(\varepsilon) > 0$ such that $E_\varepsilon(f) \cap (a, a + L) \neq \emptyset$ for any $a \in \mathbb{R}$. By $S^p_{\rm ap}(\mathbb{R}; X)$ we denote the space of all S^p-a.p. functions from $\mathbb{R}$ to X.

Let $\omega_T \equiv \omega \times (0, T)$ and $l \in \mathbb{Z},\ l \geq 0$. We introduce the following spaces:

$$\widetilde{S}^l(\mathbb{R}; X) = \left\{ u \in S^2(\mathbb{R}; X) \;\Big|\; \|u\|^2_{\widetilde{S}^l(\mathbb{R};X)} \equiv \sum_{|\alpha|=0}^{l} \left\| \mathrm{D}^\alpha_x u \right\|^2_{S^2(\mathbb{R};X)} < \infty \right\},$$

$$\widetilde{S}^l_{\rm ap}(\mathbb{R}; X) = \left\{ u \in \widetilde{S}^l(\mathbb{R}; X) \;\Big|\; \mathrm{D}^\alpha_x u \in S^2_{\rm ap}(\mathbb{R}; X),\ |\alpha| = 0, 1, \ldots, l \right\},$$

$$\widetilde{S}^{l,l/2}\left(\mathbb{R}; L^2(\omega_T)\right) = \widetilde{S}^l\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^0\left(\mathbb{R}; L^2\left(\omega; W^{l/2}_2(0, T)\right)\right),$$

$$\widetilde{S}^{l,l/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right) = \widetilde{S}^l_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^0_{\rm ap}\left(\mathbb{R}; L^2\left(\omega; W^{l/2}_2(0, T)\right)\right)$$

Moreover we define the norm

$$\|u\|^2_{\widetilde{S}^{l,l/2}_T} \equiv \|u\|^2_{\widetilde{S}^l(\mathbb{R};L^2(\omega_T))} + \|u\|^2_{\widetilde{S}^0(\mathbb{R};L^2(\omega;W^{l/2}_2(0,T)))}.$$

For simplicity, by $\|\cdot\|$, $\|\cdot\|_{S^p}$, $\|\cdot\|_{\widetilde{S}^l}$ and $|||\cdot|||_T$ we denote the norms in $S^2\left(\mathbb{R}; L^2(\omega)\right)$, $S^p\left(\mathbb{R}; L^p(\omega)\right)$, $\widetilde{S}^l\left(\mathbb{R}; L^2(\omega)\right)$ and $S^2\left(\mathbb{R}; L^2(\omega_T)\right)$, respectively; $\nabla' \equiv (\partial_1, \partial_2)$, $\Delta' \equiv \partial_1^2 + \partial_2^2$, $\partial_k = \partial/\partial x_k$, $\|\mathrm{D}^l_x \phi\|^2_{L^p(\Omega)} \equiv \sum_{|\alpha|=l} \|\mathrm{D}^\alpha_x \phi\|^2_{L^p(\Omega)}$ $(1 \leq p < \infty,\ l = 2, 3)$.

For problems (1.5), (1.3) and (1.4), (1.3) with Stepanov-almost-periodic initial data in the magnetic field direction, we have the following main theorems.

Theorem 1.1. *Let $D > 0$, $n^*(|x'|) \in W^2_2(\omega)$ and $n^*(|x'|) \geq n_*$ with n_* being a positive constant. Assume that $(\phi_0, n_0) \in \widetilde{S}^4_{\rm ap}(\mathbb{R}; L^2(\omega)) \times \widetilde{S}^2_{\rm ap}(\mathbb{R}; L^2(\omega))$ satisfies the compatibility conditions $\phi_0(x) = \Delta\phi_0(x) = n_0(x) = 0$ for $x \in \Gamma$. Then the initial boundary value problem* (1.5), (1.3) *has a unique solution $(\phi, n) \equiv (\phi^D, n^D) \in \left(\widetilde{S}^5_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^2_{\rm ap}\left(\mathbb{R}; L^2\left(\omega; W^{3/2}_2(0, T)\right)\right)\right) \times \widetilde{S}^{3,3/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)$ for any $T > 0$.*

Theorem 1.2. *Let n^* and (ϕ_0, n_0) satisfy the same assumptions as in* Theorem 1.1. *Then the initial boundary value problem* (1.4), (1.3) *has a unique solution $(\phi, n) \in \left(\widetilde{S}^4_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^2_{\rm ap}\left(\mathbb{R}; L^2\left(\omega; W^1_2(0, T)\right)\right)\right) \times \widetilde{S}^{2,1}_{\rm ap}(\mathbb{R}; L^2(\omega_T))$ for some $T > 0$.*

In Section 2 we prove Theorem 1.1 by the following steps: i) in Section 2.1 the local-in-time existence and uniqueness of Stepanov-almost-periodic solution to problem (1.5), (1.3) with Stepanov-almost-periodic initial data by the Galerkin method and the method of successive approximations; ii) in Section 2.2 *a priori* estimates depending on D of the solution and its extension up to any time interval.

In Section 3 we prove Theorem 1.2 by the following steps: i) in Section 3.1 uniform *a priori* estimates with respect to D and the extension up to T; ii) in Section 3.2 following Kato's arguments ([25,35]) the existence by passing to the limit $D \to 0$ *via* the uniform estimates; iii) in Section 3.3 Stepanov-almost-periodicity.

Throughout this paper, we denote by c and $C(t)$ constants, independent of t and dependent on t nondecreasingly, respectively, which may differ at each occurrence.

2. Proof of Theorem 1.1

2.1. Local-in-time existence and uniqueness

2.1.1. *Auxiliary lemmas*

Let X be a Hilbert space and $\psi \in S^2_{\mathrm{ap}}(\mathbb{R}; X)$. Note that for any $\xi \in \mathbb{R}$ the mean value

$$\psi_\xi = \mathcal{M}\left\{\psi(x)\,\mathrm{e}^{-\mathrm{i}\,\xi x_3}\right\} \equiv \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \psi(x)\,\mathrm{e}^{-\mathrm{i}\,\xi x_3}\,\mathrm{d}x_3$$

exists in X ([12,49]), where $\mathrm{i} = \sqrt{-1}$.

Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\xi_k \neq \xi_{k'}$ for $k \neq k'$. For each $m \in \mathbb{N}$, it is easy to obtain

$$\mathcal{M}\left\{\left\|\psi(x_3) - \sum_{k=1}^{m} \psi_{\xi_k}\,\mathrm{e}^{-\mathrm{i}\,\xi_k x_3}\right\|_X^2\right\} = \mathcal{M}\left\{\|\psi(x_3)\|_X^2\right\} - \sum_{k=1}^{m} \|\psi_{\xi_k}\|_X^2,$$

and hence

$$\sum_{k=1}^{m} \|\psi_{\xi_k}\|_X^2 \le \mathcal{M}\left\{\|\psi(x_3)\|_X^2\right\}.$$

This inequality implies that for any $\varepsilon > 0$ there correspond at most a finite number of ξ_k for which $\|\psi_{\xi_k}\|_X > \varepsilon$. From this fact it follows that every $\|\psi_{\xi_k}\|_X$ ($\neq 0$) belongs to one of the enumerable set of inequalities

$$\|\psi_{\xi_k}\|_X > 1, \quad \frac{1}{m} \ge \|\psi_{\xi_k}\|_X > \frac{1}{m+1} \quad \text{for} \quad m = 1, 2, 3, \dots,$$

and each of these inequalities is satisfied by at most a finite number of ξ_k. Therefore, only for at most countable $\xi \in \mathbb{R}$ the quantity ψ_ξ is a non-zero element of X. We call $\sigma(\psi) = \left\{\xi \in \mathbb{R} \;\middle|\; \|\psi_\xi\|_X \neq 0\right\}$ the spectrum of ψ, and the formal series $\sum_{\xi\in\sigma(\psi)} \psi_\xi\,\mathrm{e}^{\mathrm{i}\,\xi x_3}$ the Bohr–Fourier series of ψ, which is written as

$$\psi \sim \sum_{\xi\in\sigma(\psi)} \psi_\xi\,\mathrm{e}^{\mathrm{i}\,\xi x_3}.$$

Then the following lemmas hold (see [2,7,15,16]).

Lemma 2.1. *If* $\psi,\ \psi' \in S^2_{\rm ap}(\mathbb{R}; X)$ *have the same Bohr–Fourier series, then*

$$\left\| \psi - \psi' \right\|_{S^2(\mathbb{R}; X)} = 0.$$

Lemma 2.2. *For any* $\psi \in S^2_{\rm ap}(\mathbb{R}; X)$ *Parseval's identity*

$$\mathcal{M}\left\{ \|\psi(x_3)\|_X^2 \right\} = \sum_{\xi \in \sigma(\psi)} \|\psi_\xi\|_X^2$$

holds.

Let us consider a generalized trigonometric series

$$\sum_{\xi \in \Lambda} a_\xi \, e^{i\xi x}, \tag{2.1}$$

where Λ is a countable subset of $\mathbb{R}$ and $\left\{a_\xi\right\}_{\xi\in\Lambda} \subset \mathbb{C}$. Let $\left\{\gamma_j\right\}_{j\in\mathbb{N}}$ be a basis of Λ ([12]). Bochner–Fejér sum $\mathcal{S}^m(x)$ associated with (2.1) is given by

$$\mathcal{S}^m(x) = \sum_{\nu_1=-(m!)^2}^{(m!)^2} \cdots \sum_{\nu_m=-(m!)^2}^{(m!)^2} \left(1 - \frac{|\nu_1|}{(m!)^2}\right) \cdots \left(1 - \frac{|\nu_m|}{(m!)^2}\right)$$

$$\times a_\xi^* \exp\left(i \sum_{j=1}^{m} \nu_j \frac{\gamma_j}{m!} x \right),$$

where for $\xi \in \Lambda$

$$a_\xi^* = \begin{cases} a_\xi & \text{if} \quad \displaystyle\sum_{j=1}^{m} \nu_j \frac{\gamma_j}{m!} = \xi \\ 0 & \text{if} \quad \displaystyle\sum_{j=1}^{m} \nu_j \frac{\gamma_j}{m!} \neq \xi. \end{cases}$$

By introducing an increasing symmetric sequence $\{\Lambda_m\}_{m\in\mathbb{N}}$ of Λ converging to Λ, that is, $-\Lambda_m = \Lambda_m,\ \Lambda_m \subset \Lambda_{m+1}$ and $\Lambda = \cup_m \Lambda_m,\ \mathcal{S}^m(x)$ can be written as

$$\mathcal{S}^m(x) = \sum_{\xi \in \Lambda_m} d_\xi^{(m)} a_\xi \, e^{i\xi x}$$

with constants $d_\xi^{(m)}$ satisfying $0 \le d_\xi^{(m)} \le 1$ and $\lim_{m\to\infty} d_\xi^{(m)} = 1$. Note that $d_\xi^{(m)}$ depend on ξ and m, but not on a_ξ ([16]).

We say that $\mathcal{F} \subset S^p_{\rm ap}(\mathbb{R}; X)$ is S^p-equi-almost-periodic if for any $\varepsilon > 0$ there exists a relatively dense subset E_ε of $\mathbb{R}$ such that

$$\sup_{s\in\mathbb{R}} \int_s^{s+1} \|f(x+\sigma) - f(x)\|_X^p \, dx < \varepsilon \quad \text{for } f \in \mathcal{F}, \text{ and } \sigma \in E_\varepsilon.$$

It is well-known that Riesz–Fischer theorem does not hold for $S^p_{\rm ap}(\mathbb{R}; X)$, where $1 \le p < \infty$ ([6,29]), while the following lemma holds true (see [13,18]).

Lemma 2.3. *A necessary and sufficient condition for a generalized trigonometric series* (2.1) *to be a Bohr–Fourier series of a function* $f \in S^p_{\rm ap}(\mathbb{R}; X)$ $(1 < p < \infty)$ *is that a sequence of the Bochner–Fejér sums* $\{\mathcal{S}^m(x)\}_{m\in\mathbb{N}}$ *associated with the series* (2.1) *is bounded in* $S^p(\mathbb{R}; X)$ *and* S^p*-equi-almost-periodic.*

2.1.2. *Linear problems*

In this subsection we prove the following Proposition 2.4 with the help of Lemmas 2.1, 2.3 and 2.6. For that first we prepare *a priori* estimates for Galerkin approximations (Lemmas 2.8-2.11), and second we apply Lemmas 2.1 and 2.3 to our problem with the help of Lemmas 2.8-2.12. In the same way as that for Proposition 2.4 we prove Proposition 2.5 with the help of Lemmas 2.1, 2.3 and 2.7.

Proposition 2.4. *Let* $D > 0$ *and* $n^\diamond(|x'|) \in W^2_2(\omega)$. *Assume that* $(\psi_0, n_0) \in \left(\widetilde{S}^2_{\rm ap}\left(\mathbb{R}; L^2(\omega)\right)\right)^2$ *satisfies the compatibility conditions* $\psi_0(x) = n_0(x) = 0$ *for* $x \in \Gamma$ *and* $(f, g) \in \left(\widetilde{S}^{1,1/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$ *satisfies* $f(x,t) = g(x,t) = 0$ *for* $x \in \Gamma,\ t > 0$. *Then there exists a unique solution* $(\psi, n) \in \left(\widetilde{S}^{3,3/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$ *to problem*

$$\begin{cases} \dfrac{\partial\psi}{\partial t} - c_2\Delta\psi - n^\diamond\dfrac{\partial^2 n}{\partial x_3^2} = f \\ \dfrac{\partial n}{\partial t} - D\Delta n - n^\diamond\dfrac{\partial^2 n}{\partial x_3^2} = g & \text{for } x \in \Omega, \text{ and } t > 0 \\ \psi(x,0) = \psi_0(x), \quad n(x,0) = n_0(x) & \text{for } x \in \Omega \\ \psi(x,t) = n(x,t) = 0 & \text{for } x \in \Gamma, \text{ and } t > 0. \end{cases} \tag{2.2}$$

Moreover, this solution satisfies the inequality

$$\|\psi\|_{\widetilde{S}^{3,3/2}_T} + \|n\|_{\widetilde{S}^{3,3/2}_T} \le C(T)\left(\|\psi_0\|_{\widetilde{S}^2} + \|n_0\|_{\widetilde{S}^2} + \|f\|_{\widetilde{S}^{1,1/2}_T} + \|g\|_{\widetilde{S}^{1,1/2}_T}\right)$$

with $C(T)$ *being a positive constant depending increasingly on* T.

Proposition 2.5. *Assume that* $\psi \in \widetilde{S}^{3,3/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)$. *Then the problem*

$$\begin{cases} \Delta\phi = \psi & \text{for } x \in \Omega, \text{ and } t > 0 \\ \phi(x,t) = 0 & \text{for } x \in \Gamma, \text{ and } t > 0 \end{cases} \tag{2.3}$$

has a unique solution $\phi \in \widetilde{S}^5_{\rm ap}\big(\mathbb{R}; L^2(\omega_T)\big) \cap \widetilde{S}^2_{\rm ap}\big(\mathbb{R}; L^2\big(\omega; W^{3/2}_2(0,T)\big)\big)$, *which satisfies the inequality*

$$\|\phi\|_{\widetilde{S}^5(\mathbb{R}; L^2(\omega_T))} + \|\phi\|_{\widetilde{S}^2(\mathbb{R}; L^2(\omega; W^{3/2}_2(0,T)))} \le c\,\|\psi\|_{\widetilde{S}^{3,3/2}_T}.$$

Note that the following lemmas are well-known (see, for example, [28,33,41,45]).

Lemma 2.6. *Let $l \in \mathbb{R}, l \geq 0, D > 0, \xi \in \mathbb{R}$ and $n^\diamond(|x'|) \in W_2^{1+l}(\omega)$. Assume that $(\psi_0, n_0) \in \left(W_2^{1+l}(\omega)\right)^2$ satisfies the compatibility conditions up to order* $\max\{[l - 3/2], 0\}$ *and $(f, g) \in \left(W_2^{l,l/2}(\omega_T)\right)^2$ satisfies $f(x', t) = g(x', t) = 0$ for $x' \in \partial\omega,\ t > 0$. Then there exists a unique solution $(\psi, n) \in \left(W_2^{2+l,1+l/2}(\omega_T)\right)^2$ to problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - c_2(\Delta' - \xi^2)\psi + n^\diamond \xi^2 n = f & \\ \dfrac{\partial n}{\partial t} - D(\Delta' - \xi^2)n + n^\diamond \xi^2 n = g & \text{for } x' \in \omega, \text{ and } t > 0 \\ \psi(x', 0) = \psi_0(x'), \quad n(x', 0) = n_0(x') & \text{for } x' \in \omega, \\ \psi(x', t) = n(x', t) = 0 & \text{for } x' \in \partial\omega, \text{ and } t > 0. \end{cases}$$

Moreover, this solution satisfies the inequality

$$\begin{aligned} &\|\psi\|_{W_2^{2+l,1+l/2}(\omega_T)} + \|n\|_{W_2^{2+l,1+l/2}(\omega_T)} \\ \leq\ & c_\xi \left(\|\psi_0\|_{W_2^{1+l}(\omega)} + \|n_0\|_{W_2^{1+l}(\omega)} + \|f\|_{W_2^{l,l/2}(\omega_T)} + \|g\|_{W_2^{l,l/2}(\omega_T)}\right) \end{aligned}$$

with c_ξ being a positive constant depending on ξ.

Lemma 2.7. *Assume that $\psi \in W_2^{2+l,1+l/2}(\omega_T)$, $l \geq 0$ and $\xi \in \mathbb{R}$. Then the problem*

$$\begin{cases} \left(\Delta' - \xi^2\right)\phi = \psi & \text{for } x' \in \omega, \text{ and } t > 0 \\ \phi(x', t) = 0 & \text{for } x' \in \partial\omega, \text{ and } t > 0 \end{cases}$$

has a unique solution $\phi \in L^2\big(0, T; W_2^{4+l}(\omega)\big) \cap W_2^{1+l/2}\big(0, T; W_2^2(\omega)\big)$, which satisfies the inequality

$$\|\phi\|_{L^2(0,T;W_2^{4+l}(\omega))} + \|\phi\|_{W_2^{1+l/2}(0,T;W_2^2(\omega))} \leq c_\xi\ \|\psi\|_{W_2^{2+l,1+l/2}(\omega_T)}$$

with c_ξ being a positive constant depending on ξ.

Proof of Proposition 2.4. Let us fix the symmetric increasing sequence $\{\Lambda_m\}_{m\in\mathbb{N}}$ of $\Lambda \equiv \sigma(\psi_0) \cup \sigma(n_0) \cup \sigma(f) \cup \sigma(g)$ converging to Λ. For $\xi \in \Lambda$ we consider problem

$$\begin{cases} \dfrac{\partial \psi_\xi}{\partial t} - c_2(\Delta' - \xi^2)\psi_\xi + n^\diamond \xi^2 n_\xi = f_\xi \\ \dfrac{\partial n_\xi}{\partial t} - D(\Delta' - \xi^2)n_\xi + n^\diamond \xi^2 n_\xi = g_\xi & \text{for } x' \in \omega, \text{ and } t > 0 \\ \left(\psi_\xi, n_\xi\right)|_{t=0} = \left(\psi_{0\xi}, n_{0\xi}\right) & \text{for } x' \in \omega \\ \left(\psi_\xi, n_\xi\right) = (0,0) & \text{for } x' \in \partial\omega, \text{ and } t > 0, \end{cases} \tag{2.4}$$

where $(f_\xi, g_\xi, \psi_{0\xi}, n_{0\xi}) = \mathcal{M}\{(f, g, \psi_0, n_0)\,\mathrm{e}^{-\mathrm{i}\,\xi x_3}\}$. Lemma 2.6 implies that problem (2.4) has a unique solution (ψ_ξ, n_ξ). Then it is obvious that $(\mathcal{S}^m_\psi, \mathcal{S}^m_n)$ $= \sum_{\xi\in\Lambda_m} d^{(m)}_\xi (\psi_\xi, n_\xi)\,\mathrm{e}^{\mathrm{i}\,\xi x_3}$ is a solution of problem

$$\begin{cases} \dfrac{\partial \mathcal{S}^m_\psi}{\partial t} - c_2\Delta\mathcal{S}^m_\psi - n^\diamond \dfrac{\partial^2 \mathcal{S}^m_n}{\partial x_3^2} = \mathcal{S}^m_f, \\ \dfrac{\partial \mathcal{S}^m_n}{\partial t} - D\Delta\mathcal{S}^m_n - n^\diamond \dfrac{\partial^2 \mathcal{S}^m_n}{\partial x_3^2} = \mathcal{S}^m_g & \text{for } x \in \Omega, \text{ and } t > 0 \\ \left(\mathcal{S}^m_\psi, \mathcal{S}^m_n\right)\Big|_{t=0} = \left(\mathcal{S}^m_{\psi_0}, \mathcal{S}^m_{n_0}\right) & \text{for } x \in \Omega \\ \left(\mathcal{S}^m_\psi, \mathcal{S}^m_n\right) = (0,0) & \text{for } x \in \Gamma, \text{ and } t > 0, \end{cases} \tag{2.5}$$

where $(\mathcal{S}^m_f, \mathcal{S}^m_g, \mathcal{S}^m_{\psi_0}, \mathcal{S}^m_{n_0}) = \sum_{\xi\in\Lambda_m} d^{(m)}_\xi (f_\xi, g_\xi, \psi_{0\xi}, n_{0\xi})\,\mathrm{e}^{\mathrm{i}\,\xi x_3}$.

Now we derive *a priori* estimates of $(\mathcal{S}^m_\psi, \mathcal{S}^m_n)$. For that we introduce the following cut-off function $\eta_s(x_3) \in C^1(\mathbb{R})$ for $s,\ \delta \in \mathbb{R},\ \delta > 2$:

$$\eta_s = \begin{cases} 1 & \text{on } [s, s+\delta-1] \\ 0 & \text{on } (-\infty, s-\delta] \cup [s+2\delta-1, +\infty), \end{cases}$$
$$0 \le \eta_s(x_3) \le 1,$$
$$\left|\eta_s'(x_3)\right| \le \frac{c}{\delta} \tag{2.6}$$

with a constant c independent of δ.

Lemma 2.8. *For any $t \in [0, T]$*

$$\begin{aligned} &\left\|\mathcal{S}^m_\psi(t)\right\|^2 + \left\|\mathcal{S}^m_n(t)\right\|^2 + \left|\left|\left|\nabla\mathcal{S}^m_\psi\right|\right|\right|^2_t + |||\nabla\mathcal{S}^m_n|||^2_t \\ \le\ & C(t)\left(\left\|\mathcal{S}^m_{\psi_0}\right\|^2 + \left\|\mathcal{S}^m_{n_0}\right\|^2 + \left|\left|\left|\mathcal{S}^m_f\right|\right|\right|^2_t + \left|\left|\left|\mathcal{S}^m_g\right|\right|\right|^2_t\right), \end{aligned} \tag{2.7}$$

where $C(t)$ is a positive constant depending increasingly on t.

Proof. Multiplying $(2.5)_1$ by $\mathcal{S}_\psi^m\,\eta_s$ and integrating over $\Omega^s \equiv \omega\times(s-\delta, s+2\delta-1)$, we have, by integration by parts and Schwarz's inequality,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + c_2\left\|\nabla\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\le c_2\left\|\partial_3\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\left\|\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})} + c\left\|\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|_{L^2(\Omega^s)} \\
&\quad\times\left(\left\|\partial_3\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} + \left\|\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\right) \\
&\quad + \left\|\mathcal{S}_f^m(t)\right\|_{L^2(\Omega^s)}\left\|\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} \\
&\le 3\delta\left\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\partial_3\mathcal{S}_\psi^m(t)\right\|^2 + \left(\frac{c}{\varepsilon}+\frac{1}{\delta^2}\right)\left\|\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2\right. \\
&\quad\left. + c\left\|\mathcal{S}_\psi^m(t)\right\|^2 + c\left\|\mathcal{S}_f^m(t)\right\|^2\right\}
\end{aligned} \tag{2.8}$$

for any $\varepsilon > 0$. In the last inequality, we used Young's inequality, (2.6) and the inequality

$$\sup_{s\in\mathbb{R}}\|f\|_{L^p(\Omega^s)}^p \le 3\delta\,\|f\|_{S^p}^p \quad \text{for } f\in S^p(\mathbf{R}; L^p(\omega)) \quad (1\le p<\infty). \tag{2.9}$$

Similarly, from $(2.5)_2$ we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{S}_n^m(t)\sqrt{\eta_s}\|_{L^2(\Omega^s)}^2 + D\,\|\nabla\mathcal{S}_n^m(t)\sqrt{\eta_s}\|_{L^2(\Omega^s)}^2 + \left\|\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond\eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\le 3\delta\left\{\frac{1}{\delta^2}\left(D\,\|\partial_3\mathcal{S}_n^m(t)\|^2 + \left\|\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2\right)\right. \\
&\qquad\qquad\left. + c\,(1+D)\,\|\mathcal{S}_n^m(t)\|^2 + c\left\|\mathcal{S}_g^m(t)\right\|^2\right\}.
\end{aligned} \tag{2.10}$$

Adding (2.10) and (2.8) multiplied by ε^2 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + \|\mathcal{S}_n^m(t)\sqrt{\eta_s}\|_{L^2(\Omega^s)}^2\right) + \varepsilon^2 c_2\left\|\nabla\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\qquad + D\,\|\nabla\mathcal{S}_n^m(t)\sqrt{\eta_s}\|_{L^2(\Omega^s)}^2 + \left\|\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond\eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\quad\le 3\delta\left\{\varepsilon^2\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\partial_3\mathcal{S}_\psi^m(t)\right\|^2 + \frac{D}{\delta^2}\|\partial_3\mathcal{S}_n^m(t)\|^2\right. \\
&\qquad + \left(c\varepsilon+\frac{1+\varepsilon^2}{\delta^2}\right)\left\|\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2 + c\varepsilon^2\left(\left\|\mathcal{S}_\psi^m(t)\right\|^2 + \left\|\mathcal{S}_f^m(t)\right\|^2\right) \\
&\qquad\left. + c\,(1+D)\,\|\mathcal{S}_n^m(t)\|^2 + c\left\|\mathcal{S}_g^m(t)\right\|^2\right\}.
\end{aligned}$$

Here we choose $\varepsilon(>0)$ and δ in such a way that

$$\min\{c_2, 1\} > 3\delta\left((c+1)\varepsilon + \left(1+\varepsilon^2\right)\delta^{-2}\right).$$

Then, integrating this over $[0,t]$ and taking the supremum over $s \in \mathbb{R}$, we obtain (2.7) with the help of the inequality

$$\|f\|^2 \le \sup_{s\in\mathbb{R}} \|f\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} \quad \text{for } f \in S^2\left(\mathbf{R}; L^2(\omega)\right). \tag{2.11}$$

□

Lemma 2.9. *For any $t \in [0,T]$*

$$\begin{aligned}
&\left\|\nabla\mathcal{S}^m_\psi(t)\right\|^2 + \|\nabla\mathcal{S}^m_n(t)\|^2 + \left|\left|\left|\Delta\mathcal{S}^m_\psi\right|\right|\right|^2_t \\
&\quad + \left|\left|\left|\partial_t\mathcal{S}^m_\psi\right|\right|\right|^2_t + |||\Delta\mathcal{S}^m_n|||^2_t + |||\partial_t\mathcal{S}^m_n|||^2_t \\
&\le C(t)\left(\left\|\mathcal{S}^m_{\psi_0}\right\|^2_{\widetilde{S}^1} + \|\mathcal{S}^m_{n_0}\|^2_{\widetilde{S}^1} + \left|\left|\left|\mathcal{S}^m_f\right|\right|\right|^2_t + \left|\left|\left|\mathcal{S}^m_g\right|\right|\right|^2_t\right),
\end{aligned} \tag{2.12}$$

where $C(t)$ is a positive constant depending increasingly on t.

Proof. Applying the gradient to $(2.5)_1$, multiplying it by $\nabla\mathcal{S}^m_\psi\,\eta_s$ and integrating over Ω^s, we have, by integration by parts and Schwarz's inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\mathcal{S}^m_\psi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + c_2\left\|\Delta\mathcal{S}^m_\psi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} \\
&\le c_2\left\|\Delta\mathcal{S}^m_\psi(t)\right\|_{L^2(\Omega^s)}\left\|\partial_3\mathcal{S}^m_\psi(t)\right\|_{L^2(\Omega^s)}\|\eta'_s\|_{L^\infty(\mathbb{R})} \\
&\quad + \left(c\left\|\partial_3^2\mathcal{S}^m_n(t)\sqrt{n^\diamond}\right\|_{L^2(\Omega^s)} + \left\|\mathcal{S}^m_f(t)\right\|_{L^2(\Omega^s)}\right) \\
&\quad \times\left(\left\|\Delta\mathcal{S}^m_\psi(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} + \left\|\partial_3\mathcal{S}^m_\psi(t)\right\|_{L^2(\Omega^s)}\|\eta'_s\|_{L^\infty(\mathbb{R})}\right) \\
&\le 3\delta\left\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\Delta\mathcal{S}^m_\psi(t)\right\|^2 + c\left\|\partial_3\mathcal{S}^m_\psi(t)\right\|^2\right. \\
&\quad \left. + c\left(\frac{1}{\varepsilon}+\frac{1}{\delta^2}\right)\left(\left\|\partial_3^2\mathcal{S}^m_n(t)\sqrt{n^\diamond}\right\|^2 + \left\|\mathcal{S}^m_f(t)\right\|^2\right)\right\}
\end{aligned} \tag{2.13}$$

for any $\varepsilon > 0$. In the right most inequality, we used Young's inequality, (2.6) and (2.9).

Similarly, from $(2.5)_2$ we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + D\left\|\Delta\mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&+\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\le D\left\|\Delta\mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\partial_3\mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\\
&+c\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|_{L^2(\Omega^s)}\left\|\nabla\mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\\
&+\left|\int_{\Omega^s}\nabla n^\diamond\cdot\nabla\partial_3\mathcal{S}_n^m\partial_3\mathcal{S}_n^m\eta_s\,\mathrm{d}x\right|+\left|\int_{\Omega^s}\nabla n^\diamond\cdot\nabla\mathcal{S}_n^m\partial_3\mathcal{S}_n^m\partial_3\eta_s\,\mathrm{d}x\right| \qquad (2.14)\\
&+\left\|\mathcal{S}_g^m(t)\right\|_{L^2(\Omega^s)}\Big(\left\|\Delta\mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})}\\
&+\left\|\partial_3\mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\Big)\\
&\le 3\delta\bigg\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left(D\left\|\Delta\mathcal{S}_n^m(t)\right\|^2+\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2\right)\\
&+C(D,\varepsilon,\delta)\left(\left\|\nabla\mathcal{S}_n^m(t)\right\|^2+\left\|\mathcal{S}_g^m(t)\right\|^2\right)\bigg\}
\end{aligned}
$$

for any $\varepsilon > 0$, where $C(D, \varepsilon, \delta)$ is a positive constant depending on D, ε and δ. Here in the second inequality we used Young's inequality, (2.6) and the inequalities

$$
\begin{aligned}
&\left|\int_{\Omega^s}\nabla n^\diamond\cdot\nabla\partial_3\mathcal{S}_n^m\partial_3\mathcal{S}_n^m\eta_s\,\mathrm{d}x\right|\\
&\le 3\delta c\left\|\nabla n^\diamond\right\|_{L^4(\omega)}\left\|\nabla\partial_3\mathcal{S}_n^m(t)\right\|\left\|\partial_3\mathcal{S}_n^m(t)\right\|_{S^4}\|\eta_s\|_{L^\infty(\mathbb{R})}\\
&\le 3\delta\left\{\varepsilon\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2+\frac{c}{\varepsilon}\sup_{x'\in\omega}n^*(|x'|)^7\left\|\nabla n^\diamond\right\|_{L^4(\omega)}^8\left\|\partial_3\mathcal{S}_n^m(t)\right\|^2\right\}
\end{aligned}
$$

for any $\varepsilon > 0$, and

$$
\begin{aligned}
&\left|\int_{\Omega^s}\nabla n^\diamond\cdot\nabla\mathcal{S}_n^m\partial_3\mathcal{S}_n^m\partial_3\eta_s\,\mathrm{d}x\right|\\
&\le 3\delta\left\{\frac{1}{2\delta^2}\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2+c\left\|\nabla\mathcal{S}_n^m(t)\right\|^2\right\}.
\end{aligned}
$$

Adding (2.14) and (2.13) multiplied by ε^2 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\nabla\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2+\left\|\nabla\mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2\right)\\
&\quad+\varepsilon^2c_2\left\|\Delta\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2+D\left\|\Delta\mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\quad+\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\le 3\delta\left\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left(\varepsilon^2\left\|\Delta\mathcal{S}_\psi^m(t)\right\|^2+D\left\|\Delta\mathcal{S}_n^m(t)\right\|^2\right)\right.\\
&\quad+\left(c\varepsilon+\frac{1+c\varepsilon^2}{\delta^2}\right)\left\|\nabla\partial_3\mathcal{S}_n^m(t)\sqrt{n^\diamond}\right\|^2+c\varepsilon^2\left\|\partial_3\mathcal{S}_\psi^m(t)\right\|^2\\
&\quad\left.+c\left(\varepsilon+\frac{\varepsilon^2}{\delta^2}\right)\left\|\mathcal{S}_f^m(t)\right\|^2+C(D,\varepsilon,\delta)\left(\left\|\nabla\mathcal{S}_n^m(t)\right\|^2+\left\|\mathcal{S}_g^m(t)\right\|^2\right)\right\}.
\end{aligned}$$

Here again ε (>0) and δ are chosen as

$$\min\{c_2,1\}>3\delta\left((c+1)\varepsilon+\left(1+c\varepsilon^2\right)\delta^{-2}\right).$$

Then, integrating this over $[0,t]$ and taking the supremum over $s\in\mathbb{R}$, we obtain (2.12) with the help of (2.7) and (2.11).

The estimates of the derivatives of $\mathcal{S}_\psi^m$ and $\mathcal{S}_n^m$ with respect to t are easily derived from the estimates above and equation (2.5). □

The following arguments for the estimates of the higher order derivatives are formal, since the regularity of the solution is not sufficient. However, one can justify them by the method of difference quotients or mollifiers.

Lemma 2.10. *For any $t\in[0,T]$*

$$\begin{aligned}
&\left\|\Delta\mathcal{S}_\psi^m(t)\right\|^2+\left\|\Delta\mathcal{S}_n^m(t)\right\|^2+\left|\left|\left|\nabla\Delta\mathcal{S}_\psi^m\right|\right|\right|_t^2+|||\nabla\Delta\mathcal{S}_n^m|||_t^2\\
&\le C(t)\Bigg(\left\|\mathcal{S}_{\psi_0}^m\right\|_{\widetilde{S}^2}^2+\left\|\mathcal{S}_{n_0}^m\right\|_{\widetilde{S}^2}^2+\left|\left|\left|\mathcal{S}_f^m\right|\right|\right|_t^2+\left|\left|\left|\mathcal{S}_g^m\right|\right|\right|_t^2 \qquad (2.15)\\
&\quad+\left|\left|\left|\nabla\mathcal{S}_f^m\right|\right|\right|_t^2+\left|\left|\left|\nabla\mathcal{S}_g^m\right|\right|\right|_t^2\Bigg),
\end{aligned}$$

where $C(t)$ is a positive constant depending increasingly on t.

Proof. The boundary conditions on Γ in (2.5) yield that $\partial^2/\partial x_3^2$ is a tangential derivative on Γ, and hence

$$\Delta\mathcal{S}_\psi^m(x,t) = \Delta\mathcal{S}_n^m(x,t) = 0 \quad \text{for } x \in \Gamma,\ \text{and } t > 0. \tag{2.16}$$

Applying the Laplacian Δ to $(2.5)_1$, multiplying it by $\Delta\mathcal{S}_\psi^m\,\eta_s$ and integrating over Ω^s, we have, by integration by parts and (2.16),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + c_2\left\|\nabla\Delta\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\le c_2\left\|\partial_3\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\left\|\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\\
&\quad+\int_{\Omega^s}\left(n^\diamond\frac{\partial^2\Delta\mathcal{S}_n^m}{\partial x_3^2}+2\nabla n^\diamond\cdot\frac{\partial^2\nabla\mathcal{S}_n^m}{\partial x_3^2}+\Delta n^\diamond\frac{\partial^2\mathcal{S}_n^m}{\partial x_3^2}+\Delta\mathcal{S}_f^m\right)\Delta\mathcal{S}_\psi^m\,\eta_s\,\mathrm{d}x\\
&\le c_2\left\|\partial_3\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\left\|\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\\
&\quad+\|n^\diamond\|_{L^\infty(\omega)}\|\partial_3\Delta\mathcal{S}_n^m(t)\|_{L^2(\Omega^s)}\Big(\left\|\partial_3\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})}\\
&\quad+\left\|\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\Big)\\
&\quad+c\,\|\nabla n^\diamond\|_{L^4(\omega)}\left\|\nabla\partial_3^2\mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\Delta\mathcal{S}_\psi^m(t)\right\|_{L^4(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})}\\
&\quad+\Big(\|\nabla n^\diamond\|_{L^4(\omega)}\left\|\partial_3^2\mathcal{S}_n^m(t)\right\|_{L^4(\Omega^s)}+\left\|\nabla\mathcal{S}_f^m(t)\right\|_{L^2(\Omega^s)}\Big)\\
&\quad\times\Big(\left\|\nabla\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})}+\left\|\Delta\mathcal{S}_\psi^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\Big)\\
&\le 3\delta\Big\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\nabla\Delta\mathcal{S}_\psi^m(t)\right\|^2+c\left\|\Delta\mathcal{S}_\psi^m(t)\right\|^2\\
&\quad+c\left(\frac{1}{\varepsilon}+\frac{1}{\delta^2}\right)\Big(\|\partial_3\Delta\mathcal{S}_n^m(t)\|^2+\|\Delta\mathcal{S}_n^m(t)\|^2+\left\|\nabla\mathcal{S}_f^m(t)\right\|^2\Big)\Big\}
\end{aligned}\tag{2.17}$$

for any $\varepsilon > 0$. In the right most inequality, we used Gagliardo–Nirenberg and Young's inequalities, (2.6), (2.9) and the inequalities

$$\|\partial_3^2 \mathcal{S}_n^m\| \le c\|\Delta \mathcal{S}_n^m\|, \ \left\|\nabla \partial_3^2 \mathcal{S}_n^m\right\| \le c \left\|\partial_3 \Delta \mathcal{S}_n^m\right\|.$$

Similarly, from $(2.5)_2$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta \mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + D\left\|\nabla\Delta \mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\quad + \left\|\partial_3 \Delta \mathcal{S}_n^m(t)\sqrt{n^\diamond \eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\le D\left\|\partial_3\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})} \\
&\quad + \left\|n^\diamond\right\|_{L^\infty(\omega)}\left\|\partial_3\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})} \\
&\quad + \left\|\nabla n^\diamond\right\|_{L^4(\omega)}\left\|\partial_3^2 \mathcal{S}_n^m(t)\right\|_{L^4(\Omega^s)}\Big(\left\|\nabla\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} \\
&\qquad\qquad + \left\|\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\Big) \\
&\quad + \left\|\nabla n^\diamond\right\|_{L^4(\omega)}\left\|\nabla\partial_3^2 \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\Delta \mathcal{S}_n^m(t)\right\|_{L^4(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} \\
&\quad + \left\|\nabla \mathcal{S}_g^m(t)\right\|_{L^2(\Omega^s)}\Big(\left\|\nabla\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} \\
&\qquad\qquad + \left\|\Delta \mathcal{S}_n^m(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\Big) \\
&\le 3\delta\bigg\{\left(\varepsilon + \frac{1}{\delta^2}\right)cD\left\|\nabla\Delta \mathcal{S}_n^m(t)\right\|^2 + C(D,\varepsilon,\delta)\left\|\Delta \mathcal{S}_n^m(t)\right\|^2 \\
&\qquad + c\left(1 + \frac{1}{D\varepsilon}\right)\left\|\nabla \mathcal{S}_g^m(t)\right\|^2\bigg\}
\end{aligned} \tag{2.18}$$

for any $\varepsilon > 0$, where $C(D, \varepsilon, \delta)$ is a positive constant depending on D, ε, δ. In the right most inequality, we used Gagliardo–Nirenberg and Young's inequalities, Sobolev imbedding theorem, (2.6), (2.9), (2.12) and the inequality

$$\|\mathcal{S}_n^m\|_{\widetilde{\mathfrak{S}}^{2+l}} \le c\|\Delta \mathcal{S}_n^m\|_{\widetilde{\mathfrak{S}}^l} \ \text{ for } \ l = 0, 1.$$

Adding (2.18) and (2.17) multiplied by $D\varepsilon^2$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(D\varepsilon^2\left\|\Delta\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2+\left\|\Delta\mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2\right)\\
&\quad+D\varepsilon^2c_2\left\|\nabla\Delta\mathcal{S}_\psi^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2+D\left\|\nabla\Delta\mathcal{S}_n^m(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\quad+\left\|\partial_3\Delta\mathcal{S}_n^m(t)\sqrt{n^\diamond\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\le 3\delta\Bigg\{\left(\varepsilon+\frac{1}{\delta^2}\right)D\left(\varepsilon^2\left\|\nabla\Delta\mathcal{S}_\psi^m(t)\right\|^2+c\left\|\nabla\Delta\mathcal{S}_n^m(t)\right\|^2\right)\\
&\quad+Dc\left(\varepsilon+\frac{\varepsilon^2}{\delta^2}\right)\left\|\partial_3\Delta\mathcal{S}_n^m(t)\right\|^2+cD\varepsilon^2\left\|\Delta\mathcal{S}_\psi^m(t)\right\|^2\\
&\quad+C(D,\varepsilon,\delta)\left\|\Delta\mathcal{S}_n^m(t)\right\|^2+c\left(1+\frac{1}{D\varepsilon}\right)\left\|\nabla\mathcal{S}_g^m(t)\right\|^2\\
&\quad+Dc\left(\varepsilon+\frac{\varepsilon^2}{\delta^2}\right)\left(\left\|\Delta\mathcal{S}_n^m(t)\right\|^2+\left\|\nabla\mathcal{S}_f^m(t)\right\|^2\right)\Bigg\}.
\end{aligned}$$

Choose ε (>0) and δ as

$$\min\{c_2,1\}>3\delta\left((c+1)\varepsilon+\left(c+1+c\varepsilon^2\right)\delta^{-2}\right).$$

Then, integrating this over $[0,t]$ and taking the supremum over $s\in\mathbb{R}$, we have (2.15) with the help of (2.11) and (2.12). □

Let $\mathcal{V}_{\alpha\sigma}^m(x,t)=\mathcal{S}_\alpha^m\left(x',x_3+\sigma,t\right)-\mathcal{S}_\alpha^m\left(x',x_3,t\right)$ for any $\sigma\neq 0$ $(\alpha=\psi,n)$. Then $\left(\mathcal{V}_{\psi\sigma}^m,\mathcal{V}_{n\sigma}^m\right)$ satisfies

$$\begin{cases}
\dfrac{\partial\mathcal{V}_{\psi\sigma}^m}{\partial t}-c_2\Delta\mathcal{V}_{\psi\sigma}^m-n^\diamond\dfrac{\partial^2\mathcal{V}_{n\sigma}^m}{\partial x_3^2}=\mathcal{V}_{f\sigma}^m, & \\
\dfrac{\partial\mathcal{V}_{n\sigma}^m}{\partial t}-D\Delta\mathcal{V}_{n\sigma}^m-n^\diamond\dfrac{\partial^2\mathcal{V}_{n\sigma}^m}{\partial x_3^2}=\mathcal{V}_{g\sigma}^m & \text{for } x\in\Omega,\ \text{and } t>0\\
\left(\mathcal{V}_{\psi\sigma}^m,\mathcal{V}_{n\sigma}^m\right)\Big|_{t=0}=\left(\mathcal{V}_{\psi 0\sigma}^m,\mathcal{V}_{n_0\sigma}^m\right) & \text{for } x\in\Omega\\
\left(\mathcal{V}_{\psi\sigma}^m,\mathcal{V}_{n\sigma}^m\right)=(0,0) & \text{for } x\in\Gamma,\ \text{and } t>0.
\end{cases}$$

The following lemma holds in the same way as in Lemmas 2.8-2.10.

Lemma 2.11. *For any $t \in [0, T]$*

$$\left\| \mathcal{V}^m_{\psi\sigma}(t) \right\|^2 + \left\| \mathcal{V}^m_{n\sigma}(t) \right\|^2 + \left|\left|\left| \nabla \mathcal{V}^m_{\psi\sigma} \right|\right|\right|^2_t + \left|||\nabla \mathcal{V}^m_{n\sigma}|||\right|^2_t$$

$$\leq C(t) \left(\left\| \mathcal{V}^m_{\psi_0\sigma} \right\|^2 + \left\| \mathcal{V}^m_{n_0\sigma} \right\|^2 + \left|\left|\left| \mathcal{V}^m_{f\sigma} \right|\right|\right|^2_t + \left|\left|\left| \mathcal{V}^m_{g\sigma} \right|\right|\right|^2_t \right),$$

$$\left\| \nabla \mathcal{V}^m_{\psi\sigma}(t) \right\|^2 + \left\| \nabla \mathcal{V}^m_{n\sigma}(t) \right\|^2 + \left|\left|\left| \Delta \mathcal{V}^m_{\psi\sigma} \right|\right|\right|^2_t + \left|\left|\left| \partial_t \mathcal{V}^m_{\psi\sigma} \right|\right|\right|^2_t + \left|||\Delta \mathcal{V}^m_{n\sigma}|||\right|^2_t$$

$$+ \left|||\partial_t \mathcal{V}^m_{n\sigma}|||\right|^2_t \leq C(t) \left(\left\| \mathcal{V}^m_{\psi_0\sigma} \right\|^2_{\widetilde{S}^1} + \left\| \mathcal{V}^m_{n_0\sigma} \right\|^2_{\widetilde{S}^1} + \left|\left|\left| \mathcal{V}^m_{f\sigma} \right|\right|\right|^2_t + \left|\left|\left| \mathcal{V}^m_{g\sigma} \right|\right|\right|^2_t \right),$$

$$\left\| \Delta \mathcal{V}^m_{\psi\sigma}(t) \right\|^2 + \left\| \Delta \mathcal{V}^m_{n\sigma}(t) \right\|^2 + \left|\left|\left| \nabla \Delta \mathcal{V}^m_{\psi\sigma} \right|\right|\right|^2_t + \left|||\nabla \Delta \mathcal{V}^m_{n\sigma}|||\right|^2_t$$

$$\leq C(t) \Bigg(\left\| \mathcal{V}^m_{\psi_0\sigma} \right\|^2_{\widetilde{S}^2} + \left\| \mathcal{V}^m_{n_0\sigma} \right\|^2_{\widetilde{S}^2} + \left|\left|\left| \mathcal{V}^m_{f\sigma} \right|\right|\right|^2_t + \left|\left|\left| \mathcal{V}^m_{g\sigma} \right|\right|\right|^2_t$$

$$+ \left|\left|\left| \nabla \mathcal{V}^m_{f\sigma} \right|\right|\right|^2_t + \left|\left|\left| \nabla \mathcal{V}^m_{g\sigma} \right|\right|\right|^2_t \Bigg),$$

where $C(t)$ is a positive constant depending increasingly on t.

Now we prove that $\left\{ (\mathcal{S}^m_\psi, \mathcal{S}^m_n) \right\}_{m=1}^\infty$ is a bounded sequence in the space $\left(\widetilde{S}^{3,3/2}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$ and $\left(\widetilde{S}^{3,3/2}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$-equi-almost-periodic with the help of Lemmas 2.8-2.11 and the well-known fact (see [6,8,13,16])

$$\left\| \mathcal{S}^m_\psi \right\|_{S^p(\mathbf{R}; X)} \leq \|\psi\|_{S^p(\mathbf{R}; X)}, \tag{2.19}$$

$$\left\| \mathcal{S}^m_\psi - \psi \right\|_{S^p(\mathbf{R}; X)} \to 0 \quad \text{as } m \to \infty \tag{2.20}$$

for any $\psi \in S^p_{\text{ap}}(\mathbf{R}; X)$ defined on a Banach space X $(1 \leq p < \infty)$.

Indeed, the boundedness of $\left\{ (\mathcal{S}^m_\psi, \mathcal{S}^m_n) \right\}_{m=1}^\infty$ in $\left(\widetilde{S}^{3,3/2}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$ follows from

$$\left\| \mathcal{S}^m_{\psi_0} \right\|^2_{\widetilde{S}^2} + \left\| \mathcal{S}^m_{n_0} \right\|^2_{\widetilde{S}^2} + \left\| \mathcal{S}^m_f \right\|^2_{\widetilde{S}^{1,1/2}_T} + \left\| \mathcal{S}^m_g \right\|^2_{\widetilde{S}^{1,1/2}_T}$$

$$\leq \|\psi_0\|^2_{\widetilde{S}^2} + \|n_0\|^2_{\widetilde{S}^2} + \|f\|^2_{\widetilde{S}^{1,1/2}_T} + \|g\|^2_{\widetilde{S}^{1,1/2}_T},$$

which is directly derived from (2.19) and Lemmas 2.8-2.10.

Let for any $\sigma \neq 0$

$$(F_\sigma, G_\sigma)(x,t) = (f,g)(x', x_3+\sigma, t) - (f,g)(x', x_3, t),$$

$$(\Psi_{0\sigma}, N_{0\sigma})(x) = (\psi_0, n_0)(x', x_3+\sigma) - (\psi_0, n_0)(x', x_3).$$

It is easy to see that

$$\left(F_{\sigma\xi}, G_{\sigma\xi}\right) = \left(\mathrm{e}^{\mathrm{i}\xi\sigma} - 1\right)\left(f_\xi, g_\xi\right), \quad \left(\Psi_{0\sigma\xi}, N_{0\sigma\xi}\right) = \left(\mathrm{e}^{\mathrm{i}\xi\sigma} - 1\right)(\psi_{0\xi}, n_{0\xi}),$$

$$\left(\mathcal{V}^m_{f\sigma}, \mathcal{V}^m_{g\sigma}\right)(x,t) = \left(\mathcal{S}^m_{F_\sigma}, \mathcal{S}^m_{G_\sigma}\right)(x,t), \quad \left(\mathcal{V}^m_{\psi_0\sigma}, \mathcal{V}^m_{n_0\sigma}\right)(x) = \left(\mathcal{S}^m_{\Psi_{0\sigma}}, \mathcal{S}^m_{N_{0\sigma}}\right)(x).$$

Then (2.19) yields

$$\left\|\mathcal{V}^m_{\psi_0\sigma}\right\|^2_{\widetilde{S}^2} + \left\|\mathcal{V}^m_{n_0\sigma}\right\|^2_{\widetilde{S}^2} + \left\|\mathcal{V}^m_{f\sigma}\right\|^2_{\widetilde{S}^{1,1/2}_T} + \left\|\mathcal{V}^m_{g\sigma}\right\|^2_{\widetilde{S}^{1,1/2}_T}$$

$$\leq \|\Psi_{0\sigma}\|^2_{\widetilde{S}^2} + \|N_{0\sigma}\|^2_{\widetilde{S}^2} + \|F_\sigma\|^2_{\widetilde{S}^{1,1/2}_T} + \|G_\sigma\|^2_{\widetilde{S}^{1,1/2}_T}.$$

From this we find that $\left(\mathcal{S}^m_\psi, \mathcal{S}^m_n\right)$ is $\left(\widetilde{S}^{3,3/2}(\mathbb{R}; L^2(\omega_T))\right)^2$-equi-almost-periodic by virtue of Lemmas 2.11 and 2.12:

Lemma 2.12. ([7]) *Assume that* $(f,g) \in \left(\widetilde{S}^{1,1/2}_{\mathrm{ap}}(\mathbb{R}; L^2(\omega_T))\right)^2$ *and* $(\psi_0, n_0) \in \left(\widetilde{S}^2_{\mathrm{ap}}(\mathbb{R}; L^2(\omega))\right)^2$. *Then for any* $\varepsilon > 0$ *the set*

$$E_\varepsilon = \left\{\sigma \in \mathbb{R} \;\middle|\; \left(\|\Psi_{0\sigma}\|^2_{\widetilde{S}^2} + \|N_{0\sigma}\|^2_{\widetilde{S}^2} + \|F_\sigma\|^2_{\widetilde{S}^{1,1/2}_T} + \|G_\sigma\|^2_{\widetilde{S}^{1,1/2}_T}\right)^{1/2} \leq \varepsilon\right\}$$

is relatively dense in $\mathbb{R}$.

Lemmas 2.3 implies that (ψ, n) belongs to $\left(\widetilde{S}^{3,3/2}_{\mathrm{ap}}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$. Moreover (ψ, n) is unique in the same class according to Lemma 2.1. Therefore from (2.20), we have

$$\left(\mathcal{S}^m_\psi, \mathcal{S}^m_n\right) \to (\psi, n) \quad \text{in } \left(\widetilde{S}^{3,3/2}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2 \quad \text{as } m \to \infty. \tag{2.21}$$

Since

$$\left(\mathcal{S}^m_{\psi_0}, \mathcal{S}^m_{n_0}, \mathcal{S}^m_f, \mathcal{S}^m_g\right) \to (\psi_0, n_0, f, g)$$

$$\text{in } \left(\widetilde{S}^2\left(\mathbb{R}; L^2(\omega)\right)\right)^2 \times \left(\widetilde{S}^{1,1/2}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2 \quad \text{as } m \to \infty$$

follows from (2.20), we conclude from (2.5) and (2.21) that this (ψ, n) is a solution of problem (2.2). Thus the proof of Proposition 2.4 is complete.

Just in the same way as that for Proposition 2.4 we can prove Proposition 2.5 with the help of Lemmas 2.1, 2.3 and 2.7.

2.1.3. *Nonlinear problem*

In this subsection we prove local-in-time existence and uniqueness of Stepanov-almost-periodic solution to problem (1.5), (1.3) with Stepanov-almost-periodic initial data by the method of successive approximations with the help of Propositions 2.4 and 2.5.

Let $(\psi^\star, n^\star)$ be the solution of (2.2) with $f \equiv 0$, $g \equiv 0$, $\psi_0 = \Delta\phi_0$, and $\phi^\star$ be the solution of (2.3) with $\psi = \psi^\star$. Then Propositions 2.4 and 2.5 imply that $\phi^\star \in \widetilde{S}^5_{\mathrm{ap}}\big(\mathbb{R}; L^2(\omega_T)\big) \cap \widetilde{S}^2_{\mathrm{ap}}\big(\mathbb{R}; L^2\big(\omega; W_2^{3/2}(0,T)\big)\big)$, $n^\star \in \widetilde{S}^{3,3/2}_{\mathrm{ap}}\big(\mathbb{R}; L^2(\omega_T)\big)$ satisfy the inequality

$$\begin{aligned}&\left\|\phi^\star\right\|_{\widetilde{S}^5(\mathbb{R};L^2(\omega_T))} + \left\|\phi^\star\right\|_{\widetilde{S}^2(\mathbb{R};L^2(\omega;W_2^{3/2}(0,T)))} + \left\|n^\star\right\|_{\widetilde{S}^{3,3/2}_T}\\ &\le c\left(\|\phi_0\|_{\widetilde{S}^4} + \|n_0\|_{\widetilde{S}^2}\right).\end{aligned} \tag{2.22}$$

By putting $n^\diamond = c_1/n^*$, $\Phi \equiv \phi - \phi^\star$, $N \equiv n - n^\star$, the problem (1.5), (1.3) becomes

$$\begin{cases} \dfrac{\partial\Delta\Phi}{\partial t} - c_2\Delta^2\Phi - n^\diamond\dfrac{\partial^2 N}{\partial x_3^2} = -n^\diamond\dfrac{\partial^2}{\partial x_3^2}\left(\Phi+\phi^\star\right) & \\ \qquad + \left(\nabla\left(\Phi+\phi^\star\right)\times\vec{e}\right)\cdot\nabla\Delta\left(\Phi+\phi^\star\right) \equiv F(\Phi) & \\ \dfrac{\partial N}{\partial t} - D\Delta N - n^\diamond\dfrac{\partial^2 N}{\partial x_3^2} = -n^\diamond\dfrac{\partial^2}{\partial x_3^2}\left(\Phi+\phi^\star\right) & \\ \qquad + \left(\nabla\left(\Phi+\phi^\star\right)\times\vec{e}\right)\cdot\nabla\left(N+n^\star\right) \equiv G(\Phi,N) & \text{for } x\in\Omega, \text{ and } t>0 \\ (\Phi, N)\,|_{t=0} = (0,0) & \text{for } x\in\Omega \\ (\Phi, \Delta\Phi, N) = (0,0,0) & \text{for } x\in\Gamma, \text{ and } t>0. \end{cases}$$

We shall solve this problem by the method of successive approximations. Let $\left(\Phi^{(0)}, N^{(0)}\right) = (0,0)$ and $\left(\Phi^{(m+1)}, N^{(m+1)}\right)$ for $m = 0,1,2,\ldots$ be a solution of the initial boundary value problem

$$\begin{cases} \dfrac{\partial\Delta\Phi^{(m+1)}}{\partial t} - c_2\Delta^2\Phi^{(m+1)} - n^\diamond\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} = F\left(\Phi^{(m)}\right) & \\ \dfrac{\partial N^{(m+1)}}{\partial t} - D\Delta N^{(m+1)} - n^\diamond\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} = G\left(\Phi^{(m)}, N^{(m)}\right) & \text{for } x\in\Omega, \text{ and } t>0 \\ \left(\Phi^{(m+1)}, N^{(m+1)}\right)\Big|_{t=0} = (0,0) & \text{for } x\in\Omega \\ \left(\Phi^{(m+1)}, \Delta\Phi^{(m+1)}, N^{(m+1)}\right) = (0,0,0) & \text{for } x\in\Gamma, \text{ and } t>0, \end{cases}$$

for a given $\left(\Phi^{(m)}, N^{(m)}\right) \in \left(\widetilde{S}^{5}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^{2}_{\rm ap}\left(\mathbb{R}; L^2\left(\omega; W_2^{3/2}(0,T)\right)\right)\right) \times \widetilde{S}^{3,3/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)$.

Obviously this problem is equivalent to the problem

$$\begin{cases} \Delta\Phi^{(m+1)} = \Psi^{(m+1)} \\ \dfrac{\partial \Psi^{(m+1)}}{\partial t} - c_2\Delta\Psi^{(m+1)} - n^{\diamond}\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} = F\left(\Phi^{(m)}, \Psi^{(m)}\right) \\ \dfrac{\partial N^{(m+1)}}{\partial t} - D\Delta N^{(m+1)} - n^{\diamond}\dfrac{\partial^2 N^{(m+1)}}{\partial x_3^2} = G\left(\Phi^{(m)}, N^{(m)}\right) \\ \qquad\qquad\qquad\qquad \text{for } x \in \Omega, \text{ and } t > 0 \\ \left(\Phi^{(m+1)}, \Psi^{(m+1)}, N^{(m+1)}\right)\Big|_{t=0} = (0,0,0) \qquad \text{for } x \in \Omega \\ \left(\Phi^{(m+1)}, \Psi^{(m+1)}, N^{(m+1)}\right) = (0,0,0) \qquad \text{for } x \in \Gamma, \text{ and } t > 0, \end{cases} \tag{2.23}$$

where $(\Phi^{(m)}, \Psi^{(m)}, N^{(m)}) \in \left(\widetilde{S}^{5}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^{2}_{\rm ap}\left(\mathbb{R}; L^2\left(\omega; W_2^{3/2}(0,T)\right)\right)\right) \times \left(\widetilde{S}^{3,3/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$ is given and

$$F\,(\Phi, \Psi) \equiv -n^{\diamond}\frac{\partial^2\left(\Phi + \phi^{\star}\right)}{\partial x_3^2} + \left(\nabla\left(\Phi + \phi^{\star}\right) \times \vec{e}\right) \cdot \nabla\left(\Psi + \Delta\phi^{\star}\right).$$

It is easy to see $\left(F(\Phi^{(m)}, \Psi^{(m)}), G(\Phi^{(m)}, N^{(m)})\right) \in \left(\widetilde{S}^{1,1/2}_{\rm ap}\left(\mathbb{R}; L^2(\omega_T)\right)\right)^2$, so that by virtue of Propositions 2.4 and 2.5, the problem (2.23) has a unique solution $\left(\Phi^{(m+1)}, \Psi^{(m+1)}, N^{(m+1)}\right)$ satisfying the inequality

$$\begin{aligned} z^{(m+1)}(t) \equiv{}& \left\|\Phi^{(m+1)}\right\|_{\widetilde{S}^{5}\left(\mathbb{R}; L^2(\omega_t)\right)} + \left\|\Phi^{(m+1)}\right\|_{\widetilde{S}^{2}\left(\mathbb{R}; L^2\left(\omega; W_2^{3/2}(0,t)\right)\right)} \\ &+ \left\|\Psi^{(m+1)}\right\|_{\widetilde{S}_t^{3,3/2}} + \left\|N^{(m+1)}\right\|_{\widetilde{S}_t^{3,3/2}} \\ \le{}& c\left(\left\|F\left(\Phi^{(m)}, \Psi^{(m)}\right)\right\|_{\widetilde{S}_t^{1,1/2}} + \left\|G\left(\Phi^{(m)}, N^{(m)}\right)\right\|_{\widetilde{S}_t^{1,1/2}}\right) \end{aligned}$$

for any $t \in [0, T]$. It is easy to prove

Lemma 2.13. *Let $l > 1/2$, $t > 0$ and ω be a bounded domain in $\mathbf{R}^2$. Then the following inequalities hold:*

$$\|fg\|_{W_2^{1/2}(0,t)} \le c\,\|f\|_{W_2^{1/2}(0,t)}\,\|g\|_{W_2^{l}(0,t)} \text{ for } f \in W_2^{1/2}(0,t), \text{ and } g \in W_2^{l}(0,t),$$

$$\|f\nabla g\| \le c\,\|f\|_{S^6}\,\|\nabla g\|_{S^3} \le c\,\|f\|_{\widetilde{S}^1}\,\|g\|_{\widetilde{S}^2}^{3/4}\,\|g\|^{1/4}$$

$$\text{for } f \in \widetilde{S}^1(\mathbb{R}; L^2(\omega)), \text{ and } g \in \widetilde{S}^2\left(\mathbb{R}; L^2(\omega)\right).$$

From this lemma and by interpolation and Young's inequalities one can easily obtain

$$
\begin{aligned}
&\left\| F\left(\Phi^{(m)}, \Psi^{(m)}\right)\right\|_{\widetilde{S}_t^{1,1/2}} \\
&\le c\left\|\frac{\partial^2\phi^\star}{\partial x_3^2}\right\|_{\widetilde{S}_T^{1,1/2}} + \varepsilon\left(\left\|\Phi^{(m)}\right\|_{\widetilde{S}^5(\mathbb{R};L^2(\omega_T))} + \left\|\Phi^{(m)}\right\|_{\widetilde{S}^2(\mathbb{R};L^2(\omega;W_2^{3/2}(0,T)))}\right) \\
&\quad + \frac{ct}{\varepsilon}\left(\left\|\Phi^{(m)}\right\|_{\widetilde{S}^0(\mathbb{R};L^2(\omega;W_2^1(0,T)))} + \left\|\Phi^{(m)}\right\|_{\widetilde{S}^2(\mathbb{R};L^2(\omega;W_2^1(0,T)))}\right) \\
&\quad + ct^{3/8}\left\|\Phi^{(m)}+\phi^\star\right\|_{\widetilde{S}^3(\mathbb{R};L^2(\omega;W_2^1(0,T)))}\left\|\Psi^{(m)}+\Delta\phi^\star\right\|^{1/4}_{\widetilde{S}^1(\mathbb{R};L^2(\omega;W_2^1(0,T)))} \\
&\qquad \times\left\|\Psi^{(m)}+\Delta\phi^\star\right\|^{3/4}_{\widetilde{S}^3(\mathbb{R};L^2(\omega_T))} \\
&\quad + ct^{3/8}\left\|\Phi^{(m)}+\phi^\star\right\|_{\widetilde{S}^2(\mathbb{R};L^2(\omega;W_2^{3/2}(0,T)))}\left\|\Psi^{(m)}+\Delta\phi^\star\right\|^{1/4}_{\widetilde{S}^0(\mathbb{R};L^2(\omega;W_2^{3/2}(0,T)))} \\
&\qquad \times\left\|\Psi^{(m)}+\Delta\phi^\star\right\|^{3/4}_{\widetilde{S}^2(\mathbb{R};L^2(\omega;W_2^{1/2}(0,T)))}
\end{aligned}
$$

for any $\varepsilon > 0$ and any $t \in [0, T]$.

For $\left\|G\left(\Phi^{(m)}, N^{(m)}\right)\right\|_{\widetilde{S}_t^{1,1/2}}$ we can get the similar estimate.

Consequently these inequalities and (2.22) yield

$$
z^{(m+1)}(t) \le cE + cE^2t^{3/8} + \left(\varepsilon + C(\varepsilon)t + cEt^{3/8}\right)z^{(m)}(T) + ct^{3/8}z^{(m)}(T)^2
$$

for any $t \in [0, T]$, where $E \equiv \|\phi_0\|_{\widetilde{S}^4} + \|n_0\|_{\widetilde{S}^2}$. We choose first a positive constant M in such a way that $M > cE$, second ε (> 0) as $\varepsilon M < M - cE$, and finally a positive constant T' $(\le T)$ so that

$$
cE^2T'^{3/8} + \left(C(\varepsilon)T' + cET'^{3/8}\right)M + cT'^{3/8}M^2 < M - cE - \varepsilon M.
$$

Then we conclude that $z^{(m)}(T) < M$ implies $z^{(m+1)}(T') < M$. This means that the sequence $\{(\Phi^{(m)}, \Psi^{(m)}, N^{(m)})\}_{m=0}^{\infty}$ is well-defined on $(0, T')$ and for all m $z^{(m)}(T') < M$ hold.

Now let us verify the convergence of $\{(\Phi^{(m)}, \Psi^{(m)}, N^{(m)})\}_{m=0}^{\infty}$. Subtract from (2.23) the similar equations for $(\Phi^{(m)}, \Psi^{(m)}, N^{(m)})$, and set $\Phi_*^{(m+1)} \equiv \Phi^{(m+1)} -$

$\Phi^{(m)}$, $\Psi_*^{(m+1)} \equiv \Psi^{(m+1)} - \Psi^{(m)}$, $N_*^{(m+1)} \equiv N^{(m+1)} - N^{(m)}$. Then we have

$$\begin{cases} \Delta\Phi_*^{(m+1)} = \Psi_*^{(m+1)} \\ \dfrac{\partial \Psi_*^{(m+1)}}{\partial t} - c_2 \Delta \Psi_*^{(m+1)} - n^\diamond \dfrac{\partial^2 N_*^{(m+1)}}{\partial x_3^2} \\ \qquad = F\left(\Phi^{(m)}, \Psi^{(m)}\right) - F\left(\Phi^{(m-1)}, \Psi^{(m-1)}\right) \\ \dfrac{\partial N_*^{(m+1)}}{\partial t} - D\Delta N_*^{(m+1)} - n^\diamond \dfrac{\partial^2 N_*^{(m+1)}}{\partial x_3^2} \\ \qquad = G\left(\Phi^{(m)}, N^{(m)}\right) - G\left(\Phi^{(m-1)}, N^{(m-1)}\right) \\ \qquad\qquad\qquad\qquad \text{for } x \in \Omega, \text{ and } T' > t > 0 \\ \left(\Phi_*^{(m+1)}, \Psi_*^{(m+1)}, N_*^{(m+1)}\right)\Big|_{t=0} = (0,0,0) \qquad \text{for } x \in \Omega, \\ \left(\Phi_*^{(m+1)}, \Psi_*^{(m+1)}, N_*^{(m+1)}\right) = (0,0,0) \qquad \text{for } x \in \Gamma, \text{ and } T' > t > 0. \end{cases}$$

By the same way as above, we can deduce

$$\begin{aligned} Z^{(m+1)}(t) \equiv & \left\|\Phi_*^{(m+1)}\right\|_{\widetilde{S}^5\left(\mathbb{R};L^2(\omega_t)\right)} + \left\|\Phi_*^{(m+1)}\right\|_{\widetilde{S}^2\left(\mathbb{R};L^2\left(\omega;W_2^{3/2}(0,t)\right)\right)} \\ & + \left\|\Psi_*^{(m+1)}\right\|_{\widetilde{S}_t^{3,3/2}} + \left\|N_*^{(m+1)}\right\|_{\widetilde{S}_t^{3,3/2}} \\ \le & \left(\varepsilon + C(\varepsilon)t + c\left(E + z^{(m)}(t)\right)t^{3/8}\right) Z^{(m)}(t) \end{aligned} \tag{2.24}$$

for any $\varepsilon > 0$ and any $t \in [0,T']$. Since we can find positive constants ε and $T''(\le T')$ satisfying $\varepsilon + C(\varepsilon)T'' + c\,(E+M)\,T''^{3/8} < 1$, the sequence $\{(\Phi^{(m)}, \Psi^{(m)}, N^{(m)})\}$ converges uniformly on $[0, T'']$ to (Φ, Ψ, N) as $m \to \infty$. It is clear that $(\Phi,\Psi,N) \in \left(\widetilde{S}^5_{\mathrm{ap}}\left(\mathbb{R};\, L^2(\omega_{T''})\right) \cap \widetilde{S}^2_{\mathrm{ap}}\left(\mathbb{R};\, L^2\left(\omega;\, W_2^{3/2}(0,T'')\right)\right)\right) \times \left(\widetilde{S}^{3,3/2}_{\mathrm{ap}}\left(\mathbb{R};\, L^2(\omega_{T''})\right)\right)^2$ and this (Φ, N) is a solution of problem (1.5), (1.3).

The uniqueness of such a solution can be easily proved by making use of the estimate analogous to (2.24).

2.2. Global-in-time existence and uniqueness

In this subsection we prove the global-in-time existence and uniqueness of Stepanov-almost-periodic solution to problem (1.5), (1.3) with Stepanov-almost-periodic initial data. In Subsection 2.2.1 we derive the *a priori* estimates of the solution

(ϕ, n) established in Subsection 2.1. Then in Subsection 2.2.2 we prove the global-in-time existence and uniqueness of Stepanov-almost-periodic solution to our problem with the help of *a priori* estimates established in Subsection 2.2.1.

Through this subsection, we put $n^\diamond = c_1/n^*$ also and denote inessential constants which are independent of D by the same symbol c.

2.2.1. A priori *estimates*

Let (ϕ, n) be a solution of (1.5), (1.3) belonging to

$$\left(\widetilde{S}^5_{\mathrm{ap}}\left(\mathbb{R}; L^2(\omega_T)\right) \cap \widetilde{S}^2_{\mathrm{ap}}\left(\mathbb{R}; L^2\left(\omega; W_2^{3/2}(0,T)\right)\right)\right) \times \widetilde{S}^{3,3/2}_{\mathrm{ap}}\left(\mathbb{R}; L^2(\omega_T)\right)$$

for any $T > 0$.

Lemma 2.14. *For any $t \in [0, T]$*

$$\|\Delta\phi(t)\|^2 + \|n(t)\|^2 + |||\nabla\Delta\phi|||_t^2 + D\,|||\nabla n|||_t^2 + |||\partial_3 n|||_t^2 \le C_1(t), \quad (2.25)$$

where $C_1(t)$ is a positive constant which depends increasingly on t, $\|\Delta\phi_0\|$ and $\|n_0\|$, but not on D.

Proof. Multiplying $(1.5)_1$ by $\Delta\phi\,\eta_s$ and integrating over Ω^s, we have, by integration by parts and Schwarz's inequality,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + c_2\left\|\nabla\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le c_2\left\|\partial_3\Delta\phi(t)\right\|_{L^2(\Omega^s)}\left\|\Delta\phi(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})} + c\left\|\partial_3 n(t)\sqrt{n^\diamond}\right\|_{L^2(\Omega^s)}\\
&\quad\times\left(\left\|\partial_3\Delta\phi(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s\right\|_{L^\infty(\mathbb{R})} + \left\|\Delta\phi(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\right)\\
&\quad + c\left\|\partial_3^2\phi(t)\right\|_{L^2(\Omega^s)}\left\|\Delta\phi(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s\right\|_{L^\infty(\mathbb{R})}\\
&\le 3\delta\Bigg\{\left(\varepsilon + \frac{1}{\delta^2}\right)\left\|\partial_3\Delta\phi(t)\right\|^2 + \left(\frac{c}{\varepsilon} + \frac{1}{\delta^2}\right)\left\|\partial_3 n(t)\sqrt{n^\diamond}\right\|^2\\
&+ c\left\|\Delta\phi(t)\right\|^2\Bigg\}
\end{aligned} \tag{2.26}$$

for any $\varepsilon > 0$. Here we used Young's inequality, (2.6), (2.9) and the inequality $\left\|\partial_3^2\phi\right\| \le c\|\Delta\phi\|$.

Similarly, from $(1.5)_2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\nabla n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\partial_3 n(t)\sqrt{n^\diamond \eta_s}\right\|^2_{L^2(\Omega^s)}$$

$$\leq 3\delta\left\{\frac{1}{\delta^2}\left(D\left\|\partial_3 n(t)\right\|^2 + \left\|\partial_3 n(t)\sqrt{n^\diamond}\right\|^2\right)\right. \tag{2.27}$$

$$\left. + c\,(1+D)\left\|n(t)\right\|^2 + c\left\|\Delta\phi(t)\right\|^2\right\}.$$

Adding (2.27) and (2.26) multiplied by ε^2 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right) + \varepsilon^2 c_2\left\|\nabla\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}$$

$$+ D\left\|\nabla n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\partial_3 n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}$$

$$\leq 3\delta\left\{\varepsilon^2\left(\varepsilon + \frac{1}{\delta^2}\right)\left\|\partial_3\Delta\phi(t)\right\|^2 + \frac{D}{\delta^2}\left\|\partial_3 n(t)\right\|^2\right.$$

$$+ \left(c\varepsilon + \frac{1+\varepsilon^2}{\delta^2}\right)\left\|\partial_3 n(t)\sqrt{n^\diamond}\right\|^2 + c\left(\varepsilon^2+1\right)\left\|\Delta\phi(t)\right\|^2$$

$$\left. + c\,(1+D)\left\|n(t)\right\|^2\right\}.$$

Choose ε (> 0) and δ as

$$\min\{c_2, 1\} > 3\delta\left((c+1)\varepsilon + \left(1+\varepsilon^2\right)\delta^{-2}\right).$$

Then, integrating this over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, we finally obtain (2.25) with the help of (2.11). □

Lemma 2.15. *For any $t \in [0, T]$*

$$\left\|\nabla\Delta\phi(t)\right\|^2 + \left\|\nabla n(t)\right\|^2 + \left|\left|\left|\Delta^2\phi\right|\right|\right|^2_t + |||\partial_t\Delta\phi|||^2_t + D\,|||\Delta n|||^2_t$$

$$+ |||\nabla\partial_3 n|||^2_t + |||\partial_t n|||^2_t \leq C_2\left(t, \frac{1}{D}\right), \tag{2.28}$$

where $C_2(t, 1/D)$ is a positive constant depending increasingly on t, $1/D$, $\|\Delta\phi_0\|_{\mathfrak{F}^1}$ and $\|n_0\|_{\mathfrak{F}^1}$.

Proof. Apply the gradient to $(1.5)_1$, multiply it by $\nabla\Delta\phi\,\eta_s$ and integrate over Ω^s. Then we have by virtue of the integration by parts and Schwarz' inequality

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+c_2\left\|\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le c_2\|\Delta^2\phi(t)\|_{L^2(\Omega^s)}\|\partial_3\Delta\phi(t)\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\\
&\quad+c\Bigg(\|\nabla\phi(t)\|_{L^4(\Omega^s)}\|\nabla\Delta\phi(t)\|_{L^4(\Omega^s)}+\left\|\partial_3^2 n(t)\sqrt{n^\diamond}\right\|_{L^2(\Omega^s)}\\
&\quad+\left\|\partial_3^2\phi(t)\right\|_{L^2(\Omega^s)}\Bigg)\Bigg(\left\|\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})}\\
&\quad+\|\partial_3\Delta\phi(t)\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\Bigg)\\
&\le 3\delta\Bigg\{c\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\Delta^2\phi(t)\right\|^2+\left(c+\frac{1}{\delta^2}\right)\|\nabla\Delta\phi(t)\|^2\\
&\quad+c\left(1+\frac{1}{\varepsilon}\right)\left(\left\|\partial_3^2 n(t)\sqrt{n^\diamond}\right\|^2+C_1(t)\right)+C(\varepsilon,\delta)C_1(t)^9\Bigg\}
\end{aligned}\tag{2.29}$$

for any $\varepsilon>0$, where $C(\varepsilon,\delta)$ is a positive constant depending on ε and δ. In the right most inequality, we used the Gagliardo–Nirenberg and Young's inequalities, (2.6), (2.9), (2.25) and the inequality $\|\Delta\phi\|_{\widetilde{S}^2}\le c\left\|\Delta^2\phi\right\|$.

Similarly, from $(1.5)_2$ we get (*cf*. (2.14)),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+D\left\|\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\nabla\partial_3 n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left(cD\|\Delta n(t)\|^2+\left\|\nabla\partial_3 n(t)\sqrt{n^\diamond}\right\|^2\right)+c\|\nabla\Delta\phi(t)\|^2\\
&\quad+C\left(D,\varepsilon,\delta\right)\|\nabla n(t)\|^2+cC_1(t)+\frac{c}{(D\varepsilon)^7}C_1(t)^5\Bigg\}
\end{aligned}\tag{2.30}$$

for any $\varepsilon>0$, where $C(D,\varepsilon,\delta)$ is a positive constant depending on D, ε and δ.

Adding (2.30) and (2.29) multiplied by ε^2 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\nabla\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\nabla n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\\
&+\varepsilon^2 c_2\left\|\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+D\left\|\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\nabla\partial_3 n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{c\left(\varepsilon+\frac{1}{\delta^2}\right)\left(\varepsilon^2\left\|\Delta^2\phi(t)\right\|^2+D\|\Delta n(t)\|^2\right)\\
&\quad+\left(c\left(\varepsilon+\varepsilon^2\right)+\frac{1}{\delta^2}\right)\left\|\nabla\partial_3 n(t)\sqrt{n^\diamond}\right\|^2+\left(\varepsilon^2\left(c+\frac{1}{\delta^2}\right)+c\right)\|\nabla\Delta\phi(t)\|^2\\
&\quad+C\left(D,\varepsilon,\delta\right)\|\nabla n(t)\|^2+c\left(1+\varepsilon+\varepsilon^2\right)C_1(t)+\frac{c}{(D\varepsilon)^7}C_1(t)^5+C\left(\varepsilon,\delta\right)C_1(t)^9\Bigg\}.
\end{aligned}$$

Choose ε (> 0) and δ as

$$\min\{c_2, 1\} > 3\delta \left(c\varepsilon + c\varepsilon^2 + (c+1)\delta^{-2}\right).$$

Then, integrating this over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, we obtain (2.28) with the help of (2.11) and (2.25).

The estimates of the derivatives of $\Delta\phi$ and n with respect to t are easily derived from the estimates above and equation (1.5). □

Here again the following arguments are formal since the regularity of the solution is not sufficient. However, one can justify them by the method of difference quotients or mollifiers.

Lemma 2.16. *For any $t \in [0, T]$*

$$\left\|\Delta^2\phi(t)\right\|^2 + \|\Delta n(t)\|^2 + \left|\left|\left|\nabla\Delta^2\phi\right|\right|\right|_t^2 + D\,|||\nabla\Delta n|||_t^2 + |||\partial_3\Delta n|||_t^2 \le C_3\left(t, \frac{1}{D}\right), \quad (2.31)$$

where $C_3(t, 1/D)$ is a positive constant depending increasingly on t, $1/D$, $\|\Delta\phi_0\|_{\mathfrak{S}^2}$ and $\|n_0\|_{\mathfrak{S}^2}$.

Proof. Note that the boundary conditions on Γ in (1.5) imply that $\partial^2/\partial x_3^2$ and $(\nabla\phi(x,t) \times \vec{e}) \cdot \nabla$ are tangential derivatives on Γ, and hence

$$\Delta^2\phi(x,t) = \Delta n(x,t) = 0 \quad \text{for } x \in \Gamma,\ t > 0. \quad (2.32)$$

Applying the Laplacian Δ to $(1.5)_1$, multiplying it by $\Delta^2\phi\,\eta_s$ and integrating over Ω^s, we have (*cf.* (2.17)),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\phi(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + c_2\left\|\nabla\Delta^2\phi(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 \\
&\le c_2\left\|\partial_3\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\left\|\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})} \\
&\quad+\Bigg(\left\|\mathrm{D}_x^2\phi(t)\right\|_{L^4(\Omega^s)}\|\nabla\Delta\phi(t)\|_{L^4(\Omega^s)} + \|\nabla\phi(t)\|_{L^\infty(\Omega^s)}\left\|\mathrm{D}_x^2\Delta\phi(t)\right\|_{L^2(\Omega^s)} \\
&\quad+\|\nabla n^\diamond\|\left\|\partial_3^2\phi(t)\right\|_{L^\infty(\Omega^s)} + \|n^\diamond\|_{L^\infty(\omega)}\left\|\nabla\partial_3^2\phi(t)\right\|_{L^2(\Omega^s)} \\
&\quad+\|\nabla n^\diamond\|_{L^4(\omega)}\left\|\partial_3^2 n(t)\right\|_{L^4(\Omega^s)}\Bigg)\Bigg(\left\|\nabla\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} \\
&\quad+\left\|\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\Bigg) + \|n^\diamond\|_{L^\infty(\omega)}\|\partial_3\Delta n(t)\|_{L^2(\Omega^s)} \\
&\quad\times\left(\left\|\partial_3\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} + \left\|\Delta^2\phi(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\right) \\
&\quad+c\,\|\nabla n^\diamond\|_{L^4(\omega)}\left\|\nabla\partial_3^2 n(t)\right\|_{L^2(\Omega^s)}\left\|\Delta^2\phi(t)\right\|_{L^4(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Then we have with the help of Gagliardo–Nirenberg and Young's inequalities, Sobolev imbedding theorem, (2.6), (2.9), (2.28) and the inequalities

$$\|\Delta\phi\|_{\widetilde{S}^2} \le c\|\Delta^2\phi\|, \quad \|n\|_{\widetilde{S}^2} \le c\|\Delta n\|, \quad \left\|\nabla\partial_3^2 n\right\| \le c\left\|\partial_3\Delta n\right\|,$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + c_2\left\|\nabla\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\left\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\nabla\Delta^2\phi(t)\right\|^2 + C(\varepsilon,\delta)\left(1+C_2\left(t,\frac{1}{D}\right)\right)\left\|\Delta^2\phi(t)\right\|^2\right. \qquad (2.33)\\
&\quad\left.+c\left(\frac{1}{\varepsilon}+\frac{1}{\delta^2}\right)\left(\|\partial_3\Delta n(t)\|^2+\|\Delta n(t)\|^2\right)+C(\varepsilon,\delta)\left(C_2\left(t,\frac{1}{D}\right)^2+C_2\left(t,\frac{1}{D}\right)^5\right)\right\}
\end{aligned}$$

for any $\varepsilon > 0$, where $C(\varepsilon,\delta)$ is a positive constant depending on ε and δ.

Similarly, from $(1.5)_2$ we have (*cf.* (2.18)),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\nabla\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\partial_3\Delta n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\left\{\left(\varepsilon+\frac{1}{\delta^2}\right)cD\left\|\nabla\Delta n(t)\right\|^2 + C(D,\varepsilon,\delta)\left(1+C_2\left(t,\frac{1}{D}\right)\right)\|\Delta n(t)\|^2\right.\\
&\quad\left.+\left(1+\frac{1}{D\varepsilon}\right)cC_2\left(t,\frac{1}{D}\right)\right\} \qquad (2.34)
\end{aligned}$$

for any $\varepsilon > 0$, where $C(D,\varepsilon,\delta)$ is a positive constant depending on D, ε and δ. Adding (2.34) and (2.33) multiplied by $D\varepsilon^2$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(D\varepsilon^2\left\|\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\\
&\quad+D\varepsilon^2 c_2\left\|\nabla\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\nabla\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\quad+\left\|\partial_3\Delta n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\left\{D\varepsilon^2\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\nabla\Delta^2\phi(t)\right\|^2 + cD\left(\varepsilon+\frac{1+\varepsilon^2}{\delta^2}\right)\|\nabla\Delta n(t)\|^2\right.\\
&\quad+D\varepsilon^2 C(\varepsilon,\delta)\left(1+C_2\left(t,\frac{1}{D}\right)\right)\left\|\Delta^2\phi(t)\right\|^2\\
&\quad+\left(C(D,\varepsilon,\delta)\left(1+C_2\left(t,\frac{1}{D}\right)\right)+Dc\left(\varepsilon+\frac{\varepsilon^2}{\delta^2}\right)\right)\|\Delta n(t)\|^2\\
&\quad\left.+D\varepsilon^2 C(\varepsilon,\delta)\left(C_2\left(t,\frac{1}{D}\right)^2+C_2\left(t,\frac{1}{D}\right)^5\right)+\left(1+\frac{1}{D\varepsilon}\right)cC_2\left(t,\frac{1}{D}\right)\right\}.
\end{aligned}$$

Choose ε (> 0) and δ as

$$\min\{c_2, 1\} > 3\delta\left((c+1)\varepsilon + (c+1+c\varepsilon^2)\delta^{-2}\right).$$

Then, integrating this over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, we have (2.31) with the help of (2.11) and (2.28). □

By the standard arguments with the help of Lemmas 2.14-2.16 the solution (ϕ, n) established in section 2.1 can be extended up to any time interval $[0, T]$.

2.2.2. *Stepanov-almost-periodicity*

Based on *a priori* estimates in Lemmas 2.14-2.16, we shall prove that $(\phi, n) \in \left(\widetilde{S}^5_{\mathrm{ap}}(\mathbb{R}; L^2(\omega_T)) \cap \widetilde{S}^2_{\mathrm{ap}}\left(\mathbb{R}; L^2\left(\omega; W_2^{3/2}(0,T)\right)\right)\right) \times \widetilde{S}^{3,3/2}_{\mathrm{ap}}\left(\mathbb{R}; L^2(\omega_T)\right)$ for any $T > 0$. Putting

$$\begin{cases} (\phi_\sigma, n_\sigma)(x,t) = (\phi, n)(x', x_3 + \sigma, t), \\ (\phi_{0\sigma}, n_{0\sigma})(x) = (\phi_0, n_0)(x', x_3 + \sigma), \\ (\Phi_\sigma, N_\sigma)(x,t) = (\phi_\sigma - \phi, n_\sigma - n)(x,t), \\ (\Phi_{0\sigma}, N_{0\sigma})(x) = (\phi_{0\sigma} - \phi_0, n_{0\sigma} - n_0)(x) \end{cases}$$

for any $\sigma \neq 0$. Subtracting (1.5) for (ϕ_σ, n_σ) from those for (ϕ, n), we have

$$\begin{cases} \dfrac{\partial \Delta\Phi_\sigma}{\partial t} - (\nabla\phi_\sigma \times \vec{e}) \cdot \nabla\Delta\Phi_\sigma - (\nabla\Phi_\sigma \times \vec{e}) \cdot \nabla\Delta\phi \\ \qquad = -n^\diamond \dfrac{\partial^2}{\partial x_3^2}(\Phi_\sigma - N_\sigma) + c_2\Delta^2\Phi_\sigma, \\ \dfrac{\partial N_\sigma}{\partial t} - (\nabla\phi_\sigma \times \vec{e}) \cdot \nabla N_\sigma - (\nabla\Phi_\sigma \times \vec{e}) \cdot \nabla n \\ \qquad = -n^\diamond \dfrac{\partial^2}{\partial x_3^2}(\Phi_\sigma - N_\sigma) + D\Delta N_\sigma & \text{for } x \in \Omega, \text{ and } T > t > 0 \\ \Phi_\sigma(x,0) = \Phi_{0\sigma}(x) \quad N_\sigma(x,0) = N_{0\sigma}(x) & \text{for } x \in \Omega \\ \Phi_\sigma(x,t) = \Delta\Phi_\sigma(x,t) = N_\sigma(x,t) = 0 & \text{for } x \in \Gamma, \text{ and } T > t > 0. \end{cases} \tag{2.35}$$

In the rest of this subsection, we denote inessential functions determined from Lemmas 2.14-2.16 by the same symbol $C(t)$.

Lemma 2.17. *For any $t \in [0, T]$*

$$\|\Delta\Phi_\sigma(t)\|^2 + \|N_\sigma(t)\|^2 + |||\nabla\Delta\Phi_\sigma|||_t^2 + |||\nabla N_\sigma|||_t^2 \leq C(t), \tag{2.36}$$

where $C(t)$ is a positive constant depending increasingly on t, $\|\Delta\Phi_{0\sigma}\|$ and $\|N_{0\sigma}\|$.

Proof. Multiplying $(2.35)_1$ by $\Delta\Phi_\sigma\,\eta_s$ and integrating over Ω^s, we have, with the help of (2.6), (2.9) and (2.28) (*cf*. (2.26)),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + c_2\left\|\nabla\Delta\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{c\left(\varepsilon+\frac{1}{\delta^2}\right)\|\nabla\Delta\Phi_\sigma(t)\|^2+\left(\frac{c}{\varepsilon}+\frac{1}{\delta^2}\right)\left\|\partial_3 N_\sigma(t)\sqrt{n^\diamond}\right\|^2\\
&\quad + C(t)\,\|\nabla\Phi_\sigma(t)\|^2+\left(\frac{1}{\varepsilon}+1\right)C(t)\,\|\Delta\Phi_\sigma(t)\|^2\Bigg\}
\end{aligned}\tag{2.37}$$

for any $\varepsilon > 0$. Here we used the inequalities

$$\begin{aligned}
&\left|\int_{\Omega^s}(\nabla\Phi_\sigma\times\vec{e})\cdot\nabla\Delta\phi\Delta\Phi_\sigma\eta_s\,\mathrm{d}x\right|\\
&\le c\,\|\nabla\Phi_\sigma(t)\|_{L^\infty(\Omega^s)}\,\|\nabla\Delta\phi(t)\|_{L^2(\Omega^s)}\,\|\Delta\Phi_\sigma(t)\|_{L^2(\Omega^s)}\,\|\eta_s\|_{L^\infty(\mathbb{R})}\,,\\
&\qquad\|\Phi_\sigma\|_{\widetilde{S}^{2+l}}\le c\,\|\Delta\Phi_\sigma\|_{\widetilde{S}^l}\quad(l=0,1).
\end{aligned}$$

Similarly, from $(2.35)_2$ we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+D\left\|\nabla N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\partial_3 N_\sigma(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{\frac{1}{\delta^2}\left(D\,\|\partial_3 N_\sigma(t)\|^2+\left\|\partial_3 N_\sigma(t)\sqrt{n^\diamond}\right\|^2\right)+C(t)\,\|\Delta\Phi_\sigma(t)\|^2\\
&\quad +C(t)\,\|\nabla\Phi_\sigma(t)\|^2+c\,(1+D)\,\|N_\sigma(t)\|^2\Bigg\}.
\end{aligned}\tag{2.38}$$

Adding (2.38) and (2.37) multiplied by ε^2 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\Delta\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\\
&+\varepsilon^2 c_2\left\|\nabla\Delta\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+D\left\|\nabla N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&+\left\|\partial_3 N_\sigma(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{c\varepsilon^2\left(\varepsilon+\frac{1}{\delta^2}\right)\|\nabla\Delta\Phi_\sigma(t)\|^2+\frac{D}{\delta^2}\,\|\partial_3 N_\sigma(t)\|^2\\
&\quad+\left(\varepsilon c+\frac{1+\varepsilon^2}{\delta^2}\right)\left\|\partial_3 N_\sigma(t)\sqrt{n^\diamond}\right\|^2+\left(\varepsilon+\varepsilon^2+1\right)C(t)\,\|\Delta\Phi_\sigma(t)\|^2\\
&\quad+\left(\varepsilon^2+1\right)C(t)\,\|\nabla\Phi_\sigma(t)\|^2+c\,(1+D)\,\|N_\sigma(t)\|^2\Bigg\}.
\end{aligned}$$

Choose ε (> 0) and δ as

$$\min\{c_2, 1\} > 3\delta \left(c\varepsilon + \left(c + 1 + \varepsilon^2\right)\delta^{-2}\right).$$

By integrating this over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, Gronwall's lemma and (2.11) lead to (2.36). □

Lemma 2.18. *For any* $t \in [0, T]$

$$\|\nabla\Delta\Phi_\sigma(t)\|^2 + \|\nabla N_\sigma(t)\|^2 + \left|\left|\left|\Delta^2\Phi_\sigma\right|\right|\right|_t^2 + |||\Delta N_\sigma|||_t^2 \le C(t), \quad (2.39)$$

where $C(t)$ *is a positive constant depending increasingly on* t, $\|\Delta\Phi_{0\sigma}\|_{\widetilde{S}^1}$ *and* $\|N_{0\sigma}\|_{\widetilde{S}^1}$.

Proof. Applying the gradient to $(2.35)_1$, multiplying it by $\nabla\Delta\Phi_\sigma\,\eta_s$ and integrating over Ω^s, we have, with the help of (2.6), (2.9), (2.28) and (2.31) (*cf.* (2.29)),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\Delta\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + c_2\left\|\Delta^2\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} \\
&\le 3\delta\left\{c\left(\varepsilon + \frac{1}{\delta^2}\right)\left\|\Delta^2\Phi_\sigma(t)\right\|^2 + \left(C(t)\left(\frac{1}{\delta} + \frac{1}{\varepsilon}\right) + \frac{c}{\delta^2}\right)\|\nabla\Delta\Phi_\sigma(t)\|^2 \right. \quad (2.40)\\
&\left. + \left(1 + \frac{1}{\varepsilon}\right)\left(c\left\|\partial_3^2 N_\sigma(t)\sqrt{n^\diamond}\right\|^2 + C(t)\|\Delta\Phi_\sigma(t)\|^2 + C(t)\|\nabla\Phi_\sigma(t)\|^2\right)\right\}
\end{aligned}$$

for any $\varepsilon > 0$. Here we used the inequalities

$$\begin{aligned}
&\left|\int_{\Omega^s}\nabla\left[(\nabla\phi_\sigma\times\vec{e})\cdot\nabla\Delta\Phi_\sigma + (\nabla\Phi_\sigma\times\vec{e})\cdot\nabla\Delta\phi\right]\cdot\nabla\Delta\Phi_\sigma\eta_s\,\mathrm{d}x\right| \\
&\le c\Big(\|\nabla\phi_\sigma(t)\|_{L^\infty(\Omega^s)}\|\nabla\Delta\Phi_\sigma(t)\|_{L^2(\Omega^s)} + \|\nabla\Phi_\sigma(t)\|_{L^\infty(\Omega^s)}\|\nabla\Delta\phi(t)\|_{L^2(\Omega^s)}\Big) \\
&\quad\times\left(\left\|\Delta^2\Phi_\sigma(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} + \|\partial_3\Delta\Phi_\sigma(t)\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\right),
\end{aligned}$$

$$\|\Phi_\sigma\|_{\widetilde{S}^3} \le c\|\Delta\Phi_\sigma\|_{\widetilde{S}^1}, \quad \|\Delta\Phi_\sigma\|_{\widetilde{S}^2} \le c\left\|\Delta^2\Phi_\sigma\right\|.$$

Similarly, from $(2.35)_2$ we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\Delta N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&+\left\|\nabla\partial_3 N_\sigma(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left(cD\left\|\Delta N_\sigma(t)\right\|^2+\left\|\nabla\partial_3 N_\sigma(t)\sqrt{n^\diamond}\right\|^2\right)\\
&+c\left\|\nabla\Delta\Phi_\sigma(t)\right\|^2+(cD+C(t))\left\|\nabla N_\sigma(t)\right\|^2\\
&+c\left\|\Delta\Phi_\sigma(t)\right\|^2+c\left\|\nabla\Phi_\sigma(t)\right\|^2\Bigg\}
\end{aligned}\tag{2.41}$$

for any $\varepsilon > 0$.

Adding (2.41) and (2.40) multiplied by ε^2 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\nabla\Delta\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\nabla N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\\
&+\varepsilon^2 c_2\left\|\Delta^2\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+D\left\|\Delta N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&+\left\|\nabla\partial_3 N_\sigma(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{c\left(\varepsilon+\frac{1}{\delta^2}\right)\left(\varepsilon^2\left\|\Delta^2\Phi_\sigma(t)\right\|^2+D\left\|\Delta N_\sigma(t)\right\|^2\right)\\
&+\left(c\left(\varepsilon+\varepsilon^2\right)+\frac{1}{\delta^2}\right)\left\|\nabla\partial_3 N_\sigma(t)\sqrt{n^\diamond}\right\|^2\\
&+\left(C(t)\left(\varepsilon+\frac{\varepsilon^2}{\delta}\right)+\frac{c\varepsilon^2}{\delta^2}+c\right)\left\|\nabla\Delta\Phi_\sigma(t)\right\|^2\\
&+(cD+C(t))\left\|\nabla N_\sigma(t)\right\|^2\\
&+\left(\left(\varepsilon^2+\varepsilon\right)C(t)+c\right)\left(\left\|\Delta\Phi_\sigma(t)\right\|^2+\left\|\nabla\Phi_\sigma(t)\right\|^2\right)\Bigg\}.
\end{aligned}$$

Choose ε (> 0) and δ again as

$$\min\{c_2, 1\} > 3\delta\left(c\varepsilon+c\varepsilon^2+(c+1)\delta^{-2}\right).$$

By integrating this over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, Gronwall's lemma, (2.11) and (2.36) lead to (2.39). □

The following arguments are also formal because of the less regularity of the solution. However, as above one can justify them by using the method of difference quotients or mollifiers.

Lemma 2.19. *For any $t \in [0, T]$*

$$\left\|\Delta^2\Phi_\sigma(t)\right\|^2 + \|\Delta N_\sigma(t)\|^2 + \left|\left|\left|\nabla\Delta^2\Phi_\sigma\right|\right|\right|_t^2 + |||\nabla\Delta N_\sigma|||_t^2 \le C(t), \quad (2.42)$$

where $C(t)$ is a positive constant depending increasingly on t, $\|\Delta\Phi_{0\sigma}\|_{\widetilde{S}^2}$ and $\|N_{0\sigma}\|_{\widetilde{S}^2}$.

Proof. Applying the Laplacian Δ to $(2.35)_1$, multiplying it by $\Delta^2\Phi_\sigma\,\eta_s$ and integrating over Ω^s, we have with the help of (2.6), (2.9) and (2.31) (*cf.* (2.33))

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + c_2\left\|\nabla\Delta^2\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\Bigg\{\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\nabla\Delta^2\Phi_\sigma(t)\right\|^2 + \left(\frac{1}{\varepsilon}+\frac{1}{\delta}\right)C(t)\Bigg(\left\|\Delta^2\Phi_\sigma(t)\right\|^2\\
&\quad + \|\nabla\Delta\Phi_\sigma(t)\|^2 + \|\Delta\Phi_\sigma(t)\|^2 + \|\nabla\Phi_\sigma(t)\|^2\Bigg)\\
&\quad + c\left(\frac{1}{\varepsilon}+\frac{1}{\delta^2}\right)\left(\|\partial_3\Delta N_\sigma(t)\|^2 + \|\Delta N_\sigma(t)\|^2\right)\Bigg\}
\end{aligned} \quad (2.43)$$

for any $\varepsilon > 0$. Here we used the inequalities

$$\begin{aligned}
&\left|\int_{\Omega^s}\Delta\left[(\nabla\phi_\sigma\times\vec{e})\cdot\nabla\Delta\Phi_\sigma + (\nabla\Phi_\sigma\times\vec{e})\cdot\nabla\Delta\phi\right]\Delta^2\Phi_\sigma\eta_s\,\mathrm{d}x\right|\\
&\le c\Bigg(\left\|\mathrm{D}_x^2\phi_\sigma(t)\right\|_{L^\infty(\Omega^s)}\|\nabla\Delta\Phi_\sigma(t)\|_{L^2(\Omega^s)}\\
&\quad + \|\nabla\phi_\sigma(t)\|_{L^\infty(\Omega^s)}\left\|\mathrm{D}_x^2\Delta\Phi_\sigma(t)\right\|_{L^2(\Omega^s)}\\
&\quad + \left\|\mathrm{D}_x^2\Phi_\sigma(t)\right\|_{L^4(\Omega^s)}\|\nabla\Delta\phi(t)\|_{L^4(\Omega^s)}\\
&\quad + \|\nabla\Phi_\sigma(t)\|_{L^\infty(\Omega^s)}\left\|\mathrm{D}_x^2\Delta\phi(t)\right\|_{L^2(\Omega^s)}\Bigg)\\
&\quad\times\left(\left\|\nabla\Delta^2\Phi_\sigma(t)\right\|_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})} + \left\|\Delta^2\Phi_\sigma(t)\right\|_{L^2(\Omega^s)}\|\eta_s'\|_{L^\infty(\mathbb{R})}\right),\\
&\|\Delta\Phi_\sigma\|_{\widetilde{S}^2} \le c\left\|\Delta^2\Phi_\sigma\right\|, \quad \|\Phi_\sigma\|_{\widetilde{S}^{2+l}} \le c\,\|\Delta\Phi_\sigma\|_{\widetilde{S}^l} \quad (l = 0, 1),\\
&\|N_\sigma\|_{\widetilde{S}^2} \le c\,\|\Delta N_\sigma\|, \quad \left\|\nabla\partial_3^2 N_\sigma\right\| \le c\,\|\partial_3\Delta N_\sigma\|.
\end{aligned}$$

Similarly, from $(2.35)_2$ we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\nabla\Delta N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} \\
&+\left\|\partial_3\Delta N_\sigma(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)} \\
\le\ & 3\delta\Bigg\{\left(\varepsilon+\frac{1}{\delta^2}\right)cD\left\|\nabla\Delta N_\sigma(t)\right\|^2 + \left(C(D,\varepsilon,\delta)+C(t)\right)\left\|\Delta N_\sigma(t)\right\|^2 \\
&+C(t)\left\|\nabla N_\sigma(t)\right\|^2 + \left(\frac{1}{\varepsilon}+1\right)C(t)\Bigg(\left\|\nabla\Delta\Phi_\sigma(t)\right\|^2 + \left\|\Delta\Phi_\sigma(t)\right\|^2 \\
&+\left\|\nabla\Phi_\sigma(t)\right\|^2\Bigg)\Bigg\}
\end{aligned} \tag{2.44}$$

for any $\varepsilon > 0$, where $C(D,\varepsilon,\delta)$ is a positive constant depending on D, ε and δ.

Adding (2.44) and (2.43) multiplied by $D\varepsilon^2$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(D\varepsilon^2\left\|\Delta^2\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\Delta N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right) \\
&+D\varepsilon^2 c_2\left\|\nabla\Delta^2\Phi_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\nabla\Delta N_\sigma(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} \\
&+\left\|\partial_3\Delta N_\sigma(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)} \\
\le\ & 3\delta\Bigg\{D\varepsilon^2\left(\varepsilon+\frac{1}{\delta^2}\right)\left\|\nabla\Delta^2\Phi_\sigma(t)\right\|^2 + cD\left(\varepsilon+\frac{1+\varepsilon^2}{\delta^2}\right)\left\|\nabla\Delta N_\sigma(t)\right\|^2 \\
&+\left(D\left(\varepsilon+\frac{\varepsilon^2}{\delta}\right)+\frac{1}{\varepsilon}+1\right)C(t)\Bigg(\left\|\Delta^2\Phi_\sigma(t)\right\|^2 + \left\|\nabla\Delta\Phi_\sigma(t)\right\|^2 \\
&+\left\|\Delta\Phi_\sigma(t)\right\|^2 + \left\|\nabla\Phi_\sigma(t)\right\|^2\Bigg) + \left(Dc\left(\varepsilon+\frac{\varepsilon^2}{\delta^2}\right)+C(D,\varepsilon,\delta)+C(t)\right) \\
&\times\left\|\Delta N_\sigma(t)\right\|^2 + C(t)\left\|\nabla N_\sigma(t)\right\|^2\Bigg\}.
\end{aligned}$$

Here $\varepsilon\ (>0)$ and δ are chosen as

$$\min\{c_2, 1\} > 3\delta\left((c+1)\varepsilon + \left(c+1+c\varepsilon^2\right)\delta^{-2}\right).$$

Integrating this over $[0, t]$, taking the supremum over $s \in \mathbb{R}$ and applying Gronwall's lemma, we have (2.42) by virtue of (2.11) and (2.39). □

Lemma 2.20. *For any $t \in [0, T]$*

$$|||\nabla\Delta\partial_t\Phi_\sigma|||_t^2 + |||\nabla\partial_t N_\sigma|||_t^2 \le C(t), \tag{2.45}$$

where $C(t)$ is a positive constant depending increasingly on t, $\|\Delta\Phi_{0\sigma}\|_{\widetilde{S}^2}$ and $\|N_{0\sigma}\|_{\widetilde{S}^2}$.

Proof. Apply the gradient operator to $(2.35)_1$ and $(2.35)_2$, multiply them by $\nabla\Delta\partial_t\Phi_\sigma\,\eta_s$ and $\nabla\partial_t N_\sigma\,\eta_s$, respectively, and integrate over Ω^s. Then, integrating these over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, we have (2.45) with the help of (2.6), (2.9), (2.11), (2.25), (2.28), (2.31), (2.36), (2.39) and (2.42). □

Let

$$E_{0\varepsilon} = \left\{\sigma \in \mathbb{R} \;\middle|\; \left(\|\Phi_{0\sigma}\|_{\widetilde{S}^4}^2 + \|N_{0\sigma}\|_{\widetilde{S}^2}^2\right)^{1/2} \le \varepsilon\right\},$$

$$E_\varepsilon = \left\{\sigma \in \mathbb{R} \;\middle|\; \left(\|\Phi_\sigma(t)\|_{\widetilde{S}^4}^2 + \|N_\sigma(t)\|_{\widetilde{S}^2}^2 + \left|\left|\left|\nabla\Delta^2\Phi_\sigma\right|\right|\right|_t^2 + |||\nabla\Delta N_\sigma|||_t^2 \right.\right.$$
$$\left.\left. + |||\nabla\Delta\partial_t\Phi_\sigma|||_t^2 + |||\nabla\partial_t N_\sigma|||_t^2 \right)^{1/2} \le \varepsilon\right\}$$

for $\varepsilon > 0$, $t \in [0, T]$. Applying Propositions 2.4 and 2.5 to (2.35) with the help of Lemmas 2.12 and 2.17-2.20, one can see that E_ε is relatively dense in $\mathbb{R}$. Hence $(\phi, n) \in \left(\widetilde{S}^5_{\text{ap}}\big(\mathbb{R}; L^2(\omega_T)\big) \cap \widetilde{S}^2_{\text{ap}}\big(\mathbb{R}; L^2\big(\omega; W_2^{3/2}(0, T)\big)\big)\right) \times \widetilde{S}^{3,3/2}_{\text{ap}}\big(\mathbb{R}; L^2(\omega_T)\big)$.

From the above all the proof of Theorems 1.1 is complete.

3. Proof of Theorem 1.2

3.1. Uniform estimates

Note that the estimate in Lemma 2.14 holds uniformly in D. The aim of this subsection is to get the D-independent versions of Lemma 2.15 and Lemma 2.16 for n.

Lemma 3.1. *There exists a positive constant T^* independent of D such that the estimate*

$$\begin{aligned} &\|\nabla\Delta\phi(t)\|^2 + \|\nabla n(t)\|^2 + \left|\left|\left|\Delta^2\phi\right|\right|\right|_t^2 + D\,|||\Delta n|||_t^2 + |||\nabla\partial_3 n|||_t^2 \\ &\le \frac{C^\star(0)C^\star(t)}{C^\star(0) - c\,(1 + C^\star(0))\,C^\star(t)t} \equiv C_4(t) \end{aligned} \tag{3.1}$$

holds on $[0, T^*)$ *for* $(\phi, n) = \left(\phi^D, n^D\right)$. *Here* c *is a constant independent of* D *and*

$$C^\star(t) \equiv c\left(\|\nabla\Delta\phi_0\|^2 + \|\nabla n_0\|^2 + C_1(t)\,(1+t) + C_1(t)^9 t\right).$$

Proof. Applying the gradient to $(1.5)_2$, multiplying it by $\nabla n\,\eta_s$ and integrating over Ω^s, we have with the help of (2.6), (2.9) and (2.25) (*cf*. (2.30)),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\nabla\partial_3 n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\left\{\frac{D}{\delta^2}\|\Delta n(t)\|^2 + \left(\varepsilon + \frac{1}{\delta^2}\right)\left\|\nabla\partial_3 n(t)\sqrt{n^\diamond}\right\|^2 + c\varepsilon^3\left\|\Delta^2\phi(t)\right\|^2\right.\\
&\quad\left. + \frac{c}{\varepsilon^3}\|\nabla n(t)\|^4 + c\|\nabla\Delta\phi(t)\|^2 + c\left(1 + D + \frac{1}{\varepsilon}\right)\|\nabla n(t)\|^2 + cC_1(t)\right\}
\end{aligned} \tag{3.2}$$

for any $\varepsilon > 0$. Here we used the inequalities

$$\begin{aligned}
&\left|\int_{\Omega^s}\nabla\left[(\nabla\phi\times\vec{e})\cdot\nabla n\right]\cdot\nabla n\eta_s\,\mathrm{d}x\right|\\
&\le c\left\|\mathrm{D}_x^2\phi(t)\right\|_{L^\infty(\Omega^s)}\|\nabla n(t)\|^2_{L^2(\Omega^s)}\|\eta_s\|_{L^\infty(\mathbb{R})},\\
&\|\phi\|_{\widetilde{S}^3} \le c\|\Delta\phi\|_{\widetilde{S}^1}, \quad \|\phi\|_{\widetilde{S}^4} \le c\|\Delta\phi\|_{\widetilde{S}^2} \le c\left\|\Delta^2\phi\right\|.
\end{aligned}$$

Adding (3.2) and (2.29) multiplied by ε^2 yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^2\left\|\nabla\Delta\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + \left\|\nabla n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\\
&+ \varepsilon^2 c_2\left\|\Delta^2\phi(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)} + D\left\|\Delta n(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&+ \left\|\nabla\partial_3 n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)}\\
&\le 3\delta\left\{c\varepsilon^2\left(2\varepsilon + \frac{1}{\delta^2}\right)\left\|\Delta^2\phi(t)\right\|^2 + \frac{D}{\delta^2}\|\Delta n(t)\|^2\right.\\
&+ \left(c\varepsilon + \varepsilon^2 + \frac{1}{\delta^2}\right)\left\|\nabla\partial_3 n(t)\sqrt{n^\diamond}\right\|^2\\
&+ \left(c\left(1+\varepsilon^2\right) + \frac{\varepsilon^2}{\delta^2}\right)\|\nabla\Delta\phi(t)\|^2 + c\left(1 + D + \frac{1}{\varepsilon}\right)\|\nabla n(t)\|^2\\
&\left.+ \frac{c}{\varepsilon^3}\|\nabla n(t)\|^4 + c\left(1 + \varepsilon + \varepsilon^2\right)C_1(t) + C(\varepsilon,\delta)C_1(t)^9\right\},
\end{aligned} \tag{3.3}$$

where $C(\varepsilon, \delta)$ is a positive constant depending on ε and δ.

Now we choose ε (> 0) and δ as

$$\min\{c_2, 1\} > 3\delta \left(2c\varepsilon + \varepsilon^2 + (c+1)\delta^{-2}\right).$$

Then integrating this over $[0, t]$ and taking the supremum over $s \in \mathbb{R}$, we get, with the help of (2.11) and (2.25),

$$\begin{aligned}
\|\nabla n(t)\|^2 \le\ & 3\delta\Bigg\{c\left(1 + D + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^3}\right)\int_0^t \left(\|\nabla n(\tau)\|^2 + \|\nabla n(\tau)\|^4\right) \mathrm{d}\tau \\
& + c\,\|\nabla\Delta\phi_0\|^2 + c\,\|\nabla n_0\|^2 + c\left(1 + \varepsilon^2 + \frac{\varepsilon^2}{\delta^2}\right) C_1(t) \\
& + c\left(1 + \varepsilon + \varepsilon^2\right) C_1(t)t + C(\varepsilon, \delta) C_1(t)^9 t\Bigg\} \\
\equiv\ & c\int_0^t \left(\|\nabla n\,(\tau)\|^2 + \|\nabla n\,(\tau)\|^4\right) \mathrm{d}\tau + C^\star(t) \equiv V(t).
\end{aligned}$$

Differentiating $V(t)$ with respect to t, we have

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= c\left(\|\nabla n(t)\|^2 + \|\nabla n(t)\|^4\right) + \frac{\mathrm{d}C^\star(t)}{\mathrm{d}t} \\
&\le c\left(V(t) + V(t)^2\right) + \frac{\mathrm{d}C^\star(t)}{\mathrm{d}t}.
\end{aligned}$$

Since $C^\star(t)$ is increasing, one can derive from this inequality

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{V(t)}\right) &\le c\left(\frac{1}{V(t)} + 1\right) + \frac{1}{V(t)^2}\frac{\mathrm{d}C^\star(t)}{\mathrm{d}t} \\
&\le c\left(\frac{1}{C^\star(0)} + 1\right) + \frac{1}{C^\star(t)^2}\frac{\mathrm{d}C^\star(t)}{\mathrm{d}t}.
\end{aligned}$$

Integrating this inequality over $[0, t]$, we have

$$V(t) \le \frac{C^\star(0)C^\star(t)}{C^\star(0) - c\,(1 + C^\star(0))\,C^\star(t)t}.$$

Then we choose $T^* > 0$ such that

$$C^\star(0) - c\left(1 + C^\star(0)\right) C^\star(T^*)T^* = 0.$$

This and the integral of (3.3) over $[0, t]$ lead to the inequality (3.1). □

Lemmas 2.14 and 3.1 imply that the sequence $\{(\phi^D, n^D)\}_{D>0}$ has a subsequence converging to some function (ϕ, n) weakly in

$$\left(\widetilde{S}^4\left(\mathbb{R}; L^2\left(\omega_{T^*}\right)\right) \cap \widetilde{S}^2\left(\mathbb{R}; L^2\left(\omega; W_2^1(0, T^*)\right)\right)\right) \times \widetilde{S}^{2,1}\left(\mathbb{R}; L^2(\omega_{T^*})\right).$$

In order to prove the convergence of the full sequence $\{(\phi^D, n^D)\}_{D>0}$, we prepare the following lemma. For that, similarly as in Lemma 2.14, the arguments used are formal since the regularity of the solution is not sufficient. However, one can also justify them by using the method of difference quotients or mollifiers.

Lemma 3.2. *There exists a positive constant T^{**} independent of D such that the estimate*

$$\|\Delta n(t)\|^2 + D\,|||\nabla\Delta n|||_t^2 + |||\partial_3\Delta n|||_t^2 \leq \frac{C^\dagger(0)C^\dagger(t)}{C^\dagger(0) - c\left(1 + C^\dagger(0)\right)C^\dagger(t)t} \tag{3.4}$$

holds on $[0, T^*)$ for $(\phi, n) = \left(\phi^D, n^D\right)$. Here c is a constant independent of D and

$$C^\dagger(t) \equiv c\left(\|\Delta n_0\|^2 + C_4(t)\,(1+t) + C_4(t)^{\frac{4}{3}}t\right).$$

Proof. Applying the Laplacian Δ to $(1.5)_2$, multiplying it by $\Delta n\,\eta_s$ and integrating over Ω^s, we have, by integration by parts (*cf.* (2.34)),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta n(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + D\left\|\nabla\Delta n(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2 + \left\|\partial_3\Delta n(t)\sqrt{n^\diamond\eta_s}\right\|_{L^2(\Omega^s)}^2\\
&\leq D\left\|\partial_3\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\\
&\quad + \left\|n^\diamond\right\|_{L^\infty(\omega)}\left\|\partial_3\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\\
&\quad + \left(\left\|\nabla\Delta\phi(t)\right\|_{L^4(\Omega^s)}\left\|\nabla n(t)\right\|_{L^4(\Omega^s)} + 2\left\|\mathrm{D}_x^2\phi(t)\right\|_{L^\infty(\Omega^s)}\left\|\mathrm{D}_x^2 n(t)\right\|_{L^2(\Omega^s)}\right)\\
&\quad \times\left\|\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s\right\|_{L^\infty(\mathbb{R})} + \Bigg(\left\|\Delta n^\diamond\right\|_{L^2(\omega)}\left\|\partial_3\phi(t)\right\|_{L^\infty(\Omega^s)}\\
&\quad + 2\left\|\nabla n^\diamond\right\|_{L^4(\omega)}\left\|\partial_3\nabla\phi(t)\right\|_{L^4(\Omega^s)} + \left\|n^\diamond\right\|_{L^\infty(\omega)}\left\|\partial_3\Delta\phi(t)\right\|_{L^2(\Omega^s)}\\
&\quad + \left\|\Delta n^\diamond\right\|_{L^2(\omega)}\left\|\partial_3 n(t)\right\|_{L^\infty(\Omega^s)} + 2\left\|\nabla n^\diamond\right\|_{L^4(\omega)}\left\|\partial_3\nabla n(t)\right\|_{L^4(\Omega^s)}\Bigg)\\
&\quad \times\left(\left\|\partial_3\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s\right\|_{L^\infty(\mathbb{R})} + \left\|\Delta n(t)\right\|_{L^2(\Omega^s)}\left\|\eta_s'\right\|_{L^\infty(\mathbb{R})}\right).
\end{aligned}$$

Here we used the condition (2.32). Then we have, with the help of Gagliardo–Nirenberg and Young's inequalities, Sobolev imbedding theorem, (2.6), (2.9), (3.1) and the inequalities

$$\|n\|_{\widetilde{S}^2} \le c\,\|\Delta n\|\,, \quad \|\partial_3 n\|_{\widetilde{S}^2} \le c\,\|\partial_3\Delta n\|\,, \quad \|\phi\|_{\widetilde{S}^4} \le c\,\|\Delta\phi\|_{\widetilde{S}^2} \le c\,\left\|\Delta^2\phi\right\|,$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\|\Delta n(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + D\,\|\nabla\Delta n(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + \left\|\partial_3\Delta n(t)\sqrt{n^\diamond\eta_s}\right\|^2_{L^2(\Omega^s)} \\
&\le 3\delta\Bigg\{\left(\frac{D}{\delta^2} + c\left(\varepsilon + \frac{1}{\delta^2}\right)\right)\|\partial_3\Delta n(t)\|^2 + c\,\|\Delta n(t)\|^4 \\
&\quad + (cD + C(\varepsilon,\delta))\,\|\Delta n(t)\|^2 + c\left\|\Delta^2\phi(t)\right\|^2 + cC_4(t)^{\frac{4}{3}} \\
&\quad + C(\varepsilon,\delta)C_4(t)\Bigg\}
\end{aligned} \tag{3.5}$$

for any $\varepsilon > 0$, where $C(\varepsilon,\delta)$ is a positive constant depending on ε and δ.

Choose ε (> 0) and δ as

$$\min\{c_2, 1\} > 3\delta\left(c\varepsilon + (c+1)\delta^{-2}\right).$$

Then, integrating (3.5) over $[0,t]$ and taking the supremum over $s \in \mathbb{R}$, we have with the help of (2.11) and (3.1)

$$\begin{aligned}
\|\Delta n(t)\|^2 &\le 3\delta\Bigg\{(cD + C(\varepsilon,\delta))\int_0^t \left(\|\Delta n(\tau)\|^2 + \|\Delta n(\tau)\|^4\right)\mathrm{d}\tau \\
&\qquad + c\,\|\Delta n_0\|^2 + cC_4(t) + cC_4(t)^{\frac{4}{3}}t + C(\varepsilon,\delta)C_4(t)t\Bigg\} \\
&\equiv c\int_0^t \left(\|\Delta n(\tau)\|^2 + \|\Delta n(\tau)\|^4\right)\mathrm{d}\tau + C^\dagger(t) \equiv W(t).
\end{aligned}$$

Therefore, we obtain

$$W(t) \le \frac{C^\dagger(0)C^\dagger(t)}{C^\dagger(0) - c\left(1 + C^\dagger(0)\right)C^\dagger(t)t}.$$

Now we choose $T^{**} > 0$ such that

$$C^\dagger(0) - c(1 + C^\dagger(0))C^\dagger(T^{**})T^{**} = 0.$$

This and the integral of (3.5) over $[0,t]$ lead to the inequality (3.4). □

3.2. Passage to the limit $D \to 0$

Using Lemmas 2.14, 3.1 and 3.2, we prove that the sequence $\{(\phi^D, n^D)\}_{D>0}$ is a Cauchy sequence in

$$\left(\widetilde{S}^4\left(\mathbb{R}; L^2\left(\omega_{T^{**}}\right)\right) \cap \widetilde{S}^2\left(\mathbb{R}; L^2\left(\omega; W_2^1(0, T^{**})\right)\right)\right) \times \widetilde{S}^{2,1}\left(\mathbb{R}; L^2(\omega_{T^{**}})\right).$$

Subtracting (1.5) from those with $D = D'$ $(0 < D' < D \le 1)$ and denoting by $\overline{\Phi} \equiv \phi^D - \phi^{D'}$, $\overline{N} \equiv n^D - n^{D'}$, we have

$$\begin{cases} \dfrac{\partial \Delta\overline{\Phi}}{\partial t} - \left(\nabla\phi^D \times \vec{e}\right) \cdot \nabla\Delta\overline{\Phi} - \left(\nabla\overline{\Phi} \times \vec{e}\right) \cdot \nabla\Delta\phi^{D'} \\ = -n^{\diamond}\dfrac{\partial^2}{\partial x_3^2}\left(\overline{\Phi} - \overline{N}\right) + c_2\Delta^2\overline{\Phi}, \\ \dfrac{\partial \overline{N}}{\partial t} - \left(\nabla\phi^D \times \vec{e}\right) \cdot \nabla\overline{N} - \left(\nabla\overline{\Phi} \times \vec{e}\right) \cdot \nabla n^{D'} \\ = -n^{\diamond}\dfrac{\partial^2}{\partial x_3^2}\left(\overline{\Phi} - \overline{N}\right) + D'\Delta\overline{N} + \left(D - D'\right)\Delta n^D & \text{for } x \in \Omega, \text{ and } T^{**} > t > 0 \\ \overline{\Phi}(x, 0) = \overline{N}(x, 0) = 0 & \text{for } x \in \Omega \\ \overline{\Phi}(x, t) = \Delta\overline{\Phi}(x, t) = \overline{N}(x, t) = 0 & \text{for } x \in \Gamma, \text{ and } T^{**} > t > 0. \end{cases} \tag{3.6}$$

It is to be noted that (3.6) except for $\left(D - D'\right)\Delta n^D$ is of same type as (2.35). Hence we can easily prove in the same way as in [25] that the sequence $\{(\phi^D, n^D)\}_{D>0}$ is a Cauchy sequence in

$$\left(\widetilde{S}^4\left(\mathbb{R}; L^2(\omega_{T^{**}})\right) \cap \widetilde{S}^2\left(\mathbb{R}; L^2(\omega; W_2^1(0, T^{**}))\right)\right) \times \widetilde{S}^{2,1}\left(\mathbb{R}; L^2(\omega_{T^{**}})\right).$$

Therefore, $(\phi, n)(x, t) = \lim_{D \to 0}\left(\phi^D, n^D\right)(x, t)$ exists in the same function space that $\left(\phi^D, n^D\right)$ belongs to, and this (ϕ, n) is our desired solution to problem (1.4), (1.3).

The uniqueness of such a solution can be easily proved.

3.3. Stepanov-almost-periodicity

Stepanov-almost-periodicity of the solution established in subsection 3.2 can be proved in the same way as in subsection 2.2.2 by virtue of Lemmas 2.14, 3.1 and 3.2.

From the above all the proof of Theorem 1.2 is complete.

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Meiji Institue for Advanced Study
of Mathematical Scinces
Meiji University
4-21-1 Nakano, Nakano ku, Tokyo
164-8525, Japan
kondo_s@2005.jukuin.keio.ac.jp

Department of Mathematics
Keio University
3-14-1 Hiyoshi
Yokohama 223-8522, Japan
tani@math.keio.ac.jp