Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVI (2016), 277-290

Prohibiting isolated singularities in optimal transport

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Abstract. We give natural topological conditions on the support of the target measure under which solutions to the optimal transport problem with cost function satisfying the (weak) Ma, Trudinger, and Wang condition cannot have any interior isolated singular points.

Mathematics Subject Classification (2010): 35J96 (primary).

1. Main results

The optimal transport problem is the following: given *source* and *target* probability measure spaces (Ω, μ), $\left(\bar{\Omega}, \nu\right)$, and a measurable *cost function* $c : \Omega \times \bar{\Omega} \to \mathbb{R}$, find an optimal measurable mapping $T : \Omega \to \bar{\Omega}$ defined μ-a.e., minimizing

$$\int_{\Omega} c(x, F(x))\mu(dx) \tag{1.1}$$

over the set of all measurable $F : \Omega \to \bar{\Omega}$ with $F_{\#}\mu = \nu$. Here $F_{\#}\mu$ is the *pushforward measure* of μ under F, defined by $F_{\#}\mu(E) := \mu\left(F^{-1}(E)\right)$ for any measurable $E \subset \bar{\Omega}$. This theory originated with the work of Monge in the 18th century, and was further developed by Kantorovich in the 1940's. Since then it has undergone dramatic advances, finding connections to partial differential equations, geometry, probability, mathematical physics, economics, among others; for example, [38] has a good survey of modern developments.

Y.-H.Kim's research is partially supported by Natural Sciences and Engineering Research Council of Canada Discovery Grants 371642-09 and 2014-05448, as well as the Alfred P. Sloan Research Fellowship 2012–2016. Both of the authors were also supported by the Mathematical Sciences Research Institute (MSRI) as members in the thematic program "Optimal Transport: Geometry and Dynamics". Part of this work was done while Y.-H.Kim was visiting University Paris-Est Créteil (UPEC) and Korea Advanced Institute of Science and Technology (KAIST).

Received March 31, 2011; accepted in revised form November 24, 2014.

Published online March 2016.

A fundamental problem in optimal transport theory is to understand the regularity of solutions minimizing (1.1). In the classical case where the cost function is given by the quadratic cost $c(x, \bar{x}) = \frac{1}{2}|x - \bar{x}|^2$ on $\mathbb{R}^n \times \mathbb{R}^n$ (or equivalently $c(x, \bar{x}) = -\langle x, \bar{x}\rangle$), by a result of Brenier [4] it is known that the optimal map minimizing (1.1) (defined Lebesgue a.e.) is given by the gradient of some convex function u, which satisfies in a weak sense the Monge-Ampère equation

$$\det D^2 u = F(x, \nabla u(x)).$$

As a result, regularity theory for the equation can be applied to show the minimizing map T is Hölder continuous (equivalently, u is $C^{1,\alpha}$) [6–8] if the support of the target measure ν is convex, and μ and ν are absolutely continuous with respect to Lebesgue measure, with densities bounded above and below on their supports. For more general cost functions, for example, the geodesic distance squared function on a Riemannian manifold, one requires certain structural conditions: to extend the regularity theory, one requires a necessary convexity condition on the supports of the measures called c-convexity [34], and a fourth order tensorial condition on the cost, known as the Ma-Trudinger-Wang condition (MTW) [34, 37] (which is also necessary, [31]). This can also be interpreted as a condition related to the curvature of a pseudo-metric on the product space $\Omega \times \bar{\Omega}$, induced by the cost function (see [25]). The precise conditions of c-convexity and (MTW), along with other structural conditions (namely (A0), (Twist), and (Nondeg)), are contained in Section 2. Under these conditions, a minimizer T can again be constructed from the (a.e. defined) gradient of a real valued function u (known as a *Brenier solution*, see Definition 2.6), which also satisfies a fully nonlinear PDE of Monge-Ampère type. Hölder continuity of the optimal map (or again, $C^{1,\alpha}$ regularity of u) is known, under the assumption that the source and target measures have densities bounded from above and below, see [15, 16, 29, 31] (see also [21]). For smoother measures higher regularity is also known, see [30, 34, 37].

A natural question one can ask is what happens if one of the above structural conditions is violated. In particular, we focus on the geometric condition of convexity / c-convexity of the support of the target measure, where it is known that without such conditions optimal maps may not be continuous [8, 34]: in [8] there is an example where the support of the target measure fails to be convex but remains connected, yet the optimal map is discontinuous (the support of the target measure consists of two half disks joined by a very thin channel in the middle). There are partial regularity results, which show that the set of singularities (discontinuities) is a set of zero measure [10, 12, 13]. However, it is desirable to further understand the finer geometric structure of such singular sets.

As a first step in this direction, in this paper we analyze the case of *isolated* singular points (throughout this paper, by *singularity* or *singular point* we indicate a point where an optimal map is discontinuous, or equivalently, where its Brenier solution u is not differentiable). In our main theorem we prove that under the Ma-Trudinger-Wang (MTW) condition on the cost function, if the support of the target measure has no "holes" (by which we mean a bounded, open, connected component of the complement of the support of the target measure, see Definition 2.11 for a rigorous definition), then the corresponding Brenier solution cannot have an isolated singular point in the interior of the support of the source measure. In particular, our theorem prohibits isolated singularities when the target measure has bounded, contractible support; however, it should be noted that the condition under which isolated singularities *can* happen is more restrictive. Throughout this paper, we will denote the closure, interior, and boundary of a set A by A^{cl}, A^{int}, and A^{∂} respectively. Our main result is as follows:

Theorem 1.1. *Let M and $\bar{M}$ be n-dimensional Riemannian manifolds, and Ω and $\bar{\Omega}$ be bounded, open subsets in M and $\bar{M}$, respectively. Let c be a cost function $c : \Omega^{\mathrm{cl}} \times \bar{\Omega}^{\mathrm{cl}} \to \mathbb{R}$ that satisfies* (A0)*,* (Twist)*,* (Nondeg)*, and* (MTW)*, and assume that Ω and $\bar{\Omega}$ are c-convex with respect to each other. Relevant definitions can be found in Section 2. (See also Remark* 1.3*).*

Consider two absolutely continuous probability measures $\mu = f\,\mathrm{dVol}$ and $\nu = g\,\mathrm{dVol}$ on M and $\bar{M}$, respectively, with supports $\operatorname{spt}\mu \subset \Omega$ and $\operatorname{spt}\nu \subset \bar{\Omega}$. Assume that $\operatorname{spt}\nu \cap \bar{\Omega}^{\partial} = \emptyset$ and that there exists a constant $0 < \Lambda < \infty$ such that

$$\Lambda^{-1} \le f,\ g \le \Lambda \tag{1.2}$$

on their supports.

Finally, let u be a Brenier solution (see Definition 2.6*) to the optimal transport problem with cost c. For each $x_0 \in (\operatorname{spt}\mu)^{\mathrm{int}}$, if there are no holes (see Definition* 2.11*) in $\operatorname{spt}\nu$ that are c-*convex *with respect to x_0, then x_0 cannot be an isolated singular point of u.*

Remark 1.2. The preceding theorem shows that even with some topological holes, as long as the holes are not c-convex, the optimal map cannot have an interior isolated singular point. In the special case when M and $\bar{M}$ are open subsets of Euclidean space and $c(x, \bar{x}) = \frac{1}{2}|x - \bar{x}|^2$, c-convexity reduces to ordinary convexity (we will henceforth refer to this setting as the *Euclidean case*). Thus in the Euclidean case, there is no interior isolated singular point for an optimal map if the target measure has no *convex* holes in its support.

Remark 1.3. The conditions (A0), (Twist), (Nondeg) are satisfied for a large class of examples. In particular, the important example of $c(x, \bar{x}) = \frac{1}{2}d^2(x, \bar{x})$ given by the geodesic distance squared on a Riemannian manifold satisfies the various conditions outside the cut locus (we will refer to this as the *Riemannian case*). On the other hand, the condition (MTW) is more restrictive. In the Riemannian case, it is necessary that the sectional curvature be nonnegative everywhere [31] (however, this is not sufficient [24], see also [18]). Known Riemannian examples which do satisfy (MTW) include the round sphere [32] and its products and quotients [26], as well as small perturbations [11,17,20,33]. Additionally, many non-Riemannian examples can be found in [27,28,34,37]. For a survey of these results, we refer the reader to [38].

Remark 1.4. In the two dimensional Euclidean case, Figalli [12] studied the geometric structure of the singular set by giving characterizations for all of its possible behaviours, and the above result on isolated singularity follows as a special case. In higher dimensions, it seems that no result on the geometric structure of singular sets (similar to the one in [12]) is currently known. For some previous related works in the Euclidean case, see [2,23], [39], and [9, Section 5] in the case of dimension 2, and [3,35] in higher dimensions.

While the other results mentioned above consider isolated singularities of the Monge-Ampère equation, the papers [39] and [9, Section 5] deal specifically with the case of the optimal transport problem (however, still in the Euclidean case). Both results discuss the finer question of Lipschitz or C^1 propagation of singularities, but assume stronger conditions aside from just topological restrictions on the support of the target measure. Specifically, [39] assumes that all singular points have a subdifferential of affine dimension at most one, while [9] requires the support of the source measure be convex. Our main result applies to a more general class of c, and also requires no hypothesis on $\operatorname{spt}\mu$; in fact we obtain the condition required for [39] in the course of our proof (see Proposition 3.2).

2. Relevant definitions and preliminaries

In this section we gather some relevant definitions and facts about c-convex potential functions in relation to solutions of the optimal transport problem. Some good references are [22,38].

Let M and $\bar{M}$ be n-dimensional Riemannian manifolds and Ω and $\bar{\Omega}$ be bounded, open subsets in M and $\bar{M}$, respectively. Let c be a measurable cost function $c : \Omega^{\mathrm{cl}} \times \bar{\Omega}^{\mathrm{cl}} \to \mathbb{R}$. We start out by stating the various assumptions we may require on our cost function c.

<u>**Smoothness of cost**</u>:

$$c \in C^4(\Omega^{\mathrm{cl}} \times \bar{\Omega}^{\mathrm{cl}}). \tag{A0}$$

<u>**Twist**</u>:

We will say c satisfies condition (Twist) if each of the mappings

$$\begin{aligned} \bar{\Omega} \ni \bar{x} &\mapsto -Dc(x_0, \bar{x}) \in T^*_{x_0}M, \\ \Omega \ni x &\mapsto -\bar{D}c(x, \bar{x}_0) \in T^*_{\bar{x}_0}\bar{M}, \end{aligned} \tag{Twist}$$

are injective for each $x_0 \in \Omega$ and $\bar{x}_0 \in \bar{\Omega}$. Here, D, $\bar{D}$ denote the usual differential in the x or $\bar{x}$ variable.

Remark 2.1. We use the standard notation $\exp^c_{x_0}(\cdot)$ and $\exp^c_{\bar{x}_0}(\cdot)$ to denote the inverses of the above two mappings. Also, for any $A \subset \Omega$, $\bar{x} \in \bar{\Omega}$ or $\bar{A} \subset \bar{\Omega}$, $x \in \Omega$, we will write

$$\begin{aligned} [A]_{\bar{x}} &:= -\bar{D}c(A, \bar{x}), \\ \left[\bar{A}\right]_x &:= -Dc(x, \bar{A}). \end{aligned}$$

We also comment here, for the cost $c(x, \bar{x}) = -\langle x, \bar{x}\rangle$ on $\mathbb{R}^n \times \mathbb{R}^n$, these mappings are both just the identity map.

Definition 2.2 (c-convexity of a set [34]). If $A \subset \Omega$ and $\bar{x} \in \bar{\Omega}$, we say that *A is c-convex with respect to $\bar{x}$* if the set $[A]_{\bar{x}}$ is a convex subset of $T^*_{\bar{x}}\bar{M}$. If $\bar{A} \subset \bar{\Omega}$ and $x \in \Omega$, we define when *$\bar{A}$ is c-convex with respect to x* and *A and $\bar{A}$ are c-convex with respect to each other* in the obvious way.

Nondegeneracy

We say c satisfies condition (Nondeg) if, for each $x \in \Omega$ and $\bar{x} \in \bar{\Omega}$, the linear mapping

$$-\bar{D}Dc(x, \bar{x}) : T_{\bar{x}}\bar{M} \to T_{-Dc(x,\bar{x})}\big(T^*_x M\big) \cong T^*_x M \qquad \text{(Nondeg)}$$

is invertible (and consequently, so is its adjoint mapping, $-D\bar{D}c(x, \bar{x}) : T_x M \to T^*_{\bar{x}}\bar{M}$).

We say c satisfies the condition (MTW) if, for any $x \in \Omega$, $\bar{x} \in \bar{\Omega}$, and $V \in T_x M$, $\eta \in T^*_x M$ with $\eta(V) = 0$,

$$-(c_{ij,pq} - c_{ij,r}c^{r,s}c_{s,pq})c^{p,k}c^{q,l}(x, \bar{x})V^i V^j \eta_k \eta_l \geq 0. \qquad \text{(MTW)}$$

Here we fix coordinate systems on M and $\bar{M}$ and take all derivatives with respect to these coordinates; lower indices before a comma denote derivatives of c with respect to the x variable, and lower indices after a comma denote derivatives with respect to the $\bar{x}$ variable. Also, a pair of raised indices denotes the inverse of a matrix.

We next define some basic concepts of use in c-convex geometry.

Definition 2.3. A real valued function u defined on Ω is said to be *c-convex* if for any $x_0 \in \Omega$, there exists some $\bar{x}_0 \in \bar{\Omega}$ and $\lambda_0 \in \mathbb{R}$ such that

$$-c(x_0, \bar{x}_0) + \lambda_0 = u(x_0),$$
$$-c(x, \bar{x}_0) + \lambda_0 \leq u(x)$$

for all $x \in \Omega$. Any function of the form $-c(\cdot, \bar{x}_0) + \lambda_0$ is called a *c-affine function (with focus $\bar{x}_0$)*, and if it satisfies the above relations is said to *support u from below at x_0*.

We also define the c-subdifferential of a c-convex function, and the subdifferential of a semi-convex function.

Definition 2.4. The *subdifferential* of a semi-convex function u at a point $x \in (\mathrm{dom}\,(u))^{\mathrm{int}}$ is defined by the set

$$\partial u\,(x) := \big\{\bar{p} \in T^*_x M \mid u(x) + \langle v, \bar{p}\rangle + o(|v|) \leq u(\exp_x(v)),\ T_x M \ni v \to 0\big\},$$

here $\exp_x$ is the Riemannian exponential mapping on M.

Similarly, the *c-subdifferential* of a c-convex function u at a point $x \in (\mathrm{dom}\,(u))^{\mathrm{int}}$ is defined as the set

$$\partial_c u(x) := \left\{\bar{x} \in \bar{\Omega} \mid -c(y,\bar{x}) + c(x,\bar{x}) + u(x) \le u(y),\ \forall y \in \mathrm{dom}\,(u)\right\}.$$

If $A \subseteq \Omega$, we write

$$\partial_c u(A) := \bigcup_{x \in A} \partial_c u(x).$$

Remark 2.5. Note that if u is semi-convex, each $\partial u\,(x)$ is a nonempty, convex set, and for any point x where u is differentiable, we have $\partial u\,(x) = \{Du(x)\}$. Additionally, it is known that if c satisfies (A0), then a c-convex function is semi-convex, hence in particular it is differentiable a.e.

Additionally, if u is c-convex it is not difficult to see that its c-subdifferential is *c-monotone*, *i.e.* for any $x_0, x_1 \in \Omega$ and $\bar{x}_0 \in \partial_c u(x_0), \bar{x}_1 \in \partial_c u(x_1)$, we have

$$c(x_0,\bar{x}_0) + c(x_1,\bar{x}_1) \le c(x_0,\bar{x}_1) + c(x_1,\bar{x}_0).$$

Definition 2.6. Suppose c satisfies (Twist). A *Brenier solution (to the optimal transport problem with cost $c(x,\bar{x})$) pushing μ forward to ν* is a c-convex function u defined on $\mathrm{spt}\,\mu$ such that

$$T_{\#}\mu = \nu,$$
$$T\,(\mathrm{spt}\,\mu) \subseteq \mathrm{spt}\,\nu,$$

where T is the *Brenier map* defined for a.e. x (where u is differentiable) by

$$T(x) := \exp^c_x(Du(x)).$$

If u is a Brenier solution pushing μ forward to ν, then it is well known that T as defined above is optimal in (1.1).

The following result (discovered by Loeper [31] in $\mathbb{R}^n$, further developed in [19,25,33,36], and extended to Riemannian manifolds under certain conditions) details certain geometric properties of c-convex functions. It will play a key role in our main proof.

Theorem 2.7 (Loeper's maximum principle [31]). *Suppose c, Ω, and $\bar{\Omega}$ satisfy the conditions of Theorem* 1.1. *Also let $x_0 \in \Omega$, $\bar{p}_0$, $\bar{p}_1 \in \left[\bar{\Omega}\right]_{x_0}$, and $\bar{x}(t) := \exp^c_{x_0}((1-t)\bar{p}_0 + t\bar{p}_1)$. Then for any $x \in \Omega$,*

$$\begin{aligned} &-c(x,\bar{x}(t)) + c(x_0,\bar{x}(t)) \\ &\qquad \le \max\left\{-c(x,\bar{x}(0)) + c(x_0,\bar{x}(0)), -c(x,\bar{x}(1)) + c(x_0,\bar{x}(1))\right\}. \end{aligned} \tag{2.1}$$

An analogous inequality holds with the roles of Ω and $\bar{\Omega}$ reversed.

This theorem has several important consequences, we will require the following two of them later; the second of which was first observed and used in [14, 15] and [29].

Corollary 2.8 ([31, Theorem 3.1]). *Suppose c, Ω, and $\bar{\Omega}$ satisfy the same conditions as Theorem* 2.7 *above, and u is a c-convex function on Ω. Then for any $x_0 \in \Omega$,*

$$[\partial_c u(x_0)]_{x_0} = \partial u\,(x_0)\,, \tag{2.2}$$

in particular, $\partial_c u(x_0)$ is c-convex with respect to x_0.

Corollary 2.9. *Suppose c, Ω, and $\bar{\Omega}$ satisfy the same conditions as Theorem* 2.7*, and u is a c-convex function on Ω. Then, for any $\bar{x}_0 \in \bar{\Omega}$ and $\lambda_0 \in \mathbb{R}$, the* section

$$\{x \in \Omega \mid u(x) \leq -c(x, \bar{x}_0) + \lambda_0\}$$

is c-convex with respect to $\bar{x}_0$.

We also state here a fairly standard result concerning c-subdifferentials of c-*cones*.

Lemma 2.10. *Suppose c, Ω, and $\bar{\Omega}$ satisfy the conditions of Theorem* 1.1*, u is a c-convex function, m_0 is a c-affine function with focus $\bar{x}_0$, and let $S_0 := \{u \leq m_0\}$ be such that $S_0 \cap \Omega^{\partial} = \emptyset$. Fix $x_0 \in S_0^{\mathrm{int}}$ and define* the c-cone over the section S_0 with vertex x_0 *by*

$$K^c_{x_0, S_0}(x) := \sup_m m(x),$$

where the supremum is taken over all c-affine functions m satisfying $m \leq m_0$ on S_0^{∂}, and $m(x_0) \leq u(x_0)$. Then,

$$\partial_c K^c_{x_0, S_0}(x_0) \subset \partial_c u(S_0), \tag{2.3}$$

and if $\bar{x}_0 \in \bar{\Omega}^{\mathrm{int}}$,

$$-Dc(x_0, \bar{x}_0) \in \left[\partial_c K^c_{x_0, S_0}(x_0)\right]^{\mathrm{int}}_{x_0}. \tag{2.4}$$

Proof. A proof of (2.3) is contained, for example, in [21, Lemma 3.4].

We will show (2.4). By assumption, $m_0(x_0) - u(x_0) > 0$. Let us write $\bar{p}_0 := -Dc(x_0, \bar{x}_0)$, then recall that $\exp^c_{x_0}(\bar{p}_0) = \bar{x}_0$. Hence for a sufficiently small $r_0 > 0$, we have (for some $C > 0$ depending only on derivatives of the cost c) that for all $\bar{p} \in B_{r_0}(\bar{p}_0)$, the function $m_{\bar{p}}(x) := -c(x, \exp^c_{x_0}(\bar{p})) + c(x_0, \exp^c_{x_0}(\bar{p})) + u(x_0)$ satisfies

$$\begin{aligned} m_{\bar{p}}(x) &= (-c(x, \exp^c_{x_0}(\bar{p})) + c(x_0, \exp^c_{x_0}(\bar{p})) + m_0(x_0)) - m_0(x_0) + u(x_0) \\ &\leq m_0(x) + Cr_0 - (m_0(x_0) - u(x_0)) \\ &< m_0(x) \end{aligned}$$

for all $x \in S_0^\partial$. Thus $m_{\bar{p}}$ is admissible in the supremum defining $K^c_{x_0,S_0}$, and it must support the c-cone $K^c_{x_0,S_0}$ from below at x_0. In particular we have for all $\bar{p} \in B_{r_0}(\bar{p}_0)$ that $\exp^c_{x_0}(\bar{p}) \in \partial_c K^c_{x_0,S_0}(x_0)$, hence by Corollary 2.8, $\bar{p} \in \left[\partial_c K^c_{x_0,S_0}(x_0)\right]_{x_0}$, proving (2.4). □

Finally, we give the precise definition of a *hole*.

Definition 2.11. Given any set A, we say that $\mathcal{O}$ is a *hole in A* if $\mathcal{O} \neq \emptyset$ is a bounded, open, connected set such that

$$\mathcal{O} \cap A^{\mathrm{int}} = \emptyset,$$
$$\mathcal{O}^\partial \subset A^\partial.$$

3. Proof of Theorem 1.1

We begin by deriving several intermediate results. We start with stating a very useful tool in our analysis, due to Albano and Cannarsa:

Proposition 3.1 ([1, Theorem 4.2]). *Suppose that u is a semi-convex function and $x_0 \in (\mathrm{dom}\,(u))^{\mathrm{int}}$ is a point where u is not differentiable. If there exists an open neighborhood $\mathcal{N}$ of x_0 such that u is differentiable on $\mathcal{N} \setminus \{x_0\}$, then for every $p \in \partial u\,(x_0)^\partial$ there exists a sequence $x_k \to x_0$ such that $Du(x_k) \to p$ as $k \to \infty$.*

The next result excludes having a full dimensional subdifferential at an isolated singular point, when the support of the target measure contains no holes. Note that the result can be shown under just the condition (Twist), and can be strengthened under (Nondeg) and (MTW). We also comment that this will be the only place where we use the no-hole condition on $\mathrm{spt}\,\nu$, for the proof of Theorem 1.1.

Proposition 3.2. *Suppose that c is C^1 and satisfies* (Twist)*, u is a c-convex Brenier solution, and* $\mathrm{spt}\,\nu$ *contains no holes. Then u cannot have any isolated singular point $x_0 \in (\mathrm{spt}\,\mu)^{\mathrm{int}}$ with* $\mathrm{affdim}\,\partial u\,(x_0) = n$ *(here* affdim *is the affine dimension of a convex set).*

If in addition, c satisfies (A0), (Nondeg)*, and* (MTW)*, and Ω and $\bar{\Omega}$ are c-convex with respect to each other, we obtain the same conclusion under the weaker condition that* $\mathrm{spt}\,\nu$ *contains no holes c-convex with respect to x_0.*

Proof. Suppose by contradiction that $x_0 \in (\mathrm{spt}\,\mu)^{\mathrm{int}}$ is an isolated singular point of u, and the affine dimension of $\partial u\,(x_0)$ is n. Since c is C^1 and satisfies (Twist), the mapping $\exp^c_{x_0}(\cdot)$ is continuous and injective, thus Brouwer's invariance of domain theorem (see [5]) gives that $\exp^c_{x_0}(\cdot)$ is a homeomorphism between the open set $\partial u\,(x_0)^{\mathrm{int}}$ and its image. In particular, $\exp^c_{x_0}(\partial u\,(x_0)^{\mathrm{int}})$ is a nonempty, open, bounded, connected set. Then since x_0 is an isolated singularity, by Proposition 3.1 we have

$$\exp^c_{x_0}(\partial u\,(x_0)^{\partial}) \subset \operatorname{spt}\nu \cap \partial_c u(x_0), \tag{3.1}$$

as $Du(\operatorname{dom}(Du)) \subset \operatorname{spt}\nu$ for the Brenier solution u.

We now claim that

$$\exp^c_{x_0}(\partial u\,(x_0)^{\mathrm{int}}) \cap \partial_c u(x_1) = \emptyset \tag{3.2}$$

for any $x_1 \neq x_0$. First, fix such an $x_1 \in \Omega$ and define

$$F(\bar{x}) := c(x_0, \bar{x}) - c(x_1, \bar{x}),$$

which is a C^1 function satisfying $\overline{D}F(\bar{x}) \neq 0$ for any $\bar{x}$ (by (Twist)). In particular, F cannot attain its maximum over the compact set $\exp^c_{x_0}(\partial u\,(x_0))$ except at the boundary, say at $\bar{x}_0 \in \exp^c_{x_0}(\partial u\,(x_0)^{\partial}) \subset \partial_c u(x_0)$. Thus if there exists $\bar{x}_1 \in \exp^c_{x_0}(\partial u\,(x_0)^{\mathrm{int}}) \cap \partial_c u(x_1)$, this would imply that

$$\begin{aligned} & F(\bar{x}_1) < F(\bar{x}_0) \\ \iff\ & c(x_0, \bar{x}_1) - c(x_1, \bar{x}_1) < c(x_0, \bar{x}_0) - c(x_1, \bar{x}_0) \\ \iff\ & c(x_0, \bar{x}_1) + c(x_1, \bar{x}_0) < c(x_0, \bar{x}_0) + c(x_1, \bar{x}_1), \end{aligned}$$

which is a violation of c-monotonicity of the c-subdifferential of u (see Remark 2.5). As a result there cannot be such an $\bar{x}_1$, and we obtain (3.2). Since $\nu = T_{\#}\mu$, we must then have

$$\exp^c_{x_0}(\partial u\,(x_0)^{\mathrm{int}}) \cap \operatorname{spt}\nu = \emptyset.$$

However, when combined with (3.1) this exactly implies that $\exp^c_{x_0}(\partial u\,(x_0)^{\mathrm{int}})$ is a hole in $\operatorname{spt}\nu$ which contradicts our initial assumption, therefore it must be that $\operatorname{affdim}\partial u\,(x_0) < n$.

If c also satisfies (A0), (Nondeg), and (MTW), by Corollary 2.8 we have that $\exp^c_{x_0}(\partial u\,(x_0)^{\mathrm{int}}) = \partial_c u\,(x_0)^{\mathrm{int}}$ and is c-convex with respect to x_0; the conclusion thus follows from the same proof as above. □

We recall here the definition of extremal point.

Definition 3.3. If K is a convex set, a point $x \in K$ is said to be an *extremal point* of K if, whenever x can be written as $x = (1-\lambda)x_0 + \lambda x_1$ for some $\lambda \in [0, 1]$ and $x_0, x_1 \in K$, it must be that $x_0 = x$ or $x_1 = x$.

In the next lemma, we extend to c-convex functions the following easy result about convex functions. Namely, if a convex function u is equal to some affine function (supporting from below) along a line segment containing a point x_0, then either

1. the gradient of this affine function is an extremal point of the convex set $\partial u\,(x_0)$; or
2. u is not differentiable at any point in this line segment.

Our extension is, in particular, to cost functions such that Loeper's maximum principle, Theorem 2.7 holds.

Lemma 3.4. *Suppose that c satisfies* (A0), (Twist), (Nondeg), *and* (MTW) *(so that Loeper's maximum principle, Theorem* 2.7 (2.1) *and its consequences, Corollary* 2.8 (2.2) *and Corollary* 2.9 *hold). Also let u be a c-convex function on Ω and assume that x_0 is an isolated singular point of u. Then if $\bar{p}_0$ is not an extremal point of the convex set $[\partial_c u(x_0)]_{x_0}$ and $\bar{x}_0 := \exp^c_{x_0}(\bar{p}_0)$, the contact set*

$$S_0 := \{x \in \Omega \mid u(x) = -c(x, \bar{x}_0) + c(x_0, \bar{x}_0) + u(x_0)\}$$

consists only of the single point x_0.

Proof. Fix a $\bar{p}_0$ that is not an extremal point of $[\partial_c u(x_0)]_{x_0}$. There exist $\bar{p}_\pm \neq \bar{p}_0$ such that $\bar{p}_\pm \in [\partial_c u(x_0)]_{x_0}$ and $\bar{p}_0 = \frac{1}{2}(\bar{p}_+ + \bar{p}_-)$; let us write $\bar{x}_\pm := \exp^c_{x_0}(\bar{p}_\pm)$.

Now, suppose by contradiction that there exists some $x_1 \in S_0$ with $x_1 \neq x_0$. Consider the c-segment $x(\lambda) := \exp^c_{\bar{x}_0}((1-\lambda)p_0 + \lambda p_1)$, for $\lambda \in [0, 1]$ from x_0 to x_1 where $p_0 := -\overline{D}c(x_0, \overline{x}_0)$ and $p_1 := -\overline{D}c(x_1, \overline{x}_0)$; observe from Corollary 2.9 that $x(\lambda) \in S_0$ for all $\lambda \in [0, 1]$. Also using that $\bar{x}_\pm \in \partial_c u(x_0)$, we must have

$$\max\{-c(x, \bar{x}_+) + c(x_0, \bar{x}_+),\ -c(x, \bar{x}_-) + c(x_0, \bar{x}_-)\} + u(x_0) \leq u(x) \tag{3.3}$$

for all $x \in \Omega$. In particular,

$$\begin{aligned} -c(x(\lambda), \bar{x}_\pm) + c(x_0, \bar{x}_\pm) + u(x_0) &\leq u(x(\lambda)) \\ &= -c(x(\lambda), \bar{x}_0) + c(x_0, \bar{x}_0) + u(x_0) \end{aligned}$$

for all $\lambda \in [0, 1]$. At the same time by using Theorem 2.7 (2.1),

$$-c(x(\lambda),\bar{x}_0)+c(x_0,\bar{x}_0) \leq \max\{-c(x(\lambda),\bar{x}_+)+c(x_0,\bar{x}_+),-c(x(\lambda),\bar{x}_-)+c(x_0,\bar{x}_-)\},$$

thus by combining these we must have the equality

$$\max\{-c(x(\lambda), \bar{x}_+) + c(x_0, \bar{x}_+),\ -c(x(\lambda), \bar{x}_-) + c(x_0, \bar{x}_-)\} + u(x_0) = u(x(\lambda))$$

for all $\lambda \in [0, 1]$. Together with (3.3), this implies that for each $\lambda \in [0, 1]$, either $\bar{x}_+ \in \partial_c u(x(\lambda))$ or $\bar{x}_- \in \partial_c u(x(\lambda))$. Since $\bar{x}_+, \bar{x}_- \neq \bar{x}_0$ by construction, and clearly $\bar{x}_0 \in \partial_c u(x(\lambda))$ for all $\lambda \in [0, 1]$ this implies all points $x(\lambda)$ in the c-segment must be singular points, contradicting that x_0 is an isolated singular point. This proves $S_0 = \{x_0\}$. □

In order to prove the main theorem, we require a modified version of the estimate [15, Lemma 6.10] (see also [21, Theorem 4.1]) (this is proven in the same vein as [13, Proposition 1] for the Euclidean case of $c(x, \bar{x}) = -\langle x, \bar{x}\rangle$). By the notation $|\cdot|_{\mathcal{L}}$, we denote the volume of a set in M, $\bar{M}$ or an associated cotangent space, induced by the Riemannian metric on either M or $\bar{M}$ (which will be clear from context).

Lemma 3.5. *Suppose c, u, Ω, $\bar{\Omega}$, μ, and ν satisfy the conditions of Theorem 1.1. Also let m_0 be a c-affine function with focus $\bar{x}_0$, let $S_0 := \{u \leq m_0\}$ with $S_0 \cap \Omega^\partial = \emptyset$, fix two parallel planes Π^+ and Π^- in $T^*_{\bar{x}_0}\bar{M}$ supporting the (convex) set $[S_0]_{\bar{x}_0}$ from opposite sides, and let $\ell_{\Pi^\pm}$ be the length of the longest line segment orthogonal to $\Pi^\pm$ that is contained in $[S_0]_{\bar{x}_0}$. Finally, suppose that for some $\delta > 0$, $x_0 \in S_0^{\mathrm{int}}$ is such that there exists $\bar{p}_\delta \in \left[\partial_c u(x_0) \cap (\operatorname{spt}\nu)^{\mathrm{int}}\right]_{x_0}$ with $d\left(\bar{p}_\delta, \left[(\operatorname{spt}\nu)^\partial\right]_{x_0}\right) \geq \delta$. Then (writing $p_0 := -\bar{D}c(x_0, \bar{x}_0)$) there exists a constant $C > 0$ depending only on δ, n, Λ,* diam (spt ν)*, and c such that*

$$(m_0(x_0) - u(x_0))^n \leq \frac{C \min\left\{d\left(p_0, \Pi^+\right), d\left(p_0, \Pi^-\right)\right\}}{\ell_{\Pi^\pm}} |S_0|^2_{\mathcal{L}}.$$

Proof. First, one can use (1.2) and follow a proof analogous to [13, Lemma 3.4] (using Remark 2.5, and replacing the Legendre transform of a function by the *c-transform*, see [15, Section 3]), to obtain

$$\left|[\partial_c u(S_0)]_{x_0} \cap \left[\operatorname{spt}\nu\right]_{x_0}\right|_{\mathcal{L}} = C\,|\partial_c u(S_0) \cap \operatorname{spt}\nu|_{\mathcal{L}} \leq \Lambda^2 C\,|S_0|_{\mathcal{L}},$$

where $C > 0$ depends on the cost function c. Now let $K^c_{x_0,S_0}(\cdot)$ be the c-cone over the section S_0 with vertex x_0. Then, by using [13, Lemma 3.1], we calculate

$$\begin{aligned}\left|\left[\partial_c K^c_{x_0,S_0}(x_0)\right]_{x_0}\right|_{\mathcal{L}} &\leq C(\delta, \operatorname{diam}(\operatorname{spt}\nu))\left|\left[\partial_c K^c_{x_0,S_0}(x_0)\right]_{x_0} \cap B_\delta\left(\bar{p}_\delta\right)\right|_{\mathcal{L}}\\ &\leq C(\delta, \operatorname{diam}(\operatorname{spt}\nu))\left|[\partial_c u(S_0)]_{x_0} \cap \left[\operatorname{spt}\nu\right]_{x_0}\right|_{\mathcal{L}}\\ &\leq C\,|S_0|_{\mathcal{L}},\end{aligned}$$

where the final constant C depends on c, Λ, δ, and diam (spt ν). Combining this with the original proof of [15, Lemma 6.10], we immediately obtain the claim. □

With all of the preceding ingredients in hand, we are ready to prove the main theorem.

Proof of Theorem 1.1. Suppose by contradiction that u has an isolated singular point $x_0 \in (\operatorname{spt}\mu)^{\mathrm{int}}$.

We begin by a localization of u around x_0. $[\partial_c u(x_0)]_{x_0}$ is convex by Corollary 2.8 (2.2) and contains more than one point since u is singular at x_0; thus there must exist at least one non-extremal point $\bar{p}_0$ of $[\partial_c u(x_0)]_{x_0}$. Let us define a family of sections around x_0 using c-affine functions with focus $\bar{x}_0 := \exp^c_x(\bar{p}_0)$, for $h > 0$ let

$$S_h := \{x \in \Omega \mid u(x) \leq -c(x, \bar{x}_0) + c(x_0, \bar{x}_0) + u(x_0) + h\}.$$

Notice that by Lemma 3.4, it holds the section is a singleton when $h = 0$, *i.e.* $S_0 = \{x_0\}$. As a result S_h can be made sufficiently small around x_0 for small enough $h > 0$. Thus by the assumption that x_0 is an isolated singularity, we may assume $h > 0$ to be small enough that $S_h \subset (\operatorname{spt}\mu)^{\mathrm{int}}$ and u is differentiable on $S_h \setminus \{x_0\}$.

On the other hand, by Proposition 3.2 we see that the affine dimension of $\partial u\,(x_0)$ is strictly less than n, and in particular $\partial u\,(x_0) = \partial u\,(x_0)^{\partial}$. Hence by Proposition 3.1, the definition of Brenier solution, and closedness of $\operatorname{spt}\nu$, we see that

$$\partial_c u(x_0) = \exp^c_{x_0}(\partial u\,(x_0)) \subset \operatorname{spt}\nu. \tag{3.4}$$

In particular, $\bar{x}_0 \in \operatorname{spt}\nu$. Since u is differentiable on $S_h \setminus \{x_0\}$, (3.4) and the definition of Brenier solution imply

$$\partial_c u(S_h) \subset \operatorname{spt}\nu. \tag{3.5}$$

Now consider the c-cone $K^c_{x_0,S_h}(x)$ over S_h with vertex x_0 as in Lemma 2.10. From the condition $\operatorname{spt}\nu \cap \bar{\Omega}^{\partial} = \emptyset$, it holds $\bar{x}_0 \in \bar{\Omega}^{\mathrm{int}}$, therefore we can apply Lemma 2.10 (2.3) and (2.4) to see

$$-Dc(x_0, \bar{x}_0) \in \left[\partial_c K^c_{x_0,S_h}(x_0)\right]^{\mathrm{int}}_{x_0} \subset [\partial_c u(S_h)]_{x_0}\,.$$

From (3.5), this implies $-Dc(x_0, \bar{x}_0) \in \left[\operatorname{spt}\nu\right]^{\mathrm{int}}_{x_0}$.

However if this is the case, then one can follow the proof of [15, Theorem 8.3], using Lemma 3.5 above (with $\delta = d\Big(-Dc(x_0, \bar{x}_0), \left[\operatorname{spt}\nu\right]^{\partial}_{x_0}\Big) > 0$) in place of [15, Theorem 6.11], to obtain that u is differentiable at x_0; this contradicts that x_0 is a singular point, completing the proof. $\square$

References

[1] P. ALBANO and P. CANNARSA, *Structural properties of singularities of semiconcave functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 719–740.

[2] R. BEYERSTEDT, *Removable singularities of solutions to elliptic Monge-Ampère equations*, Math. Z. **208** (1991), 363–373.

[3] R. BEYERSTEDT, *Isolated singularities of elliptic Monge-Ampère equations in dimensions $n \geq 2$*, Arch. Math. (Basel) **64** (1995), 230–236.

[4] Y. BRENIER, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), 375–417.

[5] L. E. J. BROUWER, *Beweis der invarianz des n-dimensionalen gebiets*, Math. Ann. **71** (1911), 305–313.

[6] L. A. CAFFARELLI, *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity*, Ann. of Math. (2) **131** (1990), 129–134.

[7] L. A. CAFFARELLI, *Some regularity properties of solutions of Monge Ampère equation*, Comm. Pure Appl. Math. **44** (1991), 965–969.

[8] L. A. CAFFARELLI, *The regularity of mappings with a convex potential*, J. Amer. Math. Soc. **5** (1992), 99–104.

[9] P. CANNARSA and Y. YU, *Singular dynamics for semiconcave functions*, J. Eur. Math. Soc. (JEMS) **11** (2009), 999–1024.
[10] G. DE PHILIPPIS and A. FIGALLI, *Partial regularity for optimal transport maps*, Publ. Math. Inst. Hautes Études Sci. **121** (2015), 81–112.
[11] P. DELANOË and Y. GE, *Regularity of optimal transport on compact, locally nearly spherical, manifolds*, J. Reine Angew. Math. **646** (2010), 65–115.
[12] A. FIGALLI, *Regularity properties of optimal maps between nonconvex domains in the plane*, Comm. Partial Differential Equations **35** (2010), 465–479.
[13] A. FIGALLI and Y.-H. KIM, *Partial regularity of Brenier solutions of the Monge-Ampère equation*, Discrete Contin. Dyn. Syst. **28** (2010), 559–565.
[14] A. FIGALLI, Y.-H. KIM and R. J. MCCANN, *Continuity and injectivity of optimal maps for non-negatively cross-curved costs*, arXiv:0911.3952, November 2009.
[15] A. FIGALLI, Y.-H. KIM and R. J. MCCANN, *Hölder continuity and injectivity of optimal maps*, Arch. Ration. Mech. Anal. **209** (2013), 747–795.
[16] A. FIGALLI, Y.-H. KIM and R. J. MCCANN, *On supporting hyperplanes to convex bodies*, Methods Appl. Anal. **20** (2013), 261–271.
[17] A. FIGALLI and L. RIFFORD, *Continuity of optimal transport maps and convexity of injectivity domains on small deformations of* $\mathbb{S}^2$, Comm. Pure Appl. Math. **62** (2009), 1670–1706.
[18] A. FIGALLI, L. RIFFORD and C. VILLANI, *On the Ma-Trudinger-Wang curvature on surfaces*, Calc. Var. Partial Differential Equations **39** (2010), 307–332.
[19] A. FIGALLI, L. RIFFORD and C. VILLANI, *Necessary and sufficient conditions for continuity of optimal transport maps on Riemannian manifolds*, Tohoku Math. J. (2) **63** (2011), 855–876.
[20] A. FIGALLI, L. RIFFORD and C. VILLANI, *Nearly round spheres look convex*, Amer. J. Math. **134** (2012), 109–139.
[21] N. GUILLEN and J. KITAGAWA, *On the local geometry of maps with c-convex potentials*, Calc. Var. Partial Differential Equations **52** (2015), 345–387.
[22] C. E. GUTIÉRREZ, "The Monge-Ampère Equation", Progress in Nonlinear Differential Equations and their Applications, Vol. 44. Birkhäuser Boston Inc., Boston, MA, 2001.
[23] K. JÖRGENS, *Harmonische Abbildungen und die Differentialgleichung* $rt - s^2 = 1$, Math. Ann. **129** (1955), 330–344.
[24] Y.-H. KIM, *Counterexamples to continuity of optimal transport maps on positively curved riemannian manifolds*, Int. Math. Res. Not., doi: 10.1093/imrn/rnn120, 2008.
[25] Y.-H. KIM and R. J. MCCANN, *Continuity, curvature, and the general covariance of optimal transportation*, J. Eur. Math. Soc. (JEMS) **12** (2010), 1009–1040.
[26] Y.-H. KIM and R. J. MCCANN, *Towards the smoothness of optimal maps on Riemannian submersions and Riemannian products (of round spheres in particular)*, J. Reine Angew. Math. **664** (2012), 1–27.
[27] P. W. Y. LEE and J. LI, *New examples satisfying Ma-Trudinger-Wang conditions*, SIAM J. Math. Anal. **44** (2012), 61–73.
[28] P. W. Y. LEE and R. J. MCCANN, *The Ma-Trudinger-Wang curvature for natural mechanical actions*, Calc. Var. Partial Differential Equations **41** (2011), 285–299.
[29] J. LIU, *Hölder regularity of optimal mappings in optimal transportation*, Calc. Var. Partial Differential Equations **34** (2009), 435–451.
[30] J. LIU, N. S. TRUDINGER and X.-J. WANG, *Interior* $C^{2,\alpha}$ *regularity for potential functions in optimal transportation*, Comm. Partial Differential Equations **35** (2010), 165–184.
[31] G. LOEPER, *On the regularity of solutions of optimal transportation problems*, Acta Math. **202** (2009), 241–283.
[32] G. LOEPER, *Regularity of optimal maps on the sphere: the quadratic cost and the reflector antenna*, Arch. Ration. Mech. Anal. **199** (2011), 269–289.
[33] G. LOEPER and C. VILLANI, *Regularity of optimal transport in curved geometry: the nonfocal case*, Duke Math. J. **151** (2010), 431–485.

[34] X.-N. MA, N. S. TRUDINGER and X.-J. WANG, *Regularity of potential functions of the optimal transportation problem*, Arch. Ration. Mech. Anal. **177** (2005), 151–183.
[35] F. SCHULZ and L. WANG, *Isolated singularities of Monge-Ampère equations*, Proc. Amer. Math. Soc. **123** (1995), 3705–3708.
[36] N. S. TRUDINGER and X.-J. WANG, *On strict convexity and continuous differentiability of potential functions in optimal transportation*, Arch. Ration. Mech. Anal. **192** (2009), 403–418.
[37] N. S. TRUDINGER and X.-J. WANG, *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **8** (2009), 143–174.
[38] C. VILLANI, "Optimal Transport: Old and New", Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 338, Springer-Verlag, Berlin, 2009.
[39] Y. YU, *Singular set of a convex potential in two dimensions*, Comm. Partial Differential Equations **32** (2007), 1883–1894.

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