Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVI (2016), 183-276

Almost strictly pseudo-convex domains. Examples and applications

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Abstract. In this work we introduce a class of smoothly bounded domains Ω in $\mathbb{C}^n$ with few non strictly pseudo-convex points in $\partial\Omega$ with respect to a certain Minkowski dimension. We call them almost strictly pseudo-convex, **aspc**. For these domains we prove that a canonical measure associated to a separated sequence of points in Ω which projects on the set of weakly pseudo-convex points is automatically a geometric Carleson measure. This class of **aspc** domains contains of course strictly pseudo-convex domains but also pseudo-convex domains of finite type in $\mathbb{C}^2$, domains locally diagonalizable, convex domains of finite type in $\mathbb{C}^n$, domains with real analytic boundary and domains like $|z_1|^2 + \exp\{1 - |z_2|^{-2}\} < 1$, which are not of finite type.

As an application we study interpolating sequences for convex domains of finite type in $\mathbb{C}^n$. After proving a Carleson-type embedding theorem, we get that if Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $S \subset \Omega$ is a dual bounded sequence of points in $H^p(\Omega)$, if $p = \infty$ then for any $q < \infty$, S is $H^q(\Omega)$ interpolating with the linear extension property and if $p < \infty$ then S is $H^q(\Omega)$ interpolating with the linear extension property, provided that $q < \min(p,\ 2)$.

Mathematics Subject Classification (2010): 32A35 (primary); 32A50 (secondary).

Contents

Received February 11, 2011; accepted in revised form October 29, 2014.
Published online March 2016.

1. Introduction

The aim of this work is to study a classical problem in harmonic analysis and complex variables, namely the interpolating sequences in some domains in $\mathbb{C}^n$. In order to do this we shall develop notions and tools well adapted to this aim, and which may be useful in other areas.

The first notion we shall study is the existence of a good family of polydiscs.

Troughout this work, domain will mean bounded open connected pseudo-convex set with $\mathcal{C}^\infty$ smooth boundary. The Lebesgue measure on a manifold of real dimension k will be denoted σ_k. We also use the notation in formula: $A :: B \ldots$ which means A such that $B \ldots$

Let Ω be a domain in $\mathbb{C}^n$ and $a \in \mathcal{U} \subset \Omega$, where $\mathcal{U}$ is a neighborhood of $\partial\Omega$ in Ω such that we have a well defined normal projection π on $\partial\Omega$; we set $\alpha := \pi(a)$ and to this point $\alpha \in \partial\Omega$, we associate a multi-index $m(\alpha) = (1,\ m_2(\alpha),\ \ldots,\ m_n(\alpha))$, $m_1 \leq m_2 \leq \ldots \leq m_m$ and an orthonormal basis $b(\alpha) = (L_1, \ldots, L_n)$ of $\mathbb{C}^n$ such that L_1 is in the complex normal at α to $\partial\Omega$ and $(L_2, \ldots, L_n)$ is a basis of the complex tangent space at α to $\partial\Omega$. We set $m(a) := m(\pi(a))$, $d(a) := d(a, \partial\Omega)$.

Now we define a polydisc $Q_a(\delta)$ centered at a of parameter $\delta > 0$ such that it has a radius $\delta d(a)$ in the L_1 direction and radii $\delta d(a)^{1/m_j(a)}$, $j = 2, \ldots, n$ along the L_j complex direction. We shall say that we have a "good family" $\mathcal{Q}$ of polydiscs if these polydiscs reflect well the geometry of the domain, *i.e.* there is a parameter $\delta_0 > 0$ such that $\forall a \in \mathcal{U}$, $Q_a(\delta_0) \subset \Omega$ and $M(\mathcal{Q}) := \sup_{\alpha\in\partial\Omega} m_n(\alpha) < \infty$. For instance if $\alpha = \pi(a)$ is a point of strict pseudo-convexity then we have that $m_2(a) = \ldots = m_n(a) = 2$.

This notion of good family $\mathcal{Q}$ of polydiscs is strongly inspired by the work of Catlin [11].

This good family $\mathcal{Q}$ allows us to define separated sequences of points in Ω.

Definition 1.1. Let Ω be a domain in $\mathbb{C}^n$ with a good family of polydiscs $\mathcal{Q}$. We shall say that a sequence of points $S \subset \Omega$ is δ *separated* if any two distinct points in S are center of disjoint polydiscs in the family $\mathcal{Q}$ with parameter δ.

Associated to this good family $\mathcal{Q}$ we define $\mu(a) := \sum_{j=2}^n \frac{1}{m_j(a)}$. If S is a separated sequence of points in Ω, we define its canonical measure to be

$$\nu_S := \sum_{a\in S} d(a)^{1+2\mu(a)}\delta_a, \tag{1.1}$$

where δ_a is the Dirac measure at a. For instance if Ω is a strictly pseudo-convex domain, then $m(a) = (1, 2, \ldots, 2)$ and $\nu_S = \sum_{a\in S} d(a)^n\delta_a$.

We shall see that the sequences of points we are interested in are contained in the zero set of holomorphic functions.

Let u be a holomorphic function in a domain Ω, $u \in \mathcal{H}(\Omega)$, set $X := u^{-1}(0)$ its zero set and $\Theta := \partial\bar{\partial}\ln|u|$ its associated $(1, 1)$ current of integration. As usual we have that $\mathrm{Tr}\Theta(z)$ is the trace of the associated matrix and we have ([24, page 55] $\mathrm{Tr}\Theta(z) = \Delta \ln|u(z)|$.

We shall define a class of such zero sets which contains the zero sets of Nevanlinna functions.

Definition 1.2. A holomorphic divisor X in the domain Ω is in the *Blaschke class* if, with Θ its associated $(1,1)$ current of integration and $d(z) := d(z, \Omega^c)$ the euclidean distance to the boundary,

$$\|\Theta\|_B := \int_\Omega d(z)\mathrm{Tr}\Theta(z) < \infty.$$

We shall need to study sequences of points contained in such sets; let σ_k be the Lebesgue measure on manifold of real dimension k. Let Ω be a domain equipped with a good family $\mathcal{Q}$ of polydiscs and X a divisor in Ω. We set for $a \in \Omega,\ X_a := X \cap Q_a(\delta),\ X_a^j$ the projection of X_a on $E_j := \{z \in \mathbb{C}^n :: z_j = 0\}$ in the coordinates in the basis $b(\alpha)$ associated to $\alpha = \pi(a)$, and $A_j(X_a) := \sigma_{2n-2}(X_a^j)$. As usual $\sigma_{2n-2}(X_a)$ is the measure of the regular points in X_a, as defined in [24, Proposition 2.48, page 55].

We get:

Theorem 1.3 (Discretized Blaschke condition). *Let u be holomorphic in Ω, $X := u^{-1}(0)$ and $\Theta := \partial\bar{\partial}\ln|u|$ its current of integration; suppose that Θ is in the Blaschke class. Let S be a δ separated sequence in X with respect to a good family $\mathcal{Q}$ of polydiscs with parameter δ_0. Then we have*

$$\sum_{a\in S} d(a)\sigma_{2n-2}(X_a) \leq \frac{2}{\delta_0}\|\Theta\|_B.$$

To go further and get the Malliavin discretized condition, *with the right control on the constants,* we need to introduce **quasi convex domains** with respect to the good family $\mathcal{Q}$, *i.e.* $\mathcal{Q}$ **quasi convex domains**. This is a class of domains containing the convex ones and the lineally convex ones and adapted to our aim. They will be defined precisely by Definition 3.9.

Theorem 1.4 (Discretized Malliavin condition). *Let $\Omega = \{\rho < 0\}$ be a domain equipped with a good family $\mathcal{Q}$ of polydiscs with parameter δ_0 and which is $\mathcal{Q}$ quasi convex. Let Θ be a current in the Blaschke class and S a δ separated sequence in $X \cap \mathcal{U}$. Then we have*

$$\sum_{a\in S}\sum_{j=2}^{n} A_j(X_a) \leq C\|\Theta\|_B,$$

where C is a constant depending only on the $\mathcal{M}(Q)+1$ first order derivatives of ρ and on δ, δ_0, and the constant of quasi convexity.

Together these two results gives the following:

Theorem 1.5. *Let Ω be a domain equipped with a good family $\mathcal{Q}$ of polydiscs such that Ω is $\mathcal{Q}$ quasi convex; let S be a δ separated sequence of points which is contained in the Blaschke divisor X. Then*

$$\delta^{2n-2}\sum_{a\in S} d(a)^n \leq \gamma(\Omega)\|\Theta\|_B,$$

where $\gamma(\Omega)$ depends only on the $\mathcal{C}^{\mathcal{M}(\mathcal{Q})+1}$ norm of ρ, on n and δ_0, the parameter of the family $\mathcal{Q}$, and on the constant of quasi convexity.

We have that $1 + 2\mu(a) \leq n$ and equality holds for a point a such that $\pi(a)$ is a strictly pseudo-convex point, hence in general this is not enough to deal with all types of sequence in Ω. So we are lead to introduce a class of domains with "few" points non strictly pseudo-convex, *i.e.* few "bad" points. If Ω is a domain in $\mathbb{C}^n$, throughout this work $W \subset \partial\Omega$ will denote the set of non strictly pseudo-convex points of $\partial\Omega$.

Let $\alpha \in \partial\Omega$ by linear change of variables we can suppose that $\alpha = 0 \in \partial\Omega \subset \mathbb{C}^n$, $z_1 = 0$ is the equation of the complex tangent space. The projection π locally near $0 \in \partial\Omega$ can be seen as a $\mathcal{C}^\infty$ diffeomorphism $\tilde{\pi} : \partial\Omega \to T_0(\partial\Omega)$, $\tilde{\pi} := (\pi_{|T_0(\partial\Omega)})^{-1}$.

Definition 1.6. The pseudo-convex domain Ω in $\mathbb{C}^n$ is said to be *almost stricly pseudo-convex*, **aspc** at $0 \in \partial\Omega$ if there is a neighbourhood V_0 of 0 and a basis $b := \{L_1, \dots, L_n\}$ of $\mathbb{C}^n$, with L_1 a complex normal unit vector, such that, with $(z_1, \dots, z_n)$ its associated coordinates, the slices

$$\tilde{\pi}(W \cap V_0) \cap \{z_1 = 0\} \cap \{z_2 = a_2\} \cap \ldots \cap \{z_{n-1} = a_{n-1}\}$$

have homogeneous Minkowki dimension less than $2 - \beta$, $\beta > 0$.

Ω is said to be **aspc** if this is true for all points in $\partial\Omega$ with the same $\beta > 0$.

This means that we need only to find a particular coordinate system $z_1, \dots, z_n$ such that the slices of non s.p.c. points along the z_n direction of the tangent space to $\partial\Omega$ have small Minkowski dimension.

Of course the strictly pseudo-convex domains are **aspc** because $W = \emptyset$. The *homogeneous Minkowski dimension* is defined precisely in Section 4 and it quantifies the fact that bad points are few.

This class of domains contains a large family of interesting domains such as strictly pseudo-convex domains, convex domains of finite type, etc., as shown in Section 5.

And also non finite type domains as $\{z \in \mathbb{C}^2 :: |z_1|^2 + \exp(1 - |z_2|^{-2}) < 1\}$.

Usually we think that strictly pseudo-convex points are easier to deal with than non strictly pseudo-convex ones but for these domains and the properties we are interested in, this is not the case. In fact we have a good control on what happen for points projecting on weakly pseudo-convex points.

Theorem 1.7. *Let $\mathcal{Q}$ be a good family of polydiscs on a* **aspc** *domain Ω in $\mathbb{C}^n$, and S be a δ separated sequence of points in Ω. Let W be the set of non strictly pseudo-convex points on $\partial\Omega$. If $\pi(S\cap\mathcal{U})\subset V\cap W$, where V is an open set of $\partial\Omega$, then we have:*

$$\sum_{a\in S\cap\mathcal{U}} d(a)^{1+2\mu(a)} = \delta^{-2n}\sum_{a\in S\cap\mathcal{U}}\sigma_{2n}(Q_a(\delta))\le C(\Omega)\frac{\sigma_{2n-1}(V)}{\delta^2}, \tag{1.2}$$

where $C(\Omega)$ depends only on $\rho,\ n,\ \mathcal{M}(Q)$, and the constant β in the Minkowski dimension of $W\subset\partial\Omega$.

In fact this theorem says that the canonical measure associated to such a sequence is a geometric Carleson measure. So, for these domains, it remains to concentrate only on points which project on strictly pseudo-convex points on $\partial\Omega$. As an application we get:

Theorem 1.8. *Let Ω be a* **aspc** *domain in $\mathbb{C}^n$. Let $\mathcal{Q}=\{Q_a(\delta_0)\}_{a\in\Omega}$ be a good family of polydiscs for Ω and suppose that Ω is $\mathcal{Q}$ quasi convex. Let S a δ separated sequence of points contained in a divisor X of the Blaschke class of Ω which projects on the open set $\mathcal{V}\subset\partial\Omega$. Then we have, with σ the Lebesgue measure on $\partial\Omega$,*

$$\sum_{a\in S} d(a)^{1+2\mu(a)}\le\gamma(\Omega)\|\Theta_X\|_B + C(\Omega)\sigma(\mathcal{V})<\infty.$$

The interpolating sequences are defined via the Hardy spaces of the domain Ω.

Definition 1.9. Let Ω be a domain in $\mathbb{C}^n$ defined by the function

$$\rho\in\mathcal{C}^\infty(\mathbb{C}^n),\ \Omega:=\{z\in\mathbb{C}^n :: \rho(z)<0\},\ \forall z\in\partial\Omega,\ \partial\rho(z)\neq 0.$$

Let f be a holomorphic function in Ω, we say that f is in the *Hardy class $H^p(\Omega)$* if

$$\|f\|_p^p := \sup_{\epsilon>0}\int_{\{\rho(z)=-\epsilon\}}|f(z)|^p\, d\sigma_\epsilon(z)<\infty.$$

We say that f is in the *Nevanlinna class $\mathcal{N}(\Omega)$* if

$$\|f\|_{\mathcal{N}} = \sup_{\epsilon>0}\int_{\{\rho(z)=-\epsilon\}}\log^+|f(z)|\, d\sigma_\epsilon(z)<\infty.$$

Here $d\sigma_\epsilon$ is the Lebesgue measure on the smooth manifold $\{\rho(z)=-\epsilon\}$ for ϵ small enough.

These spaces are independent of the choice of the defining function [34].

As we shall see, the study of interpolating sequences is intimately linked to p Carleson measures.

Definition 1.10. Let μ be a positive Borel measure on the domain Ω and $p \geq 1$. We shall say that μ is a *p Carleson measure* in Ω if:

$$\exists C_p > 0 :: \ \forall f \in H^p(\Omega), \ \int_\Omega |f|^p \, d\mu \leq C_p^p \|f\|_{H^p}^p.$$

This means that we have a continuous embedding of $H^p(\Omega)$ in $L^p(\mu)$.

Usually we have only a geometric condition to work with:

Definition 1.11. Let μ be a positive Borel measure on the domain Ω equipped with a good family of polydiscs $\mathcal{Q}$. We shall say that μ is a *geometric Carleson measure* in Ω if:

$$\exists C > 0 :: \forall a \in \Omega, \ \mu(\Omega \cap Q_a(2)) \leq C\sigma(\partial\Omega \cap Q_a(2)).$$

So we need a way to go from geometric Carleson measures to p Carleson measures and this is why we need to restrict to convex domains of finite type. For them we have a Carleson embedding theorem.

Theorem 1.12. *Let Ω be a convex domain of finite type. If the measure μ is a geometric Carleson measure we have*

$$\forall p > 1, \ \exists C_p > 0 :: \forall f \in H^p(\Omega), \ \int_\Omega |f|^p \, d\mu \leq C_p^p \|f\|_{H^p}^p.$$

Conversely if the positive measure μ is p Carleson for a $p \in [1, \ \infty[$, then it is a geometric Carleson measure, hence it is q Carleson for any $q \in]1, \ \infty[$.

It remains to see when the canonical measure associated to a separated sequence is a geometric Carleson measure. In the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ this is done by an easy generalization of a lemma of Garnett: a measure μ is Carleson in the ball $\mathbb{B}$ iff all its images under the automorphisms of $\mathbb{B}$ are uniformly bounded measures [5]. In a general domain there is only the identity as automorphism, so we have to overcome this issue.

We do it by building sub-domains associated to a point $a \in \Omega$ and which are equivalent to Carleson windows. This can be done with the right control of the constants if Ω is a *well balanced* domain; this notion will be defined later. Convex domains, linearly convex domains are well balanced.

The space $H^2(\Omega)$ is a subspace of the Hilbert space $L^2(\partial\Omega)$ hence there is an orthogonal projection $S \ \ L^2(\partial\Omega) \to H^2(\Omega)$. We shall denote $k_a(z)$ the kernel of this (Szegö) projection, it is a reproducing kernel for $H^2(\Omega)$.

Now we have the tools needed to deal with interpolating sequences.

Definition 1.13. We say that the sequence S of points in Ω is $H^p(\Omega)$ *interpolating* if

(i) $\forall a \in S, \ k_a \in H^{p'}(\Omega)$; (this is always true if $p \geq 2$.);
(ii) $\forall \lambda \in \ell^p(S), \ \exists f \in H^p(\Omega) :: \forall a \in S, \ f(a) = \lambda_a \|k_a\|_{p'}$,

with p' the conjugate exponent of p, *i.e.* $\frac{1}{p} + \frac{1}{p'} = 1$.

We have a weaker notion than interpolation:

Definition 1.14. We shall say that the sequence S of points in Ω is *dual bounded* in $H^p(\Omega)$ if there is a bounded sequence of elements in $H^p(\Omega)$, $\{\rho_a\}_{a\in S} \subset H^p(\Omega)$ which dualizes the associated sequence of reproducing kernels, *i.e*.

(i) $\forall a \in S,\ k_a \in H^{p'}(\Omega)$; (this is always true if $p \geq 2$.)
(ii) $\exists C > 0 :: \forall a \in S,\ \|\rho_a\|_p \leq C,\ \forall a, b \in S,\ \langle \rho_a,\ k_b\rangle = \delta_{a,b}\|k_b\|_{p'}$.

Clearly if S is $H^p(\Omega)$ interpolating then S is dual bounded in $H^p(\Omega)$: just interpolate the basic sequence of $\ell^p(S)$. In the unit disc of $\mathbb{C}$ the converse is true, here we have a partial converse of this.

Theorem 1.15. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$ and let $S \subset \Omega$ be a dual bounded sequence of points in $H^p(\Omega)$, if $p = \infty$ then for any $q < \infty$, S is $H^q(\Omega)$ interpolating; if $p < \infty$ then S is $H^q(\Omega)$ interpolating, provided that $q <$* $\min(p,\ 2)$.

Let us give a rough sketch of the proof.

Take a sequence S in the convex domain Ω; to apply a general result on interpolating sequences done in [2] we need the following facts:

- a link between the $H^p(\Omega)$ norm of the reproducing kernels k_a and the geometry of the boundary of $\partial\Omega$, the p regularity of the domain Ω, which says
$$\exists C > 0 :: \forall a \in \Omega,\ \|k_a\|_p^{-p'} \leq C\sigma(\partial\Omega \cap Q_a(2)),$$
where p' is the conjugate exponent of p. We shall see that this is true for convex domain of finite type.
- Structural hypotheses for the Lebesgue measure on $\partial\Omega$. These are reverse Hölder inequalities for the norms of the reproducing kernels k_a. We shall see that this is also true for convex domain of finite type.
- The fact that the canonical measure associated to S, $\nu_S := \sum_{a\in S} d(a)^{1+2\mu(a)}\delta_a$, is q Carleson.

And this is the main difficulty. To achieve this we use the fact that a convex domain of finite type is almost strictly pseudo-convex, so, with W the set of weakly pseudo-convex points in $\partial\Omega$, we have that the measure $\nu_b := \sum_{a\in S\cap\pi^{-1}(W)} d(a)^{1+2\mu(a)}\delta_a$ is already a geometric Carleson measure in Ω by Theorem 1.7.

It remains to deal with the points which project on the strictly pseudo-convex points in $\partial\Omega$.

By assumption $S\backslash\{a\}$ is contained in the zero set of $\rho_a \in H^p(\Omega) \subset \mathcal{N}(\Omega)$. So we can use Theorem 1.5 to get, because a convex domain is quasi convex, that $\nu := \sum_{a\in S} d(a)^n\delta_a$ is a bounded measure in Ω. To prove that ν is a geometric Carleson measure we construct sub-domains Ω_a associated to points $a \in \Omega$ and which are comparable to the Carleson windows $\Omega\cap Q_a(2)$. Because we have a precise estimate of the bound of $\sum_{a\in S} d(a)^n\delta_a$ in terms of Ω and of the holomorphic function u, whose zero set contains S, we can apply what we have done to the sub-domain Ω_a and get that $\sum_{b\in S\cap\Omega_b} d(b)^n\delta_b$ is bounded by a uniform constant times $\sigma_{2n-1}(\partial\Omega_a\cap\partial\Omega)$ which means that $\nu := \sum_{a\in S} d(a)^n\delta_a$ is a geometric Carleson measure in Ω.

Now we use the Carleson embedding Theorem 1.12 to get that the measure $\nu := \sum_{a\in S} d(a)^n\delta_a$ is a q Carleson measure for any $q\in]1,\infty[$.

For "good points", *i.e.* those which project on strictly pseudo-convex ones we have that $1+2\mu(a) = n$, hence gluing with the estimate coming from the **aspc** side, we get Theorem 1.15 as an application of the notion of **aspc** domains.

The general organization is as follow.

In Section 2 we define the good family $\mathcal{Q}$ of polydiscs in a domain Ω and we give two characterisations of them:

- an analytic one in term of finite linear type;
- a geometric one in term of complex tangentially ellipsoid at every point $\alpha\in\partial\Omega$.

In Section 3 we define precisely the Blaschke class of divisors X in Ω, the notion of $\mathcal{Q}$ quasi convexity, and we prove the discretized Blaschke and Malliavin conditions.

In Section 4 we introduce the notion of almost strictly pseudo-convex domains and we use a nice theorem of Ostrowski to get Theorem 1.8.

In Section 5 we prove that domains of finite type in $\mathbb{C}^2$, locally diagonalizable domains, convex domains of finite type, domains with real analytic boundary, are all **aspc** domains, together of course with the strictly pseudo-convex domains.

In Section 6 we set the geometric properties we need for convex domain of finite type and in Section 7 we study Carleson measures in such domains and state and prove the Carleson embedding Theorem 1.12.

In Section 8 we construct the sub-domain associated to a point $a\in\Omega$ which is equivalent to the Carleson window $Q_a(2)\cap\Omega$ and which allows us to overcome the lack of automorphisms.

In Section 9 we define the notion of *p regularity* making a link between the $H^p(\Omega)$ norm of the reproducing kernels and the geometry of $\partial\Omega$. Then we prove Theorem 1.15 via a tour around properties of reproducing kernels.

Finally in the Section 10 we state and prove the facts we need from potential theory.

ACKNOWLEDGEMENTS. I am deeply grateful to the referee who not only had to deal with the mathematics in this paper but also gave me a lot of valuable suggestions on the presentation of it. Hence even if the results here are essentially the same as in the preprint *A weak notion of strict pseudo convexity*, I have done in 2009, the presentation is completely rewritten, the statements are precised and the proofs are detailed.

2. Good family of polydiscs

In this section we shall study domains with a good family of polydiscs and get some properties of these domains we shall use later.

Let Ω be a domain in $\mathbb{C}^n$, recall that here this means a bounded open connected set with a $\mathcal{C}^\infty$ smooth boundary. Let $\mathcal{U}$ be a neighbourhood of $\partial\Omega$ in Ω such that the normal projection π onto $\partial\Omega$ is a smooth well defined application. For $a \in \Omega$ set $d(a) := d(a,\ \Omega^c)$ the distance from a to the boundary of Ω.

We shall need the notion of a "good" family of polydiscs, directly inspired by the work of Catlin [11].

Let $\alpha \in \partial\Omega$ and let $b(\alpha) = (L_1,\ L_2, \dots,\ L_n)$ be an orthonormal basis of $\mathbb{C}^n$ such that $(L_2, \dots,\ L_n)$ is a basis of the tangent complex space $T_\alpha^{\mathbb{C}}$ of $\partial\Omega$ at α; hence L_1 is the complex normal at α to $\partial\Omega$.

Let $m(\alpha) = (m_1,\ m_2, \dots,\ m_n) \in \mathbb{R}^n$ be a multi-index at α with $m_1 = 1,\ \forall j \geq 2,\ m_j \geq 2$.

For $a \in \mathcal{U}$, let $\alpha = \pi(a) \in \partial\Omega,\ b(a) := b(\alpha),\ m(a) := m(\alpha)$, and $\delta > 0$; set $Q_a(\delta) := \prod_{j=1}^n \delta D_j$ the polydisc such that δD_j is the disc centered at a, parallel to $L_j(\alpha)$ with radius $\delta \times d(a)^{1/m_j(\alpha)},\ j = 1, \dots, n$.

This way we have a family of polydiscs $\mathcal{Q} := \{Q_a(\delta)\}_{a\in\mathcal{U}}$ defined by the family of basis $\{b(\alpha)\}_{\alpha\in\partial\Omega}$, the family of multi-indices $\{m(\alpha)\}_{\alpha\in\partial\Omega}$ and the number δ.

It will be useful to extend this family to the whole of Ω. In order to do so let $(z_1, \dots, z_n)$ be the canonical coordinates system in $\mathbb{C}^n$ and for $a \in \Omega\backslash\mathcal{U}$, let $Q_a(\delta)$ be the polydisc of center a, of sides parallel to the axes and radius $\delta d(a)$ in the z_1 direction and $\delta d(a)^{1/2}$ in the other directions. So the points $a \in \Omega\backslash\mathcal{U}$ have automatically a "minimal" multi-index $m(a) = (1,\ 2, \dots,\ 2)$.

Now we can set

Definition 2.1. We say that $\mathcal{Q}$ is a *"good family" of polydiscs* for Ω if the $m_j(a)$ are uniformly bounded, *i.e.* $M(\mathcal{Q}) := \sup_{j=1,\dots,n,a\in\Omega} m_j(a) < \infty$, and if there exists $\delta_0 > 0$, called the parameter of the family $\mathcal{Q}$, such that all the polydiscs $\{Q_a(\delta_0)\}_{a\in\Omega}$ of $\mathcal{Q}$ are contained in Ω. In this case we call $m(a)$ the multi-type at a of the family $\mathcal{Q}$.

We notice that, for a good family $\mathcal{Q}$, by definition the multi-type is always finite. Moreover there is no regularity assumptions on the way that the basis $b(\alpha)$ varies with respect to $\alpha \in \partial\Omega$.

We can see easily that there is always good families of polydiscs in a domain Ω in $\mathbb{C}^n$ for a point $\alpha \in \partial\Omega$, take any orthonormal basis $b(\alpha) = (L_1,\ L_2, \dots,\ L_n)$, with L_1 a complex normal vector to $\partial\Omega$, and the "minimal" multi-type $m(\alpha) = (1,\ 2, \dots,\ 2)$. Then, because Ω is of class $\mathcal{C}^2$ and relatively compact, we have the existence of a uniform $\delta_0 > 0$ such that the family $\mathcal{Q}$ is a good one.

2.1. Examples of domains with a good family of polydiscs

The stricly pseudo-convex domains in $\mathbb{C}^n$ they have a good family of polydiscs associated with the best possible multi-type, the one defined by Catlin [11], which is also the “minimal” one in this case:

$$\forall a \in \mathcal{U},\ m_1 = 1,\ \ \forall j = 2, \dots, n,\ m_j(a) = 2.$$

Moreover these polydiscs are associated to the pseudo balls of a structure of spaces of homogeneous type (Koranyi-Vagi [22], Coifman-Weiss [13]).

The finite type domains in $\mathbb{C}^2$: also here we have the best possible multi-type and a structure of spaces of homogeneous type. (Nagel-Rosay-Stein-Wainger [31].)

The bounded convex finite type domains in $\mathbb{C}^n$: again we have the best multi-type and a structure of spaces of homogeneous type. (McNeal [26].)

2.2. An analytical characterisation by linear finite type

We shall recall precisely the definition of the multi-type [11] and the linear multi-type (McNeal [25], Yu [36]). We shall take the definitions and the notation from J. Yu [36].

Let Ω be a domain in $\mathbb{C}^n$ defined by the function ρ, and let $p \in \partial\Omega$ be fixed.

Let Γ_n be the set of the n -tuples of numbers $\Lambda = (m_1, \dots, m_n)$ with $1 \leq m_j \leq \infty$ and such that

(i) $m_1 \leq m_2 \leq \dots \leq m_n$.
(ii) for all $k = 1, \dots, n$, either $m_k = +\infty$ or there are non negative integers $a_1, \dots, a_k$ such that $a_k > 0$ and $\sum_{j=1}^{k} a_j/m_j = 1$.

This condition (ii) is automatically fulfilled in the case all m_j are integers.

An element in Γ_n will be called a *weight*. The set Γ_n of weights can be ordered lexicographically:

$$\Lambda = (m_1, \dots, m_n) < \Lambda' = (m'_1, \dots, m'_n)$$

if there is a k such that $\forall j < k,\ m_j = m'_j$ and $m_k < m'_k$.

Lemma 2.2. *The entries m_j of a weight $m = (m_1, \dots, m_n)$ are rational numbers. Given $M > 0$ there is only a finite number of weights $m = (m_1, \dots, m_n)$ such that $m_n \leq M$. Moreover if $m_1 = 1$ then $m_2 \in \mathbb{N}$.*

Proof. We have by (ii) that $\exists a_1 \in \mathbb{N} :: \frac{a_1}{m_1} = 1$ hence $m_1 = a_1 \in \mathbb{N}$. Again by (ii)

$$\exists a_1, a_2 \in \mathbb{N} :: \frac{a_1}{m_1} + \frac{a_2}{m_2} = 1 \Rightarrow a_1 \leq m_1 \leq M,\ a_2 \leq m_2 \leq M,$$

hence we have only a finite number of possible $m_1,\ a_1,\ a_2$. For each of such possibility we have

$$\frac{1}{m_2} = \frac{1}{a_2}\left(1 - \frac{a_1}{m_1}\right),$$

hence only one solution and a rational one.

So we have only a finite number of solutions for m_2 and all are rational numbers. We notice that if $m_1 = 1$, then $a_1 = 0$ and $m_2 = a_2 \in \mathbb{N}$.

Suppose now that $m_1, \dots, m_k$ are in finite number, then, as we just seen, $a_1, \dots, a_k$ are also in finite number and $a_{k+1} \le m_{k+1} \le M$ so only a finite number of a_{k+1}. Now as above for each choice of $a_1, \dots, a_{k+1},\ m_1, \dots, m_k$ we have only one solution m_{k+1} for

$$\frac{1}{m_{k+1}} = \frac{1}{a_{k+1}} \left(1 - \frac{a_1}{m_1} - \dots - \frac{a_k}{m_k} \right)$$

which is rational and the lemma is proved by induction. □

A weight is said to be *distinguished* if there exist holomorphic coordinates $z_1, \dots, z_n$, in a neighbourhood of p with p mapped to the origin and such that:

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{m_i} < 1 \ \Rightarrow\ \partial^\alpha \bar{\partial}^\beta \rho(p) = 0, \tag{2.1}$$

where $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$ and $\bar{\partial}^\beta := \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \dots \partial \bar{z}_n^{\beta_n}}$.

Definition 2.3. The *multi-type* $\mathcal{M}(\partial\Omega, p)$ is the smallest weight $\mathcal{M} := (m_1, \dots, m_n)$ in Γ_n (in lexicographic sense) such that $\mathcal{M} \ge \Lambda$ for every distinguished weight Λ.

Because $\partial\Omega$ is smooth at p, we always have $m_1 = 1$.

We call a weight Λ *linearly distinguished* if there exists a complex linear change of variables near p with p mapped to the origin and such that (2.1) holds in these new coordinates.

Definition 2.4. The *linear multi-type* $\mathcal{L}(\partial\Omega,\ p)$ is the smallest weight $\mathcal{L} := (m_1, \dots, m_n)$ such that $\mathcal{L} \ge \Lambda$ for every linear distinguished weight Λ. We shall say that Ω is of *linear finite type* if

$$\exists m \in \mathbb{N} :: \forall p \in \partial\Omega,\ \mathcal{L}(\partial\Omega,\ p) \le (m, \dots, m).$$

Clearly we have $\mathcal{L}(\partial\Omega,\ p) \le \mathcal{M}(\partial\Omega,\ p)$.

If, for $p \in \partial\Omega$ fixed, Ω is of linear finite type $\mathcal{L}(\partial\Omega,\ p) = (m_1, \dots, m_n)$, then there is a $\mathbb{C}$ -linear change of variables such that [36]:

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{m_i} < 1 \ \Rightarrow\ \partial^\alpha \bar{\partial}^\beta \tilde{\rho}(0) = 0,$$

where $\tilde{\rho}$ is the defining function of Ω in these new coordinates $\zeta = (\zeta_1, \dots, \zeta_n)$. Set $m_n' := \lceil m_n \rceil = \min_{k \in \mathbb{N}, k \ge m_n} k$.

Lemma 2.5. *We have*

$$\tilde{\rho}(\zeta) = \Re\zeta_1 + \sum_{2\leq|\alpha|+|\beta|\leq m'_n} A_{\alpha\beta}\zeta^\alpha\bar{\zeta}^\beta + o(|\zeta|^{m'_n}),$$

with $A_{\alpha\beta} \neq 0$ *only if* $\sum_{i=1}^n \frac{\alpha_i+\beta_i}{m_i} \geq 1$.

Proof. We expand $\tilde{\rho}$ by Taylor formula near 0 up to order m'_n and we compute

$$\partial^\alpha\bar{\partial}^\beta\tilde{\rho}(0) = \alpha!\beta!A_{\alpha\beta}.$$

But if $\sum_{i=1}^n \frac{\alpha_i+\beta_i}{m_i} < 1$ then $\partial^\alpha\bar{\partial}^\beta\tilde{\rho}(0) = 0$ because the linear multi-type of Ω at 0 is $m = (1,\ m_2, \dots,\ m_n)$.

Because $j \geq 2 \Rightarrow m_j \geq 2$, fixing $j \geq 2$ and taking $\alpha_j = 1, \alpha_i = 0$ for $i \neq j$ and $\beta_i = 0$ for all i we get

$$\sum_{i=1}^n \frac{\alpha_i+\beta_i}{m_i} = \frac{1}{m_j} < 1$$

hence $\forall j \geq 2,\ \frac{\partial\tilde{\rho}}{\partial\zeta_j}(0) = 0$.

Replacing α_j by β_j we get $\forall j \geq 2,\ \frac{\partial\tilde{\rho}}{\partial\bar{\zeta}_j}(0) = 0$ hence the complex tangent plane to $\partial\Omega$ at 0 is still $\zeta_1 = 0$, and the $\zeta_j,\ j \geq 2$ are coordinates in the complex tangent space.

So multiplying ζ_1 by a complex constant of modulus 1 if necessary, we have

$$\tilde{\rho}(\zeta) = \Re\zeta_1 + \sum_{2\leq|\alpha|+|\beta|\leq m'_n} A_{\alpha\beta}\zeta^\alpha\bar{\zeta}^\beta + o(|\zeta|^{m'_n}),$$

with $\sum_{i=1}^n \frac{\alpha_i+\beta_i}{m_i} < 1 \Rightarrow \partial^\alpha\bar{\partial}^\beta\tilde{\rho}(0) = 0.$ □

The aim of this subsection is to show:

Theorem 2.6. *If Ω is a domain in $\mathbb{C}^n$ of finite linear type, then there is a good family $\mathcal{Q}$ of polydiscs such that the multi-type associated to $\mathcal{Q}$ is precisely the linear multi-type of Ω.*

Proof. Going back to the previous coordinates, this means that there are complex directions $v_1,\ v_2, \dots, v_n$ with v_1 the complex normal at $p,\ v_2, \dots,\ v_n$ in the complex tangent space, such that:

$$\sum_{i=1}^n \frac{\alpha_i+\beta_i}{m_i} < 1 \ \Rightarrow\ \partial_v^\alpha\bar{\partial}_v^\beta\rho(p) = 0,$$

with now $\partial_v^\alpha := \frac{\partial^{|\alpha|}}{\partial v_1^{\alpha_1} \cdots \partial v_n^{\alpha_n}}$ and $\bar{\partial}_v^\beta := \frac{\partial^{|\beta|}}{\partial \bar{v}_1^{\beta_1} \cdots \partial \bar{v}_n^{\beta_n}}$ are the derivatives in the directions v_j. We can suppose that $p = 0$.

To define the polydiscs we need to have an orthonormal basis at p and we shall built it with the vectors $v_j, \ j = 2, \ldots, v_n$.

We have already that v_1 is the complex normal direction, so choose e_1 in the direction v_1 and with norm 1. Now we use the Gram-Schmidt orthogonalisation procedure in the complex tangent plane $\mathrm{Span}(v_2, v_3, \ldots, v_n)$:

take e_n parallel to v_n and of norm 1;
in $\mathrm{Span}(e_n, \ v_{n-1})$ take e_{n-1} of norm 1 and orthogonal to e_n;

and proceed this way to get an orthonormal basis $(e_2, \ldots, e_n)$ of $T_0^{\mathbb{C}}(\partial\Omega)$ and complete it with e_1 to get an orthonormal basis $b(p) = (e_1, \ldots, e_n)$ at p $(= 0)$. By this construction we have, with ζ_j the coordinates associated to the basis $b(p)$,

$$\zeta_1 = z_1, \ \zeta_2 = b_2^2 z_2, \ \ldots, \ \zeta_n = b_n^2 z_2 + \ldots + b_n^n z_n,$$

i.e. the matrix of change of coordinates is triangular.

So the lemma gives, still with $m_n' := \lceil m_n \rceil$,

$$\tilde{\rho}(\zeta) = \Re\zeta_1 + \sum_{2 \le |\alpha|+|\beta| \le m_n'} A_{\alpha\beta}\zeta^\alpha \bar{\zeta}^\beta + o(|\zeta|^{m_n'}),$$

with $A_{\alpha\beta} \neq 0 \Rightarrow \sum_{i=1}^n \frac{\alpha_i + \beta_i}{m_i} \ge 1$.

Where now the $\zeta_j = b_j \cdot z$ are seen as linear forms on z. Fix $t > 0$ small enough so that $a := (-t, 0, \ldots, 0) \in \mathcal{U}$, hence $\pi(a) = 0 = p$. Suppose that $z \in Q_a(\delta)$ the polydisc based on $b(p)$ with δ to be fixed later; this means $t = d(a)$ and

$$\forall j = 1, \ldots, n, \ \left|z_j\right| < \delta d(a)^{1/m_j}.$$

This implies, because $m_1 = 1 \le m_2 \le \ldots \le m_n$, that

$$\begin{aligned} \left|\zeta_j\right| &\le \sum_{k=1}^j \left|b_j^k\right| |z_k| \le \sum_{k=1}^j \left|b_j^k\right| \delta d(a)^{1/m_k} \\ &\le \delta d(a)^{1/m_j} \left(\sum_{k=1}^j \left|b_j^k\right| \right) = \delta B_j d(a)^{1/m_j}, \end{aligned}$$

with $B_j := \sum_{k=1}^j \left|b_j^k\right|$. So we get

$$\left|\zeta^\alpha\right| \le \delta^{|\alpha|} B^{|\alpha|} \prod_{j=1}^n d(a)^{\frac{\alpha_j}{m_j}},$$

with $B := \max_{j=1,\dots,n} \left|B_j\right|$. Replacing α_j by β_j in the previous proof, we get

$$\left|\bar{\zeta}^{\beta}\right| \leq \delta^{|\beta|} B^{|\beta|} \prod_{j=1}^{n} d(a)^{\frac{\beta_j}{m_j}},$$

so

$$\left|\zeta^{\alpha}\bar{\zeta}^{\beta}\right| \leq \delta^{|\alpha|+|\beta|} B^{|\alpha|+|\beta|} \prod_{j=1}^{n} d(a)^{\frac{\alpha_j+\beta_j}{m_j}} = \delta^{|\alpha|+|\beta|} B^{|\alpha|+|\beta|} d(a)^{\sum_{j=1}^{n} \frac{\alpha_j+\beta_j}{m_j}}.$$

In the sum, in order to have $A_{\alpha\beta} \neq 0$, we have $\sum_{j=1}^{n} \frac{\alpha_j+\beta_j}{m_j} \geq 1$, hence

$$\begin{aligned}\rho(z) &\leq \Re z_1 + \sum_{2\leq|\alpha|+|\beta|\leq m'_n} A_{\alpha\beta}\delta^{|\alpha|+|\beta|} B^{|\alpha|+|\beta|} d(a) + o(|z|^{m'_n}) \\ &\leq \Re z_1 + \delta d(a) C + o(|z|^{m'_n})\end{aligned}$$

with

$$C := \sum_{2\leq|\alpha|+|\beta|\leq m'_n} A_{\alpha\beta}\delta^{(|\alpha|+|\beta|-1)} B^{|\alpha|+|\beta|}.$$

Moreover we have $|z_1 - d(a)| < \delta d(a)$ so

$$\rho(z) \leq -d(a) + \delta d(a) + \delta d(a) C + o(|z|^{m'_n}) = d(a)(-1 + \delta(1 + C)) + o(|z|^{m'_n}).$$

The constant C depends on a finite number of derivatives of ρ. Because the domain is of finite linear type $\exists M(\mathcal{Q}) :: \forall p \in \partial\Omega,\ m_n(p) \leq M(\mathcal{Q})$, by the compactness of $\partial\Omega$ we have $\exists D > 0,\ C = C(p) \leq D$ for any $p \in \partial\Omega$. Hence if $\delta_0(1 + D) \leq 1/2$ we have $\rho(z) < 0$ if $|z|$ is small enough to absorb the $o(|z|^{m'_n})$. This means that $Q_a(\delta_0) \subset \Omega$.

So we find a $\delta_0 > 0$ such that, shrinking $\mathcal{U}$ if necessary to absorb the $o(|z|^{m'_n})$, we get $\forall a \in \mathcal{U},\ Q_a(\delta_0) \subset \Omega$. □

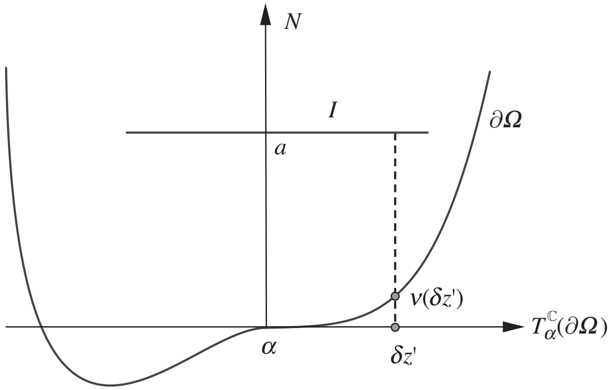
Proposition 2.7. *If Ω is equipped with a good family $\mathcal{Q}$ with multi-type $\{m(a)\}_{a\in\Omega}$, then it is of linear multi-type smaller than $\{(1, \lceil m_n(\alpha)\rceil, \dots, \lceil m_n(\alpha)\rceil)\}_{\alpha\in\partial\Omega}$.*

Proof. Let $\alpha \in \partial\Omega$ and suppose, by rotation and translation, that $\alpha = 0,\ \rho(z) = \Re z_1 + \Gamma(\Im z_1; z')$ with $z' = (z_2, \dots, z_n)$.

We have that for any point $a \in \mathcal{U}$ such that $\pi(a) = \alpha$, the polydisc $Q_a(\delta_0)$ is contained in Ω.

This means that, for $\delta < \delta_0$, the point $A_\delta := (-d(a),\ \delta z_2, \dots, \delta z_j, \dots, \delta z_n)$ with $\left|z_j\right| = d(a)^{1/m_j}$ is in Ω, hence the real segment $I := \{A_\delta\}_{\delta\in[0,\delta_0]}$ centered at a is contained in Ω.

Fix $(0, z') \in T_\alpha^{\mathbb{C}}(\partial\Omega)$ and set $\Re z_1 = \nu(\delta z'),\ \delta \in [0, \delta_0[$ the graph of $\partial\Omega$ over the segment I, *i.e.* $\nu(\delta z')$ is such that $(\nu(\delta z'), \delta z') \in \partial\Omega$; then we have that $\forall\delta \in [0, \delta_0[,\ \nu(\delta z') < d(a)$. (See the following picture, with N the *inward* normal),



So the distance from the point $\delta z' \in T_\alpha^{\mathbb{C}}(\partial\Omega)$ to $\alpha(=0)$ is $\delta\sqrt{\sum_{j=2}^n d(a)^{2/m_j}}$.

Recall that the order of contact of $\partial\Omega$ with the real direction v at $\alpha \in \partial\Omega$ is the order of vanishing of $\rho(\alpha + tv) - \rho(\alpha)$ when $t \to 0$.

Setting $t := \delta\sqrt{\sum_{j=2}^n d(a)^{2/m_j}}$, $x = d(a)$ and $u = \sum_{j=2}^n x^{2/m_j}$ we have

$$\frac{dt}{dx} = \delta\frac{\sum_{j=2}^n \frac{2}{m_j} x^{-1+2/m_j}}{2\sqrt{u}} \neq 0$$

and finite for $x \neq 0$, so by the implicit function theorem we have a smooth function $f(t)$ such that $d(a) = f(t)$ for $d(a) \neq 0$.

Now we make the change of variables $\delta z' = t\zeta'$, still with $\zeta' \in T_\alpha^{\mathbb{C}}(\partial\Omega)$, we have $\nu(\delta z') = \nu(t\zeta') \leq f(t) = d(a)$; hence the order of contact of $\partial\Omega$ with the direction z' is bigger than the order of contact of $f(t)$ at $t = 0$. So fix $\delta < \delta_0$ and let $d(a) \to 0$; because for any $a :: \pi(a) = \alpha$ we have $Q_a(\delta_0) \subset \Omega$, we get

$$t/\delta d(a)^{1/m_n(\alpha)} = \sqrt{\sum_{j=2}^n d(a)^{2/m_j - 2/m_n}}$$

and because $d(a)^{2/m_j - 2/m_n} \to 0$ if $m_j < m_n$, we get

$$t/\delta d(a)^{1/m_n(\alpha)} \to \sqrt{l(\alpha)}$$

where $l(\alpha)$ is the number of $j :: m_j = m_n$, hence $1 < l(\alpha) \leq n - 1$.

So $t \simeq \sqrt{l(\alpha)}\delta d(a)^{1/m_n(\alpha)} \Rightarrow f(t) = d(a) \simeq l(\alpha)^{-m_n(\alpha)/2}\delta^{-m_n} t^{m_n}$, hence the order of contact of $f(t)$ at 0 is $m_n(\alpha)$ hence the order of contact of $\partial\Omega$ in the real direction z' is at least $m_n(\alpha)$.

We have proved that for any real direction in $T_\alpha^{\mathbb{C}}(\partial\Omega)$ the order of contact of $\partial\Omega$ is at least $m_n(\alpha)$.

Let us make *any* linear change of variables keeping $T^{\mathbb{C}}_{\alpha}(\partial\Omega)$ and sending α to 0. Let us expand ρ in these new coordinates

$$\rho(w,\bar{w}) = \sum_{|\alpha|+|\beta|=k} A_{\alpha,\beta} w^{\alpha}\bar{w}^{\beta} + \mathcal{O}\left(|w|^{k+1}\right).$$

Set $w_1 = 0$, fix $\zeta = (0, \zeta_2, \dots, \zeta_n)$ and set $w = t\zeta \in T^{\mathbb{C}}_{\alpha}(\partial\Omega)$. Then we have

$$\rho(w,\bar{w}) = t^k \sum_{|\alpha|+|\beta|=k} A_{\alpha,\beta}\zeta^{\alpha}\bar{\zeta}^{\beta} + \mathcal{O}\left(t^{k+1}\right).$$

We already know that the order of contact of $\partial\Omega$ with any real direction of $T^{\mathbb{C}}_{\alpha}(\partial\Omega)$ is bigger than m_n, and this is still true if we change coordinates *linearly* provided that we keep $T^{\mathbb{C}}_{\alpha}(\partial\Omega)$. So the order of vanishing of ρ along the real line $t\zeta$ is bigger than $m_n(\alpha)$ hence in order to have $A_{\alpha,\beta} = \frac{\partial^{\alpha+\beta}\rho}{\partial^{\alpha}w\partial^{\beta}\bar{w}}(0)$ not all zeros for $|\alpha| + |\beta| = k$, we need to have $k \geq m_n$ hence $k \geq \lceil m_n \rceil$. This implies $\sum_{j=2}^{n} \frac{\alpha_j+\beta_j}{\lceil m_n \rceil} \geq \sum_{j=2}^{n} \frac{\alpha_j+\beta_j}{k} = 1$.

This means that for this change of variables the weight $(1, \lceil m_n \rceil, \dots, \lceil m_n \rceil)$ is linearly distinguished and hence $\partial\Omega$ at α is of finite linear type. Moreover the linear multi-type $(1, m'_2, \dots, m'_n)$ of $\partial\Omega$ at α being smaller than the weight $(1, \lceil m_n \rceil, \dots, \lceil m_n \rceil)$ by definition, we have

$$\forall j = 2, \dots, n,\ m'_j(\alpha) \leq \lceil m_n(\alpha) \rceil. \qquad \square$$

Theorem 2.6 and Proposition 2.7 give the characterization:

Corollary 2.8. *The domain Ω has a good family $\mathcal{Q}$ of polydiscs associated to $\{m(\alpha)\}_{\alpha\in\partial\Omega}$ iff the linear multi-type of Ω is smaller than $\{(1, \lceil m_n(\alpha) \rceil, \dots$ $\dots, \lceil m_n(\alpha) \rceil)\}_{\alpha\in\partial\Omega}$.*

2.3. A geometrical characterisation by existence of inner complex tangential ellipsoids

First we set tools we shall need. Recall the standard notation

$$\forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n,\ \partial^{\alpha} f(x,p) := \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}(x,p),\ x^{\alpha} := x_1^{\alpha_1}\dots x_n^{\alpha_n}.$$

Lemma 2.9. *Let $f(x,p)$ be a $\mathcal{C}^{\infty}(\mathbb{R}^n \times \mathbb{R}^m)$ function; then there exist $\mathcal{C}^{\infty}(\mathbb{R}^n \times \mathbb{R}^m)$ functions f_{α}, for $\alpha \in \mathbb{N}^n$, such that:*

$$f(x,p) = f(0,p) + \dots + \frac{1}{k!}\sum_{|\beta|=k} x^{\beta}\partial^{\beta} f(0,p) + \frac{1}{k!}\sum_{|\alpha|=k+1} x^{\alpha} f_{\alpha}(x,p).$$

The f_{α} are given explicitly by the formulas:

$$f_{\alpha}(x,p) := \int_0^1 \partial^{\alpha} f(tx,p)(1-t)^k dt.$$

Proof. Set, for $t \in \mathbb{R}$, $g(t,p) := f(tx,p)$. Then, we have

$$g^{(k)}(t,p) := \frac{\partial^k g}{\partial t^k}(t,p) = \sum_{|\alpha|=k} x^\alpha \partial^\alpha f(tx,p).$$

Apply to g the Taylor formula with integral remainder

$$g(1,p) = g(0,p) + \ldots + \frac{g^{(k)}(0,p)}{k!} + \frac{1}{k!}\int_0^1 g^{(k+1)}(t,p)(1-t)^k dt.$$

We get

$$\begin{aligned} f(x,p) = f(0,p) + \ldots &+ \sum_{|\alpha|=k} x^\alpha \partial^\alpha f(0,p) \\ &+ \frac{1}{k!}\sum_{|\beta|=k+1} x^\beta \int_0^1 \partial^\beta f(tx,p)(1-t)^k dt. \end{aligned}$$

Now set $f_\beta(x,p) := \int_0^1 \partial^\beta f(tx,p)(1-t)^k dt$, then, deriving under the integral sign, we have that f_β is $\mathcal{C}^\infty$ in the two variables x, p. □

We suppose we are given a family of orthonormal basis and multi-types $\{b_\alpha, m(\alpha)\}_{\alpha\in\partial\Omega}$.

First, without loss of generality, we make the assumption that the normal derivative of ρ is 1 at any point $\alpha \in \partial\Omega$, and $\rho \in \mathcal{C}^\infty(\mathbb{C}^n)$.

Fix $\alpha \in \partial\Omega$; by translation we can suppose $\alpha = 0$, *i.e.* with $\rho_\alpha(z) := \rho(z+\alpha)$ we have $\rho_\alpha(0) = 0$.

Now we make the rotation U_α sending the standard basis of $\mathbb{C}^n$ to b_α, *i.e.* $\rho_\alpha(z) := \rho(U_\alpha z + \alpha)$. In these new coordinates, we have that $z' = (z_2, \ldots, z_n)$ are the coordinates in the complex tangent space and $z_1 = x_1 + iy_1$ is the coordinate in the normal complex plane and x_1 is the coordinate in the real normal at α $(= 0)$.

Set $h_\alpha(z') := \rho_\alpha(0, z') \in \mathcal{C}^\infty(\mathbb{C}^{n-1})$ then $\rho_\alpha(z) - h_\alpha(z') = 0$ if $z_1 = 0$, $\forall z' \in \mathbb{C}^{n-1}$.

Set

$$g_\alpha(x_1, y_1, z') := -x_1 + \rho_\alpha(z) - h_\alpha(z') \in \mathcal{C}^\infty(\mathbb{C}^n);$$

we have $\rho_\alpha(z) = x_1 + g_\alpha(z_1, z') + h_\alpha(z')$. Recall that x_1 is the coordinate on the real normal, so $\frac{\partial\rho_\alpha}{\partial x_1}(0) = 1$ by assumption and, because y_1 is a tangent coordinate, we have $\frac{\partial\rho_\alpha}{\partial y_1}(0) = 0$, so

$$\frac{\partial g_\alpha}{\partial x_1}(0,0) = -1 + \frac{\partial\rho_\alpha}{\partial x_1}(0,0) = 0, \quad \frac{\partial g_\alpha}{\partial y_1}(0,0) = \frac{\partial\rho_\alpha}{\partial y_1}(0,0) = 0.$$

Lemma 2.10. *There is a number $R > 0$, independent of $\alpha \in \partial\Omega$, such that, after the change of coordinates above, we have the estimate*

$$\forall\alpha \in \partial\Omega,\ \forall z \in B(0,R),\ |g_\alpha(z)| \le \frac{1}{4}|z_1|$$

and the factorization

$$g_\alpha(x_1, y_1, z') = x_1 g_1(x_1, y_1, z') + y_1 g_2(x_1, y_1, z'),$$

with $\forall z \in B(0, R),\ j = 1, 2,\ \left|g_j(z)\right| < 3/10.$

Proof. We apply Lemma 2.9 to $g := g_\alpha(x_1, y_1, z')$ to order 1 with z' as the parameter p. We get

$$\begin{aligned} g(x_1, y_1, z') = g(0, 0, z') + x_1 \frac{\partial g}{\partial x_1}(0, 0, z') + y_1 \frac{\partial g}{\partial y_1}(0, 0, z') \\ + x_1^2 g_{(2,0)}(x_1, y_1, z') + x_1 y_1 g_{(1,1)}(x_1, y_1, z') + y_1^2 g_{(0,2)}(x_1, y_1, z'). \end{aligned}$$

Since $g(0, 0, z') = \rho_\alpha(0, z') - h_\alpha(z') = 0$, there remain the other terms.

Because

$$g = g_\alpha = -x_1 + \rho_\alpha(z) - h_\alpha(z') = -x_1 + \rho(U_\alpha z + \alpha) - \rho(U_\alpha(0, z') + \alpha)$$

all its derivatives are controlled by the derivatives of ρ in a neighborhood of $\bar{\Omega}$, because U_α is a rotation independent of z, so they are controlled uniformly in α. So are the integrals of them, hence the functions $g_{(j,k)}$. Because $\frac{\partial g}{\partial x_1}(0, 0, 0) = 0$, we have that $\frac{\partial g}{\partial x_1}(0, 0, z')$ is small when $\left|z'\right|$ is small and this is uniform with respect to the point $\alpha \in \partial\Omega$.

The same for $\frac{\partial g}{\partial y_1}(0, 0, z')$. Moreover the functions $g_{(j,k)}$ are bounded again uniformly with respect to the point $\alpha \in \partial\Omega$. So finally we can choose $R' > 0$ small enough and independent of the point $\alpha \in \partial\Omega$ to get

$$\left|z'\right| < R' \Rightarrow \left|\frac{\partial g}{\partial x_1}(0, 0, z')\right| < 1/10,\ \left|\frac{\partial g}{\partial y_1}(0, 0, z')\right| < 1/10.$$

Take R'' small enough to have

$$\forall z \in B(0, R''),\ i, j = 0, 1, 2,\ \ |z_1| \left|g_{(i,j)}(z)\right| < 1/10;$$

then, with $R := \min(R', R'', 1/6)$ we get

$$\forall z \in B(0, R),\ \left|g(x_1, y_1, z')\right| \le \frac{1}{10}(|x_1| + |y_1| + 3R\,|z_1|) \le \frac{|z_1|}{10}(2 + 3R) \le \frac{|z_1|}{4}.$$

Now setting

$$g_1(x_1, y_1, z') := \frac{\partial g}{\partial x_1}(0, 0, z') + x_1 g_{(2,0)}(x_1, y_1, z') + y_1 g_{(1,1)}(x_1, y_1, z')$$

and

$$g_2(x_1, y_1, z') := y_1 g_{(0,2)}(x_1, y_1, z'),$$

we have

$$\forall z \in B(0,R),\ |g_1(z)| \leq \frac{3}{10},\ |g_2(z)| \leq 1/10,$$

and the factorization

$$g(x_1,y_1,z') = g_\alpha(x_1,y_1,z') = x_1 g_1(x_1,y_1,z') + y_1 g_2(x_1,y_1,z'). \qquad \square$$

Lemma 2.11. *If Ω has a good family of polydiscs for $\alpha \in \partial\Omega$, there is a complex tangentially elliptic domain $C = C_\alpha$, with aperture $\Gamma > 0$ near the point $\alpha \in \partial\Omega$, of class $\mathcal{C}^2$ and such that*

- $C_\alpha \subset \Omega$, *near* α,
- $\bar{C}_\alpha$ *and* $\bar{\Omega}$ *meet at* α,
- $\exists A > 0,\ \forall a \in \mathcal{U} :: \pi(a) = \alpha,\ Q_a(\delta) \subset C_\alpha$ *provided that*

$$\delta^2 < \min\left(\Gamma, \frac{1}{4(n-1)A}, 1/4\right).$$

Proof. We shall build C. We make the change of variables above; then we can write

$$\rho_\alpha(z) = x_1 + g_\alpha(z) + h_\alpha(z'),$$

where $z_1 = x_1 + iy_1, z' := (z_2,\ldots,z_n)$ and $g_\alpha \in \mathcal{C}^\infty(\mathbb{C}^n), h_\alpha \in \mathcal{C}^\infty(\mathbb{C}^{n-1}),\ g_\alpha(\alpha) = h_\alpha(\alpha') = 0$. Let

$$\mu(z) := x_1 + g_\alpha(z) + A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big),$$

with $m = m(\alpha)$. We fix an aperture $\Gamma > 0$ and we shall choose A in order to have that

$$C := \{\mu < 0\} \cap \{|y_1| < -\Gamma x_1\}$$

fills the requirements of the lemma.

This domain $C = C_\alpha$ is what we shall call a "complex tangentially elliptic domain with aperture $\Gamma > 0$ ". As the referee remarks this can also be seen as the classical "approach regions" to the boundary in the strictly pseudo-convex case.

Fix $a = (-t,0,\ldots,0), t \in \mathbb{R}, (-t,z') \in B(0,R)$ in order to have $\left|g_\alpha(-t,z')\right| < t/4$, by Lemma 2.10; consider the slice S_{z_1} of C_α

$$\forall z_1 :: (z_1,z') \in B(0,R),$$
$$S_{z_1} := \left\{z' := (z_2,\ldots,z_n)\ :\ A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big) < -(\Gamma+2)x_1\right\}.$$

Then

$$z_1 = -t \Rightarrow y_1 = 0 \Rightarrow$$
$$\Rightarrow S_{-t} := \left\{z' := (z_2,\ldots,z_n) :: A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big) < (\Gamma+2)t\right\}.$$

If $z' \in S_{-t}$, then $\forall j \geq 2,\ A\left|z_j\right|^{m_j} < (\Gamma+2)t \Rightarrow \left|z_j\right| < \frac{1}{(A/(\Gamma+2))^{1/m_j}} t^{1/m_j} = \frac{d(a)^{1/m_j}}{(A/(\Gamma+2))^{1/m_j}}$.

Hence if $A \geq \frac{\Gamma+2}{\delta_0^{m_n(\alpha)}}$ then $\forall j \geq 2, (A/(\Gamma+2))^{1/m_j}\delta_0 \geq (A/(\Gamma+2))^{1/m_n}\delta_0 \geq 1$ hence

$$z' \in S_{-t} \Rightarrow \forall j = 2, \ldots, n,\ \left|z_j\right| < \frac{d(a)^{1/m_j}}{(A/(\Gamma+2))^{1/m_j}} \leq \delta_0 d(a)^{1/m_j} \Rightarrow (-t, z') \in Q_a(\delta_0) \subset \Omega.$$

So if $(-t+iy_1, z') \in C$, we have $|y_1| < \Gamma t$ and

$$\mu(-t+iy_1, z') = -t + g_\alpha(z) + A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big) < 0 \Rightarrow A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big) < t - g_\alpha(-t+iy_1, z').$$

But, in the ball $B(0, R)$, we have $\left|g_\alpha(-t+iy_1, z')\right| < \frac{|y_1|}{4} + \frac{t}{4}$, hence

$$A\big(|z_2|^{m_2} + \cdots + |z_n|^{m_n}\big) < t - g_\alpha(-t+iy_1, z') < t + t/4 + |y_1|/4 < (\Gamma+2)t,$$

then $z' \in S_{-t+iy_1}$ and $(-t+iy_1, z') \in Q_a(\delta_0) \subset \Omega \Rightarrow \rho_\alpha(-t+iy_1, z') < 0$ provided that $|y_1| < \delta_0 t$, hence we need to take the aperture $\Gamma \leq \delta_0$. To have the same A for all the boundary points, we take

$$A = \frac{\Gamma+2}{\delta_0^{M(\mathcal{Q})}} \text{ with } M(\mathcal{Q}) = \sup_{\alpha\in\partial\Omega} m_n(\alpha),$$

which is bounded because $\mathcal{Q}$ is a good family.

With this choice of A, we have that

$$(-t+iy_1, z') \in C \cap B(0, R) \Rightarrow (-t+iy_1, z') \in Q_a(\delta_0) \subset \Omega$$

so $\mu(-t+iy_1, z') < 0 \Rightarrow \rho(-t+iy_1, z') < 0$, *i.e.*

$$\mu(z) = -t + g_\alpha(-t+iy_1, z') + A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big) < 0,$$

hence

$$A\big(|z_2|^{m_2} + \cdots + |z_n|^{m_n}\big) < t - g_\alpha(-t+iy_1, z'),$$

and this implies $\rho_\alpha(z) < 0$, *i.e.*

$$h_\alpha(z') < t - g_\alpha(-t+iy_1, z'),$$

so necessarily $h_\alpha(z') \leq A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big)$, because if not suppose there is a z' such that

$$A\big(|z_2|^{m_2} + \cdots + |z_n|^{m_n}\big) < h(z'),$$

take a $t > 0$ with

$$A\big(|z_2|^{m_2} + \cdots + |z_n|^{m_n}\big) < t - g(-t, z') < h(z'),$$

then the point $(-t, z')$ belongs to C, because we have $y_1 = 0 < \Gamma t$, but not to Ω, which is a contradiction.

Hence we proved

$$z \in C_\alpha \cap B(0, R) \Rightarrow h_\alpha(z') \le A\big(|z_2|^{m_2} + \cdots + |z_n|^{m_n}\big). \tag{2.2}$$

So over any point of $B(\alpha, R) \cap \{|y_1| < -\Gamma x_1\}$ we have a domain of class $\mathcal{C}^2$, because $m_1 = 1 \Rightarrow m_2 \in \mathbb{N}$ by Lemma 2.2 hence

$$\frac{\partial^2 |z_2|^{m_2}}{\partial z_2^2} = \frac{\partial^2 (\bar{z}_2 z_2)^{m_2/2}}{\partial z_2^2} = \frac{m_2(m_2-2)}{4} |z_2|^{m_2-4}\, \bar{z}_2^2.$$

If $m_2 = 2$ then $\frac{\partial^2 |w|^m}{\partial w^2} = 0$ and this term is $\mathcal{C}^2$. If $m_2 \ge 3$ then

$$\frac{\partial^2 |z_2|^{m_2}}{\partial z_2^2} = \frac{m_2(m_2-2)}{4} |z_2|^{m_2-4}\, \bar{z}_2^2$$

and this is continuous, so again this term is $\mathcal{C}^2$.

Now we have $m_j \ge m_2$ for $j \ge 3$ hence all the other terms are also $\mathcal{C}^2$.

It remains to prove the last item of the lemma.

Take a point $a \in \Omega$, $\pi(a) = \alpha$ then $a = (-t, 0, \dots, 0)$ after the usual change of variables; fix a $\delta > 0$ to be precised later. If $(x_1 + iy_1, z') \in Q_a(\delta)$, then

$$\forall j = 2, \dots, n,\ \left|z_j\right| < \delta t^{1/m_j},\ |x_1 + t| < \delta t \Rightarrow x_1 < -t(1-\delta),\ |y_1| < \delta t,$$

so we already choose $\delta \le \Gamma$ to have $|y_1| < -\Gamma x_1$, and

$$A\big(|z_2|^{m_2} + \cdots + |z_n|^{m_n}\big) < tA \sum_{j=2}^{n} \delta^{m_j},$$

hence

$$\begin{aligned}\mu(x_1, y_1, z') &= x_1 + g(z) + A\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big)\\ &< -t(1-\delta) + tA \sum_{j=2}^{n} \delta^{m_j} + |g(z)|\,.\end{aligned}$$

Because $m_j \ge 2$, $A\delta^{m_j} \le \delta^2 A$, so

$$\mu(x_1, y_1, z') < -t(1-\delta) + (n-1)\delta^2 tA + |g(z)|\,.$$

But, by Lemma 2.10, the smooth function $g(z)$ is bounded on $B(0, R)$ by $\frac{1}{4}\ |z_1|$, and we have $|y_1| < \Gamma\,|x_1|$ so

$$\mu(x_1, y_1, z') < -t(1-\delta)+(n-1)\delta^2 tA+\frac{t}{4}(1+\Gamma)=t\left(\delta - \frac{3}{4} + (n-1)\delta^2 A + \frac{\Gamma}{4}\right),$$

if we choose $\Gamma < 1,\ (n-1)\delta^2 A < \frac{1}{5},\ \delta < \frac{1}{5}$ and $z \in B(0, R)$ we get

$$\mu(x_1, z') < t\left(-\frac{3}{4} + \frac{1}{5} + \frac{1}{5} + \frac{1}{4}\right) < 0,$$

i.e. $Q_a(\delta) \subset C$. It remains to choose δ with $\delta < \min\left(\Gamma, \frac{1}{2\sqrt{(n-1)A}}, 1/5\right)$, $\Gamma < \min(\delta_0, 1)$ and $z \in B(0, R)$ to have $Q_a(\delta) \subset C$.

The family $\{C_\alpha\}$ is determined by $\{b(\alpha),\ m(\alpha)\}_{\alpha\in\partial\Omega}$, the aperture Γ and the number A. □

It would be nice to have an actual ellipsoid domain osculating Ω at α, instead of a conic domain whose slices parallel to the complex tangent space centered on the real normal are convex ellipsoids.

But this is not true in general as shown by the following simple example in $\mathbb{C}^2$. Take $\Omega = \{\rho < 0\}$ near 0, with:

$$\rho(z) = x_1 + ay_1^2 + b\,|z_2|^m + cy_1x_2,$$

with $m \geq 3,\ c > 0$. Then there is no way to have that $C := \{\gamma < 0\} \subset \Omega$ near 0 with:

$$\mu(z) = x_1 + Ay_1^2 + B\,|z_2|^m$$

for any choice of A and B.

Just take points $y_1 = \frac{x_2}{k},\ y_2 = 0$ then $z \in \partial C \Rightarrow -x_1 = A\frac{x_2^2}{k^2} + Bx_2^m$ and at this point we have

$$\rho(z) = (a - A)\frac{x_2^2}{k^2} + (b - B)x_2^m + c\frac{x_2^2}{k}$$

and this is *not* negative for k big enough and x_2 small enough if $m \geq 3$.

Now we shall see that we have a converse to Lemma 2.11.

Lemma 2.12. *If a domain Ω contains a family of complex tangentially elliptic domains $\{C_\alpha\}_{\alpha\in\partial\Omega}$ based on $\{b(\alpha),\ m(\alpha)\}_{\alpha\in\partial\Omega}$, aperture $\Gamma > 0,\ \Gamma < 1$, and number A, then Ω possesses a good family of polydiscs still based on $\{b(\alpha),\ m(\alpha)\}_{\alpha\in\partial\Omega}$ and with parameter $\delta_0 = \min\left(\Gamma, \frac{1}{2\sqrt{(n-1)A}}, 1/5\right)$.*

Proof. This is a remake of the proof of the last item in Lemma 2.11.

Again we make the canonical change of variables associated to the basis b_α; we have

$$\rho_\alpha(z) = x_1 + g_\alpha(z) + h_\alpha(z'),$$

where $z_1 = x_1 + iy_1$, $z' := (z_2, \dots, z_n)$ and $g = g_\alpha \in \mathcal{C}^\infty(\mathbb{C}^n)$, $h = h_\alpha \in \mathcal{C}^\infty(\mathbb{C}^{n-1})$, $g(\alpha) = h(\alpha') = 0$.

Let $a = (-t, 0, \dots, 0)$ and fix a $\delta > 0$ to be precised later; if $(x_1 + iy_1, z') \in Q_a(\delta)$ then

$$\forall j = 2, \dots, n, \ \left|z_j\right| < \delta t^{1/m_j}, \ |x_1 + t| < \delta t \Rightarrow x_1 < -t(1-\delta), \ |y_1| < \delta t,$$

so we already choose $\delta \le \Gamma$ and

$$A\big(|z_2|^{m_2} + \dots + |z_n|^{m_n}\big) < tA \sum_{j=2}^{n} \delta^{m_j},$$

hence

$$\begin{aligned}\mu(x_1, y_1, z') &= x_1 + g(z) + tA\big(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\big) \\ &< -t(1-\delta) + tA \sum_{j=2}^{n} \delta^{m_j} + |g(z)|.\end{aligned}$$

We get, because $m_j \ge 2$, $A\delta^{m_j} \le \delta^2 A$,

$$\mu(x_1, y_1, z') < -t(1-\delta) + (n-1)\delta^2 A + |g(z)|.$$

But, by Lemma 2.10, the smooth function $g(z)$ is bounded on $B(0, R)$ by $\frac{1}{4}|z_1|$, with $R > 0$ independent of α. So as in the proof of the last item in Lemma 2.11, if $\Gamma < 1$, $\delta < \frac{1}{5}$, $\delta^2(n-1)A < \frac{1}{5}$, we have $Q_a(\delta) \subset C$.

It remains to choose $\Gamma < 1$, δ with $\delta < \min\Big(\Gamma, \frac{1}{2\sqrt{(n-1)A}}, 1/5\Big)$, which is independent of α, and $z \in B(0, R)$ to have $Q_a(\delta) \subset C$.

This means that

$$(z_1, z') \in B(0, R) \cap Q_a(\delta) \Rightarrow (z_1, z') \in C.$$

For $d(a) < \frac{1}{2}R^{M(\mathcal{Q})} \le \frac{1}{2}R^{m_n(\alpha)}$, because we can always choose $R \le 1$, with $M(\mathcal{Q}) = \sup_{\alpha\in\partial\Omega} m_n(\alpha) < \infty$, then $Q_a(\delta_0) \subset B(\alpha, R)$ hence in this case $Q_a(\delta_0) \subset C \subset \Omega$. □

Together these lemmas proved:

Theorem 2.13. *Let Ω be a domain in $\mathbb{C}^n$; there is a good family of polydiscs in Ω with multi-type $\{b(\alpha),\ m(\alpha)\}_{\alpha\in\partial\Omega}$ iff there is a family of complex tangentially ellipsoids $\{C_\alpha\}_{\alpha\in\partial\Omega}$ with parameters $\{b(\alpha),\ m(\alpha)\}_{\alpha\in\partial\Omega}$ such that $\forall\alpha \in \partial\Omega$, $C_\alpha \cap B(\alpha, R) \subset \Omega \cap B(\alpha, R)$, where R is given by Lemma* 2.10.

3. Divisors of the Blaschke class

Let u be a holomorphic function in a domain Ω, set $X := u^{-1}(0)$ its zero set and $\Theta := \partial\bar{\partial} \ln |u|$ its associated $(1, 1)$ current of integration. We shall define a class of such zero sets containing the zero sets of Nevanlinna functions.

Definition 3.1. A holomorphic divisor X in the domain Ω is in the *Blaschke class* if, with Θ its associated $(1, 1)$ current of integration,

$$\|\Theta\|_B := \int_\Omega d(z)\mathrm{Tr}\Theta(z) < \infty.$$

Let S be a separated sequence of points in Ω contained in the zero set X in the Blaschke class of Ω. The aim of this section is to show that the measure $\nu := \sum_{a\in S} d(a)^n \delta_a$ is finite.

We shall need the easy lemma:

Lemma 3.2. *Let $\mathcal{Q} = \{Q_a(\delta),\ a \in \Omega\}$ be a good family of polydiscs for Ω with parameter δ_0 and $\delta < \delta_0$. Then we have*

$$\forall a \in \Omega,\ \forall z \in Q_a(\delta),\ d(a) \leq \frac{1}{\delta_0 - \delta} d(z, \partial\Omega).$$

Proof. We have by definition $Q_a(\delta_0) \subset \Omega$, hence $\forall z \in Q_a(\delta),\ d(z, \Omega^c) \geq d(z,\ Q_a(\delta_0)^c)$, but because $m_j(a) \geq m_1(a) = 1, d(a) \leq d(a)^{1/m_j(a)}$ by the construction of the polydisc $Q_a(\delta)$ we have

$$\begin{aligned}\forall z \in Q_a(\delta),\ d(z, \partial\Omega) &\geq d(z,\ Q_a(\delta_0)^c)\\ &\geq \min_{j=1,\dots,n} (\delta_0 - \delta) d(a)^{1/m_j(a)} \geq (\delta_0 - \delta) d(a). \qquad \square\end{aligned}$$

3.1. The discretized Blaschke condition

Let $u \in \mathcal{H}(\Omega)$, *i.e.* u is holomorphic in Ω, and let $X := u^{-1}(0)$; put $\Theta := \partial\bar{\partial} \ln |u|$ the $(1, 1)$ current associated to X. Recall that $\Delta \ln |u(z)| = \mathrm{Tr}\,\Theta$, the trace of Θ, and Θ is a positive current, hence its trace controls all its coefficients.

We have, for any open set $\mathcal{V} \subset \Omega$, the equality (see for instance [24, page 55])

$$\int_{\mathcal{V}} \mathrm{Tr}\,\Theta = \sigma_{2n-2}(X \cap \mathcal{V}). \tag{3.1}$$

Let $E_j := \{z \in \mathbb{C}^n :: z_j = 0\}$, this is the subspace orthogonal to the z_j complex plane. Let Ω be a domain equipped with a good family $\mathcal{Q}$ of polydiscs and X a divisor in Ω. We set for $a \in \Omega,\ X_a := X \cap Q_a(\delta),\ X_a^j$ the projection of X_a on E_j and $A_j(X_a) := \sigma_{2n-2}(X_a^j)$.

We have the discretized Blaschke condition:

Theorem 3.3 (Discretized Blaschke condition). *Let u be holomorphic in Ω, $X := u^{-1}(0)$ and $\Theta := \partial\bar{\partial} \ln |u|$ its current of integration; suppose that Θ is in the Blaschke class. Let S be a δ separated sequence in X with respect to a good family $\mathcal{Q}$ of polydiscs with parameter δ_0. Then we have, provided that $\delta < \delta_0/2$,*

$$\sum_{a\in S} d(a)\sigma_{2n-2}(X_a) \leq \frac{2}{\delta_0}\|\Theta\|_B. \tag{3.2}$$

Proof. Let $a \in S$; then by Lemma 3.2 we have $\forall z \in Q_a(\delta),\ d(z) \geq \delta_0 - \delta \geq \frac{\delta_0}{2} d(a)$. Now

$$\sum_{a\in S} \int_{Q_a(\delta)} d(z) \operatorname{Tr} \Theta \leq \int_\Omega d(z) \operatorname{Tr} \Theta = \|\Theta\|_B,$$

because S is δ separated, hence the polydiscs $Q_a(\delta)$ are disjoint. Then

$$\|\Theta\|_B \geq \frac{\delta_0}{2} \sum_{a\in S} d(a) \int_{Q_a(\delta)} \operatorname{Tr} \Theta = \frac{\delta_0}{2} \sum_{a\in S} d(a)\sigma_{2n-2}(X_a). \qquad \square$$

3.2. The discretized Malliavin condition

Let us set $\gamma := i \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, we have that $\partial\gamma = \bar{\partial}\gamma = 0$ and γ is a positive $(1, 1)$ form. We shall follow the proof by H. Skoda [33, page 277].

Set $\beta := \gamma^{\wedge(n-2)}$ and apply Stokes formula to $\rho\Theta \wedge \bar{\partial}\rho \wedge \beta$

$$0 = \int_{\partial\Omega} \rho\Theta \wedge \bar{\partial}\rho \wedge \beta = \int_\Omega \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta - \int_\Omega \rho\Theta \wedge \partial\bar{\partial}\rho \wedge \beta,$$

because Θ and β are closed. Hence

$$\begin{aligned} \left| \int_\Omega \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta \right| = \left| \int_\Omega \rho\Theta \wedge \partial\bar{\partial}\rho \bigwedge \beta \right| &\leq \left\| \partial\bar{\partial}\rho \wedge \beta \right\|_\infty \int_\Omega (-\rho) \operatorname{Tr} \Theta \\ &\leq \left\| \partial\bar{\partial}\rho \right\|_\infty \|\partial\rho\|_\infty \|\beta\|_\infty \int_\Omega d(z,\ \partial\Omega) \operatorname{Tr} \Theta \\ &\leq \left\| \partial\bar{\partial}\rho \right\|_\infty \|\partial\rho\|_\infty \|\beta\|_\infty \|\Theta\|_B < \infty, \end{aligned}$$

because the trace of Θ controls all its coefficients and $(-\rho(z)) \leq \|\partial\rho\|_\infty d(z,\ \partial\Omega)$.

The norm $\|\beta\|_\infty$ is a constant depending only on the dimension n, hence we can set $C(\rho) := \left\| \partial\bar{\partial}\rho \right\|_\infty \|\partial\rho\|_\infty \|\beta\|_\infty$ which depends only on the first two derivatives of the defining function ρ.

Hence we proved

Lemma 3.4. *We have the estimate:*

$$\left| \int_\Omega \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta \right| \leq C(\rho)\|\Theta\|_B.$$

Set $m_n' := \lceil m_n \rceil$.

Lemma 3.5. *If a real smooth function $h(z)$ verifies*

$$|h(z)| \leq \sum_{j=1}^{n} |z_j|^{m_j}$$

with $m_j \geq 2$ and $|z_j|^{m_j} \leq d(a)$, then $\partial h \wedge \bar{\partial} h = d(a)\Gamma(z)$, where $\Gamma(z)$ is a positive bounded $(1, 1)$ form with its sup *norm controlled by the $m_n' + 1$ derivatives of h.*

Proof. We shall use Lemma 2.9, this time using complex variables notation, for the function h with no parameter; there are smooth functions $f_{\alpha,\beta}(z, \bar{z})$ for $|\alpha| + |\beta| = m_n'$ such that

$$h(z) = \sum_{k=0}^{m_n'-1} \sum_{\alpha,\beta,|\alpha|+|\beta|=k} a_{\alpha,\beta} z^\alpha \bar{z}^\beta + \sum_{\alpha,\beta,|\alpha|+|\beta|=m_n'} f_{\alpha,\beta}(z, \bar{z}) z^\alpha \bar{z}^\beta$$

with $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and the same for $\bar{z}^\beta$. Consider the path $t \in [0, \epsilon] \to z_j(t) := \zeta_j t^{1/m_j}$ then

$$z^\alpha = \zeta^\alpha t^{\gamma(\alpha)},$$

with $\gamma(\alpha) = \sum_{j=1}^{n} \frac{\alpha_j}{m_j}$, hence

$$\begin{aligned} h(z(t)) = & \sum_{\alpha,\beta,|\alpha|+|\beta|<m_n'} a_{\alpha,\beta} \zeta^\alpha \bar{\zeta}^\beta t^{\gamma(\alpha)+\gamma(\beta)} \\ & + \sum_{\alpha,\beta,|\alpha|+|\beta|=m_n'} f_{\alpha,\beta}(z(t), \bar{z}(t)) \zeta^\alpha \bar{\zeta}^\beta t^{\gamma(\alpha)+\gamma(\beta)}. \end{aligned}$$

We also have

$$\sum_{j=1}^{n} |z_j|^{m_j} = t \sum_{j=1}^{n} |\zeta_j|^{m_j}$$

hence let $s = \gamma(\alpha) + \gamma(\beta)$, then for $|\alpha| + |\beta| = m_n'$ we have

$$1 = \sum_{j=1}^{n} \frac{\alpha_j}{m_n'} + \sum_{j=1}^{n} \frac{\beta_j}{m_n'} \leq \sum_{j=1}^{n} \frac{\alpha_j}{m_j} + \sum_{j=1}^{n} \frac{\beta_j}{m_j} = s,$$

because $m_j \leq m_n \leq m_n'$.

The function $s = \gamma(\alpha) + \gamma(\beta)$ can take only a finite number of values, say $s_1 < \ldots < s_k$, then because $|h(z(t))| \leq t \sum_{j=1}^{n} \left|\zeta_j\right|^{m_j}$, if $s_1 < 1$,

$$\begin{aligned}\left|\sum_{\alpha,\beta,\gamma(\alpha)+\gamma(\beta)=s_1} a_{\alpha,\beta}\zeta^\alpha\bar{\zeta}^\beta\right| &\leq t^{1-s_1}\sum_{j=1}^{n}\left|\zeta_j\right|^{m_j} \\ &+ \sum_{s=s_2}^{s_k} t^{s-s_1}\left|\sum_{\alpha,\beta,\ \gamma(\alpha)+\gamma(\beta)=s} a_{\alpha,\beta}\zeta^\alpha\bar{\zeta}^\beta\right| \\ &+ \sum_{\alpha,\beta,|\alpha|+|\beta|=m'_n}\left|f_{\alpha,\beta}(z(t))\right| |\zeta|^{|\alpha|+|\beta|}\, t^{\gamma(\alpha)+\gamma(\beta)-s_1}.\end{aligned}$$

In the last sum we have $\gamma(\alpha)+\gamma(\beta) \geq 1$ because $|\alpha|+|\beta| = m'_n$ and the functions $f_{\alpha,\beta}$ are bounded. Letting $t \to 0$, we get

$$\sum_{\alpha,\beta,\ \gamma(\alpha)+\gamma(\beta)=s_1} a_{\alpha,\beta}\zeta^\alpha\bar{\zeta}^\beta = 0.$$

We can repeat the same computation for $s_2, \ldots, s_j$ provided that $s_j < 1$, and we get

$$\sum_{\alpha,\beta,\ \gamma(\alpha)+\gamma(\beta)<1} a_{\alpha,\beta}\zeta^\alpha\bar{\zeta}^\beta = 0.$$

So in the expansion of h it remains only α, β such that $\gamma(\alpha) + \gamma(\beta) \geq 1$.

Now we compute

$$\partial z^\alpha = \sum_{j=1}^{n}\alpha_j z^\alpha/z_j dz_j = z^\alpha\sum_{j=1}^{n}\frac{\alpha_j}{z_j}dz_j.$$

And

$$\bar{\partial}\bar{z}^\beta = \sum_{j=1}^{n}\beta_j\bar{z}^\beta/\bar{z}_j d\bar{z}_j = \bar{z}^\beta\sum_{j=1}^{n}\frac{\beta_j}{\bar{z}_j}d\bar{z}_j.$$

Set $\omega(z,\alpha) := \sum_{j=1}^{n}\frac{\alpha_j}{z_j}dz_j$, we have

$$\begin{aligned}\partial h = &\sum_{\alpha,\beta,1\leq\gamma(\alpha)+\gamma(\beta)<m'_n} a_{\alpha,\beta}z^\alpha\bar{z}^\beta\omega(z,\alpha) + \sum_{\alpha,\beta,|\alpha|+|\beta|=m'_n} f_{\alpha,\beta}(z,\bar{z})z^\alpha\bar{z}^\beta\omega(z,\alpha) \\ &+ \sum_{\alpha,\beta,|\alpha|+|\beta|=m'_n} z^\alpha\bar{z}^\beta\partial f_{\alpha,\beta}(z,\bar{z}).\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}h = &\sum_{\alpha,\beta,\gamma(\alpha)+\gamma(\beta)\geq 1} a_{\alpha,\beta}z^\alpha\bar{z}^\beta\bar{\omega}(z,\beta) + \sum_{\alpha,\beta,|\alpha|+|\beta|=m'_n} f_{\alpha,\beta}(z,\bar{z})z^\alpha\bar{z}^\beta\bar{\omega}(z,\beta) \\ &+ \sum_{\alpha,\beta,|\alpha|+|\beta|=m'_n} z^\alpha\bar{z}^\beta\bar{\partial} f_{\alpha,\beta}(z,\bar{z}).\end{aligned}$$

So we have as the generic term for $\partial h \wedge \bar{\partial} h$

$$A dz_j \wedge d\bar{z}_k := z^{\alpha+\alpha'}\bar{z}^{\beta+\beta'}\frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k}$$

hence

$$|A| = |z_1|^{\alpha_1+\beta_1+\alpha'_1+\beta'_1} \dots |z_n|^{\alpha_n+\beta_n+\alpha'_n+\beta'_n} \left|z_j\right|^{-1} |z_k|^{-1}$$

with $|z_l|^{m_l} \leq d(a)$ we get

$$|A| \leq d(a)^{\gamma(\alpha)+\gamma(\beta)+\gamma(\alpha')+\gamma(\beta')} d(a)^{-1/m_j} d(a)^{-1/m_k} \leq d(a),$$

because

$$\gamma(\alpha) + \gamma(\beta) + \gamma(\alpha') + \gamma(\beta') \geq 2 \text{ and } m_j \geq 2, m_k \geq 2.$$

The special terms are of the forms

$$B dz_j \wedge d\bar{z}_k := f_{\alpha,\beta}(z) z^{\alpha+\alpha'}\bar{z}^{\beta+\beta'}\frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k}$$

and, by the same argument, they verify

$$|B| \leq \left\| f_{\alpha,\beta} \right\|_\infty d(a);$$

or

$$C dz_j \wedge d\bar{z}_k := \frac{\partial f_{\alpha,\beta}}{\partial z_j} z^{\alpha+\alpha'}\bar{z}^{\beta+\beta'} dz_j \wedge \frac{d\bar{z}_k}{\bar{z}_k}$$

and they verify a fortiori $|C| \leq \left\| \partial f_{\alpha,\beta} \right\|_\infty d(a)$; or

$$D dz_j \wedge d\bar{z}_k := f_{\alpha,\beta}(z) f_{\alpha',\beta'}(z) z^{\alpha+\alpha'}\bar{z}^{\beta+\beta'}\frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k}$$

and they verify $|D| \leq \left\| f_{\alpha,\beta} f_{\alpha',\beta'} \right\|_\infty d(a)$; or

$$E dz_j \wedge d\bar{z}_k := \frac{\partial f_{\alpha,\beta}}{\partial z_j} \frac{\partial f_{\alpha',\beta'}}{\partial z_j} z^{\alpha+\alpha'}\bar{z}^{\beta+\beta'} dz_j \wedge d\bar{z}_k$$

and they verify $|E| \leq \left\| \partial f_{\alpha,\beta} \bar{\partial} f_{\alpha',\beta'} \right\|_\infty d(a)$; or

$$F dz_j \wedge d\bar{z}_k := \frac{\partial f_{\alpha,\beta}}{\partial z_j} f_{\alpha,\beta}(z) z^{\alpha+\alpha'}\bar{z}^{\beta+\beta'} dz_j \wedge \frac{d\bar{z}_k}{\bar{z}_k}$$

and they verify $|F| \leq \left\| f_{\alpha',\beta'} \partial f_{\alpha,\beta} \right\|_\infty d(a)$; and the conjugates of these expressions are also bounded. All the bounds are controlled by the $m'_n + 1$ derivatives of h and we have a finite set of such smooth coefficients so

$$\partial h \wedge \bar{\partial} h = d(a)\Gamma(z),$$

where $\Gamma(z)$ is a positive bounded $(1, 1)$ form controlled by the $m'_n + 1$ derivatives of h. □

We shall evaluate the integral $\int_{Q_a(\delta)} \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta$, and we start first with the following function μ, defining a complex tangential ellipsoid C as in the previous section, Lemmas 2.9, 2.10, but here we choose twice the previous one to ease the computations,

$$\mu(z) := 2x_1 + 2x_1 g_1(z) + 2y_1 g_2(z) + 2A(|z_2|^{m_2} + \ldots + |z_n|^{m_n}).$$

We have, with $a = (a_1, 0, \ldots, 0),\ a_1 < 0$,

$$\begin{aligned}\partial\mu(z) = &\left(1 + g_1(z) - ig_2(z) + 2x_1 \frac{\partial g_1}{\partial z_1} + 2y_1 \frac{\partial g_2}{\partial z_1}\right) dz_1 \\ &+ 2x_1 \sum_{j\geq 2} \frac{\partial g_1}{\partial z_j}(z) dz_j + 2y_1 \sum_{j\geq 2} \frac{\partial g_2}{\partial z_j}(z) dz_j \\ &+ A\left(m_2 |z_2|^{m_2-2} \bar{z}_2 dz_2 + \ldots + m_n |z_2|^{m_n-2} \bar{z}_n dz_n\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}\mu(z) = &\left(1 + g_1(z) + ig_2(z) + 2x_1 \frac{\partial g_1}{\partial \bar{z}_1} + 2y_1 \frac{\partial g_2}{\partial \bar{z}_1}\right) d\bar{z}_1 + 2x_1 \sum_{j\geq 2} \frac{\partial g_1}{\partial z_j} d\bar{z}_j \\ &+ 2y_1 \sum_{j\geq 2} \frac{\partial g_2}{\partial z_j} d\bar{z}_j + A\left(m_2 |z_2|^{m_2-2} z_2 d\bar{z}_2 + \ldots + m_n |z_n|^{m_n-2} z_n d\bar{z}_n\right),\end{aligned}$$

because

$$\frac{\partial\, |w|^m}{\partial w} = \partial_w((\bar{w}w)^{m/2}) = \frac{m}{2}(\bar{w}w)^{m/2-1}\bar{w} = \frac{m}{2}\,|w|^{m-2} \times \bar{w}.$$

Lemma 3.6. *We have*

$$\begin{aligned}\forall z \in Q_a(\delta),\ \partial\mu \wedge \bar{\partial}\mu = B(z) dz_1 \wedge d\bar{z}_1 &+ \sum_{j=2}^{n} C_j(z) \left|z_j\right|^{m_j-1} dz_1 \wedge d\bar{z}_j \\ &+ \sum_{j=2}^{n} D_j(z) \left|z_j\right|^{m_j-1} dz_j \wedge d\bar{z}_1 + d(a)\Gamma(z),\end{aligned}$$

where B, C_j, D_j are bounded with bounds depending only on the the $\mathcal{C}^1$ norms of g_1, g_2 and Γ is a $(1,1)$ form with bounded coefficients depending only on the the $\mathcal{C}^1$ norms of g_1, g_2.

Proof. Because $\forall z \in Q_a(\delta)$ we have $|z_1| \leq \delta d(a) \Rightarrow |x_1| \leq \delta d(a),\ |y_1| \leq \delta d(a)$ and $\forall j \geq 2,\ \left|z_j\right| \leq \delta d(a)^{1/m_j}$, so the terms in $\partial\mu \wedge \bar{\partial}\mu$ containing $\frac{\partial g_1}{\partial z_j}$ or $\frac{\partial g_2}{\partial z_j}$ or $\frac{\partial g_1}{\partial \bar{z}_j}$ or $\frac{\partial g_2}{\partial \bar{z}_j}$ can be put in Γ. For the terms in

$$A^2 \sum_{j,k=1,\ldots,n} m_j m_k \left|z_j\right|^{m_j-2} |z_k|^{m_k-2} \bar{z}_j z_k dz_j \wedge d\bar{z}_k$$

we have

$$\forall j, k \geq 2,\ \left|z_j\right|^{m_j-1} |z_k|^{m_k-1} \leq \delta^2 d(a)^{\frac{m_j-1}{m_j}+\frac{m_k-1}{m_k}};$$

suppose that $m_j \geq m_k$, then

$$\frac{m_j-1}{m_j} + \frac{m_k-1}{m_k} \geq \frac{m_j-1+m_k-1}{m_j} \geq \frac{m_j}{m_j} = 1,$$

because $\forall k \geq 1,\ m_k \geq 2$. Hence they also can be put in Γ. It remains

$$B(z)dz_1 \wedge d\bar{z}_1 \text{ with } B(z) := (1+g_1)^2 + g_2^2,$$

$$\sum_{j=2}^{n} C_j(z) \left|z_j\right|^{m_j-2} z_j dz_1 \wedge d\bar{z}_j \text{ with } C_j(z) := (1+g_1(z) - ig_2(z))Am_j,$$

and

$$\sum_{j=2}^{n} D_j(z) \left|z_j\right|^{m_j-2} \bar{z}_j dz_j \wedge d\bar{z}_1 \text{ with } D_j(z) := (1+g_1(z) + ig_2(z))Am_j.$$

Clearly the bounds on those terms and in Γ depend only on the $\mathcal{C}^1$ norms of g_1, g_2. □

Lemma 3.7. *Let Θ be a positive $(1,1)$ current and $F(z_j)$ a function; then for all $\eta > 0$ we have*

$$2\left|\Theta \wedge F(z_j)dz_1 \wedge d\bar{z}_j \wedge \beta\right| \leq \eta\Theta\wedge dz_1 \wedge d\bar{z}_1 \wedge \beta + \frac{1}{\eta}\Theta\wedge\left|F(z_j)\right|^2 dz_j \wedge d\bar{z}_j \wedge \beta.$$

Proof. By Cauchy-Schwarz, because $\Theta \wedge \beta$ is positive, we get

$$\left|\Theta \wedge dz_1 \wedge F(z_j)d\bar{z}_j \wedge \beta\right|^2 \leq \left|\Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta\right| \left|\Theta \wedge F(z_j)dz_j \wedge \bar{F}(z_j)d\bar{z}_j \wedge \beta\right|$$

hence, because $2ab \leq \eta a^2 + \frac{1}{\eta}b^2$,

$$2\ \left|\Theta \wedge dz_1 \wedge F(z_j)d\bar{z}_j \wedge \beta\right| \leq \eta\Theta\wedge dz_1 \wedge d\bar{z}_1 \wedge \beta + \frac{1}{\eta}\Theta\wedge\left|F(z_j)\right|^2 dz_j \wedge d\bar{z}_j \wedge \beta. \ \square$$

Let us go back to the general case. Fix $a \in \Omega,\ \alpha = \pi(a)$, we know by Lemma 2.11 that there is a complex tangential ellipsoid $C = C_\alpha$ with exponents $\{m_j(\alpha)\}$ meeting $\partial\Omega$ at α and contained in Ω. Moreover we have, after the canonical change of variables of Lemma 2.10, and multiplying by 2 the functions to make the following computations slightly easier,

$$\rho(z) = 2x_1 + 2x_1 g_1(z) + 2y_1 g_2(z) + h_\alpha(z') = 2x_1 + 2x_1 g_1(z) + 2y_1 g_2(z) + \rho(0, z'),$$

as the defining function for Ω and

$$\mu(z) := 2x_1 + 2x_1 g_1(z) + 2y_1 g_2(z) + 2A(|z_2|^{m_2} + \ldots + |z_n|^{m_n}),\ |y_1| < -\Gamma x_1,$$

as the defining functions for C_α. We notice that the functions g_j in μ are the same as the functions g_j in ρ and depend only on ρ. In particular the $\mathcal{C}^1$ norms of the g_j are controlled by the $\mathcal{C}^2$ norm of ρ hence they are uniformly bounded with respect to α.

Lemma 3.8. *We have, with Θ a positive $(1,1)$ current,*

$$\int_{Q_a(\delta)} \Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta \leq 5 \int_{Q_a(\delta)} \Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta + \Gamma d(a) \int_{Q_a(\delta)} \mathrm{Tr}\Theta.$$

with the constant Γ depending only on the $\mathcal{C}^2$ norm of ρ, on n and δ_0.

Proof. Using Lemma 3.6, we get

$$\begin{aligned}
&\Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta - B(z)\Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta \\
&\qquad = \sum_{j=2}^{n} C_j(z) \left|z_j\right|^{m_j-2} \bar{z}_j \Theta \wedge dz_1 \wedge d\bar{z}_j \wedge \beta \\
&\qquad\quad + \sum_{j=2}^{n} D_j(z) \left|z_j\right|^{m_j-2} z_j \Theta \wedge dz_j \wedge d\bar{z}_1 \wedge \beta + d(a)\Theta \wedge \Gamma \wedge \beta.
\end{aligned}$$

Hence

$$B(z)\Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta = \Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta - U - d(a)\Theta \wedge \Gamma \wedge \beta,$$

with

$$\begin{aligned}
U := &\sum_{j=2}^{n} C_j(z) \left|z_j\right|^{m_j-2} \bar{z}_j \Theta \wedge dz_1 \wedge d\bar{z}_j \wedge \beta \\
&+ \sum_{j=2}^{n} D_j(z) \left|z_j\right|^{m_j-2} z_j \Theta \wedge dz_j \wedge d\bar{z}_1 \wedge \beta
\end{aligned}$$

By Lemma 3.7 we get, with $\eta > 0$ to be fixed later,

$$\begin{aligned}
2\left|C_j(z) \left|z_j\right|^{m_j-1} \Theta \wedge dz_1 \wedge d\bar{z}_j \wedge \beta\right| &\leq \eta \Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta \\
&\quad + \frac{1}{\eta} \left|C_j\right|^2 \left|z_j\right|^{2m_j-2} \Theta \wedge dz_j \wedge d\bar{z}_j \wedge \beta.
\end{aligned}$$

But for $z \in Q_a(\delta)$ we have $\left|z_j\right|^{2m_j-2} \leq \left|z_j\right|^{m_j-2} \delta d(a)$ because $\left|z_j\right|^{m_j} \leq \delta d(a)$ hence

$$\begin{aligned}
2\left|C_j(z) \left|z_j\right|^{m_j-1} \Theta \wedge dz_1 \wedge d\bar{z}_j \wedge \beta\right| &\leq \eta \Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta \\
&\quad + \frac{1}{\eta} \left|C_j\right|^2 \left|z_j\right|^{m_j-2} \delta d(a) \Theta \wedge dz_j \wedge d\bar{z}_j \wedge \beta.
\end{aligned}$$

Set $C'_j := 2\left|C_j\right|^2\left|z_j\right|^{m_j-2}$ whose bound depend on the $\mathcal{C}^1$ norm of the g_j we get

$$\left|\sum_{j=2}^{n} C_j(z)\left|z_j\right|^{m_j-2}\bar{z}_j\Theta\wedge dz_1\wedge d\bar{z}_j\wedge\beta\right| \leq \frac{1}{2}(n-1)\eta\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta$$
$$+\frac{\delta}{\eta}d(a)\sum_{j=2}^{n}C'_j\Theta\wedge dz_j\wedge d\bar{z}_j\wedge\beta.$$

Doing exactly the same proof, with $D'_j := 2\left|D_j\right|^2\left|z_j\right|^{m_j-2}$ we get

$$\left|\sum_{j=2}^{n} D_j(z)\left|z_j\right|^{m_j-2}z_j\Theta\wedge dz_j\wedge d\bar{z}_1\wedge\beta\right| \leq \frac{1}{2}(n-1)\eta\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta$$
$$+\frac{\delta}{\eta}d(a)\sum_{j=2}^{n}D'_j\Theta\wedge dz_j\wedge d\bar{z}_j\wedge\beta.$$

So we get

$$|U| \leq (n-1)\eta\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta+\frac{\delta}{\eta}d(a)\sum_{j=2}^{n}(C'_j+D'_j)\Theta\wedge dz_j\wedge d\bar{z}_j\wedge\beta.$$

Now we choose $\eta := \frac{1}{4(n-1)}$ and we get, because $\Theta\wedge\partial\mu\wedge\bar{\partial}\mu\wedge\beta$ and $B(z)\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta$ are positive,

$$B(z)\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta \leq \Theta\wedge\partial\mu\wedge\bar{\partial}\mu\wedge\beta+|U|+d(a)\left|\Theta\wedge\Gamma\wedge\beta\right|.$$

Hence

$$B(z)\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta \leq \Theta\wedge\partial\mu\wedge\bar{\partial}\mu\wedge\beta+\frac{1}{4}\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta$$
$$+d(a)(4(n-1)\delta(a)\left|\Theta\wedge\Gamma'\wedge\beta\right|+|\Theta\wedge\Gamma\wedge\beta|).$$

with

$$\Gamma' := \sum_{j=2}^{n}C'_j dz_j\wedge d\bar{z}_j+\sum_{j=2}^{n}D'_j dz_j\wedge d\bar{z}_j.$$

Finally

$$\left(B(z)-\frac{1}{4}\right)\Theta\wedge dz_1\wedge d\bar{z}_1\wedge\beta \leq \Theta\wedge\partial\mu\wedge\bar{\partial}\mu\wedge\beta$$
$$+d(a)\Big(4(n-1)\delta(a)\left|\Theta\wedge\Gamma'\wedge\beta\right|+|\Theta\wedge\Gamma\wedge\beta|\Big).$$

Recall that $B(z) = (1+g_1(z))^2 + g_2(z)^2 \geq 0$ and we know by Lemma 2.10 that $\forall z \in B(0,R)$, $|g_1(z)| \leq \frac{3}{10}$ hence, provided that $Q_a(\delta) \subset B(0,R)$, *i.e.* $d(a) < R^{1/M(\mathcal{Q})}$, with $M(\mathcal{Q}) = \sup_{\alpha\in\partial\Omega} m_n(\alpha) < \infty$ because $\mathcal{Q}$ is a good family,

$$(1+g_1(z))^2 + g_2(z)^2 - 1/4 \geq \left(\frac{7}{10}\right)^2 - \frac{1}{4} = \frac{24}{100}.$$

So dividing by $B(z) - \frac{1}{4}$ we get

$$\begin{aligned}\Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta \leq\ & 5\Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta \\ & + 5d(a)\left(4(n-1)\delta(a)\left|\Theta \wedge \Gamma' \wedge \beta\right| + 5\left|\Theta \wedge \Gamma \wedge \beta\right|\right).\end{aligned}$$

Integrating, we get the lemma because the trace of Θ controls all its coefficients. $\square$

We shall need the following definition.

Definition 3.9. The domain Ω, equipped with a good family $\mathcal{Q}$, will be said *quasi convex* at $a \in \Omega$ if, with $\alpha = \pi(a)$, $m = m(\alpha)$, taking the coordinates associated to the basis $b(\alpha)$, centered at α, we have with ρ_α a defining function for Ω,

$$\forall z \in Q_a(2) :: \rho_\alpha(0,z') < 0,\ -\rho_\alpha(0,z') \leq \gamma(|z_2|^{m_2} + \cdots + |z_n|^{m_n}).$$

The domain will be said *quasi convex* if Ω is quasi convex at a for all $a \in \mathcal{U} \cap \Omega$ with the same constant γ.

A convex Ω or a lineally convex Ω are quasi convex because for them $\Omega \cap T_\alpha^{\mathbb{C}}(\partial\Omega) = \emptyset$ hence $\rho(0,z') \geq 0$.

We have $\Theta = \sum_{i,j=1}^n \Theta_{ij}\, dz_i \wedge \partial\bar{z}_j$ and

$$\Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta = \sum_{i,j=2}^n \Theta_{ij}\, dz_i \wedge \partial\bar{z}_j \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta.$$

In the integral $\int_{Q_a(\delta)} \Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta$, it remains precisely the sum of the σ_{2n-1} areas of the projections of X_a on the E_j, $j \geq 2$, see [24, Proposition 2.48, page 55]. So recall that for $a \in \Omega$, $X_a := X \cap Q_a(\delta)$, X_a^j the projection of X_a on E_j is denoted X_a^j and $A_j(X_a) := \sigma_{2n-2}(X_a^j)$; we get

$$\int_{Q_a(\delta)} \Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta = \sum_{j=2}^n A_j(X_a).$$

So by Lemma 3.8 we have

$$\begin{aligned}\sum_{j=2}^n A_j(X_a) =\ & \int_{Q_a(\delta)} \Theta \wedge dz_1 \wedge d\bar{z}_1 \wedge \beta \\ \leq\ & 5\int_{Q_a(\delta)} \Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta + \Gamma d(a) \int_{Q_a(\delta)} \mathrm{Tr}\Theta.\end{aligned}$$

Hence, because if $z \in Q_a(\delta)$, by Lemma 3.2, $d(a) \leq \frac{1}{\delta_0 - \delta} d(z)$, then

$$\sum_{j=2}^{n} A_j(X_a) \leq 5 \int_{Q_a(\delta)} \Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta + \frac{\Gamma}{\delta_0 - \delta} \int_{Q_a(\delta)} d(z) \mathrm{Tr}\Theta.$$

At this point we shall use equation (2.2) which says, (recall we multiply by 2 the defining function ρ of the domain and the defining function μ of the cone)

$$\forall z \in C_\alpha \cap B(0, R),\ \rho(0; z') \leq 2A \left(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\right).$$

So either $\rho(0; z') \geq 0$, then we have

$$0 \leq \rho(0; z') \leq 2A \left(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\right),$$

or $\rho(0, z') < 0$, then we use that Ω is $m(\alpha)$ quasi convex at α to get

$$-\rho(0, z') \leq \gamma \left(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\right).$$

In any case we can apply Lemma 3.5 with z' instead of z to

$$h(z') := \mu - \rho = 2A \left(|z_2|^{m_2} + \ldots + |z_n|^{m_n}\right) - \rho(0; z'),$$

to get

$$\partial h(z') \wedge \bar{\partial} h(z') = d(a)\Gamma(z'),$$

with the sup norm of Γ controlled by the $m_n(a) + 1$ derivatives of h.

So we have $\mu = \rho + h$, with $\partial h \wedge \bar{\partial} h = d(a)\Gamma(z')$, hence

$$\begin{aligned} \Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta &= \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta + \Theta \wedge \partial h \wedge \bar{\partial} h \wedge \beta \\ &\quad + \Theta \wedge \partial\rho \wedge \bar{\partial}\mu \wedge \beta + \Theta \wedge \partial h \wedge \bar{\partial}\rho \wedge \beta, \end{aligned}$$

by Cauchy-Schwartz, because $\Theta \wedge \beta$ is positive, we get

$$\left|\Theta \wedge \partial h \wedge \bar{\partial}\rho \wedge \beta\right|^2 \leq \left|\Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta\right| \left|\Theta \wedge \partial h \wedge \bar{\partial} h \wedge \beta\right|$$

hence, because $2ab \leq a^2 + b^2$,

$$2 \left|\Theta \wedge \partial h \wedge \bar{\partial}\rho \wedge \beta\right| \leq \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta + \Theta \wedge \partial h \wedge \bar{\partial} h \wedge \beta$$

and

$$\Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta \leq 2\Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta + 2\Theta \wedge \partial h \wedge \bar{\partial} h \wedge \beta.$$

Finally

$$\Theta \wedge \partial\mu \wedge \bar{\partial}\mu \wedge \beta \leq 2\Theta \wedge \partial\rho \wedge \bar{\partial}\rho \wedge \beta + 2d(a)\Theta \wedge \Gamma(z') \wedge \beta.$$

So we have, with $\Gamma' = \frac{1}{\delta_0-\delta}\sup_{z\in Q_a(\delta)}(\Gamma(z'),\gamma)$, still controlled by the $m_n(a)+1$ derivatives of h,

$$\int_{Q_a(\delta)}\Theta\wedge\partial\mu\wedge\bar\partial\mu\wedge\beta \le \int_{Q_a(\delta)}\Theta\wedge\partial\rho\wedge\bar\partial\rho\wedge\beta + \Gamma'\int_{Q_a(\delta)}d(z)\mathrm{Tr}\Theta.$$

Hence

$$\sum_{j=2}^{n}A_j(X_a)\le 5\int_{Q_a(\delta)}\Theta\wedge\partial\rho\wedge\bar\partial\rho\wedge\beta + 5\Gamma'\int_{Q_a(\delta)}d(z)\mathrm{Tr}\Theta.$$

By use of Lemma 3.4 and setting $S' := S\cap\mathcal{U}$, we have, because the polydiscs $Q_a(\delta),\ a\in S'$ are disjoint

$$\sum_{a\in S'}\sum_{j=2}^{n}A_j(X_a)\le C\|\Theta_X\|_{B(\Omega)}, \tag{3.3}$$

where $C = 5C(\rho) + 5\Gamma'$. Notice that the constant C does not depend on α and depends only on the derivatives of ρ up to order $M(\mathcal{Q})+1$, with $M(\mathcal{Q}) = \sup_{a\in\Omega} m_n(a)<\infty$, because $\mathcal{Q}$ is a good family.

So we proved the discretized Malliavin condition:

Theorem 3.10 (Discretized Malliavin condition). *Let Ω be a domain in $\mathbb{C}^n$ equipped with a good family $\mathcal{Q}$ of polydiscs with parameter δ_0, and which is $\mathcal{Q}$ quasi convex. Let Θ be a current in the Blaschke class and S a δ separated sequence in $X\cap\mathcal{U}$ with respect to the family $\mathcal{Q}$. Then we have*

$$\sum_{a\in S}\sum_{j=2}^{n}A_j(X_a)\le C\|\Theta\|_B, \tag{3.4}$$

where C is a constant depending only on the derivatives of ρ up to order $M(\mathcal{Q})+1$, on $\delta,\ \delta_0$ and on the constant of quasi convexity.

3.3. A geometrical lemma

Let Ω be a domain in $\mathbb{C}^n$. Let $a\in\mathcal{U},\ \alpha=\pi(a)$ and $Q_a(\delta)$ the polydisc of a good family $\mathcal{Q}$ associated to Ω.

Let $\mathbb{D}^n$ be the unit polydisc in $\mathbb{C}^n$, and let Φ_a be the bi-holomorphic application from $\mathbb{D}^n$ onto $Q_a(\delta)$

$$\forall z=(z_1,\dots,z_n)\in\mathbb{D}^n,\ 1\le j\le n,\ Z_j = a_j + \delta d(a)^{1/m_j(a)}z_jL_j.$$

If X is the zero set of a holomorphic function in Ω with $a\in X$, we can lift $X_a := X\cap Q_a(\delta)$ in $\mathbb{D}^n$ by Φ_a^{-1}. Set $Y_a := \Phi_a^{-1}(X_a)$, and recall that the multi-type is such that $m_1=1$ and m_n is always bounded, $m_n(a)\le M(\mathcal{Q})$. We have:

Lemma 3.11.

(i) $\sigma_{2n-2}(X_a) = \sum_{j=1}^{n} A_j(X_a)$;
(ii) $\forall j=1,\dots,n,\ A_j(X_a)=\delta^{2n-2}d(a)^{2\mu_j(a)}A_j(Y_a)$, *with* $\mu_j(a)=\sum_{k\neq j}\frac{1}{m_k(a)}$;
(iii) *with* $m_1 < m_2 \leq \dots \leq m_n$,

$$\frac{n-1}{M(\mathcal{Q})} \leq \mu_1(a) \leq \mu_2(a) \leq \dots \leq \mu_n(a) \leq n/2;$$

$$\mu_1(a) = \sum_{k=2}^{n} \frac{1}{m_k(a)} \leq \frac{n-1}{2};$$

(iv) $c_n \leq \sigma_{2n-2}(Y_a)$.

Proof. The (i) is classical ([24, Proposition 2.48, p 55]).

The application Φ_a sends $E_k = \{z_k = 0\}$ in $F_k := \{$ the orthogonal to L_k axis$\}$ and the jacobian of this restriction at the point a, $J_k\Phi$, is $J_k\Phi = \delta^{n-1}d(a)^{\mu_k(a)}$. Because the application is holomorphic, we get that the jacobian for the change of real variables is

$$|J_k|^2 = \delta^{2n-2}d(a)^{2\mu_k(a)},$$

which gives the (ii).

For the (iii) we notice that

$$2 \leq j,k \leq n,\ m_k(a) \geq 2 \Rightarrow \frac{1}{m_k(a)} \leq \frac{1}{2} \Rightarrow \sum_{k\neq j,\ 2\leq k\leq n} \frac{1}{m_k(a)} \leq \frac{n-2}{2}.$$

Hence if $2 \leq j \leq n$, $\mu_j(a) = \sum_{k\neq j, 2\leq k\leq n} \frac{1}{m_k(a)} + 1 \leq n/2$; if $j = 1$, $\mu_1(a) \leq \frac{n-1}{2} \leq n/2$. Hence (iii).

The (iv) is the Wirtinger inequality [19], adapted from the ball to the polycube as follows: $Y_a \cap B(0,1) \subset Y_a$ hence by Wirtinger inequality we get

$$c_n \leq \sigma_{2n-2}\left(Y_a \cap B(0,1)\right) \leq \sigma_{2n-2}(Y_a),$$

so the lemma is proved. □

3.4. The result

Theorem 3.12. *Let Ω be a domain in $\mathbb{C}^n$ equipped with a good family $\mathcal{Q}$ of polydiscs and which is $\mathcal{Q}$ quasi convex. Let S be a δ separated sequence of points which is contained in the Blaschke divisor X. Then*

$$\delta^{2n-2}\sum_{a\in S} d(a)^n \leq \gamma(\Omega)\|\Theta_X\|_B,$$

where $\gamma(\Omega)$ depends only on the derivatives of ρ up to order $M(\mathcal{Q})+1$, on n and δ_0, the parameter of the family $\mathcal{Q}$, and on the constant of quasi convexity.

Proof. We have by Lemma 3.11

$$\forall j = 2, \ldots, n, \; A_j(X_a) = \delta^{2n-2} d(a)^{2\mu_j(a)} A_j(Y_a),$$

but for $j \geq 2, \; 2\mu_j(a) \leq n$, hence

$$\forall j = 2, \ldots, n, \; A_j(X_a) \geq \delta^{2n-2} d(a)^n A_j(Y_a).$$

For $j = 1$ we have $1 + 2\mu_1(a) \leq n$, by Lemma 3.11, (iii), hence

$$A_1(X_a) \geq \delta^{2n-2} d(a)^n A_1(Y_a).$$

Then Theorem 3.10 gives

$$C\|\Theta_X\|_B \geq \sum_{a\in S} \sum_{j=2}^{n} A_j(X_a) \geq \delta^{2n-2} \sum_{a\in S} d(a)^n \sum_{j=2}^{n} A_j(X_a).$$

And the Blaschke condition gives

$$\begin{aligned} \frac{2}{\delta_0}\|\Theta\|_B &\geq \sum_{a\in S} d(a)\text{Area}(X_a) \\ &\geq \sum_{a\in S} d(a) A_1(X_a) \geq \delta^{2n-2} \sum_{a\in S} d(a)^{1+2\mu_1(a)} A_1(Y_a), \end{aligned}$$

hence

$$\frac{2}{\delta_0}\|\Theta\|_B \geq \delta^{2n-2} \sum_{a\in S} d(a)^n A_1(Y_a).$$

So

$$\delta^{2n-2} \sum_{a\in S} d(a)^n \left(A_1(Y_a) + \sum_{j\geq 2} A_j(Y_a) \right) \leq (C + \frac{2}{\delta_0})\|\Theta_X\|_B.$$

Now with $A_1(Y_a) + \sum_{j\geq 2} A_j(Y_a) \geq c_n$ by Wirtinger inequality, we get the theorem. □

We already have defined in the introduction (1.1) the canonical measure associated to a sequence S

$$\nu_S := \sum_{a\in S\cap\mathcal{U}} d(a)^{1+2\mu(a)} \delta_a$$

with $\mu(a) := \sum_{j=2}^{n} \frac{1}{m_j(a)}$.

The theorem says that the measure $\sum_{a\in S\cap\mathcal{U}} d(a)^n \delta_a$ is bounded which is weaker than the fact the measure ν_S is bounded, unless S is a separated sequence projecting on points of strict pseudo-convexity, because there we have $1 + 2\mu(a) = n$. In the next section we shall introduce domains for which we can control the right measure ν_S.

4. Almost strictly pseudo-convex domains

We shall introduce a family of domains Ω with "few" non strictly pseudo-convex points on $\partial\Omega$.

This family will be big enough to contain interesting cases, as convex domains of finite type for instance and will allow us to manage these "bad" points.

4.1. Minkowski dimension

4.1.1. *Definitions and first properties*

Lemma 4.1. *Let f be a function Lipschitz $\alpha > 0,\ \alpha \leq 1$, on the closed interval $I = [0,\ h]$ of $\mathbb{R}$. Then the graph*

$$G := \{(x, y) :: x \in I,\ y = f(x)\} \subset \mathbb{R}^2$$

of f can be covered by $N_r(h) \leq Chr^{\alpha-2}$ disjoint discs $D(a, r)$ centered at $a \in G$ and of radius r, provided that $r \leq h$.

Proof. This is corollary in [18, 11.2, page 147]. The proof is as follows. Let $0 < r < 1$ and m the least integer greater than or equal to h/r. We have by proposition in [18, 11.1, page 146]:

$$r^{-1}\sum_{j=0}^{m-1} R_f(jr, (j+1)r) \leq N_r(h) \leq 2m + r^{-1}\sum_{j=0}^{m-1} R_f(jr, (j+1)r), \qquad (4.1)$$

with $R_f(t_1, t_2) := \sup_{t_1<t,u<t_2} |f(t) - f(u)|$. Because f is Lipschitz α we have $R_f(t_1, t_2) \leq C\,|t_1 - t_2|^{\alpha}$ hence $R_f(jr, (j+1)r) \leq Cr^{\alpha}$. Putting this in (4.1) we get

$$N_r(h) \leq 2m + mCr^{\alpha-1}.$$

But provided that $m > 0$, *i.e.* $h \geq r$, we have $m \leq 2h/r$ so

$$N_r(h) \leq 4\frac{h}{r} + 2Chr^{\alpha-2} \leq C'hr^{\alpha-2}. \qquad \square$$

We shall define an homogeneous Minkowski dimension. Denote $\#A$ the number of points in the set A.

Definition 4.2. Let $W \subset \mathbb{R}^2$ be a bounded set and $\alpha > 0$; let $D(a, h)$ be a disc centered at a and of radius h; let $\mathcal{R}_r(W \cap D(a, h))$ be a covering of $W \cap D(a, h)$ by discs of radius r; we shall say that W has *homogeneous Minkowski* dimension α if:

$$\exists C > 0,\ \forall a \in W,\ \forall h > 0,\ \forall r > 0,\ r \leq h,$$
$$\exists \mathcal{R}_r(W \cap D(a, h)) :: \#\mathcal{R}_r(W \cap D(a, h)) \leq \max(1, Chr^{-\alpha}).$$

The number C will be called the *constant* of W with respect to the homogeneous Minkowski dimension α.

Clearly if W has homogeneous Minkowski dimension α with constant C then it has upper Minkowski dimension α (see [18]), but the converse is false as can be seen with the canonical example of $W = \{0, 1, 1/2, 1/3, \dots, 1/n, \dots\}$ which has Minkowski dimension $1/2$ but it is not homogeneous, *i.e.* $W \cap D(0, h)$ has no covering with the property above for any $h > 0$.

On the other hand, Lemma 4.1 gives examples of such sets.

The following result is a corollary of a nice theorem of Ostrowski [32].

Corollary 4.3. *Let $P(y)$ be a monic polynomial of degree d in the real variable y whose coefficients are $\mathcal{C}^\infty$ functions of $x \in \mathbb{R}$. Then the graph of the zero set of P has homogeneous Minkowski dimension less than $2 - \frac{1}{d}$.*

Proof. By a theorem of Ostrowski [32] we have that locally the roots y of the equation

$$P(y) = y^d + a_1 y^{d-1} + \ldots + a_d = 0,$$

are Lipschitz $\frac{1}{d}$ functions of the coefficients a_j. Composing with the $\mathcal{C}^\infty$ function

$$x \to a(x) := \{a_j(x),\ j = 1, \dots, d\},$$

we get that the roots $y_k(x)$, $k \le d$, are still Lipschitz $\frac{1}{d}$ and we can apply Lemma 4.1 to the graph of each root. Because there is at most d such graphs, the corollary is proved. □

4.1.2. *Domains in $\mathbb{C}^n$*

Let $D(\rho)$ be the disc in $\mathbb{C}$ of center 0 and radius ρ and denote σ_{2n} the Lebesgue measure in $\mathbb{C}^n = \mathbb{R}^{2n}$. We have the lemma:

Lemma 4.4. *Let $W \subset \mathcal{D} := D(d) \times D(R)^{n-2} \times D(h) \subset \mathbb{C}^n$ and $\alpha > 0$ such that the homogeneous Minkowski dimension of*

$$W \cap \{z_1 = a_1,\ \dots,\ z_{n-1} = a_{n-1}\}$$

is $2-\alpha$ for all $a' = (a_1, \dots, a_{n-1}) \in D(d) \times D(R)^{n-2}$. Let $S \subset W$ and let another orthonormal basis $b = \{L_1, .., L_n\}$ with $w = (w_1, \dots, w_n)$ as coordinates; let P_a be a polydisc with respect to the basis $b(a)$ centered on $a \in S$, $b(a)$ varying with $a \in S$, P_a with fixed radii $(r,\ l_2 r \dots,\ l_n r)$, and such that these polydiscs are disjoint. Let $l = \max_{j=2,\dots,n} l_j$. Then

$$\exists C :: \sum_{a \in S} \sigma_{2n}(P_a) \le C h d^2 R^{2(n-2)} l^\alpha r^\alpha = C h^{-1} \sigma_{2n}(\mathcal{D}) l^\alpha r^\alpha.$$

Proof. Denote by C the canonical basis of $\mathbb{C}^n$ with the z_j as coordinates.

First set $C(b,\ lr)$ a polycube, *i.e.* a polydisc with all its radii are equal, with respect to the canonical basis C in $\mathcal{D}$, centered at b and of radii $(lr, \ldots,\ lr)$. Any polydisc P_a with a in $C(b, lr)$ is contained in the "double" polycube $C(b, 2(2)^n lr)$, the 2^n because of the "angle" between the two bases; hence the measure of the union of all those polydiscs P_a is bounded by the measure of $C(b,\ 2^{n+1}lr)$. These polydiscs being disjoint we get

$$\sum_{a\in S\cap C(b,\ lr)} \sigma_{2n}(P_a) \le \sigma_{2n}(C(b,\ 2^{n+1}lr)) = 2^{2n+1}\pi^n l^{2n} r^{2n}.$$

Each polydisc verifies $\sigma_{2n}(P_a) = \pi^n l_2^2 \ldots l_n^2 r^{2n}$, hence the number of points N_C of S in $C(b,\ lr)$ can be estimated by:

$$N_C \le 2^{2n+1}\pi^n l^{2n} r^{2n} / \pi^n l_2^2 \cdots l_n^2 r^{2n} = 2^{2n+1}\frac{l^{2n}}{l_2^2 \cdots l_n^2}.$$

Let $b' = (b_1, \ldots, b_{n-1})$ be fixed, then the set $C((b',\ b_n),\ lr) \cap \{z' = b'\} \subset D(h)$ is a disc centered at $b_n \in D(h)$ and of radius lr. The homogeneous Minkowski assumption gives that there is a subfamily of these discs which covers S whose number n_B of elements verifies

$$n_B \le Ch(lr)^{\alpha-2}.$$

Define the slice of depth lr to be $B(b', lr) := \bigcup_{b_n\in D(h)} C((b', b_n), lr)$; then the number N_B of points of S in this slice verifies

$$N_B \le n_B \times N_C \le Ch(lr)^{\alpha-2} \times 2^{2n+1}\frac{l^{2n}}{l_2^2 \ldots l_n^2}.$$

The number of such slices, when b' varies, is bounded by $\frac{d^2 R^{2(n-2)}}{(lr)^{2(n-1)}}$, hence the total number N of points in S can be estimated by:

$$N \le \frac{N_B d^2 R^{2(n-2)}}{l^{2(n-1)} r^{2(n-1)}} \le 2^{2n+1} d^2 R^{2(n-2)} Chl^\alpha r^\alpha \frac{1}{l_2^2 \cdots l_n^2 r^{2n}}.$$

So the total measure of the polydiscs P_a is

$$\begin{aligned} A := \sum_{a\in S}\sigma_{2n}(P_a) = N \times \pi^n l_{2^2} \cdots l_{n^2} r^{2n} &\le 2^{2n+1}\pi^n d^2 R^{2(n-2)} Chl^\alpha r^\alpha \\ &= C' h^{-1}\sigma_{2n}(\mathcal{D}) l^\alpha r^\alpha, \end{aligned}$$

with $C' = 2^{2n+1}\pi^n C$ which depends only on C, the Minkowski constant of W. $\square$

4.2. Almost strictly pseudo-convex domains

Let W be the set of weakly pseudo-convex points of $\partial\Omega$, *i.e.* W is the zero set of the determinant of the Levi form $\mathcal{L}$ of $\partial\Omega$. Let π be the normal projection from Ω onto $\partial\Omega$, defined in a neighbourhood $\mathcal{U}$ of $\partial\Omega$ in Ω.

Let $\alpha \in \partial\Omega$; by linear change of variables we can suppose that $\alpha = 0 \in \partial\Omega \subset \mathbb{C}^n$, $z_1 = 0$ is the equation of the complex tangent space. The projection π locally near $0 \in \partial\Omega$ gives a $\mathcal{C}^\infty$ diffeomorphism $\tilde{\pi}\ \ \partial\Omega \to T_0(\partial\Omega)$, $\tilde{\pi} := (\pi_{|T_0(\partial\Omega)})^{-1}$.

Definition 4.5. The pseudo-convex domain Ω in $\mathbb{C}^n$ is said to be *almost stricly pseudo-convex*, **aspc**, at 0 if there is a neighbourhood V_0 of 0, a positive number β, and a basis $b := \{L_1, \dots, L_n\}$ of $\mathbb{C}^n$, still with L_1 a complex normal unit vector, such that the slices in the associated coordinates for the basis b,

$$\tilde{\pi}(W \cap V_0) \cap \{z_1 = 0\} \cap \{z_2 = a_2\} \cap \dots \cap \{z_{n-1} = a_{n-1}\}$$

have homogeneous Minkowki dimension less than $2 - \beta$, $\beta > 0$.

Ω is said to be **aspc** if this is true for all points in $\partial\Omega$ with the same $\beta > 0$ and the same underlying constant.

The basis b is in general different from the basis $b(\alpha)$ used in the definition of the good family $\mathcal{Q}$.

Of course the strictly pseudo-convex domains are **aspc** because $W = \emptyset$.

4.3. Sequences projecting on weak pseudo-convex points

We still shall use the notation:

$\forall a \in \mathcal{U}, \alpha := \pi(a)$, $m(a) := m(\alpha) = (m_1(\alpha), \dots, m_n(\alpha))$ is the multi-type of a point;

W is the set of non strictly pseudo-convex points on $\partial\Omega$;

$\forall a \in \mathcal{U}$, $\mu(a) := \sum_{j=2}^{n} \frac{1}{m_j(a)}$ is the weight exponent.

Theorem 4.6. *Let $\mathcal{Q}$ be a good family of polydiscs on a* **aspc** *domain Ω in $\mathbb{C}^n$, and S be a δ separated sequence of points in Ω. If $\pi(S \cap \mathcal{U}) \subset V \cap W$, where V is an open set of $\partial\Omega$, then we have:*

$$\sum_{a \in S\cap\mathcal{U}} d(a)^{1+2\mu(a)} = \delta^{-2n} \sum_{a \in S\cap\mathcal{U}} \sigma_{2n}(Q_a(\delta)) \leq C(\Omega)\sigma_{2n-1}(V), \tag{4.2}$$

where $C(\Omega)$ depends only on ρ, n, the good family $\mathcal{Q}$ and the constant β in the Minkowski dimension of $W \subset \partial\Omega$.

Proof. The polydisc $Q_a(\delta)$ has radius $\gamma := \delta d(a)$ in the normal direction and in its conjugate and has radii

$$\left(\delta d(a)^{1/m_2(a)}, \dots, \delta d(a)^{1/m_n(a)}\right)$$

in the complex tangent directions. Let us denote $L_2, \dots, L_n$ the complex tangent directions in the basis $b(\alpha)$ associated to $\pi(a)$ with multi-type $(m_2(a), \dots, m_n(a))$.

Now fix $\zeta \in W \subset \partial\Omega$ and let $V_\zeta := B(\zeta, \epsilon) \cap \partial\Omega$ be a neighbourhood of ζ in $\partial\Omega$ such that $\tilde{\pi}$ is a diffeomorphism from V_ζ on a neighbourhood of ζ on the (real) tangent space T_ζ. One can choose the radius of the euclidean ball $B(\zeta, \epsilon)$, $\epsilon > 0$ to be fixed independently of ζ, because $\partial\Omega$ is of class $\mathcal{C}^2$ and compact.

Because Ω is **aspc**, we know that there is a basis $b = \{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ such that $v_1 = L_1$ is still in the complex normal space, and a complex direction in the complex tangent space at ζ, say v_n, along which W is of homogeneous Minkowski dimension $2 - \beta$, $\beta > 0$. *I.e.* these two basis are different in the complex tangent space only.

Let $S \subset \Omega :: \pi(S) \subset W$ be the δ separated given sequence. First we shall prove the theorem with $V = V_\zeta$ and then complete it.

The proof will follow from several reductions.

4.3.1. *Reduction to a layer parallel to the complex tangent space*

As usual we suppose that $\zeta = 0$, $\Re z_1 = 0$ is the tangent space $T_0(\partial\Omega)$.

By use of the $\mathcal{C}^\infty$ diffeomorphism $\tilde{\pi}$, we can suppose that $\partial\Omega \simeq T_0(\partial\Omega)$ in a ball $B(0, \epsilon)$ with a uniform $\epsilon > 0$ which depends only on Ω via its defining function ρ.

Consider the polydisc, *in the basis b*, $P_0(R,h,d) := D(d) \times D(R)^{n-2} \times D(h) \subset B(0, \epsilon)$ where $D(r)$ is a disc centered at 0 and of radius r. We can manage it to have $\epsilon/2\sqrt{n} \leq d \leq h$ and still $P_0(R, h, d) \subset B(0, \epsilon)$.

In this ball $B(0, \epsilon)$ we consider Ω as a half space $T_0(\partial\Omega) \times]0, \epsilon]$ by use of the diffeomorphism $\tilde{\pi}$.

From now on we shall restrict everything to $P_0(R, h, d)$, which means, in particular, that $z \in P_0(R, h, d) \Rightarrow |z_1| \leq d$.

Let $C_\gamma \subset P_0(R, h, d)$ be a layer parallel to $T_0(\partial\Omega)$ at a distance $\gamma \leq d$ from the boundary, *i.e.*

$$a = (a_1, \dots, a_n) \in C_\gamma \iff \Re a_1 \simeq d(a) \in [(1 - \delta)\gamma, (1 + \delta)\gamma],$$

with δ the separating constant.

Now let $S_\gamma := S \cap C_\gamma \cap P_0(R, h, d)$.

4.3.2. *Reduction to a fixed multi-type*

There is only a finite set of possible multi-types for the points of S because we have a good family of polydiscs and the multi-type is uniformly bounded by Lemma 2.2. Hence it is enough to show the inequality (4.2) for the points $a \in S$ with a fixed multi-type, $m(a) = (1, m_2, \dots, m_n)$. Of course the axes of the polydisc $Q_a(\delta)$ are *dependent on a*.

We can apply Lemma 4.4 to the sequence S_γ; because $m_2 \leq \dots \leq m_n$, we set:

$$r := \gamma^{1/m_2},\ l := \gamma^{\frac{1}{m_n} - \frac{1}{m_2}}.$$

The lemma gives:

$$\sum_{a\in S_\gamma}\sigma_{2n}(Q_a)\le ChR^{2(n-2)}d^2l^\beta r^\beta = ChR^{2(n-2)}d^2\gamma^{\beta/m_n}.$$

The measure of the trace of $P_0(R,h,d)$ on the *real* tangent space $T_0(\partial\Omega)$ is

$$\sigma_{2n-1}(P_0(R,h,d)\cap T_0(\partial\Omega)) = R^{2(n-2)}h^2d,$$

because the disc $D(d)$ is in the complex normal.

So we get

$$\begin{aligned}\sum_{a\in S_\gamma}\sigma_{2n}(Q_a) &\le ChR^{2(n-2)}d^2\gamma^{\beta/m_n}\\ &= C\frac{d}{h}\gamma^{\beta/m_n}\sigma_{2n-1}(P_0(R,h,d)\cap T_0(\partial\Omega)).\end{aligned}\tag{4.3}$$

4.3.3. *Adding the layers*

Because the sequence is separated, the layers can be ordered this way $\gamma_k=\nu^k\gamma_0,\ k\in\mathbb{N}$, where $\gamma_0\le d$ is the farthest point from the boundary and $\nu=\frac{1-\delta}{1+\delta}<1$.

We have to add them and, because of inequality (4.3), we get

$$\sum_{k\in\mathbb{N}}\sum_{a\in S_{\gamma_k}}\sigma_{2n}(Q_a)\le C\frac{d}{h}\sigma_{2n-1}(P_0(R,h,d)\cap T_0(\partial\Omega))\sum_{k\in\mathbb{N}}\gamma_k^{\beta/m_n}.$$

But $\gamma_k=\nu^k\gamma_0,\ k\in\mathbb{N}$, so

$$\sum_{k\in\mathbb{N}}\gamma_k^{\beta/m_n}=\gamma_0^{\beta/m_n}\sum_{k\in\mathbb{N}}\nu^{k\beta/m_n}=\frac{\gamma_0^{\beta/m_n}}{1-\nu^{\beta/m_n}}\le\frac{d^{\beta/m_n}}{1-\nu^{\beta/m_n}}.$$

Hence we get

$$\sum_{k\in\mathbb{N}}\sum_{a\in S_{\gamma_k}}\sigma_{2n}(Q_a)\le C'\frac{d^{1+\beta/m_n}}{h}\sigma_{2n-1}(P_0(R,h,d)\cap T_0(\partial\Omega)),\tag{4.4}$$

with $C':=C\frac{1}{1-\nu^{\beta/m_n}}$.

4.3.4. *Adding for all the multi-types*

Because we have a good family of polydiscs the multi-type is bounded, hence $\forall a\in\Omega,\ m_n(a)\le M(\mathcal{Q})$, so we have that, for any multi-type,

$$C':=C\frac{1}{1-\nu^{\beta/m_n}}\le C\frac{1}{1-\nu^{\beta/M(\mathcal{Q})}}=:D,$$

hence the inequality (4.4) implies

$$\sum_{k\in\mathbb{N}}\sum_{a\in S_{\gamma_k}}\sigma_{2n}(Q_a)\leq D\frac{d^{1+\beta/m_n}}{h}\sigma_{2n-1}(P_0(R,h,d)\cap T_0(\partial\Omega)). \tag{4.5}$$

Since $d\leq h$ we have $\frac{d^{1+\beta/m_n}}{h}\leq d^{\beta/m_n}$.

Recall that $\sigma_{2n-1}(Q_a(\delta))=\delta^{-2n}d(a)^{1+2\mu(a)}$, then we get

$$\sum_{k\in\mathbb{N}}\sum_{a\in S_{\gamma_k}}d(a)^{1+2\mu(a)}\leq D2^n\delta^{-2n}d^{\beta/m_n}\sigma_{2n-1}\left(P_0(R,h,d)\cap T_0(\partial\Omega)\right).$$

Now set $V_0:=P_0(R,h,d)\cap T_0(\partial\Omega),\ d\leq h$; the number of possible multi-types being finite, we have a finite sum of finite numbers so $\sum_{a\in S}d(a)^{1+2\mu(a)}$ is finite, for $S\cap\mathcal{U}\cap\{d(a)\leq d\}\subset\pi^{-1}(V_0)$, with constant $C(\Omega)\sigma_{2n-1}(V_0)$, where $C(\Omega)$ depends only on the defining function ρ of Ω, the Minkowski constants of W and of the good family $\mathcal{Q}$.

Now let V be an open set in $\partial\Omega$; because $\partial\Omega$ is a bounded smooth manifold in $\mathbb{R}^{2n}$ we can cover it by a finite number of sets $\{V_\zeta\}_{\zeta\in\mathcal{R}}$ "almost" disjoint, *i.e.* such that

- the union $\bigcup_{\zeta\in\mathcal{R}}V_\zeta$ covers $\partial\Omega$;
- any point of $\partial\Omega$ belongs to at most N of the V_ζ.

This gives

$$V\subset\bigcup_{\zeta\in\mathcal{R}}V_\zeta\cap V.$$

Hence

$$\sigma_{2n-1}(V)\leq\sum_{\zeta\in\mathcal{R}}\sigma_{2n-1}(V_\zeta\cap V).$$

On the other hand we just proved, shrinking $\mathcal{U}$ to $\mathcal{U}\cap\{d(a)\leq d\}$ if necessary,

$$\sum_{a\in S\cap\mathcal{U}\cap\pi^{-1}(V_\zeta\cap V)}d(a)^{1+2\mu(a)}\leq C(\Omega)\sigma_{2n-1}(V_\zeta),$$

so

$$\begin{aligned}\sum_{a\in S\cap\mathcal{U}\cap\pi^{-1}(V)}d(a)^{1+2\mu(a)}&\leq\sum_{\zeta\in\mathcal{R}}\sum_{a\in S\cap\pi^{-1}(V_\zeta\cap V)}d(a)^{1+2\mu(a)}\\&\leq C(\Omega)\sum_{\zeta\in\mathcal{R}}\sigma_{2n-1}(V_\zeta)\\&\leq CN\times\sigma_{2n-1}(V),\end{aligned}$$

the last inequality because any point of V is covered at most N times. $\square$

Remark 4.7. In fact this theorem says that the measure

$$\mu_S := \sum_{a\in S\cap\mathcal{U}} d(a)^{1+2\mu(a)}\delta_a$$

associated to the a separated sequence S of points projecting on the weakly pseudo-convex points in $\partial\Omega$ is a geometric Carleson measure, as we shall see later.

4.4. Sequence of points in a Blaschke divisor

We shall glue the previous result with the one we got in Theorem 3.12 to have the control of the canonical measure ν_S associated to a separated sequence S.

Theorem 4.8. *Let Ω be a* **aspc** *domain in $\mathbb{C}^n$ equipped with a good family $\mathcal{Q}$ of polydiscs and which is $\mathcal{Q}$ quasi convex. Let S a δ separated sequence of points contained in a divisor X of the Blaschke class of Ω, with Θ as its current of integration, which projects on the open set $\mathcal{V}\subset\partial\Omega$. Then we have*

$$\sum_{a\in S} d(a)^{1+2\mu(a)} \leq \gamma(\Omega)\|\Theta\|_B + C(\Omega)\sigma_{2n-1}(\mathcal{V}) < \infty,$$

where $d(a)$ is the distance from a to the boundary of Ω and $\mu(a) := \sum_{j=2}^{n}\frac{1}{m_j(a)}$, with $(1,\ m_2(a),\dots,m_n(a))$ is the multi-type associated to the family $\mathcal{Q}$.

Moreover the constants $C(\Omega),\ \gamma(\Omega)$, depend only on the $\mathcal{C}^{M(\mathcal{Q})+1}$ norm of the defining function $\rho,\ n,\ \delta$ and δ_0 the parameter of the good family $\mathcal{Q}$, the Minkowski constants of the **aspc** *domain Ω and the constant of quasi convexity.*

Proof. Let B_S be the set of (bad) points in the sequence S, *i.e.* which project on the weakly pseudo-convex points in $\mathcal{V}\subset\partial\Omega$; let G_S be the set of (good) points in the sequence S, *i.e.* which project on the strictly pseudo-convex points in $\mathcal{V}\subset\partial\Omega$; then $S = B_S\cup G_S$ and we have by Theorem 3.12

$$\sum_{a\in G_S} d(a)^n \leq \sum_{a\in S} d(a)^n \lesssim \|\Theta\|_B \Rightarrow \sum_{a\in G_S} d(a)^{1+2\mu(a)} \leq \gamma(\Omega)\|\Theta\|_B,$$

because, for these points we have $m_1 = 1,\ m_2 = 2,\dots,\ m_n = 2$, hence $n = 1+2\mu(a)$. By Theorem 4.6 we have

$$\sum_{a\in B_S} d(a)^{1+2\mu(a)} \leq C(\Omega)\sigma_{2n-1}(\mathcal{V}) < \infty,$$

so adding these two inequalities, we get

$$\sum_{a\in S} d(a)^{1+2\mu(a)} \leq \gamma(\Omega)\|\Theta_X\|_B + C(\Omega)\sigma_{2n-1}(\mathcal{V}) < \infty. \qquad \square$$

5. Examples of almost strongly pseudo-convex domains

An example of **aspc** domain *not* of finite type is the following

$$|z_1|^2 + \exp(1 - |z_2|^{-2}) < 1,$$

because the set W of its weakly pseudo-convex points is the circle $|z_1| = 1$, $z_2 = 0$, hence it has Minkowski dimension 1.

The other examples are mainly based on the following theorem.

Let Ω be a domain in $\mathbb{C}^n$ and $\mathcal{L}$ its Levi form. Set $\mathcal{D} := \det \mathcal{L}$; then the set W of points of weak pseudo-convexity is $W := \{z \in \partial\Omega :: \mathcal{D}(z) = 0\}$.

Theorem 5.1. *Let Ω be a domain in $\mathbb{C}^n$ of finite linear type, and $\mathcal{D}$ the determinant of its Levi form. Suppose that:*

$$\forall\alpha \in \partial\Omega,\ \exists v \in T_\alpha^{\mathbb{C}}(\partial\Omega) :: \exists k \in \mathbb{N},\ \frac{\partial^k\mathcal{D}}{\partial v^k}(\alpha) \neq 0,$$

then Ω is **aspc** *and can be equipped with a family of polydiscs whose multi-type is the given linear multi-type.*

Proof. The fact that there is a good family of polydiscs associated to the linear type is given by Theorem 2.6.

It remains to verify the condition on the smallness of the set W of weakly pseudo-convex points.

Let $\alpha \in \partial\Omega$, we may suppose that $\alpha = 0$ and that the complex normal is the z_1 axis.

Because Ω fullfills the hypothesis of the theorem, there is a $j :: 1 < j \leq n$, a real direction, for instance the y_j axis, with $z_j = x_j + iy_j$, and an integer m, such that, with $\tilde{\mathcal{D}}$ being the restriction of $\mathcal{D}$ to the z_j complex plane via the diffeomorphism π, $\tilde{\mathcal{D}} := \mathcal{D} \circ \pi$

$$\frac{\partial^m\tilde{\mathcal{D}}}{\partial y_j^m}(0) = \frac{\partial^m\mathcal{D}}{\partial y_j^m}(0) \neq 0.$$

The differentiable preparation theorem of Malgrange gives that there is a polynomial with $\mathcal{C}^\infty$ coefficients,

$$P(x_j,\ y_j) = y_j^m + \sum_{k=1}^{m} a_k(x_j)y_j^{m-k}$$

and a $\mathcal{C}^\infty$ function $Q(x_j,\ y_j)$, $Q(0) \neq 0$ such that

$$\tilde{\mathcal{D}}(x_j,\ y_j) = Q(x_j,\ y_j)P(x_j,\ y_j).$$

Hence the zero set of $\tilde{\mathcal{D}}$ is the same as the one of P and we know, by Corollary 4.3, that the homogeneous Minkowski dimension of it is less than of equal to $2 - \frac{1}{m}$.

Because $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are $\mathcal{C}^\infty$ functions, $\frac{\partial^m \tilde{\mathcal{D}}}{\partial y_j^m} \neq 0$ in a neighbourhood of 0 with the same number m, hence we have that the homogeneous Minkowski dimension of $\{\tilde{\mathcal{D}} = 0\}$ is less that of equal to $2 - \frac{1}{m}$ in all the slices parallel to the z_j axis in a neighbourhood of 0, and we are done. $\square$

A natural question, asked by the referee, is: *Is the condition* $\forall \alpha \in \partial\Omega,\ \exists v \in T_\alpha^{\mathbb{C}}(\partial\Omega) :: \exists k \in \mathbb{N},\ \frac{\partial^k \mathcal{D}}{\partial v^k}(\alpha) \neq 0$ *actually necessary ?*

I have no answer to it, but we shall see that for convex domains this condition is a consequence of the linear finite type of Ω.

We shall need the definition.

Definition 5.2. Let f be a function defined on an open set $\mathcal{V} \subset \mathbb{R}^n,\ f \in \mathcal{C}^\infty(\mathcal{V})$; we shall say that f is *flat* at $a \in \mathcal{V}$ if $\forall \alpha \in \mathbb{N}^n,\ \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}(a) = 0$.

5.1. Pseudo-convex domains of finite type in $\mathbb{C}^2$

Lemma 5.3. *Let $f(z)$ be a real valued smooth function of $z \in \mathbb{D}$, the unit disc in $\mathbb{C}$; if Δf is flat at 0 then for any $m \in \mathbb{N}$ there is a harmonic function h in $\mathbb{D}$ such that $f - h = \mathcal{O}(|z|^m)$ at the origin.*

Proof. Take the Taylor expansion of f at 0:

$$f(x + iy) = \sum_{k,l=0}^{m+2} a_{kl} x^k y^l + \mathcal{O}(|z|^{m+3}).$$

We get the expansion of Δf near 0:

$$\Delta f(x + iy) = \sum_{k=2,l=0}^{m} k(k-1) a_{kl} x^{k-2} y^l + \sum_{k=0,l=2}^{m} l(l-1) a_{kl} x^k y^{l-2} + \mathcal{O}\left(|z|^{m+1}\right).$$

Hence

$$\Delta f(x+iy) = \sum_{k,l=0}^{m} \left[(k+1)(k+2) a_{k+2,l} + (l+1)(l+2) a_{k,l+2}\right] x^k y^l + \mathcal{O}\left(|z|^{m+1}\right).$$

But Δf flat at 0 means that $[(k+1)(k+2)a_{k+2,l} + (l+1)(l+2)a_{k,l+2}] = 0$, hence setting

$$h := \sum_{k,l=0}^{m+2} a_{kl} x^k y^l,$$

we have that

$$\Delta h(x+iy)=\sum_{k,l=0}^{m}[(k+1)(k+2)a_{k+2,l}+(l+1)(l+2)a_{k,l+2}]x^k y^l=0$$

because all the coefficients are zero. So we get that h is harmonic and $f-h=\mathcal{O}(|z|^{m+3})$. ☐

Theorem 5.4. *Let Ω be a domain of finite type in $\mathbb{C}^2$; then Ω is* **aspc**.

For the proof of this theorem we shall use the following lemma.

Lemma 5.5. *Let h be a real valued harmonic function in a disc $D(0,R)\subset\mathbb{C}$; then h cannot have isolated zeroes.*

Proof. Suppose that $h(0)=0,\ h\not\equiv 0$, then by the mean formula we have for any $0\le r<R$,

$$0=h(0)=\frac{1}{2\pi}\int_0^{2\pi}h(re^{i\theta})d\theta.$$

But h being real valued on the circle $C(r):=\{|z|=r\},\ h$ cannot be always positive or always negative, hence it must change sign on $C(r)$ so it must be zero at least twice, because h is continuous. This is true for any $0<r<R$, hence the lemma is proved. ☐

Proof of Theorem 5.4. Let $\Omega\subset\mathbb{C}^2$ be defined near the origin by

$$\rho(z)=\Re z_1+f(\Im z_1,z_2).$$

We have that

$$\rho(z)=\Re z_1+f(0,z_2)+(f(\Im z_1,z_2)-f(0,z_2)).$$

Suppose that $\mathcal{D}:=\Delta f(0,z_2)$ is flat at 0; then by Lemma 5.3 for any $m\in\mathbb{N}$ there is $h(z_2)$ harmonic near $z_2=0$ and such that

$$f(0,z_2)=h(z_2)+\mathcal{O}(|z_2|^m).$$

There is a conjugate $\tilde{h}$ to h such that $u:=h+i\tilde{h}$ is holomorphic in z_2 near 0 and $\tilde{h}(0)=0\Rightarrow u(0)=0$. We have $f(\Im z_1,z_2)-f(0,z_2)=\Im z_1\times g(\Im z_1,z_2)$, with g smooth as we seen in Lemma 2.9; hence we have

$$\rho(z)=\Re z_1+h(z_2)+\Im z_1\times g(\Im z_1,z_2)+\mathcal{O}(|z_2|^m).$$

Let $X:=\{z_1=-u(z_2)\}$ be this holomorphic variety. By Lemma 5.5 there is a sequence $Z:=\{w_n\}_{n\in\mathbb{N}}\subset\{z_1=0\}$ such that $\tilde{h}(w_n)=0$ and $w_n\to 0$.

Now take a point $a_n = (a_n^1, w_n) \in X \Rightarrow \Re a_n^1 = -h(w_n),\ \Im a_n^1 = -\tilde{h}(w_n)$. We have that $\Im a_n^1 = -\tilde{h}(w_n) = 0$ hence

$$\rho(a_n) = \Re a_n^1 + h(w_n) + \mathcal{O}(|z_2|^m) = -h(w_n) + h(w_n) + \mathcal{O}(|w_n|^m) = \mathcal{O}(|w_n|^m),$$

because $\Im z_1 \times g(\Im z_1, z_2) = 0$ on a_n.

Hence the distance from $\partial\Omega$ to the holomorphic variety X is $\mathcal{O}(|z_2|^m)$ near 0 along the sequence Z going to 0, so the type of $\partial\Omega$ is bigger than m at 0.

This being true for any $m \in \mathbb{N}$ we have a contradiction with the fact that Ω is of finite type in D'Angelo sense [15].

Hence $\Delta f(0, z_2)$ is not flat at 0 and we can apply directly Theorem 5.1 to get that Ω is **aspc**. □

5.2. Locally diagonalizable domains

In this context, the domains with a locally diagonalizable Levi form where introduced by C. Fefferman, J. Kohn and M. Machedon [12] in order to obtain Hölder estimates for the $\bar{\partial}_b$ operator.

Recall that Ω locally diagonalizable means that there is a neighbourhood $V_\alpha \subset \partial\Omega$ of $\alpha \in \partial\Omega$ and $(L_1, \ldots, L_n)$ a basis of $\mathbb{C}^n$ depending smoothly on $\zeta \in V_\alpha$ and diagonalizing the Levi form $\mathcal{L}$.

We shall need the following lemma.

Lemma 5.6. *Let Ω be a domain locally diagonalizable in $\mathbb{C}^n$ and of finite linear type. Then the determinant of its Levi form is not flat on the complex tangent space of $\partial\Omega$.*

Proof. Let $\alpha \in \partial\Omega$, then there is a neighbourhood V_α of α and $(L_1, \ldots, L_n)$ a basis of $\mathbb{C}^n$ depending smoothly on $z \in V_\alpha$, and diagonalizing the Levi form $\mathcal{L}$, with L_1 the complex normal direction, so we have, restricting $\mathcal{L}$ to the complex tangent space:

$$\mathcal{L}(z) = \begin{pmatrix} \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Hence $\mathcal{D} := \det \mathcal{L} = \lambda_2 \cdots \lambda_n$. Now suppose that, for any complex direction $L_j,\ j = 2, \ldots, n$, at α, there is a real direction $v_j,\ v_j \in L_j$, such that $\exists k = k_j \in \mathbb{N},\ \frac{\partial^k \lambda_j}{\partial v_j^k}(\alpha) \neq 0$, then with $k := (k_2, \ldots, k_n)$:

$$\frac{\partial^{|k|}\mathcal{D}}{\partial v_2^{k_2} \cdots \partial v_n^{k_n}}(\alpha) \neq 0,$$

and $\mathcal{D}$ is not flat at α. Hence if $\mathcal{D}$ is flat at α, we must have

$$\exists L_j,\ \forall v_j \in L_j,\ \forall k \in \mathbb{N},\ \frac{\partial^k \lambda_j}{\partial v_j^k}(\alpha) = 0.$$

Now this j is fixed and we slice Ω

$$\Omega_j := \{z_2 = \ldots = z_{j-1} = z_{j+1} = \ldots = 0\} \cap \Omega.$$

We are exactly in the situation of a domain in $\mathbb{C}^2$ and we can use the proof of Theorem 5.4 to get a contradiction with the fact that ρ has a finite order of contact with a real direction in L_j because Ω is of finite linear type. $\square$

Hence we proved

Theorem 5.7. *Let Ω be a domain locally diagonalizable in $\mathbb{C}^n$ and of finite linear type. Then Ω is* **aspc**.

5.3. Convex domains

Theorem 5.8. *Let Ω be convex in a neighborhood of $0 \in \mathbb{R}^{n+1}$. Suppose that the tangent space at 0 is $x_{n+1} = 0$ and $\partial\Omega = \{x_{n+1} = f(x_1, \ldots, x_n)\}$, with f convex. If the determinant of the Hessian of f is flat at 0 then f is flat in a direction $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ of the tangent space at 0.*

Proof. If f is not flat in any direction, we can find $\alpha > 0$ and $m \in \mathbb{N}$ such that $f(x) \geq \alpha\, |x|^{2m}$ in a ball $B(0, R) \subset \mathbb{R}^n$. Let us consider the functions

$$h(x) := \frac{\alpha}{2}\, |x|^{2m},\ g(x) := \frac{\alpha}{2}\, |x|^{2m} + \epsilon\, |x|^2 + \delta,$$

with $\epsilon > 0$ and $\delta > 0$. Denote H_f the Hessian of the function f.

Because $\det H_f$ is flat at 0, there is a ball $B(0, r) \subset \mathbb{R}^n$ such that:

$$\forall x \in B(0, r),\ \det H_f(x) \leq \det H_h(x)$$
$$\forall \epsilon > 0,\ \forall \delta > 0,\ \det H_h < \det H_g. \tag{5.1}$$

We choose ϵ and δ so small that there is a real t such that

$$r > t > \left(2\frac{\delta + \epsilon r^2}{\alpha}\right)^{1/2m}$$

then

$$\frac{\alpha}{2} t^{2m} > \epsilon t^2 + \delta \Rightarrow \alpha t^{2m} > \frac{\alpha}{2} t^{2m} + \epsilon t^2 + \delta,$$

hence,

$$\forall x :: |x| = t,\ f(x) \geq \alpha\, |x|^{2m} > g(x). \tag{5.2}$$

On the other hand, because $g(0) = \delta > f(0) = 0$, and f and g are continuous, we get

$$\exists s > 0,\ s < t :: \forall x,\ |x| < s,\ f(x) < g(x).$$

The maximum principle for the Monge-Ampère operator says [6]:

Lemma 5.9. *Let v be a convex function (*i.e. $H_v \geq 0$ *) defined in a bounded open set V and a regular function ρ such that*

$$\det H_v(x) > \det H_\rho(x),\ v \leq \rho$$

on ∂V. Then $v \leq \rho$ on V.

Because $\det H_g > \det H_f$ in $B(0, r)$ by (5.1) and $g < f$ on $\partial B(0, t)$ by (5.2), we can apply this principle, *i.e.* $g \leq f$ everywhere in $B(0,\ t)$ which is a contradiction in the ball $B(0, s)$. Hence f has to be flat in some direction. □

Corollary 5.10. *Let Ω be a convex domain in a neighbourhood of $0 \in \partial\Omega \subset \mathbb{R}^n$. If $\partial\Omega$ is flat in no direction of its tangent space at 0, then the determinant of the Hessian of Ω is not flat at 0.*

Proof. If not we have a contradiction with Theorem 5.8. □

Let us see now the case of a convex domain of finite type in $\mathbb{C}^n$. We shall need the following lemma.

Lemma 5.11. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$ then for any complex line L in the tangent complex space at $0 \in \partial\Omega$ there is at most one real direction v in L such that $\partial\Omega$ is flat in this direction at 0.*

Proof. We can choose $\rho(z) = \Re z_1 - f(\Im z_1,\ z_2, \dots, z_n)$ as defining function for Ω with f a positive real valued convex function and with the z_n axis L_n as the given L. (The complex normal direction is L_1 as usual.)

Suppose there are two such directions v_1, v_2 in L_n; this means

$$\forall k \in \mathbb{N},\ \frac{\partial^k \rho}{\partial v_j^k}(0) = 0,\ j = 1, 2.$$

The vector v_1 can be seen as a point a_1 in the complex plane $P_n = \{z_1 = z_2 = \cdots = z_{n-1} = 0\}$ and also v_2 corresponds to the point $a_2 \in P_n$. Let $t \in [0, 1],\ a_t := ta_1 + (1-t)a_2 \in P_n$, because f is convex this implies that $0 \leq f(a_t) \leq tf(a_1) + (1-t)f(a_2)$ and this means that the order of contact in the direction $v = tv_1 + (1-t)v_2$ is bigger than the minimum of the order of contact in the directions v_1 and v_2, hence

$$\forall k \in \mathbb{N},\ \frac{\partial^k \rho}{\partial v^k}(0) = 0,\ j = 1, 2,$$

with $v = tv_1 + (1-t)v_2$. This being true for any $t \in [0, 1]$ we have that f is flat in the sector of P_n between v_1 and v_2, but f being $\mathcal{C}^\infty$ this implies that f is flat at 0.

By a result of Boas and Straube [9] we have that for a convex domain the multi-type or the order of contact with real lines is the same; the multi-type of $\partial\Omega$ being finite, this means that there is a real direction in L which is not flat, hence a contradiction which gives the lemma. □

Corollary 5.12. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$ near $0 \in \partial\Omega$. There is a complex line L in the tangent complex space at 0 and a real vector $v \in \mathbb{C}L$, such that the determinant of the Levi form of a defining function for Ω near 0 is not flat in the direction v.*

Proof. Let $L_2, \dots, L_n$ be an orthonormal basis of $T_0^{\mathbb{C}}(\partial\Omega)$. Because Ω is of finite type, we know by Lemma 5.11 that in any complex direction L_j, $2 \le j \le n$, there is at most one real direction in which $\partial\Omega$ is flat; we can always take that direction to be the y_j axis without changing the ambiant complex structure. If such a direction does not exist we still take the y_j axis in the following.

We set E to be the subspace $E := \{y_2 = \cdots = y_n = 0\} \cap \{z_1 = 0\}$.

We write the defining function as usual

$$\rho(z) = \Re z_1 - f(\Im z_1, z_2, \dots, z_n),$$

hence the domain $\Omega \cap E$ has defining function

$$\tilde{\rho}(x) := -f(0, x_2, \dots, x_n).$$

Let $\mathcal{L}(z_1, \dots, z_n) := \partial\bar{\partial}\rho(z)$ be the Levi form of Ω, we have

$$\partial\bar{\partial} f(x, 0) = -\mathcal{L}(x,\ 0) = \left\{ \frac{\partial^2 f}{\partial x_j \partial x_k}(x,\ 0) \right\}_{j,k=2,\dots,n} = H_{\tilde{f}}(x) \tag{5.3}$$

with $\tilde{f}(x_2, \dots, x_n) := f(0, x)$, and the new convex set $\Omega_1 := \Omega \cap E$ still verifies the conditions of Corollary 5.10: $\tilde{\mathcal{D}}(x) := \det H_{\tilde{f}}(x)$ is not flat because we get rid of the flat directions. Hence there is a real vector v in the tangent space at 0 for $\partial\Omega_1$ such that $\tilde{\mathcal{D}}$ is not flat in the direction v. This means

$$\exists k \in \mathbb{N} :: \frac{\partial^k \tilde{\mathcal{D}}}{\partial v^k}(0) \neq 0;$$

but, using (5.3), we get

$$\frac{\partial^k \mathcal{D}}{\partial v^k}(0) = \frac{\partial^k \tilde{\mathcal{D}}}{\partial v^k}(0) \neq 0. \qquad \square$$

Theorem 5.13. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$; then Ω is* **aspc**.

Proof. By use of Corollary 5.12, it remains to apply Theorem 5.1. $\square$

5.4. Domains with real analytic boundary

Lemma 5.14. *Let Ω be a bounded domain with real analytic boundary. Then Ω is of finite linear type.*

Proof. Take a point $\alpha \in \partial\Omega$ and suppose that a real line through α has a contact of infinite order with $\partial\Omega$, then, using Lojasiewicz [35] we get that the line, which is real analytic, and $\partial\Omega$ are regularly situated, hence the line must be contained in $\partial\Omega$. But this cannot happen because $\partial\Omega$ is bounded. □

In fact we have a better result because we know, by the work of K. Diederich and J. E. Fornaess [16], that Ω is of finite type.

The function $\mathcal{D} = \det \mathcal{L}$ is also real analytic, hence if $\mathcal{D}$ is flat at a point $\alpha \in \partial\Omega$. This means in particular that $\forall v \in T_\alpha(\partial\Omega)$, $\forall k \in \mathbb{N}$, $\frac{\partial^k \mathcal{D}}{\partial v^k}(\alpha) = 0$, hence $\mathcal{D}$ is identically zero on $\partial\Omega$. This says that all the points of $\partial\Omega$ are non stricly pseudo-convex points. But this is impossible because $\partial\Omega$ is compact, hence contains at least a strictly pseudo-convex point, because of the following simple and well known lemma [23]:

Lemma 5.15. *Let Ω be a bounded domain in $\mathbb{R}^n$, with a smooth boundary of class $\mathcal{C}^3$. Then $\partial\Omega$ contains a point of strict convexity.*

Now let $\alpha \in \partial\Omega$ and suppose that $\mathcal{D}$ is flat in all the complex tangent directions of $T_\alpha^{\mathbb{C}}(\partial\Omega)$. Then, because $\partial\Omega$ is of finite type, we can recover the derivatives in the “missing direction”, namely the real direction conjugate to the normal one, by brackets of derivatives in the complex tangent directions.

Hence we have that $\mathcal{D}$ is also flat in the direction conjugate to the normal one, but this would imply that $\mathcal{D}$ is flat at the point α, and this is forbidden by the lemma. So we can apply Theorem 5.1 to conclude:

Theorem 5.16. *Let Ω be a domain in $\mathbb{C}^n$ with real analytic boundary, then Ω is* **aspc** *and of finite linear type.*

6. Convex domains of finite type

McNeal [26], introduced tools for studying the geometry of convex domains of finite type: a family of polydiscs and a related pseudo-distance which are well suited to these domains.

These tools were used and a little bit modified by different authors: McNeal and Stein [29], J. Bruna, P. Charpentier and Y. Dupain [10], K. Diederich and E. Mazzilli [17], A. Cumenge [14] and also T. Hefer [20], among others.

We start first with notation and definitions taken from Hefer [20] in order for the reader to follow easily the citations we use. This means that the polydiscs in the family seem different but we shall show, in Section 7, that they are the same than the ones defined in Section 2.

Let r be a defining function for Ω, $\Omega := \{z \in \mathbb{C}^n :: r(z) < 0\}$, where Ω is a convex domain of finite type.

Hefer uses the ϵ distance in the direction v:

$$\tau(\zeta, v, \epsilon) := \sup\{c \ \ |r(\zeta + \lambda v) - r(\zeta)| \leq \epsilon, \ \forall \lambda \in \mathbb{C} :: |\lambda| \leq c\}, \tag{6.1}$$

and builds two ϵ extremal bases (introduced in [10]), a variant of the original one of McNeal [26], which are equivalent, from which we keep one:

$$b_\epsilon(\zeta) = (v_1(\zeta, \epsilon), \dots, v_n(\zeta, \epsilon)),$$

and the ϵ distance in the direction v_k:

$$\forall k = 1, \dots, n, \ \tau_k(\zeta, \epsilon) := \tau(\zeta, v_k, \epsilon).$$

This allows him to define a family of polydiscs

$$\forall t > 0, \ tP_\epsilon(\zeta) := \left\{ z = \zeta + \sum_{k=1}^{n} w_k v_k(\zeta, \epsilon) \in \mathbb{C}^n :: \forall k = 1, \dots, n, |w_k| < t\tau_k(\zeta, \epsilon) \right\}, \tag{6.2}$$

and the **pseudo-distance** $d(z, \zeta) := \inf\{\epsilon :: z \in P_\epsilon(\zeta)\}$ associated to it. See the nice introduction in [20] to see why this definition is relevant.

From his theorem of [20, 1.7] I just keep the “geometrical” part.

Theorem 6.1. *Let $\Omega \subset \mathbb{C}^n$ be a smooth convex domain of finite type and let $(m_1, \dots, m_n)$ be its multi-type.*

If $\mathcal{U}$ is a sufficiently small compact neighborhood of $\partial\Omega$, if $\zeta \in \mathcal{U}$ and if $(m_1(\zeta), \dots, m_n(\zeta))$ is the multi-type of $\partial\Omega_\zeta := \{z \in \mathbb{C}^n :: r(z) = r(\zeta)\}$ at the point ζ, then there are constants $c, C > 0$ depending only on $\mathcal{U}$ (and on the fixed defining function r of Ω) such that

$$c\epsilon^{1/m_j(\zeta)} \leq \tau_j(\zeta, \epsilon) \leq C\epsilon^{1/m_j(\zeta)}. \tag{6.3}$$

Hence we have a family of polydiscs

$$\mathcal{P} := \{P_\epsilon(\zeta)\}_{\zeta \in \mathcal{U}, \epsilon > 0} \tag{6.4}$$

which is equivalent to the family used by McNeal and Stein [29].

We shall extract from [20, proposition 2.7] the following facts we shall need later.

$\forall t > 0, \ \exists c_t, \ \exists C_t$ depending only on t such that

$$\forall \zeta \in \mathcal{U}, \ P_{c_t\epsilon}(\zeta) \subset tP_\epsilon(\zeta) \subset P_{C_t\epsilon}(\zeta). \tag{6.5}$$

There are constants $C_1 > 1, \ c_2 < 1$ and $c_3 > 0$, independant of ζ and ϵ, such that

$$\begin{aligned} &\forall \zeta \in \mathcal{U}, \forall \epsilon > 0, \ \frac{1}{2} P_\epsilon(\zeta) \subset C_1 P_{\epsilon/2}(\zeta); \\ &\forall \epsilon > 0, \ \forall t < c_2\epsilon, \ \forall \zeta, \ C_1 P_t(\zeta) \subset P_\epsilon(\zeta); \\ &\forall \zeta \in \Omega, \ c_3 P_{|r(\zeta)|}(\zeta) \subset \Omega. \end{aligned} \tag{6.6}$$

There is a constant C_3 independent of $z,\ \zeta \in \mathcal{U}$, and independent of $s > 0$ such that

$$P_s(z) \cap P_s(\zeta) \neq \emptyset \Rightarrow P_s(z) \subset C_3 P_s(\zeta). \tag{6.7}$$

This implies, with $\sigma_{2n}(P)$ the euclidean volume of P,

$$\frac{1}{C_3^{2n}} \sigma_{2n}(P_s(\zeta)) \leq \sigma_{2n}(P_s(z)) \leq C_3^{2n} \sigma_{2n}(P_s(\zeta)).$$

But $\sigma_{2n}(P_s(\zeta)) = \tau_1(\zeta, s)^2 \prod_{j=2}^n \tau_j(\zeta,\ s)^2$ and $\tau_1(\zeta, s) \simeq s$, so we have

$$P_s(z) \cap P_s(\zeta) \neq \emptyset \Rightarrow \prod_{j=2}^{n} \tau_j(\zeta,\ s)^2 \simeq \prod_{j=2}^{n} \tau_j(z,\ s)^2. \tag{6.8}$$

If $\pi(z)$ is the projection of z to $\partial\Omega$, then we have the estimate

$$d(z,\ \pi(z)) \simeq |r(z)|;\ z \in P_\epsilon(\zeta) \Rightarrow d(z,\ \zeta) \leq \epsilon;\ z \notin P_\epsilon(\zeta) \Rightarrow d(z,\ \zeta) \gtrsim \epsilon;$$

and

$$d(z,\ \zeta) \leq \epsilon \Rightarrow z \in P_t(\zeta)$$

for all $t \gtrsim \epsilon$ and $d(z,\ \zeta) \geq \epsilon \Rightarrow z \notin P_t(\zeta)$ for all $t \lesssim \epsilon$.

6.1. Szegö and Poisson-Szegö kernels

We shall continue with notions introduced by McNeal and Stein [29]; we modify slightly the previous notation: $\forall x, y \in \partial\Omega,\ \rho(x, y) := d(x, y)$ is the pseudo-distance which, proved by McNeal [25], gives a structure of space of homogeneous type to $\partial\Omega$.

The "distance" in $\bar{\Omega},\ \rho^*(z, w)$ is defined by:

$$\rho^*(z, w) := |r(z)| + |r(w)| + \rho(\pi(z),\ \pi(w)),$$

where π is the normal projection on the boundary $\partial\Omega$ of Ω, well defined in $\mathcal{U}$, (shrinking $\mathcal{U}$ if necessary).

We have, still following McNeal and Stein [29]:

- the pseudo-balls on $\partial\Omega$ are defined by

$$\forall \alpha \in \partial\Omega,\ B(\alpha,\ \epsilon) := P_\epsilon(\alpha) \cap \partial\Omega;$$

- the "tents" are defined in $\mathcal{U} \cap \bar{\Omega}$, where $\mathcal{U}$ is a sufficiently small compact neighborhood of $\partial\Omega$ defined in Theorem 6.1, by $\forall a \in \mathcal{U} \cap \bar{\Omega},\ T_a(\epsilon) = P_\epsilon(a) \cap \bar{\Omega}$.

We shall also need this notation:

$$\forall z \in \Omega,\ \forall w \in \bar{\Omega},\ T(z, w)$$

is the smallest "tent" containing the two points z and w. The ϵ underlying this tent is equivalent to $\rho^*(z, w)$ as done in [29].

Let $S(z, w)$ be the Szegö kernel of Ω, *i.e.* the kernel associated to the orthogonal projection from $L^2(\partial\Omega)$ onto the Hardy space $H^2(\Omega)$.

We have ([29, page 521]):

$$\forall (z, w) \in \Omega\times\Omega\backslash\Delta,\ |S(z, w)| \lesssim \frac{\delta}{\sigma_{2n}(T(z, w))},\ \delta := \rho^*(z, w).$$

Keeping $z \in \Omega$ and pushing w to $\partial\Omega$, we still have

$$\forall (z, y) \in \Omega\times\partial\Omega,\ |S(z, y)| \lesssim \frac{\delta}{\sigma_{2n}(T(z, y))},\ \delta := \rho^*(z, y). \tag{6.9}$$

We also have the following estimates ([29, page 525])

$$\begin{aligned} \sigma_{2n-1}(B(x,\ \epsilon)) &\simeq \epsilon \prod_{j=2}^{n} \tau_j(x,\ \epsilon)^2; \\ \sigma_{2n}(T_z(\ \epsilon)) &\simeq \epsilon^2 \prod_{j=2}^{n} \tau_j(x,\ \epsilon)^2 \simeq \epsilon\sigma_{2n-1}(B(x, \epsilon)). \end{aligned} \tag{6.10}$$

We have, by its very definition (6.1), that $\forall k \in \mathbb{N},\ \tau_j(x,\ 2^k\epsilon) \geq \tau_j(x,\ \epsilon)$, hence using (6.10)

$$\forall k \in \mathbb{N},\ \sigma_{2n-1}(B(x,\ 2^k\epsilon)) \gtrsim 2^k\sigma_{2n-1}(B(x,\ \epsilon)). \tag{6.11}$$

Let $z \in \Omega,\ x = \pi(z) \in \partial\Omega$ be fixed and cover $\partial\Omega$ by annuli

$C_k := B(x, 2^k\delta)\backslash B(x,\ 2^{k-1}\delta),\ k \geq 1$ and $C_0 := B(x, \delta)$ with $\delta = \rho^*(z, z) = 2\,|r(z)|$.

Lemma 6.2. *With $z \in \Omega,\ x = \pi(z) \in \partial\Omega,\ \delta := \rho^*(z, z) = 2\,|r(z)|$, we have:*

$$\begin{aligned} \forall z \in \Omega, \forall y \in \partial\Omega, |S(z, y)| \lesssim\ & \frac{1}{\sigma_{2n-1}(B(x, \delta/2))} \mathbb{1}_{B(x,\delta)}(y) \\ & + \sum_{k\in\mathbb{N}} \frac{1}{\sigma_{2n-1}(B(x,\ 2^k\delta))} \mathbb{1}_{C_k}(y). \end{aligned} \tag{6.12}$$

Proof. This is a well known technique of harmonic analysis (we already used it in [4] for the same goal, for instance).

By inequality (6.9) we get, with $y \in \partial\Omega \cap C_k$, hence $\rho(x,y) \leq 2^k\delta$,

$$|S(z,y)| \lesssim \frac{\rho^*(z,y)}{\sigma_{2n}(T(z,y))} = \frac{|r(z)| + \rho(x,y)}{\sigma_{2n}(T(z,y))} \leq \frac{(1+2^k)\delta}{\sigma_{2n}(T(z,y))}.$$

So

$$|S(z,y)| \lesssim \frac{\delta}{\sigma_{2n}(T(z,y))} \mathbb{1}_{B(x,\delta)}(y) + \sum_{k\geq 1} \frac{(1+2^k)\delta}{\sigma_{2n}(T(z,y))} \mathbb{1}_{C_k}(y).$$

If $y \in C_k,\ k \geq 1$, we have $\sigma_{2n}(T(z,y)) \gtrsim \sigma_{2n}(T_z(\,2^{k-1}\delta))$ and for $y \in B(x,\delta)$ we have

$$\sigma_{2n}(T(z,y)) \gtrsim \sigma_{2n}(T_z(|r(z)|)) = \sigma_{2n}(T_z(\delta/2)),$$

so

$$|S(z,y)| \lesssim \frac{\delta}{\sigma_{2n}(T_z(\delta/2))} \mathbb{1}_{B(x,\delta)}(y) + \sum_{k\geq 1} \frac{(1+2^k)\delta}{\sigma_{2n}(T_z(2^k\delta))} \mathbb{1}_{C_k}(y).$$

Now by the equivalences (6.10) we have $\sigma_{2n}(T_z(h)) \simeq h\sigma_{2n-1}(B(x,h))$, so we get

$$|S(z,y)| \lesssim \frac{\delta}{\delta\sigma_{2n-1}(B(x,\delta/2))} \mathbb{1}_{B(x,\delta)}(y) + \sum_{k\geq 1} \frac{(1+2^k)\delta}{\delta(1+2^k)\sigma_{2n-1}(B(x,N^k\delta))} \mathbb{1}_{C_k}(y),$$

hence

$$|S(z,y)| \lesssim \frac{\mathbb{1}_{B(x,\delta)}(y)}{\sigma_{2n-1}(B(x,\delta/2))} + \sum_{k\geq 1} \frac{\mathbb{1}_{C_k}(y)}{\sigma_{2n-1}(B(x,2^{k-1}\delta))}. \qquad \square$$

Lemma 6.3. *We have, with $z \in \Omega,\ x = \pi(z) \in \partial\Omega,\ \delta := \rho^*(z,z) = 2\,|r(z)|\,$,*

$$\|S(z,\cdot)\|_p \lesssim \frac{1}{\sigma_{2n-1}(B(x,\delta))^{1/p'}},$$

where p' is the conjugate exponent of p.

Proof. Lemma 6.2 gives us

$$|S(z,y)| \lesssim \frac{\mathbb{1}_{B(x,\delta)}(y)}{\sigma_{2n-1}(B(x,\delta/2))} + \sum_{k\geq 1} \frac{\mathbb{1}_{C_k}(y)}{\sigma_{2n-1}(B(x,\ 2^{k-1}\delta))},$$

hence integrating on $\partial\Omega$, we get

$$\|S(z,\cdot)\|_p^p \lesssim \frac{\sigma_{2n-1}(B(x,\delta))}{\sigma_{2n-1}(B(x,\delta/2))^p} + \sum_{k\geq 1} \frac{\sigma_{2n-1}(C_k)}{\sigma_{2n-1}(B(x,\ 2^{k-1}\delta))^p}.$$

From $C_k \subset B(x,\ 2^k\delta)$, we get $\sigma_{2n-1}(C_k) \leq \sigma_{2n-1}(B(x,\ 2^k\delta))$, hence

$$\|S(z,\cdot)\|_p^p \lesssim \frac{\sigma_{2n-1}(B(x,\delta))}{\sigma_{2n-1}(B(x,\delta/2))^p} + \sum_{k\geq 1} \frac{1}{\sigma_{2n-1}(B(x,\ 2^{k-1}\delta))^{p-1}}.$$

Because these pseudo-balls are associated to a space of homogeneous type, there is a constant K such that $\sigma_{2n-1}(B(x,\ 2h)) \leq K\sigma_{2n-1}(B(x,h))$. Using also inequality (6.11) we get, with $t = \delta/2 = |r(z)|$

$$\begin{aligned}\|S(z,\cdot)\|_p^p &\lesssim \frac{1}{\sigma_{2n-1}(B(x,t))^{p-1}} + \sum_{k\geq 1}\frac{1}{2^{(p-1)k}}\frac{1}{\sigma_{2n-1}(B(x,\ t))^{p-1}}\\ &\lesssim \frac{1}{\sigma_{2n-1}(B(x,\ t))^{p-1}}\left(1+\sum_{k\geq 1}\frac{1}{2^{(p-1)k}}\right) \lesssim \frac{1}{\sigma_{2n-1}(B(x,\ t))^{p-1}}.\end{aligned}$$

for $p > 1$, we get the estimate:

$$\|S(z,\cdot)\|_p^p \lesssim \sigma_{2n-1}(B(x,t))^{1-p} \Rightarrow \|S(z,\cdot)\|_p \lesssim \frac{1}{\sigma_{2n-1}(B(x,\delta))^{1/p'}}. \qquad \square$$

Now let $K_\Omega(z,w)$ be the Bergman kernel of Ω, *i.e.* the kernel associated to the orthogonal projection $L^2(\Omega) \to A^2(\Omega)$, where A^2 is the Bergman space of square summable holomorphic functions in Ω.

We have a lower bound ([26, Theorem 3.4]):

$$K_\Omega(a,a) \gtrsim \prod_{j=1}^n \tau_j(a,\ \delta)^{-2} \simeq \frac{1}{\delta\sigma_{2n-1}(B(\alpha,\delta))}, \tag{6.13}$$

here with $\delta = |r(a)|$ and a in a neighbourhood $\mathcal{V}_p$ of the point $p \in \partial\Omega$ and $\alpha = \pi(a)$. We also have an upper bound ([26, Theorem 5.2]:

$$K_\Omega(a,z) \lesssim \prod_{j=1}^n \tau_j(a,\ \delta)^{-2} \simeq \frac{1}{\sigma_{2n}(T(z,a))}, \tag{6.14}$$

always in a neighbourhood of uniform size of $p \in \partial\Omega$, and here with

$$\delta = |r(a)| + |r(z)| + \rho(\pi(a),\ \pi(z)) = \rho^*(a,z).$$

So, with $\alpha \in \partial\Omega$ fixed, $\pi(a) = \alpha$, and $\mathcal{V}$ a neighbourhood of α valid for these two estimates, we have:

Lemma 6.4. *We have, with $\alpha = \pi(a),\ \delta = |r(a)|$,*

$$\|K_\Omega(a,\cdot)\|_{H^p}^p \lesssim \frac{1}{\delta^p\sigma_{2n-1}(B(\alpha,\ \delta))^{p-1}}$$

and

$$\|S(a,\cdot)\|_{H^p(\Omega)} \geq \frac{1}{\sigma_{2n-1}(B(\alpha,\delta))^{1/p'}}. \tag{6.15}$$

Proof. From the inequality (6.14) and using the annuli

$$C_k := B(x,\ 2^k\delta)\backslash B(x,\ 2^{k-1}\delta),\ k \geq 1,\ C_0 = B(x,\delta),$$

we already used in the proof of Lemma 6.2 we get, exactly as before, with $x = \pi(z)$, and $\delta = 2\,|r(z)|$,

$$|K_\Omega(z,w)| \lesssim \frac{\mathbb{1}_{B(x,\delta)}(w)}{\delta\sigma_{2n-1}(B(x,\delta/2))} + \sum_{k\geq 1} \frac{\mathbb{1}_{C_k}(w)}{2^{k-1}\delta\sigma_{2n-1}(B(x,\ 2^{k-1}\delta))}.$$

Hence, with $\alpha = \pi(a)$,

$$\begin{aligned}\int_{\mathcal{V}\cap\{r(z)=-\delta/2\}} |K_\Omega(a,z)|^p\ d\sigma(z) \lesssim{}& \frac{\sigma_{2n-1}(B(\alpha,\delta))}{\delta^p\sigma_{2n-1}(B(\alpha,\delta/2))^p} \\ &+ \sum_{k\geq 1} \frac{\sigma_{2n-1}(C_k)}{2^{p(k-1)}\delta^p\sigma_{2n-1}(B(\alpha,\ 2^k\delta)^p}.\end{aligned}$$

Hence, again as before,

$$\int_{\mathcal{V}\cap\{r(z)=-\delta/2\}} |K_\Omega(a,z)|^p\ d\sigma_{2n-1}(z) \leq \frac{1}{\delta^p\sigma_{2n-1}(B(\alpha,\delta))^{p-1}}.$$

Ouside of $\mathcal{V}$, $K_\Omega(a,\ \cdot)$ is bounded because by [28, page 178]:

$$|K_\Omega(a,\ z)| \lesssim \frac{1}{\sigma_{2n}(T(a,z))},$$

and if $z \notin \mathcal{V}$ then $1 \lesssim \sigma_{2n}(T(a,z))$ uniformly in $a \in \Omega$.

Hence

$$\begin{aligned}\|K_\Omega(a,\cdot)\|^p_{H^p} ={}& \int_{\mathcal{U}\cap\{r(z)=-\delta/2\}} |K_\Omega(a,z)|^p\ d\sigma(z) \\ &+ \int_{(\partial\Omega\backslash\mathcal{U})\cap\{r(z)=-\delta/2\}} |K_\Omega(a,z)|^p\ d\sigma(z) \\ \lesssim{}& \frac{1}{\delta^p\sigma_{2n-1}(B(\alpha,\ \delta))^{p-1}} + c \lesssim \frac{1}{\delta^p\sigma_{2n-1}(B(\alpha,\ \delta))^{p-1}},\end{aligned}$$

because c is uniformly bounded, hence

$$\|K_\Omega(a,\cdot)\|^p_{H^p} \lesssim \frac{1}{\delta^p\sigma_{2n-1}(B(\alpha,\ \delta))^{p-1}},$$

which proves the first part of the lemma.

Notice that even if K_Ω is linked to Bergman space, we have an estimate of its *Hardy* $H^p(\Omega)$ norm.

Using the lower bound (6.13) and the previous inequality, we get

$$\frac{K_\Omega(a,a)}{\|K_\Omega(a,\cdot)\|_{H^{p'}(\Omega)}} \geq \frac{1}{\delta\sigma_{2n-1}(B(\alpha,\delta))} \times \delta\sigma_{2n-1}(B(\alpha,\delta))^{1/p} \geq \frac{1}{\sigma_{2n-1}(B(\alpha,\delta))^{1/p'}}.$$

Hence, because

$$\|S(a,\ \cdot)\|_{H^p} = \sup\{|f(a)| = |\langle f,\ S(a,\ \cdot)\rangle| :: f \in H^{p'}(\Omega),\ \|f\|_{p'} = 1\}$$

we get

$$\|S(a,\cdot)\|_{H^p(\Omega)} \geq \frac{1}{\sigma_{2n-1}(B(\alpha,\delta))^{1/p'}} \tag{6.16}$$

by the choice of $f(z) := \frac{K_\Omega(a,z)}{\|K_\Omega(a,\cdot)\|_{H^{p'}}}$. □

Recall that the Poisson-Szegö kernel is

$$\forall z \in \Omega,\ y \in \partial\Omega,\ P(z,y) := \frac{|S(z,y)|^2}{\|S(z,\ \cdot)\|_{H^2}^2} = \frac{|S(z,y)|^2}{S(z,z)}.$$

We have that this kernel reproduces the holomorphic functions:

$$\forall f \in A(\Omega),\ \int_{\partial\Omega} f(y)P(z,y)\,d\sigma(y) = \frac{1}{S(z,z)}\langle fS(z,\ \cdot),\ S(z,\ \cdot)\rangle = f(z),$$

because of the reproducing property of the Szegö kernel. The kernel $P(z,y)$ is positive and has a $L^1(\partial\Omega,\ d\sigma_{2n-1})$ norm equal to one.

Also recall the Hardy-Littlewood kernel

$$\forall x,\ y \in \partial\Omega,\ P_t^0(x,\ y) := \frac{1}{\sigma_{2n-1}(B(x,\ t))}\mathbb{1}_{B(x,t)}(y).$$

We have

Lemma 6.5. *The Poisson-Szegö kernel $P(z,y)$ is dominated by the Hardy-Littlewood one: this means precisely that we have, with $x = \pi(z),\ t = |r(z)|$,*

$$\forall z \in \Omega,\ \forall y \in \partial\Omega,\ P(z,y) \lesssim P_{2t}^0(x,\ y) + \sum_{k\in\mathbb{N}} \frac{1}{2^{k+1}} P_{2^{k+1}t}^0(x,\ y).$$

Proof. Using (6.12) we get, still with $x = \pi(z),\ t = |r(z)|$,

$$|S(z,y)|^2 \lesssim \frac{\mathbb{1}_{B(x,2t)(y)}}{\sigma_{2n-1}(B(x,t))^2} + \sum_{k\geq 1} \frac{\mathbb{1}_{C_k}(y)}{\sigma_{2n-1}(B(x,\ 2^k t))^2}.$$

Because $C_k \subset B(x,\ 2^{k+1}t)$ we have $\mathbb{1}_{C_k} \leq \mathbb{1}_{B_k}$, with $B_k := B(x,\ 2^{k+1}t)$, hence setting

$$\forall x,\ y \in \partial\Omega,\ P_t^0(x,\ y) := \frac{1}{\sigma_{2n-1}(B(x,\ t))}\mathbb{1}_{B(x,t)}(y),$$

the Hardy-Littlewood kernel, we have

$$|S(z,y)|^2 \lesssim \frac{1}{\sigma_{2n-1}(B(x,\ t))}P_{2t}^0(x,\ y) + \sum_{k\geq 1}\frac{1}{\sigma_{2n-1}(B(x,\ 2^k t))}P_{2^k t}^0(x,\ y).$$

But by (6.11) we have $\sigma_{2n-1}(B(x,\ 2^k t)) \gtrsim 2^k \sigma_{2n-1}(B(x,\ t))$, hence

$$|S(z,y)|^2 \lesssim \frac{1}{\sigma_{2n-1}(B(x,\ t))}\left(P_{2t}^0(x,\ y) + \sum_{k\geq 1}\frac{1}{2^k}P_{2^k t}^0(x,\ y)\right).$$

By (6.16) we have, with $p = 2$,

$$\|S(a,\cdot)\|_{H^2}^2 = S(a,a) \geq \frac{1}{\sigma_{2n-1}(B(\alpha,\ 2t))},$$

Hence we get for the Poisson-Szegö kernel, still with $t = |r(z)|\,,\ x = \pi(z)$,

$$P(z,y) \lesssim P_{2t}^0(x,\ y) + \sum_{k\geq 1}\frac{1}{2^k}P_{2^k t}^0(x,\ y).$$

□

Combining the previous results we have:

Theorem 6.6. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$, then, with $S(z,\ y)$ its Szegö kernel we have, setting*

$$x = \pi(z),\ t = |r(z)|\,,\ C_0 := B(x,t),\ \forall k \geq 1,\ C_k := B(x,\ 2^k t)\backslash B(x,\ 2^{k-1}t),$$

- $\forall z \in \Omega, \forall y \in \partial\Omega, |S(z,y)| \lesssim \frac{1}{\sigma_{2n-1}(B(x,t))}\mathbb{1}_{B(x,\ 2t)}(y) + \sum_{k\geq 1}\frac{1}{\sigma_{2n-1}(B(x,2^k t))}\mathbb{1}_{C_k}(y);$
- $\|S(z,\cdot)\|_{H^p(\Omega)} \simeq \frac{1}{\sigma_{2n-1}(B(x,t))^{1/p'}}.$

And with $P_t^0(x,y)$ the Hardy-Littlewood kernel and $P(z,y)$ the Poisson-Szegö kernel

- $P(z,y) \lesssim P_{2t}^0(x,\ y) + \sum_{k\geq 1}\frac{1}{2^k}P_{2^k t}^0(x,\ y).$

7. Carleson measures

7.1. Harmonic analysis

We start by a "copy and paste" from [4], where we introduced the notion of Carleson measures of order α.

Let $(X,\ \rho,\ d\sigma)$ be a homogeneous type space [13]. Denote $B(x,t) := \{y \in X :: \rho(x,y) < t\}$ the pseudo-ball centered at x and of radius $t > 0$.

We define the Carleson windows (or "tents") on $\mathbb{R}^+ \times X$ in the following way: let A be an open set in X, then

$$W(A) := \{(t,x) \in \mathbb{R}^+ \times X :: B(x,t) \subset A\}.$$

We set $W(A)$ instead of $T(A)$ to differentiate notation from the case of the convex domains of finite type we seen in the previous section.

Definition 7.1. We say that the mesure μ on $\mathbb{R}^+ \times X$ is a *homogeneous geometric Carleson measure* of order α if, for any open set $A \subset X$,

$$|\mu|\,(W(A)) \leq C\sigma(A)^{\alpha}.$$

The usual homogeneous geometric Carleson measures are those with $\alpha = 1$.

We shall abbreviate homogeneous geometric Carleson measure by h.g. Carleson measure.

In the case $\alpha = 1$ it is enough to test on the sets $A = B(x,t)$ because the pseudo-balls generate all open sets in a homogeneous type space [13]. In the case $\alpha = 1$ we shall speak simply of h.g. Carleson measure.

The action of a kernel P_t on a function f will be denoted $P_t f$, precisely

$$P_t f(y) := \int_X P_t(x,y) f(x) d\sigma(x).$$

Now we have the abstract Carleson embedding theorem.

Theorem 7.2. *If the kernel P_t is dominated by the Hardy-Littlewood kernel, and if μ is an h.g. Carleson measure on $\mathbb{R}^+ \times X$, we have*

$$\forall f \in L^p(X,\ \sigma),\ \int_{X \times \mathbb{R}^+} |P_t f(x)|^p \ d\,|\mu|\,(x,t) \lesssim \|f\|^p_{L^p(\sigma)}.$$

Proof. This is quite well known and implicitly contained in Hörmander [21, Theorem 2.4]. But I shall give a proof taken from [4] where the same notation as here is used and which uses h.g. Carleson measures of order α.

Let V^0 the space of finite measure on $\mathbb{R}^+ \times X$, V^1 the space of h.g. Carleson ones and, with $\alpha := 1 - 1/p$, $W^{\alpha} := (V^0,\ V^1)_{(\alpha,p)}$ the intermediate class by the real interpolating method. We proved in [4, Proposition 1, page 30] that

$$w \in W^{\alpha} \iff \exists \mu \in V^1,\ \exists h \in L^p(\mu) :: dw = h\, d\mu. \tag{7.1}$$

Moreover the norm of w in W^{α} is equivalent to the norm of h in $L^p(\mu)$.

Because P_t, being dominated by the Hardy-Littlewood kernel, verifies the $(H1)$ hypothesis of [4, Theorem 2, page 27], we have that

$$\forall w \in W^{\alpha},\ \forall g \in L^{p'}(\sigma),\ \int_{\mathbb{R}^+\times X} |P_t g(x)|\ d\,|w|\,(t,x) \leq C_w \|g\|_{L^{p'}(\sigma)}. \tag{7.2}$$

Now let μ be a geometric Carleson measure and $f \in L^{p'}(X,\ \sigma)$; we want to prove that $P_t f(x) \in L^{p'}(\mu)$. Let $h \in L^p(\mu)$ and set $dw := h\,d\mu$ then $w \in W^{\alpha}$ by (7.1). We have by (7.2)

$$\begin{aligned}\int_{\mathbb{R}^+\times X} |P_t f(x)|\ d\,|w|\,(t,x) &= \int_{\mathbb{R}^+\times X} |P_t f(x)|\,|h|\ d\,|\mu|\,(t,x)\\ &\leq C\|h\|_{L^p(\mu)}\|f\|_{L^{p'}(\sigma)};\end{aligned}$$

but this being true for all functions h in $L^p(\mu)$, we have that $P_t f(x) \in L^{p'}(\mu)$ and the theorem is proved by exchanging p' and p. ☐

7.2. Carleson measures in convex domain of finite type

Now to define the geometric Carleson measures in our domains we have 2 possibilities for a positive Borel measure on Ω

- $\exists C > 0 :: \forall a \in \Omega,\ \epsilon := 2\,|r(a)|\,,\ \mu(T_a(\epsilon)) \leq C\sigma(\partial\Omega \cap P_{\epsilon}(a))$,

with $P_{\epsilon}(a) \in \mathcal{P}$ is the family defined in (6.4).

- $\exists C > 0 :: \forall a \in \Omega,\ \alpha = \pi(a),\ \mu(\Omega \cap W(B(\alpha,\ |r(a)|)) \leq C\sigma(B(\alpha,\ |r(a)|))$,

where $B(\alpha,\ |r(a)|)$ is the pseudo-ball on $\partial\Omega$ of center α and radius $|r(a)|$, and $W(B(\alpha,\ |r(a)|))$ is the Carleson window defined in the previous subsection. For this section we set $\sigma = \sigma_{2n-1}$.

We shall show that they are equivalent. We have that

$$\forall a \in \mathcal{U} \cap \Omega,\ \epsilon := |r(a)|\,,\ B(\alpha,\epsilon) := \partial\Omega \cap P_{\epsilon}(\alpha),$$

by definition of the family $\mathcal{P}$. Then we want to show:

Lemma 7.3. *There is a constant γ, independent of a, such that*

$$W(B(\alpha,\ |r(a)|)) \subset T_a(\ \gamma\,|r(a)|).$$

Proof. We have, by definition of the Carleson window:

$$z \in W(B(\alpha,\ |r(a)|)) \iff B(x,\ |r(z)|) \subset B(\alpha,\ |r(a)|),$$

where $x = \pi(z)$. This implies, because $\partial\Omega$ is a space of homogeneous type, that we have $|r(z)| \leq c\,|r(a)|$, with a uniform constant $c \geq 1$.

But with $\delta := |r(z)|$, $P_\delta(z) \cap B(x,\ \delta) \neq \emptyset$ hence with $\epsilon = |r(a)|$,

$$P_\delta(z) \cap B(\alpha,\ \epsilon) \neq \emptyset \Rightarrow P_\delta(z) \cap P_\epsilon(\alpha) \neq \emptyset.$$

Let $s = \max(\delta,\ \epsilon)$ then

$$P_\delta(z) \cap P_\epsilon(\alpha) \neq \emptyset \Rightarrow P_s(z) \cap P_s(\alpha) \neq \emptyset \Rightarrow P_s(z) \subset C_3 P_s(\alpha)$$

by (6.7).

But if $s = \delta$ then $s \leq c\epsilon$ and if $s = \epsilon$ then again $s \leq c\epsilon$ because $c \geq 1$; so in any case $s \leq c\epsilon$ and this implies

$$P_\delta(z) \subset P_s(z) \subset C_3 P_s(\alpha) \subset C_4 P_\epsilon(\alpha) \subset P_{\gamma\epsilon}(\alpha),$$

by (6.5) with $t = C_4,\ \gamma = C_t$.

And again because $P_\epsilon(a) \cap B(\alpha,\ \epsilon) \neq \emptyset$, we get $P_{\gamma\epsilon}(\alpha) \subset P_{\gamma\epsilon}(a)$ by (6.7) and (6.5) ; and finally $P_\delta(z) \subset P_{\gamma\epsilon}(a)$. Cutting with Ω we get

$$z \in P_\delta(z) \cap \Omega \subset P_{\gamma\epsilon}(a) \cap \Omega = T_a(\ \gamma\ |r(a)|). \qquad \square$$

We shall use the following definition for geometric Carleson measure in a convex domain of finite type to continue with the same notation.

Definition 7.4. Let μ be a positive Borel measure on the bounded convex domain of finite type Ω. We shall say that μ is a *geometric Carleson measure* in Ω if:

$$\exists C > 0 :: \forall a \in \Omega,\ \epsilon = 2\,|r(a)|\,,\ \mu(T_a(\epsilon)) \leq C\sigma(\partial\Omega \cap P_\epsilon(a)).$$

7.3. Carleson embedding

We are in position to prove a Carleson embedding theorem for convex domains of finite type.

To prove it we shall need the lemma:

Lemma 7.5. *Let $a \in \Omega,\ \alpha = \pi(a),\ \delta = |r(a)|$; there is a uniform constant $\gamma > 0$ such that*

$$\forall z \in \gamma P_\delta(a),\ |K_\Omega(z,\ a)| \geq \frac{c}{\delta\sigma_{2n-1}(B(\alpha, \delta))}. \tag{7.3}$$

Proof. We have the lower bound (6.13) of the Bergman kernel

$$K_\Omega(a, a) \gtrsim \prod_{j=1}^{n} \tau_j(a,\ \delta)^{-2} \simeq \frac{1}{\delta\sigma_{2n-1}(B(\alpha, \delta))},$$

the last equivalence by equations (6.13) and (6.14) and a upper bound of its derivatives ([26, Theorem 5.2], and [27])

$$\left|\partial_z^\mu \bar{\partial}_a^\nu K_\Omega(z, a)\right| \leq C_{\mu\nu} \prod_{j=1}^{n} \tau_j(a,\ \beta)^{-2-\mu_j-\nu_j}, \tag{7.4}$$

with $\beta = \rho^*(a,\ z)$.

Set for $t \in [0,1]$, $f(t) := K_\Omega(a+t(z-a), a)$, then f being complex valued, we have $f(t) = (f_1 + if_2)$.

Apply the mean value theorem:

$$\exists t_1, t_2 \in [0,1] :: f(1)-f(0) = (f_1'(t_1) + if_2'(t_2)) \Rightarrow |f(1)-f(0)| \leq 2 \sup_{t\in[0,1]} \left|f'(t)\right|.$$

Hence

$$\begin{aligned}|K_\Omega(z,a) - K_\Omega(a,a)| &\leq 2 \sup_{t\in[0,1]} \left| \sum_{j=1}^n (z_j - a_j) \frac{\partial K_\Omega(a+t(z-a),\ a)}{\partial \zeta_j} \right| \\ &\lesssim \sum_{j=1}^n \frac{\left|z_j - a_j\right|}{\tau_j(a,\ \beta)} \prod_{k=1}^n \tau_k(a,\ \beta)^{-2},\end{aligned}$$

by inequality (7.4); so

$$|K_\Omega(z,a) - K_\Omega(a,a)| \lesssim \frac{1}{\beta\sigma_{2n-1}(B(\alpha,\beta))} \sum_{j=1}^n \frac{\left|z_j - a_j\right|}{\tau_j(a,\ \beta)},$$

by equations (6.13) and (6.14).

Now choose z such that $\left|z_j - a_j\right| \leq \gamma\tau_j(a,\ \delta) \Rightarrow \beta \lesssim \delta$ and the homogeneous nature of Ω gives that $\tau_j(a,\ \beta) \simeq \tau_j(a,\ \delta)$ hence

$$|K_\Omega(z,a) - K_\Omega(a,a)| \lesssim \frac{1}{\beta\sigma_2 n - 1(B(\alpha,\delta))} \sum_{j=1}^n \frac{\left|z_j - a_j\right|}{\tau_j(a,\ \delta)} \lesssim \frac{n\gamma}{\delta\sigma_{2n-1}(B(\alpha,\delta))}.$$

Take γ uniformly small enough to compensate the constant in the last inequality above to get

$$|K_\Omega(z,a) - K_\Omega(a,a)| \leq \frac{1}{2} \times \frac{1}{\delta\sigma_{2n-1}(B(\alpha,\delta))},$$

this means that, for z in the polydisc $\gamma P_\delta(a)$, we have $|K_\Omega(z,\ a)| \geq \frac{c}{\delta\sigma_{2n-1}(B(\alpha,\delta))}$, the positive constants $c,\ \gamma$ being uniform. □

We shall need the definition.

Definition 7.6. Let μ be a positive Borel measure on the domain Ω and $p \geq 1$. We shall say that μ is a p *Carleson measure* in Ω if:

$$\exists C_p > 0,\ \forall f \in H^p(\Omega),\ \int_\Omega |f|^p\, d\mu \leq C_p^p \|f\|_{H^p}^p.$$

This means that we have a continuous embedding of $H^p(\Omega)$ in $L^p(\mu)$.

Now we have:

Theorem 7.7. *If the measure μ is a geometric Carleson measure we have*

$$\forall p>1,\ \exists C_p>0,\ \forall f\in H^p(\Omega),\ \int_\Omega |f|^p\,d\mu\le C_p^p\|f\|_{H^p}^p.$$

Conversely if the positive measure μ is p Carleson for a $p\in[1,\ \infty[$, then it is a geometric Carleson measure, hence it is q Carleson for any $q\in]1,\ \infty[$.

Proof. We apply Theorem 7.2 to the Poisson-Szegö kernel $P(z,y)$ which is dominated by the Hardy-Littlewood kernel. Because a function in $A(\Omega)$, the algebra of holomorphic function in Ω continuous up to $\partial\Omega$, is reproduced by $P(z,y)$ and because this algebra is dense in $H^p(\Omega)$, the first part of the theorem is proved.

Suppose now that μ is p Carleson for a $p\in[1,\ \infty[$, then we have

$$\exists C>0,\ \forall a\in\Omega,\ \int_\Omega |K_\Omega(z,\ a)|^p\,d\mu(z)\le C\|K_\Omega(\cdot,\ a)\|_{H^p}^p,$$

with $K_\Omega(z,\ a)$ the Bergman kernel at a. Using the inequality (7.3) of the lemma, we get

$$\begin{aligned}\forall a\in\Omega,\ \int_{\Omega\cap\gamma P_\delta(a)}\left(\frac{1}{\delta\sigma(B(\alpha,\delta))}\right)^p d\mu(z)&\le\int_\Omega |K_\Omega(z,\ a)|^p\,d\mu(z)\\&\le C\|K_\Omega(\cdot,\ a)\|_{H^p}^p,\end{aligned}$$

hence

$$\forall a\in\Omega,\ \left(\frac{1}{\delta\sigma(B(\alpha,\delta))}\right)^p\mu(\Omega\cap\gamma P_\delta(a))\le C\|K_\Omega(\cdot,\ a)\|_{H^p}^p.$$

We can use the estimate of $\|K_\Omega(\cdot,\ a)\|_{H^p}$ done in lemma (6.4)

$$\|K_\Omega(\cdot,\ a)\|_{H^p}^p\lesssim\frac{1}{\delta^p\sigma(B(\alpha,\ \delta))^{p-1}},$$

to get

$$\forall a\in\Omega,\ \left(\frac{1}{\delta\sigma(B(\alpha,\delta))}\right)^p\mu(\Omega\cap\gamma P_\delta(a))\le C\frac{1}{\delta^p\sigma(B(\alpha,\ \delta))^{p-1}},$$

hence

$$\forall a\in\Omega,\ \mu(\Omega\cap\gamma P_\delta(a))\le C\sigma(B(\alpha,\ \delta)).$$

Still by homogeneity we have $\gamma P_\delta(a)\supset P_{c\delta}(\alpha)$ and

$$B(\alpha,\ \delta)\subset CB(\alpha,\ c\delta)\Rightarrow\sigma(B(\alpha,\ \delta))\le C'\sigma(B(\alpha,\ c\delta)),$$

so

$$\forall a\in\Omega,\ \mu(\Omega\cap P_{c\delta}(\alpha))\le CC'\sigma(B(\alpha,\ c\delta)),$$

and the measure μ is a geometric Carleson measure, hence it is a q Carleson measure by the first part of the theorem. □

If Ω is a convex domain of finite type, with the family $\mathcal{P}$ of polydiscs of McNeal, we define a related family $\mathcal{Q}$ of polydiscs:

$$\forall a \in \mathcal{U}\backslash\partial\Omega,\ \forall t > 0,\ \epsilon := |r(a)|\,,\ Q_a(t) := tP_\epsilon(a),$$

where $tP_\epsilon(a)$ si the dilated polydisc as defined in (6.2).

Lemma 7.8. *The family $\mathcal{Q} := \{Q_a(t),\ t > 0,\ a \in \mathcal{U}\}$ is a good family of polydiscs in Ω.*

Proof. By (6.6) ((3) of Proposition 2.7 in [20]), we get that $\exists\delta_0 > 0$, such that

$$a \in \Omega \cap \mathcal{U} \Rightarrow \delta_0 P_{|r(a)|}(a) \subset \Omega;$$

because $d(a) := d(a, \Omega^c) \simeq |r(a)|$, the constants being independent of $a \in \Omega$, we have with

$$Q_a(t) := tP_\epsilon(a),\ t = \delta_0,\ \epsilon = |r(a)| \simeq d(a),$$

that

$$a \in \Omega \cap \mathcal{U} \Rightarrow Q_a(\delta_0) \subset \Omega,$$

which means precisely that the family $\mathcal{Q} = \{Q_a(t)\}_{a\in\mathcal{U}\cap\Omega,\ t>0}$ is a good family of polydiscs in the sense of Section 1. Moreover the Hefer's Theorem 6.1 gives that the size of the sides of $Q_a(t)$ are precisely equivalent to

$$|r(a)|^{1/m_j} \simeq d(a)^{1/m_j},$$

which means that the multi-type for this family in the sense of Definition 2.1 is precisely $m_j(a),\ j = 2,\ \dots,\ n.$ $\square$

So we can give a general definition for geometric Carleson measures equivalent to the one we gave in the case of convex domains of finite type.

Definition 7.9. Let μ be a positive Borel measure on the domain Ω equipped with a good family of polydiscs $\mathcal{Q}$. We shall say that μ is a *geometric Carleson measure* in Ω if:

$$\exists C > 0 :: \forall a \in \Omega,\ \mu(\Omega \cap Q_a(2)) \leq C\sigma(\partial\Omega \cap Q_a(2)).$$

8. Construction of balanced sub-domains

In the unit ball of $\mathbb{C}^n$ a measure whose images by all automorphisms of the ball is uniformly bounded is a geometric Carleson measure, and this is a fact we used for instance in [3]. Unfortunately in a general domain, even convex ones or strictly pseudo-convex ones, there is just the identity as automorphism, so we have to overcome this issue.

The aim now is to build a sub-domain Ω_a associated to a point $a \in \Omega$ near the boundary such that the restriction to it of the measure we want to study is bounded by the right bound. If the domain Ω_a is equivalent to a Carleson window, as defined at the beginning of Section 7, then it will work.

The main difficulty here is to get bounds *independent* of $a \in \Omega$. We shall start with convex domains and define later a more general kind of domains for which our methods work.

Let Ω be a $\mathcal{C}^\infty$ smooth convex domain in $\mathbb{C}^n$, $a \in \Omega$. By translation and rotation we can suppose that $a = 0$, $\alpha = \pi(a) = (d(a), 0, \dots, 0)$ and the defining function $\rho = d(a) + \Re z_1 + \Gamma(z)$, with $\Gamma(z) = \mathcal{O}(|z|^2)$. Let $\mathcal{E}_a$, $\mathcal{E}'_a$ be smooth complex ellipsoids centered at $a = 0$

$$\mathcal{E}_a := \left\{ z \in \mathbb{C}^n :: \sum_{j=1}^n \frac{|z_j|^2}{d(a)^{2/m_j}} < 4n \right\},\ \mathcal{E}'_a := \left\{ z \in \mathbb{C}^n :: \sum_{j=1}^n \frac{|z_j|^2}{d(a)^{2/m_j}} < 5n \right\}.$$

Consider the convex domain $\mathcal{E}'_a \cap \Omega$ and smooth it to get a smoothly bounded convex domain Ω_a such that $\mathcal{E}_a \cap \Omega \subset \Omega_a \subset \mathcal{E}'_a \cap \Omega$. This can be done as in [1, page 129]. Suppose that $\alpha = \pi(a) = 0$ and, as usual, $\rho(z) = \Re z_1 + f(\Im z_1, z')$, with $z' = (z_2, \dots, z_n)$. Then there is a function $S(x, y)$, convex and $\mathcal{C}^\infty(\mathbb{R}^2)$ such that a defining function ρ_a for Ω_a is given by $\rho_a := S(2|\Im z_1|^2 + |z'|^2,\ \rho)$; hence any $\mathcal{C}^k$ norm of ρ_a is controlled by the $\mathcal{C}^k$ norm of the defining function ρ of Ω, *i.e.* $\forall k \in \mathbb{N},\ \|\rho_a\|_{\mathcal{C}^k} \le C_k \|\rho\|_{\mathcal{C}^k}$. Moreover we have that the outward normal derivative $\frac{\partial\rho}{\partial\eta}$ is uniformly bounded below because of the compactness of $\partial\Omega$ and we have also $\frac{\partial\rho_a}{\partial\eta} \ge \delta\frac{\partial\rho}{\partial\eta} > 0$ independently of a, by the construction of Ω_a. We shall need this last fact when we shall apply Theorem 10.11 to interpolating sequences in Section 9; see Remark 10.12.

Let $\mathbb{S}$ be the unit sphere in $\mathbb{C}^n$ and because Ω_a is convex it is starlike with respect to a $(= 0)$, $\partial\Omega_a$ admits a spherical parametrization, *i.e.* there is a function $R(\zeta) \in \mathcal{C}^1(\mathbb{S})$, $R(\zeta) > 0$, such that:

$$\partial\Omega_a = \{z \in \mathbb{C}^n :: \exists\zeta \in \mathbb{S},\ z = R(\zeta)\zeta\}.$$

Let $\zeta \in \mathbb{S}$ and define D_ζ to be the complex plane slice through ζ

$$D_\zeta := \{tR(e^{i\theta}\zeta)e^{i\theta}\zeta,\ \theta \in [0, 2\pi],\ t \in [0, 1[\}.$$

We shall use the notation

$$\forall\zeta \in \mathbb{S},\ d_\zeta(0) = \inf_{\theta\in[0,2\pi]} R(e^{i\theta}\zeta);\ d_{\zeta\,\max}(0) = \sup_{\theta\in[0,2\pi]} R(e^{i\theta}\zeta).$$

Lemma 8.1. *We have*

$$Q_a(2) \cap \Omega \subset \Omega_a \subset Q_a(\sqrt{5n}).$$

and

$$\forall\zeta \in \partial\Omega,\ d_{\zeta\,\max}(0) \le \frac{\sqrt{5n}}{\delta_0} d_\zeta(0).$$

Proof. $z \in Q_a(2) \cap \Omega \Rightarrow \forall j = 1, \dots, n, \left|z_j\right| < 2d(a)^{1/m_j} \Rightarrow \sum_{j=1}^{n} \frac{\left|z_j\right|^2}{d(a)^{2/m_j}} < 4n \Rightarrow$ $z \in \mathcal{E}_a \cap \Omega \subset \Omega_a$. If $z \in \Omega_a \subset \mathcal{E}'_a$ then

$$\sum_{j=1}^{n} \frac{\left|z_j\right|^2}{d(a)^{2/m_j}} < 5n \Rightarrow \forall j = 1, \dots, n, \ \left|z_j\right| < \sqrt{5n}d(a)^{1/m_j} \Rightarrow z \in Q_a(\sqrt{5n}),$$

and the first assertion follows.

Let us see that a is "in the middle" of the slices D_ζ.

Choose θ such that $d_\zeta(0) = R(e^{i\theta}\zeta)$, then the real segment from 0 to $R(e^{i\theta}\zeta)e^{i\theta}\zeta \in \partial\Omega_a$ cross the boundary of $Q_a(\delta_0)$ at a point $tR(e^{i\theta}\zeta)e^{i\theta}\zeta$ with $0 < t \leq 1$ because $Q_a(\delta_0) \subset \Omega_a$.

But if $z = (z_1, \dots, z_n) \in \partial Q_a(\delta_0)$ then $\exists j :: \left|z_j\right| = \delta_0 d(a)^{1/m_j}$, so we have here

$$\exists j :: tR(e^{i\theta}\zeta)\left|\zeta_j\right| = \delta_0 d(a)^{1/m_j}$$

and because $0 < t \leq 1$ we get

$$\delta_0 d(a)^{1/m_j} \leq R(e^{i\theta}\zeta)\left|\zeta_j\right| = d_\zeta(0)\left|\zeta_j\right|.$$

On the other hand, because $\Omega_a \subset Q_a(\sqrt{5n})$ which is a polydisc with sides parallel to the axes, we have

$$\forall k = 1, \dots, n, \ \forall \varphi \in [0, 2\pi], \ R(e^{i\varphi}\zeta)\left|\zeta_k\right| \leq \sqrt{5n}d(a)^{1/m_k}$$
$$\Rightarrow d_{\zeta\,\max}(0)\left|\zeta_k\right| \leq \sqrt{5n}d(a)^{1/m_k};$$

in particular for $\varphi = \theta$ and $k = j$ we get

$$\delta_0 d(a)^{1/m_j} \leq d_\zeta(0)\left|\zeta_j\right| \leq d_{\zeta\,\max}(0)\left|\zeta_j\right| \leq \sqrt{5n}d(a)^{1/m_j}.$$

This implies $\frac{1}{\left|\zeta_j\right|} \leq \frac{d_\zeta(0)}{\delta_0 d(a)^{1/m_j}}$ and $d_{\zeta\,\max}(0) \leq \frac{\sqrt{5n}d(a)^{1/m_j}}{\left|\zeta_j\right|} \leq \frac{\sqrt{5n}}{\delta_0}d_\zeta(0)$. □

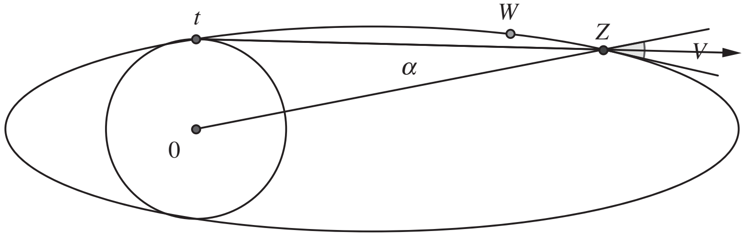
Let D be a bounded convex domain in $\mathbb{C}$; take a biggest disc contained in D, say $D(0, r)$ with $0 \in D$ being its center and $D(0, R)$ the smallest disc containing D with the same center 0.

Now parametrize the boundary ∂D of the convex D by polar coordinates $s(\theta)e^{i\theta}$ and set $\gamma := \frac{R}{r}$.

Lemma 8.2. *Let D be a convex domain in $\mathbb{C}$, $0 \in D$ with the previous notation; let s' be the derivative of s, then we have*

$$\left|\frac{s'}{s}\right| \leq \sqrt{\gamma^2 - 1}.$$

Proof. We have that $D(0,r) \subset D \subset D(0,R)$. Let $z \in \partial D$ such that $\tan V$ is minimal, where V is the angle between $(0,z)$ and the tangent at z to ∂D. Take the segment tangent T from z to t on the circle $\partial D(0,r)$; because D is convex we have $T \subset D$ and the points $w \in \partial D$ near z are such that the angle between (w,z) and $(0,z)$ is bigger than the angle α between (t,z) and $(0,z)$, hence the angle V is bigger than α.



Now we have that $|\sin\alpha| = \frac{r}{|z|} \geq \frac{r}{R}$, hence $|\tan\alpha| \geq \frac{1}{\sqrt{\gamma^2-1}}$, where $\gamma := \frac{R}{r}$.

So, because $\left|\frac{s'}{s}\right| = \frac{1}{|\tan V|}$, we have

$$\left|\frac{s'}{s}\right| = \frac{1}{|\tan V|} \leq \sqrt{\gamma^2 - 1}. \qquad \square$$

We shall apply this lemma to the slices D_ζ of Ω_a.

Recall that $U_\zeta(\theta) = R(e^{i\theta}\zeta)$ is precisely the polar coordinates parametrization of ∂D_ζ in the coordinates of $\mathbb{C}\zeta$ and $d_\zeta(0)$ is the distance from $a(=0)$ to ∂D_ζ, hence here we have $r = d_\zeta(0)$, $R = d_{\zeta\,\max}(0)$.

We shall say that Ω_a is γ balanced with respect to a (Definition 10.6) if $\forall\zeta \in \mathbb{S}$, $d_{\zeta\,\max}(a) \leq \gamma d_\zeta(a)$ and $\left|U'_\zeta(\theta)\right| \leq \gamma d_{\zeta\,\max}(a)$; with this we have

Lemma 8.3. *Because Ω_a is such that all its slices $D_\zeta = \Omega_a \cap \{z :: z = a + \lambda\zeta,\ \lambda \in \mathbb{C}\}$ are convex we have that Ω_a is γ balanced with $\gamma = \frac{\sqrt{5n}}{\delta_0}$.*

Proof. By Lemma 8.2 with $s(\theta) := U_\zeta(\theta)$, we have

$$\left|\frac{U'_\zeta}{U_\zeta}\right| \leq \sqrt{\gamma^2 - 1} \leq \gamma \Rightarrow \left|U'_\zeta\right| \leq \gamma\left|U_\zeta\right| \leq \gamma d_{\zeta\,\max}(0).$$

Now using Lemma 8.1 $\gamma = \frac{R}{r}$ hence we have that Ω_a is γ balanced with $\gamma = \frac{\sqrt{5n}}{\delta_0}$. $\square$

All we have done works as soon as the domain Ω verifies the following definition.

Definition 8.4. A smoothly $\mathcal{C}^m,\ m \geq 2$ bounded domain Ω with a good family of polydiscs is *well balanced* if $\exists R > 2,\ \exists \epsilon > 0,\ \exists \gamma > 0 :: \forall a \in \Omega,\ d(a) < \epsilon,\ \exists \Omega_a\ \gamma$ balanced such that $Q_a(2) \cap \Omega \subset \Omega_a \subset Q_a(R)$.

And we have the theorem

Theorem 8.5. *If Ω is a smoothly $\mathcal{C}^m,\ m \geq 2$, bounded convex domain in $\mathbb{C}^n$, with a good family of polydiscs then Ω is well balanced.*

Proof. This is Lemma 8.3. □

Theorem 8.6. *If Ω is well balanced, then for any $a \in \Omega,\ d(a) < \epsilon$, there is a γ balanced sub-domain $\Omega_a :: Q_a(2) \cap \Omega \subset \Omega_a \subset Q_a(R)$ with the property*

$$\forall u \in \mathcal{N}(\Omega_a),\ \ln|u(a)| = 0 \text{ then, with } \Theta := \partial\bar{\partial}\ln|u|,\ \int_{\Omega_a} d(z)\mathrm{Tr}\Theta \leq C\|u\|_{\mathcal{N}(\Omega_a)},$$

where the constant depends only on Ω and not on a.

Proof. We apply Theorem 10.11 to Ω_a, then we have that

$$\int_{\Omega_a} d(z)\mathrm{Tr}\Theta \leq C\|u\|_{\mathcal{N}(\Omega_a)},$$

where the constant C depends only on Ω. □

Remark 8.7. If Ω is locally biholomorphic to a well balanced domain, then we have an analogous result by constructing the Ω_a via the biholomorphism. Precisely let $p \in \partial\Omega$ and Φ a biholomorphism of $\Omega \cap B(p, R)$ on a well balanced domain $\Omega' \cap \Phi(B(p, R))$. Then we build the sub-domains $\Omega'_{\Phi(a)}$ and consider $\Omega_a := \Phi^{-1}(\Omega'_{\Phi(a)})$. Because Φ is biholomorphic in a neighborhood of $\bar{\Omega} \cap B(p, R)$ we get easily that Theorem 8.6 is still valid.

In particular if Ω is strictly pseudo-convex, then it works.

9. Interpolating and dual bounded sequences in $H^p(\Omega)$

Let Ω be a domain in $\mathbb{C}^n$ equipped with a good family of polydiscs. We shall study interpolating sequences in Ω and generalise previous results we got for the unit ball to convex domains of finite type.

9.1. Reproducing kernels

Let $S(z, \zeta)$ be the Szegö kernel of Ω, *i.e.* the kernel of the orthogonal projection from $L^2(\partial\Omega)$ onto $H^2(\Omega)$.

To any point $a \in \Omega$ we associate the vector $k_a(\cdot) := S(\cdot, a) = \bar{S}(a, \cdot) \in H^2(\Omega)$. This is a reproducing kernel for a because

$$\forall f \in H^2(\Omega),\ f(a) = \int_{\partial\Omega} f(\zeta)S(a,\zeta)d\sigma(\zeta)$$

by the definition of the Szegö kernel, but

$$\int_{\partial\Omega} f(\zeta)S(a,\zeta)d\sigma(\zeta) = \int_{\partial\Omega} f(\zeta)\bar{k}_a(\zeta)d\sigma(\zeta) = \langle f, k_a\rangle,$$

by the definition of k_a.

Definition 9.1. We say that the sequence S of points in Ω is $H^p(\Omega)$ *interpolating* if

(i) $\forall a \in S,\ k_a \in H^{p'}(\Omega)$; (this is always true if $p \geq 2$.)
(ii) $\forall \lambda \in \ell^p(S),\ \exists f \in H^p(\Omega) :: \forall a \in S,\ f(a) = \lambda_a \|k_a\|_{p'}$,

with p' the conjugate exponent of p, *i.e.* $\frac{1}{p} + \frac{1}{p'} = 1$.

A weaker notion is:

Definition 9.2. We shall say that the sequence S of points in Ω is *dual bounded* in $H^p(\Omega)$ if there is a bounded sequence of elements in $H^p(\Omega)$, $\{\rho_a\}_{a\in S} \subset H^p(\Omega)$ which dualizes the associated sequence of reproducing kernels, *i.e.*

(i) $\forall a \in S,\ k_a \in H^{p'}(\Omega)$; (this is always true if $p \geq 2$.)
(ii) $\exists C > 0 :: \forall a \in S,\ \|\rho_a\|_p \leq C,\ \forall a, b \in S,\ \langle \rho_a,\ k_b\rangle = \delta_{a,b}\|k_b\|_{p'}$.

Clearly if S is $H^p(\Omega)$ interpolating then S is dual bounded in $H^p(\Omega)$, just interpolate the basic sequence of $\ell^p(S)$.

Definition 9.3. We say that S has the *linear extension property* if S is $H^p(\Omega)$ interpolating and if moreover there is a bounded linear operator $E\colon \ell^p(S) \to H^p(\Omega)$ making the interpolation, *i.e.*

$$\exists C > 0,\ \forall \lambda \in \ell^p(S),\ \forall a \in S,\ E(\lambda)(a) = \lambda_a \|k_a\|_{p'}$$

and

$$\|E(\lambda)\|_{H^p(\Omega)} \leq C\|\lambda\|_p.$$

9.2. The p regularity

Let us introduce a link between the H^p norm of the reproducing kernels and the geometry of the boundary of Ω, with respect to the good family $\mathcal{Q}$.

Definition 9.4. We shall say that Ω is *p regular* with respect to the family $\mathcal{Q}$ if:

$$\exists C > 0 :: \forall a \in \Omega,\ \|k_a\|_p^{-p'} \leq C\sigma(\partial\Omega \cap Q_a(2)),$$

where p' is the conjugate exponent of p. Here we use the convention that if $k_a \notin H^p(\Omega)$, then $\|k_a\|_p = +\infty \Rightarrow \|k_a\|_p^{-p'} = 0$, so the inequality is true in this case.

Lemma 9.5. *If Ω is a convex domain of finite type in $\mathbb{C}^n$, then Ω is p regular for any $p > 1$.*

Proof. Theorem 6.6 gives

$$\begin{aligned}\|k_a\|_{H^p(\Omega)} &= \|S(a,\cdot)\|_{H^p(\Omega)} \\ &\simeq \frac{1}{\sigma_{2n-1}(B(\alpha, d(a)))^{1/p'}} \simeq \frac{1}{\sigma_{2n-1}(\partial\Omega \cap Q_a(2))^{1/p'}},\end{aligned}$$

which, by the Definition 9.4 of p regularity, implies the p regularity of Ω. □

Proposition 9.6. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$, $a \in \Omega$ and Ω_a the sub-domain associated to a. The measure $d\sigma_{2n-1|\partial\Omega_a\backslash\partial\Omega}$ is a geometric Carleson measure in Ω.*

To prove this proposition we shall use the following lemmas.

Lemma 9.7. *Let U be an open set in $\mathbb{R}^k$ and V a graph in $\mathbb{R}^{k+1}$ over U,* i.e.

$$V := \left\{(x, y) \in \mathbb{R}^{k+1} :: y = f(x),\ x = (x_1, \ldots, x_k) \in U\right\},$$

with f of class $\mathcal{C}^1(U)$. Then $\sigma_k(V) \geq \sigma_k(U)$.

Proof. We shall use the formula for the Lebesgue measure for such a graph given in [7, page 203, formula 6.4.1.1]: let (U, g) be a parametrization of V, then we have that:

$$g^*\omega = \sqrt{\det\left(\frac{\partial g}{\partial x_i} \mid \frac{\partial g}{\partial x_j}\right)} dx_1 \wedge \ldots \wedge dx_k,$$

where $M := \left(\frac{\partial g}{\partial x_i} \mid \frac{\partial g}{\partial x_j}\right)$ is the matrix of the scalar product of the vectors $\frac{\partial g}{\partial x_i}$ and $\frac{\partial g}{\partial x_j}$.

Here we have that $g(x) = (x_1, \ldots, x_k, f(x))$ hence

$$\frac{\partial g}{\partial x_j} = (0, \ldots, 0, 1, 0, \ldots, 0, f_j'(x)),$$

with the 1 at the j^{th} position and $f_j' := \frac{\partial f}{\partial x_j}$. So we get

$$\left\langle \frac{\partial g}{\partial x_i}, \frac{\partial g}{\partial x_j} \right\rangle = f_i' f_j' \text{ if } i \neq j \text{ and } \left\langle \frac{\partial g}{\partial x_j}, \frac{\partial g}{\partial x_j} \right\rangle = 1 + \left(f_j'\right)^2 \text{ if } i = j.$$

Hence the matrix M can be written $M = I + FF^t$ where F is the column vector $F := (f_1', \dots, f_k')$, and F^t is the transpose line matrix.

Clearly the matrix FF^t is positive, because for any vector $v = (v_1, \dots, v_k)$ we have

$$v^t FF^t v = (v^t F)(F^t v) = \left(\sum_{j=1}^{k} f_j' v_j\right)^2.$$

The eigenvalues λ_j of FF^t are all 0 but one because $FF^t v = 0$ as soon as $\sum_{j=1}^{k} f_j' v_j = 0$ which is a hyperplane and hence the only non zero eigenvalue, λ_k, is such that $\lambda_k = \mathrm{Tr} FF^t = \sum_{j=1}^{k} (f_j')^2$ because the sum of the eigenvalues is the trace of the matrix.

Now we have that the eigenvalues of M are $1 + \lambda_j$, hence the determinant of M is their product, so

$$\det M = 1 + \lambda_k = 1 + \sum_{j=1}^{k} (f_j')^2 \geq 1.$$

The case $k = 2$ was already done in [7, page 204], and here we provide the generalisation.

Now we have

$$\sigma_k(V) = \int_U \sqrt{\det M} dx_1 \cdots dx_k \geq \int_U dx_1 \cdots dx_k = \sigma_k(U). \qquad \square$$

Remark 9.8. In fact this lemma just says that the measure of the orthogonal projection U of V on $\mathbb{R}^k$ has a Lebesgue measure smaller than the measure of V. *I.e.*, the orthogonal projection is contracting for the Lebesgue measure, which seems quite natural.

Lemma 9.9. *Let $b \in \Omega$, $\beta = \pi(b)$ and $T_\beta(\partial\Omega)$ be the real tangent space to $\partial\Omega$ at β. Let $F_b := T_\beta(\partial\Omega) + d(b)n_\beta$, where n_β is the outward real normal at β to $\partial\Omega$ and $B_b := F_b \cap \bar{Q}_b(2)$, the "bottom" of $Q_b(2)$; a sufficient condition to have $\pi(\Omega \cap Q_b(2)) \subset \pi(\partial\Omega \cap Q_b(2))$ is that $\Omega \cap B_b = \emptyset$.*

As a consequence if Ω is convex then $\pi(\Omega \cap Q_b(2)) \subset \pi(\partial\Omega \cap Q_b(2))$.

Proof. Suppose that $\Omega \cap B_b = \emptyset$ and take $z \in \Omega$, take $\zeta = \pi_b(z)$ where π_b is the orthogonal projection on F_b; then we have $\zeta \in B_b$ hence $\zeta \notin \Omega$ so $\rho(\zeta) \geq 0$. On the other hand $z \in \Omega \Rightarrow \rho(z) < 0$, hence ρ being continuous on the real segment $[z, \zeta]$, there is a $w \in]z, \zeta]$ such that $\rho(w) = 0$, so $w \in \partial\Omega$. Now $T_\beta(\partial\Omega)$ and F_b being parallel the segment $[z, \zeta]$ is orthogonal to $T_\beta(\partial\Omega)$ hence $\pi(z) = T_\beta(\partial\Omega) \cap [z, \zeta] = \pi(w)$. Hence, because $\pi(w) \in \pi(\partial\Omega)$ we have $\pi(\Omega \cap Q_b(2)) \subset \pi(\partial\Omega \cap Q_b(2))$.

If Ω is convex then it lies on the same side of $T_\beta(\partial\Omega)$ hence we have $\Omega \cap B_b = \emptyset$. $\square$

Proof of the Proposition 9.6. We have to see that

$$\exists C > 0 :: \forall b \in \Omega,\ \sigma_{2n-1}(\partial\Omega_a \cap Q_b(2)) \le C\sigma_{2n-1}(\partial\Omega \cap Q_b(2)).$$

Of course we take b such that $(\partial\Omega_a\backslash\partial\Omega) \cap Q_b(2) \neq \emptyset$ and take the convex hull E of $(\partial\Omega_a\backslash\partial\Omega) \cap Q_b(2)$; because the domain $Q_b(2)$ is convex, $E \subset Q_b(2)$ hence, by [30, Corollary 7.2.9, page 82], we have that

$$\sigma_{2n-1}(\partial E) \le \sigma_{2n-1}(\partial Q_b(2));$$

but, because $\Omega_a \subset \Omega$ is convex,

$$(\partial\Omega_a\backslash\partial\Omega) \cap Q_b(2) \subset \partial E \Rightarrow \sigma_{2n-1}((\partial\Omega_a\backslash\partial\Omega) \cap Q_b(2)) \le \sigma_{2n-1}(\partial E).$$

We have

$$\partial Q_b(2)) = \bigcup_{j=1}^{n} (\partial D_j(b, d(b)^{1/m_j(b)}) \prod_{k\neq j, k=1}^{n} D_k(b, d(b)^{1/m_k(b)})),$$

where $D_j(b, r_j)$ is the disc of center b in the direction L_j given by the basis of the good family at the point $\pi(b)$. So

$$\begin{aligned}\sigma_{2n-1}(\partial Q_b(2)) &= \sum_{j=1}^{n} 2\pi d(b)^{1/m_j(b)} \prod_{k\neq j, k=1}^{n} \pi d(b)^{2/m_k(b)} \\ &\le 2\pi^n \sum_{j=1}^{n} d(b)^{2\mu(b)+2-1/m_j(b)}\end{aligned}$$

because $2\mu(b) = \sum_{k=2}^{n} \frac{2}{m_k}$; but $\forall j = 1, \dots, n,\ 2 - 1/m_j \ge 1$ and we can restrict ourself to b such that $d(b) \le 1$ because we need to test only with the b near the boundary. Hence

$$\sigma_{2n-1}(\partial Q_b(2)) \le 2\pi^n n d(b)^{1+2\mu(b)}.$$

So far we have

$$\begin{aligned}\sigma_{2n-1}((\partial\Omega_a\backslash\partial\Omega) \cap Q_b(2)) &\le \sigma_{2n-1}(\partial E) \le \sigma_{2n-1}(\partial Q_b(2)) \\ &\le 2\pi^n n d(b)^{1+2\mu(b)}.\end{aligned} \tag{9.1}$$

To get $d(b)^{1+2\mu(b)} \lesssim \sigma_{2n-1}(\partial\Omega \cap Q_b(2))$ we shall use Lemma 9.7. Set $k = 2n-1,\ U = Q_b(2) \cap T_\beta(\partial\Omega)$ where $\beta = \pi(b) \in \partial\Omega$ and $V = \partial\Omega \cap Q_b(2)$. For b uniformly near $\partial\Omega$, V is a graph over $U' := \pi(V) \subset U$, with π the orthogonal projection on the real tangent space $T_\beta(\partial\Omega)$, and we have by Lemma 9.7 that $\sigma_{2n-1}(V) \ge \sigma_{2n-1}(U')$ so it remains to estimate $\sigma_{2n-1}(U')$.

Recall that, by the definition of a good family, we have $Q_b(\delta_0) \subset \Omega$ hence $\pi(Q_b(\delta_0)) \subset \pi(\Omega \cap Q_b(2))$.

We apply Lemma 9.9 to Ω convex to get $\pi(Q_b(\delta_0)) \subset \pi(V) = U'$. So

$$\sigma_{2n-1}(U') \geq \sigma_{2n-1}(\pi(Q_b(\delta_0))).$$

Because the basis for Q_b is the basis at $\beta = \pi(b)$ and $T_\beta(\partial\Omega)$ is the real tangent space, the only missing direction is the real normal at β, hence we have

$$\sigma_{2n-1}(\pi(Q_b(\delta_0)) = \delta_0 d(b) \times \prod_{j=2}^{n} \delta_0^2 d(b)^{2/m_j(b)} = \delta_0^{2n+1} d(b)^{1+2\mu(b)}.$$

Finally we get

$$\delta_0^{2n+1} d(b)^{1+2\mu(b)} \leq \sigma_{2n-1}(V) = \sigma_{2n-1}(\partial\Omega \cap Q_b(2))$$

and by (9.1)

$$\sigma_{2n-1}((\partial\Omega_a \backslash \partial\Omega) \cap Q_b(2)) \leq 2\pi^n n d(b)^{1+2\mu(b)}$$

hence

$$\sigma_{2n-1}((\partial\Omega_a \backslash \partial\Omega) \cap Q_b(2)) \leq \frac{2\pi^n}{\delta_0^{2n+1}} n \sigma_{2n-1}(\partial\Omega \cap Q_b(2)),$$

which says precisely that the measure $d\sigma_{2n-1|\partial\Omega_a \backslash \partial\Omega}$ is a geometric Carleson measure in Ω. $\square$

In order to continue we shall need the easy remark:

Remark 9.10. For any smoothly bounded domain Ω we have the inequality

$$\forall f \in H^p(\Omega),\ \|f\|_{\mathcal{N}(\Omega)} \leq \sqrt[p']{\sigma_{2n-1}(\partial\Omega)} \|f\|_{H^p(\Omega)}.$$

Proof. We have $\ln^+ |f| \leq |f|$, so, with σ_ϵ the σ_{2n-1} Lebesgue measure on the manifold $r(z) = -\epsilon$,

$$\|f\|_{\mathcal{N}(\Omega)} := \sup_{\epsilon>0} \int_{r(z)=-\epsilon} \ln^+ |f(z)|\, d\sigma_\epsilon(z) \leq \sup_{\epsilon>0} \int_{r(z)=-\epsilon} |f(z)|\, d\sigma_\epsilon(z),$$

hence $\|f\|_{\mathcal{N}(\Omega)} \leq \|f\|_{H^1(\Omega)}$. But $\sigma(\partial\Omega)$ being finite, we have $\|f\|_{H^1(\Omega)} \leq \sqrt[p']{\sigma_{2n-1}(\partial\Omega)} \|f\|_{H^p(\Omega)}$. $\square$

Proposition 9.11. *If S is a H^p dual bounded sequence in a convex domain of finite type, then S is separated.*

Proof. The hypothesis on the sequence S implies that

$$\exists C > 0,\ \exists \rho_a \in H^p(\Omega) :: \|\rho_a\|_p \leq C,\ \langle \rho_a,\ k_b \rangle = 0,\ |\langle \rho_a,\ k_a \rangle| \gtrsim \|k_a\|_{p'};$$

then

$$\|k_a - k_b\|_{p'} \geq \left| \left\langle \frac{\rho_a}{\|\rho_a\|_p},\ k_a - k_b \right\rangle \right| \geq \frac{1}{C} |\langle \rho_a,\ k_a \rangle| \gtrsim \|k_a\|_{p'}. \tag{9.2}$$

Now for $\epsilon > 0$ we get the existence of γ such that, if we suppose that $b \in Q_a(t)$ for a $t < \gamma$,

$$|\langle \rho_a,\ k_a - k_b\rangle| \leq C\|k_a - k_b\|_{p'} \leq C\epsilon\|k_a\|_{p'}$$

and a contradiction with (9.2) if we choose ϵ small enough. $\square$

Theorem 9.12. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$. If the sequence of points $S \subset \Omega$ is dual bounded in $H^p(\Omega)$, then the measure $\nu := \sum_{a\in S} d(a)^n\delta_a$ is a geometric Carleson measure in Ω.*

Proof. We have to show that

$$\forall a \in \Omega,\ \nu(\Omega \cap Q_a(2)) = \sum_{b\in S\cap Q_a(2)} d(b)^n \leq C\sigma_{2n-1}(\partial\Omega \cap Q_a(2)).$$

Dual boundedness means that we have a sequence $\{\rho_a\}_{a\in S} \subset H^p(\Omega)$ such that

$$\forall a, b \in S,\ \rho_a(b) = \delta_{ab}\|k_a\|_{p'},\ \|\rho_a\|_p \leq C.$$

This implies that

$$\forall a \in S,\ \|\rho_a/\rho_a(a)\|_{H^p(\Omega)} \leq C\|k_a\|_{p'}^{-1}.$$

In the case of the unit ball [5] we used the automorphisms and a classical lemma by Garnett to pass from bounded measures to geometric Carleson ones. Here of course we have to overcome the lack of automorphisms.

Because S is dual bounded it is a separated sequence of points in Ω. Consider the sub-domain Ω_a associated to the point a, built in Section 8 and the sequence $S_a := S \cap \Omega_a \subset \Omega_a$.

Let $a \in S$ and $u := \rho_a/\rho_a(a)$; we have $u \in H^p(\Omega)$ by hypothesis. We notice that $S\backslash\{a\} \subset u^{-1}(0)$, and that $u(a) = 1$, so we get by Theorem 3.12, with $X = u^{-1}(0) \cap \Omega_a$ and Θ its (1, 1) current of integration,

$$\sum_{c\in S_a} d(c)^n \leq \Gamma(\Omega_a)\|\Theta\|_B,$$

where $\Gamma(\Omega_a)$ depends on the $\mathcal{C}^{M(\mathcal{Q})+1}$ norm of the defining function of Ω_a which, by construction of Ω_a, is controlled by the $\mathcal{C}^{M(\mathcal{Q})+1}$ norm of the defining function of Ω.

Now because Ω is convex of finite type, Ω_a is $\frac{\sqrt{5n}}{\delta_0}$ balanced by Lemma 8.3 with respect to a, hence by Theorem10.11 we get $\|\Theta\|_B \leq C\|u\|_{\mathcal{N}(\Omega_a)}$, the constant C depending only on Ω and not on a. So

$$\sum_{c\in S_a} d(c)^n \leq C\Gamma\|u\|_{\mathcal{N}(\Omega_a)},$$

and again the constants are independent of a.

By the Remark 9.10 we get

$$\sum_{c\in S_a} d(c)^n \leq C\Gamma \|u\|_{\mathcal{N}(\Omega_a)} \leq C\Gamma \sqrt[p']{\sigma_{2n-1}(\partial\Omega_a)}\|u\|_{H^p(\Omega_a)}.$$

Set $C(\Omega) := C\Gamma$ which depends only on Ω, we get

$$\sum_{c\in S_a} d(c)^n \leq C(\Omega) \sqrt[p']{\sigma_{2n-1}(\partial\Omega_a)}\|u\|_{H^p(\Omega_a)}.$$

The measure $d\sigma_{|\partial\Omega_a\backslash\partial\Omega}$ is a geometric Carleson measure in Ω by Lemma 9.6 hence by the embedding Carleson Theorem 7.7 we have

$$u \in H^p(\Omega_a) \ \text{ and } \ \|u\|_{H^p(\Omega_a)} \leq C\|u\|_{H^p(\Omega)},$$

with the constant C independent of a.

Hence

$$\forall a \in S, \ \sum_{c\in S_a} d(c)^n \leq C(\Omega) \sqrt[p']{\sigma_{2n-1}(\partial\Omega_a)}\|u\|_{H^p(\Omega_a)}.$$

The dual boundedness then gives, because $u := \rho_a/\rho_a(a)$,

$$\forall a \in S, \ \|u\|_{H^p(\Omega)} \lesssim \|k_a\|_{p'}^{-1},$$

hence

$$\forall a \in S, \ \sum_{c\in S_a} d(c)^n \leq C(\Omega)\|u\|_{H^p(\Omega)} \leq C(\Omega)\|k_a\|_{p'}^{-1},$$

where the (new) constant $C(\Omega)$ is still independent of a.

Finally the p regularity of Ω gives

$$\|k_a\|_{p'}^{-1} \lesssim (\sigma_{2n-1}(\partial\Omega \cap Q_a(2)))^{1/p},$$

hence

$$\begin{aligned}\forall a \in S, \ \sum_{c\in S_a} d(c)^n &\leq C(\Omega)(\sigma_{2n-1}(\partial\Omega_a))^{1/p'}\|k_a\|_{p'}^{-1} \\ &\leq C(\Omega)\sigma_{2n-1}(\partial\Omega \cap Q_a(2)),\end{aligned} \tag{9.3}$$

because we have that $\sigma_{2n-1}(\partial\Omega_a) \simeq \sigma_{2n-1}(\partial\Omega \cap Q_a(2))$, still with the constant independent of a.

So we have proved the correct inequality for a point $a \in S$. It remains to have it for any point in Ω.

Fix $b \in \Omega$; take a point $a_1 \in S \cap Q_b(2)$ such that $d(a_1) := d(a_1, \partial\Omega)$ is as big as possible. Now set $E_1 := Q_b(2)\backslash Q_{a_1}(2)$ and take a point $a_2 \in S \cap E_1$ such that $d(a_2) := d(a_2, \partial\Omega)$ is as big as possible; set $E_2 := E_1\backslash Q_{a_2}(2)$ and take a point

$a_3 \in S \cap E_2$ such that $d(a_3) := d(a_3,\ \partial\Omega)$ is as big as possible etc. In this way we have a sequence $G := \{a_j\}$ of points in $Q_b(2) \cap S$ with $d(a_j)$ decreasing.

Moreover we have

$$S \cap Q_b(2) = \bigcup_{j=1}^{\infty} S \cap Q_{a_j}(2).$$

For any $j = 1, \ldots$ we have, by 9.3

$$\sum_{c \in S_{a_j}} d(c)^n \lesssim \left\| k_{a_j} \right\|_{p'}^{-1} \leq C(\Omega)\sigma_{2n-1}(\partial\Omega \cap Q_{a_j}(2)).$$

Now define $P_a(\delta) := Q_a(\delta) \cap T_a(\partial\Omega)$, where $T_a(\partial\Omega)$ is the parallel hyperplane to the tangent to $\partial\Omega$ at α passing through a. We have that if $a \in \Omega \cap \mathcal{U}$ then $\sigma_{2n-1}(P_a(\delta)) \simeq \sigma_{2n-1}(\pi(P_a(\delta)))$, where the constants behind the sign $\simeq$ are independent of a, because the projection π is a diffeomorphism from $P_a(\delta)$ onto its image in $\partial\Omega$. Its jacobian J is still a smooth function, hence we have that $C = \|J\|_\infty$ is uniformly bounded by the compactness of $\partial\Omega$, and so is $\left\| J^{-1} \right\|_\infty$.

Because the sets $Q_{a_j}(\delta)$ are disjoint we get that

- $\pi(P_{a_j}(\delta))$ are disjoint and $\sigma_{2n-1}(P_{a_j}(\delta)) \simeq \sigma_{2n-1}(\pi(P_{a_j}(\delta)))$,
- $\sigma_{2n-1}(P_{a_j}(2)) \simeq \left(\frac{2}{\delta}\right)^{2n-1} \sigma_{2n-1}(P_{a_j}(\delta))$.

So

$$\sigma_{2n-1}(P_{a_j}(2)) \simeq \left(\frac{2}{\delta}\right)^{2n-1} \sigma_{2n-1}(\pi(P_{a_j}(\delta))).$$

We want to estimate

$$\sum_{c \in S_b} d(c)^n = \sum_{j=1}^{\infty} \sum_{c \in S_{a_j}} d(c)^n \lesssim \sum_{j=1}^{\infty} \sigma_{2n-1}(\partial\Omega \cap Q_{a_j}(2)),$$

but

$$\sum_{j=1}^{\infty} \sigma_{2n-1}(\pi(P_{a_j}(\delta))) \leq \sigma_{2n-1}(\partial\Omega \cap Q_b(2)),$$

because the $\pi(P_{a_j}(\delta))$ are disjoint and contained in $\partial\Omega \cap Q_b(2)$ and

$$\begin{aligned}\sum_{j=1}^{\infty} \sigma_{2n-1}(\pi(P_{a_j}(\delta))) &\gtrsim \sum_{j=1}^{\infty} \sigma_{2n-1}(P_{a_j}(\delta))) \\ &\gtrsim \sum_{j=1}^{\infty} \sigma_{2n-1}(P_{a_j}(2))) \gtrsim \sum_{j=1}^{\infty} \sigma_{2n-1}(\partial\Omega \cap Q_{a_j}(2)).\end{aligned}$$

So

$$\begin{aligned}\sum_{c \in S_b} d(c)^n &\lesssim \sum_{j=1}^{\infty} \sigma_{2n-1}(\partial\Omega \cap Q_{a_j}(2)) \lesssim \sum_{j=1}^{\infty} \sigma_{2n-1}(\pi(P_{a_j}(\delta))) \\ &\lesssim \sigma_{2n-1}(\partial\Omega \cap Q_b(2)).\end{aligned}$$

□

Theorem 9.13. *Let Ω be a convex domain of finite type in $\mathbb{C}^n$. If the sequence of points $S \subset \Omega$ is dual bounded in $H^p(\Omega)$, then the canonical measure $\nu := \sum_{a\in S} d(a)^{1+2\mu(a)}\delta_a$ is a geometric Carleson measure in Ω.*

Proof. We take advantage of the fact that a convex domain of finite type is **aspc** to separate the sequence S in two parts $S = B_S \cup G_S$. For the bad points we need not the hypothesis of dual boundedness because Theorem 4.6 gives

$$\sum_{c\in B_S\cap Q_a(2)} d(c)^{1+2\mu(c)} \leq C(\Omega)\frac{\sigma_{2n-1}(\partial\Omega\cap Q_a(2))}{\delta^2},$$

which is true for any $a \in \Omega$, and this is precisely the definition of a geometric Carleson measure, so we get that the measure $\nu_b := \sum_{a\in B_S} d(a)^{1+2\mu(a)}\delta_a$ is a geometric Carleson measure.

We have $\nu_g := \sum_{a\in G_S} d(a)^{1+2\mu(a)}\delta_a \leq \mu := \sum_{a\in S} d(a)^n\delta_a$, and μ is a geometric Carleson measure by Theorem 9.12. So adding ν_b and ν_g we get that ν is a geometric Carleson measure. ☐

9.3. Interpolating sequences

We shall need the definition

Definition 9.14. The sequence S is a *q Carleson sequence* if

$$\exists D>0,\ \forall\lambda\in\ell^q(S),\ \left\|\sum_{a\in S}\lambda_a\frac{k_a}{\|k_a\|_q}\right\|_q \leq D\|\lambda\|_{\ell^q(S)}.$$

In [2], we proved by duality that if the canonical measure $\nu := \sum_{a\in S} d(a)^{1+2\mu(a)}\delta_a$ is a q' Carleson measure and if $\|k_a\|_q^{-q'} \simeq d(a)^{1+2\mu(a)}$ then S is a q Carleson sequence. We shall do it again in this setting.

Lemma 9.15. *If Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $S \subset \Omega$ is a dual bounded sequence of points in $H^p(\Omega)$, then S is a q Carleson sequence for any $q \in]1,\ \infty[$.*

Proof. Because the Szegö projection is bounded on $L^p(\partial\Omega)$ for $1 < p < \infty$ ([29, Theorem 5.1]) we have that the dual of $H^p(\Omega)$ is $H^{p'}(\Omega)$, with p' the conjugate exponent of p. Hence we can evaluate the norm this way

$$\begin{aligned}\left\|\sum_{a\in S}\lambda_a\frac{k_a}{\|k_a\|_q}\right\|_q &\simeq \sup_{f\in H^{q'}(\Omega),\ \|f\|_{q'}\leq 1}\left|\sum_{a\in S}\lambda_a\frac{f(a)}{\|k_a\|_q}\right| \\ &\lesssim \|\lambda\|_{\ell^q(S)}\sup_{f\in H^{q'}(\Omega),\ \|f\|_{q'}\leq 1}\left|\sum_{a\in S}\frac{|f(a)|^{q'}}{\|k_a\|_q^{q'}}\right|^{1/q'},\end{aligned}$$

by Hölder. But

$$\|k_a\|_p = \|S(\cdot,\ a)\|_{H^p} \simeq \frac{1}{\sigma_{2n-1}(B(\alpha, d(a)))^{1/p'}}$$

by Theorem 6.6 and we have, by (6.10), $\sigma_{2n-1}(B(\alpha,\ \epsilon)) \simeq \epsilon \prod_{j=2}^{n} \tau_j(\alpha,\ \epsilon)^2$ and by (6.3) in Hefer's Theorem 6.1, we have $\tau_j(\zeta,\ \epsilon) \simeq \epsilon^{1/m_j(\zeta)}$, hence,

$$\sigma(B(\alpha,\ \epsilon)) \simeq \epsilon \prod_{j=2}^{n} \tau_j(\alpha,\ \epsilon)^2 \simeq \epsilon^{1+2\mu(\alpha)},\ \mu(\alpha) := \sum_{j=2}^{n} \frac{1}{m_j(\alpha)}.$$

We shall apply this with $\alpha = \pi(a),\ \epsilon = d(a)$ and, because $P_\epsilon(a) \cap P_\epsilon(\alpha) \neq \emptyset$, we have by (6.8)

$$\prod_{j=2}^{n} \tau_j(a, d(a))^2 \simeq \prod_{j=2}^{n} \tau_j(\alpha, d(a))^2 \Rightarrow d(a)^{2\mu(a)} \simeq d(a)^{2\mu(\alpha)}.$$

Putting this in $\|k_a\|_q$ we get

$$\|k_a\|_q^{-1} \simeq d(a)^{\frac{1+2\mu(\alpha)}{q'}} \Rightarrow \|k_a\|_q^{-q'} \simeq d(a)^{1+2\mu(\alpha)} \simeq d(a)^{1+2\mu(a)}.$$

Hence

$$\forall f \in H^{q'}(\Omega),\ \sum_{a\in S} \frac{|f(a)|^{q'}}{\|k_a\|_q^{q'}} \simeq \sum_{a\in S} d(a)^{1+2\mu(a)}\,|f(a)|^{q'}.$$

But Theorem 9.13 gives that the measure $\nu := \sum_{a\in S} d(a)^{1+2\mu(a)}\delta_a$ is a geometric Carleson measure. We apply the embedding Carleson Theorem 7.7 to ν to get

$$\forall q' > 1,\ \exists C_{q'} > 0,\ \forall f \in H^{q'}(\Omega),\ \int_\Omega |f|^{q'}\, d\nu \leq C_{q'}^{q'} \|f\|_{H^{q'}(\Omega)}^{q'},$$

explicitly

$$\forall f \in H^{q'}(\Omega),\ \sum_{a\in S} |d(a)|^{1+2\mu(a)}\,|f(a)|^{q'} \leq C_{q'}^{q'} \|f\|_{H^{q'}(\Omega)}^{q'},$$

hence

$$\sum_{a\in S} \frac{|f(a)|^{q'}}{\|k_a\|_q^{q'}} \lesssim \|f\|_{H^{q'}(\Omega)}^{q'}.$$

□

9.4. Structural hypotheses

We get easily the structural hypotheses [2] for the domain Ω.

Corollary 9.16. *If Ω is a convex domain of finite type in $\mathbb{C}^n$, then the structural hypotheses $\mathrm{SH}(q)$ and $\mathrm{SH}(p, s)$ are true for the Lebesgue measure σ_{2n-1} on $\partial\Omega$,* i.e. $\forall q \in]1, \infty[$,

$$\mathrm{SH}(q): \quad \|k_a\|_q \|k_a\|_{q'} \lesssim \|k_a\|_2^2,$$

and, for $\forall p, s \in [1, \infty]$, $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$,

$$\mathrm{SH}(p, s): \quad \|k_a\|_{s'} \lesssim \|k_a\|_{p'} \|k_a\|_{q'}.$$

Proof. Theorem 6.6 gives again

$$\|k_a\|_{H^p(\Omega)} = \|S(a, \cdot)\|_{H^p(\Omega)} \simeq \frac{1}{\sigma_{2n-1}(B(\alpha, d(a)))^{1/p'}}$$

hence, just replacing,

$$\|k_a\|_q \|k_a\|_{q'} \simeq \|k_a\|_2^2, \ \|k_a\|_{s'} \simeq \|k_a\|_{p'} \|k_a\|_{q'}. \qquad \square$$

Now we are in position to prove Theorem 1.15:

Theorem 9.17. *If Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $S \subset \Omega$ is a dual bounded sequence of points in $H^p(\Omega)$, if $p = \infty$ then for any $q < \infty$, S is $H^q(\Omega)$ interpolating with the linear extension property; if $p < \infty$ then S is $H^q(\Omega)$ interpolating with the linear extension property, provided that $q < \min(p,\ 2)$.*

Proof. We shall apply the main theorem from [2]: we state it in the special case of a domain $\Omega \subset \mathbb{C}^n$ and of the uniform algebra $A(\Omega)$ of holomorphic functions in Ω, continuous up to $\partial\Omega$

Theorem 9.18. *Let Ω be a domain in $\mathbb{C}^n$ with σ the Lebesgue measure on $\partial\Omega$; if we have, with $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$, that the measure σ verifies the structural hypotheses $\mathrm{SH}(q)$, $\mathrm{SH}(p,\ s)$;*

- *S is dual bounded in $H^p(\Omega)$;*
- *S is a q Carleson sequence;*

then S is $H^s(\Omega)$ interpolating and has the linear extension property, provided that either $p = \infty$ or $p \le 2$.

All the requirements of Theorem 9.18 are by now verified so we have that for any $q < p$, S is $H^q(\Omega)$ interpolating with the linear extension property, provided that $p = \infty$ or $p \le 2$. So if $p = \infty$ or $p \le 2$, the theorem is proved.

If $2 \leq p < \infty$, then S dual bounded in $H^p(\Omega)$ means $\exists\{\rho_a\}_{a\in S} \subset H^p(\Omega)$ with:

$$\exists C > 0 :: \forall a \in S,\ \|\rho_a\|_p \leq C,\ \forall a, b \in S,\ \langle \rho_a,\ k_b\rangle = \delta_{a,b}\|k_b\|_{p'}.$$

Let $s :: \frac{1}{2} = \frac{1}{p} + \frac{1}{s}$ then we set

$$\forall a \in S,\ \tilde{\rho}_a := \rho_a \times \frac{k_a}{\|k_a\|_s} \Rightarrow \|\tilde{\rho}_a\|_2 \leq C$$

and, by the reproducing property of k_a

$$\left\langle \tilde{\rho}_a,\ \frac{k_a}{\|k_a\|_2}\right\rangle = \rho_a(a) \times \frac{k_a(a)}{\|k_a\|_s} \times \frac{1}{\|k_a\|_2},$$

but $k_a(a) = \|k_a\|_2^2$, and $\rho_a(a) = \|k_a\|_{p'}$ by definition, hence

$$\left\langle \tilde{\rho}_a,\ \frac{k_a}{\|k_a\|_2}\right\rangle = \frac{\|k_a\|_{p'} \times \|k_a\|_2}{\|k_a\|_s}.$$

The structural hypotheses, by Corollary 9.16, gives

$$\mathrm{SH}(s): \quad \|k_a\|_s\|k_a\|_{s'} \lesssim \|k_a\|_2^2$$

and

$$\forall p, s \in [1, \infty],\ \frac{1}{s} = \frac{1}{p} + \frac{1}{q},\ \mathrm{SH}(p, s): \quad \|k_a\|_{s'} \lesssim \|k_a\|_{p'}\|k_a\|_{q'},$$

hence here, with the correct values of p, s

$$\|k_a\|_2 \lesssim \|k_a\|_{p'}\|k_a\|_{s'} \leq \|k_a\|_{p'} \times \frac{\|k_a\|_2^2}{\|k_a\|_s},$$

the last inequality by $\mathrm{SH}(s)$, hence

$$1 \lesssim \frac{\|k_a\|_{p'}\|k_a\|_2}{\|k_a\|_s} = \left\langle \tilde{\rho}_a,\ \frac{k_a}{\|k_a\|_2}\right\rangle.$$

So we have that S is dual bounded in $H^2(\Omega)$ and by Theorem 9.18 we have that if $\forall q < 2$ then S is $H^q(\Omega)$ interpolating with the linear extension property. $\square$

Remark 9.19. The slight improvement from Theorem 9.18 done here relies only on the structural hypotheses, so it is in fact true in the abstract setting of uniform algebras.

10. Potential

Let us recall quickly how Green formula gives us the Blaschke condition [33]:

$$\ln|u(p)| = \int_{\partial\Omega} \ln|u(\zeta)|\, P(p,\zeta)\, d\sigma(\zeta) + \int_{\Omega} \Delta \ln|u(z)|\, G(p,z)\, dm(z),$$

where $p \in \Omega,\ G(p,z)$ is the Green kernel of Ω with pole at p and $P(p,\zeta)$ is the Poisson kernel of Ω still with pole at p.

Let $p \in \Omega$ fixed such that $u(p) \neq 0$ and we suppose u normalized to have

$$u(p) = 1 \Rightarrow \ln|u(p)| = 0. \tag{10.1}$$

Taking the positive Green function (minus the usual one) we have $G \geq 0,\ 0 \leq P(p,\zeta) \leq \|P(p,\ \cdot)\|_\infty$, and we get

$$\int_\Omega \Delta \ln|u(z)|\, G(p,z)\, dm(z) = \int_{\partial\Omega} \ln^+|u|\, P(p,\zeta)\, d\sigma - \int_{\partial\Omega} \ln^-|u|\, P(p,\zeta)\, d\sigma,$$

$$\begin{aligned}\int_\Omega \Delta \ln|u(z)|\, G(p,z)\, dm(z) &\leq \int_{\partial\Omega} \ln^+|u(\zeta)|\, P(p,\zeta)\, d\sigma \\ &\leq \|P(p,\ \cdot)\|_\infty \int_{\partial\Omega} \ln^+|u(\zeta)|\ d\sigma(\zeta) \\ &\leq \|P(p,\ \cdot)\|_\infty \|u\|_{\mathcal{N}}.\end{aligned}$$

But $\Delta \ln|u(z)| = \operatorname{Tr}\Theta$, the trace of Θ, so

$$\int_\Omega G(p,z) \operatorname{Tr}\Theta(z) dm(z) \leq \|P(p,\ \cdot)\|_\infty \int_{\partial\Omega} \ln^+|u(\zeta)|\ d\sigma(\zeta). \tag{10.2}$$

We have the known estimates ([8, Proposition 2.1]):

Proposition 10.1. *Let $\Omega := \{x \in \mathbb{R}^N :: \rho(x) < 0\}$ be a bounded domain of class $\mathcal{C}^2$ in $\mathbb{R}^N$, defined by the function ρ and $a \in \Omega$ then there are constants $c,\ c_1, c_2$, depending only on the regularity of ρ up to second order, such that, with P the Poisson kernel of Ω, with $d(x)$ the distance from x to $\partial\Omega$,*

$$\forall (x,\zeta) \in \Omega \times \partial\Omega,\ c_1 \frac{d(x)}{|\zeta - x|^N} \leq P(x,\zeta) \leq c_2 \frac{d(x)}{|\zeta - x|^N}.$$

For the Green function $G(x,z)$ of Ω we have,

$$\forall (x,z) \in \Omega \times \Omega,\ G(x,z) \geq c \frac{d(z)d(x)}{|z - x|^N}.$$

Using Proposition 10.1 we get

Theorem 10.2. *Let* $\Omega := \{z \in \mathbb{C}^n :: \rho(x) < 0\}$ *be a bounded domain of class* $\mathcal{C}^2$ *in* $\mathbb{C}^n$*; let* $G(p, z)$ *be the positive Green function (minus the usual one) with pole* $p \in \Omega$ *and* u *be a holomorphic function in* Ω *such that* $u(p) = 1$*, then we have*

$$\int_\Omega d(z)\mathrm{Tr}\Theta(z)dm(z) \leq C\frac{R^{2n}}{r^{2n}}\int_{\partial\Omega} \ln^+ |u(\zeta)|\; d\sigma(\zeta)$$

where r *is the radius of the biggest ball* $B(p, r)$ *centered at* p *and contained in* Ω *and* R *is the radius of the smallest ball* $B(p, R)$ *centered at* p *and containing* Ω*. The constant* C *depends only on the regularity of* ρ *up to second order.*

Proof. We have by Proposition 10.1, $\|P(p,\cdot)\|_\infty \leq c_2 d(p)^{-2n+1}$ and $G(p, z) \geq c\frac{d(z)d(p)}{|z-p|^{2n}}$ and, because $d(p) = r$ and $|z - p| \leq R$, we get $\|P(p,\cdot)\|_\infty \leq c_2 r^{-2n+1}$, $G(p, z) \geq c\frac{d(z)r}{R^{2n}}$ so, putting this in (10.2), we get

$$c\frac{r}{R^{2n}}\int_\Omega d(z)\mathrm{Tr}\Theta(z)dm(z) \leq \frac{c_2}{r^{2n-1}}\int_{\partial\Omega} \ln^+ |u(\zeta)|\; d\sigma(\zeta),$$

which gives the theorem with $C := \frac{c_2}{c}$. □

Setting $\|u\|_{\mathcal{N}(\Omega)} := \int_{\partial\Omega} \ln^+ |u(\zeta)|\; d\sigma(\zeta)$, the Nevanlinna norm of u, this proves that the zero set of a function in the Nevanlinna class verifies the Blaschke condition.

Unfortunately the domains Ω_a we are interested in have not the euclidean ball property that $\frac{R}{r} \leq \gamma$ with a γ independent of a; in fact they have it but for complex planes slices of Ω_a with r, R depending on the slices but still with $\frac{R}{r} \leq \gamma$, γ independent of the slice, *i.e.* they have this type of property but for "ellipsoid" instead of balls. This is why the proofs are a little bit more involved.

10.1. Complex potential theory

In this section we shall use the notation $dm := d\sigma_{2n}$ for the Lebesgue measure in $\mathbb{C}^n$ and $d\sigma$ for $d\sigma_{2n-1}$.

Let Ω be a domain in $\mathbb{C}^n = \mathbb{R}^{2n}$, and $u \in \mathcal{N}(\Omega)$ a holomorphic function in the Nevanlinna class of Ω. With $\Theta := \Delta \ln |u|$ and ρ be a defining function for Ω, we have the lemma, application of the Green formula,

Lemma 10.3. *We have, with* η *the outward normal to* $\partial\Omega$,

$$\int_\Omega (-\rho)\mathrm{Tr}\Theta dm = \int_{\partial\Omega} \ln |u| \frac{\partial\rho}{\partial\eta} d\sigma - \int_\Omega \ln |u|\; \Delta\rho dm.$$

Proof. We have, by the Green formula,

$$\int_\Omega \rho\Delta v dm - \int_\Omega v\Delta\rho dm = \int_{\partial\Omega} \rho\frac{\partial v}{\partial\eta}d\sigma - \int_{\partial\Omega} v\frac{\partial\rho}{\partial\eta}d\sigma;$$

but $\rho = 0$ on $\partial\Omega$ and changing sign, we get

$$\int_\Omega (-\rho)\Delta v dm = -\int_\Omega v\Delta\rho dm + \int_{\partial\Omega} v\frac{\partial\rho}{\partial\eta}d\sigma.$$

Now setting $v = \ln|u|$ and approximating $\ln|u|$ by smooth functions as usual, we get the lemma. □

The aim is to prove, under some circumstances, that we have

$$\int_\Omega (-\rho)\mathrm{Tr}\Theta \leq C\int_{\partial\Omega} \ln^+|u|\, d\sigma,$$

with a *good control* on C.

Definition 10.4. Let $\mathbb{S}$ be the unit sphere in $\mathbb{C}^n$ and Ω a domain in $\mathbb{C}^n$, $0 \in \Omega$. We shall say that Ω si $\mathcal{C}^1$ *starlike relatively to* 0 if $\partial\Omega$ admits a spherical parametrization, *i.e*. there is a function $R(\zeta) \in \mathcal{C}^1(\mathbb{S})$, $R(\zeta) > 0$, such that:

$$\partial\Omega = \{z \in \mathbb{C}^n :: \exists\zeta \in \mathbb{S},\ z = R(\zeta)\zeta\}.$$

This implies that $\Omega = \{z = tR(\zeta)\zeta,\ \zeta \in \mathbb{S},\ t \in [0,\ 1[\}$.

Let $\zeta \in \mathbb{S}$ and define $\partial\Omega_\zeta$ to be the complex plane slice through ζ

$$\partial\Omega_\zeta := \{R(e^{i\theta}\zeta)e^{i\theta}\zeta,\ \theta \in [0, 2\pi]\}.$$

The Lebesgue measure $d\sigma_{\partial\Omega_\zeta}(\eta)$, $\eta = R(e^{i\theta}\zeta)e^{i\theta}\zeta$, on $\partial\Omega_\zeta$ and $d\theta$ on $[0, 2\pi]$ are related by

$$d\sigma_{\partial\Omega_\zeta}(\eta) = \sqrt{\left|U'_\zeta(\theta)\right|^2 + U_\zeta(\theta)^2}d\theta,$$

where $U_\zeta(\theta) := R(e^{i\theta}\zeta)$. Of course if $\eta = e^{i\varphi}\zeta$ then $\partial\Omega_\eta = \partial\Omega_\zeta$ and the measure is the same.

We set $\left\|U'_\zeta\right\|_\infty := \sup_{\theta\in[0,2\pi]}\left|U'_\zeta(\theta)\right|$. We shall use the notation

$$\forall\zeta \in \mathbb{S},\ d_\zeta(0) = \inf_{\theta\in[0,2\pi]} R(e^{i\theta}\zeta);\ d_{\zeta\,\max}(0) = \sup_{\theta\in[0,2\pi]} R(e^{i\theta}\zeta).$$

Now we have.

Lemma 10.5. *Let $\mathbb{S}$ be the unit sphere of $\mathbb{C}^n$ and Ω a domain in $\mathbb{C}^n$, $0 \in \Omega$ such Ω is $\mathcal{C}^1$ starlike with respect to* 0. *Then*

$$\int_\Omega f(z)dm(z) = c_n \frac{1}{2\pi} \int_0^1 \left(\int_{\mathbb{S}} \left(\int_{\partial\Omega_\zeta} f(t\eta) J_\zeta(\eta) t^{2n-1} d\sigma_\zeta(\eta) \right) d\sigma_{\mathbb{S}}(\zeta) \right) dt,$$

with $\frac{d_\zeta(0)^{2n}}{\sqrt{\left\| U'_\zeta \right\|_\infty^2 + d_{\zeta\,\max}(0)^2}} \leq J_\zeta(\eta) \leq \frac{d_{\zeta\,\max}(0)^{2n}}{d_\zeta(0)}.$

Proof. Integrating in spherical coordinates we get, with $c_n = 2nv_n/s_n$ where v_n is the volume of the unit ball in $\mathbb{C}^n$, s_n the area of the unit sphere in $\mathbb{C}^n$,

$$I := \int_\Omega f(z)dm(z) = c_n \int_{\mathbb{S}} \{ \int_0^{R(\zeta)} r^{2n-1} f(r\zeta)dr \} d\sigma_{2n-1}(\zeta).$$

Set $t = \frac{r}{R(\zeta)} \Rightarrow dr = R(\zeta)dt$ and

$$\forall \zeta \in \mathbb{S},\ \int_0^{R(\zeta)} r^{2n-1} f(r\zeta)dr = \int_0^1 R(\zeta)^{2n} f(tR(\zeta)\zeta) t^{2n-1} dt,$$
$$I = c_n \frac{1}{2\pi} \int_{\mathbb{S}\times[0,1]\times[0,2\pi]} R(e^{i\theta}\zeta)^{2n} f(tR(e^{i\theta}\zeta)e^{i\theta}\zeta) t^{2n-1} dt d\theta d\sigma_{2n-1}(\zeta).$$

Now we fix $\zeta \in \mathbb{S}$; we get

$$\int_{[0,1]\times[0,2\pi]} R(e^{i\theta}\zeta)^{2n} f(tR(e^{i\theta}\zeta)e^{i\theta}\zeta) t^{2n-1} dt d\theta.$$

Set $\zeta \in \mathbb{S},\ \forall \theta \in [0, 2\pi],\ \eta = R(e^{i\theta}\zeta)e^{i\theta}\zeta \in \partial\Omega_\zeta$ and $U_\zeta(\theta) := R(e^{i\theta}\zeta)$ then we have

$$\partial\Omega_\zeta = \{U_\zeta(\theta)e^{i\theta}\zeta,\ \theta \in [0, 2\pi]\}$$

and

$$d\sigma_{\partial\Omega_\zeta}(\eta) = \sqrt{\left| U'_\zeta(\theta) \right|^2 + U_\zeta(\theta)^2} d\theta,$$

so

$$\int_{[0,2\pi]} R(e^{i\theta}\zeta)^{2n} f(tR(e^{i\theta}\zeta)e^{i\theta}\zeta) d\theta = \int_{\partial\Omega_\zeta} f(t\eta) J_\zeta(\eta) d\sigma_\zeta(\eta),$$

where $J_\zeta(\eta) = \frac{R(\eta)^{2n}}{D_\zeta(\eta)}$ and $D_\zeta(\eta) = \sqrt{\left| U'_\zeta(\theta) \right|^2 + U_\zeta(\theta)^2}$, expressed in η coordinates.

So we have

$$I = c_n \frac{1}{2\pi} \int_0^1 \left(\int_{\mathbb{S}} \left(\int_{\partial\Omega_\zeta} f(t\eta) J_\zeta(\eta) t^{2n-1} d\sigma_\zeta(\eta) \right) d\sigma_{\mathbb{S}}(\zeta) \right) dt.$$

Notice that in $\partial\Omega_\zeta$ we have

$$d_\zeta(0) \leq R(\eta) \leq d_{\zeta\,\max}(0)\ ;\ d_\zeta(0) \leq D_\zeta(\eta) \leq \sqrt{\left\| U'_\zeta \right\|_\infty^2 + d_{\zeta\,\max}(0)^2}$$

hence

$$\frac{d_\zeta(0)^{2n}}{\sqrt{\left\| U'_\zeta \right\|_\infty^2 + d_{\zeta\,\max}(0)^2}} \leq J_\zeta(\eta) \leq \frac{d_{\zeta\,\max}(0)^{2n}}{d_\zeta(0)}. \qquad \square$$

Definition 10.6. The domain $\Omega \subset \mathbb{C}^n$ is said to be γ *balanced relatively to* 0 if:

- Ω is $\mathcal{C}^1$ starlike with respect to 0,
- all its slices $\partial\Omega_\zeta$ through the origin verify

$$\forall \zeta \in \mathbb{S},\ d_{\zeta\,\max}(0) \leq \gamma d_\zeta(0);\quad \left\| U'_\zeta \right\|_\infty \leq \gamma d_{\zeta\,\max}(0).$$

Set for any function v, $v^+(z) := \max(v(z), 0)\ ;\ v^-(z) := -\max(-v(z), 0)$. Then we have the lemmas.

Lemma 10.7. *Suppose that v is a subharmonic function in a γ balanced domain D in $\mathbb{C}$, such that $v(0) = 0$. Then*

$$\int_{\partial D} v^-(z) d\sigma(z) \leq \gamma^2 \frac{c_2}{c_1} \int_{\partial D} v^+(z) d\sigma(z).$$

Proof. Because v is subharmonic we have

$$0 = v(0) \leq \int_{\partial D} P(0,\zeta) v(\zeta) d\sigma(\zeta) = \int_{\partial D} P(0,\zeta) v^+(\zeta) d\sigma(\zeta) - \int_{\partial D} P(0,\zeta) v^-(\zeta) d\sigma(\zeta),$$

where $P(0, \zeta)$ is the Poisson kernel of D for $0 \in D$. So

$$\int_{\partial D} P(0, \zeta) v^-(\zeta) d\sigma(\zeta) \leq \int_{\partial D} P(0, \zeta) v^+(\zeta) d\sigma(\zeta).$$

Now we use the estimates in Proposition 10.1 $\frac{c_1 d(0)}{d_{\max}(0)^2} \leq \frac{c_1 d(0)}{|\zeta|^2} \leq P(0, \zeta) \leq \frac{c_2}{|\zeta|} \leq \frac{c_2}{d(0)}$ to get

$$\frac{c_1 d(0)}{d_{\max}(0)^2} \int_{\partial D} v^-(\zeta) d\sigma(\zeta) \leq \frac{c_2}{d(0)} \int_{\partial D} v^+(\zeta) d\sigma(\zeta),$$

hence

$$\int_{\partial D} v^-(\zeta) d\sigma(\zeta) \leq \frac{c_2 d_{\max}(0)^2}{c_1 d(0)^2} \int_{\partial D} v^+(\zeta) d\sigma(\zeta) \leq \gamma^2 \frac{c_2}{c_1} \int_{\partial D} v^+(\zeta) d\sigma(\zeta). \qquad \square$$

Lemma 10.8. *Let $\Omega \subset \mathbb{C}^n$ be a γ balanced domain and let v be a pluri subharmonic function in Ω such that $v(0) = 0$, then*

$$\int_\Omega |v(z)|\, dm(z) \leq \left(1 + 2\gamma^{2n+3}\frac{c_2}{c_1}\right) \int_\Omega v^+(z) dm(z).$$

Proof. We shall use the decomposition of Lemma 10.5

$$\int_\Omega v^-(z)dm(z) = c_n \frac{1}{2\pi} \int_0^1 \left(\int_{\mathbb{S}} \left(\int_{\partial\Omega_\zeta} v^-(t\eta) J_\zeta(\eta) t^{2n-1} d\sigma_\zeta(\eta) \right) d\sigma_{\mathbb{S}}(\zeta) \right) dt.$$

But still by Lemma 10.5 we have $J_\zeta(\eta) \leq \frac{d_{\zeta\,\max}(0)^{2n}}{d_\zeta(0)}$ hence

$$\int_{\partial\Omega_\zeta} v^-(t\eta) J_\zeta(\eta) d\sigma_\zeta(\eta) \leq \frac{d_{\zeta\,\max}(0)^{2n}}{d_\zeta(0)} \int_{\partial\Omega_\zeta} v^-(t\eta) d\sigma_\zeta(\eta).$$

Doing the same we get

$$\int_{\partial\Omega_\zeta} v^+(t\eta) J_\zeta(\eta) d\sigma_\zeta(\eta) \geq \frac{d_\zeta(0)^{2n}}{\sqrt{\left\| U'_\zeta \right\|_\infty^2 + d_{\zeta\,\max}(0)^2}} \int_{\partial\Omega_\zeta} v^+(t\eta) d\sigma_\zeta(\eta).$$

Set

$$A := \frac{d_{\zeta\,\max}(0)^{2n}}{d_\zeta(0)} \;;\; B := \frac{d_\zeta(0)^{2n}}{\sqrt{\left\| U'_\zeta \right\|_\infty^2 + d_{\zeta\,\max}(0)^2}}$$

then

$$\int_{\partial\Omega_\zeta} v^-(t\eta) J_\zeta(\eta) d\sigma_\zeta(\eta) \leq A \int_{\partial\Omega_\zeta} v^-(t\eta) d\sigma_\zeta(\eta)$$

and

$$\int_{\partial\Omega_\zeta} v^+(t\eta) J_\zeta(\eta) d\sigma_\zeta(\eta) \geq B \int_{\partial\Omega_\zeta} v^+(t\eta) d\sigma_\zeta(\eta).$$

But Lemma 10.8 gives, because v being pluri subharmonic in Ω is subharmonic in Ω_ζ,

$$\int_{\partial\Omega_\zeta} v^-(t\eta) d\sigma_\zeta(\eta) \leq \gamma^2 \frac{c_2}{c_1} \int_{\partial\Omega_\zeta} v^+(t\eta) d\sigma_\zeta(\eta),$$

so

$$\int_{\partial\Omega_\zeta} v^-(t\eta) J_\zeta(\eta) d\sigma_\zeta(\eta) \leq A \int_{\partial\Omega_\zeta} v^-(t\eta) d\sigma_\zeta(\eta) \leq A\gamma^2 \frac{c_2}{c_1} \int_{\partial\Omega_\zeta} v^+(t\eta) d\sigma_\zeta(\eta),$$

hence continuing

$$\int_{\partial\Omega_\zeta} v^-(t\eta)J_\zeta(\eta)d\sigma_\zeta(\eta) \leq A\gamma^2\frac{c_2}{c_1}\int_{\partial\Omega_\zeta} v^+(t\eta)d\sigma_\zeta(\eta)$$
$$\leq \frac{Ac_2}{Bc_1}\gamma^2\int_{\partial\Omega_\zeta} v^+(t\eta)J_\zeta(\eta)d\sigma_\zeta(\eta).$$

So

$$\int_{\partial\Omega_\zeta} v^-(t\eta)J_\zeta(\eta)d\sigma_\zeta(\eta) \leq \frac{Ac_2}{Bc_1}\gamma^2\int_{\partial\Omega_\zeta} v^+(t\eta)J_\zeta(\eta)d\sigma_\zeta(\eta).$$

Now we notice that

$$\frac{A}{B} = \frac{d_{\zeta\max}(0)^{2n}\sqrt{\left\|U'_\zeta\right\|_\infty^2 + d_{\zeta\max}(0)^2}}{d_\zeta(0)^{2n+1}}$$
$$\leq \gamma^{2n+1}\sqrt{1+\frac{\left\|U'_\zeta\right\|_\infty^2}{d_{\zeta\max}(0)^2}} \leq \gamma^{2n+1}\sqrt{\gamma^2+1}.$$

Multiplying by t^{2n-1} and integrating on $\mathbb{S}\times[0,1]$ give

$$\int_\Omega v^-(z)dm(z) \leq 2\gamma^{2n+3}\frac{c_2}{c_1}\int_\Omega v^+(z)dm(z);$$

but $|v(z)| = v^+(z) + v^-(z)$ hence

$$\int_\Omega |v(z)|\, dm(z) \leq \left(1+2\gamma^{2n+3}\frac{c_2}{c_1}\right)\int_\Omega v^+(z)dm(z). \qquad \square$$

Lemma 10.9. *Let Ω be a domain in $\mathbb{C}^n$ of class $\mathcal{C}^2$. If v is a positive subharmonic function in Ω, then*

$$\int_\Omega v(z)dm(z) \leq 2c_2\mathrm{diam}(\Omega)\int_{\partial\Omega} v(\zeta)d\sigma(\zeta).$$

Proof. Let $P(z,\zeta)$ the Poisson kernel we have, by Proposition 10.1,

$$\forall(z,\zeta)\in\Omega\times\partial\Omega,\ c_1\frac{d(z)}{|\zeta-z|^{2n}} \leq P(z,\zeta) \leq c_2\frac{d(z)}{|\zeta-z|^{2n}},$$

so, because $d(x) \leq |\zeta - x|$, we get

$$\forall(z,\zeta)\in\Omega\times\partial\Omega,\ P(z,\zeta) \leq c_2\frac{1}{|\zeta-z|^{2n-1}}.$$

Hence

$$\forall \zeta \in \partial\Omega,\ \int_{\Omega} P(z,\zeta)dm(z) \leq c_2 \int_{\Omega} \frac{dm(z)}{|\zeta - z|^{2n-1}}$$
$$\leq c_2 \int_{B(0,\ \mathrm{diam}(\Omega))} \frac{dm(z)}{|\zeta - z|^{2n-1}} \leq 2c_2 \mathrm{diam}(\Omega).$$

Because v is subharmonic we have

$$v(z) \leq \int_{\partial\Omega} P(z,\zeta)v(\zeta)d\sigma(\zeta),$$

so, by Fubini-Tonnelli, everything being positive,

$$\int_{\Omega} v(z)dm(z) \leq \int_{\Omega\times\partial\Omega} P(z,\zeta)v(\zeta)dm(z)d\sigma(\zeta)$$
$$\leq 2c_2\mathrm{diam}(\Omega) \int_{\partial\Omega} v(\zeta)d\sigma(\zeta). \qquad \square$$

Proposition 10.10. *Let Ω be a domain in $\mathbb{C}^n$ of class $\mathcal{C}^2$, γ balanced relatively to $0 \in \Omega$; if v is pluri subharmonic in Ω and $v(0) = 0$ then*

$$\int_{\Omega} |v(z)|\, dm(z) \leq 2c_2\mathrm{diam}(\Omega) \left(2\gamma^{2n+3}\frac{c_2}{c_1} + 1\right) \int_{\partial\Omega} v^+(\zeta)d\sigma(\zeta).$$

Proof. We apply successively Lemma 10.8 and Lemma 10.9, which can be done because v^+ is still pluri subharmonic in Ω. $\square$

Theorem 10.11. *Let Ω be a domain in $\mathbb{C}^n$ of class $\mathcal{C}^2$, γ balanced relatively to $0 \in \Omega$; if u is holomorphic in Ω and $|u(0)| = 1$ then, with $X := u^{-1}(0)$,*

$$\|\Theta_X\|_B := \int_{\Omega} d(z)\mathrm{Tr}\Theta \leq C \int_{\partial\Omega} \ln^+ |u|\, d\sigma(\zeta) =: C\|u\|_{\mathcal{N}(\Omega)},$$

with a constant C depending only on the constant γ and the derivatives of ρ up to order 2.

Proof. By Lemma 10.3 we have

$$\int_{\Omega} (-\rho)\mathrm{Tr}\Theta = \int_{\partial\Omega} \ln|u| \frac{\partial\rho}{\partial\eta} d\sigma - \int_{\Omega} \ln|u|\, \Delta\rho dm.$$

The function $\ln|u|$ is pluri subharmonic in Ω hence we can apply to it Proposition 10.10:

$$\int_{\Omega} |\ln|u(z)||\, dm(z) \leq 2c_2\mathrm{diam}(\Omega) \left(2\gamma^{2n+3}\frac{c_2}{c_1} + 1\right) \int_{\partial\Omega} \ln^+ |u(\zeta)|\, d\sigma(\zeta).$$

so we get, because $0 < \frac{\partial\rho}{\partial\eta}$,

$$\begin{aligned}\int_\Omega (-\rho)\mathrm{Tr}\Theta &= \int_{\partial\Omega} \ln|u|\,\frac{\partial\rho}{\partial\eta}d\sigma - \int_\Omega \ln|u|\,\Delta\rho dm\\ &\leq \left\|\frac{\partial\rho}{\partial\eta}\right\|_\infty \int_{\partial\Omega} \ln^+|u|\,d\sigma + \|\Delta\rho\|_\infty \int_\Omega |\ln|u(z)||\,dm(z)\\ &\leq A\int_{\partial\Omega} \ln^+|u(\zeta)|\,d\sigma(\zeta),\end{aligned}$$

with $A := \left\|\frac{\partial\rho}{\partial\eta}\right\|_\infty + \|\Delta\rho\|_\infty(2c_2\mathrm{diam}(\Omega)(2\gamma^{2n+3}\frac{c_2}{c_1}+1))$.

But $\frac{\partial\rho}{\partial\eta}(z)d(z) \simeq (-\rho(z))$ so, with $M := \left\|\frac{1}{\partial\rho/\partial\eta}\right\|_\infty$, we get

$$\int_\Omega d(z)\mathrm{Tr}\Theta \leq M\int_\Omega (-\rho)\mathrm{Tr}\Theta.$$

This proves the theorem with $C = MA$, a constant depending only on γ and the derivatives of ρ up to order 2. □

Remark 10.12. This theorem will be applied to the domains Ω_a built in Section 8 and for these domains the derivatives of the defining function ρ_a are controlled by the derivatives of the global function ρ; also the derivative $\frac{\partial\rho_a}{\partial\eta}$ is bounded below uniformly independently of a by $\frac{\partial\rho}{\partial\eta}$ so the constant C of Theorem 10.11 is independent of a.

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