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Intrinsic co-local weak derivatives and Sobolev spaces between manifolds

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Abstract. We define the notion of co-locally weakly differentiable maps from a manifold M to a manifold N. If $p \geq 1$ and if the manifolds M and N are endowed with a Riemannian metric, this allows us to define intrinsically the homogeneous Sobolev space $\dot{W}^{1,p}(M, N)$. This new definition is equivalent to the definition by embedding in the Euclidean space and to that of Sobolev map into a metric space. The co-local weak derivative is an approximate derivative. The co-local weak differentiability is stable under a suitable weak convergence. The Sobolev spaces can be endowed with various intrinsic distances that induce the same topology and for which the space is complete.

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Introduction

Sobolev spaces between manifolds are a natural tool to study variational problems for maps between manifolds, arising in geometry [22,45,54] or in non-linear physical models [8,15,37].

A Sobolev space of maps between the manifolds M and N can be defined for every $p \in [1, \infty)$ by [8,11,15,18,22,27–30,34,36,37,46],

$$\dot{W}^{1,p}(M, N) = \{u : M \to N : \iota \circ u \in W^{1,1}_{\mathrm{loc}}(M, \mathbb{R}^\nu) \text{ and } |D(\iota \circ u)| \in L^p(M)\} \quad (0.1)$$

where $\iota : N \to \mathbb{R}^\nu$ is an isometric embedding of the target manifold N in the Euclidean space $\mathbb{R}^\nu$. This definition is always possible, since every Riemannian manifold is isometrically embedded in a Euclidean space [49, Theorem 2] and [50, Theorem 3]. Since the embedding $\iota : N \to \mathbb{R}^\nu$ is not unique, this definition could in principle depend on the choice of the embedding ι.

This difficulty can be avoided by defining Sobolev spaces into N by using only the metric structure of N, either by composition with Lipschitz maps, [2,34,51]

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or by oscillations on balls [40,43]. These definitions are equivalent to each other [17,40] and equivalent to the definition by isometric embedding (0.1) [34, Theorem 3.2], [35, Theorem 2.17]. They are all intrinsic but they do not have any notion of weak derivative; they only provide a notion of Dirichlet integrand $|Du|$ which might differ from the one given by isometric embedding and might depend on the integrability exponent p [17]. If the manifold N has a Riemannian structure, then an approximate derivative has been constructed a posteriori almost everywhere on M [26]; in contrast with the classical theory of Sobolev spaces between Euclidean spaces the derivative is a fine property of a function that plays no role in the definition of the Sobolev maps. Several distances have been proposed for spaces of Sobolev maps between metric spaces, but the spaces are not complete for any of these distances [17].

The goal of this work is to propose a robust intrinsic definition of Sobolev maps between manifolds in which the weak derivative plays a central role and to endow with well-behaved intrinsic metrics the space of Sobolev maps. We shall proceed in three steps: first we shall define a notion of differentiability and derivative, then we shall study the integrability of the derivative and finally we shall endow these spaces with convergence and metrics. Each of these steps will require additional structure on the manifolds: at the beginning we shall simply use the differentiable structure of the manifolds, then a Riemannian metric on the manifolds and finally a Riemannian metric on their tangent bundles. Defining the derivative before the space gives immediately the independence of the derivative from the Riemannian metric or the integrability exponent p. The primary role of the derivative in our approach will be quite handy to define complete intrinsic metrics.

In the first step we define *co-locally weakly differentiable maps* as maps $u : M \to N$ for which $f \circ u$ is weakly differentiable when $f \in C^1_c(N, \mathbb{R})$ (Definition 1.1). The *co-local weak derivative* is defined as the unique morphism of bundles Du such that the chain rule $D(f \circ u) = Df \circ Du$ holds (Definition 1.2):

$$\begin{array}{ccc} TM & \xrightarrow{Du} & TN \\ & \searrow^{D(f\circ u)} & \downarrow{\scriptstyle Df} \\ & & \mathbb{R}. \end{array}$$

The co-local weak derivative has the usual non-linear properties of a weak derivative; the definition extends previous definitions of the derivatives by truncation [4,9]. The co-local weak derivative is an approximate derivative (Proposition 1.13). This follows from the Euclidean counterpart. We recover thus without any Riemannian structure the derivative of Focardi and Spadaro [26].

In the second step, we define when M and N are Riemannian manifolds, for every $p \in [1, \infty]$, the *homogeneous Sobolev space* (Definition 2.1)

$$\dot{W}^{1,p}(M, N) = \Big\{u\colon M \to N\colon u \text{ is co-locally weakly differentiable} \\ \text{and } |Du|_{g_M^* \otimes g_N} \in L^p(M)\Big\},$$

where the Euclidean norm $|\cdot|_{g_M^* \otimes g_N}$ is induced by the Riemannian metrics on M and N. This definition is equivalent to (0.1) when N is isometrically embedded in $\mathbb{R}^\nu$ (Proposition 2.7) — (0.1) is thus *a posteriori* an intrinsic definition — and with the definition of Sobolev spaces into metric spaces (Proposition 2.2). Given a co-locally weakly differentiable map $u\colon M \to N$, we characterize the quantity $|Du|_{g_M^* \otimes g_N}$ as the smallest measurable function $w\colon M \to \mathbb{R}$ such that for every $f \in C_c^1(N, \mathbb{R}^{\min(\dim(M),\dim(N))})$,

$$|D(f \circ u)| \le |f|_{\mathrm{Lip}}\, w \qquad \text{almost everywhere in } M. \tag{0.2}$$

This allows us to define a robust Dirichlet integrand; previous definitions with scalar test function $f \in C_c^1(N, \mathbb{R})$ were quite unstable [2,51]. Furthermore the inequality (0.2) might provide a robust definition of the Dirichlet integrand for Sobolev maps into metric spaces.

We study weakly convergent sequences in Section 3. This part should be useful to obtain compactness and lower semi-continuity results in the calculus of variations for maps between manifolds.

We conclude our work by giving a natural notion of strong convergence and associated intrinsic distances $\dot{\delta}_{1,p}$ and $\delta_{1,p}$. If the Riemannian manifold N is complete, Sobolev spaces are complete under such distances (Propositions 4.2 – 4.12). This was not the case for distances proposed by Chiron [17, Proposition 4.9] and this opens the study of the completion of smooth maps in Sobolev spaces. Our notion of strong convergence is equivalent to existing ones for embedded manifolds and metric spaces (Subsection 4.2).

1. Co-locally weakly differentiable maps and co-local weak derivative

1.1. Weak differentiability on a differentiable manifold

We assume that M and N are differentiable manifolds of dimensions m and n which are Hausdorff and have a countable basis [21, Section 0.5], [38, Section 1.5].

We recall various definitions of local measure-theoretical notions on a manifold. A set $E \subset M$ is *negligible* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ – that is $\psi : V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ is a diffeomorphism – such that $x \in V$ and the set $\psi(E \cap V) \subset \mathbb{R}^m$ is negligible. A map $u : M \to N$ is *measurable* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ is measurable [38, Section 3.1], [20, Section 3]. A function $u : M \to \mathbb{R}$ is *locally integrable* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and $u \circ \psi^{-1}$ is integrable on $\psi(V)$ [39, Section 6.3]. Similarly, a locally integrable map $u : M \to \mathbb{R}$ is *weakly differentiable* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and the map $u \circ \psi^{-1}$ is weakly differentiable. All these notions are independent on any particular metric or measure on the manifold M.

A Radon measure μ on M is *absolutely continuous* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and the image measure

$\psi_*(\mu)$ defined by $\psi_*(\mu)(A) = \mu(\psi^{-1}(A))$ is absolutely continuous with respect to the Lebesgue measure. The measure μ is *positive* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and the Lebesgue measure on $\psi(V)$ is absolutely continuous with respect to $\psi_*(\mu)(A) = \mu(\psi^{-1}(A))$ [42, Definition 5.1.1]. Every absolutely continuous measure on M is absolutely continuous with respect to every positive measure. A set $E \subset M$ is negligible if and only if $\mu(E) = 0$ for every positive absolutely continuous Radon measure μ. A measurable function $u : M \to \mathbb{R}$ is locally integrable if and only if there exists a positive absolutely continuous Radon measure μ such that $\int_M |u| \, d\mu < \infty$.

We first define the notion of co-locally weakly differentiable map.

Definition 1.1. A map $u \colon M \to N$ is *co-locally weakly differentiable* if u is measurable and for every $f \in C^1_c(N, \mathbb{R})$, $f \circ u$ is weakly differentiable.

When $N = \mathbb{R}$, Definition 1.1 is related to the space $GBV(M, \mathbb{R})$ of functions of generalized bounded variation introduced by L. Ambrosio and E. De Giorgi [4], [5, Definition 4.26] and to the space $\mathcal{T}^{1,1}_{\mathrm{loc}}(M)$ of Sobolev functions by truncations from M to $\mathbb{R}$ [9]. Any function in $\mathcal{T}^{1,1}_{\mathrm{loc}}(M)$ is co-locally weakly differentiable. The converse is false as it can be observed by taking a function $u \in C^1(\mathbb{R} \setminus \{0\}, \mathbb{R})$ such that $\lim_{t \overset{<}{\to} 0} u(t) = -\infty$ and $\lim_{t \overset{>}{\to} 0} u(t) = +\infty$.

When $N = \mathbb{R}^n$ and $n \ge 2$, co-locally weakly differentiable functions are closely related to functions of generalized bounded variation [5, Definition 4.26]. Indeed, if $f \in C^1(\mathbb{R}^n, \mathbb{R})$, then $\operatorname{supp}(Df)$ is compact if and only if f is constant outside a compact set.

Finally, if $u : \mathbb{R}^m \to \mathbb{R}^n$ is weakly differentiable, then u is co-locally weakly differentiable. The converse is false: for example, the function $u \colon \mathbb{R}^m \to \mathbb{R}$ defined for every $x \in \mathbb{R}^m \setminus \{0\}$ by $u(x) = |x|^{-\alpha}$ is not weakly differentiable for any $\alpha > m - 1$, but is co-locally weakly differentiable for every $\alpha \in \mathbb{R}$.

In order to define the co-local weak derivative, we denote by (TM, π_M, M) the *tangent bundle* over M, that is,

$$TM = \bigcup_{x \in M} \{x\} \times T_x M,$$

$\pi_M \colon TM \to M$ is the natural projection and for every $x \in M$, the fiber $\pi_M^{-1}(\{x\})$ is isomorphic to $\mathbb{R}^m$; a map $\upsilon \colon TM \to TN$ is a *bundle morphism* that covers $u \colon M \to N$ if

$$\begin{array}{ccc} TM & \xrightarrow{\ \upsilon\ } & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[\ u\ ]{} & N \end{array}$$

commutes, that is, $\pi_N \circ \upsilon = u \circ \pi_M$, and for every $x \in M$, $\upsilon(x) \colon \pi_M^{-1}(\{x\}) \to \pi_N^{-1}(\{u(x)\})$ is linear. The space of all bundle morphisms is denoted by $\operatorname{Hom}(TM, TN)$. In particular, if $u \colon M \to N$ is a differentiable map, then $Du \colon TM \to TN$ is a bundle morphism that covers u. By a direct covering argument, if $u : M \to \mathbb{R}$ is weakly differentiable, then there exists a bundle morphism $Du : TM \to \mathbb{R}$ such that for every local chart $\psi : V \subseteq M \to \mathbb{R}^m$,

$$D(u \circ \psi^{-1}) = Du \circ D\psi^{-1}$$

almost everywhere on $\psi(V)$. We have now all the ingredients to define the co-local weak derivative.

Definition 1.2. Let $u\colon M \to N$ be a co-locally weakly differentiable map. A map $Du\colon TM \to TN$ is a *co-local weak derivative* of u if Du is a measurable bundle morphism that covers u and for every $f \in C^1_c(N, \mathbb{R})$,

$$D(f \circ u) = Df \circ Du$$

almost everywhere in M.

Consequently, if Du is a co-local weak derivative of u, for almost every $x \in M$, $Du(x) \in \mathcal{L}(T_x M, T_{u(x)} N)$ and for each $e \in T_x M$, $D(f \circ u)(x)[e] = Df(u(x))[Du(x)[e]]$.

We first observe that this notion extends the notion of classical differentiability:

Proposition 1.3 (Equivalence of classical and co-local weak derivatives). *Let $u \in C(M, N)$. Then u has a continuous co-local weak derivative if and only if $u \in C^1(M, N)$. Moreover, the co-local weak derivative and the classical derivative coincide almost everywhere.*

Proof. Since for every $f \in C^1_c(N, \mathbb{R})$, $f \circ u$ is weakly differentiable, we can apply the equivalence of classical and weak derivatives (Du Bois-Reymond lemma) [58, Theorem 6.1.4] [44, Theorem 6.10] and local charts on M to obtain that $f \circ u \in C^1(M, \mathbb{R})$. Since f is arbitrary, the map u is continuously differentiable. □

Definition 1.4. A bundle morphism $\upsilon : TM \to TN$ that covers $u : M \to N$ is *bilocally integrable* on $A \subseteq M$ if there exist local charts $\psi : V \subseteq M \to \mathbb{R}^m$, $\varphi : U \subseteq N \to \mathbb{R}^n$ such that if $L \subset V$ and $K \subset U$ are compact, then the function $D\varphi \circ \upsilon_{|V} \circ D(\psi^{-1})$ is integrable on $\psi(A \cap L \cap u^{-1}(K))$.

If μ is a positive absolutely continuous measure on M, the morphism υ is bilocally integrable if and only if there exists a continuous norm $|\cdot|$ on $T^*M \otimes TN$ such that $\int_M |\upsilon| \, d\mu < \infty$.

The main result of the current section is that co-locally weakly differentiable maps have a co-local weak derivative.

Proposition 1.5 (Existence and uniqueness of the co-local weak derivative). *If the map $u\colon M \to N$ is co-locally weakly differentiable, then u has a unique co-local weak derivative $Du\colon TM \to TN$. Moreover, the bundle morphism Du is bilocally integrable and for every $f \in C^1(N, \mathbb{R})$ for which $f \circ u : M \to \mathbb{R}$ is weakly differentiable,*

$$D(f \circ u) = Df \circ Du \quad \textit{almost everywhere in } M.$$

This result was already known when $N = \mathbb{R}$ for functions of generalized bounded variation [5, Theorem 4.34] and for Sobolev functions by truncations [9, Lemma 2.1]; as remarked there, the co-local weak derivative need not be locally integrable.

The important geometric tool for proving Proposition 1.5 is the existence of extended local charts. This construction is reminiscent of the patch mappings to the sphere $\varphi \in C^1(N, \mathbb{S}^n)$ [50, page 60].

Lemma 1.6 (Extended local charts). *For every $y \in N$, there exist an open subset $U \subseteq N$ such that $y \in U$, and maps $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ such that*

(i) *the set* $\operatorname{supp} \varphi$ *is compact,*
(ii) *the set* $\overline{\{x \in \mathbb{R}^n : \varphi^*(x) \neq \varphi^*(\varphi(y))\}}$ *is compact,*
(iii) *the map $\varphi_{|U}$ is a diffeomorphism onto its image $\varphi(U)$,*
(iv) $\varphi^* \circ \varphi = \mathrm{id}$ *in U.*

Proof. By definition of differentiable manifold, there exists a local chart $\psi : V \subseteq N \to \psi(V) \subseteq \mathbb{R}^n$ around $y \in N$. Without loss of generality, we assume that $\psi(y) = 0$. Since the set $\psi(V) \subseteq \mathbb{R}^n$ is open, there exists $r > 0$ such that $B_{2r} \subseteq \psi(V)$. We choose a map $\theta \in C^1_c(\mathbb{R}^n, \mathbb{R})$ such that $0 \leq \theta \leq 1$ on $\mathbb{R}^n$, $\theta = 1$ on B_r and $\operatorname{supp}(\theta) \subset B_{2r}$. We take the set $U = (\psi_{|V})^{-1}(B_r)$ and the maps $\varphi : N \to \mathbb{R}^n$ defined for every $z \in N$ by

$$\varphi(z) = \begin{cases} \theta(\psi(z))\psi(z) & \text{if } z \in (\psi_{|V})^{-1}(B_{2r}), \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi^* : \mathbb{R}^n \to N$ defined for every $x \in \mathbb{R}^n$ by $\varphi^*(x) = (\psi_{|V})^{-1}(\theta(x)\, x)$. □

We begin by proving a local counterpart of Proposition 1.5.

Lemma 1.7. *If $u : M \to N$ is a co-locally weakly differentiable map and $y \in N$, then there exist an open subset $U \subseteq N$ such that $y \in U$ and a unique measurable bundle morphism $D_U u : \pi_M^{-1}(u^{-1}(U)) \to TN$ such that for every $f \in C^1(N, \mathbb{R})$ for which $f \circ u : M \to \mathbb{R}$ is weakly differentiable,*

$$D(f \circ u) = Df \circ D_U u \qquad \text{almost everywhere on } u^{-1}(U).$$

Moreover, $D_U u$ is bilocally integrable on $u^{-1}(U)$.

Proof. Let $U \subseteq N$, $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ be the extended local charts given by Lemma 1.6. Since u is co-locally weakly differentiable, the map $\varphi \circ u : M \to \mathbb{R}^n$ is weakly differentiable. Since for every $x \in u^{-1}(U)$, the linear map between tangent spaces $D\varphi(u(x)) : T_{u(x)}N \to \mathbb{R}^n$ is invertible, the map $D_U u$ is uniquely defined for almost every $x \in u^{-1}(U)$ by

$$D_U u(x) = (D\varphi(u(x))^{-1} \circ (D(\varphi \circ u)(x)).$$

If $f \in C^1(N, \mathbb{R})$ and $f \circ u : M \to \mathbb{R}$ is weakly differentiable, since $\varphi^*(\mathbb{R}^n)$ is compact and $f \circ \varphi^* \in C^1(\mathbb{R}^n, \mathbb{R})$, the chain rule for weakly differentiable functions implies (see for example [58, Theorem 6.1.13], [23, Theorem 4.2.4 (ii)]) that $f \circ \varphi^* \circ \varphi \circ u$ is weakly differentiable and

$$D(f \circ \varphi^* \circ \varphi \circ u) = D(f \circ \varphi^*) \circ D(\varphi \circ u) = D(f \circ \varphi^*) \circ D\varphi \circ D_U u.$$

Since $f \circ u = f \circ \varphi^* \circ \varphi \circ u$ on $u^{-1}(U)$ and $D(f \circ \varphi^*) \circ D\varphi = Df$ on U, we have [58, Corollary 6.1.14], [23, Theorem 4.2.4 (iv)], [44, Theorem 6.19] almost everywhere on $u^{-1}(U)$

$$D(f \circ u) = D(f \circ \varphi^*) \circ D\varphi \circ D_U u = Df \circ D_U u. \qquad \square$$

Proof of Proposition 1.5. Let $(U_i)_{i \in I}$ be an open cover of N by sets given by Lemma 1.7. Since the manifold N has a countable basis, we can assume that I is at most countable [48, Theorem 3.30]. Let $D_{U_i} u$ and $D_{U_i \cap U_j} u$ be the derivatives given by Lemma 1.7. By uniqueness, $D_{U_i \cap U_j} u = D_{U_i} u = D_{U_j} u$ almost everywhere on $u^{-1}(U_i \cap U_j)$. Since $\bigcup_{i \in I} u^{-1}(U_i) = M$ and I is countable, the bundle morphism $Du : TM \to TN$ can be defined uniquely almost everywhere by

$$Du = D_{U_i} u \text{ almost everywhere on } u^{-1}(U_i). \qquad \square$$

1.2. Properties of the co-local weak derivative

The co-local weak derivative retains some properties of weak derivatives.

Proposition 1.8 (Chain rule). *Let $\tilde{N}$ be a differentiable manifold. If the maps $u\colon M \to N$ and $f \in C^1(N, \tilde{N})$ are such that u and $f \circ u$ are co-locally weakly differentiable, then*

$$D(f \circ u) = Df \circ Du \text{ almost everywhere in } M.$$

Proof. For every $\varphi \in C^1_c(\tilde{N}, \mathbb{R})$, $\varphi \circ f \in C^1(N, \mathbb{R})$ and $\varphi \circ f$ is weakly differentiable. Therefore, by Proposition 1.5 and the classical chain rule,

$$\begin{aligned} D\varphi \circ D(f \circ u) = D(\varphi \circ f \circ u) = D\big((\varphi \circ f) \circ u\big) = D(\varphi \circ f) \circ Du \\ = D\varphi \circ (Df \circ Du). \end{aligned} \qquad \square$$

Proposition 1.9. *Let $\iota\colon N \to \tilde{N}$ be an embedding and let $u\colon M \to N$ be a map. If $\iota \circ u$ is co-locally weakly differentiable, then u is co-locally weakly differentiable and Du is the unique bundle morphism such that*

$$D(\iota \circ u) = D\iota \circ Du$$

almost everywhere on M. If moreover $\iota(N)$ is closed, then the converse holds.

Proof. Since ι is an embedding, $\iota(N)$ has a tubular neighborhood in N: there exist a vector bundle (E, π_N, N) and an embedding $\tilde{\iota} : E \to \tilde{N}$ such that $\tilde{\iota}|_N = \iota$ and $\tilde{\iota}(E)$ is open in $\tilde{N}$ [38, Theorem 4.5.2]. In particular, $\iota \circ \pi_N \circ \tilde{\iota}^{-1}$ is a retraction of the tubular neighborhood $\tilde{\iota}(E)$ on $\iota(N)$. Let $f \in C^1_c(N, \mathbb{R})$. We choose $\eta \in C^1_c(\tilde{N}, \mathbb{R})$ such that $\eta = 1$ on $\iota(\operatorname{supp} f)$ and $\operatorname{supp} \eta \subset \tilde{\iota}(E)$ and define $\tilde{f} = (f \circ \pi_N \circ \tilde{\iota}^{-1})\eta : \tilde{N} \to \mathbb{R}$. Since $\operatorname{supp} \eta \subset \tilde{\iota}(E)$, the function $\tilde{f}$ is well-defined, $\tilde{f} \in C^1_c(\tilde{N}, \mathbb{R})$ and $\tilde{f} \circ \iota = f$ on N. In particular, $f \circ u = \tilde{f} \circ \iota \circ u$ is weakly differentiable.

Conversely, if $\iota(N)$ is closed, then for every $\varphi \in C^1_c(\tilde{N}, \mathbb{R})$, $\varphi \circ \iota \in C^1_c(N, \mathbb{R})$ and $\iota \circ u$ is thus co-locally weakly differentiable. □

By the Whitney embedding theorem [57], there always exists an embedding $\iota : N \to \mathbb{R}^\nu = \mathbb{R}^{2n+1}$ such that $\iota(N)$ is closed [1, Theorem 2.6], [20, Theorem 5]. Proposition 1.9 gives thus an equivalent definition of co-local weak differentiability; the drawback of this alternate approach to co-local weak differentiability is its dependence on the Whitney embedding theorem for differentiable manifolds.

Since differentiable manifolds do not have in general any algebraic structure and since the co-locally weakly differentiable functions between Euclidean spaces do not form a vector space [5, 9], the co-local weak derivative does not have any algebraic properties of sum or product. There is however still a property of the Cartesian product of maps.

Proposition 1.10 (Product of manifolds). *Let N_1, N_2 be two differentiable manifolds. If $u_1 : M \to N_1$ and $u_2 : M \to N_2$ are co-locally weakly differentiable, then the map $u = (u_1, u_2) : M \to N_1 \times N_2$ is co-locally weakly differentiable and*

$$Du = (Du_1, Du_2) \qquad \textit{almost everywhere in } M.$$

The uniqueness property can be refined for maps that coincide on a set of positive measure:

Proposition 1.11 (Locality of derivatives). *If the maps $u, v : M \to N$ are co-locally weakly differentiable, then $Du = Dv$ almost everywhere on the set $\{x \in M : u(x) = v(x)\}$.*

Proof. Let $A = \{x \in M : u(x) = v(x)\}$. Let $U \subseteq N$ and $\varphi \in C^1(N, \mathbb{R}^n)$ be the extended local chart given by Lemma 1.6 and let $\psi : V \subseteq M \to \mathbb{R}^m$ be a local chart. Since $\varphi \circ u \circ \psi^{-1}$ and $\varphi \circ v \circ \psi^{-1}$ are weakly differentiable and $\varphi \circ u \circ \psi^{-1} = \varphi \circ v \circ \psi^{-1}$ on $\psi(A \cap V)$, $D(\varphi \circ u \circ \psi^{-1}) = D(\varphi \circ v \circ \psi^{-1})$ almost everywhere on $\psi(A \cap V)$. By definition of the co-local weak derivative, $D\varphi \circ Du \circ D\psi^{-1} = D\varphi \circ Dv \circ D\psi^{-1}$ almost everywhere on $\psi(A \cap V)$. Since $D\varphi$ is invertible on $u^{-1}(U)$, $Du = Dv$ almost everywhere on $A \cap V \cap u^{-1}(U) \cap v^{-1}(U)$. Taking open countable covers $(U_i)_{i \in I}$ of N and $(V_j)_{j \in J}$ of M, we conclude that $Du = Dv$ almost everywhere on A. □

1.3. Approximate differentiability

We study the relationship between the co-local weak derivative and the approximate derivative. For a map between differentiable manifolds, we generalize the classical definition of approximate derivative of maps between vector spaces [23, Definition 6.1], [24, 3.1.2].

Definition 1.12. Let $u\colon M \to N$ and $x \in M$. The linear map $\xi : T_x M \to T_{u(x)} N$ is an *approximate derivative* of u at x if there exist local charts $\psi : V \to \psi(V) \subseteq \mathbb{R}^m$ with $x \in V$ and $V \subseteq M$ open, and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ with $u(x) \in U$ and $U \subseteq N$ open such that for every $\varepsilon > 0$,

$$\begin{aligned}\lim_{\rho\to 0} \rho^{-m} \mathcal{L}^m \bigl(\psi(\{y \in V : u(y) \in U \\ |\varphi(u(y)) - \varphi(u(x)) - (D\varphi(u(x)) \circ \xi \circ \bigl(D\psi(x)\bigr)^{-1})[\psi(y) - \psi(x)]| \\ > \varepsilon|\psi(y) - \psi(x)|\}) \cap B_\rho(\psi(x))\bigr) = 0.\end{aligned}$$

The approximate derivative is unique and it is sufficient to establish its existence for one pair of diffeomorphisms.

If M and N are endowed with Riemannian metrics g_M and g_N respectively, it is natural to take for φ and ψ the exponential coordinates and to use the Riemannian distances d_N and d_M and measure μ_M; the approximate differentiability can be observed to be equivalent to requiring for every $\varepsilon > 0$,

$$\lim_{\rho\to 0} \rho^{-m} \mu_M\bigl(\{y \in B^M_\rho(x) : d_N(u(x), \exp_{u(x)}(\xi(\exp_x^{-1}(y))) > \varepsilon d_M(x, y)\}\bigr) = 0;$$

we recover thus in this particular case the definition of Focardi and Spadaro for maps from the Euclidean space to a Riemannian manifold [26, Definition 0.3].

Sobolev maps into Riemannian manifolds are known to have such an approximate derivative [26, Corollary 1.3]. This property is also satisfied for Sobolev maps into other non-flat targets for which a notion of weak derivative is not yet available [25, proposition 2.2], [19, Lemma 1.4].

Proposition 1.13. (Approximate differentiability of co-locally weakly differentiable maps). *If $u\colon M \to N$ is co-locally weakly differentiable, then for almost every $x \in M$, the co-local weak derivative $Du(x)$ is the approximate derivative of u at x.*

Proof. Let $\psi : V \subseteq M \to \mathbb{R}^m$ be a local chart around $x \in M$. Let $U \subseteq N$ and $\varphi \in C^1(N, \mathbb{R}^n)$ be the extended local chart given by Lemma 1.6. Since u is co-locally weakly differentiable, $\varphi \circ u \circ \psi^{-1} : \psi(V) \to \mathbb{R}^n$ is weakly differentiable and $D(\varphi \circ u \circ \psi^{-1}) = D\varphi \circ Du \circ D\psi^{-1}$ on $\psi(V)$. Since weakly differentiable maps between vector spaces are approximately differentiable [23, Theorem 6.1.4],

$\varphi \circ u \circ \psi^{-1}$ is approximately differentiable almost everywhere on $\psi(V)$, that is,

$$\lim_{\rho \to 0} \rho^{-m} \mathcal{L}^m \big(\psi(\{y \in V : |\varphi(u(y)) - \varphi(u(x)) - (D\varphi(u(x)) \circ Du(x) \circ \big(D\psi(x)\big)^{-1})[\psi(y) - \psi(x)]| > \varepsilon |\psi(y) - \psi(x)|\}) \cap B_\rho(\psi(x))\big) = 0.$$

Next, we note that since u is measurable, almost every $x \in M$ is a Lebesgue point of u. Since the set U is open, for almost every $x \in u^{-1}(U) \cap V$,

$$\lim_{\rho \to 0} \rho^{-m} \mathcal{L}^m \big(\psi(\{y \in V : u(y) \notin U\}) \cap B_\rho(\psi(x))\big) = 0.$$

Therefore we have for almost every $x \in u^{-1}(U) \cap V$,

$$\lim_{\rho \to 0} \rho^{-m} \mathcal{L}^m \big(\psi(\{y \in V : u(y) \in U \; |\varphi_{|U}(u(y)) - \varphi_{|U}(u(x)) - (D\varphi(u(x)) \circ Du(x) \circ \big(D\psi(x)\big)^{-1})[\psi(y) - \psi(x)]| > \varepsilon |\psi(y) - \psi(x)|\}) \cap B_\rho(\psi(x))\big) = 0,$$

that is, $Du(x)$ is the approximate derivative of u for almost every $x \in u^{-1}(U) \cap V$. The conclusion follows by a countable covering argument. □

2. Sobolev maps between Riemannian manifolds

Preliminaries We assume now that (M, g_M) and (N, g_N) are Riemannian manifolds. In particular the metrics on vectors of TM and TN induce a metric $g_M^* \otimes g_N$ on $T^*M \otimes TN$. This metric can be computed for every $\xi \in T_x^*M \otimes T_yN = \mathcal{L}(T_xM, T_yN)$ by

$$(g_M^* \otimes g_N)(\xi, \xi) = \sum_{i=1}^m g_N\big(\xi(e_i), \xi(e_i)\big),$$

where $(e_i)_{1 \le i \le m}$ is an orthonormal basis in $\pi_M^{-1}(\{x\})$ with respect to the Riemannian metric g_M.

We are now in a position to define the Sobolev spaces.

Definition 2.1. Let $p \in [1, \infty)$. A map $u : M \to N$ belongs to the *Sobolev space* $\dot{W}^{1,p}(M, N)$ if u is co-locally weakly differentiable and $|Du|_{g_M^* \otimes g_N} \in L^p(M)$.

Characterization of the norm of the derivative We characterize the quantity $|Du|_{g_M^* \otimes g_N}$ that appears in the definition of Sobolev spaces. We recall that the *operator norm* is defined for every $\xi \in T_x^* M \otimes T_y N = \mathcal{L}(T_x M, T_y N) = \mathrm{Hom}(TM, TN)_{x,y}$ by

$$|\xi|_{\mathcal{L}} = \sup \left\{ |\xi(e)|_{g_N} : e \in T_x M,\ |e|_{g_M} \le 1 \right\};$$

the *Lipschitz semi-norm* of $f \colon N \to \mathbb{R}^k$ is defined by

$$|f|_{\mathrm{Lip}} = \sup \left\{ \frac{|f(y) - f(z)|}{d_N(y, z)} : y, z \in N \text{ and } y \neq z \right\},$$

where d_N is the distance on N induced by the Riemannian metric g_N. For every $k \ge 1$, we denote by g_k the standard Euclidean metric on $\mathbb{R}^k$.

Proposition 2.2. *Let $k \ge \min(m, n)$. Let $u \colon M \to N$, $w \colon M \to \mathbb{R}$ be measurable maps. The following statements are equivalent.*

(i) *u is co-locally weakly differentiable and $|Du|_{g_M^* \otimes g_N} \le w$ almost everywhere in M,*

(ii) *for every $f \in C_c^1(N, \mathbb{R}^k)$, $f \circ u$ is weakly differentiable and almost everywhere in M*

$$|D(f \circ u)|_{g_M^* \otimes g_k} \le |Df(u)|_{\mathcal{L}} w,$$

(iii) *for every $f \in C_c^1(N, \mathbb{R}^k)$, $f \circ u$ is weakly differentiable and almost everywhere in M*

$$|D(f \circ u)|_{g_M^* \otimes g_k} \le |f|_{\mathrm{Lip}}\, w.$$

If moreover $w \in L^1_{\mathrm{loc}}(M)$, then for every $f \in \mathrm{Lip}(N, \mathbb{R}^k)$, $f \circ u$ is weakly differentiable and almost everywhere in M

$$|D(f \circ u)|_{g_M^* \otimes g_k} \le |f|_{\mathrm{Lip}}\, w.$$

Since the first assertion is independent of k, there is also equivalence between these statements for every $k \ge \min(m, n)$.

Proposition 2.2 implies in particular that if $p \in [1, \infty)$ and M is an open bounded subset of $\mathbb{R}^m$, Definition 2.1 is equivalent to the notion of Sobolev spaces into metric spaces [51, Theorem 5.1] and with the classical homogeneous Sobolev space when $N = \mathbb{R}^n$ (see also [9, Lemma 2.1]).

In order to prove Proposition 2.2, we shall use an approximation property of Lipschitz maps on manifolds.

Lemma 2.3 (Approximation of Lipschitz maps). *Let $f \in \mathrm{Lip}(N, \mathbb{R}^k)$. There exists a sequence $(f_\ell)_{\ell \in \mathbb{N}}$ of maps in $C_c^1(N, \mathbb{R}^k)$ that converges uniformly over compact subsets of N and such that*

$$\limsup_{\ell \to \infty} |f_\ell|_{\mathrm{Lip}} \le |f|_{\mathrm{Lip}}.$$

Sketch of the proof. Given $y \in N$ and $\theta \in C^1_c([0, \infty), \mathbb{R})$ such that $\theta = 1$ on $[0, 1]$, we define for every $\ell \in \mathbb{N}_*$,

$$\theta_\ell(z) = \theta\Big(\sqrt[\ell^2]{d_N(y, z)}\Big),$$

and observe that $\theta_\ell \in C^1_c(N, \mathbb{R})$,

$$|D\theta_\ell|_{\mathcal{L}} \le \frac{\|\theta'\|_{L^\infty}}{\ell^2},$$

and $(\theta_\ell)_{\ell \in \mathbb{N}_*}$ converges to 1 uniformly over compact subsets of N.

We also define $T_\ell : \mathbb{R}^k \to \mathbb{R}^k$ for $\ell \in \mathbb{N}_*$ and $x \in \mathbb{R}^k$ by

$$T_\ell(x) = \begin{cases} x & \text{if } |x| \le \ell, \\ \ell x/|x| & \text{if } |x| > \ell. \end{cases}$$

For every $\ell \in \mathbb{N}_*$, T_ℓ is nonexpansive and bounded by ℓ and the sequence $(T_\ell)_{\ell \in \mathbb{N}_*}$ converges uniformly to the identity over compact subsets.

If we define $\tilde{f}_\ell = \theta_\ell \cdot (T_\ell \circ f)$, we observe for every $\ell \in \mathbb{N}_*$ that $\tilde{f}_\ell \in \operatorname{Lip}(N, \mathbb{R}^k)$, that

$$|\tilde{f}_\ell|_{\mathrm{Lip}} \le |f|_{\mathrm{Lip}} + \frac{\|\theta'\|_{L^\infty}}{\ell},$$

and that the support of $\tilde{f}_\ell$ is compact; the sequence $(\tilde{f}_\ell)_{\ell \in \mathbb{N}_*}$ converges uniformly over compact subsets.

Hence the conclusion follows by approximating uniformly $\tilde{f}_\ell$ by differentiable functions with a control on the Lipschitz norm [31, Lemma 8], [6]. ☐

We shall also rely on a refined version of the extended local charts of Lemma 1.6.

Lemma 2.4 (Almost isometric extended local charts). *For every $y \in N$ and every $\varepsilon > 0$, there exist an open subset $U \subseteq N$ such that $y \in U$, and maps $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ such that*

(i) *the set* $\operatorname{supp} \varphi$ *is compact,*
(ii) *the set* $\overline{\{x \in \mathbb{R}^n : \varphi^*(x) \neq \varphi^*(\varphi(y))\}}$ *is compact,*
(iii) *the map $\varphi_{|U}$ is a diffeomorphism onto its image $\varphi(U)$,*
(iv) $\varphi^* \circ \varphi = \mathrm{id}$ *in U,*
(v) *for every $z \in N$,*

$$|D\varphi(z)|_{\mathcal{L}} \le 1 + \varepsilon \qquad \textit{and} \qquad |D\varphi^*(\varphi(z))|_{\mathcal{L}} \le 1 + \varepsilon.$$

The difference with Lemma 1.6 lies in the control (v) on the operator norms of $D\varphi$ and $D\varphi^*$.

Proof of Lemma 2.4. Since N is a differentiable manifold, there exists a local chart $\psi : V \subseteq N \to \mathbb{R}^n$ around $y \in N$. Up to an affine transformation on $\mathbb{R}^n$, we can assume that $\psi(y) = 0$ and $D\psi(y) \in \mathcal{L}(T_y N, \mathbb{R}^n)$ is an isometry.

By continuity of $D\psi$, there exists $\delta > 0$ such that if $z \in N$ and $d_N(y,z) \le \delta$, then

$$|D\psi(z)|_{\mathcal{L}} \le 1+\varepsilon \qquad \text{and} \qquad \bigl|D(\psi_{|V})^{-1}(\psi(z))\bigr|_{\mathcal{L}} \le 1+\varepsilon.$$

We choose $r > 0$ such that $B_{3r} \subseteq \psi(B_\delta(y))$. We take a map $\theta \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ such that $\theta = \mathrm{id}$ on B_r, $\mathrm{supp}(\theta) \subset B_{3r}$ and $\sup_{x\in\mathbb{R}^n} |D\theta(x)|_{\mathcal{L}} \le 1$. We finally define $\varphi = \theta \circ \psi$, $\varphi^* = (\psi_{|V})^{-1} \circ \theta$ and $U = (\psi_{|V})^{-1}(B_r)$ and we conclude as in the proof of Lemma 1.6. □

We shall also use the following lemma to compute Euclidean norms of maps.

Lemma 2.5 (Reduction of the Euclidean norm of operators). *Let $x \in M$ and let $\xi \in \mathcal{L}(T_x M, \mathbb{R}^n)$ be a linear map. If $k \ge \min(m,n)$, then*

$$|\xi|_{g^*_M \otimes g_n} = \sup\bigl\{|\rho \circ \xi|_{g^*_M \otimes g_k} : \rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k) \text{ and } |\rho|_{\mathcal{L}} \le 1\bigr\}.$$

Proof. On the one hand, if $\rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ and $|\rho|_{\mathcal{L}} \le 1$, then $|\rho \circ \xi|_{g^*_M\otimes g_k} \le |\rho|_{\mathcal{L}} |\xi|_{g^*_M \otimes g_n} \le |\xi|_{g^*_M\otimes g_n}$. On the other hand, since $\dim(\mathrm{Im}(\xi)) \le \min(m,n) \le k$, one can choose $\rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ such that ρ is an isometry on $\mathrm{Im}(\xi)$ and consequently $|\rho|_{\mathcal{L}} \le 1$ and $|\rho\circ\xi|_{g^*_M\otimes g_k} = |\xi|_{g^*_M\otimes g_n}$. □

Proof of Proposition 2.2. Let us prove that (i) implies (ii). For every $f \in C^1_c(N, \mathbb{R}^k)$, since u is co-locally weakly differentiable, $f \circ u$ is weakly differentiable. By Proposition 1.5, $D(f\circ u) = Df \circ Du$ almost everywhere in M, and so

$$|D(f\circ u)|_{g^*_M\otimes g_k} \le |Df(u)|_{\mathcal{L}} |Du|_{g^*_M\otimes g_N} \le |Df(u)|_{\mathcal{L}}\, w.$$

Since for every $f \in C^1_c(N, \mathbb{R}^k)$ and for every $y \in N$, $|Df(y)|_{\mathcal{L}} \le |f|_{\mathrm{Lip}}$, the assertion (ii) implies directly (iii).

In order to prove that (iii) implies (i) we first note that the map u is co-locally weakly differentiable, and, by Proposition 1.5, has a unique co-local weak derivative $Du \in \mathrm{Hom}(TM, TN)$. Secondly, let $U \subseteq N$, $\varphi \in C^1(N, \mathbb{R}^n)$ and $\varphi^* \in C^1(\mathbb{R}^n, N)$ be given by Lemma 2.4 for $y \in N$ and $\varepsilon > 0$. Since $u = \varphi^* \circ \varphi \circ u$ on $u^{-1}(U)$, by Proposition 1.11, $Du = D(\varphi^*\circ\varphi\circ u)$ almost everywhere on $u^{-1}(U)$. By Lemma 2.4, almost everywhere on $u^{-1}(U)$

$$|Du|_{g^*_M\otimes g_N} \le |D\varphi^*(\varphi(u))|_{\mathcal{L}} |D(\varphi\circ u)|_{g^*_M\otimes g_n} \le (1+\varepsilon)|D(\varphi\circ u)|_{g^*_M\otimes g_n}. \quad (2.1)$$

If $\rho \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ and $|\rho|_{\mathcal{L}} \le 1$, by assumption, $|D(\rho\circ\varphi\circ u)|_{g^*_M\otimes g_k} \le \mathrm{Lip}(\rho\circ\varphi)\, w$ almost everywhere in M. Since d_N is a geodesic distance, in view of Lemma 2.4, $\mathrm{Lip}(\varphi) = \sup_{z\in N} |D\varphi(z)|_{\mathcal{L}} \le 1+\varepsilon$. Hence, we have almost everywhere in M

$$|\rho \circ D(\varphi\circ u)|_{g^*_M\otimes g_k} = |D(\rho\circ\varphi\circ u)|_{g^*_M\otimes g_k} \le (1+\varepsilon) w.$$

Since the set of nonexpansive linear maps is separable, we deduce from Lemma 2.5 that

$$|D(\varphi \circ u)|_{g_M^* \otimes g_n} \leq (1+\varepsilon) w$$

almost everywhere in M. By inequality (2.1), we conclude that

$$|Du|_{g_M^* \otimes g_N} \leq (1+\varepsilon)^2 w$$

almost everywhere on $u^{-1}(U)$. We conclude by countable covering of N.

We now prove the last assertion. Let $f \in \operatorname{Lip}(N, \mathbb{R}^k)$. By the approximation Lemma 2.3, there exists a sequence $(f_\ell)_{\ell \in \mathbb{N}}$ in $C_c^1(N, \mathbb{R}^k)$ such that $(f_\ell)_{\ell \in \mathbb{N}}$ converges uniformly to f over compact subsets and $\limsup_{l \to \infty} |f_\ell|_{\mathrm{Lip}} \leq |f|_{\mathrm{Lip}}$. Since the sequence $(D(f_\ell \circ u))_{\ell \in \mathbb{N}}$ is bounded and uniformly integrable and since $(f_\ell \circ u)_{\ell \in \mathbb{N}}$ converges almost everywhere to $f \circ u$, in view of the weak compactness criterion in $L^1(M)$ [12, Corollary 4.7.19], [14, Theorem 4.30], the sequence $(D(f_\ell \circ u))_{\ell \in \mathbb{N}}$ converges weakly to $D(f \circ u)$ in $L^1_{\mathrm{loc}}(M)$ and $f \circ u \in W^{1,1}_{\mathrm{loc}}(M, \mathbb{R}^k)$. Moreover, for every $v \in C_c^1(M, TM \otimes \mathbb{R}^k)$,

$$\begin{aligned}\Big|\int_M \langle D(f \circ u), v\rangle\Big| &= \lim_{\ell \to \infty} \Big|\int_M \langle D(f_\ell \circ u), v\rangle\Big| \\ &\leq \liminf_{\ell \to \infty} |f_\ell|_{\mathrm{Lip}} \int_M |v|_{g_M^* \otimes g_k} w \leq |f|_{\mathrm{Lip}} \int_M |v|_{g_M^* \otimes g_k} w,\end{aligned}$$

therefore $|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\mathrm{Lip}} w$ almost everywhere in M. □

Thanks to Proposition 2.2, we can consider the composition of a Lipschitz map from a manifold into an other with a map of homogeneous Sobolev space.

Proposition 2.6 (Chain rule in Sobolev spaces). *Let $(\tilde{N}, g_{\tilde{N}})$ be a Riemannian manifold. Let $u \in \dot{W}^{1,p}(M, N)$ and let $f \in \operatorname{Lip}(M, \tilde{N})$. Then $f \circ u \in \dot{W}^{1,p}(M, \tilde{N})$ and*

$$|D(f \circ u)|_{g_M^* \otimes g_{\tilde{N}}} \leq |f|_{\mathrm{Lip}} |Du|_{g_M^* \otimes g_N} \text{ almost everywhere in } M.$$

This generalizes a well-known property ([14, Proposition 9.5], [44, Theorem 6.16], [58, Proposition 6.1.13]); the corresponding chain rule is more delicate [3].

We also obtain a characterization of Sobolev spaces by embeddings.

Proposition 2.7. *Let $\iota : N \to \tilde{N}$ be an isometric embedding. For every $u : M \to N$, $u \in \dot{W}^{1,p}(M, N)$ if and only if $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$.*

In contrast with Proposition 1.9, the equivalence does not require $\iota(N)$ to be closed.

Proof of Proposition 2.7. If $u \in \dot{W}^{1,p}(M, N)$, then by the chain rule (Proposition 2.6), $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$. Conversely, if $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$, then $\iota \circ u$ is co-locally weakly differentiable. By the weakly differentiable embedding property (Proposition 1.9), u is co-locally weakly differentiable. By the chain rule (Proposition 1.8), $D(\iota \circ u) = D\iota \circ Du$, and since the embedding ι is isometric, $|Du|_{g_M^* \otimes g_N} = |D(\iota \circ u)|_{g_M^* \otimes g_{\tilde{N}}}$. □

In particular our intrinsic definition is equivalent to the definition given by any embedding of N in a Euclidean space; such an embedding is always possible by the Nash embedding theorem [49,50].

3. Weak compactness and closure property

3.1. Weak compactness

A classical technique in the calculus of variations is to extract from a minimizing sequence a subsequence that converges almost everywhere.

To cover a rich spectrum of functional settings, we first give a compactness result independently of the Riemannian manifold or Sobolev space setting. Such a result should rely on a boundedness assumption on the derivatives and on the values of the mappings.

For the boundedness of the derivatives, we shall use the following assumption.

Definition 3.1. A sequence $(\upsilon_\ell)_{\ell\in\mathbb{N}}$ of bundle morphisms from TM to TN that covers a sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of maps from M to N is *bilocally L^1–bounded* if for every $x \in M$ and every $y \in N$, there exist local charts $\psi : V \subseteq M \to \mathbb{R}^m$ and $\varphi : U \subseteq N \to \mathbb{R}^n$ such that $x \in V$, $y \in U$ and

$$\sup_{\ell\in\mathbb{N}} \int_{\psi(V\cap u_\ell^{-1}(U))} |D\varphi \circ \upsilon_\ell \circ D(\psi^{-1})|_{g_m^*\otimes g_n} < \infty.$$

Equivalently, the sequence $(\upsilon_\ell)_{\ell\in\mathbb{N}}$ is bilocally L^1–bounded if there exist a positive measure μ on M and a continuous norm $|\cdot|$ on the vector bundle $T^*M \otimes TN$ such that

$$\sup_{\ell\in\mathbb{N}} \int_M |\upsilon_\ell| \, d\mu < \infty.$$

The boundedness of the values is expressed in the next condition.

Definition 3.2. A sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of maps from M to N is *locally compact in measure* if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and for every $\varepsilon > 0$ there exists a compact set $K \subseteq N$ such that for every $\ell \in \mathbb{N}$,

$$\mathcal{L}^m\big(\psi(u_\ell^{-1}(N \setminus K))\big) \le \varepsilon.$$

If N is compact, then this condition is trivially satisfied. In general, a sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of maps from M to N is locally compact in measure if and only if there exist a positive measure μ on N, $y \in N$ and a continuous distance d on N such that

$$\lim_{r\to\infty} \sup_{\ell\in\mathbb{N}} \mu\big(u_\ell^{-1}(N \setminus B_r^d(y))\big) = 0.$$

Proposition 3.3 (Compactness in measure). *Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of co-locally weakly differentiable maps from M to N. If the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally L^1–bounded and the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ is locally compact in measure, then there exist a measurable map $u : M \to N$ and a subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ that converges to u almost everywhere in M.*

Proof. Let $\big((U_i, \varphi_i)\big)_{i \in I}$ be a family of extended local charts satisfying the conclusion of Lemma 1.6 such that $\bigcup_{i \in I} U_i = N$ and I is countable. Assume that $\big(\eta_i\big)_{i \in I}$ is a C^1–partition of the unity subordinate to the covering $\big(U_i\big)_{i \in I}$. We observe that in view of Lemma 1.6, the set $\bar{U}_i$ is compact and hence $\eta_i \in C^1_c(N, \mathbb{R})$.

By assumption and by definition of the co-locally weakly differentiability, for every $i \in I$, the sequence $((\varphi_i \circ u_\ell, \eta_i \circ u_\ell))_{\ell \in \mathbb{N}}$ is bounded in $W^{1,1}_{\mathrm{loc}}(M, \mathbb{R}^{n+1})$. By the classical Rellich–Kondrashov compactness theorem [44, Theorem 8.9], [14, Theorem 9.16], [58, Theorem 6.4.6] and a diagonal argument, there exist a subsequence $(u_{\ell_k})_{k \in \mathbb{N}}$ and a negligible set $E \subset M$ such that the sequence $((\varphi_i \circ u_{\ell_k}(x), \eta_i \circ u_{\ell_k}(x)))_{k \in \mathbb{N}}$ converges in $\mathbb{R}^{n+1}$ for every $x \in M \setminus E$ and every $i \in I$.

We define the set

$$F = \Big\{x \in M \setminus E : \text{for every } i \in I,\ \lim_{k \to \infty} \eta_i(u_{\ell_k}(x)) = 0\Big\}.$$

For every compact set $K \subseteq N$, we observe that

$$F \subseteq \bigcup_{j \in \mathbb{N}} \bigcap_{k=j}^{\infty} u_{\ell_k}^{-1}(N \setminus K).$$

Since the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ is compact in measure, for every $x \in M$, there is a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and for every $\varepsilon > 0$ there exists a compact set $K \subseteq N$ such that for every $k \in \mathbb{N}$,

$$\mathcal{L}^m\big(\psi(u_{\ell_k}^{-1}(N \setminus K))\big) \le \varepsilon.$$

Therefore,

$$\mathcal{L}^m(\psi(F \cap V)) \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and M can be covered by countably many such charts, we conclude that the set F is negligible.

We conclude by showing that $(u_{\ell_k})_{k \in \mathbb{N}}$ converges everywhere in $M \setminus (E \cup F)$. For every $x \in M \setminus (E \cup F)$, by definition of the set F, there exists $i \in I$ such that $\lim_{k \to \infty} \eta_i(u_{\ell_k}(x)) > 0$. This implies that for $k \in \mathbb{N}$, large enough, $u_{\ell_k}(x) \in U_i$. Since φ_i is a diffeomorphism on U_i and since the sequence $(\varphi_i(u_{\ell_k}(x)))_{k \in \mathbb{N}}$ converges, we define $u(x) = {\varphi_i}_{|U_i}^{-1}\big(\lim_{k \to \infty} \varphi_i(u_{\ell_k}(x))\big)$ and we conclude that $(u_{\ell_k}(x))_{k \in \mathbb{N}}$ converges to $u(x)$. □

In particular, we have a Rellich–Kondrashov type compactness theorem.

Proposition 3.4 (Rellich–Kondrashov for Sobolev maps). *Let $(u_\ell)_{\ell\in\mathbb{N}}$ be a sequence of co-locally weakly differentiable maps from M to N, $v : M \to N$ be measurable. If (N, d) is complete, if there exist $p \in [1, \infty)$ such that*

$$\sup_{\ell\in\mathbb{N}} \int_M |Du_\ell|^p_{g_M^* \otimes g_N} < \infty$$

and $q \in [1, \infty)$ such that

$$\sup_{\ell\in\mathbb{N}} \int_M d(u_\ell, v)^q < \infty,$$

then there is a subsequence $(u_{\ell_k})_{k\in\mathbb{N}}$ that converges to $u : M \to N$ almost everywhere in M.

Proof. Since the metric space (N, d) is complete, for every $y \in N$ and $r \in \mathbb{R}$, the closed ball $\bar{B}^N_r(y)$ is compact. In particular the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ is locally compact in measure.

On the other hand, the sequence (Du_ℓ) is bilocally L^1–bounded, and therefore by Proposition 3.3, there exist a measurable map $u : M \to N$ and a subsequence $(u_{\ell_k})_{k\in\mathbb{N}}$ that converges to u almost everywhere in M. □

3.2. Closure property

In order to study the lower semi-continuity properties of functionals, it is interesting to have some sufficient conditions for a limit of co-locally weakly differentiable maps to be also co-locally weakly differentiable. Such a condition will be useful in the study of stronger notion of convergence.

We shall consider sequences of maps converging in measure.

Definition 3.5. A sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of maps from M to N *converges locally in measure* to a map $u : M \to N$ if for every $x \in M$ there exists a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ such that $x \in V$ and for every open set $U \subseteq N$,

$$\lim_{\ell\to\infty} \mathcal{L}^m \Big(\psi \Big(\Big(u^{-1}(U) \cap V\Big) \setminus u_\ell^{-1}(U)\Big)\Big) = 0.$$

If d is a continuous distance on N and if μ is an absolutely continuous positive finite measure on M, then the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ converges to u locally in measure if and only if for every $\varepsilon > 0$,

$$\lim_{\ell\to\infty} \mu\big(\{x \in M : d(u_\ell(x), u(x)) > \varepsilon\}\big) = 0.$$

This definition is consistent with the definition of convergence in measure of maps into a metric space [53, Définition 5.6.17], which depends only on the topology of the space [53, Théorème 5.6.21].

Since the topologies of M and of N have a countable basis, it suffices to check the condition for a countable set of charts $\psi : V \subseteq M \to \mathbb{R}^m$ and a countable collection of open sets $U \subseteq N$. Hence, the convergence in measure for maps between manifolds induces a metrizable topology. Such a metric is given by

$$\delta(u, v) = \int_M \frac{d(u, v)}{1 + d(u, v)} \, d\mu. \tag{3.1}$$

Any Cauchy sequence with respect to δ has a subsequence which is Cauchy almost everywhere [12, Exercise 4.7.60], [53, Théorème 5.8.31]. Thus the space of measurable maps from M to N endowed with δ is complete if and only if the space (N, d) is complete.

The definition and the remarks apply directly to a sequence $(\upsilon_\ell)_{\ell\in\mathbb{N}}$ of bundle morphisms between TM and TN viewed as maps from M to $T^*M \otimes TN$.

A sufficient condition for co-locally weakly differentiability is a uniform integrability assumption.

Definition 3.6. A sequence $(\upsilon_\ell)_{\ell\in\mathbb{N}}$ of bundle morphisms from TM to TN that covers a sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of maps from M to N is *bilocally uniformly integrable* if for every $x \in M$ and every $y \in N$, there exist local charts $\psi : V \subseteq M \to \mathbb{R}^m$ and $\varphi : U \subseteq N \to \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $W \subseteq \psi(V)$, $\mathcal{L}^m(W) \le \delta$ and $\ell \in \mathbb{N}$, then

$$\int_{W\cap\psi(u_\ell^{-1}(U))} |D\varphi \circ \upsilon_\ell \circ D(\psi^{-1})|_{g_m^*\otimes g_n} \le \varepsilon.$$

Equivalently, the sequence $(\upsilon_\ell)_{\ell\in\mathbb{N}}$ is bilocally uniformly integrable if there exist a positive absolutely continuous measure μ on M and a continuous norm $|\cdot|$ on the vector bundle $T^*M \otimes TN$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $V \subseteq M$, $\mu(V) \le \delta$ and $\ell \in \mathbb{N}$, then

$$\int_V |\upsilon_\ell| \le \varepsilon.$$

Proposition 3.7 (Closure property). *Let $(u_\ell)_{\ell\in\mathbb{N}}$ be a sequence of co-locally weakly differentiable maps from M to N. If the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ converges to $u : M \to N$ locally in measure and if the sequence $(Du_\ell)_{\ell\in\mathbb{N}}$ is bilocally uniformly integrable, then the map u is co-locally weakly differentiable, and for every map $f \in C_c^1(N, \mathbb{R})$, every local chart $\psi : V \subseteq M \to \mathbb{R}^m$ and every test function $v \in C_c^1(\psi(V), \mathbb{R}^m)$,*

$$\lim_{\ell\to\infty} \int_{\psi(V)} \langle D(f \circ u_\ell \circ \psi^{-1}), v\rangle = \int_{\psi(V)} \langle D(f \circ u \circ \psi^{-1}), v\rangle.$$

If moreover the sequence $(Du_\ell)_{\ell\in\mathbb{N}}$ converges to a bundle morphism $\upsilon : TM \to TN$ locally in measure, then $Du = \upsilon$.

In particular, under the additional condition of bilocally uniform integrability of the sequence of co-local weak derivatives, the map given by Proposition 3.3 is co-locally weakly differentiable.

Proof of Proposition 3.7. Following classical argument [58, Theorem 6.1.7], given a local chart $\psi : V \subseteq M \to \mathbb{R}^m$ and $f \in C^1_c(N, \mathbb{R})$, we define the linear functional $F_{f,\psi}$ on $C^1_c(\psi(V), \mathbb{R}^m)$ for every test function $v \in C^1_c(\psi(V), \mathbb{R}^m)$ by

$$\langle F_{f,\psi}, v\rangle = -\int_{\psi(V)} (f \circ u \circ \psi^{-1}) \operatorname{div} v.$$

Let $K \subset \psi(V)$ be compact. Since the sequence $(f \circ u_\ell \circ \psi^{-1})_{\ell\in\mathbb{N}}$ converges to $f \circ u \circ \psi^{-1}$ in $L^1(K)$, if $\operatorname{supp} v \subset K$,

$$\begin{aligned}|\langle F_{f,\psi}, v\rangle| &= \Big|\int_K (f \circ u \circ \psi^{-1}) \operatorname{div} v\Big| = \lim_{\ell\to\infty}\Big|\int_K (f \circ u_\ell \circ \psi^{-1}) \operatorname{div} v\Big| \\ &= \lim_{\ell\to\infty}\Big|\int_K \langle D(f \circ u_\ell \circ \psi^{-1}), v\rangle\Big| \le \|v\|_{L^\infty} \liminf_{\ell\to\infty} \int_K |f|_{\mathrm{Lip}} |Du_\ell|.\end{aligned}$$

Therefore $F_{f,\psi}$ is represented by a vector-valued Radon measure $\mu_{f,\psi}$ on $\psi(V)$:

$$\langle F_{f,\psi}, v\rangle = \int_{\psi(V)} v \, d\mu_{f,\psi}.$$

We observe that for every open set $W \subseteq \psi(V)$,

$$|\mu_{f,\psi}|(W) \le \liminf_{\ell\to\infty} \int_W |D(f \circ u_\ell \circ \psi^{-1})|_{g_m^* \otimes g_1}.$$

By the uniform integrability assumption, we conclude that the measure $|\mu_{f,\psi}|$ is absolutely continuous with respect to the Lebesgue measure on every compact set $K \subset \psi(V)$. The measure $\mu_{f,\psi}$ can thus be represented by a vector-field in $L^1_{\mathrm{loc}}(\psi(V))$. In particular the map $f \circ u \circ \psi^{-1}$ is weakly differentiable and

$$\lim_{\ell\to\infty} \int_{\psi(V)} \langle D(f \circ u_\ell \circ \psi^{-1}), v\rangle = \int_{\psi(V)} \langle D(f \circ u \circ \psi^{-1}), v\rangle.$$

Finally, if $(Du_\ell)_{\ell\in\mathbb{N}}$ converges to a bundle morphism $\upsilon : TM \to TN$ locally in measure, then the sequence $(Df \circ Du_\ell \circ D\psi^{-1})_{\ell\in\mathbb{N}}$ converges to $Df \circ \upsilon \circ D\psi^{-1}$ in measure on $\psi(V)$. Since the sequence $(Df \circ Du_\ell \circ D\psi^{-1})_{\ell\in\mathbb{N}}$ is uniformly integrable on every compact set $K \subset \psi(V)$, we conclude that [12, Theorem 4.5.4]

$$\lim_{\ell\to\infty} \int_{\psi(V)} \langle D(f \circ u_\ell \circ \psi^{-1}), v\rangle = \int_{\psi(V)} \langle Df \circ \upsilon \circ D\psi^{-1}, v\rangle.$$

Since $f \in C^1_c(N, \mathbb{R})$, the chart $\psi : V \to \mathbb{R}^m$ and $v \in C^1_c(\psi(V), \mathbb{R}^m)$ are arbitrary, $Du = \upsilon$. ☐

In the particular case of bounded sequences in Sobolev spaces, we have the following closure property, which will play an important role in the completeness of Sobolev spaces.

Proposition 3.8 (Weak closure property for Sobolev maps). *Let $p \geq 1$ and let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of co-locally weakly differentiable maps from M to N. Assume that the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges to $u : M \to N$ locally in measure, that*

$$\liminf_{\ell \to \infty} \int_M |Du_\ell|^p_{g_M^* \otimes g_N} < \infty,$$

and, if $p = 1$, that the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Then $u \in \dot{W}^{1,p}(M, N)$,

$$\int_M |Du|^p_{g_M^* \otimes g_N} \leq \liminf_{\ell \to \infty} \int_M |Du_\ell|^p_{g_M^* \otimes g_N},$$

and for every $f \in C^1_c(N, \mathbb{R})$ and every section $v : M \to TM$, such that $|v|_{g_M} \in L^{p/(p-1)}(M)$,

$$\lim_{\ell \to \infty} \int_M \langle D(f \circ u_\ell), v \rangle = \int_M \langle D(f \circ u), v \rangle.$$

If moreover the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to a bundle morphism $\upsilon : TM \to TN$ locally in measure, then $Du = \upsilon$.

The Euclidean counterpart of this property is well-known [58, Theorem 6.1.7]. The uniform integrability assumption is essential for $p = 1$: otherwise the closure property fails already for classical Sobolev maps between Euclidean spaces.

Proof of Proposition 3.8. By the boundedness and bilocally uniform integrability assumptions, the sequence of bundle morphisms $(Du_\ell)_{\ell \in \mathbb{N}}$ is bilocally uniformly integrable. Proposition 3.7 applies and it remains to prove that $|Du|_{g_M^* \otimes g_N} \in L^p(M)$. By the boundedness and bilocally uniform integrability assumptions, up to a subsequence, the sequence of functions $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges weakly to some $w \in L^p(M)$. For every $f \in C^1_c(N, \mathbb{R}^k)$ with $k = \min(m, n)$ and $\ell \in \mathbb{N}$, we have

$$|D(f \circ u_\ell)|_{g_M^* \otimes g_k} \leq |f|_{\mathrm{Lip}} |Du_\ell|_{g_M^* \otimes g_N}.$$

Hence, for every $v \in C^1_c(M, TM \otimes \mathbb{R}^k)$,

$$\begin{aligned} -\int_M \langle f \circ u_\ell, \operatorname{div} v \rangle = \int_M \langle D(f \circ u_\ell), v \rangle &\leq \int_M |D(f \circ u_\ell)|_{g_M^* \otimes g_k} |v|_{g_M^* \otimes g_k} \\ &\leq |f|_{\mathrm{Lip}} \int_M |Du_\ell|_{g_M^* \otimes g_N} |v|_{g_M^* \otimes g_k} \end{aligned}$$

and thus passing to the limit,

$$-\int_M \langle f \circ u, \operatorname{div} v \rangle \leq |f|_{\mathrm{Lip}} \int_M |v|_{g_M^* \otimes g_k} w.$$

Since the map u is co-locally weakly differentiable, we deduce that

$$\int_M \langle D(f \circ u), v\rangle \leq |f|_{\mathrm{Lip}} \int_M |v|_{g_M^* \otimes g_k} w.$$

Since $v \in C_c^1(M, TM \otimes \mathbb{R}^k)$ is arbitrary we conclude that

$$|D(f \circ u)|_{g_M^* \otimes g_k} \leq |f|_{\mathrm{Lip}} w$$

almost everywhere in M. By the characterization of the norm of the derivative of Proposition 2.2, $|Du|_{g_M^* \otimes g_N} \leq w$ almost everywhere in M and thus by lower semi-continuity of the norm under weak convergence [14, Proposition 3.5], [44, Theorem 2.11], [58, Theorem 5.4.6]

$$\int_M |Du|^p_{g_M^* \otimes g_N} \leq \int_M w^p \leq \liminf_{\ell \to \infty} \int_M |Du_\ell|^p_{g_M^* \otimes g_N}. \qquad \square$$

4. Strong convergence in Sobolev spaces

In order to characterize eventually $\dot{W}^{1,p}(M, N)$ as a completion of smooth maps in some cases, as it has been done for Sobolev spaces defined in embedded manifolds [11,33,36], it is essential to have a metric for which this space is complete.

4.1. Definition and properties

We first define a natural notion of convergence in homogeneous Sobolev spaces.

Definition 4.1. Let $p \in [1, \infty)$. The sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $\dot{W}^{1,p}(M, N)$ *converges strongly* to $u \in \dot{W}^{1,p}(M, N)$ *in* $\dot{W}^{1,p}(M, N)$ if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure and $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$.

In order to define a distance that gives this convergence, we recall the definition of the Sasaki metric G^S on $T^*M \otimes TN$ when M and N are of class C^2 [52] (see also [21, Chapter 3, Exercise 2]): for every $\nu \in T(T^*M \otimes TN)$,

$$G^S(\nu) = (g_M \otimes g_N)(D\pi_{M \times N}(\nu)) + (g_M^* \otimes g_N)(K_{T^*M \otimes TN}(\nu)),$$

where $\pi_{M \times N}\colon T^*M \otimes TN \to M \times N$ is the canonical bundle morphism, and the connection $K_{T^*M \otimes TN}$ is defined for every $v \in T(T^*M)$ and $w \in T(TN)$ by $K_{T^*M \otimes TN}(v \otimes w) = K_{T^*M}(v) \otimes K_{TN}(w)$, where $K_{T^*M}\colon T(T^*M) \to T^*M$ and $K_{TN}\colon T(TN) \to TN$ are the respective Levi-Civita connection maps on T^*M and TN [56, Definition 3.9]. The associated geodesic distance is denoted by d^S. Given two co-locally weakly differentiable maps $u, v\colon M \to N$, we note that for almost every $x \in M$,

$$d(u, v)(x) = d^S((x, u(x)), (x, v(x))) \leq d^S(Du(x), Dv(x)), \qquad (4.1)$$

with the usual identification of $M \times N$ with $M \times N \times \{0\} \subset T^*M \otimes TN$.

As a consequence, the convergence of Definition 4.1 is induced by the distance

$$\dot{\delta}_{1,p}(u,v) = \delta(Du, Dv) + \Big\| |Du|_{g_M^* \otimes g_N} - |Dv|_{g_M^* \otimes g_N} \Big\|_{L^p(M)},$$

here δ is a distance of the form (3.1) taking the continuous distance d^S on $T^*M \otimes TN$. In fact, a sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ if and only if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure and

$$\lim_{\ell \to \infty} \int_M |Du_\ell|^p_{g_M^* \otimes g_N} = \int_M |Du|^p_{g_M^* \otimes g_N}.$$

This follows from the Euclidean counterpart [12, Proposition 4.7.30], [58, Theorem 4.2.6], [14, Exercise 4.17.3].

The Sobolev space with this distance is automatically complete.

Proposition 4.2. *If N is complete, then the metric space $(\dot{W}^{1,p}(M, N), \dot{\delta}_{1,p})$ is complete.*

Proof. Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a Cauchy sequence for the distance $\dot{\delta}_{1,p}$. By the completeness of $L^p(M)$, there exists a map $w \in L^p(M)$ such that $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to w in $L^p(M)$. By the properties of the distance defined in (3.1), there is a subsequence $(Du_{\ell_k})_{k \in \mathbb{N}}$ which is Cauchy almost everywhere with respect to the distance d^S [12, Exercise 4.7.60], [53, Théorème 5.8.31] and $(|Du_{\ell_k}|_{g_M^* \otimes g_N})_{k \in \mathbb{N}}$ converges to w almost everywhere on M. By the nonexpansiveness property (4.1), $(u_{\ell_k})_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the distance d. Since N is complete, the sequence $(u_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to a measurable map $u : M \to N$. Since d^S is continuous, it is complete on every compact subset of $T^*M \otimes TN$ [48, Theorem 45.1]. Therefore there exists a measurable bundle morphism $v : TM \to TN$ such that $(Du_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to v and $|v|_{g_M^* \otimes g_N} = w$. By the weak closure property (Proposition 3.8), $v = Du$. Therefore, $(Du_{\ell_k})_{k \in \mathbb{N}}$ converges almost everywhere to Du and thus with respect to δ. Since the sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ is Cauchy with respect to the distance δ, the whole sequence $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du with respect to the distance δ. □

This notion of convergence is strong enough to imply the continuity of the pointwise distance in Sobolev spaces.

Proposition 4.3. *Let $p \in [1, \infty)$. If the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ in $\dot{W}^{1,p}(M, N)$ converges strongly to $u \in \dot{W}^{1,p}(M,N)$ in $\dot{W}^{1,p}(M,N)$, then the sequence $(d(u_\ell,u))_{\ell \in \mathbb{N}}$ converges strongly to 0 in $\dot{W}^{1,p}(M)$.*

In practice, this proposition allows to deduce the continuity of non-linear Sobolev embedding from their classical linear counterpart.

Proof of Proposition 4.3. We first observe that for the geodesic distance d, for every $\xi \in TN$,

$$\lim_{\substack{\zeta \to \xi \\ \pi_N(\zeta) \neq \pi_N(\xi)}} (Dd)(\xi, \zeta) = 0,$$

and that $Du_\ell = Du$ almost everywhere on the set where $u_\ell = u$ (Proposition 1.11). Therefore, since $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure, $(D(d(u_\ell, u)))_{\ell \in \mathbb{N}}$ converges to 0 in measure. Since for every $\ell \in \mathbb{N}$, almost everywhere in M

$$|D(d(u_l, u))| \leq |Du_\ell|_{g_M^* \otimes g_N} + |Du|_{g_M^* \otimes g_N},$$

and the sequence $(|Du_\ell|_{g_M^* \otimes g_N})_{\ell \in \mathbb{N}}$ converges to $|Du|_{g_M^* \otimes g_N}$ in $L^p(M)$, by Lebesgue's dominated convergence theorem, the sequence $(D(d(u_l, u)))_{\ell \in \mathbb{N}}$ converges to 0 in $L^p(M)$. □

4.2. Comparison with other notions of convergence

We first remark that the notion of convergence is stable under isometric embedding. Therefore the convergence of Definition 4.1 coincides with the classical convergence defined by embedding into Euclidean spaces.

Proposition 4.4. *If $\iota : N \to \tilde{N}$ is an isometric embedding, then the sequence $(u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to $u \in \dot{W}^{1,p}(M, N)$ in $\dot{W}^{1,p}(M, N)$ if and only if the sequence $(\iota \circ u_\ell)_{\ell \in \mathbb{N}}$ converges strongly to $\iota \circ u \in \dot{W}^{1,p}(M, \tilde{N})$ in $\dot{W}^{1,p}(M, \tilde{N})$.*

Proof. Since ι is an embedding, $(D(\iota \circ u_\ell))_{\ell \in \mathbb{N}}$ converges to $D(\iota \circ u)$ in measure if and only if $(Du_\ell)_{\ell \in \mathbb{N}}$ converges to Du locally in measure. As ι is isometric, for every $\ell \in \mathbb{N}$,

$$|D(\iota \circ u_\ell)|_{g_M^* \otimes g_{\tilde{N}}} - |D(\iota \circ u)|_{g_M^* \otimes g_{\tilde{N}}} = |Du_\ell|_{g_M^* \otimes g_N} - |Du|_{g_M^* \otimes g_N};$$

the conclusion follows from the definition of convergence. □

The distance $\dot{\delta}^{\iota}_{1,p}(u, v) = \dot{\delta}_{1,p}(\iota \circ u, \iota \circ v)$ gives thus the same topology as $\dot{\delta}_{1,p}$. However the completeness of $\dot{W}^{1,p}(M, N)$ will then depend on the completeness of $\iota(N)$; a necessary condition is that $\iota(N)$ should be closed. When N is complete but not compact, the original Nash embedding theorem will give $\iota(N)$ which is not closed [49, 50]; it is however always possible when N is complete to have a Nash embedding theorem with $\iota(N)$ closed [47].

We would also like to compare this notion with the metric of Chiron [17, Section 1.6][1] for $u, v \in \dot{W}^{1,p}(M, N)$:

$$\dot{\delta}^{C}_{1,p}(u, v) = \delta(u, v) + \Big(\int_M \big||Du|_{g_M^* \otimes g_N} - |Dv|_{g_M^* \otimes g_N}\big|^p\Big)^{\frac{1}{p}},$$

[1] As the modulus of the derivative has several definitions, we have in fact a family of distances out of which we take the one that uses our notion of modulus of the derivative. Instead of introducing the notion of Lebesgue spaces into metric spaces, we consider convergence in measure for maps from M to N.

where δ is a distance of the form (3.1). In order to study the topological equivalence of these metrics, we give a criterion of convergence in measure of the derivative.

Proposition 4.5 (Criterion for convergence in measure). *Let* $(u_\ell)_{\ell\in\mathbb{N}}$ *be a sequence of co-locally weakly differentiable maps from* M *to* N*. If the sequence* $(u_\ell)_{\ell\in\mathbb{N}}$ *converges locally in measure to a co-locally weakly differentiable map* $u\colon M \to N$ *and if the sequence* $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ *converges to* $|Du|_{g_M^*\otimes g_N}$ *in* $L^1_{\mathrm{loc}}(M)$*, then the sequence* $(Du_\ell)_{\ell\in\mathbb{N}}$ *converges to* Du *locally in measure.*

As an immediate consequence we have the topological equivalence between $\dot{\delta}_{1,p}$ and $\dot{\delta}^C_{1,p}$.

Proposition 4.6. *Let* $p\in[1,\infty)$*. Let* $(u_\ell)_{\ell\in\mathbb{N}}$ *be a sequence of maps in* $\dot{W}^{1,p}(M,N)$ *and* $u\in\dot{W}^{1,p}(M,N)$*. The sequence* $(u_\ell)_{\ell\in\mathbb{N}}$ *converges strongly to* u *in* $\dot{W}^{1,p}(M,N)$ *if and only if* $\lim_{\ell\to\infty}\dot{\delta}^C_{1,p}(u_\ell,u)=0$.

This proposition is due to Chiron when $N=\mathbb{R}$ and $p>1$. The proof of Proposition 4.5 relies on the Balder-Visintin criterion of strong convergence.

Proposition 4.7 (Balder–Visintin criterion of strong convergence). [7, Theorem 1], [55, Corollary 2]. *Let* $(f_\ell)_{\ell\in\mathbb{N}}$ *be a sequence in* $L^1(U,\mathbb{R}^k)$*. If the sequence* $(f_\ell)_{\ell\in\mathbb{N}}$ *converges weakly to* $f\in L^1(U,\mathbb{R}^k)$ *and for almost every* $x\in U$*, the point* $f(x)$ *is an extreme point of*

$$\bigcap_{j\in\mathbb{N}}\overline{\operatorname{co}\{f_i(x)\colon i\ge j\}},$$

then the sequence $(f_\ell)_{\ell\in\mathbb{N}}$ *converges to* f *in* $L^1(U,\mathbb{R}^k)$.

For a set $A\subseteq\mathbb{R}^k$, the set $\operatorname{co} A$ denotes the *convex hull* of A, that is, the set of convex combinations of elements of A. A point c is an *extreme point* of a convex set C if $C\setminus\{c\}$ is convex.

Proof of Proposition 4.5. Let $U\subseteq N$ and $\varphi\in C^1(N,\mathbb{R}^n)$ be the extended local chart given by Lemma 1.6 and let $K\subseteq M$ be compact. In view of Proposition 3.7, and the weak compactness criterion in $L^1_{\mathrm{loc}}(M)$, the sequence $(D(\varphi\circ u_\ell))_{\ell\in\mathbb{N}}$ converges weakly to $D(\varphi\circ u)$ in $L^1(K)$.

By taking a subsequence, we can assume that the sequences $(u_\ell)_{\ell\in\mathbb{N}}$ and $(|Du_\ell|)_{\ell\in\mathbb{N}}$ converge almost everywhere in M. In order to apply the Balder–Visintin criterion of strong convergence, we note that for every $x\in M$,

$$D(\varphi\circ u_\ell)(x)\in D\varphi(u_\ell(x))\left(\bar{B}_{|Du_\ell(x)|_{g_M^*\otimes g_N}}\right)$$

and so

$$\overline{\operatorname{co}\{D(\varphi\circ u_k)(x)\colon k\ge\ell\}}\subseteq\overline{\operatorname{co}\left\{D\varphi(u_k(x))\left(\bar{B}_{|Du_k(x)|_{g_M^*\otimes g_N}}\right)\colon k\ge\ell\right\}}.$$

Hence, since for almost every $x \in K$, $(D\varphi(u_\ell(x)))_{\ell\in\mathbb{N}}$ converges to $D\varphi(u(x))$ and $(|Du_\ell(x)|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges to $|Du(x)|_{g_M^*\otimes g_N}$, we have

$$\bigcap_{\ell\in\mathbb{N}} \overline{\operatorname{co}\{D(\varphi\circ u_k)(x)\colon k\geq \ell\}} \subseteq D\varphi(u(x))\left(\bar{B}_{|Du(x)|_{g_M^*\otimes g_N}}\right).$$

We finally observe that for every $x \in u^{-1}(U)$, $|Du(x)|_{g_M^*\otimes g_N}$ is an extremal point of $\bar{B}_{|Du(x)|_{g_M^*\otimes g_N}}$ and $D\varphi(u(x))$ is invertible, therefore $D(\varphi\circ u)(x)$ is an extremal point of $D\varphi(u(x))(\bar{B}_{|Du(x)|_{g_M^*\otimes g_N}})$ and hence of $\bigcap_{\ell\in\mathbb{N}} \overline{\operatorname{co}\{D(\varphi\circ u_k)(x)\colon k\geq \ell\}}$. Hence by the Balder–Visintin criterion of strong convergence (Proposition 4.7), the sequence $(D(\varphi\circ u_\ell))_{\ell\in\mathbb{N}}$ converges to $D(\varphi\circ u)$ in $L^1(K\cap u^{-1}(U))$ and thus in measure on $K\cap u^{-1}(U)$. By covering M and N by countably many such compact sets K and extended local charts (φ, U), we obtain the conclusion. □

The reader will observe that our argument relies on the structure of the norm that endows $T^*M\otimes TN$. More precisely our proof requires the norm on $T^*M\otimes TN$ to be uniformly convex; this is not the case when $\min(\dim(M), \dim(N)) \geq 2$ for the operator norm.

Proposition 4.8. *Let $p\in(1,\infty)$. Let $(u_\ell)_{\ell\in\mathbb{N}}$ be a sequence of maps in $\dot{W}^{1,p}(M,N)$ and $u \in \dot{W}^{1,p}(M,N)$. If $(u_\ell)_{\ell\in\mathbb{N}}$ converges to u locally in measure and*

$$\lim_{\ell\to\infty}\int_M |Du_\ell|^p_{g_M^*\otimes g_N} = \int_M |Du|^p_{g_M^*\otimes g_N},$$

then $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges to $|Du|_{g_M^*\otimes g_N}$ in $L^p(M)$.

In particular the two metrics introduced by Chiron have the same convergent sequences [17, Lemma 2] for $p > 1$. By Proposition 4.6, this notion of convergence is also equivalent to convergence in $\dot{W}^{1,p}(M,N)$. When $p = 1$, the equivalence does not hold already in the Euclidean case [17, Lemma 2]. The proof relies on the Proposition 2.2.

Proof of Proposition 4.8. Since $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ is bounded in $L^p(M)$ and $p \in (1,\infty)$, by taking a subsequence, we can assume that $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges weakly to some w in $L^p(M)$. Since $(u_\ell)_{\ell\in\mathbb{N}}$ converges to u locally in measure, for every $f \in C^1_c(M,\mathbb{R}^k)$, $|D(f\circ u)|_{g_M^*\otimes g_k} \leq |f|_{\mathrm{Lip}}\, w$ almost everywhere in M. Hence, by Proposition 2.2, $|Du|_{g_M^*\otimes g_N} \leq w$ almost everywhere in M. On the other hand, by lower semicontinuity of the norm under weak convergence [14, Proposition 3.5], [44, Theorem 2.11], [58, Theorem 5.4.6] and by our assumption,

$$\int_M w^p \leq \liminf_{\ell\to\infty}\int_M |Du_\ell|^p_{g_M^*\otimes g_N} = \int_M |Du|^p_{g_M^*\otimes g_N}$$

and so $w = |Du|_{g_M^*\otimes g_N}$ almost everywhere in M. The sequence $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges thus weakly to $|Du|_{g_M^*\otimes g_N}$ in $L^p(M)$. Since $p \in (1,\infty)$, we conclude that $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges to $|Du|_{g_M^*\otimes g_N}$ in $L^p(M)$ [12, Corollary 4.7.16], [58, Exercise 5.3], [14, Exercise 4.19]. □

When $N = \mathbb{R}$, we already known that the Sobolev space is *not complete* under the Chiron distance [17, Lemma 2]. We extend this result to Sobolev spaces between Riemannian manifolds.

Proposition 4.9. *If M and N are nonempty Riemannian manifold and $p \in [1, \infty)$ then $(\dot{W}^{1,p}(M, N), \dot{\delta}^C_{1,p})$ is not complete.*

Proof. We give the proof when $M = (0, 1)$. The reader will adapt the proof to the general case. Choose $\gamma \in C^1([0, L], N)$ such that for every $t \in [0, L]$, $|\gamma'(t)|_{g_N} = 1$ and define, following Chiron [17, Lemma 2], for every $\ell \in \mathbb{N}$, the function $u_\ell : (0, 1) \to N$ for each $t \in (0, 1)$ by

$$u_\ell(t) = \gamma\bigl(\operatorname{dist}(t, \mathbb{Z}/\ell)\bigr).$$

For every $\ell \in \mathbb{N}$, the function u_ℓ is Lipschitz and $|u'{}_\ell|_{g_N} = 1$ almost everywhere in $(0, 1)$. Moreover, the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ converges uniformly to the constant map $u = \gamma(0)$. Since

$$\lim_{\ell\to\infty} \int_0^1 |u'_\ell|^p_{g_N} = 1 \neq 0 = \int_0^1 |u'|^p,$$

the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ does not converge in $\dot{W}^{1,p}((0, 1), N)$. By Proposition 4.6, the space $(\dot{W}^{1,p}(M, N), \dot{\delta}^C_{1,p})$ is not complete. □

4.3. Intrinsic distance

A natural candidate for an intrinsic distance is for $u, v \in \dot{W}^{1,p}(M, N)$:

$$\delta_{1,p}(u, v) = \left(\int_M d^S(Du, Dv)^p\right)^{\frac{1}{p}} \in [0, \infty].$$

This distance can be infinite. This will happen for instance if M has infinite Riemannian volume and u, v are distinct constant maps.

We first note that this distance characterizes Sobolev maps.

Proposition 4.10. *If $v \in \dot{W}^{1,p}(M, N)$ then $u \in \dot{W}^{1,p}(M, N)$ and $d(u, v) \in L^p(M)$ if and only if the map u is co-locally weakly differentiable and $d^S(Du, Dv) \in L^p(M)$.*

This distance characterizes also the simultaneous convergence in $\dot{W}^{1,p}(M, N)$. In the absence of a Poincaré inequality, the convergence associated to $\delta_{1,p}$ is stronger than the convergence of Definition 4.1.

Proposition 4.11. *If $(u_\ell)_{\ell\in\mathbb{N}}$ is a sequence in $\dot{W}^{1,p}(M, N)$ and $u \in \dot{W}^{1,p}(M, N)$ then the sequence $(d^S(Du_\ell, Du))_{\ell\in\mathbb{N}}$ converges to 0 in $L^p(M)$ if and only if the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M, N)$ and the sequence $(d(u_\ell, u))_{\ell\in\mathbb{N}}$ converges to 0 in $L^p(M)$.*

Proof. First assume that $(u_\ell)_{\ell\in\mathbb{N}}$ converges strongly to u in $\dot{W}^{1,p}(M,N)$ and $(d(u_\ell,u))_{\ell\in\mathbb{N}}$ converges to 0 in $L^p(M)$. By the definition of convergence in $\dot{W}^{1,p}(M,N)$ (Definition 4.1), $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges to $|Du|_{g_M^*\otimes g_N}$ in $L^p(M)$ and $(Du_\ell)_{\ell\in\mathbb{N}}$ converges to Du locally in measure. The latter convergence implies that $(d^S(Du_\ell,Du))_{\ell\in\mathbb{N}}$ converges to 0 in measure. Since for every $\ell\in\mathbb{N}$,

$$d^S(Du_\ell,Du)\le d(u_\ell,u)+|Du_\ell|_{g_M^*\otimes g_N}+|Du|_{g_M^*\otimes g_N}$$

almost everywhere in M, the conclusion follows from Lebesgue's dominated convergence theorem [12, Theorem 2.8.5].

Conversely, if $(d^S(Du_\ell,Du))_{\ell\in\mathbb{N}}$ converges to 0 in $L^p(M)$, then $(d^S(Du_\ell, Du))_{\ell\in\mathbb{N}}$ converges to 0 in measure and thus $(Du_\ell)_{\ell\in\mathbb{N}}$ converges to Du locally in measure. Moreover, $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges to $|Du|_{g_M^*\otimes g_N}$ in measure. Since for every $\ell\in\mathbb{N}$,

$$|Du_\ell|_{g_M^*\otimes g_N}\le d^S(Du_\ell,Du)+|Du|_{g_M^*\otimes g_N}$$

almost everywhere in M, by Lebesgue's dominated convergence theorem, the sequence $(|Du_\ell|_{g_M^*\otimes g_N})_{\ell\in\mathbb{N}}$ converges to $|Du|_{g_M^*\otimes g_N}$ in $L^p(M)$. Finally, by the nonexpansiveness property (4.1), it is clear that the sequence $(d(u_\ell,u))_{\ell\in\mathbb{N}}$ converges to 0 in $L^p(M)$. □

Finally, Sobolev spaces are complete under this metric.

Proposition 4.12. *If N is complete, then the metric space $(\dot{W}^{1,p}(M,N),\delta_{1,p})$ is complete.*

Proof. Let $(u_\ell)_{\ell\in\mathbb{N}}$ be a Cauchy sequence for the metric $\delta_{1,p}$. There exists a subsequence $(Du_{\ell_k})_{k\in\mathbb{N}}$ which is a Cauchy sequence for d^S almost everywhere in M. By the nonexpansiveness property (4.1), $(u_{\ell_k})_{k\in\mathbb{N}}$ is a Cauchy sequence for d almost everywhere in M. Since N is complete, the sequence $(u_{\ell_k})_{k\in\mathbb{N}}$ converges almost everywhere to a map $u:M\to N$.

Since for every $k\in\mathbb{N}$, $|Du_{\ell_k}|_{g_M^*\otimes g_N}\le d^S(Du_{\ell_k},Du_{\ell_0})+|Du_{\ell_0}|_{g_M^*\otimes g_N}$ almost everywhere in M, we deduce that the sequence $(|Du_{\ell_k}|)_{k\in\mathbb{N}}$ is bounded almost everywhere in M and

$$\limsup_{k\to\infty}\int_M |Du_{\ell_k}|^p_{g_M^*\otimes g_N}<\infty.$$

Since the distance d^S is complete on any compact subset of $T^*M\otimes TN$, the sequence $(Du_{\ell_k})_{k\in\mathbb{N}}$ converges almost everywhere to a bundle morphism $\upsilon:TM\to TN$. By the closure property (Proposition 3.8), we deduce that $u\in\dot{W}^{1,p}(M,N)$ and $Du=\upsilon$. By Fatou's lemma, for every $k\in\mathbb{N}$,

$$\delta_{1,p}(u_{\ell_k},u)\le\liminf_{j\to\infty}\delta_{1,p}(u_{\ell_k},u_{\ell_j}),$$

and thus $\lim_{k\to\infty}\delta_{1,p}(u_{\ell_k},u)=0$. Since the sequence $(u_\ell)_{\ell\in\mathbb{N}}$ is a Cauchy sequence for $\delta_{1,p}$, the conclusion follows. □

Remark 4.13. In the above properties, the Sasaki metric can be generalized to *strongly concordant* (with the Euclidean structure $g_M^* \otimes g_N$) metrics, that is, any metric G on $T(T^*M \otimes TN)$ such that

(a) for every $v \in T(T^*M \otimes TN)$,

$$G(D\pi_{M\times N}(v)) \leq G(v);$$

(b) there exists $\kappa > 0$ such that for every $v \in T(T^*M \otimes TN)$,

$$(\kappa^{-1} D(g_M^* \otimes g_N)(v))^2 \leq 4(g_M^* \otimes g_N)(\pi_{T^*M\otimes TN}(v))G(v),$$

where $\pi_{T^*M\otimes TN} \colon T(T^*M \otimes TN) \to T^*M \otimes TN$ is the canonical bundle morphism, and for every $e \in T^*M \otimes TN$,

$$G(\mathrm{Vert}_e(e)) \leq \kappa^2 (g_M^* \otimes g_N)(e),$$

where Vert_e is the vertical lift defined for each $v \in \pi_{M\times N}^{-1}(\{\pi_{M\times N}(e)\})$ by

$$\mathrm{Vert}_e(v) = \frac{d}{dt}(e + t\, v)_{|t=0} \in T_e(T^*M \otimes TN).$$

Such metrics are natural metrics on vector bundles [10,32,41]. An another example is the Cheeger-Gromoll metric [16].

Given $\iota \colon N \to \mathbb{R}^\nu$ an isometric embedding such that $\iota(N)$ is closed, we have two natural distances under the hand : the distance $\delta_{1,p}$ and the induced distance,

$$\delta_{1,p}^\iota(u, v) = \left(\int_M \left(|\iota \circ u - \iota \circ v|^2 + |D(\iota \circ u) - D(\iota \circ v)|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

(The distance on the right is in fact the Sasaki distance on $T^*M \otimes T\mathbb{R}^\nu$.) The next two propositions show that even if they induce the same topology and they are both complete, they are not uniformly equivalent in general.

Proposition 4.14. *If M is a nonempty Riemannian manifold, $p \in [1, \infty)$ and $n \geq 2$, then the identity map*

$$i \colon \left(\dot{W}^{1,p}(M, \mathbb{S}^n), \delta_{1,p} \right) \to \left(\dot{W}^{1,p}(M, \mathbb{S}^n), \delta_{1,p}^\iota \right)$$

is not uniformly continuous.

Proof. We begin by considering the case $M = \mathbb{R}$. Choose $y \in \mathbb{S}^n$ and $\rho \colon \mathbb{S}^n \to \mathbb{S}^n$ be a non-identical isometry such that $\rho(y) = y$ and $u \in C^1(\mathbb{R}, \mathbb{S}^n)$ such that for every $x \in \mathbb{R} \setminus [-1, 1]$, $\rho(u(x)) = y$ and $\rho(u(x)) \neq u(x)$ in $(-1, 1)$.

Let $u_\lambda : \mathbb{R} \to \mathbb{S}^n$ be defined for every $t \in \mathbb{R}$ by $u_\lambda(t) = u(t/\lambda)$ and let $v_\lambda = \rho \circ u_\lambda$. Since ρ is an isometry, $D\rho\colon T\mathbb{S}^n \to T\mathbb{S}^n$ is an isometry on tangent vectors. Since $n \geq 2$, for every $e \in T\mathbb{S}^n$ there exists a path γ such that $P^\gamma(e) = D\rho(e)$; the length of γ with respect to the Sasaki metric G^S can be bounded uniformly by 2π. Therefore, we have for every $e \in T\mathbb{S}^n$, $d^S(e, D\rho(e)) \leq 2\pi$ and we deduce that

$$\delta_{1,p}(u_\lambda, v_\lambda) = \biggl(\int_{\mathbb{R}} d^S(Du_\lambda, Dv_\lambda)^p\biggr)^{\frac{1}{p}} \leq 2\pi\, \lambda^{\frac{1}{p}}.$$

On the other hand,

$$\delta^\iota_{1,p}(u_\lambda, v_\lambda) = \biggl(\int_{\mathbb{R}} \lambda\bigl(|u - \rho\circ u|^2 + \lambda^{-2}|Du - D(\rho\circ u)|^2\bigr)^{\frac{p}{2}}\biggr)^{\frac{1}{p}},$$

Consequently, $\lim_{\lambda\to 0} \delta_{1,p}(u_\lambda, v_\lambda) = 0$ while $\liminf_{\lambda\to 0} \delta^\iota_{1,p}(u_\lambda, v_\lambda) = \infty$.

In the general case, let $a \in M$ and let $\rho \in (0, \rho_i(a))$ where $\rho_i(a)$ is the injectivity radius of the Riemannian manifold M at the point a. We define then the maps $u_\lambda : M \to \mathbb{S}^n$ and $v_\lambda : M \to \mathbb{S}^n$ for $\lambda > 0$ and $x \in M$ by

$$u_\lambda(x) = u\Bigl(\frac{d(x,a) - \rho}{\lambda}\Bigr) \qquad \text{and} \qquad v_\lambda(x) = -u\Bigl(\frac{d(x,a) - \rho}{\lambda}\Bigr);$$

the non-uniform continuity follows as in the case $M = \mathbb{R}$ treated above. □

Proposition 4.15. *If M is a nonempty Riemannian manifold and $p \in [1, \dim M)$, then the identity embedding map*

$$i\colon (\dot{W}^{1,p}(M, \mathbb{S}^1), \delta^\iota_{1,p}) \to (\dot{W}^{1,p}(M, \mathbb{S}^1), \delta_{1,p})$$

is not uniformly continuous.

Proof. We define $u_\lambda = (\cos(\varphi_\lambda), \sin(\varphi_\lambda))$ and $v_\lambda = (-\cos(\varphi_\lambda), \sin(\varphi_\lambda))$, where

$$\varphi_\lambda(x) = \begin{cases} \dfrac{\pi}{2} & \text{if } d(x,y) \geq 2\lambda \\ \dfrac{d(x,y) - \lambda}{2\lambda}\pi & \text{if } \lambda \leq d(x,y) < 2\lambda \\ \lambda \sin \dfrac{(d(x,y) - \lambda)\pi}{2\lambda^{1 + \frac{\dim M}{p}}} & \text{if } d(x,y) < \lambda \end{cases}$$

and the point $y \in M$ is fixed. We observe that

$$\liminf_{\lambda\to 0} \delta_{1,p}(u_\lambda, v_\lambda) = 2 \liminf_{\lambda \to 0} \biggl(\int_M \bigl(\cos(\varphi_\lambda)^2 + |D\varphi_\lambda|^2\bigr)^{\frac{p}{2}}\biggr)^{\frac{1}{p}} > 0.$$

On the other hand, since $p < \dim M$,

$$\limsup_{\lambda\to 0} \delta^\iota_{1,p}(u_\lambda, v_\lambda) = 2 \limsup_{\lambda\to 0} \biggl(\int_M \bigl(\cos(\varphi_\lambda)^2 + \sin(\varphi_\lambda)^2|D\varphi_\lambda|^2\bigr)^{\frac{p}{2}}\biggr)^{\frac{1}{p}} = 0. \ \square$$

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