

Extensors and the Hilbert scheme

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Abstract. The Hilbert scheme $\mathbf{Hilb}_{p(t)}^n$ parametrizes closed subschemes and families of closed subschemes in the projective space \mathbb{P}^n with a fixed Hilbert polynomial $p(t)$. It can be realized as a closed subscheme of a Grassmannian or a product of Grassmannians. In this paper we consider schemes over a field k of characteristic zero and we present a new proof of the existence of the Hilbert scheme as a subscheme of the Grassmannian $\mathbf{Gr}_{p(r)}^{N(r)}$, where $N(r) = h^0(\mathcal{O}_{\mathbb{P}^n}(r))$. Moreover, we exhibit explicit equations defining it in the Plücker coordinates of the Plücker embedding of $\mathbf{Gr}_{p(r)}^{N(r)}$.

Our proof of existence does not need some of the classical tools used in previous proofs, as flattening stratifications and Gotzmann's Persistence Theorem.

The degree of our equations is $\deg p(t) + 2$, lower than the degree of the equations given by Iarrobino and Kleiman in 1999 and also lower (except for the case of hypersurfaces) than the degree of those proved by Haiman and Sturmfels in 2004 after Bayer's conjecture in 1982.

The novelty of our approach mainly relies on the deeper attention to the intrinsic symmetries of the Hilbert scheme and on some results about Grassmannian based on the notion of extensors.

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Introduction

The study of Hilbert schemes is a very active domain in algebraic geometry. The Hilbert scheme was introduced by Grothendieck [17] as the scheme representing the contravariant functor $\mathbf{Hilb}_{p(t)}^n : (\text{Schemes}/k)^\circ \rightarrow (\text{Sets})$ that associates to a scheme Z the set of flat families $X \hookrightarrow \mathbb{P}^n \times_{\text{Spec } k} Z \rightarrow Z$ whose fibers have Hilbert polynomial $p(t)$. Thus, the Hilbert scheme $\mathbf{Hilb}_{p(t)}^n$ parametrizes the universal family of subschemes in the projective space \mathbb{P}^n with Hilbert polynomial $p(t)$. It is natural

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to embed the Hilbert functor in a suitable Grassmann functor and to construct the Hilbert scheme as a subscheme of a Grassmannian $\mathbf{Gr}_{p(r)}^{N(r)}$ for a sufficiently large r , where $N(t)$ equals $\binom{n+t}{n}$.

Over the years, several authors addressed the problem of finding alternative proofs of the representability of the Hilbert functor, for the purpose of giving simpler equations for the representing scheme. This is also the aim of the present paper. In fact, we present a new proof of the existence of the Hilbert scheme as a subscheme of $\mathbf{Gr}_{p(r)}^{N(r)}$ and we exhibit explicit equations defining it in the case of a field k of characteristic 0.

There are some reasons for which we consider our work significant that concern the tools used in the proofs and the shape of the equations, in particular the degree.

In order to simplify Grothendieck's proof, a first crucial point is the concept of regularity that Mumford introduced for the choice of the degree r [26,27]. A further simplification is due to Gotzmann, whose Regularity Theorem gives a formula for the minimum r only depending on $p(t)$ [15]. Other key tools and results that usually appear in this context are flattening stratifications, fitting ideals, Gotzmann's Persistence Theorem and Macaulay's Estimates on the Growth of Ideals.

In this paper, the number r is always that given by Gotzmann's formula and in our proof we make use of Macaulay's Estimates, but we do not need any of the other quoted results. We replace them by a deeper attention to the inner symmetries of the Hilbert scheme induced by the action of the projective linear group on \mathbb{P}^n , and by exploiting the nice combinatorial properties of Borel-fixed ideals. These are far from being new ideas to study Hilbert schemes. Indeed, they play a central role in some of the more celebrated and general achievements on this topic, first of all Hartshorne's proof of connectedness [19]. However, to our knowledge, they have never been used before to prove the existence or to derive equations for $\mathbf{Hilb}_{p(t)}^n$.

The proof of the representability of the Hilbert functor given by Haiman and Sturmfels in [18] following Bayer's strategy starts with a reduction to the local case; the open cover of $\mathbf{Hilb}_{p(t)}^n$ they consider is that induced by the standard open cover of $\mathbf{Gr}_{p(r)}^{N(r)}$. We introduce a new open cover for the Grassmann functor, that we will call the *Borel open cover*. It is obtained considering only a few open subfunctors $\mathbf{G}_{\mathcal{J}}$ of the standard cover, each corresponding to a Borel-fixed ideal generated by $N(r) - p(r)$ monomials of degree r , and all the open subfunctors $\mathbf{G}_{\mathcal{J},g}$, for every $g \in \mathrm{PGL}(n+1)$, in their orbit (Proposition 3.2). The Borel open subfunctors $\mathbf{H}_{\mathcal{J},g}$ of the Hilbert functor are defined accordingly.

Restricting to each Borel subfunctor $\mathbf{G}_{\mathcal{J},g}$, the properties of J -marked sets and bases over a Borel-fixed ideal J developed in [24] (and later used in [1]) allow us to prove that $\mathbf{H}_{\mathcal{J},d}$ is representable and to obtain a new proof of the existence of $\mathbf{Hilb}_{p(t)}^n$ (Theorem 4.9).

Towards the aim of deriving equations for the Hilbert scheme, we then expand the notion of marked set to the universal element of the family

$$\mathcal{F} \hookrightarrow \mathbb{P}^n \times_{\mathrm{Spec} k} \mathbf{Gr}_{p(r)}^{N(r)} \rightarrow \mathbf{Gr}_{p(r)}^{N(r)}$$

parameterized by the Grassmannian and to its exterior powers. Indeed, exploiting the notion of an *extensor* and its properties, we obtain a description of the universal element by a set of bi-homogeneous generators of bi-degree $(r, 1)$ in $k[x, \Delta]$, where x and Δ are compact notation for the set of variables on \mathbb{P}^n and the Plücker coordinates on the Grassmannian. We also obtain a similar description (again linear w.r.t. Δ) for the exterior powers of the universal element of \mathcal{F} (Theorem 5.10). These sets of generators allow us to write explicitly a set of equations for the Hilbert scheme in the ring $k[\Delta]$ of the Plücker coordinates (Theorem 6.5).

The degree of our equations is upper bounded by $d + 2$, where $d := \deg p(t)$ is the dimension of the subschemes of \mathbb{P}^n parametrized by $\mathbf{Hilb}_{p(t)}^n$. It is interesting that the degree of the equations is so close to the geometry of the involved objects. Furthermore, $d + 2$ is lower than the degree of the other known sets of equations for the embedding of the Hilbert scheme in a single Grassmannian. We quote the equations of degree $N(r + 1) - p(r + 1) + 1$ in local coordinates given by Iarrobino and Kleiman [21, Proposition C.30], and the equations of degree $n + 1$ in the Plücker coordinates conjectured by Bayer in his thesis [3] and obtained by Haiman and Sturmfels as a special case of a more general result in [18].

By the way, we observe that our method, applied with slightly different strategies, also allows to obtain sets of equations very similar to those by Iarrobino and Kleiman and by Haiman and Sturmfels (Theorems 6.6 and 6.7).

At the end of the paper we apply our results in order to compute a set of equations defining the Hilbert schemes of 2 points in \mathbb{P}^2 , \mathbb{P}^3 and \mathbb{P}^4 and of 3 points in \mathbb{P}^2 . In particular, we illustrate in detail our method in the case of \mathbf{Hilb}_2^2 and we compare the equations we obtain with those obtained by Brodsky and Sturmfels [7]. We observe that the two sets of equations, though different, generate the same ideal, more precisely the saturated ideal of \mathbf{Hilb}_2^2 in $\mathbf{Gr}_2^6 \subset \mathbb{P}^{14}$. Our equations describe the saturated ideal also in the case of \mathbf{Hilb}_3^2 in $\mathbf{Gr}_2^{10} \subset \mathbb{P}^{44}$, \mathbf{Hilb}_2^4 in $\mathbf{Gr}_2^{15} \subset \mathbb{P}^{104}$ and \mathbf{Hilb}_3^2 in $\mathbf{Gr}_3^{10} \subset \mathbb{P}^{119}$, but we do not know if this nice property holds in general. However, the lower degree marks a significant step forward in order to compute this special ideal (see Table 7.1) and allows further experiments and investigations.

Let us now explain the structure of the paper. In Section 2, we recall some properties that we will use throughout the paper. In particular, we describe the Hilbert functor, its relation with the Grassmann functor and the standard open cover. In Section 3, we introduce the Borel open cover. Section 4 contains the generalities about marked sets and bases over Borel-fixed ideals and it ends with the first main result of the paper, namely Theorem 4.9 on the representability of the Hilbert functor. Section 5 contains the results on Grassmannians based on the theory of extensors (Theorem 5.10). In Section 6, after some new technical results about marked bases, we present the equations defining the Hilbert scheme and we prove their correctness (Theorem 6.5). In Subsections 6.1 and 6.2 we derive equations similar to those by Iarrobino-Kleiman and by Haiman-Sturmfels. In Section 7, we illustrate the constructions and results of the paper in the case of Hilbert schemes describing 2 or 3 points.

1. Notation

Let k be a field of characteristic 0. In the following $k[x]$ will denote the polynomial ring $k[x_0, \dots, x_n]$ and \mathbb{P}^n the n -dimensional projective space $\text{Proj } k[x]$. For a k -algebra A , we will denote by $A[x] := A \otimes_k k[x]$ the polynomial ring with coefficients in A and by \mathbb{P}_A^n the projective space $\text{Proj } A[x] = \mathbb{P}^n \times_{\text{Spec } k} \text{Spec } A$. As usual, for a subset E of a ring R , we denote by $\langle E \rangle$ the ideal of R generated by E and for a subset F of an R -module M , we denote by $\langle F \rangle$ the R -submodule of M generated by F ; we sometimes write ${}_R(E)$ and ${}_R(F)$ when more than one ring is involved.

Let us now consider a scheme $X \subset \mathbb{P}_A^n$. For each prime ideal \mathfrak{p} of A , we denote by $A_{\mathfrak{p}}$ the localization in \mathfrak{p} , by $k(\mathfrak{p})$ the residue field and by $X_{\mathfrak{p}}$ the fiber of the structure morphism $X \rightarrow \text{Spec } A$. The Hilbert polynomial $p_{\mathfrak{p}}(t)$ of $X_{\mathfrak{p}}$ is defined as

$$p_{\mathfrak{p}}(t) = \dim_{k(\mathfrak{p})} H^0(X_{\mathfrak{p}}, \mathcal{O}_{X_{\mathfrak{p}}}(t)) \otimes_k k(\mathfrak{p}), \quad t \gg 0.$$

If X is flat over $\text{Spec } A$ and the Hilbert polynomial $p_{\mathfrak{p}}(t)$ of every localization coincides with $p(t)$, then $p(t)$ is called the Hilbert polynomial of X (for further details see [20, III, Section 9]). There exists a positive integer r only depending on $p(t)$, called *Gotzmann number*, for which the ideal sheaf \mathcal{I}_X of each scheme X with Hilbert polynomial $p(t)$ is r -regular (in the sense of Castelnuovo-Mumford regularity). By Gotzmann's Regularity Theorem ([15, Satz (2.9)] and [21, Lemma C.23]), this implies the surjectivity of the morphism

$$H^0(\mathcal{O}_{\mathbb{P}_A^n}(r)) \xrightarrow{\phi_X} H^0(\mathcal{O}_X(r)).$$

We will denote by $N(t)$ the dimension of $k[x]_t$. The polynomial $q(t) := N(t) - p(t)$ is the Hilbert polynomial of the saturated ideal defining X and it is called the *volume polynomial* of X . In particular, for $t = r$ the Gotzmann number of $p(t)$, we set $p := p(r)$, $q := q(r)$ and $N := N(r)$.

We will use the usual notation for terms $x^\alpha := x_0^{\alpha_0}, \dots, x_n^{\alpha_n}$, where $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$. When a term order comes into play, we assume the variables ordered as $x_0 < \dots < x_n$; we will denote by $<_{\text{DegRevLex}}$ and $<_{\text{Lex}}$ the degree reverse lexicographic and the lexicographic orders. We will denote by $x^{\alpha(i)}$ the i -terms of degree r in descending DegRevLex order. For any term x^α , let $\min(x^\alpha)$ and $\max(x^\alpha)$ denote respectively the minimal and the maximal variable which divides x^α .

For any polynomial $f \in A[x]$, the support $\text{Supp}(f)$ of f is the set of terms that appear in f with non-zero coefficient and $\text{coeff}_x(f) \subset A$ is the set of coefficients of the terms in $\text{Supp}(f)$; with the obvious meaning, we use the notation $\text{coeff}_x(U)$ also if U is a subset of $A[x]$.

We loosely denote by the same letter the monomial ideals in $k[x]$ and that in $A[x]$ generated by the same set of terms. If J is a monomial ideal, we will denote by B_J its minimal monomial basis and by $\mathcal{N}(J)$ the set of terms in $k[x] \setminus J$. For

a subset V of a standard graded module $R = \bigoplus_t R_t$, V_s and $V_{\geq s}$ will denote respectively $V \cap R_s$ and $V \cap \bigoplus_{t \geq s} R_t$.

An s -multi-index $\mathcal{H} = (h_1, \dots, h_s)$ is an ordered sequence $h_1 < h_2 < \dots < h_s$ in $\{1, \dots, N\}$; its complementary \mathcal{H}^c is the $(N - s)$ -multi-index with entries in the set $\{1, \dots, N\} \setminus \mathcal{H}$. For any s -multi-index \mathcal{H} , we will denote by $\varepsilon_{\mathcal{H}} \in \{-1, 1\}$ the signature of the permutation $(1, \dots, N) \mapsto \mathcal{H}, \mathcal{H}^c$. Moreover, if $\mathcal{H} \subset \mathcal{K}$, $\varepsilon_{\mathcal{H}}^{\mathcal{K}}$ is the signature of $\mathcal{K} \mapsto \mathcal{H}, \mathcal{K} \setminus \mathcal{H}$. For every $m \leq N - p$, we will denote by $\mathcal{E}^{(m)}$ the set of all $(p + m)$ -multi-indices.

For every $\mathcal{I} \in \mathcal{E}^{(0)}$, $J(\mathcal{I})$ is the ideal generated by the terms $x^{\alpha(j)}$ corresponding to the indices $j \in \mathcal{I}^c$.

2. Hilbert and Grassmann functors

In the following, $\underline{\mathbf{Hilb}}_{p(t)}^n$ will denote the Hilbert functor $(\text{Schemes}/k)^\circ \rightarrow (\text{Sets})$ that associates to an object Z of the category of schemes over k the set

$$\underline{\mathbf{Hilb}}_{p(t)}^n(Z) = \{X \subset \mathbb{P}^n \times_{\text{Spec } k} Z \mid X \rightarrow Z \text{ flat with Hilbert polynomial } p(t)\}.$$

and to any morphism of schemes $f: Z \rightarrow Z'$ the map

$$\begin{aligned} \underline{\mathbf{Hilb}}_{p(t)}^n(f): \underline{\mathbf{Hilb}}_{p(t)}^n(Z') &\rightarrow \underline{\mathbf{Hilb}}_{p(t)}^n(Z) \\ X' &\mapsto X' \times_{Z'} Z. \end{aligned}$$

It is easy to prove that $\underline{\mathbf{Hilb}}_{p(t)}^n$ is a Zariski sheaf [27, Section 5.1.3]; hence, we can consider it as a covariant functor from the category of noetherian k -algebras [28, Lemma E.11]

$$\underline{\mathbf{Hilb}}_{p(t)}^n: (k\text{-Algebras}) \rightarrow (\text{Sets})$$

such that for every finitely generated k -algebra A

$$\underline{\mathbf{Hilb}}_{p(t)}^n(A) = \{X \subset \mathbb{P}_A^n \mid X \rightarrow \text{Spec } A \text{ flat with Hilbert polynomial } p(t)\}.$$

and for any k -algebra morphism $f: A \rightarrow B$

$$\begin{aligned} \underline{\mathbf{Hilb}}_{p(t)}^n(f): \underline{\mathbf{Hilb}}_{p(t)}^n(A) &\rightarrow \underline{\mathbf{Hilb}}_{p(t)}^n(B) \\ X &\mapsto X \times_{\text{Spec } A} \text{Spec } B. \end{aligned}$$

The Hilbert scheme $\mathbf{Hilb}_{p(t)}^n$ is defined as the scheme representing the Hilbert functor. Our notation for the Hilbert functor follows that used for instance in [18], where the functor of points of a scheme Z is denoted by \underline{Z} . Note that we are not assuming the representability of $\underline{\mathbf{Hilb}}_{p(t)}^n$ as a known fact, but we will prove it at the end of Section 4.

Let us briefly recall the strategy of the construction of the Hilbert scheme based on Castelnuovo-Mumford regularity and Gotzmann number. The following proposition suggests to look for an embedding in a representable functor and reduce to the local case.

Proposition 2.1 ([18, Proposition 2.7 and Corollary 2.8]). *Let Z be a scheme and $\eta: \mathbf{F} \rightarrow \underline{Z}$ be a natural transformation of functors $(k\text{-Algebras}) \rightarrow (\text{Sets})$, where \mathbf{F} is a Zariski sheaf. Suppose that \underline{Z} has a cover of open subsets \underline{U} such that each subfunctor $\eta^{-1}(\underline{U}) \subseteq \mathbf{F}$ is representable. Then, also \mathbf{F} is representable.*

Moreover, if the natural transformations $\eta^{-1}(\underline{U}) \rightarrow \underline{U}$, given by restricting η , are induced by closed embeddings of schemes, then so is η .

The overall strategy introduced by Bayer [3] for the construction of the Hilbert scheme uses an embedding in a Grassmann functor (for a detailed discussion we refer Section 2 of [18] and to [11, Section VI.1]). If $X \in \underline{\mathbf{Hilb}}_{p(t)}^n(A)$, then by flatness $H^0(\mathcal{O}_X(r))$ is a locally free A -module of rank $p(r)$. Hence, the surjective map $\phi_X: H^0(\mathcal{O}_{\mathbb{P}_A^n}(r)) \simeq A^N \rightarrow H^0(\mathcal{O}_X(r))$ is an element of the set defined by the Grassmann functor $\underline{\mathbf{Gr}}_p^N$ over A . Indeed, the Grassmann functor $\underline{\mathbf{Gr}}_p^N: (k\text{-Algebras}) \rightarrow (\text{Sets})$ associates to every finitely generated k -algebra A the set

$$\underline{\mathbf{Gr}}_p^N(A) = \left\{ \begin{array}{l} \text{isomorphism classes of epimorphisms} \\ \pi: A^N \rightarrow P \text{ of locally free modules of rank } p \end{array} \right\}.$$

Equivalently, we can define

$$\underline{\mathbf{Gr}}_p^N(A) = \left\{ \begin{array}{l} \text{submodules } L \subseteq A^N \text{ such that} \\ A^N/L \text{ is locally free of rank } p \end{array} \right\}. \quad (2.1)$$

In the second formulation, π is the canonical projection $\pi_L: A^N \rightarrow A^N/L$. This functor is representable and the representing scheme $\underline{\mathbf{Gr}}_p^N$ is called the Grassmannian (see [30, Section 16.7]).

We fix the canonical basis $\{a_1, \dots, a_N\}$ for A^N and the isomorphism $A^N \simeq H^0(\mathcal{O}_{\mathbb{P}_A^n}(r))$ given by $a_i \mapsto x^{\alpha(i)}$. Thus, we obtain a universal family

$$\mathcal{F} \hookrightarrow \mathbb{P}^n \times \underline{\mathbf{Gr}}_p^N \rightarrow \underline{\mathbf{Gr}}_p^N \quad (2.2)$$

parameterized by the Grassmannian and the natural transformation of functors

$$\mathcal{H}: \underline{\mathbf{Hilb}}_{p(t)}^n \rightarrow \underline{\mathbf{Gr}}_p^N$$

sending $X \in \underline{\mathbf{Hilb}}_{p(t)}^n(A)$ to $\pi_X: H^0(\mathcal{O}_{\mathbb{P}_A^n}(r)) \rightarrow H^0(\mathcal{O}_X(r)) \in \underline{\mathbf{Gr}}_p^N(A)$ (or equivalently to $L = H^0(\mathcal{I}_X(r))$).

The Grassmannian has the following well-known open cover that we call the *standard open cover* of $\underline{\mathbf{Gr}}_p^N$. Let us fix a basis $\{e_1, \dots, e_p\}$ for A^p . For every $\mathcal{I} = (i_1, \dots, i_p) \in \mathcal{E}^{(0)}$, let us consider the injective morphism

$$\begin{aligned} \Gamma_{\mathcal{I}}: A^p &\rightarrow A^N \\ e_j &\mapsto a_{i_j} \end{aligned}$$

and the subfunctor $\underline{\mathbf{G}}_{\mathcal{I}}$ that associates to every noetherian k -algebra A the set

$$\underline{\mathbf{G}}_{\mathcal{I}}(A) = \left\{ L \in \underline{\mathbf{Gr}}_p^N(A) \text{ such that } \pi_L \circ \Gamma_{\mathcal{I}} \text{ is surjective} \right\}.$$

Proposition 2.2. *For $\mathcal{I} \in \mathcal{E}^{(0)}$, the $\underline{\mathbf{G}}_{\mathcal{I}}$ are open subfunctors of $\underline{\mathbf{Gr}}_p^N$ that cover it.*

Proof. See [29, Section 22.22]. \square

Remark 2.3. For every $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ the map $\pi_L \circ \Gamma_{\mathcal{I}}$ is an isomorphism, as it is a surjective morphism from a free A -module to a locally free A -module of the same rank. Therefore, L is the kernel of the epimorphism $\phi_L := (\pi_L \circ \Gamma_{\mathcal{I}})^{-1} \circ \pi_L: A^N \rightarrow A^p$ such that $\phi_L(a_{i_j}) = e_j$ for every $i_j \in \mathcal{I}$.

On the other hand, the kernel of every surjective morphisms $\phi: A^N \rightarrow A^p$ sending a_{i_j} to e_j is by definition a module $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$: we will write ϕ_L instead of ϕ to emphasize this correspondence.

Every map ϕ_L is completely determined by the images of the $q = N - p$ elements a_h with $h \in \mathcal{I}^c$. If $\phi_L(a_h) = \sum_{j=1}^p \gamma_{hj} e_j = \phi_L \left(\sum_{j=1}^p \gamma_{hj} a_{i_j} \right)$, the kernel L contains the free A -module L' generated by the q elements $b_h := a_h - \sum_{j=1}^p \gamma_{hj} a_{i_j} \in A^N$ with $h \in \mathcal{I}^c$. Then, $A^N = L' \oplus \langle a_{i_j} \mid i_j \in \mathcal{I} \rangle \subseteq L \oplus \langle a_{i_j} \mid i_j \in \mathcal{I} \rangle \subseteq A^N$, so that $L = L'$ and A^N/L are free A -modules of rank q and p respectively.

Through the fixed isomorphism $A^N \simeq H^0(\mathcal{O}_{\mathbb{P}_A^n}(r))$ given by $a_j \mapsto x^{\alpha(j)}$, the elements b_h correspond to polynomials $f_{\alpha(h)} := x^{\alpha(h)} - \sum_{j=1}^p \gamma_{hj} x^{\alpha(i_j)} \in H^0(\mathcal{O}_{\mathbb{P}_A^n}(r))$. In this way, for $L = \mathcal{H}(A)(X) \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$, the polynomials $f_{\alpha(h)}$ generate the ideal $(I_X)_{\geq r}$, while for a general $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$, the A -module $\langle f_{\alpha(h)}, h \in \mathcal{I}^c \rangle \subseteq H^0(\mathcal{O}_{\mathbb{P}_A^n}(r))$ is free of rank q , but the Hilbert polynomial of $\text{Proj}(A[x]/\langle f_{\alpha(h)}, h \in \mathcal{I}^c \rangle)$ is not necessarily $p(t)$.

In the following, keeping in mind the above construction, we often consider the ideal $I = \langle f_{\alpha(h)}, h \in \mathcal{I}^c \rangle$ as an element of $\underline{\mathbf{G}}_{\mathcal{I}}(A)$, identifying it with the A -module $L = I_r$. In the same way, we will write $I \in \underline{\mathbf{H}}_{\mathcal{I}}(A)$ when $I \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ and the Hilbert polynomial $\text{Proj}(A[x]/I)$ is $p(t)$.

The proof of the representability of the Hilbert functor after Bayer's strategy given in [18] uses the open cover of $\underline{\mathbf{Hilb}}_{p(t)}^n$ of the subfunctors $\underline{\mathbf{H}}_{\mathcal{I}} := \mathcal{H}^{-1}(\underline{\mathbf{G}}_{\mathcal{I}}) \cap \underline{\mathbf{Hilb}}_{p(t)}^n$, that we will call the *standard open cover* of $\underline{\mathbf{Hilb}}_{p(t)}^n$.

In this paper we introduce new open covers of the Grassmann and the Hilbert functors, called *Borel open covers*, that take into account of the action of the projective linear group on the Grassmann and Hilbert functors induced by that on \mathbb{P}^n .

3. The Borel open cover

An ideal $J \subset k[x]$ is said *Borel-fixed* if it is fixed by the action of the Borel subgroup of the upper triangular matrices.

These ideals are involved in many general results about Hilbert schemes for the following reason. Galligo [14] and Bayer and Stillman [4] proved that the generic initial ideal of any ideal is Borel-fixed, which means, in the context of Hilbert schemes, that any component and any intersection of components of $\mathbf{Hilb}_{p(t)}^n$ contains at least a point corresponding to a scheme defined by a Borel-fixed ideal.

In characteristic zero, the notion of Borel-fixed ideals coincides with the notion of strongly stable ideals. An ideal J is said *strongly stable* if, and only if, it is generated by terms and for each term $x^\alpha \in J$ also the term $\frac{x_j}{x_i} x^\alpha$ is in J for all $x_i \mid x^\alpha$ and $x_j > x_i$. Moreover, the regularity of J is equal to the maximum degree of terms in its minimal monomial basis [16, Proposition 2.11]. For further details about Borel-fixed ideals see [6, 16, 25].

Notation 3.1. For any Hilbert polynomial $p(t)$ and for the related integers r, p, N, q

- \mathbb{B} is the set of the Borel-fixed ideals in $k[x]$ generated by q terms of degree r .
- $\mathbb{B}_{p(t)}$ is the set of Borel-fixed ideals in \mathbb{B} with Hilbert polynomial $p(t)$.
- A *Borel multi-index* \mathcal{I} is any multi-index in $\mathcal{E}^{(0)}$ such that $J(\mathcal{I}) \in \mathbb{B}$.
- For every element $g \in \mathrm{PGL} := \mathrm{PGL}_{\mathbb{Q}}(n+1)$, \tilde{g} denotes the automorphism induced by g on $A[x]_r$ and on the Grassmann and Hilbert functors and $g \cdot$ denotes the corresponding action on an element.

Notice that the set of Borel-fixed ideals in $\mathbb{B}_{p(t)}$ can be efficiently computed by means of the algorithm presented in [8] and subsequently improved in [23].

For any p -multi-index $\mathcal{I} \in \mathcal{E}^{(0)}$ and any $g \in \mathrm{PGL}$, we consider the following subfunctor of the Grassmann functor :

$$\underline{\mathbf{G}}_{\mathcal{I},g}(A) = \left\{ \begin{array}{l} \text{free quotient } A^N \xrightarrow{\pi_L} A^N/L \text{ of rank } p \\ \text{such that } \pi_L \circ \tilde{g} \circ \Gamma_{\mathcal{I}} \text{ is surjective} \end{array} \right\}.$$

These subfunctors are open, because the functorial automorphism of \mathbf{Gr}_p^N induced by \tilde{g} extends to $\underline{\mathbf{G}}_{\mathcal{I},g} \simeq \underline{\mathbf{G}}_{\mathcal{I},\mathrm{id}} = \underline{\mathbf{G}}_{\mathcal{I}}$. It is obvious that these subfunctors also cover \mathbf{Gr}_p^N , but in fact it is sufficient to consider a smaller subset.

Proposition 3.2. *The collection of subfunctors*

$$\left\{ \underline{\mathbf{G}}_{\mathcal{I},g} \mid g \in \mathrm{PGL}, \mathcal{I} \in \mathcal{E}^{(0)} \text{ s.t. } J(\mathcal{I}) \in \mathbb{B} \right\}$$

covers the Grassmann functor \mathbf{Gr}_p^N and the representing schemes $\mathbf{G}_{\mathcal{I},g}$ cover the Grassmannian \mathbf{Gr}_p^N .

Proof. Let $\pi: A^N \rightarrow P$ be an element of $\mathbf{Gr}_p^N(A)$. Following [29, Lemma 22.22.1], we prove the result showing that for any $\mathfrak{p} \in \mathrm{Spec} A$ there exist a multi-index \mathcal{I} and a change of coordinates g such that the morphism $\pi \circ \tilde{g} \circ \Gamma_{\mathcal{I}}$ is surjective in a neighborhood of \mathfrak{p} .

Let $A_{\mathfrak{p}}$ be the local algebra obtained by localizing in \mathfrak{p} , $\mathfrak{m}_{\mathfrak{p}}$ its maximal ideal and $k(\mathfrak{p})$ the residue field. Tensoring by $k(\mathfrak{p})$ the morphism π , we obtain the morphism of vector spaces

$$\pi_{\mathfrak{p}}: k(\mathfrak{p})^N \rightarrow P_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}P_{\mathfrak{p}}$$

whose kernel is a vector subspace of $k(\mathfrak{p}) \otimes S_r$ of dimension q .

Now, consider the ideal $I \subset k(\mathfrak{p}) \otimes S$ generated by $\ker \pi_{\mathfrak{p}}$ and let J be its generic initial ideal. We fix an element $g \in \text{PGL}$ such that $J = \text{in}(g \cdot I)$. By properties of Gröbner bases, we know that $\dim_{k(\mathfrak{p})} J_r = \dim_{k(\mathfrak{p})} (g \cdot I)_r$ (J and $g \cdot I$ have the same Hilbert function). Furthermore, the terms of degree r not belonging to J are a basis both of $(k(\mathfrak{p}) \otimes S_r)/J_r$ and $(k(\mathfrak{p}) \otimes S_r)/(g \cdot I)_r$.

Finally, the multi-index \mathcal{I} is the one such that $J(\mathcal{I}) = J$. \square

Definition 3.3. We call *Borel subfunctor* of \mathbf{Gr}_p^N any element of the collection of subfunctors of Proposition 3.2. Moreover, we denote by $\underline{\mathbf{H}}_{\mathcal{I},g}$ the open subfunctor $\mathcal{H}^{-1}(\mathbf{G}_{\mathcal{I},g}) \cap \underline{\mathbf{Hilb}}_{p(t)}^n$.

Theorem 3.4. *The collection of subfunctors*

$$\left\{ \underline{\mathbf{H}}_{\mathcal{I},g} \mid g \in \text{PGL}, \mathcal{I} \in \mathcal{E}^{(0)} \text{ s.t. } J(\mathcal{I}) \in \mathbb{B}_{p(t)} \right\} \quad (3.1)$$

covers the Hilbert functor $\underline{\mathbf{Hilb}}_{p(t)}^n$.

Proof. Consider an element $X \in \underline{\mathbf{Hilb}}_{p(t)}^n(A)$. As above, it is sufficient to prove that for any $\mathfrak{p} \in \text{Spec } A$, there exists a subfunctor $\underline{\mathbf{H}}_{\mathcal{I},g}$ such that $X_{\mathfrak{p}} = X \times_k \text{Spec } k(\mathfrak{p})$ is an element of $\underline{\mathbf{H}}_{\mathcal{I},g}(k(\mathfrak{p}))$.

Localizing at \mathfrak{p} , we obtain a scheme $X_{\mathfrak{p}}$ flat over $\text{Spec } k(\mathfrak{p})$ with Hilbert polynomial $p(t)$, as the flatness and so the Hilbert polynomial are preserved by localization. Let $I_X \subset k(\mathfrak{p}) \otimes S$ be the saturated ideal defining $X_{\mathfrak{p}}$, $I := (I_X)_{\geq r}$ and J the generic initial ideal of I . By the same argument used in the proof of Proposition 3.2, we fix a change of coordinates $g \in \text{PGL}$ such that $J = \text{in}(g \cdot I)$ and the multi-index $\mathcal{I} \in \mathcal{E}^{(0)}$ such that $J(\mathcal{I}) = J$. By construction, $J(\mathcal{I}) \in \mathbb{B}_{p(t)}$, as J and I share the same Hilbert function. \square

Definition 3.5. The *Borel cover* of $\underline{\mathbf{Hilb}}_{p(t)}^n$ is the collection of the open subfunctors (3.1) of Theorem 3.4.

In next section we will prove that the open subfunctor $\underline{\mathbf{H}}_{\mathcal{I},g}$ is empty if $J(\mathcal{I}) \in \mathbb{B} \setminus \mathbb{B}_{p(t)}$.

4. Representability

Our proof that the Hilbert functor is representable mainly uses the theory of marked sets and bases on a Borel-fixed ideal developed in [5,9,24]. We recall some of the results and notation contained in the quoted papers.

Definition 4.1 ([9, Definitions 1.3, 1.4]). A *monic marked polynomial* (marked polynomial for short) is a polynomial $f \in A[x]$ together with a specified term x^α of $\text{Supp}(f)$, called *head term* of f and denoted by $\text{Ht}(f)$. We assume furthermore that the coefficient of x^α in f is 1_A . Hence, we can write a marked polynomial as $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $x^\alpha = \text{Ht}(f_\alpha)$, $x^\gamma \neq x^\alpha$ and $c_{\alpha\gamma} \in A$.

Definition 4.2. Let J be a monomial ideal. A finite set F of homogeneous marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $\text{Ht}(f_\alpha) = x^\alpha$, is called a *J -marked set* if the head terms x^α form the minimal monomial basis B_J of J , and every x^γ is an element of $\mathcal{N}(J)$. Hence, $\mathcal{N}(J)$ generates the quotient $A[x]/(F)$ as an A -module.

A J -marked set F is a *J -marked basis* if the quotient $A[x]/(F)$ is freely generated by $\mathcal{N}(J)$ as an A -module, i.e., $A[x] = A[x](F) \oplus A\langle \mathcal{N}(J) \rangle$.

Remark 4.3. Observe that if I is generated by a J -marked basis, then $\text{Proj}(A[x]/I)$ is A -flat, since $A[x]/I$ is a free A -module.

In the following we will consider only J -marked sets F with $J \in \mathbb{B}$, i.e., of the shape

$$F = \left\{ f_\alpha := x^\alpha - \sum c_{\alpha\gamma} x^\gamma \mid x^\alpha \in J_r, x^\gamma \in \mathcal{N}(J)_r, c_{\alpha\gamma} \in A \right\}. \quad (4.1)$$

For every ideal I generated by such a J -marked set F , we have $A[x]_r = \langle F \rangle \oplus \langle \mathcal{N}(J)_r \rangle$, hence I_r is a free direct summand of rank q of $A[x]_r = H^0(\mathcal{O}_{\mathbb{P}^n(A)}(r))$ and it corresponds to an element of $\mathbf{Gr}_p^{\mathcal{N}}(A)$. In fact, if $\mathcal{I} \in \mathcal{E}^{(0)}$ is the p -multi-index such that $J(\mathcal{I}) = J$, then $I \in \mathbf{Gr}_{\mathcal{I}}(A)$.

Moreover $I \in \mathbf{H}_{\mathcal{I}}(A)$ if, and only if, the Hilbert polynomial of $A[x]/(I)$ is $p(t)$. Now we will prove that this happens if, and only if, $J \in \mathbb{B}_{p(t)}$ and F is a J -marked basis.

We need some more properties concerning Borel-fixed ideals and marked bases.

Lemma 4.4. *If $J \in \mathbb{B}$, then $\text{rk}(J_t) \geq q(t)$ and $k[x_{d+1}, \dots, x_n]_t \subset J_t$ for all $t \geq r$.*

Proof. The first assertion follows by Macaulay's Estimates on the Growth of Ideals [16, Theorem 3.3]. Thus, the degree of the Hilbert polynomial of $A[x]/J$ is at most $d = \deg p(t)$. By [8, Proposition 2.3], we have $x_{d+1}^r \in J$ that implies $k[x_{d+1}, \dots, x_n]_t \subset J_t$ for $t \geq r$, by the strongly stable property. \square

Proposition 4.5 ([12, Lemma 1.1], [5, Lemma 1.2]). *Let J be a strongly stable ideal and let B_J be its minimal monomial basis.*

- (i) *Each term x^α can be written uniquely as a product $x^\gamma x^\delta$ with $x^\gamma \in B_J$ and $\min x^\gamma \geq \max x^\delta$. Hence, $x^\delta <_{\text{Lex}} x^\eta$ for every term x^η such that $x^\eta \mid x^\alpha$ and $x^{\alpha-\eta} \notin J$. We will write $x^\alpha = x^\gamma *_J x^\delta$ to refer to this unique decomposition.*
- (ii) *If $x^\alpha \in J \setminus B_J$ and $x_j = \min x^\alpha$, then $x^\alpha/x_j \in J$.*
- (iii) *If $x^\beta \notin J$, while $x^\delta x^\beta \in J$, then $x^\delta x^\beta = x^\alpha *_J x^{\delta'}$ with $x^\alpha \in B_J$ and $x^\delta >_{\text{Lex}} x^{\delta'}$ (possibly $x^{\delta'} = 1$). In particular, if $x_i x^\beta \in J$, then either $x_i x^\beta \in B_J$ or $x_i > \min x^\beta$.*

Definition 4.6. Let $J \in \mathbb{B}$ and I be the ideal generated by a J -marked set F in $A[x]$. We consider the following sets of polynomials:

- $F^{(s)} := \{x^\delta f_\alpha \mid \deg(x^\delta f_\alpha) = t, f_\alpha \in F, \min x^\alpha \geq \max x^\delta \text{ (i.e. } x^{\alpha+\delta} = x^\alpha *_J x^\delta)\}$;
- $\widehat{F}^{(s)} := \{x^\delta f_\alpha \mid \deg(x^\delta f_\alpha) = t, f_\alpha \in F, \min x^\alpha < \max x^\delta\}$;
- $\mathcal{N}(J, I) := I \cap \langle \mathcal{N}(J) \rangle$.

Note that for $s = r$, we have $F^{(r)} = F, \langle F^{(r)} \rangle = I_r$ and $\mathcal{N}(J, I)_r = 0$.

Theorem 4.7 ([24, Theorems 1.7, 1.10]). For $J \in \mathbb{B}$, let I be the ideal generated by a J -marked set F in $A[x]$. Then, for every $s \geq r$,

- (i) $I_s = \langle F^{(s)} \rangle + \langle \widehat{F}^{(s)} \rangle$;
- (ii) the A -module $\langle F^{(s)} \rangle$ is free of rank equal to $|F^{(s)}| = \text{rk}(J_s)$;
- (iii) $I_s = \langle F^{(s)} \rangle \oplus \mathcal{N}(J, I)_s$.

Moreover, the following conditions are equivalent:

- (iv) F is a J -marked basis;
- (v) for all $s \geq r, I_s = \langle F^{(s)} \rangle$;
- (vi) $\mathcal{N}(J, I)_{r+1} = 0$;
- (vii) $I_{r+1} = \langle F^{(r+1)} \rangle$;
- (viii) $\bigwedge^{Q+1} I_{r+1} = 0$, where $Q := \text{rk}(J_{r+1})$.

Proof. This result is proved in a more general context in [24]. We only observe that the conditions “ $\mathcal{N}(J, I)_s = 0$ and $I_s = \langle F^{(s)} \rangle$ for every $s \leq \text{reg}(J) + 1$ ” appearing in [24] are equivalent to (vi) and (vii), since in the present hypotheses J is generated in degree r and r is its regularity. With respect to [24], the only new item is (viii), which is obviously equivalent to (vi) and (vii). In fact, by (ii) and (iii) we have $I_{r+1} = \langle F^{(r+1)} \rangle \oplus \mathcal{N}(J, I)_{r+1}$ and $\text{rk}\langle F^{(r+1)} \rangle = \text{rk}(J_{r+1}) = Q$. \square

Corollary 4.8. Let $\mathcal{I} \in \mathcal{E}^{(0)}$ be such that $J(\mathcal{I}) \in \mathbb{B}$ and let $g \in \text{PGL}$. Then:

$$\underline{\mathbf{H}}_{\mathcal{I},g} \text{ is not empty} \iff J(\mathcal{I}) \in \mathbb{B}_{p(t)}.$$

Moreover, for $J = J(\mathcal{I}) \in \mathbb{B}_{p(t)}$ and any k -algebra A

$$\underline{\mathbf{H}}_{\mathcal{I},g}(A) = \{g \cdot I \text{ s.t. } I \text{ is generated by a } J\text{-marked basis in } A[x]\}.$$

Proof. It is sufficient to prove the result for $g = \text{id}$, i.e., for $\underline{\mathbf{H}}_{\mathcal{I}}$.

Let A be any k -algebra. If $J = J(\mathcal{I}) \in \mathbb{B}_{p(t)}$, then $J \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ and the Hilbert polynomial of $\text{Proj}(A[x]/J)$ is $p(t)$; hence $J \in \underline{\mathbf{H}}_{\mathcal{I}}(A)$.

On the other hand, if $J = J(\mathcal{I}) \in \mathbb{B} \setminus \mathbb{B}_{p(t)}$ and $I \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$, then I is generated by a J -marked set and $\text{rk}(J_s) > q(s)$ for every $s \gg 0$ (Lemma 4.4). By Theorem 4.7, the A -module I_s contains a free submodule of rank equal to that of J_s , hence $I \notin \underline{\mathbf{H}}_{\mathcal{I}}(A)$.

The second statement directly follows from Theorem 4.7(ii) and the equivalence (iv) \Leftrightarrow (v). \square

In [24] a functor $\mathbf{Mf}_J: (\text{Rings}) \rightarrow (\text{Sets})$ is defined for a strongly stable ideal J in $\mathbb{Z}[x_0, \dots, x_n]$, by taking for a ring A

$$\mathbf{Mf}_J(A) = \{\text{ideals } I \text{ generated by } J\text{-marked bases in } A[x_0, \dots, x_n]\}.$$

Therefore, the open subfunctor $\mathbf{H}_{\mathcal{I}}$ is the restriction of \mathbf{Mf}_J to the sub-category (k -Algebras).

The marked functor \mathbf{Mf}_J is represented by a closed subscheme of the affine space $\mathbb{A}_{\mathbb{Z}}^D$, for a suitable D . For the main features of \mathbf{Mf}_J and the proof of its representability see [24]. Here, we are only interested in the case $J \in \mathbb{B}$. Under this condition, any J -marked set has the shape (4.1) and is uniquely determined by the list of $D = q(r) \cdot p(r)$ coefficients $c_{\alpha\gamma}$. Among the ideals generated by marked sets, those generated by marked bases are given for instance by the closed condition (viii) (or even (vi)) of Theorem 4.7.

Theorem 4.9. *The Hilbert functor $\mathbf{Hilb}_{p(t)}^n$ is the functor of points of a closed subscheme $\mathbf{Hilb}_{p(t)}^n$ of the Grassmannian \mathbf{Gr}_p^N .*

Proof. By Proposition 2.1, it suffices to check the representability on an open cover of \mathbf{Gr}_p^N and $\mathbf{Hilb}_{p(t)}^n$: we choose the Borel open cover (Definitions 3.3 and 3.5). For every $\mathcal{I} \in \mathcal{E}^{(0)}$ such that $J := J(\mathcal{I}) \in \mathbb{B}_{p(t)}$ and for every $g \in \text{PGL}$, $\mathbf{H}_{\mathcal{I},g}$ is naturally isomorphic to $\mathbf{H}_{\mathcal{I}}$. Moreover, $\mathbf{H}_{\mathcal{I}}$ is the functor of points of the k -scheme $\mathbf{H}_{\mathcal{I}} := \mathbf{Mf}_J \times_{\text{Spec } \mathbb{Z}} \text{Spec } k$. Indeed, the scheme $\mathbf{H}_{\mathcal{I}}$ is the subscheme of $\mathbb{A}_k^D = \mathbf{G}_{\mathcal{I}}$ (where $D = p(r) \cdot q(r)$) defined by the closed equivalent conditions of Theorem 4.7. Hence $\mathbf{H}_{\mathcal{I}}$ is the functor of points of a closed subscheme $\mathbf{H}_{\mathcal{I}}$ of $\mathbf{G}_{\mathcal{I}}$.

On the other hand if $J(\mathcal{I}) \in \mathbb{B} \setminus \mathbb{B}_{p(t)}$, then $\mathbf{H}_{\mathcal{I}}$ is empty (Corollary 4.8), hence it is the functor of points of a closed subscheme of $\mathbf{G}_{\mathcal{I}}$. By Proposition 3.2 and the second part of Proposition 2.1, we conclude that $\mathbf{Hilb}_{p(t)}^n$ is the functor of points of a closed subscheme $\mathbf{Hilb}_{p(t)}^n$ of \mathbf{Gr}_p^N . \square

The next sections are devoted to describe how to determine equations defining the Hilbert scheme $\mathbf{Hilb}_{p(t)}^n$ as subscheme of the Grassmannian \mathbf{Gr}_p^N .

5. Extensors and the Plücker embedding

In this section we consider any Grassmann functor, that we will denote by \mathbf{Gr}_p^N . In next sections, we will apply the tools developed to the study of the Hilbert functor and scheme. However, all the results of this section hold true for every p and N , not only for those obtained starting from an Hilbert polynomial $p(t)$ of subschemes of \mathbb{P}^n .

In this section, we think at $\mathbf{Gr}_p^N(A)$ as presented in (2.1); furthermore, our arguments allow us to restrict to the open subfunctors $\mathbf{G}_{\mathcal{I}}$, introduced in Section 3. Thus, the elements of $\mathbf{Gr}_p^N(A)$ we are mainly interested in are free submodules L of A^N of rank q , such that A^N/L is free of rank p .

We begin stating some well-know notions and results about exterior algebras.

Definition 5.1. Given a free A -module M , an *extensor* of step m in M is an element of $\wedge^m M$ of the form $\mu_1 \wedge \cdots \wedge \mu_m$ with μ_1, \dots, μ_m in M .

Notice that $\mu_1 \wedge \cdots \wedge \mu_m$ vanishes whenever the submodule generated by μ_1, \dots, μ_m has rank lower than m .

Lemma 5.2. Let $\phi: P \rightarrow Q$ be a linear morphism of A -modules.

(i) For any m , there exists a unique map $\wedge^m P \rightarrow \wedge^m Q$ such that

$$p_1 \wedge \cdots \wedge p_m \mapsto \phi(p_1) \wedge \cdots \wedge \phi(p_m).$$

We denote this morphism by $\phi^{(m)}$.

(ii) If ϕ is an isomorphism (respectively surjective), then $\phi^{(m)}$ is an isomorphism (respectively surjective) for every m .

(iii) If ϕ is injective and P is free, then $\phi^{(m)}$ is injective for every m [13, Theorems 1, 8].

(iv) If Q is free with basis $\{l_1, \dots, l_s\}$, then for every $1 \leq m \leq s$, the exterior algebra $\wedge^m Q$ is free of rank $\binom{s}{m}$ with basis $\{l_{i_1} \wedge \cdots \wedge l_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq s\}$.

In particular, all the extensors of step $s = \text{rk } Q$ associated to different bases of Q are equal up to multiplication by an invertible element of A [10, Corollary A2.3].

(v) If $M = P \oplus Q$, then $\wedge^m (P \oplus Q) = \bigoplus_{r+s=m} \wedge^r P \otimes \wedge^s Q$.

Remark 5.3. As in the previous sections, a_1, \dots, a_N is a fixed basis of the A -module A^N . We also fix the isomorphism $\wedge^N A^N \simeq A$ sending $a_1 \wedge \cdots \wedge a_N$ to 1_A . For any m -multi-index $\mathcal{J} = (j_1, \dots, j_m)$, we will denote by $a_{\mathcal{J}}$ the extensor $a_{j_1} \wedge \cdots \wedge a_{j_m}$ of $\wedge^m A^N$. By Lemma 5.2(iv), these extensors give a basis of $\wedge^m A^N$. We observe that $a_{\mathcal{J}} \wedge a_{\mathcal{H}} = 0$ if $\mathcal{H} \cap \mathcal{J} \neq \emptyset$, while $a_{\mathcal{J}} \wedge a_{\mathcal{J}^c} = \varepsilon_{\mathcal{J}} a_1 \wedge \cdots \wedge a_N$, where $\varepsilon_{\mathcal{J}}$ is the signature of $\mathcal{J}, \mathcal{J}^c$. Taking into account the fixed isomorphism, we will simply write $a_{\mathcal{J}} \wedge a_{\mathcal{J}^c} = \varepsilon_{\mathcal{J}}$.

Every A -module $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ has the special set of generators

$$\mathcal{B}_{\mathcal{I}}(L) := \left\{ b_s := a_s - \sum_{i \in \mathcal{I}} \gamma_{si} a_i \mid s \in \mathcal{I}^c \right\}$$

described in Remark 2.3. We will call it the \mathcal{I} -marked set of L , extending the terminology we use in the special case of interest in this paper (Definition 4.2)¹.

¹ The marked set $\mathcal{B}_{\mathcal{I}}(L)$ is in fact a basis for L ; however, we do not call it “marked basis”, because in the case of a Grassmannian containing a Hilbert scheme, this terminology refers only to the points of the Hilbert scheme.

Definition 5.4. For every $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ and $\mathcal{S} = (s_1, \dots, s_m) \subset \mathcal{I}^c$, we denote by $b_{\mathcal{S}}$ the extensor $b_{s_1} \wedge \dots \wedge b_{s_m} \in \wedge^m L$. The \mathcal{I} -marked set of $\wedge^m L$ is the free set of generators

$$\mathcal{B}_{\mathcal{I}}^{(m)}(L) := \{b_{\mathcal{S}} \mid \mathcal{S} \subset \mathcal{I}^c, |\mathcal{S}| = m\}.$$

In particular, $\mathcal{B}_{\mathcal{I}}^{(1)}(L) = \mathcal{B}_{\mathcal{I}}(L)$.

The aim of the present section is that of determining a unified writing in terms of the Plücker coordinates of $\underline{\mathbf{Gr}}_p^N$ of a set of generators of $\wedge^m L$, where $1 \leq m \leq q$ and $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$. This set of generators will also contain the \mathcal{I} -marked set of $\wedge^m L$.

By Lemma 5.2(iii), there is a natural inclusion $\wedge^m L \subseteq \wedge^m A^N$ for every $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$. Hence, every element $f \in \wedge^m L$ has a unique writing $f = \sum c_{\mathcal{J}} a_{\mathcal{J}}$, with coefficients $c_{\mathcal{J}} \in A$.

Lemma 5.5. Let $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$.

(i) If $b_{\mathcal{S}} \in \mathcal{B}_{\mathcal{I}}^{(m)}(L)$ and $\mathcal{K} := \mathcal{S} \cup \mathcal{I}$, then

$$b_{\mathcal{S}} = a_{\mathcal{S}} + \varepsilon_{\mathcal{S}}^{\mathcal{K}} \sum \varepsilon_{\mathcal{H}}^{\mathcal{K}} (b_{\mathcal{I}^c} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) a_{\mathcal{H}} \quad (5.1)$$

where the sum is over the m -multi-indices $\mathcal{H} \neq \mathcal{S}$ such that $\mathcal{H} \subseteq \mathcal{K}$, and $\varepsilon_{\mathcal{H}}^{\mathcal{K}}$ is the signature of the permutation $\mathcal{K} \mapsto \mathcal{H}, \mathcal{K} \setminus \mathcal{H}$.

(ii) If $f = \sum c_{\mathcal{J}} a_{\mathcal{J}}$ is any non-zero element of $\wedge^m L \subset \wedge^m A^N$, then there is at least one non-zero coefficient $c_{\mathcal{J}}$ with $\mathcal{J} \subset \mathcal{I}^c$.

Proof. Up to a permutation, we may assume that $\mathcal{K}^c, \mathcal{S}, \mathcal{I} = (1, \dots, N)$. Hence, $\varepsilon_{\mathcal{S}}^{\mathcal{K}} = 1$.

(i) We use the distributive law with $b_{s_j} = a_{s_j} - \sum_{i \in \mathcal{I}} \gamma_{s_j i} a_i$ and immediately see that the coefficient of $a_{\mathcal{S}}$ in $b_{\mathcal{S}}$ is 1_A , as $\mathcal{I} \cap \mathcal{S} = \emptyset$, and the other extensors $a_{\mathcal{H}} \neq a_{\mathcal{S}}$ that can appear with non-zero coefficient are those given in the statement. As a consequence, note that $b_{\mathcal{S}} \wedge a_{\mathcal{T}} = 0$ if \mathcal{T} is an $(N - m)$ -multi-index and $\mathcal{T}^c \not\subseteq \mathcal{K}$, i.e., $\mathcal{T} \not\subseteq \mathcal{K}^c$.

Now we prove the given formula for the coefficients, focusing on each m -multi-index \mathcal{H} . Let us denote by $\gamma_{\mathcal{H}}$ the coefficient of $a_{\mathcal{H}}$ in $b_{\mathcal{S}}$. Applying again the distributive law on $a_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}$, the only non-zero summand is $\gamma_{\mathcal{H}} (a_{\mathcal{K}^c} \wedge a_{\mathcal{H}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) = \gamma_{\mathcal{H}} \varepsilon_{\mathcal{H}}^{\mathcal{K}}$, hence, $\gamma_{\mathcal{H}} = \varepsilon_{\mathcal{H}}^{\mathcal{K}} (a_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}})$ and

$$b_{\mathcal{S}} = a_{\mathcal{S}} + \sum \varepsilon_{\mathcal{H}}^{\mathcal{K}} (a_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) a_{\mathcal{H}} \quad (5.2)$$

with $\mathcal{H} \subseteq \mathcal{K}$, $|\mathcal{H}| = m$, $\mathcal{H} \neq \mathcal{S}$.

It remains to verify that $(a_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) = (b_{\mathcal{I}^c} \wedge a_{\mathcal{K} \setminus \mathcal{H}})$. Applying (5.2), we can write $a_{\mathcal{K}^c} = b_{\mathcal{K}^c} - \sum \gamma'_{\mathcal{H}'} a_{\mathcal{H}'}$ where $|\mathcal{H}'| = |\mathcal{K}^c|$ and $\mathcal{H}' \neq \mathcal{K}^c$. We substitute and get

$$(a_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) = (b_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) - \sum_{\mathcal{H}'} \gamma'_{\mathcal{H}'} (a_{\mathcal{H}'} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}).$$

All the summands on \mathcal{H}' vanish. Indeed, this is obvious if \mathcal{H}' and $\mathcal{K} \setminus \mathcal{H}$ are not disjoint. On the other hand, if they are and we denote by \mathcal{T} their union, then $b_{\mathcal{S}} \wedge a_{\mathcal{T}} = 0$ since $\mathcal{K}^c \cap \mathcal{K} \setminus \mathcal{H} = \emptyset$ and $\mathcal{H}' \not\supseteq \mathcal{K}^c$. Finally, $(a_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) = (b_{\mathcal{K}^c} \wedge b_{\mathcal{S}} \wedge a_{\mathcal{K} \setminus \mathcal{H}}) = (b_{\mathcal{I}^c} \wedge a_{\mathcal{K} \setminus \mathcal{H}})$.

(ii) As $f \in \wedge^m L$, we can also write $f = \sum d_{\mathcal{S}} b_{\mathcal{S}}$, with $\mathcal{S} \subset \mathcal{I}^c$ and $d_{\mathcal{S}} \neq 0$. By the previous item, $a_{\mathcal{S}}$ appears only in the writing of $b_{\mathcal{S}}$, hence its coefficient $c_{\mathcal{S}}$ is $d_{\mathcal{S}} \neq 0$. \square

We would like to rewrite the coefficients appearing in the writing of the extensors $b_{\mathcal{S}}$ given in (5.1) in terms of the Plücker coordinates of L . Then, let us recall how they are defined.

The projective space \mathbb{P}^E can be seen as the scheme representing the functor

$$\underline{\mathbb{P}}^E: (k\text{-Algebras}) \rightarrow (\text{Sets})$$

that associates to any k -algebra A the set

$$\underline{\mathbb{P}}^E(A) = \left\{ \begin{array}{l} \text{isomorphism classes of epimorphisms} \\ \pi: A^{E+1} \rightarrow Q \text{ of locally free modules of rank 1} \end{array} \right\}.$$

Hence, we can consider the natural transformation of functors $\underline{\mathcal{P}}: \underline{\mathbf{Gr}}_p^N \rightarrow \underline{\mathbb{P}}^E$ given by:

$$\begin{aligned} \underline{\mathcal{P}}(A): \left(\phi_L: A^N \xrightarrow{\pi} A^N/L \right) &\in \underline{\mathbf{Gr}}_p^N(A) \\ \longmapsto \left(\phi_L^{(p)}: \wedge^p A^N \xrightarrow{\pi^{(p)}} \wedge^p (A^N/L) \right) &\in \underline{\mathbb{P}}^E(A) \end{aligned}$$

where $\wedge^p A^N$ is free of rank $\binom{N}{p} = E+1$ and $\wedge^p A^N/L$ is locally free of rank 1.

The collection of open subfunctors $\underline{\mathbf{G}}_{\mathcal{I}}$ of Proposition 2.2 is exactly that induced by the transformation $\underline{\mathcal{P}}$ and the standard affine cover of the projective space \mathbb{P}^E corresponding to the basis $\{a_{\mathcal{J}} \mid \mathcal{J} \in \mathcal{E}^{(0)}\}$ of $\wedge^p A^N$.

We denote by Δ the variables of \mathbb{P}^E and we index them using the multi-indices $\mathcal{I} \in \mathcal{E}^{(0)}$ so that $\underline{\mathbf{G}}_{\mathcal{I}}$ be the open subscheme of the Grassmannian defined by the condition $\Delta_{\mathcal{I}} \neq 0$. The Grassmannian $\underline{\mathbf{Gr}}_p^N$ is a closed subscheme of $\mathbb{P}^E = \text{Proj } k[\Delta]$ defined by the Plücker relations, that are generated by homogeneous polynomials of degree 2: we will denote by $k[\Delta]$ the coordinate ring of $\underline{\mathbf{Gr}}_p^N$, *i.e.*, the quotient of $k[\Delta]$ under the Plücker relations, so that $\underline{\mathbf{Gr}}_p^N = \text{Proj } k[\Delta] \subset \mathbb{P}^E = \text{Proj } k[\Delta]$ (see for instance [22]).

We can also associate Plücker coordinates to each module $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$. Upon fixing an isomorphism $i: \wedge^p (A^N/L) \simeq A$, $\underline{\mathcal{P}}(A)(L)$ can be seen as the map $i \circ \phi_L^{(p)}$ or, equivalently, as the function

$$i \circ \phi_L^{(p)}: \{a_{\mathcal{J}} \mid \mathcal{J} \in \mathcal{E}^{(0)}\} \rightarrow A \text{ given by } a_{\mathcal{J}} \mapsto \Delta_{\mathcal{J}}(L) := i(\phi_L^{(p)}(a_{\mathcal{J}})).$$

Since two isomorphisms $i, i' : \wedge^p A^N / L \rightarrow A$ only differ by the multiplication by a unit $u \in A$, the Plücker coordinates of L are defined up to invertible elements in A .

By definition of $\underline{\mathbf{G}}_{\mathcal{I}}(A)$, we have the decomposition as direct sum $A^N = L \oplus \langle a_i \mid i \in \mathcal{I} \rangle$, so that $\phi_L^{(p)}$ factors through $\wedge^p A^N \rightarrow \wedge^N A^N \rightarrow \wedge^p (A^N / L)$ given by $a_{\mathcal{J}} \mapsto b_{\mathcal{I}^c} \wedge a_{\mathcal{J}} \mapsto \overline{a_{\mathcal{J}}}$, where $b_{\mathcal{I}^c}$ is the only element of the \mathcal{I} -marked set $\mathcal{B}_{\mathcal{I}}^{(q)}(L)$ of $\wedge^q L$.

Hence, the Plücker coordinates of L are

$$\left(\Delta_{\mathcal{J}}(L) = b_{\mathcal{I}^c} \wedge a_{\mathcal{J}} \mid \mathcal{J} \in \mathcal{E}^{(0)} \right). \quad (5.3)$$

We identify $b_{\mathcal{I}^c} \wedge a_{\mathcal{J}}$ with elements of A by fixing the isomorphisms $\wedge^N A^N \simeq A$ and $i : \wedge^p A^N / L \rightarrow A$. For the first one we fixed that sending $a_{(1, \dots, N)}$ to 1; if we choose $i : \overline{a_{\mathcal{I}}} \mapsto 1$, then (5.3) gives the representative of the Plücker coordinates with $\Delta_{\mathcal{I}}(L) = 1$. Indeed, in our setting $\varepsilon_{\mathcal{I}^c} = 1$ and $b_{\mathcal{I}^c} \wedge a_{\mathcal{I}} = a_{\mathcal{I}^c} \wedge a_{\mathcal{I}} = a_{(1, \dots, N)}$ by Lemma 5.5.

Therefore, Plücker coordinates of L can be obtained as the maximal minors of the $q \times N$ matrix whose rows contain the elements of $\mathcal{B}_{\mathcal{I}}(L)$. More precisely, $\Delta_{\mathcal{J}}(L)$ is the minor corresponding to the columns with indices in \mathcal{J}^c , up to a sign given by the signature $\varepsilon_{\mathcal{J}}$.

Using (5.3) we can finally rewrite the coefficients appearing in (5.1) in terms of the Plücker coordinates of L .

Corollary 5.6. *Let $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$, $b_{\mathcal{S}}$ be any extensor in $\mathcal{B}_{\mathcal{I}}^{(m)}(L)$ and $\mathcal{K} := \mathcal{S} \cup \mathcal{I}$. Then*

$$\varepsilon_{\mathcal{S}}^{\mathcal{K}} \Delta_{\mathcal{I}}(L) b_{\mathcal{S}} = \varepsilon_{\mathcal{S}}^{\mathcal{K}} \Delta_{\mathcal{I}}(L) a_{\mathcal{S}} + \sum \varepsilon_{\mathcal{H}}^{\mathcal{K}} \Delta_{\mathcal{K} \setminus \mathcal{H}}(L) a_{\mathcal{H}}$$

where the sum is over the m -multi-indices \mathcal{H} such that $\mathcal{H} \subseteq \mathcal{K}$, $\mathcal{H} \neq \mathcal{S}$.

Definition 5.7. For every $1 \leq m \leq q$, we define the following subset of $k[\Delta]^N$:

$$\mathcal{B}^{(m)} := \left\{ \delta_{\mathcal{K}}^{(m)} := \sum_{\substack{\mathcal{H} \subseteq \mathcal{K} \\ |\mathcal{H}|=m}} \varepsilon_{\mathcal{H}}^{\mathcal{K}} \Delta_{\mathcal{K} \setminus \mathcal{H}} a_{\mathcal{H}} \mid \mathcal{K} \in \mathcal{E}^{(m)} \right\}.$$

Moreover, for every $\mathcal{I} \in \mathcal{E}^{(0)}$, we define $\mathcal{B}_{\mathcal{I}}^{(m)} := \left\{ \delta_{\mathcal{K}}^{(m)} \mid \mathcal{K} \in \mathcal{E}^{(m)}, \mathcal{K} \supseteq \mathcal{I} \right\}$.

Remark 5.8. For every m -multi-index \mathcal{S} that does not intersect \mathcal{I} , $a_{\mathcal{S}}$ appears in a single element of $\mathcal{B}_{\mathcal{I}}^{(m)}$, the one with index $\mathcal{K} = \mathcal{I} \cup \mathcal{S}$. Moreover, $\delta_{\mathcal{K}}^{(m)} - \varepsilon_{\mathcal{S}}^{\mathcal{K}} \Delta_{\mathcal{I}} a_{\mathcal{S}}$ is the sum $\sum \varepsilon_{\mathcal{H}}^{\mathcal{K}} \Delta_{\mathcal{K} \setminus \mathcal{H}} a_{\mathcal{H}}$, where all the m -multi-indices \mathcal{H} intersect \mathcal{I} .

Hence, for every element $f \in \wedge^m k[\Delta]^N$ we can write $\Delta_{\mathcal{I}} f$ as a sum $f_1 + f_2$ with $f_1 \in \langle \mathcal{B}_{\mathcal{I}}^{(m)} \rangle$ and $f_2 \in \langle a_{\mathcal{H}} \text{ s.t. } |\mathcal{H}| = m \text{ and } \mathcal{H} \cap \mathcal{I} \neq \emptyset \rangle$.

We will now evaluate the elements $\delta_{\mathcal{K}}^{(m)}$ at $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$; of course, such evaluations are defined only up to units of A . Through evaluation at L , we can also see that the notations of Definition 5.7 are consistent with those introduced in Definition 5.4 to denote \mathcal{I} -marked sets of $\wedge^m L$.

Theorem 5.9. *Let $\mathcal{I} \in \mathcal{E}^{(0)}$, A be a k -algebra and L be a module in $\underline{\mathbf{G}}_{\mathcal{I}}(A)$. Then, for every $1 \leq m \leq q$,*

- (i) *the evaluation of $\mathcal{B}_{\mathcal{I}}^{(m)}$ at L is the \mathcal{I} -marked set $\mathcal{B}_{\mathcal{I}}^{(m)}(L)$ of $\wedge^m L$;*
- (ii) *the evaluation $\mathcal{B}^{(m)}(L)$ of $\mathcal{B}^{(m)}$ at L is a set of generators of $\wedge^m L$.*

Proof. (i) Let \mathcal{K} be a $(q+m)$ -multi-index containing \mathcal{I} and let $\mathcal{S} = \mathcal{K} \setminus \mathcal{I}$. As a straightforward consequence of Lemma 5.5 and Corollary 5.6, we see that $\delta_{\mathcal{K}}^{(m)}(L)$ is equal (up to units of A) to the element $b_{\mathcal{S}}$ of the \mathcal{I} -marked set of $\wedge^m L$. Note that for $L \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ and $\mathcal{H} = \mathcal{S}$ we may set $\Delta_{\mathcal{K} \setminus \mathcal{S}}(L) = \Delta_{\mathcal{I}}(L) = 1$.

(ii) By the previous item, it suffices to prove that $\mathcal{B}^{(m)}(L) \subset \wedge^m L$.

Let us consider any $\delta_{\mathcal{K}'}^{(m)} \in \mathcal{B}^{(m)}$ and write $\Delta_{\mathcal{I}} \delta_{\mathcal{K}'}^{(m)} = \delta_1 + \delta_2$ as in Remark 5.8 with $\delta_1 \in \langle \mathcal{B}_{\mathcal{I}}^{(m)} \rangle$ and $\delta_2 \in \langle a_{\mathcal{H}} \text{ s.t. } |\mathcal{H}| = m \text{ and } \mathcal{H} \cap \mathcal{I} \neq \emptyset \rangle$. Under our assumption, $\Delta_{\mathcal{I}}(L)$ is a unit in A ; therefore, we need to prove that $\delta_2(L) = 0$.

If there is a p -multi-index $\mathcal{I}' \subset \mathcal{K}'$ such that $L \in \underline{\mathbf{G}}_{\mathcal{I}'}(A)$, then it follows by (i) that $\delta_{\mathcal{K}'}^{(m)}(L) \in \mathcal{B}_{\mathcal{I}'}^{(m)}(L) \subset \wedge^m L$, so that also $\delta_2(L) \in \wedge^m L$ and we get $\delta_2(L) = 0$ by Lemma 5.5(ii).

Therefore, δ_2 vanishes over the non-empty open subfunctor $\underline{\mathbf{G}}_{\mathcal{I}}(A) \cap \underline{\mathbf{G}}_{\mathcal{I}'}(A)$ of the Grassmann functor, hence it vanishes on \mathbf{Gr}_p^N . \square

We will use the results of this section in order to compute equations defining globally the Hilbert scheme as subscheme of the Grassmannian, starting from those defining $\mathbf{H}_{\mathcal{I}}$ in $\mathbf{G}_{\mathcal{I}}$. Then in the following the elements of basis a_1, \dots, a_N of A^N will correspond to the terms $x^{\alpha(1)}, \dots, x^{\alpha(N)}$ in $k[x]_r$. We can reformulate Theorem 5.9 in this special setting.

Theorem 5.10. *The universal family $\mathcal{F} \hookrightarrow \mathbb{P}^n \times \mathbf{Gr}_p^N \rightarrow \mathbf{Gr}_p^N$ parameterized by the Grassmannian, given in (2.2), is generated by the set of bi-homogeneous elements in $k[\Delta, x]$*

$$\left\{ \delta_{\mathcal{K}}^{(1)} = \sum_{h \in \mathcal{K}} \varepsilon_{\{h\}}^{\mathcal{K}} \Delta_{\mathcal{K} \setminus \{h\}} x^{\alpha(h)} \mid \forall \mathcal{K} \in \mathcal{E}^{(1)} \right\}$$

and the m -th exterior power of the universal element is generated by

$$\left\{ \delta_{\mathcal{K}}^{(m)} = \sum_{\substack{\mathcal{H} \subset \mathcal{K} \\ |\mathcal{H}|=m}} \varepsilon_{\mathcal{H}}^{\mathcal{K}} \Delta_{\mathcal{K} \setminus \mathcal{H}} x^{\alpha(h_1)} \wedge \dots \wedge x^{\alpha(h_m)} \mid \forall \mathcal{K} \in \mathcal{E}^{(m)} \right\}.$$

6. Equations

In this section we will obtain global equations defining the Hilbert scheme. In particular, the new set of equations has degree lower than the other known equations. Towards this aim we need to refine some results of Section 4, in particular Theorem 4.7.

These results concern any J -marked set F , where J is a Borel-fixed ideal generated by q terms of degree r ; we do not assume that the Hilbert polynomial $p_J(t)$ of $A[x]/J$ is $p(t)$. However, we know that r is the regularity of J and, by Lemma 4.4, $k[x_{d+1}, \dots, x_n]_{\geq r} \subset J$ and $\deg p_J(t) \leq d = \deg p(t)$; hence $\mathcal{N}(J)_{\geq r} \subset (x_0, \dots, x_d)$. In particular, the support of every polynomial $f_\alpha \in F$ is contained in (x_0, \dots, x_d) , except for only one possible term, the head term $\text{Ht}(f_\alpha) = x^\alpha$.

Definition 6.1. Let $J \in \mathbb{B}$ and let $I \subset A[x]$ be an ideal generated by a J -marked set F . Making reference (and in addition) to Definition 4.6, we set:

- $F' := \{x_i f_\alpha \in F^{(r+1)} \mid i = d+1, \dots, n\} = F^{(r+1)} \setminus (x_0, \dots, x_d)$;
- $F'' := \{x_i f_\alpha \in F^{(r+1)} \mid i = 0, \dots, d\} = F^{(r+1)} \cap (x_0, \dots, x_d)$;
- $S := \{x_j f_\beta - x_i f_\alpha \mid \forall x_j f_\beta \in \widehat{F}^{(r+1)}, x_i f_\alpha \in F^{(r+1)} \text{ s.t. } x_j x^\beta = x_i x^\alpha\}$;
- $q' := \dim_k k[x_{d+1}, \dots, x_n]_{r+1}$;
- $q'' := q(r+1) - q'$;
- $I'' := I_{r+1} \cap (x_0, \dots, x_d)$;
- $I^{(1)} := \langle x_h I_r \mid \forall h = 0, \dots, d \rangle \subseteq I''$.

Theorem 6.2. Let $J \in \mathbb{B}$ and $I \subset A[x]$ be an ideal generated by a J -marked set F . Then,

- (i) $\langle F' \rangle$ is a free A -module of rank q' ;
- (ii) $\langle F'' \rangle$ is a free A -module contained in $I^{(1)}$ of rank $\geq q''$;
- (iii) $I_{r+1} = \langle F' \rangle \oplus I''$;
- (iv) $I'' = \langle F'' \rangle \oplus \mathcal{N}(J, I)_{r+1} = \langle F'' \rangle + \langle S \rangle$.

Moreover, the following conditions are equivalent:

- (v) $J \in \mathbb{B}_{p(t)}$ and F is a J -marked basis;
- (vi) $\wedge^{q(r+1)+1} I_{r+1} = 0$;
- (vii) $\wedge^{q''+1} I'' = 0$;
- (viii) $\wedge^{q''+1} I^{(1)} = 0$ and $(\wedge^{q''} I^{(1)}) \wedge I'' = 0$.

Proof. (i) It is sufficient to recall that F' is a subset of the set of linearly independent polynomials $F^{(r+1)}$, hence the A -module $\langle F' \rangle$ is free of rank equal to $|F'|$. Moreover $|F'| = q'$ by Lemma 4.4.

(ii) We can prove that $\langle F'' \rangle$ is free with rank $|F''|$ by the same argument used for (i). Moreover, by definition and Lemma 4.4, $\langle F'' \rangle = |F^{(r+1)}| - |F'| = \text{rk}(J_{r+1}) - |F'| \geq q(r+1) - q'$.

(iii), (iv) We obtain the equality $I_{r+1} = \langle F' \rangle \oplus \langle F'' \rangle \oplus \mathcal{N}(J, I)_{r+1}$ as a consequence of Theorem 4.7(iii) and the fact that $F^{(r+1)}$ is the disjoint union of F' and F'' . It is obvious by the definition that $\langle F'' \rangle \oplus \mathcal{N}(J, I)_{r+1} \subseteq I''$. Then, to prove $I_{r+1} = \langle F' \rangle \oplus I''$ it suffices to verify that the sum $\langle F' \rangle + I''$ is direct. If h is any element $h = \sum d_{i\alpha} x_i f_\alpha \in \langle F' \rangle$ with $d_{i\alpha} \in A$, $d_{i\alpha} \neq 0$, then $x_i x^\alpha \in \text{Supp}(h)$, since the head terms of the monic marked polynomials $x_i f_\alpha \in F'$ are distinct terms in $k[x_{d+1}, \dots, x_n]_{r+1}$, while $x_i f_\alpha - x_i x^\alpha \in (x_0, \dots, x_n)$. Therefore, we also get $I'' = \langle F'' \rangle \oplus \mathcal{N}(J, I)_{r+1}$,

Let us consider the set of generators $F' \cup F'' \cup \widehat{F}^{(r+1)}$ of the A -module I_{r+1} . For every element $x_j f_\beta \in \widehat{F}^{(r+1)}$, we can find an element $x_i f_\alpha \in F' \cup F''$ such that $x_i x^\alpha = x_j x^\beta$ and $h_{j\beta} := x_j f_\beta - x_i f_\alpha \in S$. Then, we get a new set of generators replacing $\widehat{F}^{(r+1)}$ by S . The union of the three sets F' , F'' and S generates the A -module I_{r+1} and, in particular, $F'' \cup S$ generates I'' , since $S \subseteq I''$.

(v) \Leftrightarrow (vi) If $J \in \mathbb{B}_{p(t)}$, then the statement is given by Theorem 4.7(iv) \Leftrightarrow (viii), as $\text{rk}(J_{r+1}) = q(r+1)$. On the other hand, if $J \notin \mathbb{B}_{p(t)}$, then by Gotzmann's Persistence Theorem we have $\text{rk}(J_{r+1}) > q(r+1)$, so that $\wedge^{q(r+1)+1} I_{r+1} \neq 0$ by Theorem 4.7(ii).

(vi) \Leftrightarrow (vii) \Leftrightarrow (viii) are straightforward consequences of previous items. \square

Proposition 6.3. *In the setting of Theorem 6.2, let B be any set of polynomials of I_r containing F and consider the following two subsets of I_{r+1} :*

- 1) $\bigcup_{i=0}^s x_i B$;
- 2) $\{x_i f - x_j g \mid \forall f, g \in B \text{ such that } x_i f - x_j g \in (x_0, \dots, x_d)\}$.

For $s = n$ the elements in 1) generate I_{r+1} , while for $s = d$ they generate $I^{(1)}$. Moreover, the first set for $s = d$ and the second set generate I'' .

Proof. The first and second assertions are straightforward by the definitions of I_{r+1} and $I^{(1)}$. For the latter one, we observe that the polynomials in these two sets are contained in $I'' = (F)_{r+1} \cap (x_0, \dots, x_d)$. Thus, it suffices to prove the statement in the case $B = F$.

By Theorem 6.2(iv), the A -module I'' is generated by $F'' \cup S$. Obviously, F'' is contained in the set given in 1). Moreover, S is contained in the set given in 2). Indeed, by definition of J -marked set and Lemma 4.4, for every $f_\alpha \in F$ we have $f_\alpha - x^\alpha \subset \langle \mathcal{N}(J) \rangle_r \subset (x_0, \dots, x_d)$. Then, $f_\alpha \in (x_0, \dots, x_d)$ if, and only if, $x^\alpha \in (x_0, \dots, x_d)$. \square

Remark 6.4. For every ideal $I \in \underline{\mathbf{G}}_{\mathcal{I}}(A)$ with $J(\mathcal{I}) \in \mathbb{B}$, we will apply the previous results considering $J(\mathcal{I})$ as J and the set of generators $\mathcal{B}^{(1)}(I)$ (where I stands for I_r) as B . Note that $\mathcal{B}^{(1)}(I)$ contains the \mathcal{I} -marked set $\mathcal{B}_{\mathcal{I}}^{(1)}(I)$, which is monic since $\Delta_{\mathcal{I}}(I)$ is a unit in A and we may set $\Delta_{\mathcal{I}}(I) = 1$.

In order to apply to I the equivalent conditions (v), ..., (viii) of Theorem 6.2 we need to consider exterior products of the type $\wedge^m \langle x_0 I_r, \dots, x_s I_r \rangle$ for some integers

$1 \leq m \leq q(r+1) + 1$ and $0 \leq s \leq N$. The set of generators for this module we use is

$$\left\{ \bigwedge_{\substack{0 \leq i \leq s \\ m_i > 0}} x_i \delta_{\mathcal{K}_i}^{(m_i)}(I) \mid \forall \delta_{\mathcal{K}_i}^{(m_i)} \in \mathcal{B}^{(m_i)} \text{ s.t. } \sum m_i = m \right\}.$$

This set is obtained considering the decomposition of $\wedge^m \langle x_0 I_r, \dots, x_s I_r \rangle$ as the sum of the submodules $(x_0 \wedge^{m_0} I_r) \wedge \dots \wedge (x_s \wedge^{m_s} I_r)$ over the sequences of non-negative integers (m_0, \dots, m_s) with sum m . Note that in this writing we assume that the i -th piece $x_i \wedge^{m_i} I_r$ is missing whenever $m_i = 0$; the number of factors is at most s and the maximum is reached only if all the integers m_i are positive.

We are now able to exhibit the ideal \mathfrak{H} in the ring of Plücker coordinates $k[\Delta]$ that globally defines the Hilbert scheme as a subscheme of the Grassmannian. First, we set

$$\mathfrak{h}_1 := \text{coeff}_x \left\{ \bigwedge_{\substack{0 \leq i \leq d \\ m_i > 0}} x_i \delta_{\mathcal{K}_i}^{(m_i)} \mid \forall \delta_{\mathcal{K}_i}^{(m_i)} \in \mathcal{B}^{(m_i)} \text{ s.t. } \sum m_i = q'' + 1 \right\} \quad (6.1)$$

$$\mathfrak{h}_2 := \text{coeff}_x \left\{ \left(\bigwedge_{\substack{0 \leq i \leq d \\ m_i > 0}} x_i \delta_{\mathcal{K}_i}^{(m_i)} \right) \wedge \left(x_j \delta_{\mathcal{H}}^{(1)} \pm x_k \delta_{\overline{\mathcal{H}}}^{(1)} \right) \mid \right. \\ \left. \begin{array}{l} \forall \delta_{\mathcal{K}_i}^{(m_i)} \in \mathcal{B}^{(m_i)} \text{ s.t. } \sum m_i = q'' \\ \forall x_j \delta_{\mathcal{H}}^{(1)} \pm x_k \delta_{\overline{\mathcal{H}}}^{(1)} \in \mathcal{W} \end{array} \right\} \quad (6.2)$$

where \mathcal{W} is the set of polynomials $x_j \delta_{\mathcal{H}}^{(1)} \pm x_k \delta_{\overline{\mathcal{H}}}^{(1)}$ such that

- $\mathcal{H} = (\mathcal{H} \cap \overline{\mathcal{H}}) \cup \{h\}$ and $\overline{\mathcal{H}} = (\mathcal{H} \cap \overline{\mathcal{H}}) \cup \{\overline{h}\}$, i.e. the polynomial $\delta_{\mathcal{H}}^{(1)}$ contains the term $\Delta_{\mathcal{H} \cap \overline{\mathcal{H}}} x^{\alpha(h)}$ and $\delta_{\overline{\mathcal{H}}}^{(1)}$ contains $\Delta_{\mathcal{H} \cap \overline{\mathcal{H}}} x^{\alpha(\overline{h})}$;
- the pair $(x_j, -x_k)$ is a syzygy for the monomials $x^{\alpha(h)}$ and $x^{\alpha(\overline{h})}$, i.e. $x_j x^{\alpha(h)} = x_k x^{\alpha(\overline{h})}$ and the sign \pm is chosen in order for the terms $\Delta_{\mathcal{H} \cap \overline{\mathcal{H}}} x_j x^{\alpha(h)}$ and $\Delta_{\mathcal{H} \cap \overline{\mathcal{H}}} x_k x^{\alpha(\overline{h})}$ to cancel;
- $x\text{-supp}(x_j \delta_{\mathcal{H}}^{(1)} \pm x_k \delta_{\overline{\mathcal{H}}}^{(1)}) \subset (x_0, \dots, x_d)$.

Moreover, we set $\mathfrak{h} := \mathfrak{h}_1 \cup \mathfrak{h}_2$ and consider for every $g \in \text{PGL}$ the set of equations $g \cdot \mathfrak{h}$ obtained by the action of g on the elements of \mathfrak{h} . Finally we define the ideal

$$\mathfrak{H} := \left(\bigcup_{g \in \text{PGL}} g \cdot \mathfrak{h} \right).$$

Theorem 6.5. *Let $p(t)$ be an admissible Hilbert polynomial for subschemes of \mathbb{P}^n of degree d .*

The homogeneous ideal \mathfrak{H} in the ring of Plücker coordinates $k[\Delta]$ of the Plücker embedding $\mathbf{Gr}_p^N \hookrightarrow \mathbb{P}^E$ is generated in degree $\leq d+2$ and defines $\mathbf{Hilb}_{p(t)}^n$ as a closed subscheme of \mathbf{Gr}_p^N .

Proof. By definition, \mathfrak{H} is the smallest ideal in $k[\Delta]$ that contains the union of the two sets of equations \mathfrak{h}_1 and \mathfrak{h}_2 , given in (6.1) and (6.2), and is invariant by the action of PGL. Since the action of PGL does not modify the degree of polynomials, in order to prove the first part of the statement it suffices to recall that each $\delta_{\mathcal{K}}^{(m)}$ is linear in the Plücker coordinates (Theorem 5.10); hence, the degree of each polynomial in (6.1) is at most $d+1$ and the degree of each polynomial in (6.2) is at most $d+2$. In both cases equality is achieved only when all the integers m_i are strictly positive.

For convenience, we denote by \mathcal{Z} the subscheme of \mathbf{Gr}_p^N defined by \mathfrak{H} and by \mathfrak{D} the saturated ideal in $k[\Delta]$ that defines $\mathbf{Hilb}_{p(t)}^n$ as a closed subscheme of \mathbf{Gr}_p^N . We have to prove that $\mathcal{Z} = \mathbf{Hilb}_{p(t)}^n$. Note that \mathfrak{H} does not need to be saturated and coincide with \mathfrak{D} .

As equality of subschemes is a local property, we may check the equality locally. The proof is divided in two steps.

- Step 1. For every Borel multi-index \mathcal{I} such that $J(\mathcal{I}) \in \mathbb{B}$, the ideal generated by \mathfrak{h} defines $\mathbf{H}_{\mathcal{I}}$ as closed subscheme of $\mathbf{G}_{\mathcal{I}}$.
- Step 2. For every (closed) point I of \mathbf{Gr}_p^N , \mathcal{Z} and $\mathbf{Hilb}_{p(t)}^n$ coincide on a neighborhood of I .

Proof of Step 1. We have to prove that for every k -algebra A and ideal I in $\mathbf{G}_{\mathcal{I}}(A)$, I is contained in $\mathbf{H}_{\mathcal{I}}(A)$ if, and only if, the polynomials in \mathfrak{h} vanish when evaluated at I .

Referring to Theorem 6.2 and Proposition 6.3, the vanishing at I of the polynomials of \mathfrak{h}_1 is equivalent to $\wedge^{q''+1} I^{(1)} = 0$ and that of the polynomials of \mathfrak{h}_2 to $\wedge^{q''} I^{(1)} \wedge I'' = 0$. The equivalence (v) \Leftrightarrow (viii) of Theorem 6.2 and the definition of marked basis allow to conclude.

Proof of Step 2. Both ideals \mathfrak{H} and \mathfrak{D} are invariant under the action of PGL, \mathfrak{H} by construction and \mathfrak{D} because $\mathbf{Hilb}_{p(t)}^n$ is.

Due to the noetherianity of the ring of Plücker coordinates $k[\Delta]$, we can choose $h_1, \dots, h_m \in \bigcup_{g \in \text{PGL}} g \cdot \mathfrak{h}$ that generate \mathfrak{H} . If $h_i \in g_i \cdot \mathfrak{h}$, then we get

$$(g_1 \cdot \mathfrak{h} \cup \dots \cup g_m \cdot \mathfrak{h}) = \mathfrak{H}.$$

By the invariance of \mathfrak{H} under the action of PGL, we also get, for each $g \in \text{PGL}$

$$(gg_1 \cdot \mathfrak{h} \cup \dots \cup gg_m \cdot \mathfrak{h}) = g \cdot (g_1 \cdot \mathfrak{h} \cup \dots \cup g_m \cdot \mathfrak{h}) = g \cdot \mathfrak{H} = \mathfrak{H}.$$

On the other hand, if we restrict to the open subset $\mathbf{G}_{\mathcal{I},gg_1} \cap \cdots \cap \mathbf{G}_{\mathcal{I},gg_m}$, then by *Step 1* and by the invariance of \mathfrak{D} under the action of PGL, we see that the ideal

$$\mathfrak{D} = (gg_1 \cdot \mathfrak{D} \cup \cdots \cup gg_m \cdot \mathfrak{D})$$

defines the same subscheme as $\mathfrak{H} = (gg_1 \cdot \mathfrak{h} \cup \cdots \cup gg_m \cdot \mathfrak{h})$. Therefore,

$$\mathbf{Hilb}_{p(t)}^n \cap (\mathbf{G}_{\mathcal{I},gg_1} \cap \cdots \cap \mathbf{G}_{\mathcal{I},gg_m}) = \mathcal{Z} \cap (\mathbf{G}_{\mathcal{I},gg_1} \cap \cdots \cap \mathbf{G}_{\mathcal{I},gg_m}).$$

It remains to prove that for every $I \in \mathbf{Gr}_p^N$, we can find suitable $g \in \text{PGL}$ and $J(\mathcal{I}) \in \mathbb{B}$, such that $I \in \mathbf{G}_{\mathcal{I},gg_1} \cap \cdots \cap \mathbf{G}_{\mathcal{I},gg_m}$.

By Proposition 3.2, there are $J(\mathcal{I}) \in \mathbb{B}$ and \bar{g} such that $I \in \mathbf{G}_{\mathcal{I},\bar{g}}$. The orbit of I under the action of PGL is almost completely contained in $\mathbf{G}_{\mathcal{I},\bar{g}}$; let U be an open subset of PGL such that $(g')^{-1} \cdot I \in \mathbf{G}_{\mathcal{I},\bar{g}}$, i.e. $I \in \mathbf{G}_{\mathcal{I},g'\bar{g}}$. Therefore, for a general $g \in \text{PGL}$, it holds $gg_1\bar{g}^{-1}, \dots, gg_m\bar{g}^{-1} \in U$ and $I \in \mathbf{G}_{\mathcal{I},gg_1} \cap \cdots \cap \mathbf{G}_{\mathcal{I},gg_m}$ as wanted. \square

For sake of completeness we now show how our strategy also allows to mimic the construction of equations for $\mathbf{Hilb}_{p(t)}^n$ presented in the well-known papers by Iarrobino and Kleiman [21] and by Haiman and Sturmfels [18].

6.1. Equations of higher degree

Let A be a k -algebra and I be an ideal in $\mathbf{G}_{\mathcal{I}}(A)$. Exploiting Theorem 5.10, we obtain a set of generators for I_{r+1} evaluating at I the following set of polynomials

$$x_0\mathcal{B}^{(1)} \cup \cdots \cup x_n\mathcal{B}^{(1)} = \left\{ x_i \delta_{\mathcal{K}}^{(1)} \mid i = 0, \dots, n, \mathcal{K} \in \mathcal{E}^{(1)} \right\}.$$

By Theorem 6.2(v) \Leftrightarrow (vi), we know that $I \in \mathbf{H}_{\mathcal{I}}(A)$ if, and only if, $\wedge^{q(r+1)+1} I_{r+1}$ vanishes. The exterior power $\wedge^{q(r+1)+1} I_{r+1}$ is generated by all the possible exterior products of order $q(r+1)+1$ among the given set of generators of I_{r+1} . Therefore, the conditions $I \in \mathbf{H}_{\mathcal{I}}(A)$ is given by the vanishing at I of the x -coefficients in the wedge products

$$\bigwedge_{j=1}^{q(r+1)+1} x_{i_j} \delta_{\mathcal{K}_j}^{(1)}, \quad \forall 0 \leq i_1 \leq \dots \leq i_{q(r+1)+1} \leq n, \quad \forall \mathcal{K}_j \in \mathcal{E}^{(1)}.$$

The open subfunctors $\mathbf{G}_{\mathcal{I}}$ cover the Grassmann functor and each $\mathbf{H}_{\mathcal{I}}$ is representable, so that we can apply Proposition 2.1. The natural transformations $\mathcal{H}_{\mathcal{I}}: \mathbf{H}_{\mathcal{I}} \rightarrow \mathbf{G}_{\mathcal{I}}$ are induced by closed embeddings of schemes, hence the same holds true for $\mathcal{H}: \mathbf{Hilb}_{p(t)}^n \rightarrow \mathbf{Gr}_p^N$.

Theorem 6.6 (Iarrobino-Kleiman-like equations for the Hilbert scheme).

The subscheme of \mathbf{Gr}_p^N representing the Hilbert functor $\mathbf{Hilb}_{p(t)}^n$ can be defined by an ideal generated by homogeneous elements of degree $q(r+1)+1$ in the ring $k[\Delta]$ of the Plücker coordinates.

The above equations of degree $q(r + 1) + 1$ coincides on each open subscheme $\mathbf{G}_{\mathcal{I}}$ of the standard open cover of the Grassmannian with the set of equations obtained by Iarrobino and Kleiman in local coordinates. We could also exploit this same argument using the Borel open cover of \mathbf{Gr}_p^N , instead of the standard one and obtain a different set of equations of the same degree.

6.2. Equations of degree $n + 1$

As pointed out by Haiman and Sturmfels, if $I = (I_r)$ is generated by a set of polynomials B , then the matrix M_{r+1} that represents the generators $x_0B \cup \dots \cup x_nB$ of the module I_{r+1} contains $n + 1$ copies of the matrix M_r corresponding to B . Hence, some minors of M_{r+1} are also minors of M_r and every minor of M_{r+1} can be obtained as the sum of products of at most $n + 1$ minors of M_r .

This observation suggests to expand $\wedge^{q(r+1)+1} I_{r+1}$ as done in Remark 6.4 and take the x -coefficients of

$$\bigwedge_{\substack{0 \leq i \leq n \\ m_i > 0}} x_i \delta_{\mathcal{K}_i}^{(m_i)}, \quad \forall \delta_{\mathcal{K}_i}^{(m_i)} \in \mathcal{B}^{(m_i)} \text{ s.t. } \sum m_i = m.$$

Theorem 6.7 (Bayer-Haiman-Sturmfels-like equations for the Hilbert scheme). *The subscheme of \mathbf{Gr}_p^N representing the Hilbert functor $\mathbf{Hilb}_{p(t)}^n$ can be defined by an ideal generated by homogeneous elements of degree $\leq n + 1$ in the ring $k[\Delta]$ of the Plücker coordinates.*

In this case, if we use the standard open cover of \mathbf{Gr}_p^N , we obtain the same global equations given by Haiman and Sturmfels, while using the Borel open cover we obtain a different set of equations with maximum degree $n + 1$.

7. Examples: Hilbert schemes of points

7.1. The Hilbert scheme \mathbf{Hilb}_2^2

The Gotzmann number of the Hilbert polynomial $p(t) = 2$ is $r = 2$, hence $N(r) = 6$ and $p(r) = 2$. We identify $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ with k^6 by setting $a_i = x^{\alpha(i)}$ where $x^{\alpha(i)}$ is the i -th term in the sequence $(x_2^2, x_2x_1, x_1^2, x_2x_0, x_1x_0, x_0^2)$. In this way we obtain the natural transformation of functors $\mathbf{Hilb}_2^2 \rightarrow \mathbf{Gr}_2^6$.

Standard open cover of \mathbf{Gr}_2^6 There are $\binom{6}{2} = 15$ open subfunctor $\mathbf{G}_{\mathcal{I}}$ in the standard open cover, each corresponding to a 2-multi-index $\mathcal{I} \subset \{1, 2, 3, 4, 5, 6\}$. Not every element of $\mathbf{Gr}_2^6(A)$ is contained in one of them (not even the free ones), if A is not a field or even a local ring.

Let us consider for instance $A := k[t]$ and

$$\pi : A^6 \xrightarrow{\begin{pmatrix} 1-t & 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1+t & 1 \end{pmatrix}} A^2. \tag{7.1}$$

This map is surjective, since $(1, 0) = (1 + t)\pi(a_1) + \pi(a_3) - \pi(a_6)$ and $(0, 1) = \pi(a_6)$. Its kernel is the free A -module $L = \langle t^2 a_1 + (t-1)a_3 - (t-1)a_6, a_2, a_4, a_5 - (t+1)a_6 \rangle$.

Thus, the quotient $Q := A^6 / \ker \pi$ is isomorphic to A^2 and $Q \in \underline{\mathbf{Gr}}_2^6(A)$. Notice that the set of non-zero maximal minors $\{1-t, 1-t^2, t^2+t^3, t^2\}$ of the matrix defining π generates A , but none of them alone does, so that Q does not belong to any $\underline{\mathbf{Gr}}_{\mathcal{I}}(A)$.

On the other hand, for the local k -algebra $A' := k[t]_{(t)}$, the A' -module $Q' := Q \otimes_A A'$ is in $\underline{\mathbf{Gr}}_{\mathcal{I}}(A')$ for $\mathcal{I} = (1, 3)$. The \mathcal{I} -marked set of the A' -module L' such that $Q' = A'^6 / L'$ is

$$\begin{aligned} b_2 &= a_2 & b_5 &= a_5 + \frac{t^3+t^2}{1-t} a_1 - (1+t)a_3 \\ b_4 &= a_4 & b_6 &= a_6 + \frac{t^2}{1-t} a_1 - a_3 \end{aligned}$$

The Plücker coordinates of L' (with $\Delta_{13}(L') = 1$) are given by Corollary 5.6

$$\begin{aligned} \Delta_{12}(L') &= 0, & \Delta_{23}(L') &= 0, & \Delta_{35}(L') &= \frac{t^3+t^2}{1-t}, \\ \Delta_{13}(L') &= 1, & \Delta_{24}(L') &= 0, & \Delta_{36}(L') &= \frac{t^2}{1-t}, \\ \Delta_{14}(L') &= 0, & \Delta_{25}(L') &= 0, & \Delta_{45}(L') &= 0, \\ \Delta_{15}(L') &= (1+t), & \Delta_{26}(L') &= 0, & \Delta_{46}(L') &= 0, \\ \Delta_{16}(L') &= 1, & \Delta_{34}(L') &= 0, & \Delta_{56}(L') &= 0. \end{aligned}$$

The generators $\mathcal{B}^{(m)}$ For $m = 1$ there are 20 elements in $\mathcal{B}^{(1)}$, since there are $\binom{6}{3} = 20$ multi-indices $\mathcal{K} \in \mathcal{E}^{(1)}$. For instance for $\mathcal{K} \in \mathcal{E}_{13}^{(1)}$ we get

$$\begin{aligned} \delta_{123}^{(1)} &= \Delta_{23} a_1 - \Delta_{13} a_2 + \Delta_{12} a_3 \\ \delta_{134}^{(1)} &= \Delta_{34} a_1 - \Delta_{14} a_3 + \Delta_{13} a_4 \\ \delta_{135}^{(1)} &= \Delta_{35} a_1 - \Delta_{15} a_3 + \Delta_{13} a_5 \\ \delta_{136}^{(1)} &= \Delta_{36} a_1 - \Delta_{16} a_3 + \Delta_{13} a_6 \end{aligned}$$

and for $\mathcal{K} = (3, 5, 6)$ we get

$$\delta_{356}^{(1)} = \Delta_{56} a_3 - \Delta_{36} a_5 + \Delta_{35} a_6.$$

They are not independent. For instance there is the relation

$$\Delta_{13} \delta_{356}^{(1)} + \Delta_{36} \delta_{135}^{(1)} - \Delta_{35} \delta_{136}^{(1)} = (\Delta_{13} \Delta_{56} - \Delta_{15} \Delta_{36} + \Delta_{16} \Delta_{35}) a_3 = 0$$

(note that the expression in the round brackets is a Plücker relation).

For $m = 2$, $\mathcal{B}^{(2)}$ contains $\binom{6}{4} = 15$ elements. For instance,

$$\delta_{1356}^{(2)} = \Delta_{13} a_5 \wedge a_6 - \Delta_{15} a_3 \wedge a_6 + \Delta_{16} a_3 \wedge a_5 + \Delta_{35} a_1 \wedge a_6 - \Delta_{36} a_1 \wedge a_5 + \Delta_{56} a_1 \wedge a_3.$$

Finally, $\mathcal{B}^{(3)}$ contains $\binom{6}{5} = 6$ elements and $\mathcal{B}^{(4)}$ has a unique element.

Borel open cover of \mathbf{Gr}_2^6 It is easy to check that there is only one Borel multi-index of two elements, namely $\mathcal{I} = (5, 6)$.

As the minor of the matrix (7.1) corresponding to the last two columns is identically zero, for every $\mathfrak{p} \in \text{Spec } k[t]$, $Q_{\mathfrak{p}}$ is not contained in $\mathbf{G}_{56}(k[t]_{\mathfrak{p}})$. We can apply Proposition 3.2 and determine for $\mathfrak{p} = (1-t)$ a change of coordinates $g \in \text{PGL}_{\mathbb{Q}}(3)$ such that $Q_{\mathfrak{p}}$ is contained in $\mathbf{G}_{56,g}(k(\mathfrak{p}))$. Tensoring by the residue field $k(\mathfrak{p}) \simeq k$, we obtain the following surjective morphism of vector spaces

$$k^6 \xrightarrow{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{pmatrix}} k^2$$

whose kernel is the vector space $\langle x_2^2, x_2x_1, x_2x_0, x_1x_0 - x_0^2 \rangle$. The generic initial ideal of the ideal $I = (x_2^2, x_2x_1, x_2x_0, x_1x_0 - x_0^2)$ is $J = (x_2^2, x_2x_1, x_1^2, x_2x_0)$. A change of coordinates g such that $g \cdot I = J$ is, for instance, the automorphism swapping x_1 and x_0 . Indeed,

$$g \cdot (x_2^2, x_2x_1, x_2x_0, x_1x_0 - x_0^2) = (x_2^2, x_2x_0, x_2x_1, x_1^2 - \frac{1}{2}x_1x_0)$$

and

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{so that} \quad \pi_{\mathfrak{p}} \circ \tilde{g} \circ \Gamma_{56} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

is surjective. Notice that this change of coordinates does not work for all localizations. Indeed

$$\pi \circ \tilde{g} \circ \Gamma_{56} = \begin{pmatrix} 0 & t^2 \\ t+1 & 1 \end{pmatrix}$$

is not surjective in the localizations $k[t]_{(t+1)}$ and $k[t]_{(t)}$ as the determinant is not invertible.

New equations Let us finally show how to determine the equations of degree 2 defining the scheme representing the Hilbert functor \mathbf{Hilb}_2^2 in the Grassmannian \mathbf{Gr}_2^6 .

As $d = 0$, $I^{(1)} = x_0I_2$ and its rank is equal to $q(2) = q'' = 4$. Therefore, the set of equations \mathfrak{h}_1 of (6.1) is empty.

The set of equations \mathfrak{h}_2 ensures that $\bigwedge^4 I^{(1)} \wedge I'' = 0$ and contains the x -coefficients of the products between $x_0\delta_{123456}^{(4)}$ and each element of $\mathcal{W} = \{x_2\delta_{245}^{(1)} - x_1\delta_{145}^{(1)}, x_2\delta_{246}^{(1)} - x_1\delta_{146}^{(1)}, x_2\delta_{256}^{(1)} - x_1\delta_{156}^{(1)}, x_2\delta_{345}^{(1)} - x_1\delta_{245}^{(1)}, x_2\delta_{346}^{(1)} - x_1\delta_{246}^{(1)}, x_2\delta_{356}^{(1)} - x_1\delta_{256}^{(1)}, x_1\delta_{456}^{(1)} - x_0\delta_{256}^{(1)}, x_2\delta_{456}^{(1)} - x_0\delta_{156}^{(1)}, x_1\delta_{456}^{(1)} + x_0\delta_{346}^{(1)}, x_2\delta_{456}^{(1)} + x_0\delta_{246}^{(1)}\}$. We

notice that this set gives redundant relations, indeed for instance

$$\begin{aligned} x_0\delta_{123456}^{(4)} \wedge \left(x_1\delta_{456}^{(1)} - x_0\delta_{256}^{(1)} \right) &= x_0\delta_{123456}^{(4)} \wedge x_1\delta_{456}^{(1)} - x_0 \left(\delta_{123456}^{(4)} \wedge \delta_{256}^{(1)} \right) \\ &= x_0\delta_{123456}^{(4)} \wedge x_1\delta_{456}^{(1)} = x_0\delta_{123456}^{(4)} \wedge x_1\delta_{456}^{(1)} + x_0 \left(\delta_{123456}^{(4)} \wedge \delta_{346}^{(1)} \right) \\ &= x_0\delta_{123456}^{(4)} \wedge \left(x_1\delta_{456}^{(1)} + x_0\delta_{346}^{(1)} \right) \end{aligned}$$

as each exterior product of order greater than 4 vanishes (in fact the x -coefficients we obtain from such products of order 5 are contained in the ideal generated by the Plücker relations). Hence, in order to determine the equations of \mathfrak{h}_2 we consider the set of polynomials $\widehat{\mathcal{W}} = \{x_2\delta_{245}^{(1)} - x_1\delta_{145}^{(1)}, x_2\delta_{246}^{(1)} - x_1\delta_{146}^{(1)}, x_2\delta_{256}^{(1)} - x_1\delta_{156}^{(1)}, x_2\delta_{345}^{(1)} - x_1\delta_{245}^{(1)}, x_2\delta_{346}^{(1)} - x_1\delta_{246}^{(1)}, x_2\delta_{356}^{(1)} - x_1\delta_{256}^{(1)}, x_1\delta_{456}^{(1)}, x_2\delta_{456}^{(1)}\}$. We get 48 equations which are reduced to 30 by Plücker relations.

To obtain the equation defining $\mathbf{Hilb}_2^2 \subset \mathbf{Gr}_2^6$, we need to determine the orbit of these polynomials with respect to the action of $\mathrm{PGL}_{\mathbb{Q}}(3)$. However, in this special case, we discover that the ideal generated by the Plücker relations and by \mathfrak{h}_2 is already $\mathrm{PGL}_{\mathbb{Q}}(3)$ invariant, *i.e.*, the equations in \mathfrak{h}_2 define the Hilbert scheme. The Plücker relations and the following 30 equations define \mathbf{Hilb}_2^2 as subscheme of \mathbb{P}^{14} :

$$\begin{aligned} &\Delta_{13}\Delta_{14} - \Delta_{12}\Delta_{24} - \Delta_{12}\Delta_{15}, \Delta_{13}\Delta_{24} - \Delta_{12}\Delta_{34} - \Delta_{12}\Delta_{25}, \Delta_{23}\Delta_{24} - \Delta_{12}\Delta_{35}, \\ &\Delta_{14}\Delta_{24} - \Delta_{14}\Delta_{15} + \Delta_{12}\Delta_{16}, \Delta_{24}^2 - \Delta_{14}\Delta_{25} + \Delta_{12}\Delta_{26}, \Delta_{23}\Delta_{34} + \Delta_{23}\Delta_{25} - \Delta_{13}\Delta_{35}, \\ &\Delta_{14}\Delta_{34} - \Delta_{14}\Delta_{25} + \Delta_{12}\Delta_{45} + \Delta_{12}\Delta_{26}, \Delta_{24}\Delta_{34} - \Delta_{14}\Delta_{35} + \Delta_{12}\Delta_{36}, \\ &\Delta_{34}^2 - \Delta_{15}\Delta_{35} - \Delta_{23}\Delta_{45} + \Delta_{13}\Delta_{36}, \Delta_{15}^2 - \Delta_{14}\Delta_{25} - \Delta_{13}\Delta_{16} + \Delta_{12}\Delta_{26}, \\ &\Delta_{24}\Delta_{25} - \Delta_{14}\Delta_{35}, \Delta_{15}\Delta_{25} - \Delta_{14}\Delta_{35} - \Delta_{13}\Delta_{26} + \Delta_{12}\Delta_{36}, \Delta_{25}^2 - \Delta_{15}\Delta_{35} - \Delta_{23}\Delta_{45}, \\ &\Delta_{24}\Delta_{35} - \Delta_{15}\Delta_{35} - \Delta_{23}\Delta_{45} + \Delta_{23}\Delta_{26}, \Delta_{34}\Delta_{35} - \Delta_{25}\Delta_{35} + \Delta_{23}\Delta_{36}, \\ &\Delta_{14}\Delta_{45} - \Delta_{12}\Delta_{46}, \Delta_{24}\Delta_{45} - \Delta_{12}\Delta_{56}, \Delta_{34}\Delta_{45} + \Delta_{23}\Delta_{46} - \Delta_{13}\Delta_{56}, \\ &\Delta_{15}\Delta_{45} - \Delta_{13}\Delta_{46} + \Delta_{12}\Delta_{56}, \Delta_{25}\Delta_{45} - \Delta_{23}\Delta_{46}, \Delta_{35}\Delta_{45} - \Delta_{23}\Delta_{56}, \\ &\Delta_{45}^2 - \Delta_{25}\Delta_{46} + \Delta_{15}\Delta_{56}, \Delta_{24}\Delta_{26} - \Delta_{14}\Delta_{36} + \Delta_{12}\Delta_{56}, \\ &\Delta_{25}\Delta_{26} - \Delta_{15}\Delta_{36} - \Delta_{23}\Delta_{46} + \Delta_{13}\Delta_{56}, \Delta_{26}^2 - \Delta_{16}\Delta_{36} - \Delta_{25}\Delta_{46} + \Delta_{15}\Delta_{56}, \\ &\Delta_{24}\Delta_{46} - \Delta_{14}\Delta_{56}, \Delta_{34}\Delta_{46} + \Delta_{25}\Delta_{46} - \Delta_{24}\Delta_{56} - \Delta_{15}\Delta_{56}, \Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56}, \\ &\Delta_{45}\Delta_{46} - \Delta_{26}\Delta_{46} + \Delta_{16}\Delta_{56}, \Delta_{36}\Delta_{46} - \Delta_{45}\Delta_{56} - \Delta_{26}\Delta_{56}. \end{aligned}$$

Furthermore, we check that the ideal they generate is saturated, then it is *the saturated ideal* of \mathbf{Hilb}_2^2 . Its Hilbert polynomial is $\frac{21}{4!}t^4 + \frac{15}{4}t^3 + \frac{45}{8}t^2 + \frac{15}{4}t + 1$, hence $\mathbf{Hilb}_2^2 \subset \mathbb{P}^{14}$ is a subscheme of dimension 4 (as expected) and degree 21, as already proved in [7, 18].

Iarrobino-Kleiman equations Let us now see how to compute the Iarrobino-Kleiman equations for \mathbf{Hilb}_2^2 . The universal element of \mathbf{Gr}_2^6 is generated by $\mathcal{B}^{(1)}$. In order to compute $\wedge^{q(r+1)+1} I_{r+1} = \wedge^9 I_3$ we use the set of generators $x_0\mathcal{B}^{(1)} \cup x_1\mathcal{B}^{(1)} \cup x_2\mathcal{B}^{(1)}$ of I_3 . The x -coefficients of any exterior product of order 9 are

expressions of degree 9 in the Plücker coordinates. Their union defines $\mathbf{Hilb}_2^2 \subset \mathbf{Gr}_2^6$.

For instance, considering the 9 elements $x_2\delta_{126}^{(1)}, x_2\delta_{156}^{(1)}, x_2\delta_{234}^{(1)}, x_2\delta_{356}^{(1)}, x_1\delta_{123}^{(1)}, x_1\delta_{345}^{(1)}, x_0\delta_{146}^{(1)}, x_0\delta_{234}^{(1)}, x_0\delta_{456}^{(1)}$, the x -coefficients of their exterior product are the maximal minors of the following matrix

$$\begin{array}{c} x_2^3 \quad x_2^2x_1 \quad x_2x_1^2 \quad x_1^3 \quad x_2^2x_0 \quad x_2x_1x_0 \quad x_1^2x_0 \quad x_2x_0^2 \quad x_1x_0^2 \quad x_0^3 \\ \left(\begin{array}{cccccccccc} \Delta_{26} & -\Delta_{16} & 0 & 0 & 0 & 0 & 0 & \Delta_{12} & 0 & 0 \\ \Delta_{56} & 0 & 0 & 0 & 0 & -\Delta_{16} & 0 & \Delta_{15} & 0 & 0 \\ 0 & \Delta_{34} & -\Delta_{24} & 0 & \Delta_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_{56} & 0 & 0 & -\Delta_{36} & 0 & \Delta_{35} & 0 & 0 \\ 0 & \Delta_{23} & -\Delta_{13} & \Delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{45} & 0 & -\Delta_{35} & \Delta_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_{46} & 0 & 0 & -\Delta_{16} & 0 & \Delta_{14} \\ 0 & 0 & 0 & 0 & 0 & \Delta_{34} & -\Delta_{24} & \Delta_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{56} & -\Delta_{46} & \Delta_{45} \end{array} \right) \end{array}.$$

Bayer-Haiman-Sturmfels equations In order to lower the degree of the equations, we can impose the vanishing of the exterior power $\wedge^9 I_3$ by considering $\wedge^9 I_3$ generated by all exterior products $x_0\delta_{\mathcal{J}_0}^{(m_0)} \wedge x_1\delta_{\mathcal{J}_1}^{(m_1)} \wedge x_2\delta_{\mathcal{J}_2}^{(m_2)}$ for (m_0, m_1, m_2) such that $m_0 + m_1 + m_2 = 9$ and $0 \leq m_0, m_1, m_2 \leq 4$. For $m_0 = 3, m_1 = 2, m_2 = 4$, we get for instance $x_0\delta_{23456}^{(3)} \wedge x_1\delta_{1346}^{(2)} \wedge x_2\delta_{123456}^{(4)}$, where

$$\begin{aligned} x_0\delta_{23456}^{(3)} &= \Delta_{56}x_2x_1x_0 \wedge x_1^2x_0 \wedge x_2x_0^2 - \Delta_{46}x_2x_1x_0 \wedge x_1^2x_0 \wedge x_1x_0^2 + \Delta_{45}x_2x_1x_0 \wedge x_1^2x_0 \wedge x_0^3 \\ &\quad + \Delta_{36}x_2x_1x_0 \wedge x_2x_0^2 \wedge x_1x_0^2 - \Delta_{35}x_2x_1x_0 \wedge x_2x_0^2 \wedge x_0^3 + \Delta_{34}x_2x_1x_0 \wedge x_1x_0^2 \wedge x_0^3 \\ &\quad - \Delta_{26}x_1^2x_0 \wedge x_2x_0^2 \wedge x_1x_0^2 + \Delta_{25}x_1^2x_0 \wedge x_1x_0^2 \wedge x_0^3 - \Delta_{24}x_1^2x_0 \wedge x_1x_0^2 \wedge x_0^3 \\ &\quad + \Delta_{23}x_2x_0^2 \wedge x_1x_0^2 \wedge x_0^3, \\ x_1\delta_{1346}^{(2)} &= \Delta_{46}x_2^2x_1 \wedge x_1^3 - \Delta_{36}x_2^2x_1 \wedge x_2x_1x_0 + \Delta_{34}x_2^2x_1 \wedge x_1x_0^2 + \Delta_{16}x_1^3 \wedge x_2x_1x_0 \\ &\quad - \Delta_{14}x_1^3 \wedge x_1x_0^2 + \Delta_{13}x_2x_1x_0 \wedge x_1x_0^2, \\ x_2\delta_{123456}^{(4)} &= \Delta_{56}x_2^3 \wedge x_2^2x_1 \wedge x_2x_1^2 \wedge x_2^2x_0 - \Delta_{46}x_2^3 \wedge x_2^2x_1 \wedge x_2x_1^2 \wedge x_2x_1x_0 \\ &\quad + \Delta_{45}x_2^3 \wedge x_2^2x_1 \wedge x_2x_1^2 \wedge x_2x_0^2 + \Delta_{36}x_2^3 \wedge x_2^2x_1 \wedge x_2^2x_0 \wedge x_2x_1x_0 \\ &\quad - \Delta_{35}x_2^3 \wedge x_2^2x_1 \wedge x_2^2x_0 \wedge x_2x_0^2 + \Delta_{34}x_2^3 \wedge x_2^2x_1 \wedge x_2x_1x_0 \wedge x_2x_0^2 \\ &\quad - \Delta_{26}x_2^3 \wedge x_2x_1^2 \wedge x_2^2x_0 \wedge x_2x_1x_0 + \Delta_{25}x_2^3 \wedge x_2x_1^2 \wedge x_2^2x_0 \wedge x_2x_0^2 \\ &\quad - \Delta_{24}x_2^3 \wedge x_2x_1^2 \wedge x_2x_1x_0 \wedge x_2x_0^2 + \Delta_{23}x_2^3 \wedge x_2^2x_0 \wedge x_2x_1x_0 \wedge x_2x_0^2 \\ &\quad + \Delta_{16}x_2^2x_1 \wedge x_2x_1^2 \wedge x_2^2x_0 \wedge x_2x_1x_0 - \Delta_{15}x_2^2x_1 \wedge x_2x_1^2 \wedge x_2^2x_0 \wedge x_2x_0^2 \\ &\quad + \Delta_{14}x_2^2x_1 \wedge x_2x_1^2 \wedge x_2x_1x_0 \wedge x_2x_0^2 - \Delta_{13}x_2^2x_1 \wedge x_2^2x_0 \wedge x_2x_1x_0 \wedge x_2x_0^2 \\ &\quad + \Delta_{12}x_2x_1^2 \wedge x_2^2x_0 \wedge x_1^2x_0 \wedge x_2x_0^2. \end{aligned}$$

Its x -coefficients are the following polynomials of degree 3 in the Plücker coordinates:

$$\begin{aligned}
& \Delta_{26}^2 \Delta_{46} - \Delta_{25} \Delta_{46}^2 - \Delta_{16} \Delta_{26} \Delta_{56} + \Delta_{14} \Delta_{56}^2, \\
& \Delta_{25} \Delta_{26} \Delta_{46} - \Delta_{25} \Delta_{45} \Delta_{46} - \Delta_{16} \Delta_{25} \Delta_{56}, \\
& \Delta_{24} \Delta_{26} \Delta_{46} - \Delta_{16} \Delta_{24} \Delta_{56} - \Delta_{14} \Delta_{45} \Delta_{56}, \\
& \Delta_{23} \Delta_{26} \Delta_{46} + \Delta_{25} \Delta_{34} \Delta_{46} - \Delta_{16} \Delta_{23} \Delta_{56} - \Delta_{14} \Delta_{35} \Delta_{56}, \\
& \Delta_{25} \Delta_{26} \Delta_{34} - \Delta_{24} \Delta_{25} \Delta_{36} - \Delta_{25} \Delta_{34} \Delta_{45} + \Delta_{13} \Delta_{25} \Delta_{56}, \\
& \Delta_{16} \Delta_{24} \Delta_{45} + \Delta_{14} \Delta_{45}^2 + \Delta_{24}^2 \Delta_{46} - \Delta_{14} \Delta_{25} \Delta_{46}, \\
& \Delta_{24} \Delta_{25} \Delta_{46} - \Delta_{14} \Delta_{25} \Delta_{56}, \\
& \Delta_{16} \Delta_{24} \Delta_{35} - \Delta_{14} \Delta_{25} \Delta_{36} + \Delta_{14} \Delta_{35} \Delta_{45} + \Delta_{23} \Delta_{24} \Delta_{46}, \\
& \Delta_{16} \Delta_{24} \Delta_{25} - \Delta_{14} \Delta_{25} \Delta_{26} + \Delta_{14} \Delta_{25} \Delta_{45}, \\
& \Delta_{15} \Delta_{16} \Delta_{24} - \Delta_{14} \Delta_{16} \Delta_{25} + \Delta_{14} \Delta_{15} \Delta_{45} - \Delta_{12} \Delta_{24} \Delta_{46}.
\end{aligned}$$

In Table 7.1, there is a comparison between the number of generators of the ideal defining the Hilbert scheme obtained according to the three different strategies.

	New equations	B-H-S equations	I-K equations
$\mathbf{Hilb}_2^2 \subset \mathbf{Gr}_2^6 \subset \mathbb{P}^{14}$ (15 Plücker relations)	Degree of equations: 2 $\dim(I_{\mathbf{Hilb}_2^2})_2 = 45$ Number of equations: $ \mathfrak{h} = 24$ Changes of coordinates: $8 + 0$	Degree of equations: 3 $\dim(I_{\mathbf{Hilb}_2^2})_3 = 445$ Number of equations: ~ 8160	Degree of equations: 9 $\dim(I_{\mathbf{Hilb}_2^2})_9 = 808225$ Number of equations: $\sim 9 \cdot 10^{10}$
$\mathbf{Hilb}_2^3 \subset \mathbf{Gr}_2^{10} \subset \mathbb{P}^{44}$ (210 Plücker relations)	Degree of equations: 2 $\dim(I_{\mathbf{Hilb}_2^3})_2 = 570$ Number of equations: $ \mathfrak{h} = 140$ Changes of coordinates: $15 + 1$	Degree of equations: ≤ 4 $\dim(I_{\mathbf{Hilb}_2^3})_4 = 185390$ Number of equations: $\sim 2 \cdot 10^{11}$	Degree of equations: 19 $\dim(I_{\mathbf{Hilb}_2^3})_{19} \sim 6 \cdot 10^{15}$ Number of equations: $\sim 9 \cdot 10^{34}$
$\mathbf{Hilb}_2^4 \subset \mathbf{Gr}_2^{15} \subset \mathbb{P}^{104}$ (1365 Plücker relations)	Degree of equations: 2 $\dim(I_{\mathbf{Hilb}_2^4})_2 = 3575$ Number of equations: $ \mathfrak{h} = 480$ Changes of coordinates: $24 + 3$	Degree of equations: ≤ 5 $\dim(I_{\mathbf{Hilb}_2^4})_5 = 116461170$ Number of equations: $\sim 4 \cdot 10^{22}$	Degree of equations: 34 $\dim(I_{\mathbf{Hilb}_2^4})_{34} \sim 2 \cdot 10^{32}$ Number of equations: $\sim 10^{77}$
$\mathbf{Hilb}_3^2 \subset \mathbf{Gr}_3^{10} \subset \mathbb{P}^{119}$ (2310 Plücker relations)	Degree of equations: 2 $\dim(I_{\mathbf{Hilb}_3^2})_2 = 5425$ Number of equations: $ \mathfrak{h} = 720$ Changes of coordinates: $8 + 2$	Degree of equations: ≤ 3 $\dim(I_{\mathbf{Hilb}_3^2})_3 = 283245$ Number of equations: $\sim 6 \cdot 10^8$	Degree of equations: 13 $\dim(I_{\mathbf{Hilb}_3^2})_{13} \sim 3 \cdot 10^{17}$ Number of equations: $\sim 3 \cdot 10^{28}$

Table 7.1. A comparison among the characteristics of the different sets of equations defining the Hilbert schemes discussed in Section 7. The set \mathfrak{h} of new equations in this table contains the equations obtained considering only Borel multi-indices. In order to determine the $\mathrm{PGL}_{\mathbb{Q}}(n+1)$ -invariant ideal, we have always applied $(n+1)^2 - 1$ (the first summand) special changes of coordinates and the second summand corresponds to the needed random changes of coordinates (see http://www.paololella.it/EN/Publications_files/EquationsHilbertSchemesOfPoints.m2 for the explicit computation). The values of the Hilbert function of the ideals defining the Hilbert schemes have been computed from the ideal generated by the new equations. Notice the large redundancy of Bayer-Haiman-Sturmfels equations and Iarrobino-Kleiman equations which do not take into account the symmetries of the Hilbert scheme.

7.2. The Hilbert scheme \mathbf{Hilb}_2^3

The Hilbert scheme of 2 points in the projective space \mathbb{P}^3 can be constructed as subscheme of the Grassmannian $\mathbf{Gr}_2^{10} \subset \mathbb{P}^{44}$. The set \mathfrak{h}_1 is empty (this happens for every Hilbert scheme of points) and the set \mathfrak{h}_2 contains 600 equations of degree 2 that can be reduced to 330 modulo the Plücker relations. The ideal generated by \mathfrak{h}_2 and by the Plücker relations is not $\mathrm{PGL}_{\mathbb{Q}}(4)$ -invariant. To obtain the equations defining $\mathbf{Hilb}_2^3 \subset \mathbf{Gr}_2^{10}$, we need to determine the orbits of these polynomials with respect to the action of $\mathrm{PGL}_{\mathbb{Q}}(4)$. From a computational point of view, we consider a random element of $\mathrm{PGL}_{\mathbb{Q}}(4)$, apply to our set of equations the induced automorphism on the ring of Plücker coordinates of \mathbf{Gr}_2^{10} , add the new equations to the previous set and repeat this process until the generated ideal stabilizes.

The ideal we obtain is again saturated and its Hilbert polynomial is

$$\frac{370}{6!}t^6 + \frac{83}{24}t^5 + \frac{86}{9}t^4 + \frac{335}{24}t^3 + \frac{823}{72}t^2 + \frac{61}{12}t + 1$$

so that \mathbf{Hilb}_2^3 turns out to be a subscheme of \mathbb{P}^{44} of dimension 6 and degree 370, defined by 570 quadratic equations (210 of them are Plücker relations).

7.3. The Hilbert scheme \mathbf{Hilb}_2^4

The Hilbert scheme of 2 points in \mathbb{P}^4 is constructed as subscheme of the Grassmannian $\mathbf{Gr}_3^{15} \subset \mathbb{P}^{104}$. From the computational point of view, the hardest part is the computation of the orbit of the equations $g \cdot \mathfrak{h}$ for a given change of coordinates $g \in \mathrm{PGL}_{\mathbb{Q}}(5)$. A first trick is to start considering simple changes of coordinates, for instance change of sign of a variable ($x_i \rightarrow -x_i$), swap of two variables ($x_i \leftrightarrow x_j$) and sum of two variables ($x_i \rightarrow x_i + x_j$). These changes of coordinates are easier to compute and bring us closer to the $\mathrm{PGL}_{\mathbb{Q}}(5)$ -invariant ideal of the Hilbert scheme, but in general they are not sufficient. In this case, a generic (random) change of coordinates $g \in \mathrm{PGL}_{\mathbb{Q}}(5)$ induces a change of coordinates of \mathbf{Gr}_3^{15} described by a dense 105×105 matrix, so that computing the action of g on a monomial of degree 2 in the Plücker coordinates requires more than 10000 multiplications, as each variable is replaced by a linear form with 105 terms. Therefore, it would be better to avoid redundancy in the equations of \mathfrak{h} . It is possible to reduce the redundancy replacing the set $\mathcal{B}^{(mi)}$ with the union $\bigcup \mathcal{B}_{\mathcal{I}}^{(mi)}$, with \mathcal{I} a Borel multi-index, in the definition of equations (6.1) and (6.2). In the case of 2 points, there is a unique Borel multi-index and for instance in the case of \mathbb{P}^4 , the set $\mathcal{B}^{(1)}$ contains $\binom{15}{3}$ polynomials while $\mathcal{B}_{14,15}^{(1)}$ has only 13 polynomials. Applying these two tricks, we are able to compute the equations of $\mathbf{Hilb}_2^4 \subset \mathbb{P}^{104}$. The equations contained in \mathfrak{h} obtained considering only the Borel multi-index are 480 and besides 24 simple changes of coordinates we need 3 random changes of coordinates to obtain the

$\mathrm{PGL}_{\mathbb{Q}}(5)$ -invariant ideal. Finally, the Hilbert scheme turns out to be a subscheme defined by 3575 quadratic equations (1365 of them are Plücker relations) with Hilbert polynomial

$$\frac{6125}{8!}t^8 + \frac{452}{288}t^7 + \frac{4027}{576}t^6 + \frac{635}{36}t^5 + \frac{31703}{1152}t^4 + \frac{7849}{288}t^3 + \frac{4849}{288}t^2 + \frac{145}{24}t + 1,$$

i.e., \mathbf{Hilb}_2^4 is a subscheme of \mathbb{P}^{104} of dimension 8 and degree 6125.

7.4. The Hilbert scheme \mathbf{Hilb}_3^2

The Hilbert scheme \mathbf{Hilb}_3^2 can be defined as subscheme of the Grassmannian $\mathbf{Gr}_3^{10} \subset \mathbb{P}^{119}$. There are two Borel-fixed ideals defining 3 points in the plane: (x_2, x_1^3) and (x_2^2, x_2x_1, x_1^3) , so that in this case we can restrict to Borel multi-indices considering the elements of $\mathcal{B}_{7,9,10}^{(1)}$ and $\mathcal{B}_{8,9,10}^{(1)}$. In this way, the set \mathfrak{h} contains 720 equations and we obtain a $\mathrm{PGL}_{\mathbb{Q}}(3)$ -invariant ideal applying 10 changes of coordinates (8 special and 2 random). The ideal defining the Hilbert scheme is generated by 5425 quadratic equations (2310 Plücker relations) and \mathbf{Hilb}_3^2 is a subscheme of \mathbb{P}^{119} of dimension 6 and degree 3309, as its Hilbert polynomial is

$$\frac{3309}{6!}t^6 + \frac{1557}{80}t^5 + \frac{553}{16}t^4 + \frac{543}{16}t^3 + \frac{2381}{120}t^2 + \frac{33}{5}t + 1.$$

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