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Differentiable structures on metric measure spaces: a primer

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Abstract. This is an exposition of the theory of differentiable structures on metric measure spaces, in the sense of Cheeger and Keith.

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1. Introduction

1.1. Overview

A key result of geometric function theory is Rademacher's theorem: any real-valued Lipschitz function on $\mathbb{R}^n$ is differentiable almost everywhere. In [6], Cheeger found a far-reaching generalization of this result in the context of doubling metric measure spaces that satisfy a Poincaré inequality. The goal of this primer is to give a streamlined account of his construction, to provide an accessible introduction to this area of active research. Our exposition is based on Cheeger's work, and incorporates a number of simplifications due to Keith [14], as well as several of our own.

1.2. Differentiable structures

In order to generalize Rademacher's theorem, one first has to come up with an appropriate definition of differentiability. Functions between Euclidean spaces are differentiable if their infinitesimal behavior is linear. For certain non-Euclidean metric measure spaces such as Carnot groups, there is a natural substitute for linear maps, namely group homomorphisms; this leads to notion of differentiability and a generalization of Rademacher's theorem [16, 17] that has important applications in geometric group theory and bilipschitz embedding [21]. However, this approach to generalization is limited by the fact that it relies on special structure that is absent in general metric measure spaces.

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One of Cheeger's first achievements was to see that it is possible to define a notion of differentiability in a metric space without any additional algebraic structure. A real valued function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point p_0 if there is a linear combination L of the coordinate functions so that f and L agree to first order near p_0:

$$f(p) - f(p_0) = L(p) - L(p_0) + o(\|p - p_0\|) .$$

(Recall that $A(y) = o(B(y))$ near x if $A(y)/B(y) \to 0$ as $y \to x$.)

Cheeger observed that this definition of differentiability with respect to a set of coordinate functions makes sense for real valued functions on general metric measure spaces, where the role of the coordinate functions is played by suitable tuples of real valued Lipschitz functions.

Definition 1.1. Suppose $f : X \to \mathbb{R}$ and $\phi = (\phi_1, \dots, \phi_N) : X \to \mathbb{R}^N$ are Lipschitz functions on a metric measure space (X, d, μ). Then f *is differentiable with respect to* ϕ *at* $x_0 \in X$ if there is a unique $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ such that f and the linear combination $a \cdot \phi = \sum_i a_i \phi_i$ agree to first order near x_0:

$$f(x) - f(x_0) = a \cdot (\phi(x) - \phi(x_0)) + o(d(x, x_0)) ; \tag{1.1}$$

the tuple $(a_1, \dots, a_N)$ is the *derivative of* f *with respect to* ϕ and will be denoted $\partial_\phi f(x_0)$.

In analogy with the definition of a differentiable structure on a manifold, to formulate Rademacher's theorem, Cheeger's idea is to express differentiability with respect to an abundant supply of Lipschitz maps as in Definition 1.1.

Definition 1.2. A *chart (of dimension* N*)* on a metric measure space (X, d, μ) is a pair (U, ϕ) where:

- $U \subset X$ is a measurable subset, and $\phi : X \to \mathbb{R}^N$ is Lipschitz.
- Every Lipschitz function $f : X \to \mathbb{R}$ is differentiable with respect to ϕ at μ-almost every $x_0 \in U$, and the derivative defines a measurable function $\partial_\phi f : U \to \mathbb{R}^N$.

A (*measurable*) *differentiable structure on* (X, d, μ) is a countable collection $\{(U_\alpha, \phi_\alpha)\}$ of charts with uniformly bounded dimension, such that $X = \cup_\alpha U_\alpha$.

1.3. The main theorem

Having formalized differentibility in Definition 1.2, one would then like to know when a metric measure space (X, d, μ) has a measurable differentiable structure. We can now state the main theorem, which gives a sufficient condition for the existence of such a differentiable structure. (See [14, Theorem 2.3.1] and [6, Theorem 4.38].)

Theorem 1.3. *If* (X, d, μ) *is a metric measure space that is doubling (Definition* 1.4*) and supports a* p*-Poincaré inequality with constant* $L \geq 1$ *for some* $p \geq 1$ *(see Definition* 6.1*), then* X *admits a measurable differentiable structure with dimension bounded above by a constant depending only on* L *and the doubling constant.*

We now discuss the hypotheses in this theorem.

Definition 1.4. A Borel regular measure μ on a metric space X is *doubling* if there exists some constant C so that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for every $x \in X$ and $r > 0$.

The doubling property is a kind of geometric finite dimensionality condition that is natural to impose here, in view of the definition of differentiability, which asserts that near generic points one only needs finitely many functions to approximate an arbitrary function to first order. Doubling metric measure spaces are also called spaces of homogeneous type, and have a well developed theory of analysis, see [8]. However, one cannot drop the Poincaré inequality in Theorem 1.3, because the doubling condition alone is too weak: there are doubling metric measure spaces that do not have a differentiable structure in the sense of Definition 1.2, such as the standard Cantor set $C \subset [0, 1]$ with the usual probability measure (see Proposition B.1).

We defer the precise definition of the Poincaré inequality to Section 6. Roughly speaking it requires that the local behavior of a Lipschitz function is controlled, in an appropriate sense, by its infinitesimal behaviour. There are many examples of doubling metric measure spaces satisfying a Poincaré inequality, for which Theorem 1.3 guarantees the existence of a differentiable structure:

1. **Euclidean spaces:** As a consequence of Rademacher's theorem, the metric measure space $\mathbb{R}^n$ (with the usual Euclidean metric and Lebesgue measure), has a measurable differentiable structure given by a single chart $(\mathbb{R}^n, \phi)$, where the components of ϕ are the usual coordinate functions on $\mathbb{R}^n$.
2. **Carnot groups:** As a specific example of a Carnot group, consider the Heisenberg group $\mathbb{H}$ of real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

As a set, $\mathbb{H}$ can be described by $\mathbb{R}^3 = \{(x, y, z)\}$ with a Carnot-Carathèodory metric and the usual Lebesgue measure. As a consequence of a theorem of Pansu [17], this space carries a measurable differentiable structure with a single chart given by $(x, y) : \mathbb{H} \to \mathbb{R}^2$. In particular, the dimension of the differentiable structure is two, the topological dimension of the space is three, and the Hausdorff dimension of the space is four, showing that all three may differ.
3. **Glued spaces:** Consider the Heisenberg group $\mathbb{H} = \{(x, y, z)\}$ as above, and $\mathbb{R}^4 = \{(a, b, c, d)\}$ with its usual metric and measure. Note that these are both Ahlfors 4-regular metric measure spaces. (Recall that a metric measure space X is Ahlfors Q-regular if the measure of every ball $B(x, r) \subset X$ is comparable to r^Q, provided $r \leq \mathrm{Diam}(X)$.) Choose an isometrically embedded copy of $\mathbb{R}^1$ in each — for example, the x-axis in $\mathbb{H}$, and the a-axis in $\mathbb{R}^4$ — and let X be the space formed by gluing $\mathbb{H}$ and $\mathbb{R}^4$ along these subsets.

   There is a natural geodesic path metric d on X, and the measures combine to give an Ahlfors 4-regular measure μ on (X, d). By [12, Example 6.19(a)],

X admits a p-Poincaré inequality for $p > 3$. After extending the functions x, y, a, b, c, d to Lipschitz functions on all of X, the space (X, d, μ) has a measurable differentiable structure with the two charts, $(\mathbb{H}, (x, y))$ and $(\mathbb{R}^4, (a, b, c, d))$. Notice that these charts are of different dimensions.

4. **Laakso spaces:** For every $Q \geq 1$, Laakso builds an Ahlfors Q-regular space that admits a 1-Poincaré inequality [15]. These fractal spaces have topological dimension one.
5. **Bourdon-Pajot spaces:** These spaces arise as the boundary at infinity of certain Fuchsian buildings that are important examples in geometric group theory. They are all homeomorphic to the Menger sponge, and admit a 1-Poincaré inequality [3].
6. **Limit spaces:** The Gromov-Hausdorff limit of a sequence of Riemannian manifolds with Ricci curvature uniformly bounded from below, and diameter uniformly bounded from above, will admit a 1-Poincaré inequality, even though it may no longer be a manifold [5].
7. **Spaces satisfying generalized Ricci curvature bounds:** Rajala shows that metric measure spaces satisfying curvature type conditions defined using optimal transport, are doubling and satisfy a Poincaré inequality [18].

1.4. Further developments

The scope of this primer is limited to the foundational results obtained in the first part of Cheeger's paper. For a broader discussion of the historical and mathematical context of this result, we refer the reader to the papers of Cheeger and Keith referenced above, and to the survey of Heinonen [11]. We would also like to mention a few recent papers: [4] shows that a pointwise version of the doubling condition is necessary for the existence of a differentiable structure, while the papers [1,19,20] contain a wealth of results, including several new characterizations of differentiable structures.

1.5. Organization of the paper

In Section 2 we give an overview of the proof; readers with background in analysis on metric spaces may prefer to skip this, and refer back to it for definitions as needed. The proof of Theorem 1.3 is given in Sections 3-6. In Appendix A we give a simpler proof of the well known result of Semmes [6, Appendix A] that a Poincaré inequality on a complete, doubling metric space implies that the space is quasiconvex. (That is, for all $x, y \in X$ there is a path joining x to y of length at most $Cd(x, y)$, for some uniform constant C.)

Theorem A.1. *Suppose X admits a p-Poincaré inequality (with constant $L \geq 1$) for some $p \geq 1$. Then X is C-quasiconvex, where C depends only on L and the doubling constant.*

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2. Overview of the proof of Theorem 1.3

Our purpose in this section is to give a nontechnical presentation of the proof of Theorem 1.3, providing motivation, and a treatment more accessible to readers from other areas. At the end of the section we make some brief remarks about how our approach here compares to those of Cheeger and Keith.

2.1. Finite dimensionality yields a measurable differentiable structure

The first step in the proof of Theorem 1.3 is a rather general argument showing that a σ-finite metric measure space has a measurable differentiable structure provided it satisfies a certain finite dimensionality condition. This involves two definitions:

Definition 2.1. An N-tuple of functions $\mathbf{f} = (f_1, \ldots, f_N)$, where $f_i : X \to \mathbb{R}$ for $1 \le i \le N$, is *dependent (to first order)* at $x \in X$ if there exists $\lambda \in \mathbb{R}^n \setminus \{0\}$ so that

$$\lambda \cdot \mathbf{f}(y) - \lambda \cdot \mathbf{f}(x) = o(d(x, y)) \tag{2.1}$$

as y goes to x.

We denote the set where $\mathbf{f}$ is not dependent by $\mathrm{Ind}(\mathbf{f})$.

Definition 2.2. We say that in (X, d, μ) *the differentials have dimension at most N* if every $(N+1)$-tuple of Lipschitz functions is dependent almost everywhere. We say that the *differentials have finite dimension* if they have dimension at most N for some $N \in \mathbb{N}$.

With these definitions, the first step of the proof is the following:

Proposition 4.1. *If the differentials have dimension at most N_0, then X admits a measurable differentiable structure whose dimension is at most N_0.*

The proof of Proposition 4.1 is a selection argument analogous to the proof that a spanning subset of a vector space contains a basis. It works in considerable generality, *e.g.* for any σ-finite metric measure space.

The converse to Proposition 4.1 is also true, as we discuss in Appendix B.

2.2. Blow-up arguments, tangent spaces and tangent functions

The remainder of the proof is devoted to showing that under the conditions of Theorem 1.3, the differentials have finite dimension. To do this, one is faced with analyzing the behavior of a tuple $(f_1, \ldots, f_N)$ of Lipschitz functions near a typical point in X, in order to produce nontrivial linear combinations satisfying (2.1). Following [14], we approach this using a blow-up argument. Blow-up arguments occur in many places in geometry and analysis; the common features are a rescaling procedure which normalizes some quantity of interest, combined with a compactness result which allows one to pass to a limiting object which reflects the asymptotic behavior of the rescaled quantity. Then one proceeds by studying the limiting object in order to derive a contradiction, or to establish a desired estimate. We point out that the blow-up argument is not essential to this proof; it is possible to work directly in the space itself. However, in our view, the blow-up argument clarifies and streamlines the proof.

For readers who are unfamiliar with this setting and/or blow-up arguments, we first illustrate the ideas using a single function.

To fix terminology and notation, we recall that a function $f : Y \to Z$ between metric spaces (Y, d_Y) and (Z, d_Z) is *C-Lipschitz* if

$$d_Z(f(p), f(q)) \leq C\, d_Y(p, q) \tag{2.2}$$

for all $p, q \in Y$, while the *Lipschitz constant of f*

$$\mathrm{LIP}(f) = \sup_{p,q\in Y,\ p\neq q} \frac{d_Z(f(p), f(q))}{d_Y(p, q)}$$

is the infimal such C. We let $\mathrm{LIP}(Y)$ denote the collection of real-valued Lipschitz functions $f : Y \to \mathbb{R}$.

Now suppose $f \in \mathrm{LIP}(X)$ is a Lipschitz function, and $x \in X$. To study the behavior of f near x, we may choose a sequence of scales $\{r_k\}$ tending to 0, and consider the corresponding sequence of rescalings of (X, d), *i.e.* the sequence of metric spaces $\{(X_k, d_k)\}$, where $X_k = X$ and $d_k = \frac{1}{r_k} d$. One then defines a sequence of functions $\{f_k : X_k \to \mathbb{R}\}$ by rescaling f accordingly: $f_k = \frac{1}{r_k} f$. Then f_k has the same Lipschitz constant as f, and the behavior of f in the ball $B(x, r_k)$ corresponds to the behavior of f_k on the unit ball $B(x, 1) \subset (X_k, d_k)$.

Next, by passing to a subsequence, and using a suitable notion of convergence, we may assume that the metric spaces (X_k, d_k) converge to a (Gromov-Hausdorff) tangent space (X_∞, d_∞), and the functions $f_k : X_k \to \mathbb{R}$ converge to a tangent function $f_\infty : X_\infty \to \mathbb{R}$ which is $\mathrm{LIP}(f)$-Lipschitz. We will suppress the details for now, and refer the reader to Section 3 for the notion of convergence (pointed Gromov-Hausdorff convergence) and the relevant compactness theorems. The space X_∞ comes with a specified basepoint $x_\infty \in X_\infty$, and for any $R > 0$ the restriction of f_∞ to the ball $B(x_\infty, R)$ is a limit of the restrictions $f_k\big|_{B(x_k,R)}$.

2.3. Pointwise Lipschitz constants and tangent functions

The tangent function f_∞ is $\mathrm{LIP}(f)$-Lipschitz. However, since f_∞ only reflects the behavior of the original function f near x, one is led to consider localized versions of the Lipschitz constant, as in the following definitions.

Definition 2.3. (Variation and pointwise Lipschitz constants)
Suppose Y is a metric space, $x \in Y$, and $u \in \mathrm{LIP}(Y)$.

1. *The variation of u on a ball $B(x, r) \subset Y$* is

$$\mathrm{var}_{x,r}\, u := \sup \left\{ \frac{|u(y) - u(x)|}{r} \mid y \in B(x, r) \right\}. \tag{2.3}$$

We always have $\mathrm{var}_{x,r}\, u \leq \mathrm{LIP}(u)$.

2. The *lower pointwise Lipschitz constant of u at x* is

$$\operatorname{lip}_x u := \liminf_{r \to 0} \operatorname{var}_{x,r} u \, .$$

3. The *upper pointwise Lipschitz constant of u at x* is

$$\operatorname{Lip}_x u := \limsup_{r \to 0} \operatorname{var}_{x,r} u \, .$$

For any function $u : Y \to \mathbb{R}$, and $x \in Y$, we have $\operatorname{lip}_x u \le \operatorname{Lip}_x u$. In general, $\operatorname{lip}_x u$ and $\operatorname{Lip}_x u$ need not be comparable. However, in the special case of $Y = \mathbb{R}^n$, if x is a point of differentiability of u, observe that $\operatorname{lip}_x u = \operatorname{Lip}_x u = |\nabla u(x)|$.

Returning to the tangent function $f_\infty : X_\infty \to \mathbb{R}$, one observes that for any $R > 0$ the restriction of f_∞ to the ball $B(x_\infty, R) \subset X_\infty$ is the limit of the sequence $\{f_k\big|_{B(x_k,R)}\}$, which, in turn, arises from rescaling $f\big|_{B(x,Rr_k)}$. This leads to the bound

$$\operatorname{lip}_x f \le \operatorname{var}_{x_\infty,R} f_\infty \le \operatorname{Lip}_x f$$

for all $R \in [0, \infty)$; in other words, the lower and upper pointwise Lipschitz constants of f at x control the variation of the tangent function f_∞ on balls centered at x_∞.

Using the fact that the measure on X is doubling, one can strengthen this assertion to: For almost every $x \in X$, every tangent function f_∞ of f at x satisfies

$$\operatorname{lip}_x f \le \operatorname{var}_{y,r} f_\infty \le \operatorname{Lip}_x f \tag{2.4}$$

for every $y \in X_\infty$ and $r \in [0, \infty)$. The second inequality is equivalent to $\operatorname{LIP}(f_\infty) \le \operatorname{Lip}_x f$. However, for a general doubling metric measure space, the quantity $\operatorname{var}_{x,r} f$ can fluctuate wildly as $r \to 0$, which means that one could have $\operatorname{LIP}(f_\infty) \ll \operatorname{Lip}_x f$. A key observation of Keith—based on a closely related earlier observation of Cheeger—is that when (X, d, μ) satisfies a Poincaré inequality, then this bad behavior can only occur when $x \in X$ belongs to a set of measure zero.

Definition 2.4 ([14, (5)]). We say X is a *K-Lip-lip space* if for every $f \in \operatorname{LIP}(X)$,

$$\operatorname{Lip}_x f \le K \operatorname{lip}_x f \tag{2.5}$$

for μ-a.e. $x \in X$. If X is a K-Lip-lip space for some $K > 0$, we say that X *is a Lip-lip space*.

Proposition 6.3. [14, Proposition 4.3.1] *Suppose* (X, d, μ) *is doubling and admits a p-Poincaré inequality for some* $p \ge 1$. *(See Section* 6 *for the definition.) Then X has a K-Lip-lip bound* (2.5)*, where K depends only the constants in the doubling and Poincaré inequalities.*

By Proposition 6.3, to prove Theorem 1.3 it suffices to show that the differentials have finite dimension in any Lip-lip space.

2.4. Tangent functions in Lip-lip spaces, and quasilinearity

By (2.4), if (X, d, μ) is a K-Lip-lip space, and $f \in \mathrm{LIP}(X)$, then for μ-a.e. $x \in X$, every tangent function $f_\infty : X_\infty \to \mathbb{R}$ of f at x, and every $y \in X_\infty$, $r \in (0, \infty)$, one has

$$\mathrm{lip}_x f \leq \mathrm{var}_{y,r} f_\infty \leq \mathrm{LIP}(f_\infty) \leq \mathrm{Lip}_x f \leq K \, \mathrm{lip}_x f ,$$

so in particular

$$\mathrm{var}_{y,r} f_\infty \geq \frac{1}{K} \mathrm{LIP}(f_\infty) .$$

Thus for any ball $B(y, r) \subset X_\infty$, the variation of f_∞ on $B(y, r)$ agrees with the global Lipschitz constant $\mathrm{LIP}(f_\infty)$ to within a factor of K. This leads to:

Definition 2.5. A Lipschitz function $u : Y \to \mathbb{R}$ on a metric space Y is *L-quasilinear* if the variation of u on every ball $B(y, r) \subset Y$ satisfies

$$\mathrm{var}_{y,r}\, u \geq \frac{1}{L} \mathrm{LIP}(u) .$$

For example, linear functions $\mathbb{R}^n \to \mathbb{R}$ are 1-quasilinear.

In summary: when X satisfies the K-Lip-lip condition, then for every $f \in \mathrm{LIP}(X)$ and μ-a.e. $x \in X$, every tangent function of f at x is K-quasilinear.

We need another version of the doubling condition appropriate to metric spaces:

Definition 2.6. A metric space Z is *C-doubling* if every ball can be covered by at most C balls of half the radius. A metric space is *doubling* if it is C-doubling for some C.

The last key ingredient in the proof is:

Lemma 5.5. *For every K, C there is an $N \in \mathbb{N}$ such that the space of K-quasilinear functions on a C-doubling metric space Z has dimension at most N.*

The Gromov-Hausdorff tangent spaces X_∞ arising from a doubling metric measure space X are all C-doubling for a fixed $C \in [1, \infty)$. Therefore, by Lemma 5.5 there is a uniform upper bound on the dimension of any space of K-quasilinear functions on any Gromov-Hausdorff tangent space of X.

A related finite dimensionality result appears in [6]. We would like to point out that a similar idea appears in the earlier finite dimensionality theorem of Colding-Minicozzi [7], also in the setting of spaces which satisfy a doubling condition and a Poincaré inequality (in [7] the spaces are Riemannian manifolds, though the smooth structure is not used in an essential way). In their paper, the quasilinearity condition is replaced by a condition which compares the size of a function on a ball (measured in terms of normalized energy) with its size on subballs, and uses this together with the Poincaré inequality and doubling property to bound the dimension of a space of harmonic functions.

To complete the proof that the differentials have finite dimension in a K-Lip-lip space, we fix an n-tuple of Lipschitz functions $\mathbf{f} = (f_1, \ldots, f_n)$ for some $n \in \mathbb{N}$. Amplifying the above reasoning, there will be a full measure set of points $x \in X$

such that every set of tangent functions $\mathbf{f}_\infty = (f_{1,\infty}, \ldots, f_{n,\infty})$ at x spans a space of K-quasilinear functions. Thus when n is larger than the dimension bound coming from Lemma 5.5, there will be a nontrivial linear relation $\lambda \cdot \mathbf{f}_\infty = 0$ for some $\lambda \in \mathbb{R}^n \setminus \{0\}$. This implies that $f_1, \ldots, f_n$ are dependent at x.

2.5. Comparisons with the work of Cheeger and Keith

As mentioned in the Overview, this exposition is based on the work of Cheeger [6] and Keith [14]. The overall outline of our proof is similar to that of Keith, but with significant simplifications and clarifications. For example, at the following points we believe that our approach is both shorter and clearer: the proof that a Poincaré inequality implies a lip-Lip inequality (Proposition 6.3), the dimension bound on spaces of quasilinear functions (Lemma 5.5), and the construction of good tangent functions in Subsection 5.1. Our proof of the quasi-convexity of spaces with a Poincaré inequality (Theorem A.1) uses similar ideas to Keith [13], but again our exposition is simpler.

Keith considers "chunky" measures, which are more general than the doubling measures we consider here. While many of our arguments would work in this context, for clarity of exposition we have restricted to the case of doubling measures.

3. Preliminaries

3.1. Lipschitz constants

Recall that we work inside a metric measure space (X, d, μ), where μ is a Borel regular measure on X.

We begin by making some observations about $\operatorname{lip}_x f$ and $\operatorname{Lip}_x f$ (see Definition 2.3).

Lemma 3.1. *If $f : X \to \mathbb{R}$ is Lipschitz, then $\operatorname{lip}_x f$ and $\operatorname{Lip}_x f$ are Borel measurable functions of x.*

Proof. For fixed $r > 0$, we see that $\operatorname{var}_{x,r} f$ is a lower-semicontinuous function of x. (Note that f is Lipschitz, so the variation over open balls $B(y, r)$ cannot jump up as $y \to x$.)

We can rewrite $\operatorname{Lip}_x f$ as follows:

$$\operatorname{Lip}_x f = \lim_{r\to 0} \sup\{\operatorname{var}_{x,s} f \mid s < r\} = \lim_{r\to 0} \sup\{\operatorname{var}_{x,s} f \mid s < r, s \in \mathbb{Q}\}. \tag{3.1}$$

The first equality holds by definition, and the second from the inequalities

$$(s - \epsilon) \operatorname{var}_{x,(s-\epsilon)} f \le s \operatorname{var}_{x,s} f \le (s + \epsilon) \operatorname{var}_{x,(s+\epsilon)} f.$$

A countable supremum of measurable functions is measurable, and a pointwise limit of measurable functions is also measurable. Therefore, by equation (3.1), we see that $\operatorname{Lip}_x f$ is a measurable function of x. An analogous argument gives the same conclusion for $\operatorname{lip}_x f$. □

In fact, for any $x \in X$, $\mathrm{Lip}_x(\cdot)$ defines a seminorm on $\mathrm{LIP}(X)$.

Lemma 3.2. *If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are Lipschitz, then for all $x \in X$ we have* $\mathrm{Lip}_x(f+g) \le \mathrm{Lip}_x f + \mathrm{Lip}_x g$.

Proof. Fix $x \in X$. Suppose we are given $\epsilon > 0$. By equation (3.1) there exists $r > 0$ so that for all $y \in B(x,r)$ we have

$$\frac{|f(y)-f(x)|}{d(x,y)} \le \mathrm{Lip}_x f + \epsilon \quad \text{and} \quad \frac{|g(y)-g(x)|}{d(x,y)} \le \mathrm{Lip}_x g + \epsilon.$$

We can find $y \in B(x,r)$ so that

$$\mathrm{Lip}_x(f+g) \le \frac{|(f+g)(y)-(f+g)(x)|}{d(x,y)} + \epsilon,$$

and applying the triangle inequality we see that

$$\mathrm{Lip}_x(f+g) \le (\mathrm{Lip}_x f + \epsilon) + (\mathrm{Lip}_x g + \epsilon) + \epsilon. \qquad \square$$

Definition 3.3. Suppose $A \subset X$ is measurable. A point $x \in X$ is a *point of density of A* if

$$\lim_{r\to 0} \frac{\mu(B(x,r) \setminus A)}{\mu(B(x,r))} = 0.$$

A function $f : X \to \mathbb{R}$ is *approximately continuous* at $x \in X$ if there exists a measurable set A, for which x is a point of density, so that $f\big|_A$ is continuous at x.

Lemma 3.4 ([9, Theorem 2.9.13]). *Assume μ is doubling. If $A \subset X$ is measurable, then almost every point of A is a point of density for A.*

If $f : X \to \mathbb{R}$ is measurable, then f is approximately continuous almost everywhere.

For the first part of this lemma, see also [10, Theorem 1.8]. The second part follows from Lusin's theorem.

3.2. Gromov-Hausdorff convergence

In this subsection we deal with metric spaces that do not a priori come with a doubling measure; however, they are doubling metric spaces (see Definition 2.6). Every metric measure space with a doubling measure is also a doubling metric space. (For complete metric spaces the converse is also true, but much less obvious.)

Definition 3.5. A sequence $\{(X_i, d_i, x_i)\}$ of pointed metric spaces *Gromov-Hausdorff converges* to a pointed metric space (X,d,x) if there is a sequence of maps $\{\Phi_i : X \to X_i\}$, with $\Phi_i(x) = x_i$ for all i, such that for all $R \in [0,\infty)$ we have

$$\lim_{i\to\infty} \sup\Big\{ |d_i(\Phi_i(y),\Phi_i(z)) - d(y,z)| \ \Big| \ y,z \in B(x,R) \subset X \Big\} = 0,$$

and

$$\forall \delta > 0, \lim_{i\to\infty} \sup \left\{ d_i(y, \Phi_i(B(x, R+\delta))) \;\middle|\; y \in B(x_i, R) \subset X_i \right\} = 0.$$

Such a sequence of maps is called a *Hausdorff approximation*.

Theorem 3.6. *Every sequence of C-doubling pointed metric spaces $\{(X_i, x_i)\}$ has a subsequence which Gromov-Hausdorff converges to a complete C-doubling pointed metric space (X, x).*

This follows from an Arzelà-Ascoli type of argument. For each $\epsilon > 0$ and radius $r > 0$ we can approximate $B(x_i, r) \subset X_i$ by a maximal ϵ-separated net whose cardinality is independent of i. By repeatedly choosing subsequences we can ensure that these nets converge in the limit to a net of at most the same cardinality. To finish the proof, take further subsequences as $\epsilon \to 0$ and $r \to \infty$. For more details see [2, Theorem 7.4.15].

Definition 3.7. Let $\{(X_i, d_i, x_i)\}$ be a sequence of pointed metric spaces. For a fixed countable index set $\mathcal{A}$, suppose that $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is a sequence of collections of functions indexed by $\mathcal{A}$:

$$\mathcal{F}_i = \{f_{i,\alpha} : X_i \to \mathbb{R}\}_{\alpha\in\mathcal{A}}\,.$$

Then the sequence of tuples $\{(X_i, d_i, x_i, \mathcal{F}_i)\}_{i\in\mathbb{N}}$ *Gromov-Hausdorff converges* to a tuple $(X, d, x, \mathcal{F})$, where $\mathcal{F} = \{f_\alpha : X \to \mathbb{R}\}_{\alpha\in\mathcal{A}}$, if there is a Hausdorff approximation $\{\Phi_i : X \to X_i\}$ such that for all $x \in X$ and $\alpha \in \mathcal{A}$, we have

$$\lim_{i\to\infty} f_{i,\alpha}(\Phi_i(x)) = f_\alpha(x).$$

Suppose $\{(X_i, d_i, x_i)\}$ is a sequence of C-doubling metric spaces, and $\{\mathcal{F}_i = \{f_{i,\alpha} : X_i \to \mathbb{R}\}_{\alpha\in\mathcal{A}}\}$ is a sequence such that for every $\alpha \in \mathcal{A}$, both the Lipschitz constants of the family $\{f_{i,\alpha}\}$ and the values $\{f_{i,\alpha}(x_i)\}$ are uniformly bounded. Then, extending Theorem 3.6, we can pass to a subsequence so that the sequence of tuples $\{(X_i, d_i, x_i, \mathcal{F}_i)\}_{i\in\mathbb{N}}$ Gromov-Hausdorff converges.

Definition 3.8. Suppose $X = (X, d)$ is a metric space, and $x \in X$.

1. A pointed metric space $(X_\infty, d_\infty, x_\infty)$ is a *Gromov-Hausdorff (GH) tangent space* to X at x if it is the Gromov-Hausdorff limit of the pointed metric spaces $\{(X, d_i, x)\}_{i\in\mathbb{N}}$, where each $d_i = \frac{1}{r_i}d$ is the original metric d rescaled by $r_i > 0$, and the sequence $\{r_i\}$ converges to zero.
2. Suppose now that $\mathcal{F} = \{f_\alpha : X \to \mathbb{R}\}_{\alpha\in\mathcal{A}}$ is a (countable) collection of functions on X. Then
$$\mathcal{U} = \{u_{f_\alpha} : X_\infty \to \mathbb{R}\}_{\alpha\in\mathcal{A}}$$
is a collection of *tangent functions* of the functions $f_\alpha \in \mathcal{F}$ at $x \in X$ if $(X_\infty, d_\infty, x_\infty, \mathcal{U})$ is the Gromov-Hausdorff limit of the sequence of tuples $\{(X, d_i, x, \mathcal{F}_i)\}_{i\in\mathbb{N}}$, where
$$\mathcal{F}_i = \left\{f_{i,\alpha} : (X, d_i, x) \to \mathbb{R}\right\}_{\alpha\in\mathcal{A}} \quad \text{and} \quad f_{i,\alpha}(\cdot) = \frac{f_\alpha(\cdot) - f_\alpha(x)}{r_i}.$$

Since we used the same Hausdorff approximation and scaling factors for every $f_\alpha \in \mathcal{F}$, we say that the tangent functions are *compatible*.

We caution the reader that the terminology used for GH tangent spaces varies: Cheeger calls them tangent cones, and other objects tangent spaces, while Keith just calls them tangent spaces.

In general, the GH tangent spaces and functions one sees are highly dependent on the sequence of scales chosen.

Since rescaling preserves doubling and Lipschitz constants, our previous discussion has the following consequences:

Corollary 3.9.

1. *Doubling metric spaces have (doubling) GH tangent spaces at every point.*
2. *Any countable collection $\mathcal{F}$ of uniformly Lipschitz functions on a doubling metric space X has a compatible collection of tangent functions $\mathcal{U}$ at every point of X.*

4. Finite dimensionality implies a measurable differentiable structure

Our goal in this section is to prove:

Proposition 4.1 (*cf.* [14, Proposition 7.3.1]). *If the differentials have dimension at most N_0 (see Definition* 2.2*), then X admits a measurable differentiable structure whose dimension is at most N_0.*

Proof. We have N_0 fixed by the hypotheses.

Lemma 4.2. *We assume the hypothesis of Proposition* 4.1*. Then, given any measurable $A \subset X$ with positive measure, we can find a measurable $U \subset A$ with positive measure and a function $\phi : U \to \mathbb{R}^N$, for some $N \leq N_0$, so that (U, ϕ) is a chart.*

We now complete the proof, assuming Lemma 4.2. Since X is a doubling metric measure space it is σ-finite, so without loss of generality we may assume it has finite measure. Applying Lemma 4.2, we construct a sequence of charts $(U_1, \phi_1), \ldots, (U_i, \phi_i), \ldots$ inductively as follows. Given $i \geq 0$ and charts $(U_1, \phi_1), \ldots, (U_i, \phi_i)$, if the union $\cup_{j\leq i} U_j$ has full measure in X, we stop; otherwise, let $\mathcal{C}$ be the collection of charts (U, ϕ) with $U \subset X \setminus \cup_{j\leq i} U_j$, and choose $(U_{i+1}, \phi_{i+1}) \in \mathcal{C}$ such that $\mu(U_{i+1}) \geq \frac{1}{2} \sup\{\mu(U) \mid (U, \phi) \in \mathcal{C}\}$. If the resulting sequence of charts $\{U_j\}$ is infinite, then we have $\mu(U_j) \to 0$ as $j \to \infty$, because $\mu(X) < \infty$. The union $\cup_j U_j$ has full measure, else we could choose a chart (U, ϕ) where U is a positive measure subset of $X \setminus \cup_j U_j$, and this contradicts the choice of the U_j's. $\square$

It remains to prove Lemma 4.2. Before proceeding with this we note that for a chart (U, ϕ) and $f \in \mathrm{LIP}(X)$, f has derivative $\partial_\phi f(x_0)$ with respect to ϕ at $x_0 \in X$ if the following holds (compare (1.1)):

$$\mathrm{Lip}_{x_0}\big(f(\cdot) - \partial_\phi f(x_0) \cdot \phi(\cdot)\big) = 0. \tag{4.1}$$

(Notice that we do not need to require $x_0 \in U$ because we are using Definition 1.1.)

Proof of Lemma 4.2. Consider the maximal N so that there exists some positive measure set $U \subset A$, and some N-tuple of Lipschitz functions ϕ, so that $U \subset \mathrm{Ind}(\phi)$, the set where ϕ is not dependent. (Because of finite dimensionality, we have $0 \le N \le N_0$.)

We want to show that (U, ϕ) is a chart. Take any Lipschitz function $f \in \mathrm{LIP}(X)$, and consider the $(N+1)$-tuple of functions (ϕ, f). By the maximality of N this is dependent almost everywhere in U, so for μ-almost every $x \in U$ there exists $\lambda(x) \in \mathbb{R}$ and $\partial_\phi f(x) \in \mathbb{R}^N$ so that

$$\mathrm{Lip}_x\big(\lambda(x) f(\cdot) - \partial_\phi f(x) \cdot \phi(\cdot)\big) = 0. \tag{4.2}$$

Since $U \subset \mathrm{Ind}(\phi)$, we know that $\lambda(x) \neq 0$ almost everywhere, so, without loss of generality, we may assume that $\lambda(x) = 1$ everywhere.

The uniqueness of $\partial_\phi f$, up to sets of measure zero, follows from the fact that $\mathrm{Lip}_x(\cdot)$ is a semi-norm on the space of Lipschitz functions (Lemma 3.2). Indeed, suppose that $\partial_\phi f_1 : U \to \mathbb{R}^N$ and $\partial_\phi f_2 : U \to \mathbb{R}^N$ both satisfy (4.2) for almost every x. Then

$$\begin{aligned}
&\mathrm{Lip}_x\Big(\big(\partial_\phi f_1(x) - \partial_\phi f_2(x)\big) \cdot \phi(\cdot)\Big) \\
&\qquad \le \mathrm{Lip}_x\Big(f(\cdot) - \partial_\phi f_1(x) \cdot \phi(\cdot)\Big) + \mathrm{Lip}_x\Big(f(\cdot) - \partial_\phi f_2(x) \cdot \phi(\cdot)\Big) \\
&\qquad = 0, \ \text{ for } \mu\text{-a.e. } x.
\end{aligned}$$

So, if $\partial_\phi f_1$ and $\partial_\phi f_2$ differed on a set of positive measure, then ϕ would be dependent on that same set, but this is not possible. Therefore $\partial_\phi f_1 = \partial_\phi f_2$ almost everywhere.

It only remains to show that $\partial_\phi f$ is measurable. This follows if $(\partial_\phi f)^{-1}(K)$ is measurable for each compact $K \subset \mathbb{R}^N$. We fix such a K for the remainder of the proof.

Consider the function $h_x : \mathbb{R}^N \to \mathbb{R}$ given by

$$h_x(\lambda) := \mathrm{Lip}_x(f(\cdot) - \lambda \cdot \phi(\cdot)).$$

The triangle inequality for $\mathrm{Lip}_x(\cdot)$ (Lemma 3.2) implies that h_x is continuous; in fact, writing $\phi = (\phi_i)$, for $\lambda, \lambda' \in \mathbb{R}^N$,

$$\begin{aligned}
\left|h_x(\lambda) - h_x(\lambda')\right| &\leq \mathrm{Lip}_x((\lambda - \lambda') \cdot \phi) \\
&\leq \sum_{1 \leq i \leq N} \left|\lambda_i - \lambda_i'\right| \mathrm{Lip}_x(\phi_i) \\
&\leq \left(N \max_{1 \leq i \leq N} \mathrm{LIP}(\phi_i)\right) \left|\lambda - \lambda'\right|.
\end{aligned}$$

Now set

$$E := \{x \in U \mid \exists \lambda \in K \text{ s.t. } h_x(\lambda) = 0\}.$$

As we have seen, $\partial_\phi f$ is uniquely defined up to a set of measure zero, so $(\partial_\phi f)^{-1}(K)$ equals E less a set of measure zero. Consequently, it suffices to show that E is measurable. Fix a dense countable subset K' of K, and observe that

$$\begin{aligned}
E &= \left\{x \in U \mid \exists (\lambda_n)_{n \in \mathbb{N}} \subset K', \lambda \in K \text{ s.t. } h_x(\lambda_n) \to 0, \lambda_n \to \lambda\right\} \\
&= \bigcap_{n \in \mathbb{N}} \bigcup_{\lambda \in K'} \left\{x \in U \mid h_x(\lambda) < \tfrac{1}{n}\right\}.
\end{aligned}$$

The first equality follows from the continuity of h_x and the density of K' in K. The second equality follows from the compactness of K. Note that $h_x(\lambda)$ is a measurable function of x for fixed $\lambda \in \mathbb{R}^N$ (applying Lemma 3.1). Therefore, E is a measurable set, and we are done. □

We note one consequence of the above proof.

Lemma 4.3. *Suppose (X, d, μ) is a Borel regular metric measure space, and that ϕ is an N-tuple of real-valued Lipschitz functions on X. Then* $\mathrm{Ind}(\phi)$*, the set where ϕ is not dependent to first order, is a measurable set.*

Proof. This follows from the same argument that we used to prove that E was measurable in the previous lemma. Notice that

$$\begin{aligned}
X \setminus \mathrm{Ind}(\phi) &= \left\{x \in X \mid \exists \lambda \in \mathbb{R}^N \setminus \{0\} \text{ s.t. } \mathrm{Lip}_x(\lambda \cdot \phi) = 0\right\} \\
&= \bigcup_{n \in \mathbb{N}} E_n,
\end{aligned}$$

where

$$E_n = \left\{x \in X \mid \exists \lambda \in \mathbb{R}^N, \text{ s.t. } \tfrac{1}{n} \leq |\lambda| \leq n, \text{ and } \mathrm{Lip}_x(\lambda \cdot \phi) = 0\right\}.$$

Since the annulus $\{\lambda \in \mathbb{R}^N \mid \frac{1}{n} \leq |\lambda| \leq n\}$ is compact, the argument at the end of the proof of Lemma 4.2 shows that E_n is measurable, and this completes the proof. □

5. A Lip-lip inequality implies finite dimensionality

In this section we prove the following statement, which perhaps is the heart of the theorem. Throughout this section, (X, d, μ) is a doubling metric measure space with a K-Lip-lip bound, for fixed $K > 0$.

Proposition 5.1 ([14, Proposition 7.2.2]). *There exists an N_0, depending only on K and the doubling constant, so that any $(N_0 + 1)$-tuple $\mathbf{f}$ of Lipschitz functions is dependent almost everywhere.*

In other words, (X, d, μ) is finite dimensional.

Suppose we fix N Lipschitz functions $\mathbf{f} = (f_1, \ldots, f_N)$. By Lemma 4.3, we know that $\mathrm{Ind}(\mathbf{f})$, the set of points where $\mathbf{f}$ is not dependent, is measurable, and we assume that it has positive measure. The proposition will be proved if we can find a bound $N \leq N_0$.

Let $\mathcal{F}$ be the countable collection of all rational linear combinations

$$\mathcal{F} = \left\{ \lambda \cdot \mathbf{f} \mid \lambda \in \mathbb{Q}^N \right\} \subset \mathrm{LIP}(X).$$

This is a $\mathbb{Q}$-vector space. The rough idea is that we can take tangents to X and $\mathcal{F}$ at a suitable point to get a vector space of uniformly *quasilinear* functions, that is, Lipschitz functions whose variation on any ball is comparable to their Lipschitz constant. The doubling condition then provides an an upper bound for the size of this vector space, and hence of N.

5.1. Finding good tangent functions

Definition 5.2. If f is a Lipschitz function and $\epsilon > 0$, a subset $Y \subset X$ is *ϵ-good for f* if there is an $r_0 \in (0, \infty)$ such that if $r \in (0, r_0)$ and $x \in Y$, then

$$\frac{1}{K} \mathrm{Lip}_x f - \epsilon \leq \mathrm{lip}_x f - \epsilon \leq \mathrm{var}_{x,r}\ f \leq \mathrm{Lip}_x f + \epsilon\,. \tag{5.1}$$

The set Y is *good for f* if it is ϵ-good for f, for all $\epsilon > 0$. If $\mathcal{F}$ is a collection of functions, then the set Y is *ϵ-good for $\mathcal{F}$* (respectively *good for $\mathcal{F}$*) if it is ϵ-good (respectively good) for every $f \in \mathcal{F}$.

Lemma 5.3. *Suppose $Y_0 \subset X$ is a measurable subset of finite measure and $\epsilon > 0$. Given a Lipschitz function f, for all $\delta > 0$ there exists $Y \subset Y_0$ so that $\mu(Y_0 \setminus Y) < \delta$ and Y is ϵ-good for f.*

Consequently, given a countable collection of Lipschitz functions $\mathcal{F}$, neglecting a set of arbitrarily small measure we can find $Y \subset Y_0$ so that Y is good for $\mathcal{F}$.

Proof of Lemma 5.3. The first inequality of (5.1) follows, almost everywhere, from the Lip-lip inequality (2.5).

We saw $\mathrm{Lip}_x f$ was a measurable function of x using the pointwise convergence of functions in equation (3.1). (A similar equation holds for $\mathrm{lip}_x f$.) By

Egoroff's theorem, after neglecting a subset of arbitrarily small measure, we may obtain a measurable set $Y \subset Y_0$ where the convergence is uniform. This completes the proof of (5.1). □

As in the introduction to this section, we fix N Lipschitz functions $f_1, \ldots, f_N$, and let $\mathcal{F}$ be the countable collection of all rational linear combinations of these functions.

Let $Y_0 \subset X$ be a finite measure subset. By the above reasoning, and Lusin's theorem, after neglecting a subset of arbitrarily small measure, we may obtain a measurable subset $Y \subset Y_0$ such that

- for all $f \in \mathcal{F}$, the restriction of $\operatorname{Lip}_x f : X \to \mathbb{R}$ to Y is continuous as a function of x, and
- the set Y is good for $\mathcal{F}$.

Lemma 5.4. *Suppose $x \in Y$ is a density point of the above set Y. Let X_∞ denote a tangent of X at x, and $\{u_f : X_\infty \to \mathbb{R} \mid f \in \mathcal{F}\}$ denote a compatible collection of tangent functions. Then, for every $f \in \mathcal{F}$,*

1. $\operatorname{LIP} u_f \leq \operatorname{Lip}_x f$.
2. *For every $p \in X_\infty$, and every $r \in (0, \infty)$,*

$$\operatorname{Lip}_x f \leq K \operatorname{var}_{p,r} u_f .$$

Thus the functions u_f are uniformly quasilinear (Definition 2.5*), and have global Lipschitz constant comparable to* $\operatorname{Lip}_x f$.

Proof. Fix a Hausdorff approximation

$$\{\Phi_i : (X_\infty, d_\infty, x_\infty) \to (X, d_i, x)\}_{i \in \mathbb{N}} ,$$

where $d_i = \frac{1}{r_i} d$ and $r_i \to 0$. As x is a point of density for Y, and μ is doubling, we can find maps

$$\{\Phi'_i : (X_\infty, d_\infty, x_\infty) \to (Y, d_i, x)\}_{i \in \mathbb{N}},$$

so that $d_i(\Phi_i(\cdot), \Phi'_i(\cdot))$ converges to zero uniformly on compact sets.

Suppose we fix $p \neq q$ in X_∞, $f \in \mathcal{F}$, and $\epsilon > 0$. Let $p_i = \Phi'_i(p), q_i = \Phi'_i(q) \in Y$. Notice that $d(p_i, q_i) \to 0$ as $i \to \infty$.

For all sufficiently large i we have

$$\frac{\left|u_f(p) - u_f(q)\right|}{d_\infty(p, q)} \leq \frac{\left|\frac{1}{r_i} f(p_i) - \frac{1}{r_i} f(q_i)\right|}{\frac{1}{r_i} d(p_i, q_i)} + \epsilon. \tag{5.2}$$

Since Y is ϵ-good for f, there exists r_0 so that (5.1) holds. To prove (1), use (5.2) to see that

$$\begin{aligned}\frac{\left|u_f(p) - u_f(q)\right|}{d_\infty(p, q)} &\leq (1 + \epsilon) \operatorname{var}_{p_i,(1+\epsilon)d(p_i,q_i)} f + \epsilon \\ &\leq (1 + \epsilon) \operatorname{Lip}_{p_i} f + 2\epsilon + \epsilon^2, \text{by (5.1).}\end{aligned}$$

Since the restriction of $\operatorname{Lip}_x f$ to Y is continuous, and $p_i \to x$ in the metric d, we see that

$$\frac{\left|u_f(p) - u_f(q)\right|}{d_\infty(p,q)} \le (1+\epsilon)\operatorname{Lip}_x f + 2\epsilon + \epsilon^2,$$

but ϵ was arbitrary, and so were p and q, so (1) is proved.

To see (2), fix $\epsilon > 0$ and take p_i as before. Now choose $a_i \in B(p_i, (r-\epsilon)r_i) \subset (X,d)$ so that

$$\operatorname{var}_{p_i,(r-\epsilon)r_i} f \le \frac{|f(p_i) - f(a_i)|}{(r-\epsilon)r_i} + \epsilon.$$

For sufficiently large i, at a cost of adding another ϵ to the right hand side, we can assume that $a_i \in Y$, and that $a_i = \Phi_i'(v_i)$, for some $v_i \in B(p,r)$. Furthermore, since $f \circ \Phi_i' : X_\infty \to \mathbb{R}$ converges to u_f pointwise, and these functions are uniformly Lipschitz, the convergence is uniform on compact sets. Since X_∞ is doubling and complete, closed balls are compact. Therefore, for sufficiently large i,

$$\operatorname{var}_{p_i,(r-\epsilon)r_i} f \le \frac{\left|u_f(p) - u_f(v_i)\right|}{r-\epsilon} + 3\epsilon \le \frac{r}{r-\epsilon}\operatorname{var}_{p,r} u_f + 3\epsilon. \tag{5.3}$$

But by the continuity of $\operatorname{Lip}_x f$, as a function of $x \in Y$, and equation (5.1),

$$\operatorname{Lip}_x f = \lim_{i\to\infty} \operatorname{Lip}_{p_i} f \le \lim_{i\to\infty} K\left(\operatorname{var}_{p_i,(r-\epsilon)r_i} f + \epsilon\right). \tag{5.4}$$

Since $\epsilon > 0$ was arbitrary, after combining (5.3) and (5.4), we are done. □

5.2. Bounding the dimension of the space of tangent functions

We say that $T \subset X$ is a *c-net* if the c-neighborhood of T is X. If in addition every two distinct points of T are at least c apart, we say that T is a *(maximal) c-separated net*.

Lemma 5.5. *Suppose V is a linear space of K-quasilinear functions on a metric space Z.*

1. *If some r-ball in Z contains a finite $\frac{r}{4K}$-net T, then* $\dim V \le |T|$.
2. *If Z is C-doubling, for some $C \ge 2$, then* $\dim V \le (16K)^{\log_2 C}$.

Proof of 1. After rescaling, we may assume that $r = 1$. Let $B = B(x,r) = B(x,1)$, and let $T \subset B$ be the given $\frac{1}{4K}$-net.

Suppose $u \in V$ is in the kernel of the restriction map $V \subset L^\infty(B) \to L^\infty(T)$. If $x \in B$, there is a $t \in T$ with $d(t,x) < \frac{1}{4K}$, so

$$\begin{aligned}|u(x)| &= |u(x) - u(t)| \le \operatorname{LIP}(u)\, d(x,t) \\ &\le K(\operatorname{var}_B u)\cdot\frac{1}{4K} \le \frac{1}{2}\left\|u|_B\right\|_{L^\infty}.\end{aligned}$$

This implies that

$$\left\|u\big|_B\right\|_{L^\infty} \leq \frac{1}{2}\left\|u\big|_B\right\|_{L^\infty},$$

forcing $\|u|_B\|_{L^\infty} = 0$. As u is Lipschitz, $u|_B = 0$, hence $\mathrm{LIP}(u) = 0$ by quasi-linearity, and so $u \equiv 0$. Thus the restriction map is injective, and $\dim V \leq \dim L^\infty(T) = |T|$.

Proof of 2. The C-doubling condition implies that if $B \subset Z$ is a unit ball, there is a $\frac{1}{4K}$-net $T \subset B$ with $|T| \leq (16K)^{\log_2 C}$. Then Part 1 applies. □

5.3. Bounding the dimension of the differentials

As stated in the introduction to this section, we assume that $\mathrm{Ind}(\mathbf{f})$ is a measurable set of positive measure.

Using Lemmas 5.3 and 5.4 (applied to $Y_0 = \mathrm{Ind}(\mathbf{f})$) we can take GH tangents to X and $\mathcal{F}$ at some density point $x \in \mathrm{Ind}(\mathbf{f})$ to find a GH tangent space $Z = X_\infty$ with a compatible family of tangent functions $\{u_f \mid f \in \mathcal{F}\}$. Note that this family is the span over $\mathbb{Q}$ of $\{u_{f_1}, \ldots, u_{f_N}\}$. Since these are all K-quasilinear for a fixed K, the same is true of the span over $\mathbb{R}$ of $\{u_{f_1}, \ldots, u_{f_N}\}$.

We suppose for a contradiction that $N > (16K)^{\log_2 C}$. By Lemma 5.5, the functions $\{u_{f_1}, \ldots, u_{f_N}\}$ satisfy a nontrivial linear relation $\sum_i b_i u_{f_i} = 0$ with real coefficients. Approximating the vector $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$ with a sequence of rational vectors $(a_{1,k}, \ldots, a_{N,k}) \in \mathbb{Q}^n$, we get that the sequence of linear combinations $\{\sum_i a_{i,k}\, u_{f_i}\}$ tends to zero uniformly on bounded subsets of X_∞. From the construction of the u_f's and (5.1), this means that $\mathrm{Lip}_x(\sum_i a_{i,k}\, f_i) \to 0$. But then

$$\begin{aligned}\mathrm{Lip}_x\Big(\sum_i b_i\, f_i\Big) &\leq \limsup_{k\to\infty}\left(\mathrm{Lip}_x\Big(\sum_i a_{i,k}\, f_i\Big) + \mathrm{Lip}_x\Big(\sum_i (b_i - a_{i,k})\, f_i\Big)\right) \\ &\leq \limsup_{k\to\infty}\left(\mathrm{Lip}_x\Big(\sum_i a_{i,k}\, f_i\Big) + \sum_i |b_i - a_{i,k}|\ \mathrm{LIP}\, f_i\right) = 0\,.\end{aligned}$$

Hence the f_i's are dependent to first order at x, contradicting our assumption.

6. A Poincaré inequality implies a Lip-lip inequality

We define a Poincaré inequality on a metric space as follows. (Recall that μ is assumed to be doubling.)

Definition 6.1. Fix $p \geq 1$. A metric measure space (X, d, μ) admits a p-Poincaré inequality (with constant $L \geq 1$) if every ball in X has positive and finite measure, and for every $f \in \mathrm{LIP}(X)$ and every ball $B = B(y, r)$

$$⨍_B |f - f_B|\,d\mu \leq Lr\left(⨍_{LB} (\mathrm{lip}_x f)^p\, d\mu(x)\right)^{1/p}. \tag{6.1}$$

Here $u_B = ⨍_B u\,d\mu = \frac{1}{\mu(B)}\int_B u\,d\mu$ and $LB = B(x, Lr)$.

Remark 6.2. Cheeger's definition of a Poincaré inequality [6, (4.3)] follows Heinonen and Koskela [12] in requiring (6.1) to hold where $\mathrm{lip}_x f$ is replaced by any "upper gradient" for f. Cheeger observed that $\mathrm{lip}_x f$ is an upper gradient for f [6, Proposition 1.11], so Definition 6.1 is a weaker condition than [6, (4.3)]. It turns out that in the context of complete, doubling measure spaces, the two definitions are equivalent [13, Theorem 2].

The goal of this section is the following proposition.

Proposition 6.3 ([14, Proposition 4.3.1]). *Suppose X admits a p-Poincaré inequality (with constant $L \geq 1$) for some $p \geq 1$. Then X has a K-Lip-lip bound* (2.5), *where K depends only on L and the doubling constant C_μ of μ.*

We will use the following:

Lemma 6.4. *The space (X, d, μ) is given as above. Suppose $A < \infty$ and $\epsilon > 0$ are fixed constants. If $u : X \to \mathbb{R}$ is a Lipschitz function, and $x \in X$ is an approximate continuity point for* $\mathrm{lip}\, u : X \to \mathbb{R}$, *then there exists $r_0 = r_0(u, x, A, \epsilon) > 0$ such that if $r \leq r_0$, $y, y' \in B(x, Ar) \subset X$ and $d(y, y') \leq r$, then*

$$\left| \fint_B u - \fint_{B'} u \right| \leq C_1 r \,(\mathrm{lip}_x u + \epsilon), \tag{6.2}$$

where $B := B(y, r)$, $B' := B(y', r)$, and where $C_1 = C_1(C_\mu, L) < \infty$ is a suitable constant.

Proof. Set $\hat{B} := B(y, 2r)$, so $B, B' \subset \hat{B}$. Now $|u_B - u_{B'}| \leq |u_B - u_{\hat{B}}| + |u_{\hat{B}} - u_{B'}|$, and without loss of generality we assume that $|u_B - u_{B'}| \leq 2|u_{\hat{B}} - u_{B'}|$. Here we have

$$|u_{B'} - u_{\hat{B}}| = \left| \fint_{B'} u - u_{\hat{B}} \right| \leq \frac{\mu(\hat{B})}{\mu(B')} \fint_{\hat{B}} |u - u_{\hat{B}}|,$$

and so

$$C_2 \left| \fint_B u - \fint_{B'} u \right| \leq \fint_{\hat{B}} |u - u_{\hat{B}}| \leq 2Lr \left(\fint_{L\hat{B}} (\mathrm{lip}\, u)^p \right)^{\frac{1}{p}}, \tag{6.3}$$

where $C_2 > 0$ depends only on the doubling constant C_μ, and the second inequality comes from the Poincaré inequality for (X, μ). Since $\mathrm{lip}\, u \leq \mathrm{LIP}(u)$ everywhere, and x is an approximate continuity point of $\mathrm{lip}\, u$, when r is sufficiently small we have

$$\left(\fint_{L\hat{B}} (\mathrm{lip}\, u)^p \right)^{\frac{1}{p}} \leq \mathrm{lip}_x u + \epsilon. \tag{6.4}$$

Combining (6.3) and (6.4) gives the lemma. □

Proof of Proposition 6.3. Since $\mathrm{lip} f$ is Borel it is approximately continuous almost everywhere. Let $x \in X$ be an approximate continuity point for $\mathrm{lip}\, f$, and fix $\lambda \in (0, 1), \epsilon \in (0, 1)$.

Since (X, μ) is doubling, its completion $\bar{X}$ equipped with the measure $\bar{\mu}$ defined by $\bar{\mu}(Y) = \mu(Y \cap X)$ is also doubling, with a constant depending only on C_μ.

Moreover $(\bar{X}, \bar{\mu})$ satisfies a p-Poincaré inequality, as we now show. Suppose we have Lipschitz $u : \bar{X} \to \mathbb{R}$ and $\bar{B} = B(\bar{x}, r) \subset \bar{X}$. We may find $x \in B(\bar{x}, r) \cap X$, and set $B = B(x, 2r) \subset X$. Without loss of generality, we assume that $u_B \leq u_{\bar{B}}$. Then using the doubling property of $\bar{\mu}$ and the p-Poincaré inequality for X, we get

$$\begin{aligned}
\fint_{\bar{B}} |u - u_{\bar{B}}| d\bar{\mu} &= \frac{2}{\bar{\mu}(\bar{B})} \int_{\{x \in \bar{B} : u(x) \geq u_{\bar{B}}\}} (u - u_{\bar{B}}) d\bar{\mu} \\
&\leq \frac{2}{\bar{\mu}(\bar{B})} \int_{\{x \in \bar{B} : u(x) \geq u_{\bar{B}}\}} (u - u_B) d\bar{\mu} \\
&\leq \frac{2}{\bar{\mu}(\bar{B})} \int_B |u - u_B| d\mu \\
&\leq 2CLr \left(\fint_{B(x, 2Lr)} (\operatorname{lip}_x u)^p d\mu(x) \right)^{\frac{1}{p}} \\
&\leq 2C^2 Lr \left(\fint_{B(\bar{x}, (2L+1)r)} (\operatorname{lip}_x u)^p d\bar{\mu}(x) \right)^{\frac{1}{p}},
\end{aligned}$$

where C depends only on C_μ. Observe that the constant in the Poincaré inequality for $\bar{X}$ only depends on C_μ and the constant L in the Poincaré inequality for X.

As $(\bar{X}, \bar{\mu})$ satisfies a p-Poincaré inequality and $\bar{\mu}$ is doubling, X is quasiconvex by Theorem A.1, with constant depending only on L and C_μ. Therefore, given $r > 0$ and $y \in B(x, r)$, by the quasiconvexity of $\bar{X}$, there is a chain of points $x = p_1, \ldots, p_k = y$ in X, where $d(p_i, p_{i+1}) \leq \lambda r$ and $k \leq \frac{Q}{\lambda}$, for some Q that depends only on L and C_μ. Set $B_i := B(p_i, \lambda r)$. Then

$$\begin{aligned}
&|f(y) - f(x)| \\
&\qquad \leq \left| f(x) - \fint_{B_1} f \right| + \sum_{1 \leq i < k} \left| \fint_{B_{i+1}} f - \fint_{B_i} f \right| + \left| \left(\fint_{B_k} f \right) - f(y) \right|. \quad (6.5)
\end{aligned}$$

The first and last terms in this sum are each bounded by $\lambda r \operatorname{LIP}(f)$. By Lemma 6.4, applied with "r" replaced by $r\lambda$ and with $A = 1/\lambda$, when r is sufficiently small we have

$$\left| \fint_{B_{i+1}} f - \fint_{B_i} f \right| \leq C_1 \lambda r (\operatorname{lip}_x f + \epsilon),$$

so

$$\begin{aligned}|f(y) - f(x)| &\leq \left(\frac{Q}{\lambda}\right)(C_1\lambda r(\operatorname{lip}_x f + \epsilon)) + 2\lambda r \operatorname{LIP}(f)\\ &= \big(QC_1(\operatorname{lip}_x f + \epsilon) + 2\lambda \operatorname{LIP}(f)\big)\, r.\end{aligned}$$

Thus $\operatorname{Lip}_x f \leq QC_1 \operatorname{lip}_x f + QC_1\epsilon + 2\lambda \operatorname{LIP}(f)$ and, since $\lambda, \epsilon > 0$ were arbitrary, this proves the proposition. □

Appendix

A. A Poincaré inequality implies quasiconvexity

As mentioned in the introduction, in this appendix we give a simpler proof of the following theorem of Semmes [6, Appendix A]. A similar argument can be found in [13, Section 6].

Theorem A.1. *Let (X, d, μ) be a complete, doubling metric measure space satisfying a Poincaré inequality. Then X is λ-quasiconvex, where λ depends only on the doubling constant of μ and the constant in the Poincaré inequality.*

The main step in the proof of Theorem A.1 is:

Lemma A.2. *There is a constant $C \in (0, \infty)$ depending only on the doubling constant of μ and the constant in the Poincaré inequality, such that if $y, z \in X$, and $r = d(y, z)$, then there is a path of length at most $C\,r$ from $B(y, \frac{r}{4})$ to $B(z, \frac{r}{4})$.*

Assuming the lemma, the proof goes as follows. Pick $x, x' \in X$, and apply the lemma to obtain a path γ of length at most $Cd(x, x')$, such that the "total gap" $d(x, \gamma) + d(\gamma, x')$ is at most $\frac{1}{2}d(x, x')$. Now apply the lemma to each of the gaps, to get two new paths, and so on. The total gap at each step is at most half the total gap at the previous step, and the total additional path produced is at most C-times the gap left after the previous step. The closure of the union of the resulting collection of paths contains a path from p to q of length at most $2\,C\,d(x, x')$.

Before proving Lemma A.2, we make the following definition:

Definition A.3. An *ϵ-path* in a metric space X is a sequence of points $x_0, \ldots, x_k \in X$ such that $d(x_{i-1}, x_i) < \epsilon$ for all $i \in \{1, \ldots, k\}$; the *length* of the ϵ-path is $\sum_i d(x_{i-1}, x_i)$.

Proof of Lemma A.2. We show that for all $\epsilon \in (0, \infty)$, there is an ϵ-path from $B(y, \frac{r}{4})$ to $B(z, \frac{r}{4})$ of length at most $C\ d(y, z)$; then a variant of the Arzelà-Ascoli theorem applied to a sequence of discrete paths implies that there is a path of length at most $C\,d(y, z)$ from $B(y, \frac{r}{4})$ to $B(z, \frac{r}{4})$.

Fix $\epsilon \in (0, \infty)$, and define $u : X \to [0, \infty]$ by setting $u(x)$ equal to the infimal length of an ϵ-path from $B(y, \frac{r}{4})$ to x. For $A \in (0, \infty)$, let $u_A := \min(u, A)$. Then u_A is is a continuous function which is zero on $B(y, \frac{r}{4})$, and is locally 1-Lipschitz; in particular $\operatorname{lip}_x u_A \leq 1$ for all $x \in X$. The Poincaré inequality applied to u_A and $B(y, \frac{5r}{4})$ implies that u_A is $\leq C\,r$ somewhere in $B(z, \frac{r}{4})$, where C depends only on the doubling constant of μ and the constant L of the Poincaré inequality. Since this is true for $A > Cr$, the desired ϵ-path exists. □

B. A space without a differentiable structure

In this appendix we give a short proof that the standard middle-third Cantor set $C_{1/3} \subset \mathbb{R}$, with its usual probability measure, does not admit a measurable differentiable structure.

First, observe that the converse to Proposition 4.1 is true: if a metric measure space admits a measurable differentiable structure, then the differentials have finite dimension. Suppose $(f_1, \dots, f_{N_0+1})$ is a $(N_0 + 1)$-tuple of Lipschitz functions on X. For each chart $(U, \phi : X \to \mathbb{R}^N)$, for almost every $x \in U$ each f_i is differentiable with respect to ϕ at x. The $N_0 + 1$ different vectors $\partial_\phi f_i(x) \in \mathbb{R}^N$ must be linearly dependent, and so there exists $\lambda = (\lambda_i) \in \mathbb{R}^{N_0+1} \setminus \{0\}$ so that

$$\sum_{i=1}^{N_0+1} \lambda_i \, \partial_\phi f_i(x) = 0.$$

This same λ certifies that $(f_1, \dots, f_{N_0+1})$ is dependent to first order at x.

Proposition B.1. *The differentials on $C_{1/3}$ do not have finite dimension (Definition 2.2), and so $C_{1/3}$ does not admit a measurable differentiable structure.*

Proof. For each $k \in \mathbb{N}$, let U_k be the union of the 2^{k-1} disjoint intervals of length 3^{-k} which are removed at the kth stage of the construction of $C_{1/3}$.

Given a function $a : \mathbb{N} \to [0, 1]$, define $u_a : [0, 1] \to [0, 1]$ by setting $u_a \equiv a(k)$ on U_k for each k, and setting $u_a \equiv 0$ elsewhere. Define $v_a : [0, 1] \to [0, 1]$ by

$$v_a(x) = \int_0^x u_a(t)\, dt.$$

Suppose $v_{a_1}, \dots, v_{a_N}$ are dependent to first order almost everywhere, and let $x \in C_{1/3}$ be a point where they are dependent, with $\lambda = (\lambda_i) \in \mathbb{R}^N \setminus \{0\}$ coming from (2.1). We claim that

$$b(k) := \sum_{i=1}^{N} \lambda_i a_i(k) \to 0 \text{ as } k \to \infty. \tag{B.1}$$

Given $\epsilon > 0$, choose $\delta > 0$ so that if $d(x, y) \leq \delta$, the right-hand-side of (2.1) is at most $\epsilon d(x, y)$. Let $k_0 \in \mathbb{N}$ be minimal so that $2 \cdot 3^{-k_0} \leq \delta$.

Given $k \geq k_0$, there exist y', y which are endpoints of an interval of U_k, y' lies between x and y in $\mathbb{R}$, so that $d(x, y) \leq 2 \cdot 3^{-k}$, and so that $d(y, y') \geq \frac{1}{2} d(x, y)$. Now (2.1) gives

$$\begin{aligned}
&\left|\sum_{i=1}^{N} \left(\lambda_i v_{a_i}(y) - \lambda_i v_{a_i}(y')\right)\right| \\
&\leq \left|\sum_{i=1}^{N} \left(\lambda_i v_{a_i}(y) - \lambda_i v_{a_i}(x)\right)\right| + \left|\sum_{i=1}^{N} \left(\lambda_i v_{a_i}(y') - \lambda_i v_{a_i}(x)\right)\right| \\
&\leq 2\epsilon d(x, y),
\end{aligned}$$

but on the other hand

$$\left|\sum_{i=1}^{N} \left(\lambda_i v_{a_i}(y) - \lambda_i v_{a_i}(y')\right)\right| = d(y, y') \left|\sum_{i=1}^{N} \lambda_i a_i(k)\right| \geq \frac{1}{2} d(x, y) |b(k)|.$$

Thus $|b(k)| \leq 4\epsilon$ for all $k \geq k_0$. Since ϵ was arbitrary, we have (B.1).

For any N, it is easy to find functions $a_1, \ldots, a_N$ so that for no choice of λ is (B.1) satisfied. □

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