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The geometry of Grassmannian manifolds and Bernstein-type theorems for higher codimension

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Abstract. This paper continues the line of research where one shows a Bernstein-type theorem, namely, that a complete minimal submanifold of a Euclidean space has to be affine linear, by proving that its Gauss map is constant. This Gauss map is a harmonic map into some Grassmann manifold. Therefore, we need geometric conditions on the target of the Gauss map which imply that, being harmonic, it has to be constant. For this purpose, we identify a region $\mathbb{W}_{\frac{1}{3}}$ in a Grassmann manifold $\mathbf{G}_{n,m}$, not covered by a usual matrix coordinate chart, with the following important property. For a complete $n-$submanifold M of $\mathbb{R}^{n+m}$ with $n \geq 3$, and $m \geq 2$, with parallel mean curvature whose image under the Gauss map is contained in a compact subset $K \subset \mathbb{W}_{\frac{1}{3}} \subset \mathbf{G}_{n,m}$, we can construct strongly subharmonic functions on M and derive *a priori* estimates for the harmonic Gauss map, to eventually show that it is constant. While we do not know yet how close our region is to being optimal in this respect, it is substantially larger than what could be achieved previously with other methods. Consequently, this enables us to obtain substantially stronger Bernstein-type theorems in higher codimension than previously known. We also provide some new explicit computations for the example of Lawson and Osserman [18].

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1. Introduction

The classical Bernstein theorem states that any complete minimal graph M^2 in $\mathbb{R}^3$ has to be an affine plane. Equivalently, all tangent planes are parallel to each other, and the Gauss map $\gamma : M \to S^2$ is constant. This scheme extends to higher dimensions, that is, minimal graphs in $\mathbb{R}^{n+1}$, although the result becomes somewhat weaker for $n \geq 7$. In this work, we are interested in such a scheme for higher codimensional minimal submanifolds of Euclidean space.

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We now make this more precise. Let M^n be a complete n-dimensional submanifold of Euclidean space $\mathbb{R}^{n+m}$. The Grassmann manifold $\mathbf{G}_{n,m}$ is the target manifold of the Gauss map $\gamma : M \to \mathbf{G}_{n,m}$ that assigns to each point of M the direction of its tangent space in $\mathbb{R}^{n+m}$. By the Ruh-Vilms theorem [20] the Gauss map is a harmonic map into the Grassmann manifold if and only if M has parallel mean curvature. In particular, the Gauss map is harmonic for minimal submanifolds of Euclidean space. When $m = 1$, the Grassmann manifold reduces to the sphere S^n, and investigating the Gauss map has identified many conditions under which a minimal submanifold has to be affine. In this scheme, the key is to show that the Gauss map is constant. The aim of the present paper then is to obtain Bernstein-type result for higher codimension by using the geometric properties of the Grassmann manifolds $\mathbf{G}_{n,m}$ and studying the corresponding harmonic Gauss maps into $\mathbf{G}_{n,m}$.

This method was introduced by Hildebrandt-Jost-Widman [13]. The distance function from a fixed point $P_0 \in \mathbf{G}_{n,m}$ in a geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ and centered at P_0 is convex. Using this fact one obtains a strongly subharmonic function on M by composing the distance function with the Gauss map γ under the assumption that the image of the Gauss map is contained in a closed subset of the geodesic convex ball. Suppressing some basic technical difficulties in this introduction, an application of the maximum principle should then yield that this subharmonic function is constant, and hence so then is the Gauss map. Using this local convex geometry of Grassmann manifolds and advanced harmonic map regularity theory to overcome the indicated technical difficulties, Hildebrandt-Jost-Widman obtained Bernstein-type results for higher dimension and codimension. Somehow, however, while a geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ is the largest convex *ball* for $m \geq 2$ (for $m = 1$, we can take the ball of the radius $\frac{1}{2}\pi$, that is, a hemisphere), it is not the largest convex *set*, and therefore the result of [13] is not yet the best possible, and there seems opportunity for improvement. In [15], then, the largest such geodesic convex set was found, and stronger results were obtained. But, as we shall explore here, there is still further opportunity for improvement, via a deeper understanding of the convex geometry of Grassmann manifolds.

Since the geometry of general Grassmann manifolds is not as easy to visualize as the one for the special case $m = 1$ where the Grassmannian reduces to the sphere S^n, let us briefly discuss the situation for the latter. In general, the domain of a strictly convex function cannot contain any closed geodesic, and therefore, a closed hemipshere in S^n cannot support a strictly convex function. Thus, no ball with a radius $\geq \frac{\pi}{2}$ can support a strictly convex function. Nevertheless, there exist larger open sets that contain an open hemisphere, but still do not contain any closed geodesic. In fact, on S^2, we can take the complement U of a semicircle. In [16], we have shown that any compact subset of U supports a strictly convex function. (It is, however, not true that this convex supporting property holds for any open subset of S^n containing no closed geodesic.) First of all, this inspires us to look for such convex supporting sets in general Grassmann manifolds. Secondly, the reason why we need convex functions is that the postcomposition of a harmonic map (in our case, the Gauss map of a minimal submanifold) with a convex function is subharmonic

so that we can apply the maximum principle. When suitable technical conditions are met (their verification is part of our technical achievements), this tells us that the composition, being subharmonic, is constant. When the function is nontrivial, we can then conclude that the Gauss map itself is constant. Bingo – the minimal submanifold is affine! Now, however, it might happen that the postcomposition of the Gauss map with a function F turns out to be subharmonic without F being convex. Here, the more subtle geometry of the Grassmann manifolds for $m \geq 2$ enters. An intuition about the geometry comes from the following observation. We can rotate the plane $e_1 \wedge e_2$ in $\mathbb{R}^4$ either into the plane $e_1 \wedge e_3$ or into the plane $e_3 \wedge e_4$. Each of these rotations yields a geodesic arc in $\mathbf{G}_{2,2}$. However, these two geodesics stop diverging from their neighboring geodesics at different distances $\left(\frac{\sqrt{2}}{4}\pi \text{ vs. } \frac{\pi}{2}\right)$ (equivalently, the first conjugate point is reached after a distance of $\frac{\sqrt{2}}{2}\pi$ vs. π), and therefore, a maximal geodesically convex set is larger than a ball of radius $\frac{\sqrt{2}}{4}\pi$. While this geometric intuition will be important for the present paper, nevertheless, due to more subtle features of Grassmannian geometry, we can construct suitable functions that are no longer convex, but still give the required subharmonicity on even larger sets than such maximal geodesically convex sets. The function that we use is defined in a geometric manner from the Plücker embedding, or equivalently, in terms of Jordan angles. It is designed to take into account the phenomenon just described, namely that rotations of subspaces of Euclidean space are geometrically different depending on how many independent normal directions are involved. Again, this is a feature particular for $m \geq 2$, not yet apparent on a sphere. In fact, and this is a main point of this paper, we shall find such functions F here, defined on larger subsets of $\mathbf{G}_{n,m}$ than in previous work (as we shall explain in more detail below, these subsets are larger and more natural than the ones identified in our previous work [17]). Therefore, we can obtain a larger possible range of the Gauss map of a minimal submanifold that still implies that it is constant. Thus, we obtain stronger Bernstein-type results than previously known. Actually, our approach, being quantitative, also yields estimates for the second fundamental form of minimal submanifolds under suitable geometric conditions.

Of course, there are limits how far one can push Bernstein theorems in higher codimension. For codimension 1, we have Moser's theorem [19] that any entire minimal graph with bounded slope is affine linear (for $n \leq 7$, the bounded slope condition is not needed – this is Simons' theorem [21]). In higher codimensions, the situation is not so good. After all, there is the important example of Lawson-Osserman [18] of a nontrivial minimal graph (to be analyzed in detail in the Appendix of the present paper) of bounded slope that sets a limit for how far one can go. The ultimate aim then is to narrow the gap between the range of Bernstein theorems and the counterexamples as far as possible. Our paper is a further step in this direction, improving our previous result from [17]. We do not know yet whether one can still go further, or whether there exist even more striking examples than the one of [18].

One more remark: Most of what we do in this paper holds for submanifolds of parallel mean curvature, and not only for those of vanishing mean curvature, the

minimal ones. Since, however, according to mathematical tradition, the minimal submanifolds are the most interesting ones, in this introduction, we mostly restrict ourselves to discuss those minimal ones in place of the more general ones of parallel mean curvature.

Let us now describe the results in more precise terms. The Grassmann manifold $\mathbf{G}_{n,m}$ can be imbedded into Euclidean space by the Plücker imbedding. This simply means that we consider an oriented n-plane in $\mathbb{R}^{n+m}$ as an element of $\Lambda^n(\mathbb{R}^{n+m})$. In this space, we also have a scalar product. We then can introduce suitable functions, called w-functions and v-functions, on $\mathbf{G}_{n,m}$. The w-function is simply given by the above scalar product with some fixed reference n-plane, and v is the inverse of w. By precomposition with the Gauss map, we then get corresponding functions on M. The region of convexity for the v-function is the same as for the corresponding distance function [26], but in our previous work [17] we have already found that there is a larger region in $\mathbf{G}_{n,m}$ on which the v-function is no longer convex, but where its composition with the Gauss map is still strongly subharmonic by rather delicate estimates. We then obtain the following:

Theorem 1.1 ([17]). *Let $z^\alpha = f^\alpha(x^1, \dots, x^n)$, for $\alpha = 1, \dots, m$, be smooth functions defined everywhere in $\mathbb{R}^n$, with $n \geq 3$, and $m \geq 2$. Suppose their graph $M = (x, f(x))$ is a submanifold with parallel mean curvature in $\mathbb{R}^{n+m}$, and there exists a number $\beta_0 < 3$ such that*

$$\Delta_f = \left[\det\left(\delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}\right)\right]^{\frac{1}{2}} \leq \beta_0. \tag{1.1}$$

Then $f^1, \dots, f^m$ have to be affine linear, i.e., *they represent an affine n-plane.*

Note that, for this graphical situation, the quantity Δ_f corresponds to the v-function of tangent n-planes with respect to the coordinate n-plane $\{x_{n+1} = \cdots = x_{n+m} = 0\}$.

In our previous work (see [15, 17, 25, 26]) the image under the Gauss map for a submanifold of Euclidean space is contained in an $(n \times m)-$matrix chart for a Grassmann manifold. There is, however, still further room for improvement, and here, we consider a more general situation. Recall that a Grassmann manifold can be viewed as a minimal submanifold of the Euclidean sphere via the Plücker imbedding. This leads us to employ a technique from our previous work [16] for the case of codimension $m = 1$, where the convex geometry of the Euclidean sphere has been thoroughly investigated.

We shall introduce new notions of *S-orthogonality* and of *S-maps* on a Grassmann manifold. Let $P, Q \in \mathbf{G}_{n,m}$ be $S-$orthogonal to each other. This means that their intersection is of dimension $n-1$ and one is obtained from the other by rotating a single tangent vector into a normal direction by an angle of $\frac{\pi}{2}$. Using the w-function we define an S-map $\mathscr{S}\colon \mathbf{G}_{n,m} \to \overline{\mathbb{D}}$ relative to P and Q. This is a map onto the closed unit disk. It associates to $S \in \mathbf{G}_{n,m}$ the point in $\overline{\mathbb{D}}$ with coordinates given by the w-function respectively with P and Q.

Define

$$\mathbb{W}_c := \mathscr{S}^{-1}\Big(\overline{\mathbb{D}}\backslash\big(\overline{\mathbb{D}}_c \cup \{(a,0) : a \leq 0\}\big)\Big), \tag{1.2}$$

with $\overline{\mathbb{D}}_c = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq c^2\}$.

Now, for our method to apply, the image of our Gauss map can be any compact subset of $\mathbb{W}_{\frac{1}{3}}$. This is a global region in $\mathbf{G}_{n,m}$ that is not contained in any matrix coordinate chart. More precisely, even when the Gauss image is somewhat larger, we still can find subharmonic functions on our submanifold by combining some tricks in our previous work in [16] and [17]. In this way, we obtain the following Bernstein-type theorem that improves [17]:

Theorem 1.2. *Let $z^\alpha = f^\alpha(x^1, \ldots, x^n)$, for $\alpha = 1, \ldots, m$, be smooth functions defined everywhere in $\mathbb{R}^n$ with $n \geq 3$, and $m \geq 2$, such that their graph $M = (x, f(x))$ is a submanifold with parallel mean curvature in $\mathbb{R}^{n+m}$. Suppose that there exist $\beta_0 < +\infty$ and $\beta_1 < 3$, such that*

$$\Delta_f := \Big[\det\Big(\delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}\Big)\Big]^{\frac{1}{2}} \leq \beta_0. \tag{1.3}$$

and for certain α and i

$$\Delta_f \leq \beta_1\Big(1 + \big(\frac{\partial f^\alpha}{\partial x^i}\big)^2\Big)^{\frac{1}{2}}. \tag{1.4}$$

Then $f^1, \ldots, f^m$ have to be affine linear (representing an affine n-plane).

In fact, we can prove more general results, namely for minimal submanifolds satifying the so-called DVP-condition (see Theorem 6.9 in Section 6), but the preceding statement perhaps best highlights the main achievement of this paper.

In the Appendix we see that for the Lawson-Osserman cone the $v-$function is the constant 9. The Jordan angles between the Gauss image of this cone and the coordinate $n-$ plane are constant. The term $\Big(1 + \Big(\frac{\partial f^2}{\partial x^1}\Big)^2\Big)^{\frac{1}{2}}$ reaches $\sqrt{6}$, corresponding to the largest Jordan angle. Hence the Lawson-Osserman example shows that the Theorem 1.2 cannot hold with $\beta_1 = \frac{9}{\sqrt{6}} \approx 3.7$.

The paper is organized as follows. We will describe in Section 2 that the basic geometry of Grassmann manifolds and define *S-orthogonality* and *S-maps* and show their properties. Section 3 will be devoted to the computation of $\Delta \log w$. For the general case the results come from [26]. We also compute it for minimal submanifolds with rank of the Gauss map less than or equal 2. We obtain a formula more general than that in [6]. In Section 4 we will construct subharmonic functions F on our submanifolds by using the techniques in [16]. The level sets of F coincide with those of $\log w$. We first prove an important transition lemma (Lemma 4.3). Then, we can estimate Hess F in terms of Hess $\log w$, which is already computed in the previous section. Once we have subharmonic functions, the extrinsic rigidity results for compact minimal submanifolds in the sphere (Theorem 4.8) follows immediately. Using strongly subharmonic functions we can also

study complete submanifolds in Euclidean space by using harmonic map regularity theory, as Hildebrandt-Jost-Widman in [13]. In Section 5 we will obtain curvature estimates (Lemma 5.5). In Section 6 the iteration method will be used and the quantitatively controlled Gauss image shrinking properties will be obtained (see Lemma 6.6-6.7). Finally, in this section we prove our main results, Theorem 6.9 and Theorem 6.10 of which Theorem 1.2 is a direct corollary. Finally, in the Appendix, we provide computations for the Lawson-Osserman cone and related coassociated $4-$manifolds in $\mathbb{R}^7$.

2. Geometry of Grassmann manifolds

Let $\mathbf{G}_{n,m}$ be the Grassmann manifold consisting of the oriented linear n-subspaces of the $(n+m)$-Euclidean space $\mathbb{R}^{n+m}$. The canonical Riemannian structure on $\mathbf{G}_{n,m}$ makes it a natural generalization of the Euclidean sphere. Moreover $\mathbf{G}_{n,m} = SO(n+m)/SO(n) \times SO(m)$ is an irreducible symmetric space of compact type.

For every $P \in \mathbf{G}_{n,m}$, we choose an oriented basis $\{u_1, \ldots, u_n\}$ of P, and let

$$\psi(P) := u_1 \wedge \cdots \wedge u_n \in \Lambda^n \left(\mathbb{R}^{n+m}\right). \tag{2.1}$$

A different basis for P shall give a different exterior product, but the two products differ only by a positive scalar; $\psi(P)$ is called the *Plücker coordinate* of P, which is a homogenous coordinate.

Via the Plücker embedding, $\mathbf{G}_{n,m}$ can be viewed as a submanifold of some Euclidean space $\mathbb{R}^N (N = C_{n+m}^n)$. The restriction of the Euclidean inner product is denoted by $w : \mathbf{G}_{n,m} \times \mathbf{G}_{n,m} \to \mathbb{R}$

$$w(P, Q) = \frac{\langle \psi(P), \psi(Q) \rangle}{\langle \psi(P), \psi(P) \rangle^{\frac{1}{2}} \langle \psi(Q), \psi(Q) \rangle^{\frac{1}{2}}}. \tag{2.2}$$

If $\{e_1, \ldots, e_n\}$ is an oriented orthonormal basis of P and $\{f_1, \ldots, f_n\}$ is an oriented orthonormal basis of Q, then

$$w(P, Q) = \langle e_1 \wedge \cdots \wedge e_n, f_1 \wedge \cdots \wedge f_n \rangle = \det W$$

with the *W-matrix* $W = \left(\langle e_i, f_j \rangle\right)$. It is well-known that

$$W^T W = O^T \Lambda O$$

with an orthogonal matrix O and

$$\Lambda = \begin{pmatrix} \mu_1^2 & & \\ & \ddots & \\ & & \mu_n^2 \end{pmatrix}.$$

Here each $0 \leq \mu_i^2 \leq 1$. Putting $p := \min\{m, n\}$, then at most p elements in $\{\mu_1^2, \cdots, \mu_n^2\}$ are not equal to 1. Without loss of generality, we can assume $\mu_i^2 = 1$ whenever $i > p$. We also note that the μ_i^2 can be expressed as

$$\mu_i^2 = \frac{1}{1 + \lambda_i^2} \tag{2.3}$$

with $\lambda_i \in [0, +\infty)$.

The *Jordan angles* between P and Q are critical values of the angle θ between a nonzero vector u in P and its orthogonal projection u^* in Q as u runs through P. Let θ_i be a Jordan angle between P and Q determined by a unit vector e_i and its projection e_i^* in Q, we call e_i an *angle direction* of P relative to Q, and the 2-plane spanned by e_i and e_i^* an *angle 2-plane* between P and Q (see [22]). A direct calculation shows there are n Jordan angles $\theta_1, \ldots, \theta_n$, with $\theta_{p+1} = \cdots = \theta_n = 0$ and

$$\theta_i = \arccos(\mu_i) \qquad 1 \leq i \leq p.$$

Thus

$$|w| = \left(\det(W^T W)\right)^{\frac{1}{2}} = \det(\Lambda)^{\frac{1}{2}} = \prod_{i=1}^{n} \cos\theta_i \tag{2.4}$$

and (2.3) becomes

$$\lambda_i = \tan\theta_i. \tag{2.5}$$

If $w(P, Q) > 0$, arrange all the Jordan angles between P and Q as

$$\frac{\pi}{2} > \theta_1 \geq \theta_2 \geq \cdots \geq \theta_r > \theta_{r+1} = \cdots = \theta_n = 0$$

with $0 \leq r \leq p$. (If $\theta_i = \frac{\pi}{2}$ for some i, then (2.4) implies $w(P, Q) = 0$, which contradicts $w(P, Q) > 0$.) Then one can find an orthonormal basis $\{e_1, \ldots, e_{n+m}\}$ of $\mathbb{R}^{n+m}$, such that P is spanned by $\{e_1, \ldots, e_n\}$, which are angle directions of P relative to Q, and $\{e_i \wedge e_{n+i} : 1 \leq i \leq r\}$ are angle 2-planes between P and Q. Denote

$$f_i := \begin{cases} \cos\theta_i e_i + \sin\theta_i e_{n+i} & 1 \leq i \leq r, \\ e_i & r+1 \leq i \leq n \end{cases} \tag{2.6}$$

and

$$f_{n+\alpha} := \begin{cases} -\sin\theta_\alpha e_\alpha + cos\theta_\alpha e_{n+\alpha} & 1 \leq \alpha \leq r, \\ e_{n+\alpha} & r+1 \leq \alpha \leq m; \end{cases} \tag{2.7}$$

then $\{f_1, \ldots, f_{n+m}\}$ is also an orthonormal basis of $\mathbb{R}^{n+m}$, and $f_1 \wedge \cdots \wedge f_n$ is a Plücker coordinate of Q.

The distance between P and Q is defined by

$$d(P, Q) = \sqrt{\sum \theta_i^2}. \tag{2.8}$$

It is a natural generalization of the canonical distance of Euclidean spheres.

Now we fix $P_0 \in \mathbf{G}_{n,m}$. We represent it by n vectors ϵ_i, which are complemented by m vectors $\epsilon_{n+\alpha}$. Denote

$$\mathbb{U} := \{P \in \mathbf{G}_{n,m} : w(P, P_0) > 0\}. \tag{2.9}$$

We can span an arbitrary $P \in \mathbb{U}$ by n vectors e_i:

$$e_i = \epsilon_i + Z_{i\alpha}\epsilon_{n+\alpha}. \tag{2.10}$$

Here $Z = (Z_{i\alpha})$ could be regarded as the $(n \times m)$-matrix coordinate of P. The canonical Riemannian metric in $\mathbb{U}$ can be described as

$$ds^2 = \operatorname{tr}\left(\left(I_n + ZZ^T\right)^{-1} dZ \left(I_m + Z^T Z\right)^{-1} dZ^T\right), \tag{2.11}$$

where I_m (respectively I_n) denotes the $(m \times m)$-identity (respectively $n \times n$) matrix. In [23, Section 7.1] it is shown that (2.11) can be derived from (2.8).

Here and in the sequel, we use the summation convention and agree on the ranges of indices:

$$1 \le \alpha, \beta \le m, \ 1 \le i, j, k \le n.$$

We shall now introduce the new concepts of *S-orthogonal* and *S-map*, which will play a crucial role for our investigations.

Proposition 2.1. *For any $P, Q \in \mathbf{G}_{n,m}$, the following statements are equivalent:*

(a) $\dim(P \cap Q) = n - 1$ *and* $w(P, Q) = 0$*;*
(b) $\dim(P + Q) = n + 1$ *and* $w(P, Q) = 0$*, where* $P + Q = \{u + v : u \in P, v \in Q\}$*;*
(c) *There exists a single Jordan angle between P and Q being $\frac{\pi}{2}$, and the other $(n-1)$ Jordan angles are all* 0*;*
(d) *Q is the nearest point to P in $\partial\mathbb{U}$, where $\mathbb{U}$ is the $(n \times m)$-matrix coordinate chart centered at P;*
(e) *There exists an orthonormal basis $\{e_i, e_{n+\alpha}\}$ of $\mathbb{R}^{n+m}$, such that the Plücker coordinates of P and Q are $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $e_{n+1} \wedge e_2 \wedge \cdots \wedge e_n$, respectively.*

Proof.
(a)$\Longleftrightarrow$(b) follows from $\dim(P + Q) = \dim P + \dim Q - \dim(P \cap Q)$.
(a)$\Longleftrightarrow$(c) is an immediate corollary of the definition of Jordan angles and (2.4).

$\partial\mathbb{U}$ consists of all $S \in \mathbf{G}_{n,m}$ satisfying $w(P, S) = 0$; it follows from (2.4) that at least 1 Jordan angle between S and P is $\frac{\pi}{2}$; hence one can obtain

$$d(P, S) \ge \frac{\pi}{2} \qquad \text{for all } S \in \partial\mathbb{U}$$

from (2.8), and the equality holds if and only if the other $(n-1)$ Jordan angles all vanish. Hence (c)$\Longleftrightarrow$(d).

(e)$\Longrightarrow$(a) is trivial. To prove (a)$\Longrightarrow$(e), it suffices to choose $\{e_2, \ldots, e_n\}$ as an orthonormal basis of $P \cap Q$, and put e_1 (or e_{n+1}) to be the unit vector in P (or Q) that is orthogonal to $P \cap Q$. $\square$

Definition 2.2. Two points P and Q in a Grassmann manifold are called *S-orthogonal*, if they satisfy one of the properties in Proposition 2.1.

Remark 2.3. There is only one Jordan angle between any two points in the sphere S^n. If it equals $\frac{\pi}{2}$, they are orthogonal each other. The notion of S-orthogonal is a natural generalization of orthogonality in S^n.

Let $P, Q \in \mathbf{G}_{n,m}$ be S-orthogonal. By Proposition 2.1, there exists an orthonormal basis $\{e_i, e_{n+\alpha}\}$ of $\mathbb{R}^{n+m}$, such that $\psi(P) = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\psi(Q) = e_{n+1} \wedge e_2 \wedge \cdots \wedge e_n$. Let $\gamma : \mathbb{R}/(2\pi\mathbb{Z}) \to \mathbf{G}_{n,m}$ be a closed curve, such that $\gamma(t)$ is spanned by $\{\cos t\ e_1 + \sin t\ e_{n+1}, e_2, \ldots, e_n\}$. Then whenever $|t - s| < \frac{\pi}{2}$, there exists one and only one Jordan angle between $\gamma(t)$ and $\gamma(s)$ being $|t - s|$, and the other Jordan angles are all 0; hence by (2.8),

$$d(\gamma(t), \gamma(s)) = |t - s|\,, \qquad \text{whenever } |t - s| < \frac{\pi}{2}.$$

It implies γ is the closed geodesic that is extended from the minimal geodesic between P and Q. In the following, we write $P_t = \gamma(t)$; in particular, $P_0 = P$ and $P_{\frac{\pi}{2}} = Q$.

The *S-map* $\mathscr{S}\colon \mathbf{G}_{n,m} \to \mathbb{R}^2$ is defined by

$$S \mapsto \left(w(S, P_0), w\left(S, P_{\frac{\pi}{2}}\right)\right). \tag{2.12}$$

If there exists a nonzero vector in $P_0 \cap P_{\frac{\pi}{2}}$ orthogonal to S, then $w(S, P_0) = w(S, P_{\frac{\pi}{2}}) = 0$, and hence $\mathscr{S}(S) = 0$. Otherwise, the orthogonal projection p from $P_0 \cap P_{\frac{\pi}{2}}$ to S has rank $n - 1$, which enables us to find a unit vector f_1 in S which is orthogonal to the $(n - 1)$-dimensional projective image $p(P_0 \cap P_{\frac{\pi}{2}})$. Certainly, one can get

$$\begin{aligned} w(S, P_0) &= \langle f_1, e_1\rangle w\left(p\left(P_0 \cap P_{\frac{\pi}{2}}\right), P_0 \cap P_{\frac{\pi}{2}}\right), \\ w(S, P_{\frac{\pi}{2}}) &= \langle f_1, e_{n+1}\rangle w\left(p\left(P_0 \cap P_{\frac{\pi}{2}}\right), P_0 \cap P_{\frac{\pi}{2}}\right). \end{aligned} \tag{2.13}$$

It follows that

$$w^2(S, P_0) + w^2(S, P_{\frac{\pi}{2}}) \le \langle f_1, e_1\rangle^2 + \langle f_1, e_{n+1}\rangle^2 \le 1$$

and the equality holds if and only if $p(P_0 \cap P_{\frac{\pi}{2}}) = P_0 \cap P_{\frac{\pi}{2}}$ and $f_1 \in \operatorname{span}\{e_1, e_{n+1}\}$; *i.e.* $S = P_t$ for some $t \in [-\pi, \pi)$. Hence $\mathscr{S}$ is a smooth map onto the closed unit 2-disk $\overline{\mathbb{D}}$. Although $\mathscr{S}$ is not one-to-one, any point on $\partial\overline{\mathbb{D}}$ has one and only one preimage. For simplicity we write

$$x_1 = w(\cdot, P_0), \qquad x_2 = w(\cdot, P_{\frac{\pi}{2}}). \tag{2.14}$$

Then from (2.13) one can obtain

$$\begin{aligned} (\cos t\ x_1 + \sin t\ x_2)(S) &= \cos t\ w(S, P_0) + \sin t\ w\left(S, P_{\frac{\pi}{2}}\right) \\ &= \langle f_1, \cos t\ e_1 + \sin t\ e_{n+1}\rangle w\left(p\left(P_0 \cap P_{\frac{\pi}{2}}\right), P_0 \cap P_{\frac{\pi}{2}}\right) \\ &= w(S, P_t). \end{aligned} \tag{2.15}$$

3. Laplacian of $\log w$

Let $M^n \to \bar{M}^{n+m}$ be an isometric immersion with second fundamental form B, which can be viewed as a cross-section of the vector bundle $\mathrm{Hom}(\odot^2 TM, NM)$ over M, where TM and NM denote the tangent bundle and the normal bundle along M, respectively. The connection on TM and NM (denoted by ∇) can be induced naturally from the Levi-Civita connection on $\bar{M}$ (denoted by $\bar{\nabla}$). We define the mean curvature H as the trace of the second fundamental form. It is a normal vector field on M in $\bar{M}$. If $\nabla H = 0$, we say M has *parallel mean curvature*; moreover if H vanishes on M everywhere, it is called a *minimal submanifold*.

The second fundamental form, the curvature tensor of the submanifold (denoted by R), the curvature tensor of the normal bundle (denoted by R^N) and that of the ambient manifold (denoted by $\bar{R}$) satisfy the Gauss equations, the Codazzi equations and the Ricci equations (see [23] for details, for example). Here and in the sequel, $\{e_i\}$ denotes a local orthonormal tangent frame field and $\{\nu_\alpha\}$ is a local orthonormal normal frame field of M;

$$h_{\alpha,ij} := \langle B(e_i, e_j), \nu_\alpha \rangle$$

are the coefficients of the second fundamental form B of M in $\bar{M}$.

Now we consider a submanifold M^n in Euclidean space $\mathbb{R}^{n+m}$ with parallel mean curvature.

Let 0 be the origin of $\mathbb{R}^{n+m}$, $SO(n+m)$ be the Lie group consisting of all orthonormal frames $(0; e_i, \nu_\alpha)$. $TF = \{(x; e_1, \dots, e_n) : x \in M, e_i \in T_pM, \langle e_i, e_j \rangle = \delta_{ij}\}$ be the principal bundle of orthonormal tangent frames over M, and $NF = \{(x; \nu_1, \dots, \nu_m) : x \in M, \nu_\alpha \in N_xM\}$ be the principal bundle of orthonormal normal frames over M. Then $\bar{\pi} : TF \oplus NF \to M$ is the projection with fiber $SO(n) \times SO(m)$.

The Gauss map $\gamma : M \to \mathbf{G}_{n,m}$ is defined by

$$\gamma(x) = T_xM \in \mathbf{G}_{n,m}$$

via the parallel translation in $\mathbb{R}^{n+m}$ for every $p \in M$. Then the following diagram commutes

$$\begin{array}{ccc} TF \oplus NF & \xrightarrow{\ i\ } & SO(n+m) \\ \bar{\pi}\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{\ \gamma\ } & \mathbf{G}_{n,m} \end{array}$$

where i denotes the inclusion map and $\pi : SO(n+m) \to \mathbf{G}_{n,m}$ is defined by

$$(0; e_i, \nu_\alpha) \mapsto e_1 \wedge \cdots \wedge e_n.$$

From the above diagram we know that the energy density of the Gauss map is given by (see Section 3.1 of [23])

$$e(\gamma) = \frac{1}{2}\langle \gamma_* e_i, \gamma_* e_i \rangle = \frac{1}{2}|B|^2.$$

Ruh-Vilms proved that the mean curvature vector of M is parallel if and only if its Gauss map is a harmonic map [20].

Let P_0 be a fixed n-plane in $\mathbf{G}_{n,m}$. We define

$$w := w(\cdot, P_0) \circ \gamma. \tag{3.1}$$

Let $\{\varepsilon_i\}$ be an orthonormal basis of P_0, then $w = \langle e_1 \wedge \cdots \wedge e_n, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle$ at every point $x \in M$. From the Codazzi equations, we shall now get basic formulas for the function w.

Lemma 3.1 ([6,24]). *If M is a submanifold of $\mathbb{R}^{n+m}$, then*

$$\nabla_{e_i} w = h_{\alpha,ij} \langle e_{j\alpha}, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle \tag{3.2}$$

with

$$e_{j\alpha} = e_1 \wedge \cdots \wedge \nu_\alpha \wedge \cdots \wedge e_n \tag{3.3}$$

that is obtained by replacing e_j by ν_α in $e_1 \wedge \cdots \wedge e_n$. Moreover if M has parallel mean curvature, then

$$\Delta w = -|B|^2 w + \sum_i \sum_{\alpha \neq \beta, j \neq k} h_{\alpha,ij} h_{\beta,ik} \langle e_{j\alpha,k\beta}, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle \tag{3.4}$$

with

$$e_{j\alpha,k\beta} = e_1 \wedge \cdots \wedge \nu_\alpha \wedge \cdots \wedge \nu_\beta \wedge \cdots \wedge e_n \tag{3.5}$$

that is obtained by replacing e_j by ν_α and e_k by ν_β in $e_1 \wedge \cdots \wedge e_n$, respectively.

In our main Theorem there are two different statement: the general case and the special case. The latter allows a weaker assumption on the range of the Gauss map. In the following we prove Proposition 3.3 for the special case and Proposition 3.5 for the general case.

Firstly, we compute $\Delta \log w$ under the additional assumption that $H = 0$ and the rank of the Gauss map γ is at most 2. Without loss of generality, one can assume $h^\alpha_{ij} = 0$ whenever $i \geq 3$ or $j \geq 3$. Then, we need to calculate $\langle e_{1\alpha,2\beta}, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle$. The following identity shall play an important role.

Lemma 3.2. *Let $A = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ be the Plücker coordinate of P_0, then for any distinct indices α, β,*

$$\langle e_1 \wedge \cdots \wedge e_n, A \rangle \langle e_{1\alpha,2\beta}, A \rangle - \langle e_{1\alpha}, A \rangle \langle e_{2\beta}, A \rangle + \langle e_{1\beta}, A \rangle \langle e_{2\alpha}, A \rangle = 0. \tag{3.6}$$

Proof. Let $Q \in \mathbf{G}_{n-2,m+2}$ spanned by $\{e_i : 3 \leq i \leq n\}$. If there is a nonzero vector in Q which is orthogonal to the n-dimensional space P_0 spanned by $\{\varepsilon_i\}$, then by the definition of the inner product on $\Lambda^n(\mathbb{R}^{n+m})$, all the terms on the left hand side of (3.6) equal 0 and (3.6) trivially holds true.

Otherwise the orthogonal projection $p : Q \to P_0$ has rank $n-2$. Without loss of generality we assume $p(Q)$ is spanned by $\{\varepsilon_i : 3 \leq i \leq n\}$; the Jordan angles

between Q and $p(Q)$ are denoted by $\theta_3, \dots, \theta_n$; and θ_i is determined by e_i and ε_i. Hence $\langle e_i, \varepsilon_j \rangle = \cos\theta_i \delta_{ij}$ whenever $i \geq 3$, and moreover

$$\begin{aligned}\langle e_1 \wedge \cdots \wedge e_n, A\rangle &= \begin{vmatrix} (e_1,\varepsilon_1) & (e_1,\varepsilon_2) & & * & \\ (e_2,\varepsilon_1) & (e_2,\varepsilon_2) & & & \\ & & \cos\theta_3 & & \\ & \mathbf{0} & & \ddots & \\ & & & & \cos\theta_n \end{vmatrix} \\ &= \langle e_1 \wedge e_2, \varepsilon_1 \wedge \varepsilon_2\rangle \prod_{i=3}^{n} \cos\theta_i \end{aligned} \tag{3.7}$$

Similarly,

$$\langle e_{1\alpha,2\beta}, A\rangle = \langle \nu_\alpha \wedge \nu_\beta, \varepsilon_1 \wedge \varepsilon_2\rangle \prod_{i=3}^{n} \cos\theta_i \tag{3.8}$$

and

$$\langle e_{1\gamma}, A\rangle = \langle \nu_\gamma \wedge e_2, \varepsilon_1 \wedge \varepsilon_2\rangle \prod_{i=3}^{n} \cos\theta_i \tag{3.9}$$

$$\langle e_{2\gamma}, A\rangle = \langle e_1 \wedge \nu_\gamma, \varepsilon_1 \wedge \varepsilon_2\rangle \prod_{i=3}^{n} \cos\theta_i \tag{3.10}$$

for $\gamma = \alpha$ or β. A direct calculation shows

$$\begin{aligned}&\langle e_1 \wedge e_2, \varepsilon_1 \wedge \varepsilon_2\rangle\langle \nu_\alpha \wedge \nu_\beta, \varepsilon_1 \wedge \varepsilon_2\rangle - \langle \nu_\alpha \wedge e_2, \varepsilon_1 \wedge \varepsilon_2\rangle\langle e_1 \wedge \nu_\beta, \varepsilon_1 \wedge \varepsilon_2\rangle \\ &\quad + \langle \nu_\beta \wedge e_2, \varepsilon_1 \wedge \varepsilon_2\rangle\langle e_1 \wedge \nu_\alpha, \varepsilon_1 \wedge \varepsilon_2\rangle \\ &= +(e_1,\varepsilon_1)(e_2,\varepsilon_2)(\nu_\alpha,\varepsilon_1)(\nu_\beta,\varepsilon_2) + (e_2,\varepsilon_1)(e_1,\varepsilon_2)(\nu_\beta,\varepsilon_1)(\nu_\alpha,\varepsilon_2) \\ &\quad - (e_1,\varepsilon_1)(e_2,\varepsilon_2)(\nu_\beta,\varepsilon_1)(\nu_\alpha,\varepsilon_2) - (e_2,\varepsilon_1)(e_1,\varepsilon_2)(\nu_\alpha,\varepsilon_1)(\nu_\beta,\varepsilon_2) \\ &\quad - (e_1,\varepsilon_1)(e_2,\varepsilon_2)(\nu_\alpha,\varepsilon_1)(\nu_\beta,\varepsilon_2) - (e_2,\varepsilon_1)(e_1,\varepsilon_2)(\nu_\beta,\varepsilon_1)(\nu_\alpha,\varepsilon_2) \\ &\quad + (e_1,\varepsilon_2)(e_2,\varepsilon_2)(\nu_\alpha,\varepsilon_1)(\nu_\beta,\varepsilon_1) + (e_1,\varepsilon_1)(e_2,\varepsilon_1)(\nu_\alpha,\varepsilon_2)(\nu_\beta,\varepsilon_2) \\ &\quad + (e_1,\varepsilon_1)(e_2,\varepsilon_2)(\nu_\beta,\varepsilon_1)(\nu_\alpha,\varepsilon_2) + (e_2,\varepsilon_1)(e_1,\varepsilon_2)(\nu_\alpha,\varepsilon_1)(\nu_\beta,\varepsilon_2) \\ &\quad - (e_1,\varepsilon_2)(e_2,\varepsilon_2)(\nu_\alpha,\varepsilon_1)(\nu_\beta,\varepsilon_1) - (e_1,\varepsilon_1)(e_2,\varepsilon_1)(\nu_\alpha,\varepsilon_2)(\nu_\beta,\varepsilon_2) \\ &= 0.\end{aligned} \tag{3.11}$$

From (3.7)-(3.11), (3.6) immediately follows. □

Now $H = 0$ implies $0 = h_{\alpha,ii} = h_{\alpha,11} + h_{\alpha,22}$ for every $1 \leq \alpha \leq m$, thus

$$\begin{aligned} h_{\alpha,i1}h_{\beta,i2} =& h_{\alpha,11}h_{\beta,12} + h_{\alpha,21}h_{\beta,22} \\ =& -h_{\alpha,22}h_{\beta,12} - h_{\alpha,21}h_{\beta,11} = -h_{\alpha,i2}h_{\beta,i1} \end{aligned} \tag{3.12}$$

and in particular $h_{\alpha,i1}h_{\alpha,i2} = 0$. Therefore (3.4) can be rewritten as

$$\begin{aligned}\Delta w &= -|B|^2 w + \sum_{\alpha\neq\beta} h_{\alpha,i1}h_{\beta,i2}\langle e_{1\alpha,2\beta}, A\rangle + \sum_{\alpha\neq\beta} h_{\alpha,i2}h_{\beta,i1}\langle e_{1\beta,2\alpha}, A\rangle \\ &= -|B|^2 w + 2\sum_{\alpha,\beta} h_{\alpha,i1}h_{\beta,i2}\langle e_{1\alpha,2\beta}, A\rangle.\end{aligned} \tag{3.13}$$

By (3.2),

$$|\nabla w|^2 = \sum_i |\nabla_{e_i} w|^2 = h_{\alpha,ij}h_{\beta,ik}\langle e_{j\alpha}, A\rangle\langle e_{k\beta}, A\rangle. \tag{3.14}$$

With the aid of (3.6) and (3.12) one can get

$$\begin{aligned}w\Delta w - |\nabla w|^2 &= -|B|^2 w^2 + 2h_{\alpha,i1}h_{\beta,i2}\langle e_1 \wedge \cdots \wedge e_n, A\rangle\langle e_{1\alpha,2\beta}, A\rangle \\ &\quad - h_{\alpha,ij}h_{\beta,ik}\langle e_{j\alpha}, A\rangle\langle e_{k\beta}, A\rangle \\ &= -|B|^2 w^2 + 2h_{\alpha,i1}h_{\beta,i2}\Big(\langle e_{1\alpha}, A\rangle\langle e_{2\beta}, A\rangle - \langle e_{1\beta}, A\rangle\langle e_{2\alpha}, A\rangle\Big) \\ &\quad - h_{\alpha,ij}h_{\beta,ik}\langle e_{j\alpha}, A\rangle\langle e_{k\beta}, A\rangle \\ &= -|B|^2 w^2 + 2h_{\alpha,i1}h_{\beta,i2}\langle e_{1\alpha}, A\rangle\langle e_{2\beta}, A\rangle \\ &\quad + 2h_{\alpha,i2}h_{\beta,i1}\langle e_{2\alpha}, A\rangle\langle e_{1\beta}, A\rangle \\ &\quad - h_{\alpha,i1}h_{\beta,i1}\langle e_{1\alpha}, A\rangle\langle e_{1\beta}, A\rangle - h_{\alpha,i2}h_{\beta,i2}\langle e_{2\alpha}, A\rangle\langle e_{2\beta}, A\rangle \\ &\quad - h_{\alpha,i1}h_{\beta,i2}\langle e_{1\alpha}, A\rangle\langle e_{2\beta}, A\rangle - h_{\alpha,i2}h_{\beta,i1}\langle e_{2\alpha}, A\rangle\langle e_{1\beta}, A\rangle \\ &= -|B|^2 w^2 - \sum_i \Big(\sum_\alpha \big(h_{\alpha,i1}\langle e_{1\alpha}, A\rangle - h_{\alpha,i2}\langle e_{2\alpha}, A\rangle\big)\Big)^2.\end{aligned} \tag{3.15}$$

Noting that $\Delta \log w = w^{-2}(w\Delta w - |\nabla w|^2)$, we arrive at:

Proposition 3.3. *Let M be a minimal submanifold of $\mathbb{R}^{n+m}$, with the rank of Gauss map at most 2. If $w > 0$ at a point, then locally*

$$\Delta \log w \leq -|B|^2. \tag{3.16}$$

Remark 3.4. If N is a 2-dimensional minimal submanifold of S^{2+m}, then the cone CM over M is a 3-dimensional minimal submanifold of $\mathbb{R}^{3+m}$. The rank of the Gauss map from CM is no more than 2, as will be shown in the next section. Hence Proposition 3.3 is a generalization of Proposition 2.2 in [6]. We note that Lemma 3.2 is also a generalization of Lemma 2.1 in [6].

Next, we consider the general case. For any $x \in M$ satisfying $w(x) > 0$, then $w(P, P_0) > 0$ with $P = \gamma(x)$. Let $\theta_1, \cdots, \theta_n$ be the Jordan angles between P and P_0 which are arranged as

$$\frac{\pi}{2} > \theta_1 \geq \cdots \geq \theta_r > \theta_{r+1} = \cdots = \theta_n = 0 \tag{3.17}$$

with $0 \le r \le \min\{n, m\}$. As shown in Section 2, one can find an orthonormal basis $\{\varepsilon_i, \varepsilon_{n+\alpha}\}$ of $\mathbb{R}^{n+m}$, such that $\psi(P_0) = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$; if we denote

$$e_i := \begin{cases} \cos\theta_i \varepsilon_i + \sin\theta_i \varepsilon_{n+i} & 1 \le i \le r \\ \varepsilon_i & r+1 \le i \le n \end{cases} \tag{3.18}$$

and

$$\nu_\alpha := \begin{cases} -\sin\theta_\alpha \varepsilon_\alpha + \cos\theta_\alpha \varepsilon_{n+\alpha} & 1 \le \alpha \le r \\ \varepsilon_{n+\alpha} & r+1 \le \alpha \le m \end{cases} \tag{3.19}$$

then $\{e_i, \nu_\alpha\}$ is also an orthonormal basis of $\mathbb{R}^{n+m}$ and $\psi(P) = e_1 \wedge \cdots \wedge e_n$.

Now we use the notation

$$v := w^{-1} \tag{3.20}$$

as in [26] and [17]; the function v is well-defined in a neighborhood of x. By a direct calculation based on metric forms in terms of matrix coordinates (2.11), one can derive

$$\Delta v = v|B|^2 + v \sum_{i,j} 2\lambda_j^2 h_{j,ij}^2 + v \sum_i \sum_{j \ne k} \lambda_j \lambda_k \left(h_{j,ij} h_{k,ik} + h_{k,ij} h_{j,ik}\right) \tag{3.21}$$

as in [17]; where $\lambda_j = \tan\theta_j$ and $h_{\alpha,ij} = \langle B(e_i, e_j), \nu_\alpha \rangle$. Grouping the terms according to different types of the indices of the coefficients of the second fundamental form, one can proceed as in the proof of Proposition 3.1 in [17] to get

$$v^{-1}\Delta v \ge C_1 |B|^2 \qquad \text{whenever } v \le \beta \text{ for some } \beta < 3. \tag{3.22}$$

Noting that $v = \exp(-\log w)$, one can immediately get:

Proposition 3.5. *Let M be a submanifold of $\mathbb{R}^{n+m}$ with parallel mean curvature. If there exists $x \in M$ and a positive number δ, such that $w(x) \ge \frac{1}{3} + \delta$, then*

$$\Delta \log w - |\nabla \log w|^2 \le -C_1 |B|^2 \qquad \text{at } x \tag{3.23}$$

with a positive constant C_1 depending only on δ.

4. Subharmonic functions and extrinsic rigidity problem

We already defined the *S-map* $\mathscr{S} : \mathbf{G}_{n,m} \to \overline{\mathbb{D}}$ in Section 2. For every $(x_1, x_2) \in \overline{\mathbb{D}} \backslash \{(a, 0) : a \le 0\}$ that is obtained by deleting a radius from the disk, there exist unique $r \in (0, 1]$ and $\theta \in (-\pi, \pi)$, such that

$$(x_1, x_2) = (r\cos\theta, r\sin\theta).$$

Here r and θ can be seen as smooth functions on $\mathscr{S}^{-1}\Big(\overline{\mathbb{D}}\backslash\big\{(a,0) : a \leq 0\big\}\Big)$. For any $c \in [0,1)$, we put

$$\mathbb{W}_c := \mathscr{S}^{-1}\Big(\overline{\mathbb{D}}\backslash\big(\overline{\mathbb{D}}_c \cup \big\{(a,0) : a \leq 0\big\}\big)\Big), \tag{4.1}$$

with $\overline{\mathbb{D}}_c = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq c^2\big\}$, then the function r takes values between c and 1 on $\mathbb{W}_c$.

Remark 4.1. If $m = 1$, then $\mathbf{G}_{n,m} = S^n$ and $\mathbb{W}_0$ is the complement of a half-equator $\overline{S}_+^{n-1}$; in particular, if $n = 2$, then $\mathbb{W}_0$ can be obtained by deleting half of a great circle between 2 antipodal points from S^2. It has been proved in [16] that $\mathbb{W}_0$ is a maximal convex-supporting set in S^n.

Remark 4.2. Obviously, $P_t, P_{t+\pi} \in \mathbb{W}_c$ consist of an antipodal pair in $\mathbf{G}_{n,m}$ for each $t \in (-\frac{\pi}{2}, 0)$, which implies $w(P_t, S) = -w(P_{t+\pi}, S)$ for arbitrary $S \in \mathbf{G}_{n,m}$. Hence $\mathbb{W}_c$ cannot lie in an $(n \times m)$-matrix coordinate chart centered at any point of $\mathbf{G}_{n,m}$. Furthermore, they are conjugate to each other. Since the Grassmann manifolds are simply connected symmetric spaces of compact type, the cut point of P_t along a geodesic coincides with the first conjugate point along the geodesic. Therefore, $\mathbb{W}_c$ contains a cut point of P_t at least.

On $\mathbb{W}_c$ there are smooth functions r and θ. As in the previous work (see Section 3 in [16]), we concern a function F constructed from r and θ and we need to estimate Hess F. Hopefully, the level set of $F = t$ coincides with level set of $w(\cdot, P_t)$. For estimating Hess F in terms of Hess $w(\cdot, P_t)$ we need the following *transition Lemma*, which is crucial in our construction, but may also be useful in other contexts.

Lemma 4.3. *Let (N, h) be a Riemannian manifold, F be a smooth function on N, and $\{H(\cdot, t) : t \in \mathbb{R}\}$ be a smooth family of smooth functions on N. Assume that there are smooth real functions $t(s)$ and $l(s)$; for all $s \in \mathbb{R}$, the level set $F_s := \{p \in N : F(p) = s\}$ coincides with $\{p \in N : H\big(p, t(s)\big) = l(s)\}$; ∇F and $\nabla H\big(\cdot, t(s)\big)$ are nonzero normal vector fields on F_s, pointing in the same direction. Then for arbitrary $\varepsilon > 0$, there exists a smooth nonpositive function λ on N, depending on ε, such that*

$$\text{Hess } F \geq \frac{|\nabla F|}{\big|\nabla H\big(\cdot, t(s)\big)\big|}\text{Hess } H\big(\cdot, t(s)\big) - \varepsilon h + \lambda dF \otimes dF \tag{4.2}$$

on N.

Proof. For any $s \in \mathbb{R}$, F_s can be regarded as a hypersurface of N. Denote by ∇ and ∇^s the Levi-Civita connection on N and F_s, respectively. We put $\nu = \frac{\nabla F}{|\nabla F|}$, then ν is the unit normal vector field on F_s; and denote by B the second fundamental form of F_s in N.

For every $X \in T(F_s)$,

$$\begin{aligned}\text{Hess } F(X, X) &= \nabla_X \nabla_X F - (\nabla_X X)F \\ &= -(\nabla^s_X X)F - \langle B(X, X), \nu\rangle\langle \nu, \nabla F\rangle \\ &= -|\nabla F|\langle B(X, X), \nu\rangle.\end{aligned} \tag{4.3}$$

Similarly,

$$\text{Hess } H\big(\cdot, t(s)\big)(X, X) = -\big|\nabla H\big(\cdot, t(s)\big)\big|\langle B(X, X), \nu\rangle. \tag{4.4}$$

Hence

$$\left(\text{Hess } F - \frac{|\nabla F|}{\big|\nabla H\big(\cdot, t(s)\big)\big|}\text{Hess } H\big(\cdot, t(s)\big)\right)(X, X) = 0 \qquad \text{for all } X \in T(F_s). \tag{4.5}$$

It implies the existence of a smooth 1-form ω on N, such that

$$\text{Hess } F - \frac{|\nabla F|}{\big|\nabla H\big(\cdot, t(s)\big)\big|}\text{Hess } H\big(\cdot, t(s)\big) = \omega \otimes dF + dF \otimes \omega \tag{4.6}$$

on F_s. Denote by ω^* the associated vector field of ω with respect to h, *i.e.*

$$\langle Y, \omega^*\rangle = \omega(Y) \qquad \text{for all } Y \in TN.$$

Then

$$\begin{aligned}&(\omega \otimes dF + dF \otimes \omega)(Y, Y) \\ &= 2\omega(Y)dF(Y) = 2\langle \omega^*, Y\rangle dF(Y) \\ &\geq -2|\omega^*||Y|dF(Y) \geq -\varepsilon|Y|^2 - \varepsilon^{-1}|\omega^*|^2 dF(Y)^2.\end{aligned} \tag{4.7}$$

Let $\lambda = -\varepsilon^{-1}|\omega^*|^2$, then substituting (4.7) into (4.6) yields (4.2). □

Given a compact subset K of $\mathbb{W}_c$, there exists a positive number δ, such that $r \geq c + 2\delta$ on K. Hence the following function

$$F = \theta - \arccos\left(\frac{c+\delta}{r}\right) \tag{4.8}$$

is well-defined on K. Once $F(S) = t$, one can deduce that $\theta - t = \arccos(\frac{c+\delta}{r}) \in (0, \frac{\pi}{2})$, and

$$\begin{aligned}c + \delta &= r(\cos(\theta - t)) = r(\cos t \cos\theta + \sin t \sin\theta) \\ &= (\cos t\, x_1 + \sin t\, x_2)(S) = w(S, P_t).\end{aligned} \tag{4.9}$$

In other words, the level set F_t of F overlaps with the level set $\{S \in \mathbf{G}_{n,m} : -\log w(S, P_t) = -\log(c + \delta)\}$.

To apply Lemma 4.3, we need gradient estimates for the functions $-\log w(\cdot, P_t)$ and F along the common level set.

Let $\mathbb{U}$ be the matrix coordinate chart centered at P_t. Recall that $w(\cdot, P_t)$ has an expression

$$w = \left[\operatorname{tr}\left(I_n + ZZ^T\right)\right]^{-1} \tag{4.10}$$

in terms of matrix coordinates (see (2.10) and Section 2 in [26]). It is easily seen that both w and the metric on $\mathbb{U}$ are invariant under $SO(n) \times SO(m)$-actions. Hence for any $S \in \mathbb{U}$, without loss of generality one can assume

$$Z(S) = (\lambda_i \delta_{i\alpha}) = (\tan\theta_i \delta_{i\alpha})$$

with $\{\theta_i\}$ being the Jordan angles between S and P_t. As in [26], one can obtain

$$dw(\cdot, P_t) = -\sum_{1\le i\le p} \lambda_i w(\cdot, P_t)\omega_{ii} \qquad \text{at } S$$

with $p = \min\{n, m\}$ and $\{\omega_{i\alpha}\}$ is the dual basis of $\left\{(1+\lambda_i^2)^{\frac{1}{2}}(1+\lambda_\alpha^2)^{\frac{1}{2}}E_{i\alpha}\right\}$, which is an orthogonal basis of $T_S\mathbf{G}_{n,m}$. (Here $E_{i\alpha}$ is the matrix with 1 in the intersection of row i and column α and 0 otherwise.) Therefore

$$\left|\nabla \log w(\cdot, P_t)\right|^2 = \sum_{1\le i\le p} \lambda_i^2 \ge p\left(w(\cdot, P_t)^{-\frac{2}{p}} - 1\right) \tag{4.11}$$

and the equality holds if and only if $\lambda_1^2 = \cdots = \lambda_p^2 = w(\cdot, P_t)^{-\frac{2}{p}} - 1$. In particular, on F_t,

$$\left|\nabla \log w(\cdot, P_t)\right|^2 \ge p\left((c+\delta)^{-\frac{2}{p}} - 1\right) := C_2(p, \delta). \tag{4.12}$$

By (2.15),

$$\begin{aligned}\nabla w(\cdot, P_t) &= \cos t\nabla x_1 + \sin t\nabla x_2 = \cos t\nabla(r\cos\theta) + \sin t\nabla(r\sin\theta)\\ &= \cos(t-\theta)\nabla r + r\sin(t-\theta)\nabla\theta\end{aligned}$$

and moreover by (4.9),

$$\begin{aligned}-\nabla \log w(\cdot, P_t) &= -w^{-1}(\cdot, P_t)\nabla w(\cdot, P_t) = -(c+\delta)^{-1}\nabla w(\cdot, P_t)\\ &= -r^{-1}\nabla r + \tan(\theta - t)\nabla\theta\end{aligned} \tag{4.13}$$

on F_t. On the other hand, from (4.8) we have

$$\begin{aligned}\nabla F &= \nabla\theta - \left(1 - \left(\frac{c+\delta}{r}\right)^2\right)^{-\frac{1}{2}} \frac{c+\delta}{r^2}\nabla r\\ &= \nabla\theta - r^{-1}\cot(\theta - t)\nabla r.\end{aligned} \tag{4.14}$$

Comparing (4.13) with (4.14) gives

$$-\nabla \log w(\cdot, P_t) = \tan(\theta - t)\nabla F. \tag{4.15}$$

$c+2\delta \le r \le 1$ gives $\theta - t = \arccos\left(\frac{c+\delta}{r}\right) \in \left[\arccos\left(\frac{c+\delta}{c+2\delta}\right), \arccos(c+\delta)\right]$; hence $-\nabla \log w(\cdot, P_t)$ and ∇F point in the same direction, and

$$\frac{|\nabla \log w(\cdot, P_t)|}{|\nabla F|} = \tan(\theta - t) \le \frac{\sqrt{1-(c+\delta)^2}}{c+\delta} := C_3(\delta). \tag{4.16}$$

Now we put $H(\cdot, t) = -\log w(\cdot, P_t)$, then by (4.12) and (4.16), one can apply Lemma 4.3 to get the following conclusion:

Proposition 4.4. *For a compact set $K \subset \mathbb{W}_c$, put $\delta := \frac{1}{2}\inf_K(r-c)$ and $F := \theta - \arccos\left(\frac{c+\delta}{r}\right)$, then for arbitrary $\varepsilon > 0$, there exists a smooth nonpositive function λ on K, depending on ε, such that*

$$\text{Hess } F \ge -C_3^{-1}\text{Hess } \log w(\cdot, P_t) - \varepsilon g + \lambda dF \otimes dF \tag{4.17}$$

on the level set F_t. Here g is the canonical metric on $\mathbf{G}_{n,m}$ and C_3 is a positive constant depending only on δ.

Remark 4.5. Put

$$\begin{aligned}\phi(t)(a_i e_i + b_\alpha e_{n+\alpha}) =& (a_1 \cos t + b_1 \sin t)e_1 + (-a_1 \sin t + b_1 \cos t)e_{n+1} \\ &+ \sum_{i\ge 2} a_i e_i + \sum_{\alpha \ge 2} b_\alpha e_{n+\alpha};\end{aligned}$$

then $t \in \mathbb{R} \mapsto \phi(t)$ is a 1-parameter subgroup of $SO(n+m)$, which induces a 1-parameter isometry group of $\mathbf{G}_{n,m}$. Since $w(\cdot, \cdot)$ is invariant under the action of $SO(n+m)$, it is easily-seen that F_t is the orbit of F_0 under the action of $\phi(t)$. This observation can explain why the level sets of F share similar properties in terms of exterior geometry.

Combining Propositions 3.3-3.5 and Proposition 4.4, we can find a strongly subharmonic function on a submanifold with parallel mean curvature under the assumptions on the Gauss image.

Proposition 4.6. *Let M be an oriented submanifold of $\mathbb{R}^{n+m}$ with parallel mean curvature, if its Gauss image is contained in a compact subset K of $\mathbb{W}_{1/3}$, then there exists a positive bounded function f on M, such that*

$$\Delta f \ge K_0 |B|^2 \tag{4.18}$$

with a positive constant K_0 depending only on K. Moreover, if M is minimal and the rank of the Gauss map is not larger than 2, then the above conclusion still holds when $\mathbb{W}_{1/3}$ is replaced by $\mathbb{W}_0$.

Proof. We only prove the general case, since the proof for the case with the additional assumption on the rank of the Gauss map is similar.

Using the definition of F in (4.8) with $c = \frac{1}{3}$ and combining (4.17) and (4.16) gives

$$\begin{aligned}
&\mu_0^{-1}\exp(\mu_0 F)\mathrm{Hess}\exp(\mu_0 F)\\
&\geq -C_3^{-1}\mathrm{Hess}\log w(\cdot, P_t) - \varepsilon g + (\lambda+\mu_0)dF\otimes dF \qquad (4.19)\\
&\geq -C_3^{-1}\mathrm{Hess}\log w(\cdot, P_t) - \varepsilon g + C_3^{-2}|\lambda+\mu_0| d\log w(\cdot, P_t)\otimes d\log w(\cdot, P_t)
\end{aligned}$$

where μ_0 and ε are positive constants to be chosen. Define

$$f = \exp(\mu_0 F)\circ\gamma \qquad (4.20)$$

with γ the Gauss map into $K \subset \mathbb{W}_{1/3}$, then using the composition formula yields

$$\begin{aligned}
\mu_0^{-1} f^{-1}\Delta f =& \mu_0^{-1}\exp(\mu_0 F)\Big(\mathrm{Hess}\exp(\mu_0 F)(\gamma_* e_i, \gamma_* e_i) + d\exp(\mu_0 F)(\tau(\gamma)\Big)\\
\geq& -C_3^{-1}\mathrm{Hess}\ \log w(\cdot, P_t)(\gamma_* e_i, \gamma_* e_i) - \varepsilon\langle\gamma_* e_i, \gamma_* e_i\rangle\\
&+ C_3^{-2}|\lambda+\mu_0|\sum_i\Big(\nabla_{e_i}\big|\log w(\cdot, P_t)\circ\gamma\big|\Big)^2 \qquad (4.21)\\
=& -C_3^{-1}\Delta\big(\log w(\cdot, P_t)\circ\gamma\big) - \varepsilon|B|^2 + C_3^{-2}|\lambda\\
&+ \mu_0|\Big|\nabla\big(\log w(\cdot, P_t)\circ\gamma\big)\Big|^2
\end{aligned}$$

on the level set $\{x \in M : f(x) = \exp(\mu_0 t)\}$. Here $\{e_i\}$ is an orthonormal basis of the tangent space at the considered point of M, and $\tau(\gamma)$ is the tension field of the Gauss map, which is identically 0, due to the harmonicity of γ. Since $w(\cdot, P_t)$ equals $\frac{1}{3}+\delta$ everywhere on the level set F_t (see (4.9)), one can apply Proposition 3.5 to obtain

$$\Delta\big(\log w(\cdot, P_t)\circ\gamma\big) - \Big|\nabla\big(\log w(\cdot, P_t)\circ\gamma\big)\Big|^2 \leq -C_1|B|^2 \qquad (4.22)$$

with a positive constant C_1 depending only on δ. Substituting this into (4.21) yields

$$\mu_0^{-1} f^{-1}\Delta f \geq (C_3^{-1}C_1 - \varepsilon)|B|^2 + \big(C_3^{-2}|\lambda+\mu_0| - C_3^{-1}\big)\Big|\nabla\big(\log w(\cdot, P_t)\circ\gamma\big)\Big|^2.$$

By letting $\varepsilon := \frac{1}{2}C_3^{-1}C_1$ and $\mu_0 := \sup_K|\lambda| + C_3$ we arrive at (4.18) with $K_0 = \frac{1}{2}C_3^{-1}C_1\mu_0\exp(\mu_0\inf_K F)$. □

We can immediately get the following Bernstein-type theorem with the aid of the strongly subharmonic functions.

Theorem 4.7. *If M is a parabolic, oriented minimal surface in $\mathbb{R}^{2+m}$, and if its Gauss image is contained in a compact subset of $\mathbb{W}_0$, then M has to be an affine linear subspace.*

Proof. One can find a strongly subharmonic function f on M satisfying (4.18) by applying Proposition 4.6, since the mean curvature vector field vanishes and the rank of the Gauss map is ≤ 2. Since a parabolic surface cannot admit any nonconstant bounded subharmonic function, f has to be constant; hence $\Delta f \equiv 0$ and moreover $|B|^2 \equiv 0$, which means M is totally geodesic. □

We now study the extrinsic rigidity problem, initiated by J. Simons [21], finding better conditions on the Gauss image for a compact minimal submanifold M in S^{n+m} to be an equator.

As pointed out and utilized by J. Simons [21], the properties of the (minimal) submanifolds M in the sphere are closely related to those of the cone CM generated by M, which is the image under the map of $M \times [0, +\infty) \to \mathbb{R}^{n+m+1}$ defined by $(x, t) \mapsto tx$. To avoid the singularity $t = 0$, we also consider the truncated cone CM_ε ($\varepsilon > 0$) that is the image of $M \times (\varepsilon, +\infty)$ under the same map.

We choose a local orthonormal frame field $\{e_i, \nu_\alpha\}$ of S^n along M, then by parallel translating along rays issuing from the origin we obtain local vector fields $\{E_i\}$ and $\{\mathcal{N}_\alpha\}$; obviously $E_i = \frac{1}{t}e_i$ and $\mathcal{N}_\alpha = \frac{1}{t}\nu_\alpha$. Let τ be the unit tangent vector along the rays, *i.e.* $\tau = \frac{\partial}{\partial t}$, then $\{E_i, \mathcal{N}_\alpha, \tau\}$ is a local orthonormal frame field in $\mathbb{R}^{n+m+1}$ and $\{E_i, \tau\}$ is a frame field in CM_ε.

Let B and B^c denote the second fundamental form of $M \subset S^{n+m}$ and $CM_\varepsilon \subset \mathbb{R}^{n+m+1}$, respectively. Then a straightforward calculation shows (see [21] or [23, page 18])

$$\left\langle B^c(E_i, E_j), \mathcal{N}_\alpha \right\rangle = \frac{1}{t}\left\langle B(e_i, e_j), \nu_\alpha \right\rangle \tag{4.23}$$

and

$$B^c(E_i, \tau) = B^c(\tau, \tau) = 0. \tag{4.24}$$

Hence CM_ε is a minimal submanifold of $\mathbb{R}^{n+m+1}$ if and only if M is a minimal submanifold of S^{n+m}.

The *normal Gauss map* $\gamma^N : M \to \mathbf{G}_{m,n+1}$ is defined by

$$\gamma^N(x) = N_x M$$

via parallel translation in $\mathbb{R}^{n+m+1}$. Let η be the natural isometry between $\mathbf{G}_{n+1,m}$ and $\mathbf{G}_{m,n+1}$ which maps each $(n+1)$-dimensional oriented linear subspace to its orthogonal complementary m-space. It is easily seen that the Gauss map $\gamma : CM_\varepsilon \to \mathbf{G}_{n+1,m}$ is a *cone-like map*; more precisely, for every $t \in (\varepsilon, +\infty)$ and $x \in M$,

$$(\eta \circ \gamma)(tx) = \gamma^N(x).$$

Thus $\gamma_* \tau = 0$ and the rank of γ at tx equals the rank of γ^N at x for all $x \in M$ and any $t \in (\varepsilon, +\infty)$.

One can assume $\nabla e_i = 0$ at the considered point without loss of generality, then by computing (see [23] p.18),

$$\nabla^c_{E_i} E_j = -\frac{1}{t}\delta_{ij}\tau \tag{4.25}$$

with ∇^c the Levi-Civita connection on CM_ε. Let f be a cone-like real function on CM_ε, then

$$\Delta^c f = \nabla^c_{E_i}\nabla^c_{E_i} f - (\nabla^c_{E_i} E_i) f = \frac{1}{t^2}\nabla_{e_i}\nabla_{e_i} f_1 + \frac{1}{t}\nabla^c_\tau f = \frac{1}{t^2}\Delta f_1. \tag{4.26}$$

Here Δ^c is the Laplace-Beltrami operator on CM_ε and f_1 is a function on M satisfying $f(tx) = f_1(x)$ for every $x \in M$ and $t \in (\varepsilon, +\infty)$.

Based on the strongly subharmonic functions constructed in Proposition 4.6, we can derive the following extrinsic rigidity result.

Theorem 4.8. *Let M^n be a compact, oriented minimal submanifold of S^{n+m}, $P, Q \in \mathbf{G}_{m,n+1}$ that are S-orthogonal to each other. Assume*

$$w(N, P)^2 + w(N, Q)^2 > \begin{cases} 0 & \text{if } rank(\gamma^N) \le 2 \\ \frac{1}{9} & \text{otherwise} \end{cases} \tag{4.27}$$

holds for all normal m-vectors N of M, and there exists no point $x \in M$, such that $w(N, Q) = 0$ and $w(N, P) < 0$ at x. Then M has to be an equator.

Proof. As above, we only give the proof for the general case without the assumptions on the rank of the Gauss map.

By the definition of η, we have

$$\psi\big(\eta(S)\big) = *\big(\psi(S)\big)$$

with ψ denoting the Plücker embedding and $*$ being the Hodge star operator. $*$ keeps the inner product invariant, *i.e.*

$$\langle *A, *B\rangle = \langle A, B\rangle \qquad \text{for } A, B \in \Lambda^m(\mathbb{R}^{n+m+1}).$$

Hence

$$w\big(\eta(S_1), \eta(S_2)\big) = w(S_1, S_2) \qquad \text{for } S_1, S_2 \in \mathbf{G}_{m,n+1}.$$

If S_1, S_2 are S-orthogonal to each other, then $w(S_1, S_2) = 0$ and $\dim(S_1 \cap S_2) = m - 1$, which implies $w\big(\eta(S_1), \eta(S_2)\big) = 0$ and $\dim\big(\eta(S_1) + \eta(S_2)\big) = n + m + 1 - \dim(S_1 \cap S_2) = n + 2$, hence $\eta(S_1)$ and $\eta(S_2)$ are S-orthogonal to each other (see Proposition 2.1).

Denote

$$\mathbb{W}_{\frac{1}{3}} = \Big\{S \in \mathbf{G}_{n+1,m} : \big(w(S, \eta(P)), w(S, \eta(Q))\big) \in \mathbb{D}\backslash\big(\overline{\mathbb{D}}_{\frac{1}{3}} \cup \{(a, 0) : a \le 0\}\big)\Big\},$$

then the assumption on the normal Gauss map γ^N implies that the image under the Gauss map of CM_ε is contained in a compact subset of $\mathbb{W}_{1/3}$. This enables us to find a strongly subharmonic function f satisfying

$$\Delta^c f \geq K_0 |B^c|^2. \tag{4.28}$$

Noting that f is the composition of the Gauss map γ and a function on K, it should be a cone-like function. Denote $f_1 = f|_M$, then combining (4.28), (4.26) and (4.23) gives

$$\Delta f_1 = t^2 \Delta^c f \geq K_0 t^2 |B^c|^2 = K_0 |B|^2. \tag{4.29}$$

Integrating both sides along M implies

$$0 = \int_M \Delta f_1 * 1 \geq K_0 \int_M |B|^2 * 1.$$

Thus $|B|^2 \equiv 0$ and M has to be totally geodesic. □

Remark 4.9. Obviously $w(N, P) > 0$ implies $w(N, P)^2 + w(N, Q)^2 > 0$. Hence for the case of dimension 2, our result is an improvement of extrinsic rigidity theorems of Barbosa [2] and Fischer-Colbrie [6].

5. Curvature estimates

In this section, we shall give a brief summary on the curvature estimates of minimal submanifolds satisfying the DVP-condition. The proofs are the same as in [16,17].

In the previous section, we have found a strongly subharmonic function f on a submanifold $M^n \subset \mathbb{R}^{n+m}$ with parallel mean curvature under the assumption on the Gauss image. To make Stokes' theorem applicable, we take a $H^{1,2}$-function η with compact support; multiplying η with both sides of (4.18) and then integrating it along M gives

$$\begin{aligned} K_0 \int_M |B|^2 \eta * 1 \leq \int_M \eta \Delta f * 1 &= \int_M \big(\operatorname{div}(\eta \nabla f) - \langle \nabla \eta, \nabla f \rangle\big) * 1 \\ &= -\int_M \langle \nabla \eta, \nabla f \rangle * 1. \end{aligned} \tag{5.1}$$

In order to obtain *a priori* estimates for $|B|^2$, we have to choose a 'good' test function η.

We shall use the Green test function technique employed in [12] and [13]. Unfortunately, this method cannot be applied for arbitrary noncompact Riemannian manifolds. We have to impose a so-called DVP-condition as in [16].

Definition 5.1. Let (M, g) be a Riemannian manifold (not necessarily complete) satisfying:
(D) There is a distance function d on M, such that the metric topology induced by d is equivalent to the Riemannian topology of M, and $d(x, y) \leq r(x, y)$ for any $x, y \in M$, where r is the distance function induced by the metric tensor of M;
(V) Let $B_R(x_0) = \{x \in M : d(x, x_0) < R\}$, $V(x_0, R) = \mathrm{Vol}\big(B_R(x_0)\big)$, then there exists a positive constant K_1 not depending on R and x_0, such that

$$V(x_0, 2R) \leq K_1\, V(x_0, R) \qquad \text{whenever } B_{2R}(x_0) \Subset M;$$

(P) For any $x_0 \in M$ and $R > 0$ satisfying $B_R(x_0) \Subset M$, the following Neumann-Poincaré inequality

$$\int_{B_R(x_0)} |v - \bar{v}_R|^2 * 1 \leq K_2 R^2 \int_{B_R(x_0)} |\nabla v|^2 * 1$$

holds with $\bar{v}_R$ denoting the average value of v in $B_R(x_0)$ and K_2 being a positive constant not depending on x_0 and R. Then we say that (M, g) satisfies a DVP-condition.

Remark 5.2. If $M = \mathbb{D}^n(r_0) \subset \mathbb{R}^n$ ($r_0 \in \mathbb{R}^+$) with metric $g = g_{ij}dx^i dx^j$, and the eigenvalues of (g_{ij}) are uniformly bounded from below by a positive constant λ and from above by a positive constant μ, then by denoting $d(x, y) = \lambda|x - y|$ with $|\cdot|$ the standard Euclidean norm, we can show (M, g) satisfies the DVP-condition with $K_1 = \big(\frac{4\mu}{\lambda}\big)^n$ and $K_2 = 4\pi^{-2}\big(\frac{\mu}{\lambda}\big)^{n+2}$ (see [16]). Thus every Riemannian manifold has a coordiate chart centered at any point that satisfies the DVP-condition. Similarly, one can prove any simple manifold is a complete Riemannian manifold satisfying the DVP-condition (see [16]). Other complete Riemannian manifolds satisfying the DVP-condition include (M, g) with nonnegative Ricci curvature (see [4] for the proof of the Neumann-Poincaré inequality) and area-minimizing embedded hypersurfaces in $\mathbb{R}^{n+1}$. (Here d is taken to be the extrinsic distance function, see [3] for the proof of the Neumann-Poincaré inequality.)

Based on the DVP-condition, one can apply Moser's iteration [19] to derive the Harnack inequality for superharmonic functions on M, which implies the following estimates:

Lemma 5.3 ([16, Corollary 4.1]). *Let M be a Riemannian manifold satisfying a DVP-condition, then there exists a constant $C_4 \in (0, 1)$, only depending on K_1 and K_2, such that*

$$v_{+,\frac{R}{2}} \leq (1 - C_4)v_{+,R} + C_4\bar{v}_{\frac{R}{2}} \tag{5.2}$$

for any subharmonic function v on $B_R(x_0)$ ($B_{2R}(x_0) \Subset M$). Here $v_{+,R}$ denotes the supremum of v on $B_R(x_0)$.

Let Ω be an open domain of M and $\rho > 0$. Using the Lax-Milgram Theorem, it is easy to prove that there is a unique $H_0^{1,2}(\Omega)$-function $G^\rho(\cdot, x)$, such that

$$\int_\Omega \langle \nabla G^\rho(\cdot, x), \nabla\phi\rangle * 1 = \fint B_\rho(x)\phi \qquad \text{for any } \phi \in H_0^{1,2}(\Omega) \tag{5.3}$$

whenever $B_\rho(x) \subset \Omega$. Here

$$\fint B_\rho(x)\phi = \frac{\int_{B_\rho(x)} \phi}{V(x,\rho)}.$$

G^ρ is called the *mollified Green function*. As in [9] and [16], we can obtain *a priori* estimates for mollified Green functions:

Lemma 5.4 ([16, Lemma 4.3]). *Let (M, g) be a Riemannian manifold (dimension ≥ 3) satisfying a DVP-condition. Let $x_0 \in M$ and $R > 0$ satisfying $B_{2R}(x_0) \Subset M$. Denote*

$$\omega^R = \frac{V(x_0, \frac{R}{2})}{R^2} G^{\frac{R}{2}}(\cdot, x_0) \tag{5.4}$$

with $G^{\frac{R}{2}}$ being the mollified Green function on $B_R(x_0)$, then

$$\omega^R \leq c_1 \qquad \text{on } B_R(x_0) \tag{5.5}$$

$$\omega^R \geq c_2 \qquad \text{on } B_{\frac{R}{2}}(x_0) \tag{5.6}$$

with c_1 and c_2 denoting positive constants depending only on K_1 and K_2, not depending on x_0 and R.

Choosing $\eta = (\omega^R)^2$ as a test function in (5.1), we can derive the following curvature estimates with the aid of Lemmas 5.3 and 5.4, which is a version of the telescoping lemma of Giaquinta-Giusti [7]. See Section 4 in [16] and Section 4 in [17] for details.

Lemma 5.5 (Curvature estimates). *Let M^n be a submanifold of $\mathbb{R}^{n+m}$ ($n \geq 3$) with parallel mean curvature. Assume there is a distance function d on M, the metric ball $B_{4R_0}(x_0)$ defined by d satisfies the DVP-condition for some $x_0 \in M$ and $R_0 \in (0, +\infty]$, and the Gauss image of $B_{4R_0}(x_0)$ is contained in a compact set $K \subset \mathbb{W}_{1/3}$. Then there exists a positive constant C_5, depending only on K, K_1, K_2, such that for arbitrary $R < R_0$,*

$$\frac{R^2}{V(x_0, R)} \int_{B_{\frac{R}{2}}(x_0)} |B|^2 * 1 \leq C_5(f_{+,R} - f_{+,\frac{R}{2}}) \tag{5.7}$$

with f being the strongly subharmonic function constructed in Proposition 4.6. Moreover, there exists a positive constant C_6 depending only on K, K_1, K_2, such that for arbitrary $\varepsilon > 0$, we can find $R \in \left[\exp(-C_6\varepsilon^{-1})R_0, R_0\right]$, such that

$$\frac{R^2}{V(x_0, R)} \int_{B_R(x_0)} |B|^2 * 1 \leq \varepsilon. \tag{5.8}$$

If M is a minimal submanifold with $rank(\gamma) \leq 2$, then the condition on the Gauss image can be relaxed to $\gamma(B_{4R_0}(x_0)) \subset K \subset \mathbb{W}_0$.

6. Gauss image shrinking lemmas and Bernstein-type theorems

Let u be a harmonic map from M and H be a smooth function on the target manifold, then $H \circ u$ defines a smooth function on M. Take a cut-off function η supported in the interior of $B_R(x_0)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\frac{R}{2}}(x_0)$ and $|\nabla \eta| \le c_0 R^{-1}$; and denote by G^ρ the mollified Green function in $B_R(x_0)$. Then by Stokes' theorem

$$\int_{B_R(x_0)} \operatorname{div}\big(\eta G^\rho(\cdot, x)\nabla(H \circ u)\big) * 1 = 0 \qquad \text{whenever } B_\rho(x) \subset B_R(x_0). \tag{6.1}$$

After careful calculation as in [16] [17], we can arrive at a pointwise estimate for $H \circ u$:

Proposition 6.1. *Let M be as in Lemma 5.5. Let u be a harmonic map of $B_{4R_0}(x_0) \subset M$ into a convex domain $V \subset \mathbb{R}^l$ with metric $h_{ij}dx^i dx^j$, such that*

$$K_3|\xi|^2 \le \xi^i h_{ij}\xi^j \le K_4|\xi|^2$$

everywhere on V. Then for any smooth function H on V,

$$\begin{aligned} H \circ u(x) \le & H(\bar{u}_R) + C_7 \sup_V |\nabla H| \Big(\frac{R^2}{V(x_0, R)} \int_{B_R(x_0)} |du|^2 * 1\Big)^{\frac{1}{2}} \\ & - \liminf_{\rho \to 0^+} \int_{B_R(x_0)} G^\rho(\cdot, x)\eta\Delta(H \circ u) * 1 \end{aligned} \tag{6.2}$$

for all $x \in B_{\frac{R}{4}}(x_0)$ with $R < R_0$. Here $\bar{u}_R \in \overline{V}$ (the closure of V) denotes the average value of u in $B_R(x_0)$ and C_7 is a positive constant depending only on K, K_1, K_2, K_3, K_4, but independent of H, x_0 and R.

If we take u to be the harmonic Gauss map, then $|du|^2 = |d\gamma|^2 = |B|^2$. We shall make use of Lemma 5.5 to infer that the second term on the right-hand side of (6.2) becomes arbitrarily small when R is sufficiently small. In order to prove the Gauss image shrinking property as in [16,17] it remains to show that:

- $\mathbb{W}_0$ is diffeomorphic to a convex domain in Euclidean space;
- For any given $S \in K \subset \mathbb{W}_{1/3}$, one can find a smooth function H, such that $H \circ \gamma$ is a subharmonic function on $B_R(x_0)$, and the sublevel set $\big(H \le H(S)\big)$ is contained in a matrix coordinate chart of some point in $\mathbf{G}_{n,m}$.

In the sequel, $x_1, x_2, r, \theta, e_i, e_{n+\alpha}$ and P_t will be as in Section 4.

Lemma 6.2. *There is a diffeomorphism $\phi = (\phi_1, \phi_2) : \mathbb{W}_0 \to (-\pi, \pi) \times \mathbb{R}^{nm-1}$ defined by*

$$S \mapsto \big(\phi_1(S), \phi_2(S)\big),$$

such that

$$\phi_1 = \theta, \qquad \big|\phi_2(S)\big| = \frac{1}{r} - 1. \tag{6.3}$$

Proof. As shown in Section 4, θ is a smooth $(-\pi, \pi)$-valued function on $\mathbb{W}_0$, and we shall investigate the level sets of θ.

For any $t \in (-\pi, \pi)$, when $\theta(S) = t$ we have $w(S, P_{t+\frac{\pi}{2}}) = 0$ and $w(S, P_t) > 0$ by (2.15). Now we write $y_1 = w(\cdot, P_t)$, $y_2 = w\Big(\cdot, P_{t+\frac{\pi}{2}}\Big)$, then

$$y_1 = \cos t\, x_1 + \sin t\, x_2 \qquad y_2 = -\sin t\, x_1 + \cos t\, x_2. \tag{6.4}$$

Denote $\mathbb{U}_t = \big\{S \in \mathbf{G}_{n,m} : w(S, P_t) > 0\big\}$. Let $\chi_t : \mathbb{U}_t \to M_{n\times m}$ be the diffeomorphism mapping $S \in \mathbb{U}_t$ to its $(n \times m)$-matrix coordinate, then the Plücker coordinate of S is

$$\psi(S) = (f_1 + Z_{1\alpha} f_{n+\alpha}) \wedge \cdots \wedge (f_n + Z_{n\alpha} f_{n+\alpha}) \qquad \text{with } Z = \chi_t(S). \tag{6.5}$$

Here $f_1 = \cos t\, e_1 + \sin t\, e_{n+1}$, $f_{n+1} = -\sin t\, e_1 + \cos t\, e_{n+1}$, $f_i = e_i$ for all $2 \le i \le n$ and $f_{n+\alpha} = e_{n+\alpha}$ for every $2 \le \alpha \le m$. Noting that $\psi(P_t) = f_1 \wedge \cdots \wedge f_n$ and $\psi(P_{t+\frac{\pi}{2}}) = f_{n+1} \wedge f_2 \wedge \cdots \wedge f_n$. A direct calculation shows that for $S \in U_t$

$$y_1(S) = w(S, P_t) = \frac{\langle \psi(S), \psi(P_t)\rangle}{\langle \psi(S), \psi(S)\rangle^{\frac{1}{2}} \langle \psi(P_t), \psi(P_t)\rangle^{\frac{1}{2}}} = (I_n + ZZ^T)^{-\frac{1}{2}}, \tag{6.6}$$

$$\begin{aligned} y_2(S) = w\Big(S, P_{t+\frac{\pi}{2}}\Big) &= \frac{\langle \psi(S), \psi(P_{t+\frac{\pi}{2}})\rangle}{\langle \psi(S), \psi(S)\rangle^{\frac{1}{2}} \langle \psi(P_{t+\frac{\pi}{2}}), \psi(P_{t+\frac{\pi}{2}})\rangle^{\frac{1}{2}}} \\ &= Z_{11}(I_n + ZZ^T)^{-\frac{1}{2}}. \end{aligned} \tag{6.7}$$

Hence, $y_2(R) = 0$ if and only if $Z_{11} = 0$, *i.e.*

$$\{S \in \mathbb{W}_0 : \theta(S) = t\} = \chi_t^{-1}\{Z \in M_{n\times m} : Z_{11} = 0\}. \tag{6.8}$$

A straightforward calculation gives

$$\begin{aligned} \nabla y_1 &= \cos(t-\theta)\nabla r + r\sin(t-\theta)\nabla\theta, \\ \nabla y_2 &= -\sin(t-\theta)\nabla r + r\cos(t-\theta)\nabla\theta. \end{aligned}$$

Hence, along the level set $(\theta = t)$

$$r = y_1, \quad \nabla y_1 = \nabla r, \quad \nabla y_2 = r\nabla\theta.$$

Differentiating both sides of (6.7) and noting (6.8), we have

$$\nabla\theta = r^{-1}\nabla y_2 = r^{-1}\nabla(Z_{11}y_1) = r^{-1}(y_1\nabla Z_{11} + Z_{11}\nabla y_1) = \nabla Z_{11}, \tag{6.9}$$

whenever $\theta(S) = t$. In U_t let $E_{i\alpha}$ be the matrix with 1 in the intersection of row i and column α and 0 otherwise. Denote $g_{i\alpha,j\beta} = \langle E_{i\alpha}, E_{j\beta}\rangle$ and let $(g^{i\alpha,j\beta})$ be the

inverse matrix of $(g_{i\alpha,j\beta})$. A standard computation shows $|\nabla Z_{11}|^2 = g^{11,11}$. It has been shown in [26] that the eigenvalues of $(g^{i\alpha,j\beta})$ are

$$\left\{\left(1+\lambda_i^2\right)\left(1+\lambda_\alpha^2\right) : 1 \le i \le n, 1 \le \alpha \le m\right\}$$

with $\lambda_i = \tan\theta_i$ and $\{\theta_i\}$ being the Jordan angles between S and P_t. Hence

$$1 \le g^{11,11} \le \max_{(i,\alpha)}\left(\left(1+\lambda_i^2\right)\left(1+\lambda_\alpha^2\right)\right) \le w^{-4}(\cdot, P_t) = r^{-4};$$

i.e.

$$1 \le |\nabla\theta|^2 \le r^{-4}. \tag{6.10}$$

From (6.8) and (6.10), we see that θ is a non-degenerate function on $\mathbb{W}_0$, and each level set of θ is diffeomorphic to $\mathbb{R}^{nm-1}$. $T : M_{n\times m} \to M_{n\times m}$ defined by

$$Z \mapsto \left(\det(I_n + ZZ^T)^{\frac{1}{2}} - 1\right)\frac{Z}{\left(\mathrm{tr}(ZZ^T)\right)^{\frac{1}{2}}}$$

obviously is a diffeomorphism. Note that here $\left(\mathrm{tr}(ZZ^T)\right)^{\frac{1}{2}} = \left(\sum_{i,\alpha} Z_{i\alpha}^2\right)^{\frac{1}{2}}$ equals the Euclidean norm of Z when Z is treated as a vector in $\mathbb{R}^{nm}$. Thus, $T \circ \chi_t$ is a diffeomorphism between the level set $(\theta = t)$ and $\mathbb{R}^{nm-1}$, moreover

$$|T \circ \chi_t| = w^{-1}(\cdot, P_t) - 1 = r^{-1} - 1.$$

Therefore, $\phi_1 = \theta$ and $\phi_2 = T \circ \chi_t$ are the required mappings. □

Now, we show there are sufficiently many subharmonic functions for Gauss image shrinking process being possible. See Proposition 6.3 and Proposition 6.4 for the general case and the special case, respectively.

Proposition 6.3. *For any $c \in \left(\frac{1}{3}, 1\right)$ and any compact subset Θ of $(-\pi, \pi)$, there exists a smooth family of nonnegative, smooth functions $H(\cdot, t)(t \in \Theta)$ on*

$$\overline{\mathbb{W}}_{c,\Theta} := \{S \in \mathbb{W}_0 : r(S) \ge c, \theta(S) \in \Theta\},$$

such that:

(i) *$H(S, t) = 0$ if and only if $S = P_t$;*
(ii) *$H(S, t) \le 1$ (respectively $H(S, t) < 1$) if and only if $w(S, P_t) \ge \frac{3}{4}c + \frac{1}{12}$ (respectively $w(S, P_t) > \frac{3}{4}c + \frac{1}{12}$);*
(iii) *For any submanifold M in $\mathbb{R}^{n+m}$ with parallel mean curvature, if the Gauss image of M is contained in $\overline{\mathbb{W}}_{c,\Theta}$, then $H(\cdot, t) \circ \gamma$ is a subharmonic function on M for all $t \in \Theta$.*

Proposition 6.4. *For any $c \in (0,1)$ and any compact subset Θ of $(-\pi,\pi)$, there exists a smooth family of nonnegative, smooth functions $H(\cdot,t)(t \in \Theta)$ on $\overline{\mathbb{W}}_{c,\Theta}$, such that:*

(i) *$H(S,t) = 0$ if and only if $S = P_t$;*
(ii) *$H(S,t) \le 1$ (or $H(S,t) < 1$) if and only if $w(S,P_t) \ge \frac{3}{4}c$ (or $w(S,P_t) > \frac{3}{4}c$); is a minimal submanifold of $\mathbb{R}^{n+m}$ with $rank(\gamma) \le 2$, then $H(\cdot,t)\circ\gamma$ is a subharmonic function on M for all $t \in \Theta$ if the Gauss image of M is contained in $\overline{\mathbb{W}}_{c,\Theta}$.*

Proof. We only prove Proposition 6.3, since the proof of Proposition 6.4 is similar.

Let φ be a smooth function on $[0,+\infty)$ satisfying

$$\begin{cases} \varphi(u) = 0 & u \in \left[0, \frac{11}{12} - \frac{3}{4}c\right] \\ \varphi(u) = u - \frac{5}{6} + \frac{1}{2}c & u \in \left[\frac{3}{4} - \frac{1}{4}c, +\infty\right) \\ 0 \le \varphi' \le 1. \end{cases}$$

Then one can define Ψ on $\overline{\mathbb{W}}_{c,\Theta} \times (-\pi,\pi) \times [0,+\infty)$ by

$$(S,t,u) \mapsto \begin{cases} r\cos(\theta - t - \varphi(u)) + u - \varphi(u) - 1 & \text{if } \theta(S) \ge t \\ r\cos(\theta - t + \varphi(u)) + u - \varphi(u) - 1 & \text{if } \theta(S) < t. \end{cases} \tag{6.11}$$

Now we fix t and denote $\Psi_t(S,u) = \Psi(S,t,u)$. For arbitrary $S \in \overline{\mathbb{W}}_{c,\Theta}$, we put

$$I_S := \Big\{u \in [0,+\infty) : \max\{0, |\theta - t| - \pi\} \le \varphi(u) \le |\theta - t|\Big\},$$

then obviously I_S is a closed interval, $I_S := [m_S, M_S]$. If $\theta(S) \ge t$, then

$$\partial_2\Psi_t = 1 - \big(1 + r\sin(\theta - t - \varphi(u))\big)\varphi'(u) \ge 0$$

and the equality holds if and only if $\varphi(u) = |\theta - t|$ or $|\theta - t| - \pi$ and $\varphi'(u) = 1$; which implies $\varphi(u) = m_S$ or M_S. Thus

$$\partial_2\Psi_t(S,\cdot) > 0 \qquad \text{on } (m_S, M_S). \tag{6.12}$$

Similarly (6.12) holds when $\theta(S) < t$.

When $|\theta(S) - t| \le \pi$, we have $m_S = 0$ and hence

$$\Psi_t(S, m_S) = r\cos(\theta - t) - 1 \le 0. \tag{6.13}$$

Otherwise $|\theta(S) - t| > \pi$ and m_S satisfies $\varphi(m_S) = |\theta - t| - \pi$, thus

$$\begin{aligned} \Psi_t(S, m_S) &= -r + m_S - \varphi(m_S) - 1 \le -r - 1 + \lim_{u\to+\infty}\big(u - \varphi(u)\big) \\ &= -r - 1 + \frac{5}{6} - \frac{1}{2}c \le -\frac{3}{2}c - \frac{1}{6} < 0. \end{aligned} \tag{6.14}$$

Here the first inequality follows from $\big(u - \varphi(u)\big)' = 1 - \varphi'(u) \ge 0$.

Obviously $\varphi(M_S) = |\theta - t|$. By definition, φ cannot be identically zero on a neighborhood of M_S. Thus $M_S \geq \frac{11}{12} - \frac{3}{4}c$, and

$$\Psi_t(S, M_S) = r + M_S - \varphi(M_S) - 1 \geq r - 1 + \frac{11}{12} - \frac{3}{4}c \geq \frac{1}{4}c - \frac{1}{12} > 0. \quad (6.15)$$

By (6.12)-(6.15), for each $S \in \overline{\mathbb{W}}_{c,\Theta}$, there exists a unique $\tilde{H} = \tilde{H}(S,t) \in [m_S, M_S)$, such that

$$\Psi\big(S, t, \tilde{H}(S,t)\big) = \Psi_t\big(S, \tilde{H}(S,t)\big) = 0. \quad (6.16)$$

Denote

$$\Omega := \left\{(S,t) \in \overline{\mathbb{W}}_{c,\Theta} \times (-\pi, \pi) : \theta(S) \neq t\right\}; \quad (6.17)$$

then Ψ is obviously smooth on $\Omega \times [0, +\infty)$. Hence the implicit theorem implies $\tilde{H}$ is smooth on Ω. To show smoothness of $\tilde{H}$, it remains to prove that $\tilde{H}$ is smooth on a neighborhood of $\Omega^c = \left\{(S,t) \in \overline{\mathbb{W}}_{c,\Theta} \times (-\pi, \pi) : \theta(S) = t\right\}$. Denote

$$\Omega_0 := \left\{(S,t) \in \overline{\mathbb{W}}_{c,\Theta} : w(S, P_t) \geq \frac{1}{12} + \frac{3}{4}c\right\}; \quad (6.18)$$

then $\Omega^c \Subset \Omega_0$ since $\theta(S) = t$ implies $w(S, P_t) = r \geq c > \frac{1}{12} + \frac{3}{4}c$. For all $(S,t) \in \Omega_0$, $1 - w(S, P_t) \leq 1 - (\frac{1}{12} + \frac{3}{4}c) = \frac{11}{12} - \frac{3}{4}c$ and hence $\varphi\big(1 - w(S, P_t)\big) = 0$. Noting that $w(S, P_t) > 0$ implies $|\theta - t| < \pi$, we have $1 - w(S, P_t) \in [m_S, M_S)$, and moreover

$$\Psi_t\big(S, 1 - w(S, P_t)\big) = r\cos(\theta - t) + \big(1 - w(S, P_t)\big) - 1 = 0.$$

Therefore

$$\tilde{H}(S,t) = 1 - w(S, P_t) \qquad \text{for all } (S,t) \in \Omega_0 \quad (6.19)$$

and the smoothness of $\tilde{H}$ follows.

Now we put

$$U_t := \left\{S \in \overline{\mathbb{W}}_{c,\Theta} : w(S, P_t) \geq \frac{1}{12} + \frac{3}{4}c\right\}; \quad (6.20)$$

then obviously $0 \leq \tilde{H}(S,t) \leq \frac{11}{12} - \frac{3}{4}c$ whenever $S \in U_t$. On the other hand, for arbitrary $S \in \overline{\mathbb{W}}_{c,\Theta} \backslash U_t$, one of the following two cases has to occur: (I) $|\theta(S) - t| \geq \pi$; (II) $|\theta(S) - t| \leq \pi$ and $w(S, P_t) < \frac{1}{12} + \frac{3}{4}c$. If Case (I) holds, then $\tilde{H}(S,t) \geq m_S > \frac{11}{12} - \frac{3}{4}c$. For the second case, since

$$\Psi_t\left(s, \frac{11}{12} - \frac{3}{4}c\right) = r\cos(\theta - t) + \frac{11}{12} - \frac{3}{4}c - 1 < 0,$$

one can deduce that $\tilde{H}(S,t) > \frac{11}{12} - \frac{3}{4}c$ due to the monotonicity of Ψ_t with respect to the u variable. Therefore

$$U_t = \left\{ S \in \overline{\mathbb{W}}_{c,\Theta} : \tilde{H}(S,t) \le \frac{11}{12} - \frac{3}{4}c \right\}. \tag{6.21}$$

Similarly

$$\operatorname{int}(U_t) = \left\{ S \in \overline{\mathbb{W}}_{c,\Theta} : \tilde{H}(S,t) < \frac{11}{12} - \frac{3}{4}c \right\}. \tag{6.22}$$

It is easily seen from (6.19) and (6.21) that

$$\begin{aligned} \operatorname{Hess}\tilde{H}(\cdot,t) \ge \left(\frac{1}{12} + \frac{3}{4}c\right)\Big(&- \operatorname{Hess}\log w(\cdot, P_t) \\ &+ d\log w(\cdot, P_t) \otimes d\log w(\cdot, P_t)\Big) \qquad \text{on } U_t. \end{aligned} \tag{6.23}$$

For each $a \ge \frac{11}{12} - \frac{3}{4}c$ and $S \in \overline{\mathbb{W}}_{c,\Theta}$ satisfying $\theta(S) > t$, $\tilde{H}(S,t) = a$ if and only if

$$0 = \Psi_t(S,a) = r\cos(\theta - t - \varphi(a)) + a - \varphi(a) - 1;$$

i.e.

$$w(S, P_{t+\varphi(a)}) = 1 + \varphi(a) - a \tag{6.24}$$

with

$$\frac{1}{12} + \frac{3}{4}c \ge 1 + \varphi(a) - a \ge 1 + \lim_{u\to+\infty}(\varphi(u) - u) = \frac{1}{6} + \frac{1}{2}c > \frac{1}{3}.$$

Hence

$$N^+_{t,a} := \left\{ S \in \overline{\mathbb{W}}_{c,\Theta} : \theta(S) > t,\ \tilde{H}(S,t) = a \right\} \tag{6.25}$$

overlaps the level set $\left\{ S \in \mathbf{G}_{n,m} : -\log w(S, P_{t+\varphi(a)}) = -\log(1 + \varphi(a) - a) \right\}$.

From (6.16) we have $\tilde{\Psi}_t := \Psi_t\big(S, \tilde{H}(S,t)\big) \equiv 0$. Differentiating both sides implies

$$0 = \nabla_\nu \tilde{\Psi}_t = \nabla_\nu \Psi_t(\cdot, a) + (\partial_2 \Psi_t)\nabla_\nu \tilde{H}(\cdot, t) \qquad \text{on } N^+_{t,a} \tag{6.26}$$

with ν the unit normal vector field on $N^+_{t,a}$. Noting that $\Psi_t(\cdot, a) = w(\cdot, P_{t+\varphi(a)}) - (1 + \varphi(a) - a)$ and $\partial_2\Psi_t > 0$, (6.26) tells us $\nabla\tilde{H}(\cdot,t)$ and $-\nabla\log w(\cdot, P_{t+\varphi(a)})$ are both nonzero normal vector fields on $N^+_{t,a}$, pointing in the same direction. The compactness of $\bigcup_{a \ge \frac{11}{12} - \frac{3}{4}c} N^+_{t,a}$ and Θ implies the existence of a positive constant C_8 not depending on a and t, such that

$$\frac{|\nabla\tilde{H}(\cdot,t)|}{\left|\nabla\log w(\cdot, P_{t+\varphi(a)})\right|} \ge C_8 \qquad \text{on } N^+_{t,a}. \tag{6.27}$$

Hence applying Lemma 4.3 gives

$$\begin{aligned}\operatorname{Hess}\tilde{H}(\cdot,t) \geq -C_8 \log w(\cdot, P_{t+\varphi(a)}) - \varepsilon g \\ + \lambda d\tilde{H}(\cdot,t)\otimes d\tilde{H}(\cdot,t) \qquad \text{on } N^+_{t,a}\end{aligned} \tag{6.28}$$

with g being the canonical metric on $\mathbf{G}_{n,m}$, ε being a positive constant to be chosen and λ denoting a continuous nonpositive function depending on ε.

Similarly,

$$N^-_{t,a} := \{S \in \overline{\mathbb{W}}_{c,\Theta} : \theta(S) < t, \tilde{H}(S,t) = a\} \tag{6.29}$$

overlaps the level set $\left\{S \in \mathbf{G}_{n,m} : \log w(S, P_{t-\varphi(a)}) = -\log(1+\varphi(a)-a)\right\}$ for each $a \geq \frac{11}{12} - \frac{3}{4}c$. On it $\nabla\tilde{H}(\cdot,t)$ and $-\nabla \log w(\cdot, P_{t-\varphi(a)})$ are both nonzero normal vector fields pointing in the same direction. Again using Lemma 4.2 yields

$$\begin{aligned}\operatorname{Hess}\tilde{H}(\cdot,t) \geq -C_8 \log w(\cdot, P_{t-\varphi(a)}) - \varepsilon g \\ + \lambda d\tilde{H}(\cdot,t)\otimes d\tilde{H}(\cdot,t) \qquad \text{on } N^-_{t,a}.\end{aligned} \tag{6.30}$$

Let M be a submanifold of $\mathbb{R}^{n+m}$ with parallel mean curvature whose Gauss image is contained in $\overline{\mathbb{W}}_{c,\Theta}$, denote

$$\tilde{h}(\cdot,t) = \exp(\mu_0\tilde{H}(\cdot,t))\circ\gamma$$

where μ_0 is a positive constant to be chosen. Once $\gamma(x) \in U_t$, combining (6.23) and (3.23) gives

$$\begin{aligned}\mu_0^{-1}\tilde{h}(\cdot,t)^{-1}\tilde{h}(\cdot,t) &\geq -\left(\frac{1}{12}+\frac{3}{4}c\right)\Delta\left(\log w(\cdot,P_t)\circ\gamma\right) \\ &\quad + \left(\frac{1}{12}+\frac{3}{4}c+\mu_0\right)\left|\nabla\left(\log w(\cdot,P_t)\circ\gamma\right)\right|^2 \\ &\geq \left(\frac{1}{12}+\frac{3}{4}c\right)C_1|B|^2 + \mu_0\left|\nabla\left(\log w(\cdot,P_t)\circ\gamma\right)\right|^2\end{aligned} \tag{6.31}$$

at $x \in M$, where C_1 is a positive constant depending only on c. If $\gamma(x) \in N^+_{t,a}$, based on (6.28) and (3.23), one can proceed as in the proof of Proposition 4.6 to obtain

$$\begin{aligned}\mu_0^{-1}\tilde{h}(\cdot,t)^{-1}\Delta\tilde{h}(\cdot,t) \geq (C_8C_1-\varepsilon)|B|^2 + (C_8^2|\lambda \\ + \mu_0| - C_8)\left|\nabla(\log w(\cdot,P_{t+\varphi(a)})\circ\gamma)\right|^2\end{aligned} \tag{6.32}$$

at x. Similarly, once $\gamma(x) \in N^-_{t,a}$,

$$\begin{aligned}\mu_0^{-1}\tilde{h}(\cdot,t)^{-1}\Delta\tilde{h}(\cdot,t) \geq (C_8C_1-\varepsilon)|B|^2 \\ + (C_8^2|\lambda + \mu_0| - C_8)\left|\nabla(\log w(\cdot,P_{t-\varphi(a)})\circ\gamma)\right|^2.\end{aligned} \tag{6.33}$$

Now we put

$$\varepsilon := C_8 C_1, \qquad \mu_0 := \sup|\lambda| + C_8^{-1} \tag{6.34}$$

(where $\sup|\lambda| < +\infty$, since λ is a continuous function on a compact set $(\overline{\mathbb{W}}_{c,\Theta} \times \Theta)\backslash \mathrm{int}(\Omega_0)$), then from (6.31)-(6.33) we see $\tilde{h}$ is a subharmonic function on M. Therefore

$$H(\cdot, t) := \frac{\exp\left(\mu_0 \tilde{H}(\cdot, t)\right) - 1}{\exp\left((\frac{11}{12} - \frac{3}{4}c)\mu_0\right) - 1} \tag{6.35}$$

are required functions satisfying (i)-(iii). □

Remark 6.5. The auxiliary function φ in the above proof can be easily obtained from the standard bump function. Choose ξ_1 to be a nonnegative smooth function on $\mathbb{R}$, whose support is $\left(\frac{11}{12} - \frac{3}{4}c, \frac{3}{4} - \frac{1}{4}c\right)$, then

$$\xi_2(u) := \frac{\int_{\frac{11}{12}-\frac{3}{4}c}^{u} \xi_1}{\int_{\frac{11}{12}-\frac{3}{4}c}^{\frac{3}{4}-\frac{1}{4}c} \xi_1}$$

is a smooth function on $\mathbb{R}$ satisfying $0 \le \xi_2 \le 1$, $\xi_2(u) = 0$ whenever $u \le \frac{11}{12} - \frac{3}{4}c$ and $\xi_2(u) = 1$ whenever $u \ge \frac{3}{4} - \frac{1}{4}c$. $\alpha \in (0, +\infty) \mapsto \int_{\frac{11}{12}-\frac{3}{4}c}^{\frac{3}{4}-\frac{1}{4}c} \xi_2^{\alpha}$ is a strictly decreasing function, which converges to 0 as $u \to +\infty$ and converges to $-\frac{1}{6} + \frac{1}{2}c$ as $u \to 0$. There exists a unique $\beta \in (0, +\infty)$, such that

$$\int_{\frac{11}{12}-\frac{3}{4}c}^{\frac{3}{4}-\frac{1}{4}c} \xi_2^{\beta} = -\frac{1}{12} + \frac{1}{4}c.$$

Then

$$\varphi(u) := \int_0^u \xi_2^{\beta}$$

is the required auxiliary function.

Based on Lemma 6.2 and Proposition 6.3-6.4, one can derive a Gauss image shrinking property for submanifolds with parallel mean curvature as in [16,17]:

Lemma 6.6. *Let M^n be a submanifold of $\mathbb{R}^{n+m}$ ($n \ge 3$) with parallel mean curvature. Assume there is a distance function d on M, the metric ball $B_{4R_0}(x_0)$ given by d satisfies DVP-condition for some $x_0 \in M$ and $R_0 \in (0, +\infty]$, and the Gauss image of $B_{4R_0}(x_0)$ is contained in a compact set $K \subset \mathbb{W}_{1/3}$. Then there exists a constant $\delta_1 \in (0, 1)$, depending only on K, K_1, K_2, not depending on x_0 and R_0, such that the image of $B_{\delta_1 R_0}(x_0)$ under the Gauss map is contained in $\{S \in \mathbf{G}_{n,m} : w(S, P) \ge \frac{1}{12} + \frac{3}{4}c\}$ for some $P \in \mathbf{G}_{n,m}$, where $c := \inf_K r$.*

Lemma 6.7. *Let M^n be a minimal submanifold of $\mathbb{R}^{n+m}$ ($n \geq 3$) with $rank(\gamma) \leq 2$ (γ denotes the Gauss map). Assume there is a distance function d on M, the metric ball $B_{4R_0}(x_0)$ given by d satisfies the DVP-condition for some $x_0 \in M$ and $R_0 \in (0, +\infty]$, and the Gauss image of $B_{4R_0}(x_0)$ is contained in a compact set $K \subset \mathbb{W}_0$. Then there exists a constant $\delta_1 \in (0, 1)$, depending only on K, K_1, K_2, not depending on x_0 and R_0, such that the image of $B_{\delta_1 R_0}(x_0)$ under the Gauss map is contained in $\{S \in \mathbf{G}_{n,m} : w(S, P) \geq \frac{3}{4}c\}$ for some $P \in \mathbf{G}_{n,m}$, where $c := \inf_K r$.*

Proof. As above, we only prove Lemma 6.6.

Since $K \subset \mathbb{W}_{1/3}$ is compact, $c := \inf_K r > \frac{1}{3}$ and $\Theta := \left\{t \in (-\pi, \pi) : t = \theta(S) \text{ for some } S \in K\right\}$ is a compact subset of $(-\pi, \pi)$. Obviously $K \subset \overline{\mathbb{W}}_{c,\Theta}$.

By Lemma 6.2, $\overline{\mathbb{W}}_{c,\Theta}$ is diffeomorphic to $\Theta \times \overline{\mathbb{D}}^{nm-1}_{c^{-1}-1}$. Thus $\overline{\mathbb{W}}_{c,\Theta}$ can be seen as a bounded convex domain in $\mathbb{R}^{nm}$ equipped with the induced metric. The eigenvalues of the metric matrices are bounded from below by K_3 and from above by K_4, where K_3, K_4 are positive constants depending only on K.

The Ruh-Vilms Theorem [20] implies that γ is a harmonic function. Putting $u = \gamma$ in (6.2) gives $|du|^2 = |B|^2$. Let $\{H(\cdot, t) : t \in \Theta\}$ be a family of smooth functions on $\overline{\mathbb{W}}_{c,\Theta}$ as constructed in Proposition 6.3. Choosing one of the functions as a test function in (6.2) yields

$$H(\gamma(x), t) \leq H(\bar{\gamma}_R, t) + C_7 C_9 \left(\frac{R^2}{V(x_0, R)} \int_{B_R(x_0)} |B|^2 * 1 \right)^{\frac{1}{2}}. \tag{6.36}$$

for arbitrary $R \leq R_0$ and all $x \in B_{\frac{R}{4}}$. Here

$$C_9 := \sup_{\Theta} \sup_{\overline{\mathbb{W}}_{c,\Theta}} \left|\nabla H(\cdot, t)\right| \tag{6.37}$$

and the last term in (6.2) has been thrown out, since $H(\cdot, t) \circ \gamma$ is a subharmonic function.

For arbitrary $S \in \overline{\mathbb{W}}_{c,\Theta}$, $w(S, P_{\theta(S)}) = r(S) \geq c > \frac{3}{4}c + \frac{1}{12}$, which implies $H(S, \theta(S)) < 1$ by Proposition 6.3(ii). Since $\overline{\mathbb{W}}_{c,\Theta}$ is compact, there exists a positive constant ε_1, such that

$$H(S, \theta(S)) \leq 1 - \varepsilon_1 \qquad \text{for all } S \in \overline{\mathbb{W}}_{c,\Theta}. \tag{6.38}$$

By Lemma 5.5 (curvature estimates), if we denote

$$\delta_1 := \frac{1}{4} \exp(-C_6 C_7^2 C_9^2 \varepsilon_1^{-2}) \tag{6.39}$$

then there exists $R \in [4\delta_1 R_0, R_0]$, such that

$$\frac{R^2}{V(x_0, R)} \int_{B_R(x_0)} |B|^2 * 1 \leq C_7^{-2} C_9^{-2} \varepsilon_1^2. \tag{6.40}$$

Letting $t = \theta(\bar{\gamma}_R)$ and substituting (6.38) and (6.40) into (6.36) gives

$$H\big(\gamma(x), \theta(\bar{\gamma}_R)\big) \le H(\bar{\gamma}_R, \theta(\bar{\gamma}_R)) + C_7 C_9 (C_7^{-2} C_9^{-2} \varepsilon_1^2)^{\frac{1}{2}} \le 1 - \varepsilon_1 + \varepsilon_1 = 1 \quad (6.41)$$

for all $x \in B_{\frac{R}{4}}(x_0)$. Hence by Proposition 6.3(ii),

$$\gamma\big(B_{\delta_1 R_0}(x_0)\big) \subset \gamma\big(B_{\frac{R}{4}}(x_0)\big) \subset \left\{ S \in \mathbf{G}_{n,m} : w(S, P_{\theta(\bar{\gamma}_R)}) \ge \frac{1}{12} + \frac{3}{4}c \right\}. \qquad \square$$

Given $\gamma\big(B_{\delta_1 R_0}(x_0)\big) \subset \big\{ S \in \mathbf{G}_{n,m} : w(S, P) \ge \frac{1}{12} + \frac{3}{4}c > \frac{1}{3} \big\}$, one can start an iteration as in [17] to get *a priori* estimates for the Gauss image:

Lemma 6.8. *Let M^n be a submanifold of $\mathbb{R}^{n+m}$ ($n \ge 3$) with parallel mean curvature. Assume there is a distance function d on M, the metric ball $B_{4R_0}(x_0)$ defined by d satisfies the DVP-condition for some $x_0 \in M$ and $R_0 \in (0, +\infty]$, and the Gauss image of $B_{4R_0}(x_0)$ is contained in a compact set $K \subset \mathbb{W}_{1/3}$. Then for arbitrary $\varepsilon > 0$, there exists a constant $\delta_2 \in (0, 1)$, depending only on K, K_1, K_2, ε, not depending on x_0 and R_0, such that*

$$w(\gamma(x), \gamma(x_0)) \ge 1 - \varepsilon \qquad \text{on } B_{\delta_2 R_0}(x_0). \quad (6.42)$$

In particular, if M is a minimal submanifold with $rank(\gamma) \le 2$, then the condition on the Gauss image can be relaxed to $\gamma(B_{4R_0}(x_0)) \subset K \subset \mathbb{W}_0$.

Letting $R_0 \to +\infty$ we arrive at a Bernstein-type theorem.

Theorem 6.9. *Let M^n be a submanifold of $\mathbb{R}^{n+m}$ ($n \ge 3$) with parallel mean curvature. Assume there is a distance function d on M, such that M satisfies the DVP-condition and the diameter of M with respect to d is infinite; and there exist $P, Q \in \mathbf{G}_{n,m}$ that are S-orthogonal to each other, such that $\big(w(\gamma(x), P), w(\gamma(x), Q)\big)$ is contained in a compact subset K of $\overline{\mathbb{D}} \backslash \big(\overline{\mathbb{D}}_{1/3} \cup \{(a, 0) : a \le 0\}\big)$ for all $x \in M$. Then M has to be an affine linear subspace. In particular, if M is a minimal submanifold with $rank(\gamma) \le 2$, then the assumptions on the Gauss image can be replaced by $\big(w(\gamma(x), P), w(\gamma(x), Q)\big) \in K \subset \overline{\mathbb{D}} \backslash \{(a, 0) : a \le 0\}$.*

The above general theorem can be applied to graphic submanifolds as follows:

Theorem 6.10. *Let $z^\alpha = f^\alpha(x^1, \dots, x^n)$, for $\alpha = 1, \dots, m$, be smooth functions defined everywhere in $\mathbb{R}^n$ with $n \ge 3$, and $m \ge 2$, such that their graph $M = (x, f(x))$ is a submanifold with parallel mean curvature in $\mathbb{R}^{n+m}$. Suppose that there exist $\beta_0 < +\infty$ and $\beta_1 < 3$, such that*

$$\Delta_f := \left[\det \left(\delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} \le \beta_0 \quad (6.43)$$

$$\Delta_f \le \beta_1 \left(1 + \left(\frac{\partial f^1}{\partial x^1} \right)^2 \right)^{\frac{1}{2}}. \quad (6.44)$$

Then $f^1, \dots, f^m$ have to be affine linear (representing an affine n-plane).

Remark 6.11. Obviously $\beta_1\Big(1+\big(\frac{\partial f^1}{\partial x^1}\big)^2\Big)^{\frac{1}{2}} \geq \beta_1$ on the right-hand side of (6.44). Hence Theorem 6.10 improves Theorem 6.2 in [17]. We note that $\frac{\partial f^1}{\partial x^1}$ in (6.44) can be replaced by $\frac{\partial f^\alpha}{\partial x^i}$ for arbitrary $1 \leq \alpha \leq m$ and $1 \leq i \leq n$.

Proof. The map $F : \mathbb{R}^n \to M$ defined by

$$x \mapsto (x, f(x))$$

is obviously a diffeomorphism. Thus M can be viewed as an n-dimensional Euclidean space equipped with metric $g = g_{ij}dx^i dx^j$. Here

$$g_{ij} = \Big\langle F_*\frac{\partial}{\partial x^i}, F_*\frac{\partial}{\partial x^j}\Big\rangle = \Big\langle \varepsilon_i + \frac{\partial f^\alpha}{\partial x^i}\varepsilon_{n+\alpha}, \varepsilon_j + \frac{\partial f^\alpha}{\partial x^j}\varepsilon_{n+\alpha}\Big\rangle = \delta_{ij} + \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}$$

with $\{\varepsilon_i, \varepsilon_{n+\alpha}\}$ being the canonical orthonormal basis of $\mathbb{R}^{n+m}$. Let $Df := (\frac{\partial f^\alpha}{\partial x^i})$ be an $(n \times m)$-matrix valued function on $\mathbb{R}^n$, then

$$(g_{ij}) = I_n + Df(Df)^T.$$

Hence $(g_{ij}) \geq I_n$. In conjunction with $\det(g_{ij}) = \Delta_f^2 \leq \beta_0^2$, we can deduce that all the eigenvalues of (g_{ij}) take values between 1 and β_0^2. Denote $d : M \times M \to \mathbb{R}$

$$d\big(F(x), F(y)\big) = |x - y|,$$

then the diameter of M is $+\infty$ and it has been shown in Remark 5.2 that M satisfies a DVP-condition.

Let $P, Q \in \mathbf{G}_{n,m}$ whose Plücker coordinates are $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ and $\varepsilon_{n+1} \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_n$, respectively. Obviously, P, Q are S-orthogonal to each other. Denote by ψ the Plücker embedding, then

$$\psi \circ \gamma = \left(\varepsilon_1 + \frac{\partial f^\alpha}{\partial x^1}\varepsilon_{n+\alpha}\right) \wedge \cdots \wedge \left(\varepsilon_n + \frac{\partial f^\alpha}{\partial x^n}\varepsilon_{n+\alpha}\right)$$

and by a direct computation,

$$w(\gamma, P) = \frac{\langle \psi \circ \gamma, P\rangle}{\langle \psi \circ \gamma, \psi \circ \gamma\rangle^{\frac{1}{2}}\langle P, P\rangle^{\frac{1}{2}}} = \Delta_f^{-1},$$

$$w(\gamma, Q) = \frac{\langle \psi \circ \gamma, Q\rangle}{\langle \psi \circ \gamma, \psi \circ \gamma\rangle^{\frac{1}{2}}\langle Q, Q\rangle^{\frac{1}{2}}} = \frac{\partial f^1}{\partial x^1}\Delta_f^{-1}.$$

Hence $\big(w(\gamma, P), w(\gamma, Q)\big) \in K \subset \overline{\mathbb{D}}\backslash\big(\overline{\mathbb{D}}_{1/3} \cup \{(a, 0) : a \leq 0\}\big)$ if and only if

$$\beta_1^{-2} \leq w(\gamma, P)^2 + w(\gamma, Q)^2 = \left(1 + \left(\frac{\partial f^1}{\partial x^1}\right)^2\right)\Delta_f^{-2}$$

for a constant $\beta_1 < 3$. This is equivalent to (6.44). So the Bernstein-type result follows from Theorem 6.9. □

7. Appendix

Appendix Let $\mathbb{H}$ denote the quaternions with the standard basis $1, i, j, k$ and $\mathbb{C} = \mathbb{R} + \mathbb{R}i$, $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. Let $q = z_1 - \bar{z}_2 j \in \mathbb{H}$ with $z_1, z_2 \in \mathbb{C}$. Then $\bar{q} = \bar{z}_1 + \bar{z}_2 j$ and

$$\begin{aligned} qi\bar{q} &= (z_1 - \bar{z}_2 j)i(\bar{z}_1 + \bar{z}_2 j) \\ &= (\bar{z}_2 k + z_1 i)(\bar{z}_1 + \bar{z}_2 j) \\ &= \bar{z}_2 k\bar{z}_1 + z_1 i\bar{z}_1 + \bar{z}_2 k\bar{z}_2 j + z_1 i\bar{z}_2 j \\ &= z_1\bar{z}_2 k + |z_1|^2 i - |z_2|^2 i + z_1\bar{z}_2 k \\ &= (|z_1|^2 - |z_2|^2)i + 2z_1\bar{z}_2 k \end{aligned}$$

which coincides with the usual Hopf map

$$\begin{aligned} \eta &: \mathbb{H} \to \operatorname{Im}\mathbb{H} \\ \eta &= ((|z_1|^2 - |z_2|^2,\ 2z_1\bar{z}_2) : \mathbb{R}^4 \to \mathbb{R}^3 \end{aligned}$$

with $\eta(S^3) \in S^2$. Let

$$\zeta(x) = s(r)\eta\left(\frac{x}{r}\right) = s(r)r^{-2}\eta(x) := \tilde{s}(r)\eta(x)$$

with $r = |x|$. It was shown in ([11, Theorem 3.2, page 135]) that $x \mapsto (x, \zeta(x))$ defines a coassociative 4-submanifold of $\mathbb{R}^7$ invariant under S^3, provided

$$s(4s^2 - 5r^2)^2 = C, \qquad C \in \mathbb{R}.$$

Those are smooth area-minimizing submanifolds except in the case $C = 0$. When $C = 0$ then $s(r) = \frac{\sqrt{5}}{2}r$ and then $\tilde{s}(r) = \frac{\sqrt{5}}{2r}$, the function $\zeta(x) : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$\zeta = \frac{\sqrt{5}}{2r}\eta(x)$$

defines a cone over the entire $\mathbb{R}^4$. This was discovered by Lawson and Osserman [18]. This LO-cone shows that Moser's theorem that entire minimal graphs of bounded slope are affine linear cannot be extended to the case of dimension 4 and codimension 3. In this appendix, we compute some important geometric quantities of this remarkable example.

Put

$$x = (z_1, z_2) \in \mathbb{C}^2 = \mathbb{R}^4, \quad z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}, \quad |x|^2 = r^2 = r_1^2 + r_2^2.$$

The flat metric on $\mathbb{R}^4$ reads

$$ds^2 = dr_1^2 + r_1^2 d\theta_1^2 + dr_2^2 + r_2^2 d\theta_2^2.$$

The orthonormal basis on $T_x\mathbb{R}^4$ is given by $\{e_0, e_1, e_2, e_3\}$, where

$$e_0 = \frac{\partial}{\partial r} = \frac{r_1}{r}\frac{\partial}{\partial r_1} + \frac{r_2}{r}\frac{\partial}{\partial r_2}, \quad e_1 = \frac{r_2}{r}\frac{\partial}{\partial r_1} - \frac{r_1}{r}\frac{\partial}{\partial r_2}$$

$$e_2 = \frac{r_2}{r_1 r}\frac{\partial}{\partial \theta_1} - \frac{r_1}{r_2 r}\frac{\partial}{\partial \theta_2}, \quad e_3 = \frac{1}{r}\left(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}\right).$$

Now, we have

$$\eta(x) = \left(r_1^2 - r_2^2,\ 2r_1 r_2 e^{i(\theta_1-\theta_2)}\right) : \mathbb{R}^4 \to \mathbb{R}^3.$$

Since

$$\frac{\partial}{\partial r_1}\tilde{s}(r) = \tilde{s}'\frac{r_1}{r}, \quad \frac{\partial}{\partial r_2}\tilde{s}(r) = \tilde{s}'\frac{r_2}{r},$$

then

$$\zeta_* e_0 = \left(\frac{r_1}{r}\frac{\partial}{\partial r_1} + \frac{r_2}{r}\frac{\partial}{\partial r_2}\right)(\tilde{s}\eta) = \left(\tilde{s}' + \frac{2\tilde{s}}{r}\right)\eta,$$

$$\zeta_* e_1 = \frac{r_2}{r}\frac{\partial}{\partial r_1}(\tilde{s}\eta) - \frac{r_1}{r}\frac{\partial}{\partial r_2}(\tilde{s}\eta) = \frac{\tilde{s}}{r}\left(4r_1 r_2,\ 2(r_2^2 - r_1^2)e^{i(\theta_1-\theta_2)}\right).$$

Since

$$\zeta_*\frac{\partial}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}(\tilde{s}\eta) = \tilde{s}\left(0,\ 2ir_1 r_2 e^{i(\theta_1-\theta_2)}\right),$$

$$\zeta_*\frac{\partial}{\partial \theta_2} = \frac{\partial}{\partial \theta_2}(\tilde{s}\eta) = \tilde{s}\left(0,\ -2ir_1 r_2 e^{i(\theta_1-\theta_2)}\right),$$

then

$$\zeta_* e_2 = \frac{r_2}{r_1 r}\zeta_*\frac{\partial}{\partial \theta_1} - \frac{r_1}{r_2 r}\zeta_*\frac{\partial}{\partial \theta_2} = \tilde{s}r\left(0,\ 2ie^{i(\theta_1-\theta_2)}\right)$$

$$\zeta_* e_3 = 0.$$

Put

$$\rho_0^2 = \frac{1}{1+|\zeta_* e_0|^2} = \frac{1}{1+(r^2\tilde{s}' + 2r\tilde{s})^2}, \quad \rho_1^2 = \frac{1}{1+|\zeta_* e_1|^2} = \frac{1}{1+4r^2\tilde{s}^2}$$

$$\rho_2^2 = \frac{1}{1+|\zeta_* e_2|^2} = \frac{1}{1+4r^2\tilde{s}^2}, \qquad \rho_3^2 = \frac{1}{1+|\zeta_* e_3|^2)} = 1.$$

Then the Gauss map γ for the coassociate 4−submanifold is expressed by

$$(e_0, \zeta_* e_0) \wedge (e_1, \zeta_* e_1) \wedge (e_2, \zeta_* e_2) \wedge (e_3, \zeta_* e_3)$$

and the correspoinding W−matrix relative to $e_0 \wedge e_1 \wedge e_2 \wedge e_3$ is

$$W = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The nonzero Jordan angles are

$$\theta_0 = \arccos \rho_0,\ \theta_1 = \arccos \rho_1,\ \theta_2 = \arccos \rho_2.$$

In particular, the Jordan angles of the image under the Gauss map for the *LO-cone* can be obtained by substituting $\tilde{s} = \frac{\sqrt{5}}{2} r^{-1}$ in the above expressions. Those are the following constants

$$\theta_0 = \arccos \frac{2}{3}, \quad \theta_1 = \theta_2 = \arccos \frac{\sqrt{6}}{6}.$$

The $w-$function is identically $\frac{1}{9}$ and hence the $v-$function, denoted also by Δ_f for the graphic case, equals 9. This fact was originally verified by a computer program by Lawson-Osserman in [18]. The LO-cone defined by (f^1, f^2, f^3) on $\mathbb{R}^4$, where

$$\begin{aligned} f^1 &= \frac{\sqrt{5}}{2} \frac{(x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}}, \\ f^2 &= \frac{\sqrt{5}}{2} \frac{2(x^1 x^3 + x^2 x^4)}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}}, \\ f^3 &= \frac{\sqrt{5}}{2} \frac{2(x^1 x^3 - x^2 x^4)}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}}. \end{aligned}$$

At $x^3 \neq 0$ and $x^1 = x^2 = x^4 = 0$, we have $\frac{\partial f^2}{\partial x^1} = \sqrt{5}$ and $\left(1 + \left(\frac{\partial f^2}{\partial x^1}\right)^2\right)^{\frac{1}{2}} = \sqrt{6}$. Therefore, the Theorem 1.2 cannot hold with $\beta_1 = \frac{9}{\sqrt{6}} \approx 3.7$.

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