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On a singular perturbed problem in an annulus

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Abstract. In this paper we prove the conjecture due to Ruf–Srikanth [14]. We prove the existence of positive solution under Dirichlet and Neumann boundary conditions, which concentrate near the inner boundary and outer boundary of an annulus respectively as $\varepsilon \to 0$. In fact, our result is independent of the dimension of $\mathbb{R}^N$.

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1. Introduction

There has been a considerable interest in understanding the behavior of positive solutions of the elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 \quad \text{or } \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, f is a superlinear nonlinearity and Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $F(u) = \int_0^u f(t)dt$. We consider the problem when $f(0) = 0$ and $f'(0) = 0$. This type of equations arise in various mathematical models derived from population theory, chemical reactor theory see Gidas-Ni-Nirenberg [6]. In the Dirichlet case, Ni – Wei showed in [19] that the least energy solutions of equation (1.1) concentrate, for $\varepsilon \to 0$, to single peak solutions, whose maximum points P_ε converge to a point P with maximal distance from the boundary $\partial\Omega$. In the Neumann case, Ni–Takagi [17] showed that for sufficiently small $\varepsilon > 0$, the least energy solution is a single boundary spike and has only one local maximum $P_\varepsilon \in \partial\Omega$. Moreover, in [18], they prove that $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$ where $H(P)$ is the mean curvature of $\partial\Omega$ at P. A simplified proof was

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given by del Pino–Felmer in [3], for a wider class of nonlinearities using a method of symmetrization.

We mention some nice results on the multiple boundary and interior peaked solutions for the Neumann case of (1.1). For the single and multiple boundary spikes, Gui [8] constructed multiple boundary spike solutions at multiple local maximum points of $H(P)$, using a variational method. Wei [21] and Wei–Winter [22] constructed single and multiple boundary spike solutions at multiple non-degenerate critical points of $H(P)$, using the Lyapunov–Schmidt reduction method. Later on Y.Y. Li [10] and del Pino–Felmer–Wei [4] constructed single and multiple boundary spikes in the degenerate case. For a detailed bibliography on this topic, we refer to the review article by Ni [16].

Higher dimensional concentrating solutions was studied by Ambrosetti–Malchiodi–Ni in [1,2]; they consider solutions which concentrate on spheres, *i.e.* on $(N-1)$- dimensional manifolds. They studied the problem

$$\begin{cases} \varepsilon^2\Delta u - V(r)u + f(u) = 0 & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A, \end{cases} \tag{1.2}$$

in an annulus $A = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$, $V(r)$ is a smooth radial potential bounded below by a positive constant. They introduced a modified potential $M(r) = r^{N-1}V^{\theta}(r)$, with $\theta = \frac{p+1}{p-1} - \frac{1}{2}$, satisfying $M'(b) < 0$ (respectively $M'(a) > 0$), then there exists a family of radial solutions which concentrates on $|x| = r_\varepsilon$ with $r_\varepsilon \to b$ (respectively $r_\varepsilon \to a$) as $\varepsilon \to 0$. In fact, they conjectured that in $N \geq 3$ there could exist also solutions concentrating to some manifolds of dimension k with $1 \leq k \leq N-2$. Moreover, in $\mathbb{R}^2$, concentration of positive solutions on curves in the general case was proved by del Pino–Kowalczyk–Wei [5]. The Neumann case was studied by Malchiodi-Montenegro [11,12].

In Esposito *et al.* [7], the asymptotic behavior of radial solutions for the singularly perturbed elliptic problem (1.2) was studied using the Morse index information to describe the complete description of the blow-up behavior. As a result, they exhibit sufficient conditions which guarantee that radial ground state solutions blow-up and concentrate at the inner or outer boundary of the annulus. For more, interesting consequences, see Pacella-Srikanth [13] and Ruf-Srikanth [14,15].

In this paper, we consider the following two singular perturbed problems,

$$\begin{cases} \varepsilon^2\Delta u - u + u^p = 0 & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A, \end{cases} \tag{1.3}$$

$$\begin{cases} \varepsilon^2\Delta u - u + u^p = 0 & \text{in } A \\ u > 0 & \text{in } A \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial A, \end{cases} \tag{1.4}$$

where A is an annulus in $\mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^K$ with $A = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$, $\varepsilon > 0$ is a small number and ν denotes the unit normal to ∂A and $N \geq 2$. In this paper, we are interested in finding solution $u(x) = u(r, s)$ where $r = \sqrt{x_1^2 + x_2^2 + \cdots x_M^2}$ and $s = \sqrt{x_{M+1}^2 + x_{M+2}^2 + \cdots x_K^2}$.

Let us consider the conjecture due to Ruf and Srikanth:

Does there exist a solution for the problems (1.3) *and* (1.4), *which concentrates on* $\mathbb{R}^{M+K-1}$ *dimensional subsets as* $\varepsilon \to 0$?

Theorem 1.1. *For* $\varepsilon > 0$ *sufficiently small, there exists a solution of* (1.3) *which concentrates near the inner boundary of* A.

Theorem 1.2. *For* $\varepsilon > 0$ *sufficiently small, there exists a solution of* (1.4) *which concentrates near the outer boundary of* A.

2. Set up for the approximation

Note that, under symmetry assumptions, A can be reduced to a subset of $\mathbb{R}^2$ where $\mathcal{D} = \{(r, s) : r > 0, s > 0, a^2 < r^2 + s^2 < b^2\}$. Let $P_\varepsilon = (P_{1,\varepsilon}, P_{2,\varepsilon})$ be a point of maximum of u_ε in A, then $u_\varepsilon(P_\varepsilon) \geq 1$. From (1.3) we obtain

$$\varepsilon^2 u_{rr} + \varepsilon^2 u_{ss} + \varepsilon^2 \frac{(M-1)}{r} u_r + \varepsilon^2 \frac{(K-1)}{s} u_s - u + u^p = 0 \tag{2.1}$$

Let $\mathcal{D}_1, \mathcal{D}_2$ are the inner and outer boundary of $\mathcal{D}$ respectively and $\mathcal{D}_3, \mathcal{D}_4$ are the horizontal and vertical boundary of $\mathcal{D}$ respectively.

If $P = (P_1, P_2)$ be a point in $\mathcal{D}$ such that $\operatorname{dist}(P, \mathcal{D}_1) = d$, then we can express

$$P_1 = (a + d) \cos\theta;\ P_2 = (a + d) \sin\theta \tag{2.2}$$

where θ is the angle between the x-axis and the line joining P. Furthermore, if $\operatorname{dist}(P, \mathcal{D}_2) = d$, then we can express

$$P_1 = (b - d) \cos\theta;\ P_2 = (b - d) \sin\theta. \tag{2.3}$$

See Figure 2.1 and Figure 2.2.

The functional associated to the problem is

$$I_\varepsilon(u) = \int_{\mathcal{D}} r^{M-1} s^{K-1} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u^{p+1} \right) dr\, ds. \tag{2.4}$$

Moreover, (1.3) reduces to

$$\begin{cases} \varepsilon^2 u_{rr} + \varepsilon^2 u_{ss} + \varepsilon^2 \frac{(M-1)}{r} u_r + \varepsilon^2 \frac{(K-1)}{s} u_s - u + u^p = 0 & \text{in } \mathcal{D} \\ u = 0 & \text{on } \mathcal{D}_1 \cup \mathcal{D}_2 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathcal{D}_3 \cup \mathcal{D}_4. \end{cases}$$

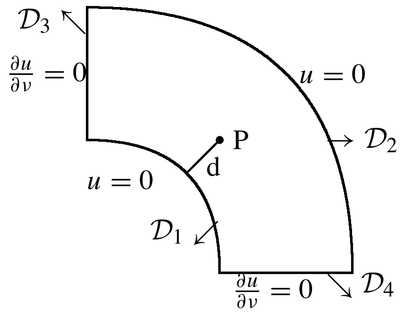

Figure 2.1. Dirichlet case

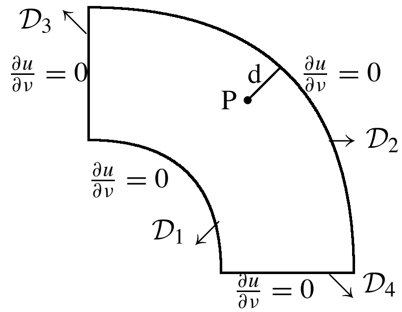

Figure 2.2. Neumann Case

Re-scaling about the point P, we obtain in A_ε

$$u_{rr} + u_{ss} + \varepsilon\frac{(M-1)}{P_1 + \varepsilon r}u_r + \varepsilon\frac{(K-1)}{P_2 + \varepsilon s}u_s - u + u^p = 0. \tag{2.5}$$

The entire solution associated to (2.1) where U satisfies

$$\begin{cases} \Delta_{(r,s)}U - U + U^p = 0 & \text{in } \mathbb{R}^2 \\ U(r,s) > 0 & \text{in } \mathbb{R}^2 \\ U(r,s) \to 0 & \text{as } |(r,s)| \to \infty. \end{cases} \tag{2.6}$$

Moreover U is non-degenerate, which means

$$\operatorname{Ker}\left[\Delta_{(r,s)} - 1 + pU^{p-1}\right] = \left\{\frac{\partial U}{\partial r}, \frac{\partial U}{\partial s}\right\}. \tag{2.7}$$

Let $z = (r,s)$. Moreover, $U(z) = U(|z|)$ and the asymptotic behavior of U at infinity is given by

$$\begin{cases} U(z) = A|z|^{-\frac{1}{2}}e^{-|z|}\left(1 + O\left(\frac{1}{|z|}\right)\right) \\ U'(z) = -A|z|^{-\frac{1}{2}}e^{-|z|}\left(1 + O\left(\frac{1}{|z|}\right)\right) \end{cases} \tag{2.8}$$

for some constant $A > 0$.

Let $K(z)$ denote the fundamental solution of $-\Delta_{(r,s)} + 1$ centered at 0. Then, for $|z| \geq 1$, we have

$$\begin{cases} U(z) = \left(B + O\left(\frac{1}{|z|}\right)\right)K(z) \\ U'(z) = \left(-B + O\left(\frac{1}{|z|}\right)\right)K(z) \end{cases} \tag{2.9}$$

for some positive constant B.

Let $U_{\varepsilon,P}(z) = U(|\frac{z-P}{\varepsilon}|)$. Now we construct the projection map for the Dirichlet case as

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} PU_{\varepsilon,P} - PU_{\varepsilon,P} + U^p_{\varepsilon,P} = 0 & \text{in } \mathcal{D} \\ PU_{\varepsilon,P}(r,s) > 0 & \text{in } \mathcal{D} \\ PU_{\varepsilon,P}(r,s) = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{2.10}$$

and the projection in the Neumann case as

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} QU_{\varepsilon,P} - QU_{\varepsilon P} + U^p_{\varepsilon,P} = 0 & \text{in } \mathcal{D} \\ QU_{\varepsilon,P}(r,s) > 0 & \text{in } \mathcal{D} \\ \dfrac{QU_{\varepsilon,P}}{\partial \nu}(r,s) = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{2.11}$$

If $v_\varepsilon = U_{\varepsilon,P} - PU_{\varepsilon,P}$ and $w_\varepsilon = U_{\varepsilon,P} - QU_{\varepsilon,P}$, then we have

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} v_\varepsilon - v_\varepsilon = 0 & \text{in } \mathcal{D} \\ v_\varepsilon = U_{\varepsilon,P} & \text{on } \partial\mathcal{D}, \end{cases} \tag{2.12}$$

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)} w_\varepsilon - w_\varepsilon = 0 & \text{in } \mathcal{D} \\ \dfrac{\partial w_\varepsilon}{\partial \nu} = \dfrac{\partial U_{\varepsilon,P}}{\partial \nu} & \text{on } \partial\mathcal{D}. \end{cases} \tag{2.13}$$

Consider the function $s(\theta) = \cos^{M-1}\theta \sin^{K-1}\theta$ in $[0, \frac{\pi}{2}]$. Then neither $\theta_0 = 0$ nor $\theta_0 = \frac{\pi}{2}$ are points of maxima of s. But $s > 0$ and hence θ_0 lies in $(0, \frac{\pi}{2})$.

Furthermore, consider the function $h(d) = d + e^{-\frac{d}{\varepsilon}}$ in $0 < d < 1$. Then h attains its minimum at a point $d = \varepsilon|\ln\varepsilon|$.

For any $\theta \in [\theta_0 - \delta, \theta_0 + \delta]$; we define the configuration space for the Dirichlet and Neumann case as

$$\Lambda_{\varepsilon,D} = \left\{ P \in \mathcal{D} : \text{dist}(P, \mathcal{D}_1) \geq \frac{k}{2}\varepsilon \ln \frac{1}{\varepsilon} \right\} \tag{2.14}$$

and

$$\Lambda_{\varepsilon,N} = \left\{ P \in \mathcal{D} : \text{dist}(P, \mathcal{D}_2) \geq \frac{k}{2}\varepsilon \ln \frac{1}{\varepsilon} \right\} \tag{2.15}$$

respectively for some $k > 0$ small.

We develop the following lemma similar to Lin, Ni and Wei [9].

Lemma 2.1. *Assuming that* $\frac{k}{2}\varepsilon|\ln\varepsilon| \leq d(P, \mathcal{D}_1) \leq \delta$, *then we obtain*

$$v_\varepsilon(z) = (B + o(1))K\left(\frac{|z - P^\star|}{\varepsilon}\right) + O(\varepsilon^{2+\sigma}), \tag{2.16}$$

where $P^\star = P + 2d(P, \mathcal{D}_1)\nu_{\overline{P}}$ *and* $\overline{P} \in \mathcal{D}_1$ *is a unique point such that* $d(P, \overline{P}) = 2d(P, \mathcal{D}_1)$, *while* σ *is a small positive number;* δ *is sufficiently small. Moreover,* $\nu_{\overline{P}}$ *is the outer unit normal at* $\overline{P}$.

Proof. Define

$$\begin{cases} \varepsilon^2\Delta_{(r,s)}\Psi_\varepsilon - \Psi_\varepsilon = 0 & \text{in } \mathcal{D} \\ \Psi_\varepsilon > 0 & \text{in } \mathcal{D} \\ \Psi_\varepsilon = 1 & \text{on } \partial\mathcal{D}. \end{cases} \tag{2.17}$$

Then for sufficiently small ε, Ψ_ε is uniformly bounded.

But for $z \in \partial\mathcal{D}$, we obtain

$$U_{\varepsilon,P}(z) = U\left(\frac{|z-P|}{\varepsilon}\right) = (A+o(1))\varepsilon^{\frac{1}{2}}|z-P|^{-\frac{1}{2}}e^{-\frac{|z-P|}{\varepsilon}}.$$

First, we have

$$U_{\varepsilon,P}(z) = (B+o(1))K\left(\frac{|z-P|}{\varepsilon}\right).$$

Hence by the comparison principle we obtain, for some $\sigma > 0$ small,

$$v_\varepsilon \le C\varepsilon^{2+\sigma}\Psi_\varepsilon \text{ whenever } d(P,\mathcal{D}_1) \ge 2\varepsilon|\ln\varepsilon|.$$

Therefore, it remains to check whether (2.16) holds in

$$\frac{k}{2}\varepsilon|\ln\varepsilon| \le d(P,\mathcal{D}_1) \le 2\varepsilon|\ln\varepsilon|. \tag{2.18}$$

Define the function

$$\phi_1(z) = (B-\varepsilon^{\frac{1}{4}})K\left(\frac{|z-P^\star|}{\varepsilon}\right) + \varepsilon^{2+\sigma}\Psi_\varepsilon. \tag{2.19}$$

Then ϕ_1 satisfies

$$\varepsilon^2\Delta_{(r,s)}\phi_1 - \phi_1 = 0. \tag{2.20}$$

For any z in $\mathcal{D}_1$ with $|z-P| \le \varepsilon^{\frac{3}{4}}$ we have

$$\frac{|z-P|}{\varepsilon} = \left(1+O\left(\varepsilon^{\frac{1}{2}}\right)|\ln\varepsilon|\right)\frac{|z-P^\star|}{\varepsilon} \tag{2.21}$$

and hence

$$v_\varepsilon \le \phi_1.$$

For any $z \in \mathcal{D}_1$ with $|z-P| \ge \varepsilon^{\frac{3}{4}}$ we have

$$v_\varepsilon(z) \le Ce^{-\varepsilon^{-\frac{1}{4}}} \le \varepsilon^{2+\sigma} \le \phi_1.$$

Summarizing, we obtain,

$$v_\varepsilon \le \phi_1 \text{ for all } z \in \mathcal{D}_1.$$

Similarly, we obtain the lower bound for $z \in \mathcal{D}_1$,

$$v_\varepsilon(z) \ge (B+\varepsilon^{\frac{1}{4}})K\left(\frac{|z-P^\star|}{\varepsilon}\right) - \varepsilon^{2+\sigma}\Psi_\varepsilon. \tag{2.22}$$

□

Corollary 2.2. *Assume that* $\frac{k}{2}\varepsilon|\ln\varepsilon| \le d(P, \mathcal{D}_2) \le \delta$ *where* δ *is sufficiently small. Then*

$$w_\varepsilon(z) = -(B + o(1))K\left(\frac{|z - P^\star|}{\varepsilon}\right) + O\left(\varepsilon^{2+\sigma}\right), \tag{2.23}$$

where $P^\star = P + 2d(P, \mathcal{D}_2)\nu_{\overline{P}}$ *and* $\overline{P} \in \mathcal{D}_2$ *is a unique point such that* $d(P, \overline{P}) = 2d(P, \mathcal{D}_2)$*, while* σ *is a small positive number. Moreover,* $\nu_{\overline{P}}$ *is the outer unit normal at* $\overline{P}$*.*

3. Refinement of the projection

Define

$$H_0^1(\mathcal{D}) = \left\{u \in H^1 : u(x) = u(r,s), u = 0 \text{ in } \mathcal{D}_1 \text{ and } \mathcal{D}_2; \frac{\partial u}{\partial \nu} = 0 \text{ in } \mathcal{D}_3 \text{ and } \mathcal{D}_4\right\}.$$

Define a norm on $H_0^1(\mathcal{D})$ as

$$\|v\|_\varepsilon^2 = \int_{\mathcal{D}} r^{M-1} r^{K-1}\left[\varepsilon^2 |\nabla v|^2\, dx + v^2\right] drds. \tag{3.1}$$

In this section we will refine the projection to incorporate the Neumann boundary condition on $\mathcal{D}_3$ and $\mathcal{D}_4$. We define a new projection as

$$V_{\varepsilon,P} = \eta P U_{\varepsilon,P}, \tag{3.2}$$

where $0 \le \eta \le 1$ is smooth cut off function

$$\eta(x) = \begin{cases} 1 & \text{in } \mathcal{D} \cap B_{d/2}(P) \\ 0 & \text{in } \mathcal{D} \setminus B_d(P). \end{cases} \tag{3.3}$$

Here $d = \text{dist}(P, \partial\mathcal{D})$ is dependent on ε. We will choose d at the end of the proof.

We define

$$u_\varepsilon = V_{\varepsilon,P} + \varphi_{\varepsilon,P}. \tag{3.4}$$

Let $\varphi_{\varepsilon,P} = \varphi$. Using this ansatz, (1.3) reduces to

$$\begin{cases} \varepsilon^2 \Delta_{(r,s)}\varphi - \varphi + \varepsilon^2 \frac{(M-1)}{r}\varphi_r + \varepsilon^2 \frac{(K-1)}{s}\varphi_r + f'(V_{\varepsilon,P})\varphi = h & \text{in } \mathcal{D} \\ \varphi = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where $h = -S_\varepsilon[V_{\varepsilon,P_\varepsilon}] + N_\varepsilon[\varphi]$, while

$$\begin{aligned} S_\varepsilon[V_{\varepsilon,P}] = {} & \varepsilon^2 \Delta_{(r,s)} V_{\varepsilon,P} + \varepsilon^2 \frac{(M-1)}{r} V_{\varepsilon,P,r} + \varepsilon^2 \frac{(K-1)}{s} V_{\varepsilon,P,s} \\ & - V_{\varepsilon,P} + f(V_{\varepsilon,P}) \end{aligned} \tag{3.5}$$

and

$$N_\varepsilon[\varphi_\varepsilon] = \{f(V_{\varepsilon,P_\varepsilon} + \varphi) - f(V_{\varepsilon,}) - f'(V_{\varepsilon,P_\varepsilon})\varphi\}.$$

Let

$$E_{\varepsilon,P} = \left\{\omega \in H_0^1(\mathcal{D}), \left\langle \omega, \frac{\partial V_{\varepsilon,P}}{\partial r}\right\rangle_\varepsilon = \left\langle \omega, \frac{\partial V_{\varepsilon,P}}{\partial s}\right\rangle_\varepsilon = 0\right\}.$$

Let $\mathcal{D}_\varepsilon = \{z : \varepsilon z + P \in \mathcal{D}\}$.

Lemma 3.1. *For any $z \in \mathcal{D}_\varepsilon \setminus B_{d/2\varepsilon}(P)$ we have*

$$PU_{\varepsilon,P}(z) = O(\varepsilon^k). \tag{3.6}$$

Proof. For any $z \in \mathcal{D}_\varepsilon \setminus B_{d/2\varepsilon}(P)$ we have

$$\begin{aligned} PU_{\varepsilon,P}(z) &\le \left| U\left(|z - \frac{P}{\varepsilon}|\right) - v_{\varepsilon,P}(\varepsilon z)\right| \\ &= O(e^{-|x-\frac{P}{\varepsilon}|} + e^{-|x-\frac{P^\star}{\varepsilon}|} + \varepsilon^{2+\sigma}) \\ &= O(e^{-\frac{d(P,P^\star)}{\varepsilon}} + \varepsilon^{2+\sigma}) \\ &= O(e^{-\frac{2d(P,\partial\mathcal{D}_1)}{\varepsilon}} + \varepsilon^{2+\sigma}) = O(\varepsilon^k). \end{aligned} \tag{3.7}$$

□

Lemma 3.2. *Let $P \in \Lambda_{\varepsilon,D}$. Then the energy expansion is given by*

$$\begin{aligned} I_\varepsilon(V_{\varepsilon,P}) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}\left(\frac{\varepsilon^2}{2}|\nabla V_{\varepsilon,P}|^2 + \frac{1}{2}V_{\varepsilon,P}^2 - \frac{1}{p+1}V_{\varepsilon,P}^{p+1}\right)drds \\ &= \gamma\varepsilon^2 P_1^{M-1}P_2^{K-1} + \gamma_1\varepsilon^2 P_1^{M-1}P_2^{K-1}U\left(\frac{2d(P,\partial\mathcal{D}_1)}{\varepsilon}\right) + o(\varepsilon^2)U\Big(k|\ln\varepsilon|\Big) \end{aligned}$$

where $\gamma = \frac{p-1}{2(p+1)}\int_{\mathbb{R}^2} U^{p+1}drds$ and $\gamma_1 = \frac{1}{2}\int_{\mathbb{R}^2} U^p e^{-r}drds$.

Proof. We obtain

$$\begin{aligned} I_\varepsilon(V_{\varepsilon,P}) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}\left(\frac{\varepsilon^2}{2}|\nabla V_{\varepsilon,P}|^2 + \frac{1}{2}V_{\varepsilon,P}^2 - \frac{1}{p+1}V_{\varepsilon,P}^{p+1}\right)drds \\ &= \int_{\mathcal{D}} \eta^2 r^{M-1}s^{K-1}\left(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2 + \frac{1}{2}PU_{\varepsilon,P}^2 - \frac{1}{p+1}PU_{\varepsilon,P}^{p+1}\right)drds \\ &\quad + \frac{1}{p+1}\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big(\eta^2 - \eta^{p+1}\Big)PU_{\varepsilon,P}^{p+1}drds \\ &\quad + \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}\eta\nabla\eta PU_\varepsilon\nabla PU_\varepsilon drds \\ &\quad + \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}|\nabla\eta|^2(PU_{\varepsilon,P})^2drds \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.8}$$

Hence we have

$$
\begin{aligned}
J_1 = &\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2 + \frac{1}{2}PU_{\varepsilon,P}^2 - \frac{1}{p+1}PU_{\varepsilon,P}^{p+1}\Big)drds \\
&- \int_{\mathcal{D}}(1-\eta^2)r^{M-1}s^{K-1}\Big(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2 + \frac{1}{2}PU_{\varepsilon,P}^2 - \frac{1}{p+1}PU_{\varepsilon,P}^{p+1}\Big)drds \\
= &\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big(\frac{1}{2}U_{\varepsilon,P}^{p}PU_{\varepsilon,P} - \frac{1}{p+1}PU_{\varepsilon,P}^{p+1}\Big)drds \\
&+ \varepsilon^2\int_{\partial B_d(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r} + \frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds \\
&- \varepsilon^2\int_{\partial B_{d/2}(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r} + \frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds \\
= &\,\varepsilon^2\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathcal{D}_\varepsilon}(P_1+\varepsilon r)^{M-1}(P_2+\varepsilon s)^{K-1}U^{p+1}(z)drds \\
&+ \frac{1}{2}\int_{\mathcal{D}} U_{\varepsilon,P}^{p}v_\varepsilon r^{M-1}s^{K-1}drds + O\left(\int_{\mathcal{D}} U_{\varepsilon,P}^{p-1}v_\varepsilon^2 r^{M-1}s^{K-1}drds\right) \\
&+ \varepsilon^2\int_{\partial B_d(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r} + \frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds \\
&- \varepsilon^2\int_{\partial B_{d/2}(P)} r^{M-1}s^{K-1}\Big(\frac{\partial PU_{\varepsilon,P}}{\partial r} + \frac{\partial PU_{\varepsilon,P}}{\partial s}\Big)PU_{\varepsilon,P}drds \\
&+ \int_{\mathcal{D}\setminus B_{d/2}} r^{M-1}s^{K-1}\Big(\frac{\varepsilon^2}{2}|\nabla PU_{\varepsilon,P}|^2 + \frac{1}{2}PU_{\varepsilon,P}^2 - \frac{1}{p+1}PU_{\varepsilon,P}^{p+1}\Big)drds.
\end{aligned}
\tag{3.9}
$$

Now we estimate

$$
\begin{aligned}
&\varepsilon^2\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathcal{D}_\varepsilon}(P_1+\varepsilon r)^{M-1}(P_2+\varepsilon s)^{K-1}U^{p+1}(z)drds \\
&\quad = \frac{p-1}{2(p+1)}\varepsilon^2 P_1^{M-1}P_2^{K-1}\int_{\mathbb{R}^2} U^{p+1}(r,s)drds \\
&\qquad + o(\varepsilon^2)U(k|\ln\varepsilon|).
\end{aligned}
\tag{3.10}
$$

From Lemma 2.1, we compute the interaction term

$$
\begin{aligned}
&\int_{\mathcal{D}} U_{\varepsilon,P}^{p} v_{\varepsilon} r^{M-1} s^{K-1} dr ds \\
&= \varepsilon^2 \int_{\mathcal{D}_\varepsilon} U^p U\left(\left|z - \frac{P-P^\star}{\varepsilon}\right|\right)(P_1+\varepsilon r)^{M-1}(P_2+\varepsilon s)^{K-1} dr ds \\
&= \varepsilon^2 P_1^{M-1} P_2^{K-1} U\left(\left|\frac{P-P^\star}{\varepsilon}\right|\right)(\gamma_1 + o(1)) + O(\varepsilon^{2+\sigma}) \qquad (3.11)\\
&= \varepsilon^2 P_1^{M-1} P_2^{K-1} U\left(\frac{2d(P,\partial\mathcal{D}_1)}{\varepsilon}\right)(\gamma_1 + o(1)) + O(\varepsilon^{2+\sigma}) \\
&= \varepsilon^2 P_1^{M-1} P_2^{K-1} U\left(\frac{2d(P,\partial\mathcal{D}_1)}{\varepsilon}\right)(\gamma_1 + o(1)) + o(\varepsilon^2) U(k|\ln\varepsilon|).
\end{aligned}
$$

Note that we have used the fact that $\frac{|P-P^\star|}{\varepsilon} \gg |z|$. Moreover, we obtain

$$
J_2 = \int_{\mathcal{D}} r^{M-1} s^{K-1} \left(\eta^2 - \eta^{p+1}\right) PU_{\varepsilon,P}^{p+1} dr ds = o(\varepsilon^2) U\left(k|\ln\varepsilon|\right).
$$

Furthermore, we have

$$
\varepsilon^2 \int_{\partial B_d(P)} r^{M-1} s^{K-1} \left(\frac{\partial PU_{\varepsilon,P}}{\partial r} + \frac{\partial PU_{\varepsilon,P}}{\partial s}\right) PU_{\varepsilon,P} dr ds = o(\varepsilon^2) U\left(k|\ln\varepsilon|\right),
$$

$$
\varepsilon^2 \int_{\partial B_{d/2}(P)} r^{M-1} s^{K-1} \left(\frac{\partial PU_{\varepsilon,P}}{\partial r} + \frac{\partial PU_{\varepsilon,P}}{\partial s}\right) PU_{\varepsilon,P} dr ds = o(\varepsilon^2) U\left(k|\ln\varepsilon|\right),
$$

$$
J_3 = \varepsilon^2 \int_{\mathcal{D}} r^{M-1} s^{K-1} \eta \nabla\eta PU_\varepsilon \nabla PU_\varepsilon dr ds = o(\varepsilon^2) U\left(k|\ln\varepsilon|\right),
$$

and

$$
J_4 = \varepsilon^2 \int_{\mathcal{D}} r^{M-1} s^{K-1} |\nabla\eta|^2 (PU_{\varepsilon,P})^2 dr ds = o(\varepsilon^2) U\Big(k|\ln\varepsilon|\Big).
$$

Hence we obtain the result. □

Remark 3.3. From lemma 3.2 we have

$$
\begin{aligned}
I_\varepsilon(V_{\varepsilon,P}) &= \int_{\mathcal{D}} r^{M-1} s^{K-1} \left(\frac{\varepsilon^2}{2}|\nabla V_{\varepsilon,P}|^2 + \frac{1}{2} V_{\varepsilon,P}^2 - \frac{1}{p+1} V_{\varepsilon,P}^{p+1}\right) dr ds \\
&= \gamma \varepsilon^2 P_1^{M-1} P_2^{K-1} + \gamma_1 \varepsilon^2 P_1^{M-1} P_2^{K-1} U\left(\frac{2d}{\varepsilon}\right) + o(\varepsilon^2) U\Big(k|\ln\varepsilon|\Big).
\end{aligned}
$$

So if we expand the expression in d and θ we have

$$\varepsilon^{-2}I_\varepsilon(V_{\varepsilon,P})=\left[\gamma a^{M+K-2}+\gamma a^{M+K-3}d+\gamma_1 a^{M+K-2}U\left(\frac{2d}{\varepsilon}\right)\right]\cos^{M-1}\theta\sin^{K-1}\theta$$
$$+o(\varepsilon^2)U\Big(k|\ln\varepsilon|\Big)+O(\varepsilon^2d^2).$$

Note that the right-hand side is a function of d and θ only and achieves its minimum in d at a point $d\sim\varepsilon|\ln\varepsilon|$ provided $\cos^{M-1}\theta\sin^{K-1}\theta\neq 0$. This is the main reason of choosing the configuration space (2.14).

4. The reduction

In this section we will reduce the proof of Theorem 1.1 to finding a solution of the form $u_\varepsilon = V_{\varepsilon,P}+\varphi$ for (1.3) to a finite dimensional problem. We will prove that for each $P\in\Lambda_{\varepsilon,D}$, there is a unique $\varphi\in E_{\varepsilon,P}$ such that

$$\left\langle I'_\varepsilon\Big(V_{\varepsilon,P}+\varphi\Big),\eta\right\rangle_\varepsilon=0\ \forall\eta\in E_{\varepsilon,P}.$$

Let

$$J_\varepsilon(\varphi)=I_\varepsilon\Big(V_{\varepsilon,P}+\varphi\Big).$$

We expand $J_\varepsilon(\varphi)$ near $\varphi_{\varepsilon,P}=0$ as

$$J_\varepsilon(\varphi)=J_\varepsilon(0)+l_{\varepsilon,P}(\varphi)+\frac{1}{2}Q_{\varepsilon,P}(\varphi,\varphi)+R_\varepsilon(\varphi)$$

where

$$\begin{aligned} l_{\varepsilon,P}(\varphi)&=\int_{\mathcal{D}}r^{M-1}s^{K-1}\left[\varepsilon^2\nabla V_{\varepsilon,P}\nabla\varphi+V_{\varepsilon,P}\varphi-V^p_{\varepsilon,P}\varphi\right]drds\\ &=\int_{\mathcal{D}}r^{M-1}s^{K-1}S_\varepsilon[V_{\varepsilon,P}]\varphi drds, \end{aligned}\tag{4.1}$$

$$Q_{\varepsilon,P}(\varphi,\psi)=\int_{\mathcal{D}}r^{M-1}s^{K-1}\left[\varepsilon^2\nabla\varphi\nabla\psi+\varphi\psi-pV^{p-1}_{\varepsilon,P}\varphi\psi\right]drds,\tag{4.2}$$

and

$$\begin{aligned} R_\varepsilon(\varphi)=\frac{1}{p+1}\int_{\mathcal{D}}r^{M-1}s^{K-1}\Bigg[\Big(V_{\varepsilon,P}+\varphi\Big)^{p+1}-\Big(V_{\varepsilon,P}\Big)^{p+1}\\ -(p+1)\Big(V_{\varepsilon,P}\Big)^p\varphi-\frac{p(p+1)}{2}\Big(V_{\varepsilon,P}\Big)^{p-1}\varphi^2\Bigg]drds. \end{aligned}\tag{4.3}$$

We will prove in Lemma 4.1 that $l_{\varepsilon,P}(\varphi)$ is a bounded linear functional in $E_{\varepsilon,P}$. Hence by the Riesz representation theorem, there exists $l_{\varepsilon,P}\in E_{\varepsilon,P}$ such that

$$\langle l_{\varepsilon,P},\varphi\rangle_\varepsilon=l_{\varepsilon,P}(\varphi)\ \forall\varphi\in E_{\varepsilon,P}.$$

In Lemma 4.2 we will prove that $Q_{\varepsilon,P}(\varphi,\eta)$ is a bounded linear operator from $E_{\varepsilon,P}$ to $E_{\varepsilon,P}$ such that

$$\langle Q_{\varepsilon,P}\varphi,\eta\rangle_\varepsilon = Q_{\varepsilon,P}(\varphi,\eta)\ \forall\,\varphi,\eta\in E_{\varepsilon,P}.$$

Thus finding a critical point of $J_\varepsilon(\varphi)$ is equivalent to solving the problem in $E_{\varepsilon,P}$:

$$l_{\varepsilon,P} + Q_{\varepsilon,P}\varphi + R'_\varepsilon(\varphi) = 0. \tag{4.4}$$

We will prove in Lemma 4.3 that the operator $Q_{\varepsilon,P}$ is invertible in $E_{\varepsilon,P}$. In Lemma 4.5, we will prove that, if φ belongs to a suitable set, $R'_\varepsilon(\varphi)$ is a small perturbation term in (4.4). Thus we can use the contraction mapping theorem to prove that (4.4) has a unique solution for each fixed $P\in\Lambda_{\varepsilon,D}$.

Lemma 4.1. *The functional* $l_{\varepsilon,P}: H_0^1(\mathcal{D})\to\mathbb{R}$ *defined in* (4.1) *is a bounded linear functional. Moreover, we have*

$$\|l_{\varepsilon,P}\|_\varepsilon = o(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}.$$

Proof. We have

$$\begin{aligned}
l_{\varepsilon,P}(\varphi) &= \int_{\mathcal{D}} r^{M-1}s^{K-1}S_\varepsilon[V_{\varepsilon,P}]\varphi\,dr\,ds\\
&= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2\Delta_{(r,s)}V_{\varepsilon,P} + \varepsilon^2\frac{(M-1)}{r}V_{\varepsilon,P,r} + \varepsilon^2\frac{(K-1)}{s}V_{\varepsilon,P,s}\\
&\qquad -V_{\varepsilon,P} + f(V_{\varepsilon,P})\Big]\varphi\\
&= \int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\varepsilon^2\Delta_{(r,s)}\eta PU_{\varepsilon,P} + \varepsilon^2\frac{(M-1)}{r}(\eta PU_{\varepsilon,P})_r\\
&\qquad + \varepsilon^2\frac{(K-1)}{s}(\eta PU_{\varepsilon,P})_s - \eta PU_{\varepsilon,P} + f(\eta PU_{\varepsilon,P})\Big]\varphi\\
&= \int_{\mathcal{D}} \eta r^{M-1}s^{K-1}\Big[\varepsilon^2\Delta_{(r,s)}PU_{\varepsilon,P}\\
&\qquad +\varepsilon^2\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \varepsilon^2\frac{(K-1)}{s}PU_{\varepsilon,P,s} - PU_{\varepsilon,P} + f(PU_{\varepsilon,P})\Big]\varphi\\
&\qquad + \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}[PU_{\varepsilon,P}\Delta_{(r,s)}\eta + \nabla PU_{\varepsilon,P}\nabla\eta]\varphi\\
&\qquad + \int_{\mathcal{D}} r^{M-1}s^{K-1}(\eta-\eta^p)PU^p_{\varepsilon,P}\varphi\\
&\qquad +\varepsilon^2\int_{\mathcal{D}} \eta r^{M-1}s^{K-1}\Big[\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \frac{(K-1)}{s}PU_{\varepsilon,P,s}\Big]\varphi\\
&\qquad + \varepsilon^2\int_{\mathcal{D}} r^{M-1}s^{K-1}\Big[\eta_r\frac{(M-1)}{r}PU_{\varepsilon,P,r} + \eta_s\frac{(K-1)}{s}PU_{\varepsilon,P}\Big]\varphi
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{D}} \eta r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi \\
&+\varepsilon^2 \int_{\mathcal{D}} \eta r^{M-1} s^{K-1}\Big[\frac{(M-1)}{r} PU_{\varepsilon,P,r} + \frac{(K-1)}{s} PU_{\varepsilon,P,s}\Big]\varphi dr ds \\
&+\varepsilon^2 \int_{\mathcal{D}} r^{M-1} s^{K-1}\Big[\eta_r \frac{(M-1)}{r} PU_{\varepsilon,P,r} + \eta_s \frac{(K-1)}{s} PU_{\varepsilon,P}\Big]\varphi \\
&+ \int_{\mathcal{D}} r^{M-1} s^{K-1} (\eta - \eta^p) PU^p_{\varepsilon,P}\varphi dr ds \\
&+\varepsilon^2 \int_{\mathcal{D}} r^{M-1} s^{K-1} [PU_{\varepsilon,P}\Delta_{(r,s)}\eta + \nabla PU_{\varepsilon,P}\nabla\eta]\varphi dr ds.
\end{aligned}
$$

Let

$$
\begin{aligned}
I_1 &= \int_{\mathcal{D}} \eta r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi dx \\
&= \int_{B_{d/2}(P)} r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi \\
&\quad + \int_{B_d \setminus B_{d/2}(P)} r^{M-1} s^{K-1}\Big[f(PU_{\varepsilon,P}) - f(U_{\varepsilon,P})\Big]\varphi.
\end{aligned}
$$

Then using the decay estimates in (2.16), we obtain

$$
\begin{aligned}
I_1 &\le C\int_{B_d} \Big(U_{\varepsilon,P}\Big)^{p-1} v_\varepsilon \varphi dx \\
&\le C\varepsilon \sqrt{U\Big(\frac{P-P^\star}{\varepsilon}\Big)}\Big(\int_{\mathcal{D}} |\varphi|^2 r^{M-1} s^{k-1} dr ds\Big)^{\frac{1}{2}} \\
&= o(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}\|\varphi\|_\varepsilon.
\end{aligned}
$$

Also it is easy to check that, all the other terms are of $o(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}\|\varphi\|_\varepsilon$. Hence we obtain

$$
|l_{\varepsilon,P}(\varphi)| = o(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}\|\varphi\|_\varepsilon
$$

and as a result

$$
\|l_{\varepsilon,P}\|_\varepsilon = o(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}. \qquad \square
$$

Lemma 4.2. *The bilinear form $Q_{\varepsilon,P}(\varphi,\eta)$ defined in* (4.2) *is a bounded linear. Furthermore,*

$$
|Q_{\varepsilon,P}(\varphi,\eta)| \le C\|\varphi\|_\varepsilon \|\eta\|_\varepsilon
$$

where C is independent of ε.

Proof. Using the Hölder's inequality, there exists $C > 0$ such that

$$\int_{\mathcal{D}} r^{M-1} s^{K-1} V_{\varepsilon,P}^{p-1} \varphi\eta \, dr ds \leq C \int_{\mathcal{D}} r^{M-1} s^{K-1} |\varphi||\eta| \leq C\|\varphi\|_\varepsilon \|\eta\|_\varepsilon$$

and

$$\left| \int_{\mathcal{D}} r^{M-1} s^{K-1} [\varepsilon^2 \nabla\varphi\nabla\eta + \varphi\eta] dr ds \right| \leq C\|\varphi\|_\varepsilon \|\eta\|_\varepsilon. \qquad \square$$

Lemma 4.3. *There exists $\rho > 0$ independent of ε, such that*

$$\|Q_{\varepsilon,P}\varphi\|_\varepsilon \geq \rho\|\varphi\|_\varepsilon \ \forall\varphi \in E_{\varepsilon,P}, \ P \in \Lambda_{\varepsilon,P}.$$

Proof. Suppose there exists a sequence $\varepsilon_n \to 0$, $\varphi_n \in E_{\varepsilon_n,P}$, $P \in \Lambda_{\varepsilon,D}$ such that $\|\varphi_n\|_{\varepsilon_n} = \varepsilon_n$ and

$$\|Q_{\varepsilon_n}\varphi_n\|_{\varepsilon_n} = o(\varepsilon_n).$$

Let $\tilde{\varphi}_n(z) = \varphi_n(\varepsilon_n z + P)$ and $\mathcal{D}_n = \{y : \varepsilon_n z + P \in \mathcal{D}\}$ such that

$$\int_{\mathcal{D}_n} r^{M-1} s^{K-1} [|\nabla\tilde{\varphi}_n|^2 + \tilde{\varphi}_n^2] = \varepsilon_n^{-2} \int_{\mathcal{D}} r^{M-1} s^{K-1} [\varepsilon_n^2 |\nabla\varphi_n|^2 + \varphi_n^2] = 1. \quad (4.5)$$

Hence there exists $\varphi \in H^1(\mathbb{R}^2)$ such that $\tilde{\varphi}_n \rightharpoonup \varphi \in H^1(\mathbb{R}^2)$ and hence $\tilde{\varphi}_n \to \varphi \in L^2_{\text{loc}}(\mathbb{R}^2)$. We claim that

$$\Delta_{(r,s)}\varphi - \varphi + pU^{p-1}\varphi = 0 \ \text{ in } \mathbb{R}^2$$

that is, for all $\zeta \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} r^{M-1} s^{K-1} \nabla\varphi\nabla\zeta + \int_{\mathbb{R}^2} r^{M-1} s^{K-1} \varphi\zeta = p \int_{\mathbb{R}^2} r^{M-1} s^{K-1} U^{p-1} \varphi\zeta. \quad (4.6)$$

Now

$$\begin{aligned} \int_{\mathcal{D}} r^{M-1} s^{K-1} \left[\varepsilon^2 D\varphi_n D\zeta + \varphi_n\zeta - pV_{\varepsilon,P}^{p-1} \varphi_n\zeta \right] &= \langle Q_{\varepsilon_n,P}\varphi_n, \zeta\rangle_\varepsilon \\ &= o(\varepsilon_n)\|\zeta\|_{\varepsilon_n} \end{aligned}$$

which implies

$$\int_{\mathcal{D}_\varepsilon} r^{M-1} s^{K-1} \left[\nabla\tilde{\varphi}_n \nabla\tilde{\zeta} + \tilde{\varphi}_n\tilde{\zeta} - p\tilde{V}_{\varepsilon,P}^{p-1} \tilde{\varphi}_n\tilde{\zeta} \right] = o(1)\|\tilde{\zeta}\|,$$

where

$$
\begin{aligned}
\tilde{V}_{\varepsilon_n,P}(z) &= V_{\varepsilon_n,P_n}(\varepsilon_n z + P),\\
\tilde{PU}_{\varepsilon_n,P}(z) &= PU_{\varepsilon_n,P_n}(\varepsilon_n z + P),\\
\|\tilde{\zeta}\|^2 &= \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\Big[|\nabla\tilde{\zeta}|^2 + |\tilde{\zeta}|^2\Big],\\
\tilde{E}_{\varepsilon_n,P} &= \Big\{\tilde{\zeta} : \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\nabla\tilde{\zeta}\nabla\tilde{W}_{n,r} + r^{M-1}s^{K-1}\tilde{\zeta}\tilde{W}_{n,r}\\
&\qquad = 0 = \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\nabla\tilde{\zeta}\nabla\tilde{W}_{n,s} + r^{M-1}s^{K-1}\tilde{\zeta}\tilde{W}_{n,s}\Big\},
\end{aligned}
$$

and $\tilde{W}_{n,r} = \varepsilon_n \frac{\partial V_{\varepsilon_n,P_n}(\varepsilon_n y+P_n)}{\partial r}$, $\tilde{W}_{n,s} = \varepsilon_n \frac{\partial V_{\varepsilon_n,P_n}(\varepsilon_n y+P_n)}{\partial s}$. Let $\zeta \in C_0^\infty(\mathbb{R}^2)$. Then we can choose $a_1, a_2 \in \mathbb{R}$ such that

$$
\tilde{\zeta}_n = \zeta - [a_{1,n}\tilde{W}_{n,r} + a_{2,n}\tilde{W}_{n,s}].
$$

Note that $\tilde{W}_{n,r}$ satisfies the problem

$$
\begin{cases} -\Delta_{(r,s)}\tilde{W}_{n,r} + \tilde{W}_{n,r} = p\eta U^{p-1}(y)\frac{\partial U}{\partial r} + \Phi_n(y) & \text{in } \mathcal{D}_n\\ \tilde{W}_{n,r} = 0 & \text{on } \partial\mathcal{D}_n \end{cases} \tag{4.7}
$$

where $\Phi_n(y) = \varepsilon_n \frac{\partial\eta}{\partial r}(U^p - \tilde{P}U_{\varepsilon,P}) + \varepsilon_n\frac{\partial}{\partial r}\big[2\nabla_{(r,s)}\eta\nabla\tilde{P}U_{\varepsilon,P} + \Delta_{(r,s)}\eta\tilde{P}U_{\varepsilon,P}\big]$.

Then we claim that $\tilde{W}_{n,r}$ is bounded in $H_0^1(\mathcal{D}_n)$. Using the Hölder's inequality, we have

$$
\begin{aligned}
\int_{\mathcal{D}_n} r^{M-1}s^{N-1}[\nabla\tilde{W}_{n,r}|^2 + \tilde{W}_{n,r}^2] &= p\int_{\mathcal{D}_n} r^{M-1}s^{N-1}\eta U^{p-1}\frac{\partial U}{\partial r}\tilde{W}_{n,r}\\
&\quad + \int_{\mathcal{D}_n} r^{M-1}s^{N-1}\Phi_n\tilde{W}_{n,r}\\
&\le C\Big(\int_{\mathcal{D}_n} r^{M-1}s^{k-1}\tilde{W}_{n,r}^2\Big)^{\frac{1}{2}}\\
&\le C\Big(\int_{\mathcal{D}_n} r^{M-1}s^{N-1}[\nabla\tilde{W}_{n,r}|^2 + \tilde{W}_{n,r}^2]\Big)^{\frac{1}{2}}.
\end{aligned} \tag{4.8}
$$

Hence $\int_{\mathcal{D}_n} r^{M-1}s^{N-1}\big[|\nabla\tilde{W}_{n,r}|^2 + \tilde{W}_{n,r}^2\big]$ is uniformly bounded and as a result there exists W_r such that

$$
\tilde{W}_{n,r} \rightharpoonup W_r \ \text{ in } H^1(\mathbb{R}^2)
$$

up to a subsequence. Hence

$$
\tilde{W}_{n,r} \to W_r \ \text{ in } L^2_{\text{loc}}.
$$

Note that W_r satisfies the problem,

$$\begin{cases} -\Delta_{(r,s)} W_r + W_r = pU^{p-1}\dfrac{\partial U}{\partial r} \qquad \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} r^{M-1}s^{K-1}[|\nabla W_r|^2 + |W_r|^2] = p\int_{\mathbb{R}^2} r^{M-1}s^{K-1}U^{p-1}\frac{\partial U}{\partial r}W_r. \end{cases} \tag{4.9}$$

We claim that $\tilde{W}_{n,r} \to W_r$ in $H^1(\mathbb{R}^2)$. First note that

$$\begin{aligned} \int_{\mathcal{D}_n} r^{M-1}s^{K-1}[|\nabla \tilde{W}_{n,r}|^2 + |\tilde{W}_{n,r}|^2] &= p\int_{\mathcal{D}_n} r^{M-1}s^{K-1}U^{p-1}\frac{\partial U}{\partial r}\tilde{W}_{n,r} \\ &\quad + \int_{\mathcal{D}_n} r^{M-1}s^{K-1}\Phi_n\tilde{W}_{n,r} \\ &\quad \to p\int_{\mathbb{R}^2} r^{M-1}s^{K-1}U^{p-1}\frac{\partial U}{\partial r}W_r \\ &= \int_{\mathbb{R}^2} r^{M-1}s^{K-1}[|\nabla W_r|^2 + |W_r|^2]\,dr\,ds. \end{aligned} \tag{4.10}$$

Here we have used that $\tilde{W}_{n,r}$ converges weakly in L^2. Hence $\tilde{W}_{n,r} \to W_r = \frac{\partial U}{\partial r}$ in H^1 strongly. Similarly, we can show that $\tilde{W}_{n,s} \to W_s = \frac{\partial U}{\partial s}$ in H^1 strongly. Now if we plug the value $\tilde{\zeta}_n$ in (4.7) and let $n \to \infty$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\zeta - pU^{p-1}\varphi\zeta + \varphi\zeta\Big] \\ &= a_1\Big(\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\frac{\partial U}{\partial r} + \varphi\frac{\partial U}{\partial r} - pU^{p-1}\varphi\frac{\partial U}{\partial r}\Big]\Big) \\ &\quad + a_2\Big(\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\frac{\partial U}{\partial s} + \varphi\frac{\partial U}{\partial s} - pU^{p-1}\varphi\frac{\partial U}{\partial s}\Big]\Big) \end{aligned}$$

where $a_i = \lim_{n\to\infty} a_{i,n}$.

Using the non-degeneracy condition (2.7) we obtain

$$\int_{\mathbb{R}^N} r^{M-1}s^{K-1}\Big[\nabla\varphi\nabla\zeta + \varphi\zeta - pU^{p-1}\varphi\zeta\Big] = 0.$$

Hence we have (4.6).

Since $\varphi \in H^1(\mathbb{R}^2)$, it follows by non-degeneracy

$$\varphi = b_1\frac{\partial U}{\partial r} + b_2\frac{\partial U}{\partial s}.$$

Since $\tilde{\varphi}_n \in \tilde{E}_{\varepsilon_n,P}$, letting $n \to \infty$ in (4.7), we have

$$\begin{aligned} &\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\nabla\varphi\nabla\frac{\partial U}{\partial r} = 0 \\ &\int_{\mathbb{R}^2} r^{M-1}s^{K-1}\nabla\varphi\nabla\frac{\partial U}{\partial s} = 0, \end{aligned}$$

which implies $b_1 = b_2 = 0$. Hence $\varphi = 0$ and for any $R > 0$ we have

$$\int_{B_{\varepsilon_n R}(P)} r^{M-1} s^{K-1} \varphi_n^2 dr ds = o(\varepsilon_n^2).$$

Hence

$$\begin{aligned} o(\varepsilon_n^2) &\geq \langle Q_{\varepsilon_n,P}(\varphi_n), \varphi_n \rangle_{\varepsilon_n} \geq \|\varphi_n\|_{\varepsilon_n}^2 - p\int_{\mathcal{D}} (V_{\varepsilon_n,P})^{p-1}\varphi_n^2 \\ &\geq \varepsilon_n^2 - o(1)\varepsilon_n^2 \end{aligned}$$ □

which implies a contradiction.

Lemma 4.4. *Let $R_\varepsilon(\varphi)$ be the functional defined by* (4.3). *Let $\varphi \in H_0^1(\mathcal{D})$, then*

$$|R_\varepsilon(\varphi)| = \varepsilon^\tau \|\varphi\|_\varepsilon^2 \tag{4.11}$$

and

$$\|R'_\varepsilon(\varphi)\|_\varepsilon = \varepsilon^\tau \|\varphi\|_\varepsilon. \tag{4.12}$$

for some $\tau > 0$ small.

Proof. We have

$$\begin{aligned} |R_\varepsilon(\varphi)| &\leq o\left(\int_{\mathcal{D}} r^{M-1} s^{K-1} V_{\varepsilon,P}^{p-1}\varphi^2\right) \\ &\leq o(1)\int_{\mathcal{D}} r^{M-1} s^{K-1} V_{\varepsilon,P}^{p-1}\varphi^2 = o(1)\|\varphi\|_\varepsilon^2. \end{aligned}$$

Choosing $o(1) = \varepsilon^\tau$, we obtain the first estimate. The second estimate follows in a similar way. □

Lemma 4.5. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, there exists a C^1 map $\varphi : E_{\varepsilon,P} \to H$, such that $\varphi \in E_{\varepsilon,P}$ we have*

$$\left\langle I'_\varepsilon\left(V_{\varepsilon,P} + \varphi\right), \eta\right\rangle_\varepsilon = 0, \quad \forall \eta \in E_{\varepsilon,P}.$$

Moreover, we have

$$\|\varphi\|_\varepsilon = o(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}.$$

Proof. We have $l_{\varepsilon,P} + Q_{\varepsilon,P}\varphi + R'_\varepsilon(\varphi) = 0$. As $Q_{\varepsilon,P}^{-1}$ exists, the above equation is equivalent to solving

$$Q_{\varepsilon,P}^{-1} l_{\varepsilon,P} + \varphi + Q_{\varepsilon,P}^{-1} R'_\varepsilon(\varphi) = 0.$$

Define

$$\mathcal{G}(\varphi) = -Q_{\varepsilon,P}^{-1} l_{\varepsilon,P} - Q_{\varepsilon,P}^{-1} R'_\varepsilon(\varphi) \qquad \forall \varphi \in \Lambda_{\varepsilon,D}.$$

Hence the problem is reduced to finding a fixed point of the map $\mathcal{G}$.
For any $\varphi_1 \in E_{\varepsilon,P}$ and $\varphi_2 \in E_{\varepsilon,P}$ with $\|\varphi_1\|_\varepsilon \le o(\varepsilon^{1-\tau})\sqrt{U(k|\ln\varepsilon|)}$, $\|\varphi_2\|_\varepsilon \le o(\varepsilon^{1-\tau})\sqrt{U(k|\ln\varepsilon|)}$

$$\|\mathcal{G}(\varphi_1) - \mathcal{G}(\varphi_2)\|_\varepsilon \le C\|R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2)\|_\varepsilon.$$

From Lemma 4.4, we have

$$\langle R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2), \eta\rangle_\varepsilon \le o(1)\|\varphi_1 - \varphi_2\|_\varepsilon \|\eta\|_\varepsilon.$$

Hence we have

$$\|R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2)\|_\varepsilon \le o(1)\|\varphi_1 - \varphi_2\|_\varepsilon.$$

Hence $\mathcal{G}$ is a contraction as

$$\|\mathcal{G}(\varphi_1) - \mathcal{G}(\varphi_2)\|_\varepsilon \le Co(1)\|\varphi_1 - \varphi_2\|_\varepsilon.$$

Also for $\varphi \in E_{\varepsilon,P}$ with $\|\varphi\|_\varepsilon \le o(\varepsilon^{1-\tau})\sqrt{U(k|\ln\varepsilon|)}$, and $\tau > 0$ sufficiently small

$$\begin{aligned}\|\mathcal{G}(\varphi)\|_\varepsilon &\le C\|l_{\varepsilon,P}\|_\varepsilon + C\|R'_\varepsilon(\varphi)\|_\varepsilon \\ &\le Co(\varepsilon)\sqrt{U(k|\ln\varepsilon|)} + Co(\varepsilon^{1-\tau+\tau})\sqrt{U(k|\ln\varepsilon|)} \\ &\le Co(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}.\end{aligned} \tag{4.13}$$

Hence

$$\mathcal{G} : \Lambda_{\varepsilon,D} \cap B_{o(\varepsilon^{1-\tau})\sqrt{U(k|\ln\varepsilon|)}}(0) \to \Lambda_{\varepsilon,D} \cap B_{o(\varepsilon^{1-\tau})\sqrt{U(k|\ln\varepsilon|)}}(0)$$

is a contraction map. Hence, by the contraction mapping principle there exists a unique $\varphi \in \Lambda_{\varepsilon,D} \cap B_{o(\varepsilon^{1-\tau})\sqrt{U(k|\ln\varepsilon|)}}(0)$ such that $\varphi = \mathcal{G}(\varphi)$ and

$$\|\varphi\|_\varepsilon = \|\mathcal{G}(\varphi)\|_\varepsilon \le Co(\varepsilon)\sqrt{U(k|\ln\varepsilon|)}.$$

□

We write $u_\varepsilon = V_{\varepsilon,P} + \varphi$. Then we have

$$\begin{aligned}I_\varepsilon(u_\varepsilon) = {} & I_\varepsilon(V_{\varepsilon,P}) \\ & + \int_D r^{M-1}s^{K-1}(\varepsilon^2\nabla V_{\varepsilon,P}\nabla\varphi - V_{\varepsilon,P}\varphi + f(V_{\varepsilon,P})\varphi)drds \\ & + \frac{1}{2}\left(\int_D r^{M-1}s^{K-1}\left[\varepsilon^2|\nabla\varphi|^2 - \varphi^2 + f'(V_{\varepsilon,P})\varphi^2\right]drds\right) \\ & - \int_D r^{M-1}s^{K-1}\left[F(V_{\varepsilon,P}+\varphi) - F(V_{\varepsilon,P}) - \varepsilon f(V_{\varepsilon,P})\varphi - \frac{1}{2}f'(V_{\varepsilon,P})\varphi^2\right]drds\end{aligned}$$

which can be expressed as

$$\begin{aligned}
I_\varepsilon(u_\varepsilon) &= I_\varepsilon(V_{\varepsilon,P}) \\
&\quad + \int_{\mathcal{D}} S_\varepsilon[V_{\varepsilon,P}]\varphi r^{M-1}s^{K-1}drds \\
&\quad + \frac{1}{2}\left(\int_{\mathcal{D}}[\varepsilon^2|\nabla\varphi|^2dx-\varphi^2+f'(V_{\varepsilon,P})\varphi^2]r^{M-1}s^{K-1}drds\right) \\
&\quad - \int_{\mathcal{D}} r^{M-1}s^{K-1}\left[F(V_{\varepsilon,P}+\varphi)-F(V_{\varepsilon,P})-f(V_{\varepsilon,P})\varphi-\frac{1}{2}f'(V_{\varepsilon,P})\varphi^2\right]drds \\
&= I_\varepsilon\left(V_{\varepsilon,P}\right)+O(\|l_{\varepsilon,P}\|_\varepsilon\|\varphi\|_\varepsilon+\|\varphi\|_\varepsilon^2+R_\varepsilon(\varphi)) \\
&= I_\varepsilon\left(V_{\varepsilon,P}\right)+o(\varepsilon^2)U(k|\ln\varepsilon|).
\end{aligned}$$

5. The reduced problem: min-max procedure

Proof of Theorem 1.1. Let $\mathcal{G}_\varepsilon(P)=\mathcal{G}_\varepsilon(d,\theta)=I_\varepsilon(u_\varepsilon)$. Consider the problem

$$\min_{d\in\Lambda_{\varepsilon,P}}\ \max_{\theta_0-\delta\le\theta\le\theta_0+\delta}\mathcal{G}_\varepsilon(d,\theta).$$

To prove that $\mathcal{G}_\varepsilon(P)=I_\varepsilon\big(V_{\varepsilon,P}+\varphi\big)$ is a solution of (1.1), we need to prove that P is a critical point of $\mathcal{G}_\varepsilon$, in other words we are required to show that P is a interior point of $\Lambda_{\varepsilon,D}$.

For any $P\in\Lambda_{\varepsilon,P}$, from Lemma 4.3 we obtain

$$\begin{aligned}
\mathcal{G}_\varepsilon(P) &= I_\varepsilon(V_{\varepsilon,P})+O(\|l_{\varepsilon,P}\|_\varepsilon\|\varphi\|_\varepsilon+\|\varphi\|_\varepsilon^2+R_\varepsilon(\varphi)) \\
&= I_\varepsilon(V_{\varepsilon,P})+o(\varepsilon^2)U(k|\ln\varepsilon|) \\
&= \varepsilon^2\gamma P_1^{M-1}P_2^{K-1}+\varepsilon^2\gamma_1P_1^{M-1}P_2^{K-1}U\left(\frac{2d(P,\mathcal{D}_1)}{\varepsilon}\right) \\
&\quad +o(\varepsilon^2)U(k|\ln\varepsilon|).
\end{aligned} \tag{5.1}$$

We have the expansion

$$\begin{aligned}
\mathcal{G}_\varepsilon(d,\theta) &= \gamma\varepsilon^2\bigg[a^{M+K-2}+a^{M+K-3}d+\gamma^{-1}\gamma_1a^{M+K-2}U\left(\frac{2d(P,\mathcal{D}_1)}{\varepsilon}\right) \\
&\qquad +O(d^2)\bigg]\cos^{M-1}\theta\sin^{K-1}\theta+o(\varepsilon^2)U(k|\ln\varepsilon|) \\
&= \gamma\varepsilon^2\left[a^{M+K-2}+a^{M+K-1}d+\gamma^{-1}\gamma_1a^{M+K-2}U\left(\frac{2d}{\varepsilon}\right)\right] \\
&\quad \times\cos^{M-1}\theta\sin^{K-1}\theta+o(\varepsilon^2)U(k|\ln\varepsilon|)+O(\varepsilon^2d^2).
\end{aligned}$$

It is clear that the maximum is attained at some interior point $\theta' \in (\theta_0 - \delta, \theta_0 + \delta)$. Moreover, for this θ', the minimum is attained at a interior point of $\Lambda_{\varepsilon,D}$. This finishes the proof. $\square$

6. The reduced problem: max-max procedure

Proof of Theorem 1.2. Here we obtain the critical point using a max-max procedure. The projection in the Neumann case is just $Q_{\varepsilon,P}$. Hence the reduced problem

$$\begin{aligned}\mathcal{R}_\varepsilon(P) = \varepsilon^2\gamma P_1^{M-1}P_2^{K-1} - \varepsilon^2\gamma_1 P_1^{M-1}P_2^{K-1}U\left(\frac{2d(P,\mathcal{D}_2)}{\varepsilon}\right) \\ + o(\varepsilon^2)U(k|\ln\varepsilon|).\end{aligned} \tag{6.1}$$

Consider

$$\max_{d\in\Lambda_{\varepsilon,N}} \max_{\theta_0-\delta\le\theta\le\theta_0+\delta} \mathcal{R}_\varepsilon(d,\theta). \tag{6.2}$$

We have the expansion

$$\begin{aligned}\mathcal{R}_\varepsilon(d,\theta) &= \gamma\varepsilon^2\bigg[b^{M+K-2} - b^{M+K-3}d - \gamma^{-1}\gamma_1 b^{M+K-2}U\left(\frac{2d(P,\mathcal{D}_2)}{\varepsilon}\right) \\ &\qquad + O(d^2)\bigg]\cos^{M-1}\theta\sin^{K-1}\theta + o(\varepsilon^2)U(k|\ln\varepsilon|) \\ &= \gamma\varepsilon^2\bigg[b^{M+K-2} - b^{M+K-3}d - \gamma^{-1}\gamma_1 b^{M+K-2}U\left(\frac{2d}{\varepsilon}\right)\bigg] \\ &\qquad \times\cos^{M-1}\theta\sin^{K-1}\theta + o(\varepsilon^2)U(k|\ln\varepsilon|) + O(\varepsilon^2 d^2).\end{aligned}$$

It is clear that the maximum in θ is attained at some interior point $\theta' \in (\theta_0 - \delta, \theta_0 + \delta)$. Moreover, for this θ', the maximum is attained at a interior point d of $\Lambda_{\varepsilon,N}$ if we choose $k > 0$ to be sufficiently small. Hence Theorem 1.2 is proved. $\square$

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