Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XV (2016), 657-662

Finite groups with many metacyclic subgroups

ADOLFO BALLESTER-BOLINCHES AND JOHN COSSEY

Abstract. The aim of this paper is to characterise the finite non-nilpotent groups in which every 2-generator subgroup is metacyclic.

Mathematics Subject Classification (2010): 20D10 (primary); 20D20 (secondary).

1. Introduction

The recovery of information about the structure of a finite group from information about its subgroups has a long history. Here we will be concerned with the influence of 2-generator subgroups on the structure of a finite group. If a group has 2-generator subgroups in a class $\mathcal{X}$ which is subgroup-closed and minimal non-$\mathcal{X}$-groups are 2-generator then the group is in $\mathcal{X}$. This is true for the classes of soluble [8, 14], supersoluble [6] and nilpotent [9, Satz III.5.2] groups. Minimal non-metacyclic p-groups (p a prime) have been classified by Blackburn [2, Theorem 3.2]. These groups are all 3-generator and so the class of groups with all 2-generator subgroups metacyclic contains non-metacyclic groups. For convenience, we denote the class of finite groups with 2-generator subgroups metacyclic by $\mathcal{M}$. Note that $\mathcal{M}$ is a subgroup and quotient closed class.

For odd primes p, the classification of p-groups in $\mathcal{M}$ is easy. If $G \in \mathcal{M}$, then G can not contain a non-abelian section of order p^3 and exponent p, since such a section is not metacyclic. It then follows by [13, Lemma 2.3.3] that G is a modular group. Conversely it follows from [13, Theorem 2.3.1] that a modular p-group has all 2-generator subgroups metacyclic. Mann [12] showed that the class of monotone 2-groups coincides with the class of 2-groups in $\mathcal{M}$. These groups have been classified by Crestani and Menegazzo [3] and we refer the reader to that paper for details.

Received January 24, 2014; accepted in revised form July 14, 2014.
Published online February 2016.

Our aim here is to complete the classification of finite groups in $\mathcal{M}$ by classifying all the non-nilpotent finite groups in this class.

If p is a prime, we denote a Sylow p-subgroup of a group G by G_p and if π is a set of primes we denote a Hall π-subgroup by G_π.

Theorem 1.1. *A non-nilpotent group G has 2-generator subgroups metacyclic if and only if*

(i) *G is supersoluble and metabelian, with Sylow subgroups modular for odd primes and monotone groups for the prime 2,*
(ii) *$N = G^{\mathfrak{N}}$ (the nilpotent residual of G) is abelian (and $\neq 1$) and so $G = NH$, $N \cap H = 1$ for every Carter subgroup H of G,*
(iii) *H acts on N as power automorphisms and if π is the set of primes dividing N then H_p is cyclic if $p \in \pi$ and $H_{\pi'}/C_{H_{\pi'}}(N)$ is cyclic.*
(iv) *If $q \in \pi'$, $x \notin C_{H_q}(N)$ and $y \in C_{H_q}(N)$ then $K = \langle x, y\rangle = U\langle x\rangle$ with U cyclic, normal in K and contained in $C_H(N)$.*

We will need the following result, essentially due to Gaschütz: the version here is due to Lucchini and Tamburini [11]. Let $d(G)$ denote the minimum number of generators of a supersoluble group G and for each isomorphism type of chief factor A (as G-module) let $\delta_G(A)$ denote the number of complemented chief factors of G isomorphic to A in a fixed chief series and let $\Omega(G)$ denote the set of non-isomorphic complemented chief factors.

Lemma 1.2. *With the notation above*

$$d(G) = \max_{A\in\Omega(G)} h_G(A)$$

where $h_G(A) = (\delta_G(A) + 1 - \theta_G(A))$ and $\theta_G(A) = 1$ if A is a trivial G-module, and $\theta_G(A) = 0$ otherwise.

2. The proof of Theorem 1

Suppose first that $G \in \mathcal{M}$.

(i) Since metacyclic groups are supersoluble, we have that G is supersoluble ([6]). We defer the proof that G is metabelian until later. The structure of the Sylow subgroups is covered by the remarks above.

(ii) Since G is supersoluble the nilpotent residual N of G is nilpotent of odd order. Let p be a prime. Suppose that the Sylow p-subgroup M of N is non-trivial, and let K be a Hall p'-subgroup of G. We show that all G-chief factors of $M/\Phi(M)$ are isomorphic as K-modules. Assume this is not true and let A/B be a G-chief factor such that $\Phi(M) \leq B < A \leq M$, all G-chief factors between M and A are K-isomorphic and A/B is not K-isomorphic to any G-chief factor between A and M. Suppose that A is a proper subgroup of M. Then M/B is an elementary abelian non-cyclic p-group and every G-chief factor between B and M is non-central in G. By [7, Theorem A. 11.5], M/B is a semisimple K-module. Let C/B be a G-chief factor below M such that $A \cap C = B$. Write $D = AC$. Since $K/C_K(A/B)$ and $K/C_K(C/B)$ are both cyclic and $C_K(A/B) \cup C_K(C/B) \neq K$ we can find an element $k \in K$ such that $C_K(A/B)\langle k\rangle = C_K(C/B)\langle k\rangle = K$. Then A/B and C/B are non-isomorphic as $\langle k\rangle$-modules. We now have $D\langle k\rangle/B$ a 2-generator group by Lemma 1. Since $(D\langle h\rangle/B)' = C/B$ is not cyclic, this group is not metacyclic, a contradiction. Assume now that $M = A$. Then $\Phi(M)$ is a proper subgroup of B and there exists a normal subgroup Z of G such that B/Z and M/B are two non-central chief factors of G which are not K-isomorphic. Then M/Z is 2-generated. Arguing as above, there exists an element $h \in K$ such that $C_K(M/B)\langle k\rangle = C_K(B/Z)\langle h\rangle = K$ and $M\langle h\rangle/Z$ is a 2-generated group which is not metacyclic, a contradiction. Therefore, all G-chief factors of $M/\Phi(M)$ are isomorphic as K-modules and, by [1, Lemma 2.1.3], K acts as power automorphisms on $M/\Phi(M)$.

Suppose that M is non-abelian. By [13, Lemma 2.3.33], there exist characteristic subgroups R and S of M such that $\Phi(M) \leq S < R$ and $[R, Aut(M)] \leq S$. In particular, every G-chief factor between $\Phi(M)$ and M is central in G. This is not possible. Hence M is abelian. It now follows from [5, Lemma 2] (see also [1, Lemma 3.3.39]) that all chief factors of G contained in M are isomorphic as G-modules. By [1, Lemma 2.1.3], p'-elements of G act on M as power automorphisms.

Since N is abelian it is complemented by every Carter subgroup H of G [7, Theorem IV.5.18].

(iii) Let N_p and H_p be Sylow p-subgroups of N and H respectively. If both N_p and H_p are non-trivial, we show that H_p is cyclic and acts on N_p as power automorphisms. To see this suppose first that H_p is non-cyclic. Since p is odd, H_p contains an elementary abelian subgroup $U = \langle u, v\rangle$ of order p^2. Let $Z = \langle z\rangle \leq N_p$ be a minimal normal subgroup of G. Since then $Z \leq Z(G_p) \cap N_p$ we have $V = \langle u, v, z\rangle$ elementary abelian of order p^3. Consider a Hall p'-subgroup $H_{p'}$ of H. If $h \in H_{p'} \backslash C_H(Z)$ set $K = \langle hu, vz\rangle$. We claim that $V \leq K$. Since h and u commute, u and h are both powers of hu so that $u, h \in K$. Next $[h, vz] = [h, z] = z^\alpha$ and so $z \in K$. This gives $v \in K$ also and so $V \leq K$. We now have K is a 2-generator which is not metacyclic, a contradiction. Thus H_p is cyclic. If G_p is abelian then H_p acts as the trivial power automorphism on N_p so we suppose that G_p is non-abelian and $H_p = \langle u\rangle$. Since $G_p = N_p H_p$, N_p is abelian and G_p is a modular group we must have $u \notin Z(G_p)$ and hence acting as a power automorphism on N_p.

Since we now have H acts on N_p as power automorphisms for each prime in π it then follows from [4, Theorem 2.3.1] that H acts as power automorphisms on N. Since $H_{\pi'}/C_{H_{\pi'}}(N_p)$ is cyclic for each $p \in \pi$ by [4, Theorem 3.4.1], $H_{\pi'}/C_{H_{\pi'}}(N)$ is abelian. Since $C_{H_{\pi'}}(N)$ is normal in G to show $H_{\pi'}/C_{H_{\pi'}}(N)$ is cyclic we may assume that $C_{H_{\pi'}}(N) = 1$. If $H_{\pi'}$ is not cyclic it must contain an elementary abelian subgroup U of order q^2 for some prime $q \in \pi'$. It follows from [4, Theorem 3.4.1] that, for any prime $p \in \pi$, $U/C_U(N_p)$ is cyclic and so we can find primes p_1 and p_2 in π such that $1 < C_U(N_{p_i}) = V_i < U, i = 1, 2$

and $V_1V_2 = U$. If W_i is a non-trivial cyclic subgroup of N_{p_i}, $i = 1, 2$, then $T = (W_1 \times W_2)U \cong (W_1V_2) \times (W_2V_1)$. Clearly $W_1 \times W_2$ is a maximal abelian subgroup of T. Suppose $T = AB$ with A and B cyclic and A normal in T. Then $W_1 \times W_2 \leq T' \leq A$. Since T/A is cyclic, $W_1 \times W_2$ is a proper subgroup of A, a contradiction. Thus $H_{\pi'}/C_{H_{\pi'}}(N)$ is cyclic.

(iv) Let $K = \langle x, y\rangle$ be as in the statement of the theorem. Then $K = C_K(N)\langle x\rangle$. If $\langle y\rangle$ is normal in K, set $U = \langle y\rangle$ and (iv) is satisfied. Hence suppose that $\langle y\rangle$ is not normal in K. Let $M = \langle m\rangle$ be a cyclic subgroup of N chosen so that $C_K(M) = C_K(N)$. By (iii) x acts as a non-trivial power automorphism on M. Since MK is generated by ym and x, MK is metacyclic. Thus $MK = ST$ with S cyclic and normal in MK and T cyclic. If $S = \langle s\rangle$ then we may assume that $s = mu$ with $u \in C_K(M)$ and $T \leq K$ since $M \leq (MK)' \leq S$. Let $U = \langle u\rangle$. Then $K = UT$. If $T = \langle t\rangle$ then $x = t^\alpha u^\beta$. If α is a multiple of q then $\langle x, y\rangle < K$, a contradiction. We now have $t^\alpha = xu^{-\beta}$ and so $K = U\langle x\rangle$, completing the proof of (iv).

(i) continued. If $G \in \mathcal{M}$, then $G = NH$ is a subdirect product of G/N and $(N_pH)/C_H(N_p)$ for each prime p dividing $|N|$. It is an easy deduction from the list of 2-groups in $\mathcal{M}$ to see that these groups are metabelian. Since modular p-groups are metabelian, we have G/N metabelian. Since $H/C_H(N_p)$ is cyclic, $(N_pH)/C_H(N_p)$ is metabelian. Hence G is metabelian.

Suppose now that G satisfies the hypotheses (i)-(iv) and let X be a 2-generator subgroup of G. If X is nilpotent then each Sylow subgroup of X is metacyclic and so X is metacyclic. If X is not nilpotent, let $Y = X^{\mathfrak{N}}$. Since $Y \leq N$ and G acts as power automorphisms on N, X acts as power automorphisms on Y. If Y is not cyclic then a Sylow p-subgroup Y_p of Y is not cyclic for some prime p and so as X-module $Y_p/\Phi(Y_p)$ is the direct product of j isomorphic irreducibles, $j \geq 2$. Now Lemma 1 gives that X has at least $j + 1$ generators, a contradiction. Thus Y is cyclic. Let Z be a complement to Y in X. Then Z is a 2-generator nilpotent subgroup of G and hence is metacyclic. If $(Z_p, Y_p) \neq 1$ then it follows from (iii) that Z_p is cyclic. Thus if π is the set of primes dividing $|Y|$ we have $Z = S \times T$, where S is a cyclic π-group and T is a metacyclic π'-group. If T is cyclic, then Z is cyclic and we are finished. If T is not cyclic and $[T, Y] = 1$ then $T = UV$ with U cyclic and normal in T (and hence in Z) and V cyclic. But then YU is a cyclic normal subgroup of X and SV is cyclic, so that $X = (YU)(SV)$ is metacyclic, a contradiction.

Thus T is not cyclic and not contained in $C_Z(Y)$. From (iii) we have $T/C_T(Y)$ cyclic. We can choose generators x, y for T with $x \notin C_T(Y)$ and $y \in C_T(Y)$. To show this it will be enough to prove it for the Sylow subgroups of T. Let T_q be the Sylow q-subgroup of T and let a, b be generators of T_q. Since $T_q/C_{T_q}(Y)$ is cyclic we must have either a or b generates T_q modulo $C_{T_q}(Y)$; suppose a. Then for some integer α, $ba^\alpha \in C_{T_q}(Y)$. If $x_q = a$ and $y_q = ba^\alpha$ then $T_q = \langle x_q, y_q\rangle$. Put $x = \prod_{q\in\pi'} x_q$ and $y = \prod_{q\in\pi'}$. Then $T = \langle x, y\rangle$ with $y \in C_T(Y)$.

From (iv) we have $T = UV$ with $U \leq C_T(Y)$ cyclic and normal in T and V cyclic. Now UY is cyclic and normal in X, VS is cyclic and $X = (UY)(VS)$ is metacyclic, a final contradiction.

3. Examples

The first two examples show that if a metacyclic q-group acts on a p-group as a group of non-trivial power automorphisms the action plays an important role in whether the extension is metacyclic or not.

Let $N = \langle m \rangle$ be cyclic of order 7 and let H be the nonabelian group of order 27 and exponent 9, generated by x of order 9 and y of order 3.

Suppose first that x acts trivially on N and y acts non-trivially on N and let $G = NH$. Then $G = ST$ with $S = N \times \langle x \rangle$ and $T = \langle y \rangle$, with S and T cyclic and S normal in G. Thus G is metacyclic.

Now suppose that y acts trivially on N and X acts non-trivially on N. We then have $C_H(N) = \langle x^3, y \rangle$. The unique maximal cyclic normal subgroup of $G = NH$ is $U = \langle x^3, N \rangle$ and G/U is not cyclic. Thus G is not metacyclic.

Since in Theorem 1 the Sylow q-subgroups are modular for q odd, it is tempting to conjecture that x acts as a power automorphism on $C_H(N)$. The following example shows this need not be true.

Let $X = \langle x \rangle$ be a cyclic group of order 27 and let $Y = \langle y \rangle \leq Aut(X)$ with $x^y = x^{10}$. Let $N = \langle n \rangle$ be a group of order 7 and let $H = XY$. Define an action of H on N by $n^y = n$ and $n^x = n^2$. Let $G = NH$. We have $C_H(N) = \langle x^3, y \rangle$ and x does not act as a power automorphism on $C_H(N)$. However G is metacyclic since $U = N\langle yx^3 \rangle$ is a cyclic normal subgroup of G and $UX = G$.

References

[1] A. BALLESTER-BOLINCHES, R. ESTEBAN-ROMERO and M. ASAAD, "Products of Finite Groups", de Gruyter Expositions in Mathematics 53, de Gruyter, Berlin, New York, 2010.

[2] N. BLACKBURN, *Generalizations of certain elementary theorems on p-groups*, Proc. London Math. Soc. (3) **11**, 1961, 1–22.

[3] E. CRESTANI and F. MENEGAZZO, *On monotone 2-groups*, J. Group Theory **15** (2012), 359–383.

[4] C. D. H. COOPER, *Power automorphisms of a group*, Math. Z. **107** (1968), 335–356.

[5] J. COSSEY and Y. WANG, *Finite dinilpotent groups of small derived length*, J. Austral. Math. Soc. (A) **67** (1999), 318–328.

[6] K. DOERK, *Minimal nicht überauflösbare, endliche Gruppen*, Math. Z. **91** (1966), 198–205.

[7] K. DOERK and T. HAWKES, "Finite Soluble Groups", De Gruyter Expositions in Mathematics 4, De Gruyter, Berlin, New York, 1992.

[8] P. FLAVELL, *Finite groups in which every two elements generate a soluble subgroup*, Invent. Math. **121** (1995), 279-285.

[9] B. HUPPERT, "Endliche Gruppen I", Springer Verlag, Berlin, Heidelberg, New York, 1967.

[10] B. HUPPERT and N. BLACKBURN, "Finite Groups II", Springer Verlag, Berlin, Heidelberg, New York, 1982.

[11] A. LUCCHINI and M. C. TAMBURINI, *Minimal generation of finite soluble groups by projectors and normalizers*, Glasg. Math. J. **41** (1999), 303–312.

[12] A. MANN, *The number of generators of finite p-groups*, J. Group Theory **8** (2005), 317–337.

[13] R. SCHMIDT, "Subgroup Lattices of Groups", Walter de Gruyter, Berlin, New York, 1994.

[14] J. G. THOMPSON, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. **74** (1968), 383–437.

Departament d'Àlgebra
Universitat de València
Dr. Moliner, 50
46100 Burjassot (València), Spain
Adolfo.Ballester@uv.es

Department of Mathematics
Mathematical Sciences Institute
Australian National University
Canberra ACT 2601, Australia
John.Cossey@anu.edu.au