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Boundary trace of positive solutions of supercritical semilinear elliptic equations in dihedral domains

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Abstract. We study the generalized boundary value problem for (E) $-\Delta u + |u|^{q-1}u = 0$ in a dihedral domain Ω, when $q > 1$ is supercritical. The value of the critical exponent can take only a finite number of values depending on the geometry of Ω. When μ is a bounded Borel measure in a k-wedge, we give necessary and sufficient conditions in order it be the boundary value of a solution of (E). We also give conditions which ensure that a boundary compact subset is removable. These conditions are expressed in terms of Bessel capacities $B_{s,q'}$ in $\mathbb{R}^{N-k}$ where s depends on the characteristics of the wedge. This allows us to describe the boundary trace of a positive solution of (E).

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1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $q > 1$. A long-term research on the equation

$$-\Delta u + |u|^{q-1}u = 0 \text{ in } \Omega, \tag{1.1}$$

has been carried out for more than twenty years by probabilistic and/or analytic methods. Much of the research was focused on three main problems in domains of class C^2:

(i) The Dirichlet problem for (1.1) with boundary data given by a finite Borel measure on $\partial\Omega$.

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(ii) The characterization of removable singular subsets of $\partial\Omega$ relative to positive solutions of (1.1).
(iii) The characterization of arbitrary positive solutions of (1.1) via an appropriate notion of boundary trace.

Consider the Dirichlet problem

$$-\Delta u + |u|^{q-1}u = 0 \quad \text{in} \quad \Omega, \; u = \mu \quad \text{in} \quad \partial\Omega \tag{1.2}$$

where $\mu \in \mathfrak{M}(\partial\Omega)$ (= space of finite Borel measures on $\partial\Omega$). Following [24], a (weak) solution $u := u_\mu$ of (1.2) is a function $u \in L^q_\rho(\Omega)$ such that,

$$\int_\Omega \left(-u\Delta\eta + \eta|u|^{q-1}u\right) dx = -\int_\Omega \mathbb{K}[\mu]\Delta\eta dx, \tag{1.3}$$

for every in $\eta \in X(\Omega)$, where

$$X(\Omega) = \left\{\eta : \; \rho^{-1}\Delta\eta \in L^\infty(\Omega)\right\}. \tag{1.4}$$

Here $\mathbb{K}[\mu]$ is the harmonic function in Ω with boundary trace μ and ρ is the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega)$ normalized so that $\max_\Omega \rho = 1$. We also denote by λ the corresponding eigenvalue. We recall that, if Ω is Lipschitz $\mathbb{K}[\mu] \in L^1_\rho(\Omega)$; if Ω is of class C^2, $\mathbb{K}[\mu] \in L^1(\Omega)$.

A measure μ is a *q-good measure* if (1.2) has a solution. The space of q-good measures is denoted by $\mathfrak{M}_q(\partial\Omega)$. It is known that, if μ is q-good, the solution is unique. Furthermore, if μ satisfies the condition

$$\int_\Omega \mathbb{K}[|\mu|]^q \rho dx < \infty, \tag{1.5}$$

then it is q-good. When μ satisfies this condition we say that it is a *q-admissible measure*.

When Ω is a domain of class C^2, $\mathbb{K}[\mu] \in L^q_\rho$ for every $q \in (1, \frac{N+1}{N-1})$ and every $\mu \in \mathfrak{M}(\partial\Omega)$. Therefore, for q in this range, every measure in $\mathfrak{M}(\partial\Omega)$ is q-good and there is no removable boundary set (except for the empty set). Problem (iii), for q in this range, was resolved by Le Gall [16] (for $N = q = 2$) and Marcus and Véron [19] (for $1 < q < \frac{N+1}{N-1}$, $N \geq 3$).

The number $q_c = \frac{N+1}{N-1}$ is called the *critical value* for (1.1). If q is supercritical, *i.e.* $q \geq q_c$, point singularities are removable. In particular there is no solution of (1.2) when $\mu = \delta_y$ (= a Dirac measure concentrated at a point $y \in \partial\Omega$).

In the supercritical case, problems (i)-(iii), Ω of class C^2, have been resolved in several stages. We say that a compact set $E \subset \partial\Omega$ is removable relative to equation (1.1) if there exists no positive solution vanishing on $\partial\Omega \setminus E$. We say that E is conditionally removable if any solution u of (1.2), with $\mu \in \mathfrak{M}(\partial\Omega)$, such that $u = 0$ on $\partial\Omega \setminus E$ must vanish in Ω.

With respect to problem (ii) it was shown that a compact set $E \subset \partial\Omega$ is removable if and only if $C_{\frac{2}{q},q'}(E) = 0$, $q' = q/(q-1)$. Here $C_{\alpha,p}$ denotes the Bessel capacity, with the indicated indexes on $\partial\Omega$. (see Subsection 4.2 for an overview of Bessel capacities). This result was obtained by Le Gall [16] for $q = 2$, Dynkin and Kuznetsov [8] for $1 < q \leq 2$, Marcus and Véron [20] for $q > 2$. For a unified analytic proof, covering all $q \geq q_c$ see [21].

The above result implies that every q-good measure μ must vanish on sets of $C_{\frac{2}{q},q'}$ capacity zero. On the other hand a result of Baras and Pierre [3] implies that every positive measure $\mu \in \mathfrak{M}(\partial\Omega)$ that vanishes on sets of $C_{\frac{2}{q},q'}$ capacity zero is the limit of an increasing sequence of admissible measures and therefore q-good. In conclusion: a measure $\mu \in \mathfrak{M}(\partial\Omega)$ is q-good if and only if it vanishes on sets of $C_{\frac{2}{q},q'}$ capacity zero. This takes care of problem (i).

Problem (iii) has been treated in several papers, with various definitions of a generalized boundary trace for positive solutions of (1.1), see [9] and [23]. Finally a full characterization of positive solutions was obtained by Mselati [25] for $q = 2$, Dynkin [7] for $1 < q < 2$ and Marcus [18] for every $q \geq q_c$. In [7, 25] the restriction to $q \leq 2$ was dictated by their use of probabilistic techniques that do not apply to $q > 2$. In [18] the proof is purely analytic.

If Ω is Lipschitz, $\xi \in \partial\Omega$, we say that q_ξ is the critical value for (1.1) at ξ if, for $1 < q < q_\xi$, problem (1.2) with $\mu = \delta_\xi$ has a solution, but for $q > q_\xi$ no such solution exists.

In contrast to the case of smooth domains, when Ω is Lipschitz, q_ξ may vary with the point. For every compact set $F \subset \partial\Omega$ there exists a number $q(F) > 1$ such that, for $1 < q < q(F)$, every measure in $\mathfrak{M}(\partial\Omega)$ supported in F is q-good. Obviously $q(F) \leq \min\{q_\xi : \xi \in F\}$ but it is not clear if equality holds.

In the special case when Ω is a polyhedron, the function $\xi \to q_\xi$ obtains only a finite number of values (in fact, it is constant on each open face and each open edge) and, if $q \geq q_\xi$, an isolated singularity at ξ is removable. Furthermore, the assumption $1 < q < \min\{q_\xi : \xi \in \partial\Omega\}$ implies that every measure in $\mathfrak{M}(\partial\Omega)$ is q-good. For this and related results see [24].

In the present paper we study problem (1.2) when Ω is a polyhedron and q is supercritical, *i.e.* $q \geq \min\{q_\xi : \xi \in \partial\Omega\}$. Following is a description of the main results.

A. *On the action of Poisson-type kernels with fractional dimension*

In preparation for the study of supercritical boundary value problem s we establish an harmonic analytic result, extending a well known result on the action of Poisson kernels on Besov spaces with negative index (see [28, 1.14.4.] and [4]). We first quote the classical result for comparison purposes.

Proposition 1.1. *Let* $1 < q < \infty$ *and* $s > 0$. *Then, for any bounded Borel measure* μ *in* $\mathbb{R}^{n-1}$,

$$I(\mu) = \int_{\mathbb{R}^n_+} |\mathbb{K}_n[\mu](y)|^q \, e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|^q_{B^{-s,q}(\mathbb{R}^{n-1})}. \tag{1.6}$$

Here $\mathbb{K}_n[\mu]$ denotes the Poisson potential of μ in $\mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, namely,

$$\mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{n-1}} \frac{d\mu(z)}{\left(y_1^2 + |\zeta - z|^2\right)^{n/2}} \quad \forall y = (y_1, \zeta) \in \mathbb{R}^n_+ \tag{1.7}$$

where γ_n is a constant depending only on n.

Notation. Let m be a positive integer and let ν be a real number, $\nu \geq m + 1$. Denote,

$$\mathbb{K}_{\nu,m}[\mu](\tau, \zeta) := \int_{\mathbb{R}^m} \frac{\tau^{\nu-m} d\mu(z)}{\left(\tau^2 + |\zeta - z|^2\right)^{\nu/2}} \quad \forall \tau \in (0, \infty),\ \zeta \in \mathbb{R}^m. \tag{1.8}$$

Note that

$$\mathbb{K}_n[\mu] = \gamma_n \mathbb{K}_{n,n-1}[\mu].$$

Theorem 1.2. *Let m and ν be as above. Then, for every $q > 1$ and every $s \in (0, m/q')$, $q' = q/(q-1)$, there exists a positive constant c such that, for every positive measure $\mu \in \mathfrak{M}(\mathbb{R}^m)$ supported in $B_{R/2}(0)$ for some $R > 1$,*

$$\begin{aligned} \frac{1}{c} \|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} &\leq \int_0^R \left(\int_{|\zeta|<R} \left|\mathbb{K}_{\nu,m}[\mu](\tau, \zeta)\right|^q d\zeta \right) \tau^{sq-1} d\tau \\ &\leq c R^{(s+\nu-m)q+1} \|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)}. \end{aligned} \tag{1.9}$$

This also holds when $s = m/q'$, provided that the diameter of supp μ *is sufficiently small.*

This is proved in Section 3 (see Theorem 3.8) using a slightly different notation.

B. *The critical value and the characterization of q-good measures in a k-wedge*

The next step towards the study of boundary value problem s in a polyhedron is the treatment of such problems in a k-wedge (or k-dihedron) *i.e.*, the domain defined by the intersection of k hyperplanes in $\mathbb{R}^N$, $1 < k < N$. The edge is an $(N-k)$ dimensional space.

We note that if $k = N$ the "edge" is a point and the corresponding wedge is a cone with vertex at this point. If $k = 1$ the wedge is a half space. Both of these cases have been treated in [24].

Let A be a Lipschitz domain in S^{k-1}. If

$$S_A := \left\{ x \in \mathbb{R}^N : |x| = 1,\ x \in A \times \prod_{j=k}^{N-1} [0, \pi] \right\} \subset S^{N-1} \tag{1.10}$$

then

$$D_A := \{x = (r, \sigma) : r > 0, \sigma \in S_A\}$$

is a k-wedge in $\mathbb{R}^N$ whose "edge" d_A may be identified with $\mathbb{R}^{N-k}$ and its "opening" is A.

Let λ_A be the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S_A)$ and denote by $\kappa_\pm$ the roots of the equation,

$$\kappa^2 + (N-2)\kappa - \lambda_A = 0. \tag{1.11}$$

Put

$$q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2} \tag{1.12}$$

and

$$q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+}. \tag{1.13}$$

Let $C_{\alpha,p}^{N-k}$ denote the Bessel capacity with the indicated indices in $\mathbb{R}^{N-k}$. The next theorem provides a characterization of q-good measures supported on d_A.

Theorem 1.3.

(a) *If $1 < q < q_c$ every measure in $\mathfrak{M}(d_A)$ is q-good relative to D_A. In fact every such measure is q-admissible.*
(b) *If $q \geq q_c^*$, the only q-good measure in $\mathfrak{M}(d_A)$ is the zero measure.*
(c) *If $q_c \leq q < q_c^*$, a measure $\mu \in \mathfrak{M}(d_A)$ is q-good relative to D_A if and only if μ vanishes on every Borel set $E \subset d_A$ such that$C_{s,q'}^{N-k}(E) = 0$, $s = 2 - \frac{k+\kappa_+}{q'}$.*

The characterization of q-good measures in a polyhedron follows as an easy consequence of the above theorem (see Theorem 4.6 below).

C. *Characterization of removable sets*

Let Ω be an N-dimensional polyhedron. Theorem 1.3 provides a necessary and sufficient condition for the removability of a singular set E relative to the family of solutions u such that

$$\int_\Omega |u|^q \rho \, dx < \infty.$$

The next result provides a necessary and sufficient condition for *removability* in the sense that the only non-negative solution $u \in C(\bar{\Omega} \setminus E)$ which vanishes on $\bar{\Omega} \setminus E$ is the trivial solution $u = 0$.

Let L denote a face or edge or vertex of Ω and put $k := \operatorname{codim} L$. If $1 < k < N$ let d_L denote the linear space spanned by L, such that L is an open subset of d_L. Let Q_L denote the k-wedge with boundary d_L such that, for some neighborhood M of L, $\Omega \cap M = Q_L \cap M$ and let A_L denote the opening of Q_L. If $k = N$, Q_L is a cone with vertex L. Let $q_c(L)$ and $q_c^*(L)$ be defined as in (1.12) and (1.13) for $A = A_L$. Finally let

$$s(L) = 2 - \frac{k + \kappa_+}{q'}$$

where $\kappa_\pm$ are the roots of (1.11) for $A = A_L$. If $k = N$, Q_L is a cone with vertex L. In this case $q_c(L) = q_c^*(L) = 1 - \frac{2}{\kappa_-}$. If $k = 1$ $q_c(L) = q_c^*(L) = (N+1)/(N-1)$.

Theorem 1.4. *Let Ω be a polyhedron in $\mathbb{R}^N$. A compact set $E \subset \partial\Omega$ is removable if and only if, for every L as above such that $E \cap L \neq \emptyset$, the following conditions hold:*

- *if $1 \leq k < N$: either $q_c(L) \leq q < q_c^*(L)$ and $C^{N-k}_{s(L),q'}(E \cap L) = 0$ or $q \geq q_c^*(L)$;*
- *if $k = N$: $q \geq q_c(L)$.*

The present paper is part of an article, "Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains" arXiv:0907.1006 (2009). The first part of this article was published in [24]. The second and last part are presented here. The characterization of q-good measures, here established in polyhedrons, was recently established in [2], for arbitrary Lipschitz domains and a general family of nonlinearities. However the full removability result, Theorem 4.11, has not been superseded. (In [2] the authors provided - in the generality mentioned above - a characterization of *conditional removability* but not of full removability.) The methods of proof in the two papers are completely different. In the present paper, the characterization of q-good measures is based on an extension of a result of [4] and [28, 1.14.4.] on the action of Poisson kernels on Besov spaces with negative index. The use of Poisson-type kernels with fractional dimension has recently appeared in [12] to be a fundamental tool for the study of the boundary trace problem for semilinear elliptic equations with critical Hardy potentials depending on the distance to the boundary in the supercritical case. In [2] the proof relies on a relation between elliptic semilinear equations with absorption and linear Schrödinger equations.

2. The Martin kernel and critical values in a k-dimensional dihedron.

2.1. The geometric framework

An N-dim polyhedron P is a bounded domain bordered by a finite number of hyperplanes. Thus the boundary of P is the union of a finite number of sets $\{L_{k,j} : k = 1, \dots, N, \ j = 1, \dots, n_k\}$ where $\{L_{1,j}\}$ is the set of open faces of P, $\{L_{k,j}\}$ for $k = 2, \dots, N-1$, is the family of relatively open $N-k$-dimensional edges and $\{L_{N,j}\}$ is the family of vertices of P. An $N-k$-dimensional edge is a relatively open set in the intersection of k hyperplanes; it will be described by the characteristic angles of these hyperplanes.

We recall that the spherical coordinates in $\mathbb{R}^N = \{x = (x_1, \dots, x_N)\}$ are expressed by

$$\begin{cases} x_1 = r \sin\theta_{N-1} \sin\theta_{N-2} \cdots \sin\theta_2 \sin\theta_1 \\ x_2 = r \sin\theta_{N-1} \sin\theta_{N-2} \cdots \sin\theta_2 \cos\theta_1 \\ x_3 = r \sin\theta_{N-1} \sin\theta_{N-2} \cdots \cos\theta_2 \\ \vdots \\ x_{N-1} = r \sin\theta_{N-1} \cos\theta_{N-2}, \\ x_N = r \cos\theta_{N-1} \end{cases} \tag{2.1}$$

where $r = |x|$, $\theta_1 \in [0, 2\pi]$ and $\theta_\ell \in [0, \pi]$ for $\ell = 2, 3, \dots, N-1$. We denote $\sigma = (\theta_1, \dots \theta_{N-1})$. Thus in spherical coordinates $x = (r, \sigma)$.

We consider an unbounded *non-degenerate k-dihedron*, $2 \leq k \leq N$ defined as follows. Let A be given by

$$A = \begin{cases} (0, \alpha_1) \times \prod_{j=2}^{k-1} \left(\alpha_j, \alpha'_j\right) & \text{if } k > 2 \\ (0, \alpha_1) & \text{if } k = 2 \end{cases}$$

where

$$0 < \alpha_1 < 2\pi, \quad 0 \leq \alpha_j < \alpha'_j < \pi \quad j = 2, \dots, k-1.$$

We denote by S_A the spherical domain

$$S_A = \left\{ x \in \mathbb{R}^N : |x| = 1,\ \sigma \in A \times \prod_{j=k}^{N-1} [0, \pi] \right\} \subset S^{N-1} \tag{2.2}$$

and by D_A the corresponding k-dihedron,

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}.$$

The *edge* of D_A is the $(N-k)$-dimensional space

$$d_A = \{x : x_1 = x_2 = \ldots = x_k = 0\}. \tag{2.3}$$

2.2. On the Martin kernel and critical values in a cone

We recall here some elements of local analysis when $\Omega = C_A \cap B_1$, A is a Lipschitz domain in S^{N-1} and C_A is the cone with vertex 0 and opening A.

Denote by λ_A the first eigenvalue and by ϕ_A the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(A)$ (normalized by $\max \phi_A = 1$). Let κ_- be the negative root of (1.11) and put

$$\Phi_1(x) := \frac{1}{\gamma} |x|^{\kappa_-} \phi_A(x/\,|x|)$$

where γ is a positive number. Then Φ_1 is a harmonic function in C_A vanishing on $\partial C_A \setminus \{0\}$. We choose $\gamma = \gamma_A$ so that the boundary trace of Φ_1 is δ_0 (=Dirac measure on with mass 1 at the origin).

(i) If $q \geq 1 - \frac{2}{\kappa_-}$, there is no solution of (1.1) in Ω_S with isolated singularity at 0 (see [10]).

(ii) If $1 < q < 1 - \frac{2}{\kappa_-}$, then for any $k > 0$ there exists a unique solution $u := u_k$ to problem (1.2) with $\mu = k\delta_0$ and

$$u_k(x) = k\Phi_1(x)(1 + o(1)) \qquad \text{as } x \to 0. \tag{2.4}$$

The function $u_\infty = \lim_{k\to\infty} u_k$ is a positive solution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies

$$u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_A(x/|x|)(1 + o(1)) \qquad \text{as } x \to 0 \tag{2.5}$$

where ω_A is the (unique) positive solution of

$$-\Delta'\omega - a_{N,q}\omega + |\omega|^{q-1}\omega = 0 \tag{2.6}$$

on S^{N-1}. Here Δ' is the Laplace-Beltrami operator and

$$a_{N,q} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right). \tag{2.7}$$

(iii) If $u \in C(\bar{\Omega}_A \setminus \{0\})$ is a positive solution of (1.1) vanishing on $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$, then either u satisfies (2.4) for some $k > 0$ or u satisfies (2.5). In particular there exists a unique positive solution vanishing on $(\partial C_A \cap B_{r_0}(0)) \setminus \{0\}$ with strong singularity at 0. (For (ii) and (iii) see [24, Theorem 5.7].)

2.3. Separable harmonic functions and the Martin kernel in a k-dihedron, $2 \le k < N$

In the system of spherical coordinates, the Laplacian takes the form

$$\Delta u = \partial_{rr}u + \frac{N-1}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^{N-1}}u$$

where the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ is expressed by induction by

$$\begin{aligned}\Delta_{S^{N-1}}u =&\frac{1}{(\sin\theta_{N-1})^{N-2}}\frac{\partial}{\partial\theta_{N-1}}\left(\left(\sin\theta_{N-1}\right)^{N-2}\frac{\partial u}{\partial\theta_{N-1}}\right)\\ &+\frac{1}{\left(\sin\theta_{N-1}\right)^2}\Delta_{S^{N-2}}u,\end{aligned} \tag{2.8}$$

and

$$\Delta_{S^1}u = \partial_{\theta_1\theta_1}u. \tag{2.9}$$

If we compute the positive harmonic functions in the k-dihedron D_A of the form

$$v(x) = v(r,\sigma) = r^\kappa\omega(\sigma) \quad \text{in } D_A, \quad v = 0 \quad \text{in } \partial D_A \setminus \{0\},$$

we find that ω must be a positive eigenfunction corresponding to the first eigenvalue, λ_A, of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S_A)$,

$$\begin{cases}\Delta_{S^{N-1}}\omega + \lambda_A\omega = 0 & \text{in } S_A\\ \omega = 0 & \text{on } \partial S_A\end{cases} \tag{2.10}$$

and κ must be a root of the algebraic equation (1.11) with λ_A as above. Thus $\kappa = \kappa_\pm$ where

$$\begin{aligned}\kappa_+ &= \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\lambda_A}\right)\\ \kappa_- &= \frac{1}{2}\left(2 - N - \sqrt{(N-2)^2 + 4\lambda_A}\right).\end{aligned} \tag{2.11}$$

Since

$$S^{N-1} = \left\{\sigma = (\sigma_2 \sin\theta_{N-1}, \cos\theta_{N-1}) : \sigma_2 \in S^{N-2},\ \theta_{N-1} \in (0,\pi)\right\},$$

we look for a solution $\omega = \omega^{\{1\}}$ of (2.10) of the form

$$\omega^{\{1\}}(\sigma) = (\sin\theta_{N-1})^{\kappa_+}\omega^{\{2\}}(\sigma_2), \quad \theta_{N-1} \in (0,\pi), \quad \sigma_2 \in S^{N-2}.$$

Here $S^{N-2} = S^{N-1} \cap \{x_N = 0\}$ and we denote

$$S_A^{\{N-2\}} = S_A \cap \{x_N = 0\}, \quad D_A^{\{N-2\}} := D_A \cap \{x_N = 0\} \subset \mathbb{R}^{N-1}.$$

Then (2.11) jointly with relation (2.8) implies

$$\begin{cases} \Delta_{S^{N-2}}\omega^{\{2\}} + (\lambda_A - \kappa_+)\omega^{\{2\}} = 0 & \text{on } S_A^{\{N-2\}} \\ \omega^{\{2\}} = 0 & \text{on } \partial S_A^{\{N-2\}}. \end{cases} \tag{2.12}$$

Since we are interested in $\omega^{\{2\}}$ positive, $\lambda_A^{\{2\}} := \lambda_A - \kappa_+$ must be the first eigenvalue of $-\Delta_{S^{N-2}}$ in $W_0^{1,2}(S_A^{\{N-2\}})$.

Next we look for positive harmonic functions $\tilde{u}$ in $D_A^{\{N-2\}}$ such that

$$\tilde{u}(x_1,\dots,x_{N-1}) = r^{\kappa'}\omega(\sigma_2), \quad \tilde{u} = 0 \text{ on } \partial D_A^{\{N-2\}}.$$

The algebraic equation which gives the exponents is

$$(\kappa')^2 + (N-3)\kappa' - \lambda_A^{\{2\}} = 0.$$

Denote by κ'_+ the positive root of this equation. By the definition of $\lambda_A^{\{2\}}$,

$$\kappa_+^2 + (N-3)\kappa_+ - \lambda_A^{\{2\}} = \kappa_+^2 + (N-2)\kappa_+ - \lambda_A = 0.$$

Therefore $\kappa'_+ = \kappa_+$. Accordingly, if $k \geq 3$, we set

$$\omega^{\{2\}}(\sigma_2) = (\sin\theta_{N-2})^{\kappa_+}\omega^{\{3\}}(\sigma_3),$$

and find that $\omega^{\{3\}}$ satisfies

$$\begin{cases} \Delta_{S^{N-3}}\omega^{\{3\}} + (\lambda_A - 2\kappa_+)\omega^{\{3\}} = 0 & \text{in } S_A^{\{N-3\}} \\ \omega^{\{3\}} = 0 & \text{on } \partial S_A^{\{N-3\}}, \end{cases} \tag{2.13}$$

where

$$S_A^{\{N-3\}} = S_A \cap \{x_N = x_{N-1} = 0\}.$$

Performing this reduction process $N-k$ times, we obtain the following results.

(i) If $k > 2$ then $\omega = \omega^{N-k}(\sigma)$ is given by

$$\omega(\sigma) = (\sin\theta_{N-1}\sin\theta_{N-2}\dots\sin\theta_k)^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1}) \tag{2.14}$$

where

$$\sigma_{N-k+1} \in S^{k-1} = S^{N-1} \cap \{x_N =, x_{N-1} = \cdots = x_{k+1} = 0\}$$

and $\omega' := \omega^{\{N-k+1\}}$ satisfies

$$\begin{cases} \Delta_{S^{k-1}}\omega' + (\lambda_A - (N-k)\kappa_+)\omega' = 0, & \text{in } S_A^{\{k-1\}} \\ \omega' = 0, & \text{on } \partial S_A^{\{k-1\}}, \end{cases} \tag{2.15}$$

where $S_A^{\{k-1\}} = S_A \cap \{x_N = x_{N-1} = \ldots = x_{k+1} = 0\} \approx A$ and $\lambda_A - (N-k)\kappa_+$ is the first eigenvalue of the problem.

(ii) If $k = 2$ then

$$\omega(\sigma) = (\sin\theta_{N-1}\sin\theta_{N-2}\dots\sin\theta_2)^{\kappa_+}\omega^{\{N-1\}}(\theta_1) \tag{2.16}$$

where $\sigma_{N-1} \in S^1 \approx \theta_1 \in (0, 2\pi)$, and $\omega^{\{N-1\}}$ satisfies

$$\begin{cases} \Delta_{S^1}\omega^{\{N-1\}} + (\lambda_A - (N-2)\kappa_+)\omega^{\{N-1\}} = 0 & \text{on } S_A^{\{1\}} \\ \omega^{\{N-1\}} = 0 & \text{on } \partial S_A^{\{1\}}, \end{cases} \tag{2.17}$$

with $\partial S_A^{\{1\}} \approx (0, \alpha)$. In this case

$$\kappa_+ = \frac{\pi}{\alpha}, \quad \omega^{\{N-1\}}(\theta_1) = \sin(\pi\theta_1/\alpha), \tag{2.18}$$

and, by (1.11),

$$\lambda_A - (N-2)\kappa_+ = \frac{\pi^2}{\alpha^2} \Longrightarrow \lambda_A = \frac{\pi^2}{\alpha^2} + (N-2)\frac{\pi}{\alpha}. \tag{2.19}$$

Observe that $\frac{1}{2} \leq \kappa_+$ with equality holding only in the degenerate case $\alpha = 2\pi$ (which we exclude).

In either case, we find a positive harmonic function v_A in D_A, vanishing on ∂D_A, of the form

$$v_A(x) = |x|^{\kappa_+}\,\omega(x/|x|) \tag{2.20}$$

with ω as in (2.14) (for $k > 2$) or (2.18) (for $k = 2$). Furthermore, if Ω is a domain in $\mathbb{R}^N$ such that, for some $R > 0$, $\Omega \cap B_R(0) = D_A \cap B_R(0)$ and w is a positive harmonic function in Ω vanishing on $d_A \cap B_R(0)$ then $w \sim v_A$ in $\Omega \cap B_{R'}(0)$ for every $R' \in (0, R)$.

Similarly we find a positive harmonic function in D_A vanishing on $\partial D_A \setminus \{0\}$, singular at the origin, of the form

$$K'_A(x) = |x|^{\kappa_-}\, \omega(x/|x|).$$

If Ω is a domain as above and z is a positive harmonic function in Ω vanishing on $d_A \cap B_R(0) \setminus \{0\}$ then $z \sim K'_A$ in $\Omega \cap B_{R'}(0) \setminus \{0\}$ for every $R' \in (0, R)$.

As K'_A is a kernel function of $-\Delta$ at 0 it follows that K'_A is, up to a multiplicative constant c_A, the Martin kernel of $-\Delta$ in D_A, with singularity at 0. The Martin kernel, with singularity at a point $z \in d_A$, is given by

$$K_A(x,z) = c_A \frac{(\sin\theta_{N-1}\sin\theta_{N-2}\dots\sin\theta_k)^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x-z|^{N-2+\kappa_+}} \tag{2.21}$$

for every $x \in D_A$. From (2.1)

$$\sin\theta_{N-1}\sin\theta_{N-2}\dots\sin\theta_k = |x-z|^{-1}\sqrt{x_1^2+x_2^2+\dots+x_k^2}.$$

Therefore, if we write $x \in \mathbb{R}^N$ in the form $x = (x', x'')$, $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \cdots, x_N)$, we obtain the formula,

$$\begin{aligned} K_A(x,z) &= c_A \frac{|x'|^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x-z|^{(N-2+2\kappa_+)}} \\ &= c_A \frac{|x'|^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1})}{(|x'|^2+|x''-z|^2)^{(N-2+2\kappa_+)/2}}. \end{aligned} \tag{2.22}$$

Therefore, the Poisson potential of a measure $\mu \in \mathfrak{M}(d_A)$ is expressed by

$$\begin{aligned} \mathbb{K}_A[\mu](x) &= c_A |x'|^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1}) \\ &\quad \times \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(|x'|^2+|x''-z|^2)^{(N-2+2\kappa_+)/2}}. \end{aligned} \tag{2.23}$$

2.4. The admissibility condition

Consider the boundary value problem

$$\begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } D_A \\ u = \mu \in \mathfrak{M}(\partial D_A). \end{cases} \tag{2.24}$$

Let

$$\Gamma_R = \left\{x = \left(x', x''\right) : \left|x'\right| \le R, \left|x''\right| \le R\right\}, \quad D_{A,R} := D_A \cap \Gamma_R \tag{2.25}$$

and let $\rho_{R,A}$ denote the first (positive) eigenfunction in $D_{A,R} := D_A \cap \Gamma_R$. In the rest of this section we drop the index A in K_A, $\rho_{A,R}$ etc., except for D_A, $D_{A,R}$ and d_A.

First we observe that a positive Radon measure on d_A is q-good relative to D_A if and only if, for every compact set $F \subset d_A$, $\mu\,\chi_F$ is q-good in D_A

Now suppose that μ is compactly supported in d_A and denote its support by F. We claim that μ is q-good in D_A if and only if it is q-good relative to $D_{A,R}$ for all sufficiently large R. Let R be such that $F \subset B^{N-k}_{R/2}(0)$. Assume that μ is q-good in $D_{A,R}$. Let v_R be the solution of (1.1) in $D_{A,R}$ such that $v_R = \mu$ on $d_A \cap \Gamma_R$, $v_R = 0$ on $\partial D_{A,R} \setminus d_A$. Then v_R increases with R and $v = \lim_{R\to\infty} v_R$ is a solution of (1.1) in D_A with boundary data μ. This proves our claim in one direction; the other direction is obvious.

The condition for μ to be q-admissible in $D_{A,R}$ is

$$\int_{D_{A,R}} \mathbb{K}^R[|\mu|](x)^q \rho_R(x)dx < \infty, \tag{2.26}$$

where K^R is the Martin kernel of $-\Delta$ in $D_{A,R}$. If R is sufficiently large then, in a neighborhood of F, $K^R \sim K$ and $\rho^R \sim \rho \sim v_A$. Therefore, a sufficient condition for μ to be q-good in D_A is

$$\int_{\Gamma_R \cap D_A} \mathbb{K}[|\mu|](x)|^q \rho(x)dx < \infty \quad \forall R > 0. \tag{2.27}$$

By the first observation in this subsection, it follows that the previous statement remains valid for any positive Radon measure supported on d_A.

By (2.21),

$$\mathbb{K}[|\mu|](x) \le \; c_A (r')^{\kappa_+} \int_{\mathbb{R}^{N-k}} j(x', x'' - z)d|\mu|(z) \tag{2.28}$$

where

$$j(x) = |x|^{-N+2-2\kappa_+} \quad \forall x \in \mathbb{R}^N. \tag{2.29}$$

Therefore, using (2.20), condition (2.27) becomes

$$\int_0^R \int_{|x''|<R} \left(\int_{\mathbb{R}^{N-k}} j(x', x'' - z)d|\mu|(z) \right)^q (r')^{(q+1)\kappa_+ + k-1} dx'' dr' < \infty \tag{2.30}$$

for every $R > 0$.

2.5. The critical values

Relative to the equation

$$-\Delta u + |u|^{q-1}u = 0 \tag{2.31}$$

there exist two thresholds of criticality associated with the edge d_A.

The first is the value q_c^* such that, for $q_c^* \le q$ the whole edge d_A is removable but for $1 < q < q_c^*$ there exist non-trivial solutions in D_A which vanish on $\partial D_A \backslash d_A$. The second $q_c < q_c^*$ corresponds to the removability of points on d_A. For $q \ge q_c$ points on d_A are removable while for $1 < q < q_c$ there exist solutions with isolated point singularities on d_A. In the next two propositions we determine these critical values.

Proposition 2.1. *Assume $q > 1$, $1 \le k < N$. Then the condition*

$$q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+} \tag{2.32}$$

is necessary and sufficient for the existence of a non-trivial solution u of (2.31) *in D_A which vanishes on $\partial D_A \setminus d_A$. Furthermore, when this condition holds, there exist non-trivial positive bounded measures μ on d_A such that $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$.*

Remark. The statement remains true for $k = N$, which is the case of the cone. In this case $q_c = q_c^* = 1 - (2/\kappa_-)$ and a straightforward computation yields:

$$q_c = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}}. \tag{2.33}$$

Proof. Recall that $\lambda_A - (N-k)\kappa_+$ is the first eigenvalue in $S_A^{\{k-1\}}$ (see (2.15) and the remarks following it). Let κ'_+, κ'_- be the two roots of the equation

$$X^2 + (k-2)X - (\lambda_A - (N-k)\kappa_+) = 0,$$

i.e.,

$$\kappa'_\pm = \frac{1}{2}\left(2 - k \pm \sqrt{(k-2)^2 + 4(\lambda_A - (N-k)\kappa_+}\right).$$

Then, by [24, Theorem 5.7], recalled in Subsection 2.2, if $1 < q < 1 - (2/\kappa'_-)$ there exists a unique solution of (2.31) in the cone $C_{S_A^{k-1}}$ *i.e.* the cone with opening $S_A^{k-1} \subset S^{k-1} \subset \mathbb{R}^k$ with trace $a\delta_0$ (where δ_0 denotes the Dirac measure at the vertex of the cone and $a > 0$). By (2.5) this solution satisfies

$$u_a(x) = a\,|x|^{-\alpha}\,\phi(x/\,|x|)(1 + o(1)) \quad \text{as } x \to 0, \tag{2.34}$$

where ϕ is the first positive eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_A^{k-1})$ normalized so that u_1 possesses trace δ_0.

The function u given by

$$\tilde{u}_a(x', x'') = u_a(x') \quad \forall (x', x'') \in D_A = C_{S_A^{k-1}} \times \mathbb{R}^{N-k},$$

is a nonzero solution of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$ and has bounded trace on d_A.

A simple calculation shows that $1 - (2/\kappa'_-)$ equals q_c^* as given in (2.32).

Next, assume that $q \geq q_c^*$ and let u be a solution of (2.31) in D_A which vanishes on $\partial D_A \setminus d_A$.

Given $\epsilon > 0$ let v_ϵ be the solution of (2.31) in $D_A^{\{N-k-1\}} \setminus \{x' \in \mathbb{R}^k : |x'| \leq \epsilon\}$ such that

$$v_\epsilon(x') = \begin{cases} 0, & \text{if } x' \in \partial D_A^{\{N-k-1\}},\ |x'| > \epsilon, \\ \infty, & \text{if } |x'| = \epsilon. \end{cases}$$

Given $R > 0$ let w_R be the maximal solution in $\{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$.

Then the function u^* given by

$$u^*(x', x'') = v_\epsilon(x') + w_R(x'')$$

is a supersolution of (2.31) in $D_A \setminus \{(x', x'') : |x'| > \epsilon,\ |x''| < R\}$ and it dominates u in this domain. But $w_R(x'') \to 0$ as $R \to \infty$ and, by [10], $v_\epsilon(x') \to 0$ as $\epsilon \to 0$. Therefore $u_+ = 0$ and, by the same token, $u_- = 0$. □

Proposition 2.2. *Let A be defined as before. Then*

$$\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A) \quad \forall \mu \in \mathfrak{M}(d_A), \quad \forall R > 0 \tag{2.35}$$

if and only if

$$1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}. \tag{2.36}$$

This statement is equivalent to the following:

Condition (2.36) *is necessary and sufficient in order that the Dirac measure* $\mu = \delta_P$, *supported at a point* $P \in d_A$, *satisfy* (2.35).

Proof. It is sufficient to prove the result relative to the family of measures μ such that μ is positive, has compact support and $\mu(d_A) = 1$. Let $R > 1$ be sufficiently large so that the support of μ is contained in $\Gamma_{R/2}$. The measure μ can be approximated (in the sense of weak convergence of measures) by a sequence $\{\mu_n\}$ of convex combinations of Dirac measures supported in $d_A \cap \Gamma_{R/2}$. For such a sequence $\mathbb{K}[\mu_n] \to \mathbb{K}[\mu]$ pointwise and $\{\mathbb{K}[\mu_n]\}$ is uniformly bounded in $D_A \setminus \Gamma_{3R/4}$. Therefore it is sufficient to prove the result when $\mu = \delta_0$. In this case the admissibility condition (1.5)) is

$$\int_0^R \int_{|x''|<R} j(x)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty,$$

i.e.,

$$\int_0^R \int_0^R |x|^{q(2-N-2\kappa_+)} \left(r'\right)^{(q+1)\kappa_+ + k - 1} \left(r''\right)^{N-k-1} dr'' dr' < \infty.$$

Substituting $\tau := r''/r'$ the condition becomes

$$\int_0^R \int_0^{R/r'} \left(1+\tau^2\right)^{\frac{q}{2}(2-N-2\kappa_+)} \left(r'\right)^{q(2-N-\kappa_+)+\kappa_++N-1} \tau^{N-k-1} d\tau\, dr' < \infty.$$

This holds if and only if $q < (\kappa_+ + N)/(\kappa_+ + N - 2)$. □

Remark. It is interesting to notice that k does not appear explicitly in (2.36). Furthermore, we observe that

$$\frac{2}{q_c-1}\left(\frac{2q_c}{q_c-1} - N\right) = \lambda_A \Longleftrightarrow \kappa_+(\kappa_+ + N - 2) = \lambda_A, \tag{2.37}$$

which follows from (2.11). This implies that there does not exist a nontrivial solution of the nonlinear eigenvalue problem

$$\begin{aligned} -\Delta_{S^{N-1}}\psi - \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\psi + |\psi|^{q-1}\psi = 0 \quad &\text{in } S_{D_A} \\ \psi = 0 \quad &\text{in } \partial S_{D_A} \end{aligned} \tag{2.38}$$

which, in turn, implies that there does not exists a nontrivial solution of (2.31) of the form $u(x) = u(r, \sigma) = |x|^{-2/(q-1)}\psi(\sigma)$, and also no solution of this equation in D_A which vanishes on $\partial D_A \setminus \{0\}$. This is the classical ansatz for the removability of isolated singularities in d_A.

3. The harmonic lifting of a Besov space $B^{-s,p}(d_A)$

Denote by $W^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0, 1 \le p \le \infty$) the Sobolev spaces over $\mathbb{R}^\ell$. In order to use interpolation, it is useful to introduce the Besov space $B^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$). If σ is not an integer then

$$B^{\sigma,p}\left(\mathbb{R}^\ell\right) = W^{\sigma,p}\left(\mathbb{R}^\ell\right). \tag{3.1}$$

If σ is an integer the space is defined as follows. Put

$$\Delta_{x,y} f = f(x+y) + f(x-y) - 2f(x).$$

Then

$$B^{1,p}\left(\mathbb{R}^\ell\right) = \left\{ f \in L^p\left(\mathbb{R}^\ell\right) : \frac{\Delta_{x,y} f}{|y|^{1+\ell/p}} \in L^p\left(\mathbb{R}^\ell \times \mathbb{R}^\ell\right) \right\}, \tag{3.2}$$

with norm

$$\|f\|_{B^{1,p}} = \|f\|_{L^p} + \left(\iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|\Delta_{x,y} f|^p}{|y|^{\ell+p}} dx\, dy \right)^{1/p}, \tag{3.3}$$

(with standard modification if $p=\infty$) and

$$B^{m,p}(\mathbb{R}^\ell)=\Big\{f\in W^{m-1,p}(\mathbb{R}^\ell):\\ D^\alpha_x f\in B^{1,p}(\mathbb{R}^\ell)\ \forall\alpha\in\mathbb{N}^\ell,\ |\alpha|=m-1\Big\} \tag{3.4}$$

with norm

$$\|f\|_{B^{m,p}}=\|f\|_{W^{m-1,p}}+\left(\sum_{|\alpha|=m-1}\iint_{\mathbb{R}^\ell\times\mathbb{R}^\ell}\frac{|D^\alpha_x\Delta_{x,y}f|^p}{|y|^{\ell+p}}dx\,dy\right)^{1/p}. \tag{3.5}$$

We recall that the following inclusions hold ([27, p 155])

$$\begin{aligned} W^{m,p}(\mathbb{R}^\ell)\subset B^{m,p}(\mathbb{R}^\ell) \quad &\text{if } p\geq 2\\ B^{m,p}(\mathbb{R}^\ell)\subset W^{m,p}(\mathbb{R}^\ell) \quad &\text{if } 1\leq p\leq 2. \end{aligned} \tag{3.6}$$

When $1<p<\infty$, the dual spaces of $W^{s,p}$ and $B^{m,p}$ are respectively denoted by $W^{-s,p'}$ and $B^{-m,p'}$.

The following is the main result of this section.

Theorem 3.1. *Suppose that $q_c<q<q_c^*$ and let A be defined as in Subsection* 2.1. *Then there exist positive constants c_1, c_2, depending on q, N, k, κ_+, such that for any $R>1$ and any $\mu\in\mathfrak{M}_+(d_A)$ with support in $B_{R/2}$*:

$$\begin{aligned} &c_1\,\|\mu\|^q_{B^{-s,q}(\mathbb{R}^{N-k})}\\ &\leq\int_{D_{A,R}}\mathbb{K}[|\mu|]^q(x)\rho(x)dx\leq c_2(1+R)^\beta\,\|\mu\|^q_{B^{-s,q}(\mathbb{R}^{N-k})}, \end{aligned} \tag{3.7}$$

where $s=2-\frac{\kappa_++k}{q'}$, $\beta=(q+1)\kappa_++k-1$ and $D_{A,R}=D_A\cap\Gamma_R$. If $q=q_c$ the estimate remains valid for measures μ such thatthe diameter of supp μ *is sufficiently small* (*depending on the parameters mentioned before*).

Remark. When $q\geq 2$ the norms in the Besov space may be replaced by the norms in the corresponding Sobolev spaces.

Recall the admissibility condition for a measure $\mu\in\mathfrak{M}_+(d_A)$:

$$\int_{D_{A,R}}\mathbb{K}[\mu]^q(x)\rho(x)dx<\infty\quad\forall R>0$$

and the equivalence (see (2.27)-(2.30))

$$\begin{aligned} &\int_{D_{A,R}}\mathbb{K}[\mu]^q(x)\rho(x)dx\approx J^{A,R}(\mu)\\ &:=\int_0^R\int_{B''_R}\left(\int_{\mathbb{R}^{N-k}}\frac{d\mu(z)}{(\tau^2+|x''-z|^2)|)^{(N-2+2\kappa_+)/2}}\right)^q\tau^{(q+1)\kappa_++k-1}dx''d\tau, \end{aligned} \tag{3.8}$$

where $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\tau = |x'|$ and $B''_R = \{x'' \in \mathbb{R}^{N-k} : |x''| < R\}$. We denote

$$\nu = N - 2 + 2\kappa_+. \tag{3.9}$$

If $2\kappa_+$ is an integer, it is natural to relate (3.8) to the Poisson potential of μ in $\mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}_{n-1}$ where $n = N - 2 + 2\kappa_+$. We clarify this statement below.

Assuming that $2 \leq n + k - N$, denote

$$y = \big(y_1, \widetilde{y}, y''\big) \in \mathbb{R}^n, \quad \widetilde{y} = (y_2, \dots, y_{n+k-N}), \quad y'' = (y_{n+k-N+1}, \dots, y_n).$$

The Poisson kernel in $\mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}_{n-1}$ is given by

$$P_n(y) = \gamma_n y_1 |y|^{-n} \quad y_1 > 0, \tag{3.10}$$

for some $\gamma_n > 0$, and the Poisson potential of a bounded Borel measure μ with support in

$$\mathbf{d} := \big\{y = \big(0, y''\big) \in \mathbb{R}^n : y'' \in \mathbb{R}_{N-k}\big\}$$

is

$$\mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{\big(y_1^2 + |\widetilde{y}|^2 + |y'' - z|^2\big)^{n/2}} \quad \forall y \in \mathbb{R}^n_+. \tag{3.11}$$

In particular, for $\widetilde{y} = 0$,

$$\mathbb{K}_n[\mu](y_1, 0, y'') = \gamma_n y_1 \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{\big(y_1^2 + |y'' - z|^2\big)^{n/2}}. \tag{3.12}$$

The integral in (3.12) is precisely the same as the inner integral in (3.8).

In fact, it will be shown that, if we set

$$n := \{\nu\} = \inf\{m \in \mathbb{N} : m \geq \nu\}, \tag{3.13}$$

this approach also works when $2\kappa_+$ is not an integer. We note that, for n given by (3.13),

$$n - N + k \geq 2, \tag{3.14}$$

with equality only if $k = 3$ and $\kappa_+ \leq 1/2$ or $k = 2$ and $\kappa_+ \in (1/2, 1]$. Indeed,

$$n - N + k = k + \{2\kappa_+\} - 2$$

and (as $\kappa_+ > 0$) $\{2\kappa_+\} \geq 1$. If $k = 2$ then $\kappa_+ > 1/2$ and consequently $\{2\kappa_+\} \geq 2$. These facts imply our assertion.

We also note that κ_+ is strictly increasing relative to λ_A and

$$\kappa_+ \begin{cases} = 1 & \text{if } D_A = \mathbb{R}^N_+ \\ < 1 & \text{if } D_A \subsetneq \mathbb{R}^N_+ \\ > 1 & \text{if } D_A \supsetneq \mathbb{R}^N_+. \end{cases} \tag{3.15}$$

Finally we observe that $\gamma := \lambda_A - (N-k)\kappa_+ > 0$ (see (2.15)) and, by (2.11) and (2.32),

$$\gamma = \kappa_+^2 + (k-2)\kappa_+, \quad q_c^* = 1 + \frac{-(k-2)+\sqrt{(k-2)^2+4\gamma}}{\gamma}. \tag{3.16}$$

Therefore q_c^* is strictly decreasing relative to γ and consequently also relative to κ_+.

The proof of the theorem is based on the following important result proved in [28, 1.14.4.]

Proposition 3.2. *Let* $1 < q < \infty$ *and* $s > 0$. *Then for any bounded Borel measure* μ *in* $\mathbb{R}^{n-1}$ *there holds*

$$I(\mu) = \int_{\mathbb{R}_+^n} |\mathbb{K}_n[\mu](y)|^q \, e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|^q_{B^{-s,q}(\mathbb{R}^{n-1})}. \tag{3.17}$$

In the first part of the proof we derive inequalities comparing $I(\mu)$ and $J^{A,R}(\mu)$. Actually, it is useful to consider a slightly more general expression than $I(\mu)$, namely:

$$I^{m,j}_{\nu,\sigma}(\mu) := \int_{\mathbb{R}_+^{m+j}} \left| \int_{\mathbb{R}^m} \frac{y_1 d\mu(z)}{\left(y_1^2 + |\widetilde{y}|^2 + |y''-z|^2\right)^{\nu/2}} \right|^q e^{-y_1} y_1^{\sigma q-1} dy, \tag{3.18}$$

where ν is an arbitrary number such that$\nu > m$, $j \geq 1$ and $\sigma > 0$. A point $y \in \mathbb{R}_+^{m+j}$ is written in the form $y = (y_1, \widetilde{y}, y'') \in \mathbb{R}_+ \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. We assume that μ is supported in $\mathbb{R}^m$. Note that,

$$I(\mu) = \gamma_n^q I^{m,j}_{n,s} \quad \text{where } m = N-k, \quad j = n-m = n-N+k. \tag{3.19}$$

Put

$$F_{\nu,m}[\mu](\tau) := \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{\left(\tau^2 + |y''-z|^2\right)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0,\infty). \tag{3.20}$$

With this notation, if $j \geq 2$ then

$$I^{m,j}_{\nu,\sigma}(\mu) := \int_0^\infty \int_{\mathbb{R}^{j-1}} F_{\nu,m}[\mu]\left(\sqrt{y_1^2+|\widetilde{y}|^2}\right) e^{-y_1} y_1^{(\sigma+1)q-1} d\widetilde{y}\, dy_1 \tag{3.21}$$

and if $j = 1$

$$I^{m,1}_{\nu,\sigma}(\mu) := \int_0^\infty F_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{(\sigma+1)q-1}\, dy_1. \tag{3.22}$$

Lemma 3.3. *Assume that $m < \nu$, $0 < \sigma$, $2 \le j$ and $1 < q < \infty$. Then there exists a positive constant c, depending on m, j, ν, σ, q, such that, for every bounded Borel measure μ with support in $\mathbb{R}^m$:*

$$\frac{1}{c}\int_0^\infty F_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau \le I^{m,j}_{\nu,\sigma}(\mu) \le c\int_0^\infty F_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau, \quad (3.23)$$

where $F_{\nu,m}$ is given by (3.20) *and, for every $\tau > 0$,*

$$h_{\sigma,j}(\tau) = \begin{cases} \dfrac{\tau^{(\sigma+1)q+j-2}}{(1+\tau)^{(\sigma+1)q}}, & \text{if } j \ge 2, \\ e^{-\tau}\tau^{(\sigma+1)q-1}, & \text{if } j = 1. \end{cases} \quad (3.24)$$

Proof. There is nothing to prove in the case $j = 1$. Therefore we assume that $j \ge 2$.

We use the notation $y = (y_1, \tilde{y}, y'') \in \mathbb{R}\times\mathbb{R}^{j-1}\times\mathbb{R}^m$. The integrand in (3.21) depends only on y_1 and $\rho := |\tilde{y}|$. Therefore, $I^{m,j}_{\nu,\sigma}$ can be written in the form

$$I^{m,j}_{\nu,\sigma}(\mu) = c_{m,j}\int_0^\infty\int_0^\infty F_{\nu,m}[\mu]\left(\sqrt{y_1^2+\rho^2}\right)e^{-y_1}y_1^{(\sigma+1)q-1}\,dy_1\rho^{j-2}d\rho.$$

We substitute $y_1 = (\tau^2-\rho^2)^{1/2}$, then change the order of integration and finally substitute $\rho = r\tau$. This yields,

$$\begin{aligned} &c^{-1}_{m,j}I^{m,j}_{\nu,\sigma}(\mu) \\ &= \int_0^\infty\int_\rho^\infty F_{\nu,m}[\mu](\tau)\rho^{j-2}e^{-\sqrt{\tau^2-\rho^2}}(\tau^2-\rho^2)^{(\sigma+1)q/2-1}\tau\,d\tau\,d\rho \\ &= \int_0^\infty\int_0^\tau F_{\nu,m}[\mu](\tau)\rho^{j-2}e^{-\sqrt{\tau^2-\rho^2}}(\tau^2-\rho^2)^{(\sigma+1)q/2-1}\tau\,d\rho\,d\tau \\ &= \int_0^\infty\int_0^1 F_{\nu,m}[\mu](\tau)\tau^{j-2+(\sigma+1)q}e^{-\tau\sqrt{1-r^2}}f(r)dr\,d\tau, \end{aligned}$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

We denote

$$I^j_\sigma(\tau) = \int_0^1 e^{-\tau\sqrt{1-r^2}}f(r)dr,$$

so that

$$I^{m,j}_{\nu,\sigma}(\mu) = c_{m,j}\int_0^\infty F_{\nu,m}[\mu](\tau)\tau^{j-2+(\sigma+1)q}I^j_\sigma(\tau)d\tau. \quad (3.25)$$

To complete the proof we estimate I_σ^j. Since $j \geq 2$, $f \in L^1(0,1)$ and I_σ^j is continuous in $[0,\infty)$ and positive everywhere. Hence, for every $\alpha > 0$, there exists a positive constant $c_\alpha = c_\alpha(\sigma)$ such that

$$\frac{1}{c_\alpha} \leq I_\sigma^j \leq c_\alpha \ \text{ in } \ [0,\alpha). \tag{3.26}$$

Next we estimate I_σ^j for large τ. Since $j \geq 2$,

$$I_\sigma^j \leq 2^{(\sigma+1)q/2-1} \int_0^1 (1-r)^{(\sigma+1)q/2-1} e^{-\tau\sqrt{1-r}}\, dr.$$

Substituting $r = 1 - t^2$ we obtain,

$$I_\sigma^j \leq 2^{(\sigma+1)q/2} \int_0^1 t^{(\sigma+1)q-1} e^{-t\tau}\, dt = c(\sigma,q)\tau^{-(\sigma+1)q}. \tag{3.27}$$

On the other hand, if $\tau \geq 2$,

$$\begin{aligned} I_\sigma^j(\tau) &= \int_0^1 \left(1-t^2\right)^{(j-3)/2} t^{(\sigma+1)q-1} e^{-\tau t}\, dt \\ &= \tau^{-(\sigma+1)q} \int_0^\tau \left(1-(s/\tau)^2\right)^{(j-3)/2} s^{(\sigma+1)q-1} e^{-s}\, ds \\ &\geq \tau^{-(\sigma+1)q} 2^{-(j-3)} \int_0^1 s^{(\sigma+1)q-1} e^{-s}\, ds. \end{aligned} \tag{3.28}$$

Combining (3.25) with (3.26)-(3.28) we obtain (3.23). □

Next we derive an estimate in which integration over $\mathbb{R}^n_+ = \mathbb{R}^j_+ \times \mathbb{R}^m$ is replaced by integration over a bounded domain, for measures supported in a fixed bounded subset of $\mathbb{R}^m$.

Let $B^j_R(0)$ and $B^m_R(0)$ denote the balls of radius R centered at the origin, in $\mathbb{R}^j$ and $\mathbb{R}^m$ respectively. Denote

$$F^R_{\nu,m}[\mu](\tau) = \int_{B^m_R} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y''-z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0,\infty) \tag{3.29}$$

and, if $j \geq 2$,

$$I^{m,j}_{\nu,\sigma}(\mu;R) = \int_{B^j_R \cap \{0<y_1\}} F^R_{\nu,m}[\mu]\left(\sqrt{y_1^2 + |\widetilde{y}|^2}\right) e^{-y_1} y_1^{\sigma q-1}\, d\widetilde{y}\, dy_1, \tag{3.30}$$

where $(y_1,\widetilde{y}) \in \mathbb{R} \times \mathbb{R}^{j-1}$. If $j = 1$ we denote

$$I^{m,1}_{\nu,\sigma}(\mu;R) = \int_0^R F^R_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{\sigma q-1}\, dy_1. \tag{3.31}$$

Similarly to Lemma 3.3 we obtain the following:

Lemma 3.4. *If $j \geq 1$, there exists a positive constant c such that, for any bounded Borel measure μ with support in $\mathbb{R}^m \cap B_R$*

$$c^{-1}\int_0^R F^R_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau \leq I^{m,j}_{\nu,\sigma}(\mu;R) \leq c\int_0^R F^R_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau \quad (3.32)$$

with $h_{\sigma,j}$ as in (3.24).

Proof. In the case $j=1$ there is nothing to prove. Therefore we assume that $j \geq 2$. From (3.30) we obtain

$$I^{m,j}_{\nu,\sigma}(\mu;R) = c_{m,j}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} F^R_{\nu,m}[\mu]\left(\sqrt{y_1^2+\rho^2}\right)e^{-y_1}y_1^{(\sigma+1)q-1}dy_1\rho^{j-2}d\rho.$$

Substituting $y_1 = (\tau^2-\rho^2)^{1/2}$, then changing the order of integration and finally substituting $\rho = r\tau$ we obtain

$$c^{-1}_{m,j}I^{m,j}_{\nu,\sigma}(\mu;R) = \int_0^R\int_0^1 F^R_{\nu,\mu}[\mu](\tau)\tau^{j-2+(\sigma+1)q}e^{-\tau\sqrt{1-r^2}}f(r)dr\,d\tau.$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

The remaining part of the proof is the same as for Lemma 3.3. □

Lemma 3.5. *Let $1 < q$, $0 < \sigma$ and assume that $m < \nu q$ and $0 \leq j-1 < \nu$. Then there exists a positive constant $\bar{c}$, depending on j, m, q, σ, ν, such that, for every $R \geq 1$ and every bounded Borel measure μ with support in $B_{R/2}(0)\cap\mathbb{R}^m$,*

$$\left|\int_0^\infty F_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau - \int_0^R F^R_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau\right| \leq \bar{c}R^{(\sigma+1-\nu)q+m+j-1}\|\mu\|^q_{\mathfrak{M}} \quad (3.33)$$

with $h_{\sigma,j}$ as in (3.24).

Proof. We estimate

$$\left|\int_0^\infty F_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau - \int_0^R F^R_{\nu,m}[\mu](\tau)h_{\sigma,j}(\tau)d\tau\right| \leq \int_R^\infty \left|F_{\nu,m}[\mu]\right|(\tau)h_{\sigma,j}(\tau)d\tau + \int_0^R \left|F_{\nu,m}[\mu]-F^R_{\nu,m}[\mu]\right|(\tau)h_{\sigma,j}(\tau)d\tau. \quad (3.34)$$

For every $\tau > 0$,

$$\left|F_{\nu,m}[\mu]\right|(\tau) \leq \tau^{-\nu q}\|\mu\|^q_{\mathfrak{M}}. \quad (3.35)$$

Since $j-1<\nu q$, it follows that

$$\begin{aligned}\int_R^\infty \left|F_{\nu,m}[\mu]\right|(\tau)h_{\sigma,j}(\tau)d\tau &\leq \|\mu\|^q_{\mathfrak{M}}\int_R^\infty \tau^{-\nu q}h_{\sigma,j}(\tau)d\tau\\ &\leq c(\sigma,q)\,\|\mu\|^q_{\mathfrak{M}}\int_R^\infty \frac{\tau^{(\sigma+1)q+j-2-\nu q}}{(1+\tau)^{(\sigma+1)q}}d\tau \\ &\leq \frac{c(\sigma,q)}{\nu q-j+1}\,\|\mu\|^q_{\mathfrak{M}}R^{j-1-\nu q}.\end{aligned}\tag{3.36}$$

Since, by assumption, $\operatorname{supp}\mu\subset B_{R/2}$, we have

$$\begin{aligned}&\int_0^R \left|F_{\nu,m}[\mu]-F^R_{\nu,m}[\mu]\right|(\tau)h_{\sigma,j}(\tau)d\tau\\ &\quad\leq \int_0^R\int_{|y''|>R}\left|\int_{\mathbb{R}^m}\frac{d\mu(z)}{\left(\tau^2+|y''-z|^2\right)^{\nu/2}}\right|^q dy''h_{\sigma,j}(\tau)d\tau\\ &\quad\leq \|\mu\|^q_{\mathfrak{M}}\int_0^R\int_{|\zeta|>R/2}\left(|\tau^2+|\zeta|^2\right)^{-\nu q/2}d\zeta\; h_{\sigma,j}d\tau\\ &\quad\leq c(m,q)\,\|\mu\|^q_{\mathfrak{M}}\int_0^R\int_{R/2}^\infty\left(\tau^2+\rho^2\right)^{-\nu q/2}\rho^{m-1}\,d\rho\, h_{\sigma,j}d\tau\\ &\quad\leq c(m,q)\,\|\mu\|^q_{\mathfrak{M}}\int_0^R\tau^{m-\nu q}\int_{R/2\tau}^\infty\left(1+\eta^2\right)^{-\nu q/2}\eta^{m-1}\,d\eta\, h_{\sigma,j}d\tau\\ &\quad\leq \frac{c(m,q)}{\nu q-m}\,\|\mu\|^q_{\mathfrak{M}}R^{m-\nu q}\int_0^R\tau^{(\sigma+1)q+j-2}\,d\tau\\ &\quad\leq \frac{c(m,q)}{(\nu q-m)((\sigma+1)q+j-1)}\,\|\mu\|^q_{\mathfrak{M}}R^{(\sigma+1)q+j-1+m-\nu q}.\end{aligned}\tag{3.37}$$

Combining (3.34)-(3.37) we obtain (3.33). □

Corollary 3.6. *For every $R>0$ put*

$$J^{m,j}_{\nu,\sigma}(\mu;R):=\int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(\sigma+1)q+j-2}d\tau.\tag{3.38}$$

Then

$$\begin{gathered}\frac{1}{c}I^{m,j}_{\nu,\sigma}(\mu)-\bar{c}R^\beta\,\|\mu\|^q_{\mathfrak{M}}\leq J^{m,j}_{\nu,\sigma}(\mu;R)\leq cR^{(\sigma+1)q}I^{m,j}_{\nu,\sigma}(\mu),\\ \beta=(\sigma+1-\nu)q+j+m-1,\end{gathered}\tag{3.39}$$

for every $R>1$ and every bounded Borel measure μ with support in $B^m_{R/2}(0):=B_{R/2}(0)\cap\mathbb{R}^m$.

Proof. This is an immediate consequence of Lemma 3.5 and Lemma 3.3. □

Lemma 3.7. *Let m, j be positive integers such that $j \geq 1$ and let $1 < q$, $0 < \sigma$. Put $n := m + j$.*

Then there exist positive constants $c, \bar{c}$, depending on j, m, q, σ, such that, for every $R > 1$ and every measure $\mu \in \mathfrak{M}_+(B^m_{R/2}(0))$,

$$\begin{aligned} &\frac{1}{c}\,\|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} - \bar{c}R^{q\left(\sigma-\frac{n-1}{q'}\right)}\,\|\mu\|^q_{\mathfrak{M}} \leq J^{m,j}_{n,\sigma}(\mu; R) \\ &\leq cR^{(\sigma+1)q}\,\|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})}. \end{aligned} \tag{3.40}$$

If $\sigma < \frac{n-1}{q'}$, there exists $R_0 > 1$ such that, for all $R > R_0$,

$$\frac{1}{2c}\,\|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} \leq J^{m,j}_{n,\sigma}(\mu; R). \tag{3.41}$$

*If $\sigma = \frac{n-1}{q'}$ then, there exists $a > 0$ such thatthe inequality remains valid for measures μ such that*diam(supp μ) $\leq a$.

If, in addition, $\frac{j-1}{q'} < \sigma$ then

$$\frac{1}{2c}\,\|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} \leq J^{m,j}_{n,\sigma}(\mu; R) \leq cR^{(\sigma+1)q}\,\|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)}, \tag{3.42}$$

where $s := \sigma - \frac{j-1}{q'}$.

Remark. Assume that $\mu \geq 0$. Then:

(i) If $\mu \in B^{-\sigma,q}(\mathbb{R}^{n-1})$ and $\frac{j-1}{q'} \geq \sigma$ then $\mu(\mathbb{R}^m) = 0$.

(ii) If $\mu \in B^{-s,q}(\mathbb{R}^m)$ and $\sigma > (n-1)/q'$ then $s > m/q'$ and therefore $B^{s,q'}(\mathbb{R}^m)$ can be embedded in $C(\mathbb{R}^m)$.

Proof. Inequality (3.40) follows from (3.39) and Proposition 3.2 (see also (3.19)).

For positive measures μ,

$$\|\mu\|_{\mathfrak{M}} = \mu\big(\mathbb{R}^{n-1}\big) \leq \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})}.$$

Therefore, if $\sigma < \frac{n-1}{q'}$, (3.40) implies that there exists $R_0 > 1$ such that(3.41) holds for all $R > R_0$.

If $\sigma = \frac{n-1}{q'}$ (3.40) implies that

$$\frac{1}{c}\,\|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} - \bar{c}\,\|\mu\|^q_{\mathfrak{M}} \leq J^{m,j}_{n,\sigma}(\mu; R).$$

But if μ is a positive bounded measure such thatdiam(supp μ) $\leq a$ then

$$\|\mu\|_{\mathfrak{M}} / \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} \to 0 \;\text{ as } a \to 0.$$

The last inequality follows from the imbedding theorem for Besov spaces according to which there exists a continuous trace operator $T : B^{\sigma,q'}(\mathbb{R}^{n-1}) \mapsto B^{s,q'}(\mathbb{R}^m)$ and a continuous lifting $T' : B^{s,q'}(\mathbb{R}^m) \mapsto B^{\sigma,q'}(\mathbb{R}^{n-1})$ where $s = \sigma - \frac{n-m-1}{q'}$. □

If $\nu \in \mathbb{N}$ and $\sigma = s + \frac{\nu-m-1}{q'}$,

$$J^{m,\nu-m}_{\nu,\sigma}(\mu; R) = \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(\sigma+1)q+\nu-m-2}\,d\tau$$
$$= \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(s+\nu-m)q-1}\,d\tau.$$

However, if μ is positive, the expression

$$M^m_{\nu,s}(\mu; R) := \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(s+\nu-m)q-1}\,d\tau, \tag{3.43}$$

is meaningful for any real $\nu > m$ and $s > 0$. Furthermore, as shown below, the results stated in Lemma 3.7 can be extended to this general case.

Theorem 3.8. *Let $1 < q$, $\nu \in \mathbb{R}$ and m a positive integer. Assume that $1 \leq \nu - m$ and $0 < s < m/q'$. Then there exists a positive constant c such that, for every bounded positive measure μ supported in $\mathbb{R}^m \cap B_{R/2}(0)$, $R > 1$,*

$$\frac{1}{c}\|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} \leq M^m_{\nu,s}(\mu; R) \leq cR^{(s+\nu-m)q+1}\|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)}. \tag{3.44}$$

This also holds when $s = m/q'$, provided that the diameter of supp μ *is sufficiently small.*

Proof. If ν is an integer and $j := \nu - m$ then this statement is part of Lemma 3.7. Indeed the condition $s > 0$ means that $\sigma = s + \frac{j-1}{q'} > \frac{j-1}{q'}$ and the condition $s < m/q'$ means that $\sigma < \frac{n-1}{q'}$.

Therefore we assume that $\nu \notin \mathbb{N}$. Let $n := \{\nu\}$ and $\theta := n - \nu$ so that $0 < \theta < 1$. Our assumptions imply that $1 \leq n - m - 1$ because (as ν is not an integer) $\nu - m > 1$ and consequently $n - m \geq 2$.

If a, b are positive numbers, put

$$A_\nu := \frac{a^{(s+\nu-m)q-1}}{(a^2+b^2)^{\nu q/2}}.$$

Obviously A_ν decreases as ν increases. Therefore, $A_n \leq A_\nu \leq A_{n-1}$ which in turn implies,

$$M^m_{n,s} \leq M^m_{\nu,s} \leq M^m_{n-1,s}.$$

By Lemma 3.7, the assertions of the theorem are valid in the case that $\nu = n$ or $\nu = n - 1$. Therefore the previous inequality implies that the assertions hold for any real ν subject to the conditions imposed. □

By (3.8),

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau)\tau^{(q+1)\kappa_+ + k - 1} d\tau,$$

where $m = N - k$ and $\nu = N - 2 + 2\kappa_+$. Consequently , by (3.38),

$$J^{A,R} = M_{\nu,s}^m$$

where s is determined by,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

It follows that

$$sq = -(k - 2 + 2\kappa_+)q + (q + 1)\kappa_+ + k = k(1 - q) + 2q - \kappa_+(q - 1)$$

and therefore

$$s = 2 - \frac{k + \kappa_+}{q'}.$$

Proof of Theorem 3.1. Put

$$\nu := N - 2 + 2\kappa_+, \quad s := 2 - \frac{\kappa_+ + k}{q'}, \quad m := N - k. \tag{3.45}$$

Recall that in the case $k = 2$ we have $\kappa_+ > 1/2$. Therefore

$$\nu - m - 1 = k - 3 + 2\kappa_+ > 0. \tag{3.46}$$

Furthermore,

$$(s + \nu - m)q - 1 = (q + 1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa.$$

Thus

$$J^{A,R} = \int_0^R F_{\nu,m}^R(\tau)\tau^{(q+1)\kappa_+ + k - 1} d\tau = M_{\nu,s}^m.$$

Next we show that $0 < s \le m/q'$. More precisely we prove

$$0 < s \le m/q' \iff q_c \le q < q_c^*. \tag{3.47}$$

Let μ be a bounded non-negative Borel measure in $B^{-s,q}(\mathbb{R}^m)$. If $s \le 0$, $B^{-s,q}(\mathbb{R}^m) \subset L^q(\mathbb{R}^m)$. Therefore, in this case, every bounded Borel measure on $\mathbb{R}^m$ is admissible *i.e.* satisfies (2.35). Consequently , by Proposition 2.2, $q < q_c$. As we assume $q \ge q_c$ it follows that $s > 0$.

If $s > 0$ and $sq' - m \ge 0$ then $C_{s,q'}(K) = 0$ for every compact subset of $\mathbb{R}^m$ and consequently $\mu(K) = 0$ for any such set. Conversely, if $sq' - m < 0$ then there exist non-trivial positive bounded measures in $B^{-s,q}(\mathbb{R}^m)$. Therefore, by Proposition 2.1, $sq' < m$ if and only if $q < q_c^*$.

In conclusion, $0 < s \le m/q'$ and $\nu - m \ge 1$; therefore Theorem 3.1 is a consequence of Theorem 3.8. □

Remark. Note that the critical exponent for the imbedding of $B^{2-\frac{\kappa_++k}{q'},q'}(\mathbb{R}^{N-k})$ into $C(\mathbb{R}^{N-k})$ is again

$$q = q_c = \frac{N+\kappa_+}{N+\kappa_+-2}.$$

4. Supercritical equations in a polyhedral domain

In this section q is a real number larger than 1 and P an N-dim polyhedral domain as described in Subsection 6.1. Denote by $\{L_{k,j} : k = 1,\dots,N,\ \ j = 1,\dots,n_k\}$ the family of faces, edges and vertices of P. In this notation, $L_{1,j}$ denotes one of the open faces of P; for $k = 2,\dots,N-1$, $L_{k,j}$ denotes a relatively open $(N-k)$-dimensional edge and $L_{N,j}$ denotes a vertex. For $1 \le k < N$, the $(N-k)$-dimensional space which contains $L_{k,j}$ is denoted by $\mathbb{R}^{N-k}_j$. If $1 < k < N$, the cylinder of radius r around the axis $\mathbb{R}^{N-k}_j$ will be denoted by $\Gamma^\infty_{k,j,r}$ and the subset $A_{k,j}$ of S^{k-1} is defined by

$$\lim_{r\to 0}\frac{1}{r}\left(\partial\Gamma^\infty_{k,j,r} \cap P\right) = L_{k,j} \times A_{k,j}.$$

$A_{k,j}$ is the 'opening' of P at the edge $L_{k,j}$. For $k = N$ we replace in this definition the cylinder $\Gamma^\infty_{N,j,r}$ by the ball $B_r(L_{N,j})$. For $1 < k \le N$ and $A = A_{k,j}$ we use d_A as an alternative notation for $\mathbb{R}^{N-k}_j$ and denote by D_A the k-dihedron with edge d_A and opening A as in Subsection 6.1 (with S_A defined as in (2.2)). For $k = 1$, D_A stands for the half space $\mathbb{R}^{N-1}_j \times (0,\infty)$.

4.1. Definitions and auxiliary results

Let Ω be a bounded Lipschitz domain. We say that $\{\Omega_n\}$ is a *Lipschitz exhaustion* of Ω if, for every n, Ω_n is Lipschitz and

$$\Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup\Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \to \mathbb{H}_{N-1}(\partial\Omega). \tag{4.1}$$

If ω_n (respectively ω) is the harmonic measure in Ω_n (respectively Ω) relative to $x_0 \in \Omega_1$, then, for every $Z \in C(\bar{\Omega})$,

$$\lim_{n\to\infty}\int_{\partial\Omega_n} Z\,d\omega_n = \int_{\partial\Omega} Z\,d\omega. \tag{4.2}$$

[24, Lemma 2.1]. Furthermore, if μ is a bounded Borel measure on $\partial\Omega$ and $v := \mathbb{K}^\Omega[\mu]$, there holds

$$\lim_{n\to\infty}\int_{\partial\Omega_n} Zv\,d\omega_n = \int_{\partial\Omega} Z\,d\mu, \tag{4.3}$$

[24, Lemma 2.2]. If v is a positive solution and (4.3) holds we say that μ is the *boundary trace* of v.

The following estimates are proved in [24, Lemma 2.3]:

Proposition 4.1. *Let μ be bounded Borel measures on $\partial\Omega$. Then $\mathbb{K}[\mu] \in L^1_\rho(\Omega)$ and there exists a constant $C = C(\Omega)$ such that*

$$\|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)} \leq C \|\mu\|_{\mathfrak{M}(\partial\Omega)} . \tag{4.4}$$

In particular if $h \in L^1(\partial\Omega; \omega)$ then

$$\|\mathbb{P}[h]\|_{L^1_\rho(\Omega)} \leq C \|h\|_{L^1(\partial\Omega;\omega)} . \tag{4.5}$$

The nest result will be used in deriving estimates in a k-dimensional dihedron when the boundary data is concentrated on the edge.

Proposition 4.2. *We denote by G^{Ω_n} (respectively G^{Ω}) the Green function in Ω_n (respectively Ω). Let v be a positive harmonic function in Ω with boundary trace μ. Let $Z \in C^2(\overline{\Omega})$ and let $\tilde{G} \in C^\infty(\Omega)$ be a function that coincides with $x \mapsto G(x, x_0)$ in $Q \cap \Omega$ for some neighborhood Q of $\partial\Omega$ and some fixed $x_0 \in \Omega$. In addition assume that there exists a constant $c > 0$ such that*

$$|\nabla Z \cdot \nabla\tilde{G}| \leq c\rho. \tag{4.6}$$

Under these assumptions, if $\zeta := Z\tilde{G}$ then

$$-\int_\Omega v\Delta\zeta\, dx = \int_{\partial\Omega} Z d\mu. \tag{4.7}$$

Proof. Let $\{\Omega_n\}$ be a C^1 exhaustion of Ω. We assume that $\partial\Omega_n \subset Q$ for all n and $x_0 \in \Omega_1$. Let $\tilde{G}_n(x)$ be a function in $C^1(\Omega_n)$ such that$\tilde{G}_n$ coincides with $G^{\Omega_n}(\cdot, x_0)$ in $Q\cap\Omega_n$, $\tilde{G}_n(\cdot, x_0) \to \tilde{G}(\cdot, x_0)$ in $C^2(\Omega\setminus Q)$ and $\tilde{G}_n(\cdot, x_0) \to \tilde{G}(\cdot, x_0)$ in Lip (Ω). If $\zeta_n = Z\tilde{G}_n$ we have,

$$\begin{aligned}
-\int_{\Omega_n} v\Delta\zeta_n\, dx &= \int_{\partial\Omega_n} v\partial_{\mathbf{n}}\zeta\, dS = \int_{\partial\Omega_n} vZ\partial_{\mathbf{n}}\tilde{G}_n(\xi, x_0)\, dS \\
&= \int_{\partial\Omega_n} vZP^{\Omega_n}(x_0, \xi)\, dS = \int_{\partial\Omega_n} vZ\, d\omega_n.
\end{aligned}$$

By (4.3),

$$\int_{\partial\Omega_n} vZ\, d\omega_n \to \int_{\partial\Omega} Z\, d\mu.$$

On the other hand, in view of (4.6), we have

$$\Delta\zeta_n = \tilde{G}_n\Delta Z + Z\Delta\tilde{G}_n + 2\nabla Z \cdot \nabla\tilde{G}_n \to \Delta Z$$

in $L^1_\rho(\Omega)$; therefore,

$$-\int_{\Omega_n} v\Delta\zeta_n\, dx \to -\int_\Omega v\Delta\zeta\, dx. \qquad \square$$

We denote by $\mathfrak{M}_q = \mathfrak{M}_q(\partial\Omega)$ the set of q-good measures on the boundary. A positive solution u of (1.1) in Ω possesses a boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$ if and only if

$$\int_{\Omega} u^q \rho dx < \infty \tag{4.8}$$

[24, Proposition 4.1]. In this case $\mu \in \mathfrak{M}_q$.

The following statements can be proved in the same way as in the case of smooth domains. For the proof in that case see [20].

I. $\mathfrak{M}_q(\partial\Omega)$ is a linear space and

$$\mu \in \mathfrak{M}_q(\partial\Omega) \iff |\mu| \in \mathfrak{M}_q(\partial\Omega).$$

II. If $\{\mu_n\}$ is an increasing sequence of measures in $\mathfrak{M}_q(\partial\Omega)$ and $\mu := \lim \mu_n$ is a finite measure, then $\mu \in \mathfrak{M}_q(\partial\Omega)$.

Proposition 4.3. *Let μ be a bounded measure on ∂P. (μ may be a signed measure.) For $i = 1, \dots, N$, $j = 1, \dots, n_i$, we define the measure $\mu_{k,j}$ on $d_{A_{k,j}}$ by,*

$$\mu_{k,j} = \mu \text{ on } L_{k,j}, \qquad \mu_{k,j} = 0 \text{ on } d_{A_{k,j}} \setminus L_{k,j}.$$

Then $\mu \in \mathfrak{M}_q(\partial P)$, i.e., the problem

$$-\Delta u + u^q = 0 \text{ in } P, \; u = \mu \text{ on } \partial P \tag{4.9}$$

possesses a solution, if and only if $\mu_{k,j}$ is a q-good measure relative to $D_{A_{k,j}}$ for all (k, j) as above.

Proof. In view of statement **I** above, it is sufficient to prove the proposition in the case that μ is non-negative. This is assumed hereafter. If $\mu \in \mathfrak{M}_q(\partial P)$ then any measure ν on ∂P such that$0 \le \nu \le \mu$ is a q-good measure relative to P. Therefore

$$\mu \in \mathfrak{M}_q(\partial P) \Longrightarrow \mu'_{k,j} := \mu \chi_{L_{k,j}} \in \mathfrak{M}_q(\partial P).$$

Assume that $\mu \in \mathfrak{M}_q(\partial P)$ and let $u_{k,j}$ be the solution of (4.9) when μ is replaced by $\mu'_{k,j}$. Denote by $u'_{k,j}$ the extension of $u_{k,j}$ by zero to the k-dihedron $D_{A_{k,j}}$. Then $u'_{k,j}$ is a subsolution of (1.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. In the present case there always exists a supersolution, *e.g.* the maximal solution of (1.1) in $D_{A_{k,j}}$ vanishing outside $d_{A_{k,j}} \setminus \bar{L}_{k,j}$. Therefore there exists a solution $v_{k,j}$ of this equation in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$, *i.e.*, $\mu_{k,j}$ is q-good relative to $D_{A_{k,j}}$.

Next assume that $\mu \in \mathfrak{M}(\partial P)$ and that $\mu_{k,j}$ is q-good relative to $D_{A_{k,j}}$ for every (k, j) as above. Let $v_{k,j}$ be the solution of (1.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. Then $v_{k,j}$ is a supersolution of problem (4.9) with μ replaced by $\mu'_{k,j}$ and consequently there exists a solution $u_{k,j}$ of this problem. It follows that

$$w := \max\{u_{k,j} : k = 1, \dots, N, \; j = 1, \dots, n_k\}$$

is a subsolution while

$$\bar{w} := \sum_{\substack{k=1,\dots,N,\\ j=1,\dots,n_k}} u_{k,j}$$

is a supersolution of (4.9). Consequently there exists a solution of this problem, *i.e.*, $\mu \in \mathfrak{M}_q(\partial P)$. □

4.2. Removable singular sets and 'good measures', I

We first introduce some standard elements associated to the Bessel capacities which are the natural way to characterize good measures or removable sets. For $\alpha \in \mathbb{R}$, we denote by G_α the Bessel kernel of order α, defined by

$$G_\alpha(\xi) = \mathcal{F}^{-1}\left((1 + |\cdot|^2)^{-\frac{\alpha}{2}}\right)(\xi), \tag{4.10}$$

where $\mathcal{F}$ is the Fourier transform in the space $\mathcal{S}'(\mathbb{R}^\ell)$ of moderate distributions in $\mathbb{R}^\ell$. For $1 \le p \le \infty$, the Bessel space $L_{\alpha,p}(\mathbb{R}^\ell)$ is defined by

$$L_{\alpha,p}(\mathbb{R}^\ell) = \{f : f = G_\alpha * g, : g \in L^p(\mathbb{R}^\ell)\}, \tag{4.11}$$

with norm

$$\|f\|_{L_{\alpha,p}} = \|g\|_{L_p} = \|G_{-\alpha} * f\|_{L_p}.$$

For $\alpha, \beta \in \mathbb{R}$ and $1 < p < \infty$, the mapping $f \mapsto G_\beta * f$ is an isomorphism from $L_{\alpha,p}(\mathbb{R}^\ell)$ into $L_{\alpha+\beta,p}(\mathbb{R}^\ell)$. Finally the Bessel spaces are connected to Besov and Sobolev spaces: when $\alpha > 0$ and $1 < p < \infty$, it is known that if $\alpha \in \mathbb{N}$, $L_{\alpha,p}(\mathbb{R}^\ell) = W^{\alpha,p}(\mathbb{R}^\ell)$ and if $\alpha \notin \mathbb{N}$, then $L_{\alpha,p}(\mathbb{R}^\ell) = B^{\alpha,p}(\mathbb{R}^\ell)$, with equivalent norms (see *e.g.* [5,27]).

The Bessel capacity $C^{\mathbb{R}^\ell}_{\alpha,p}$ ($\alpha > 0$, $p \ge 1$) is defined by the following rules: if $K \subset \mathbb{R}^\ell$ is compact

$$C^{\mathbb{R}^\ell}_{\alpha,p}(K) = \inf\left\{\|f\|^p_{L_{\alpha,p}} : f \in \mathcal{S}(\mathbb{R}^\ell),\ f \ge \chi_K\right\}. \tag{4.12}$$

If G is open

$$C^{\mathbb{R}^\ell}_{\alpha,p}(G) = \sup\left\{C^{\mathbb{R}^\ell}_{\alpha,p}(K) : K \subset G,\ K \text{ compact}\right\}. \tag{4.13}$$

If A is any set

$$C^{\mathbb{R}^\ell}_{\alpha,p}(A) = \inf\left\{C^{\mathbb{R}^\ell}_{\alpha,p}(G) : A \subset G,\ G \text{ open}\right\}. \tag{4.14}$$

Note that the capacity of any non-empty set is positive if and only if $\alpha > \frac{\ell}{p}$ because of Sobolev-Besov embedding theorem.

Proposition 4.4. *Let A be a Lipschitz domain on S^{k-1}, $2 \leq k \leq N-1$, and let D_A be the k-dihedron with opening A. Let $\mu \in \mathfrak{M}(\partial D_A)$ be a positive measure with compact support contained in d_A (= the edge of D_A). Assume that μ is q-good relative to D_A. Let $R > 1$ be large enough so that* $\operatorname{supp}\mu \subset B_R^{N-k}(0)$ *and let u be the solution of* (1.1) *in D_A^R with trace μ on d_A^R and trace zero on $\partial D_A^R \setminus d_A^R$. Then*:

(i) *For every non-negative $\eta \in C_0^\infty(B_{3R/4}^{N-k}(0))$,*

$$\left(\int_{d_A^R} \eta^{q'} d\mu\right) \leq cM^{q'} \int_{D_A^R} u^q \rho dx + cM^{q'} \left(\int_{D_A^R} u^q \rho dx\right)^{\frac{1}{q}} \left(1 + M^{-1} \|\eta\|_{L^{q'}(d_A^R)}\right). \tag{4.15}$$

where $M = \|\eta\|_{L^\infty}$ and ρ is the first eigenfunction of $-\Delta$ in D_A^R normalized by $\rho(x_0) = 1$ at some point $x_0 \in D_A^R$. The constant c depends only on $N, q, k, x_0, \lambda_1, R$ where λ_1 is the first eigenvalue.

(ii) *For any compact set $E \subset d_A$,*

$$C_{s,q}^{N-k}(E) = 0 \Longrightarrow \mu(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'}, \tag{4.16}$$

where $C_{s,q}^{N-k}$ denotes the Bessel capacity with the indicated indices in $\mathbb{R}^{N-k}$.

Remark. If we replace D_A^R by $D_A \cap B_{\tilde{R}}^k(0) \cap B_R^{N-k}(0)$, $\tilde{R} > 1$, then the constant c in (i) depends on $\tilde{R}$ but *not* on R.

Proof. We identify d_A with $\mathbb{R}^{N-k}$ and use the notation

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad y = |x'|.$$

Let $\eta \in C_0^\infty(\mathbb{R}^{N-k})$ and let R be large enough so that $\operatorname{supp}\eta \subset B_{R/2}^{N-k}(0)$. Let $w = w_R(t, x'')$ be the solution of the following problem in $\mathbb{R}_+ \times B_R^{N-k}(0)$:

$$\begin{aligned} \partial_t w - \Delta_{x''} w &= 0 && \text{in } \mathbb{R}^+ \times B_R^{N-k}(0), \\ w\big(0, x''\big) &= \eta\big(x''\big) && \text{in } B_R^{N-k}, \\ w\big(t, x''\big) &= 0 && \text{on } \partial B_R^{N-k}(0). \end{aligned} \tag{4.17}$$

Thus $w_R(t, \cdot) = S_R(t)[\eta]$ where $S_R(t)$ is the semi-group operator corresponding to the above problem. Denote,

$$H_R[\eta]\big(x', x''\big) = w_R\big(|x'|^2, x''\big) = S_R\big(y^2\big)[\eta]\big(x''\big), \quad y := \big|x'\big|. \tag{4.18}$$

We assume, as we may, that $R > 1$. Let ρ^R be the first eigenfunction of $-\Delta_{x''}$ in the ball $B_R^{N-k}(0)$ normalized by $\rho^R(0) = 1$ and let ρ_A be the first eigenfunction of $-\Delta_{x'}$ in C_A (where C_A denotes the cone with opening A in $\mathbb{R}^k$) normalized so that $\rho_A(x'_0) = 1$ at some point $x'_0 \in S_A$. Then $\rho^R\rho_A$ is the first eigenfunction of $-\Delta$ in $\{x \in D_A : |x''| < R\}$. Note that $\rho^R \leq 1$ and $\rho^R \to 1$ as $R \to \infty$ in $C^2(I)$ for any bounded set $I \subset \mathbb{R}^{N-k}$.

Let $h \in C^\infty(\mathbb{R})$ be a monotone decreasing function such that$h(t) = 1$ for $t < 1/2$ and $h(t) = 0$ for $t > 3/4$. Put

$$\psi_R(x') = h\big(|x'|/R\big)$$

and

$$\zeta_R := \rho_A\psi_R H_R[\eta]^{q'}. \tag{4.19}$$

If ρ_A^R is the first eigenfunction (normalized at x_0) of $D_A^R := D_A \cap \Gamma_R$ (Γ_R as in (2.25)) then

$$\rho_A\psi_R \leq c\rho_A^R \tag{4.20}$$

and $\rho^R\rho_A^R$ is the first eigenfunction in D_A^R.

Hereafter we shall drop the index R in ζ_R, H_R, w_R but keep it in the other notations in order to avoid confusion.

We shall verify that $\zeta \in D_A^R$. To this purpose we compute,

$$\begin{aligned}\Delta\zeta =& -\lambda_1(\rho_A\psi_R)H[\eta]^{q'} + (\rho_A\psi_R)\Delta H[\eta]^{q'} + 2\nabla(\rho_A\psi_R)\cdot\nabla H[\eta]^{q'}\\ =& -\lambda_1\zeta + q'(\rho_A\psi_R)(H[\eta])^{q'-1}\Delta H[\eta]\\ &+ q\big(q'-1\big)(\rho_A\psi_R)(H[\eta])^{q'-2}|\nabla H[\eta]|^2\\ &+ 2q'\big(H[\eta]\big)^{q'-1}\nabla(\rho_A\psi_R)\cdot\nabla H[\eta].\end{aligned} \tag{4.21}$$

In addition,

$$\begin{aligned}\nabla H[\eta] &= \nabla_{x'}H[\eta] + \nabla_{x''}H[\eta] = \partial_y H[\eta]\frac{x'}{y} + \nabla_{x''}H[\eta]\\ &= 2y\partial_t w\big(y^2, x''\big)\frac{x'}{y} + \nabla_{x''}H[\eta]\big(x', x''\big)\end{aligned}$$

and consequently (recall that y stands for $|x'|$),

$$\begin{aligned}&\nabla H[\eta]\cdot\nabla(\rho_A\psi_R)\\ &= 2\partial_t w\big(y^2, x''\big)x'\cdot\left(\psi_R\left(|x'|^{\kappa_+-1}\left(\kappa_+\frac{x'}{y}\omega_k\big(x'/y\big) + |x'|\nabla\omega_k\big(x'/y\big)\right)\right) + \rho_A\nabla\psi_R\right)\\ &= 2\kappa_+\partial_t w\big(y^2, x''\big)|x'|^{\kappa_+}\omega_k\big(x'/y\big) = 2\partial_t w\big(y^2, x''\big)\big(\kappa_+\rho_A\psi_R + \rho_A x'\cdot\nabla\psi_R\big).\end{aligned}$$

Since $w = w_R$ vanishes for $|x''| = R$ and $\eta = 0$ in a neighborhood of this sphere, $|\partial_t w(y^2, x'')| \le c\rho^R$. As ψ_R vanishes for $|x'| > 3R/4$ we have $\rho_A \nabla\psi_R \le c\rho_A^R$. Therefore

$$|\nabla H[\eta] \cdot \nabla\rho_A| \le c\rho^R \rho_A^R$$

and, in view of (4.21),

$$|\Delta\zeta| \le c\rho^R \rho_A^R. \tag{4.22}$$

Thus $\zeta \in X(D_A^R)$ and consequently

$$\int_{D_A^R} \left(-u\Delta\zeta + u^q\zeta\right) dx = -\int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta\, dx. \tag{4.23}$$

Since $q(q'-1)\rho_A(H[\eta])^{q'-2}|\nabla H[\eta]|^2 \ge 0$, we have

$$\begin{aligned}
&\left|\int_{D_A^R} u\Delta\zeta\, dx\right| \\
&\le \int_{D_A^R} u\left(\lambda_1\zeta + q'(H[\eta])^{q'-1}\left(\rho|\Delta H[\eta]| + 2|\nabla\rho.\nabla H[\eta]|\right)\right) dx \\
&\le \int_{D_A^R} u\left(\lambda_1\zeta + q'\zeta^{1/q}\left(\rho^{1/q'}|\Delta H[\eta]| + 2\rho^{-1/q}|\nabla\rho.\nabla H[\eta]|\right)\right) dx \\
&\le \left(\int_{D_A^R} u^q\zeta\, dx\right)^{\frac{1}{q}} \left(\lambda_1\left(\int_{D_A^R}\zeta\, dx\right)^{\frac{1}{q'}} + q'\,\|L[\eta]\|_{L^{q'}(D_A^R)}\right)
\end{aligned} \tag{4.24}$$

where

$$L[\eta] = \rho^{1/q'}|\Delta H[\eta]| + 2\rho^{-1/q}|\nabla\rho.\nabla H[\eta]|. \tag{4.25}$$

By Proposition 4.2

$$-\int_{D_A^R} \mathbb{K}[\mu]\Delta\zeta\, dx = \int_{d_A^R} \eta^{q'} d\mu. \tag{4.26}$$

Therefore

$$\begin{aligned}
\left(\int_{d_A^R} \eta^{q'} d\mu\right) &\le \int_{D_A^R} u^q\zeta\, dx \\
&\quad + \left(\int_{D_A^R} u^q\zeta\, dx\right)^{\frac{1}{q}} \left(\lambda_1\left(\int_{D_A^R}\zeta\, dx\right)^{\frac{1}{q'}} + q'\,\|L[\eta]\|_{L^{q'}(D_A^R)}\right).
\end{aligned} \tag{4.27}$$

Next we prove that

$$\|L[\eta]\|_{L^{q'}(D_A^R)} \le C\,\|\eta\|_{W^{s,q'}(\mathbb{R}^{N-k})} \tag{4.28}$$

starting with the estimate of the first term on the right hand side of (4.25).

$$\begin{aligned}\Delta H[\eta] &= \Delta_{x'} H[\eta] + \Delta_{x''} H[\eta] = \partial_y^2 H[\eta] + \frac{k-1}{y}\partial_y H[\eta] + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w\big(y^2, x''\big) + k\partial_t w\big(y^2, x''\big) + \Delta_{x''} H[\eta] \\ &= 2y^2 \partial_{tt} w\big(y^2, x''\big) + (k+1)\partial_t w\big(y^2, x''\big).\end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^N} \rho\, |\Delta H[\eta]|^{q'}\, dx &\le c\int_0^1 \int_{\mathbb{R}^{N-k}} \big|\partial_{tt} w\big(y^2, x''\big)\big|^{q'}\, dx'' y^{\kappa_+ + 2q' + k - 1} dy \\ &\quad + c\int_0^1 \int_{\mathbb{R}^{N-k}} \big|\partial_t w\big(y^2, x''\big)\big|^{q'}\, dx'' y^{\kappa_+ + k - 1} dy \\ &\le c\int_0^1 \int_{\mathbb{R}^{N-k}} \big|\partial_{tt} w\big(t, x''\big)\big|^{q'}\, dx'' t^{(\kappa_+ + k)/2 + q'} \frac{dt}{t} \\ &\quad + c\int_0^1 \int_{\mathbb{R}^{N-k}} \big|\partial_t w\big(t, x''\big)\big|^{q'}\, dx'' t^{(\kappa_+ + k)/2} \frac{dt}{t} \\ &\le c\int_0^1 \left\| t^{2-(1-\frac{\kappa_+ + k}{2q'}))} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \\ &\quad + c\int_0^1 \left\| t^{1-(1-\frac{\kappa_+ + k}{2q'})} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t}.\end{aligned}$$

Put $\beta = \frac{\kappa_+ + k}{2q'}$ and note that $0 < \beta = \frac{1}{2}(2-s) < 1$. By standard interpolation theory,

$$\int_0^1 \left\| t^{1-(1-\beta)} \frac{dS(t)[\eta]}{dt} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \approx \|\eta\|^{q'}_{\left[W^{2,q'}, L^{q'}\right]_{1-\beta, q'}} \approx \|\eta\|^{q'}_{W^{2(1-\beta),q'}(\mathbb{R}^{N-k})},$$

and

$$\int_0^1 \left\| t^{2-(1-\beta))} \frac{d^2 S(t)[\eta]}{dt^2} \right\|_{L^{q'}(\mathbb{R}^{N-k})}^{q'} \frac{dt}{t} \approx \|\eta\|^{q'}_{\left[W^{4,q'}, L^{q'}\right]_{\frac{1}{2}(1-\beta), q'}} \approx \|\eta\|^{q'}_{W^{2(1-\beta),q'}(\mathbb{R}^{N-k})}.$$

The second term on the right hand side of (4.25) is estimated in a similar way:

$$\int_{\mathbb{R}^N} \rho^{-q'/q} \left|\nabla H[\eta] \cdot \nabla \rho\right|^{q'} dx \leq c \int_0^1 \int_{\mathbb{R}^{N-k}} \left|\partial_t w(y^2, x'')\right|^{q'} dx' y^{\kappa_+ + k-1} dy$$
$$\leq c \int_0^1 \int_{\mathbb{R}^{N-k}} \left|\partial_t w(t, x'')\right|^{q'} dx' t^{\frac{\kappa_+ + k}{2}} \frac{dt}{t}$$
$$\leq c \int_0^1 \left\| t^{1-(\frac{1}{2}-\beta)} \frac{dS(t)[\eta]}{dt} \right\|^{q'}_{L^{q'}(\mathbb{R}^{N-k})} \frac{dt}{t}$$
$$\approx \|\eta\|^{q'}_{W^{2(1-\beta),q'}(\mathbb{R}^{N-k})}.$$

This proves (4.28). Further, (4.27) and (4.28) imply (4.15).

We turn to the proof of part (ii). Let E be a closed subset of $B^{N-k}_{R/2}(0)$ such that $C^{N-k}_{s,q'}(E) = 0$. Then there exists a sequence $\{\eta_n\}$ in $C_0^\infty(d_A)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of E (which may depend on n), $\operatorname{supp} \eta_n \subset B^{N-k}_{3R/4}(0)$ and $\|\eta_n\|_{W^{s,q'}} \to 0$. Then, by (4.28),

$$\|L[\eta_n]\|_{L^{q'}(D^R_A)} \to 0.$$

Furthermore

$$\|w\|_{L^{q'}((0,R)\times B^{N-k}_R(0))} \leq c \|\eta_n\|_{L^{q'}(B^{N-k}_R(0))}$$

and consequently

$$H[\eta_n] \to 0 \ \text{ in } \ L^{q'}\big(D^R_A\big).$$

(Here we use the fact that $k \geq 2$.) In addition

$$0 \leq H[\eta_n] \leq 1, \quad H[\eta_n] \leq c(R - |x'|)$$

with a constant c independent of n. Hence (see (4.20))

$$\zeta_{n,R} := \rho_A \psi_R H[\eta_n]^{q'} \leq \rho^R \rho_A \psi_R H[\eta_n]^{q'-1} \leq \rho^R \rho^R_A H[\eta_n]^{q'-1}.$$

As $u^q \rho^R \rho^R_A \in L^1(D^R_A)$ we obtain

$$\lim_{n\to\infty} \int_{D_A} u^q \zeta_n dx = 0.$$

This fact and (4.27) imply that

$$\int_{d^R_A} \eta_n^{q'} d\mu \to 0.$$

As $\eta_n = 1$ on a neighborhood of E in $\mathbb{R}^{N-k}$ it follows that $\mu(E) = 0$. □

Proposition 4.5. *Let D_A be a k-dihedron,* $1 \leq k < N$. *Let k_+ be as in* (2.11) *and let q_c^* and q_c be as in Proposition* 2.1 *and Proposition* 2.2 *respectively. Assume that $q_c \leq q < q_c^*$. A measure $\mu \in \mathfrak{M}(\partial D_A)$, with compact support contained in d_A, is q-good relative to D_A if and only if μ vanishes on every Borel set $E \subset d_A$ such that$C_{s,q'}(E) = 0$, where $s = 2 - \frac{k+\kappa_+}{q'}$.*

Remark. We shall use the notation $\mu \prec C_{s,q'}$ to say that μ vanishes on any Borel set $E \subset (d_A)$ such that$C_{s,q'}(E) = 0$.

In the case $k = N$: $D_A = C_A$ (= the cone with vertex 0 and opening A in $\mathbb{R}^k$) and $q_c = q_c^*$. By [24] (specifically the results quoted in Subsection 2.2) $q_c = 1 - \frac{2}{\kappa_-} = \frac{N+\kappa_+}{N+\kappa_+-2}$ and if $1 < q < q_c$ then there exist solutions for every measure $\mu = k\delta_P$, $P \in d_A$.

In the case $k = 1$, $q_c^* = \infty$, $\kappa_+ = 1$ and $q_c = \frac{N+1}{N-1}$. Thus $s = 2/q$ and the statement of the theorem is well known (see [21]).

Proof. In view of the last remark, it remains to deal only with $2 \leq k \leq N-1$. We shall identify d_A with $\mathbb{R}^{N-k}$.

It is sufficient to prove the result for positive measures because $\mu \prec C_{s,q'}$ if and only if $|\mu| \prec C_{s,q'}$. In addition, if $|\mu|$ is a q-good measure then μ is a q-good measure.

First we show that if μ is non-negative and q-good then $\mu \prec C_{s,q'}$. If E is a Borel subset of $\partial\Omega$ then $\mu\chi_E$ is q-good. If E is compact and $C_{s,q'}(E) = 0$ then, by Proposition 4.4, E is a removable set. This means that the only positive solution of (1.1) in D_A such that$\mu(\partial\Omega \setminus E) = 0$ is the zero solution. This implies that $\mu\chi_E = 0$, *i.e.*, $\mu(E) = 0$. If $C_{s,q'}(E) = 0$ but E is not compact then $\mu(E') = 0$ for every compact set $E' \subset E$. Therefore, we conclude again that $\mu(E) = 0$.

Next, assume that μ is a positive measure in $\mathfrak{M}(\partial D_A)$ supported in a compact subset of $\mathbb{R}^{N-k}$.

If $\mu \in B^{-s,q}(\mathbb{R}^{N-k})$ then, by Theorem 3.1, μ is admissible relative to $D_A \cap \Gamma_{k,R}$, for every $R > 0$. (As before $\Gamma_{k,R}$ is the cylinder with radius R around the "axis" $\mathbb{R}^{N-k}$.) This implies that μ is q-good relative to D_A.

If $\mu \prec C_{s,q'}$ then, by a theorem of Feyel and de la Pradelle [11] (see also [3]), there exists a sequence $\{\mu_n\} \subset (B^{-s,q}(\mathbb{R}^{N-k}))_+$ such that$\mu_n \uparrow \mu$. As μ_k is q-good, it follows that μ is q-good. □

Theorem 4.6. *Let P be an N-dimensional polyhedron as described in Proposition* 4.3. *Let μ be a bounded measure on ∂P,* (*may be a signed measure*). *Let $k = 1, \ldots, N$, $j = 1, \ldots, n_k$, and let $L_{k,j}$ and $A_{k,j}$ be defined as at the beginning of this section. Further, put*

$$s(k,j) = 2 - \frac{k + (\kappa_+)_{k,j}}{q'}, \tag{4.29}$$

where $(\kappa_+)_{k,j}$ is defined as in (2.11) *with $A = A_{k,j}$. Then $\mu \in \mathfrak{M}_q(\partial P)$, i.e., μ is a good measure for* (1.1) *relative to P, if and only if, for every pair (k,j) as above and every Borel set $E \subset L_{k,j}$:*

- *If* $1 \le k < N$ *then*

$$\begin{aligned} (q_c)_{k,j} \le q < \left(q_c^*\right)_{k,j},\ C^{N-k}_{s(k,j),q'}(E) = 0 &\Longrightarrow \mu(E) = 0 \\ q \ge \left(q_c^*\right)_{k,j} &\Longrightarrow \mu\big(L_{N,j}\big) = 0 \end{aligned} \tag{4.30}$$

and if $k = N$, *i.e.,* L *is a vertex,*

$$q \ge (q_c)_{k,j} = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}} \Longrightarrow \mu(L) = 0. \tag{4.31}$$

Here $(q_c^*)_{k,j}$ *and* $(q_c)_{k,j}$ *are defined as in* (2.32) *and* (2.36) *respectively, with* $A = A_{k,j}$.
- *If* $1 < q < (q_c)_{k,j}$ *then there is no restriction on* $\mu\chi_{L_{k,j}}$.

Proof. This is an immediate consequence of Proposition 4.3 and Proposition 4.5 (see also the Remark following it). In the case $k = N$, $L_{N,j}$ is a vertex and the condition says merely that for $q \ge (q_c)_{N,j}$, μ does not charge the vertex. □

4.3. Removable singular sets, II

Proposition 4.7. *Let* A *be a Lipschitz domain on* S^{k-1}, $2 \le k \le N-1$, *and let* D_A *be the* k*-dihedron with opening* A. *Let* u *be a positive solution of* (1.1) *in* D_A^R, *for some* $R > 0$. *Suppose that* $F = \mathcal{S}(u) \subset d_A^R$ *and let* Q *be an open neighborhood of* F *such that* $\bar{Q} \subset d_A^R$. *(Recall that* $d_A^R = d_A \cap B_R^{N-k}(0)$ *is an open subset of* d_A.*) Let* μ *be the trace of* u *on* $\mathcal{R}(u)$.

Let $\eta \in W_0^{s,q'}(d_A^R)$ *such that*

$$0 \le \eta \le 1, \quad \eta = 0 \ \text{ on } Q. \tag{4.32}$$

Employing the notation in the proof of Proposition 4.4, *put*

$$\zeta := \rho_A \psi_R H_R[\eta]^{q'}. \tag{4.33}$$

Then

$$\int_{D_A^R} u^q \zeta \, dx \le c\left(1 + \|\eta\|_{W^{s,q'}(d_A)}\right)^{q'} + \mu\left(d_A^R \setminus Q\right)^q, \tag{4.34}$$

c independent of u *and* η.

Proof. First we prove (4.34) for $\eta \in C_0^\infty(d_A^R)$. Let σ_0 be a point in A and let $\{A_n\}$ be a Lipschitz exhaustion of A. If $0 < \epsilon < \operatorname{dist}(\partial A, \partial A_n) = \bar{\epsilon}_n$ then

$$\epsilon\sigma_0 + C_{A_n} \subset C_A.$$

Denote

$$D_A^{R',R''} = D_A \cap \big[|x'| < R'\big] \cap \big[|x''| < R''\big].$$

Pick a sequence $\{\epsilon_n\}$ decreasing to zero such that$0 < \epsilon_n < \min(\bar{\epsilon}_n/2^n, R/8)$. Let u_n be the function given by

$$u_n\big(x'x''\big) = u\big(x' + \epsilon_n\sigma_0, x''\big) \quad \forall x \in D_{A_n}^{R_n,R}, \quad R_n = R - \epsilon_n.$$

Then u_n is a solution of (1.1) in $D_{A_n}^{R_n,R}$ belonging to $C^2(\bar{D}_{A_n}^{R_n,R})$ and we denote its boundary trace by h_n. Let

$$\zeta_n := \rho_{A_n}\psi_R H_R[\eta]^{q'},$$

with ψ_R and $H_R[\eta]$ as in the proof of Proposition 4.4. By Proposition 4.2

$$-\int_{D_{A_n}^{R_n,R}} \mathbb{P}[h_n]\Delta\zeta_n dx = \int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n \tag{4.35}$$

where ω_n is the harmonic measure on $d_{A_n}^R$ relative to $D_{A_n}^{R_n,R}$. (Note that $d_{A_n}^R = d_A^R$ and we may identify it with $B_R^{N-k}(0)$.) Hence

$$\int_{D_{A_n}^{R_n,R}} \big(-u_n\Delta\zeta_n + u_n^q\zeta_n\big)\, dx = -\int_{B_R^{N-k}(0)} \eta^{q'} h_n\, d\omega_n. \tag{4.36}$$

Further,

$$\int_{B_R^{N-k}(0)} \eta^{q'} h_n\, d\omega_n \to \int_{B_R^{N-k}(0)} \eta^{q'} d\mu \leq \mu(d_A^R \setminus Q),$$

because $\eta = 0$ in Q. By (4.24), (4.28) we obtain,

$$\begin{aligned} &\left|\int_{D_{A_n}^{R_n,R}} u_n\Delta\zeta_n\, dx\right| \\ &\leq c\left(\int_{D_{A_n}^{R_n,R}} u_n^q\zeta_n dx\right)^{\frac{1}{q}} \left(\left(\int_{D_{A_n}^{R_n,R}} \zeta_n dx\right)^{\frac{1}{q'}} + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\right). \end{aligned} \tag{4.37}$$

From the definition of ζ_n it follows that

$$\int_{D_{A_n}^{R_n,R}} \zeta_n\, dx \leq \int_{D_{A_n}^{R_n,R}} \rho_n\, dx \to \int_{D_A^R} \rho\, dx,$$

where ρ (respectively ρ_n) is the first eigenfunction of $-\Delta$ in D_A^R (respectively $D_{A_n}^{R_n,R}$) normalized by 1 at some $x_0 \in D_{A_1}^{R_1,R}$. Therefore, by (4.36),

$$\int_{D_{A_n}^{R_n,R}} u_n^q\zeta_n dx \leq c\left(\int_{D_{A_n}^{R_n,R}} u_n^q\zeta_n dx\right)^{\frac{1}{q}} \left(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\right) + \mu\left(d_A^R \setminus Q\right).$$

This implies

$$\int_{D_{A_n}^{R_n,R}} u_n^q \zeta_n dx \leq c\big(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\big)^{q'} + \mu\big(d_A^R \setminus Q\big)^q. \tag{4.38}$$

To verify this fact, put

$$m = \left(\int_{D_{A_n}^{R_n,R}} u_n^q \zeta_n dx\right)^{1/q}, \;\; b = \mu(d_A^R \setminus Q), \;\; a = c\left(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\right)$$

so that (4.38) becomes

$$m^q - am - b \leq 0.$$

If $b \leq m$ then

$$m^{q-1} - a - 1 \leq 0.$$

Therefore,

$$m \leq (a+1)^{\frac{1}{q-1}} + b$$

which implies (4.38). Finally, by the lemma of Fatou we obtain (4.34) for $\eta \in C_0^\infty$. By continuity we obtain the inequality for any $\eta \in W_0^{s,q'}$ satisfying (4.32). □

Theorem 4.8. *Let A be a Lipschitz domain on S^{k-1}, $2 \leq k \leq N-1$, and let D_A be the k-dihedron with opening A. Let E be a compact subset of d_A^R and let u be a non-negative solution of* (1.1) *in D_A^R (for some $R > 0$) such thatu vanishes on $\partial D_A^R \setminus E$. Then*

$$C_{s,q'}^{N-k}(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'} \Longrightarrow u = 0, \tag{4.39}$$

where $C_{s,q'}^{N-k}$ denotes the Bessel capacity with the indicated indices in $\mathbb{R}^{N-k}$.

Proof. By Proposition 4.4, (4.39) holds under the additional assumption

$$\int_{D_A^R} u^q \rho_R \rho_A^R dx < \infty. \tag{4.40}$$

Indeed, by [24, Proposition 4.1], (4.40) implies that the solution u possesses a boundary trace μ on ∂D_A^R. By assumption, $\mu(\partial D_A^R \setminus E) = 0$. Therefore, by Proposition 4.5, the fact that $C_{s,q'}^{N-k}(E) = 0$ implies that $\mu(E) = 0$. Thus $\mu = 0$ and hence $u = 0$.

We show that, under the conditions of the theorem, if $C_{s,q'}^{N-k}(E) = 0$ then (4.40) holds.

By Proposition 4.7, for every $\eta \in W_0^{s,q'}(d_A^R)$ such that$0 \le \eta \le 1$ and $\eta = 0$ in a neighborhood of E,

$$\int_{D_A^R} u^q \zeta \, dx \le c \left(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\right)^{q'}, \tag{4.41}$$

for ζ as in (4.33). (Here we use the assumption that $u = 0$ on $\partial D_A^R \setminus E$.)

Let $a > 0$ be sufficiently small so that $E \subset B_{(1-4a)R}^{N-k}(0)$. Pick a sequence $\{\phi_n\}$ in $C_0^\infty(\mathbb{R}^{N-k})$ such that, for each n, there exists a neighborhood Q_n of E, $\bar{Q}_n \subset B_{(1-3a)R}^{N-k}(0)$ and

$$\begin{aligned}
&0 \le \phi_n \le 1 \text{ everywhere}, \ \phi_n = 1 \text{ in } Q_n \\
&\tilde{\phi}_n := \phi_n \chi_{[|x''|<(1-2a)R]} \in C_0^\infty\left(\mathbb{R}^{N-k}\right) \\
&\left\|\tilde{\phi}_n\right\|_{W^{s,q'}(\mathbb{R}^{N-k})} \to 0 \text{ as } n \to \infty \\
&\eta_n := (1-\phi_n)\lfloor_{[|x''|<R]} \in C_0^\infty\left(d_A^R\right) \\
&\eta_n = 0 \text{ in } \left[(1-a)R < |x''| < R\right].
\end{aligned} \tag{4.42}$$

Such a sequence exists because $C_{s,q'}^{N-k}(E) = 0$. Applying (4.41) to η_n we obtain,

$$\sup \int_{D_A^R} u^q \zeta_n \, dx \le c < \infty, \tag{4.43}$$

where $\zeta_n = \rho_A \psi_R H_R^{q'}[\eta_n]$ (see (4.33)). By taking a subsequence we may assume that $\{\eta_n\}$ converges (say to η) in $L^{q'}(B_R^{N-k}(0))$ and consequently $H[\eta_n] \to H[\eta]$ in the sense that

$$H_R[\eta_n](x', \cdot) = w_{n,R}(y^2, \cdot) \to w_R(y^2, \cdot) = H_R[\eta](x', \cdot) \text{ in } L^{q'}$$

uniformly with respect to$y = |x'|$. It follows that

$$\int_{D_A^R} u^q \zeta \, dx < \infty, \quad \zeta = \rho_A \psi_R H_R^{q'}[\eta]. \tag{4.44}$$

As $\tilde{\phi}_n \to 0$ in $W^{s,q'}(\mathbb{R}^{N-k})$ it follows that $\phi_n \to 0$ and hence $\eta_n \to 1$ a.e. in $B_{(1-2a)R}^{N-k}(0)$. Thus $\eta = 1$ in this ball, $\eta = 0$ in $[(1-a)R < |x''| < R]$ and $0 \le \eta \le 1$ everywhere.

Consequently , given $\delta > 0$, there exists an N-dimensional neighborhood O of $d_A \cap B_{(1-2a)R}^{N-k}(0)$ such that

$$1 - \delta < H_R[\eta] < 1 \text{ and } 1 - \delta < \psi_R/\rho_A^R < 1 \text{ in } O.$$

Therefore (4.44) implies that

$$\int_{D_A^{(1-3a)R}} u^q \rho^R \rho_A^R \, dx \leq c < \infty. \tag{4.45}$$

Recall that the trace of u on $\partial D_A^R \setminus d_A^{(1-4a)R}$ is zero. Therefore u is bounded in $D_A^R \setminus D_A^{(1-3a)R}$. This fact and (4.45) imply (4.40). □

Definition 4.9. Let Ω be a bounded Lipschitz domain. Denote by ρ the first eigenfunction of $-\Delta$ in Ω normalized by $\rho(x_0) = 1$ for a fixed point $x_0 \in \Omega$.

For every compact set $K \subset \partial\Omega$ we define

$$M_{\rho,q}(K) = \left\{\mu \in \mathfrak{M}(\partial\Omega) : \mu \geq 0,\ \mu(\partial\Omega \setminus K) = 0,\ \mathbb{K}[\mu] \in L^q_\rho(\Omega)\right\}$$

and

$$\tilde{C}_{\rho,q'}(K) = \sup\left\{\mu(K)^q :\ \mu \in M_{\rho,q}(K),\ \int_\Omega \mathbb{K}[\mu]^q \rho \, dx = 1\right\}.$$

Finally we denote by $C_{\rho,q'}$ the outer measure generated by the above functional.

The following statement is verified by standard arguments:

Lemma 4.10. *For every compact $K \subset \partial\Omega$, $C_{\rho,q'}(K) = \tilde{C}_{\rho,q'}(K)$. Thus $C_{\rho,q'}$ is a capacity and,*

$$C_{\rho,q'}(K) = 0 \iff M_{\rho,q}(K) = \{0\}. \tag{4.46}$$

Theorem 4.11. *Let Ω be a bounded polyhedron in $\mathbb{R}^N$. A compact set $K \subset \partial\Omega$ is removable if and only if*

$$C_{s(k,j),q'}\big(K \cap L_{k,j}\big) = 0, \tag{4.47}$$

for $k = 1, \cdot, N$ $j = 1, \ldots, n_k$, where $s(k, j)$ is defined as in (4.29). *This condition is equivalent to*

$$C_{\rho,q'}(K) = 0. \tag{4.48}$$

A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is q-good if and only if it does not charge sets with $C_{\rho,q'}$-capacity zero.

Proof. The first assertion is an immediate consequence of Proposition 4.3 and Theorem 4.8. The second assertion follows from the fact that

$$C_{\rho,q'}\big(K \cap L_{k,j}\big) = C_{s(k,j),q'}\big(K \cap L_{k,j}\big).$$

The third assertion follows from Theorem 4.6 and the previous statement. □

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