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Asymptotics and regularity of flat solutions to fully nonlinear elliptic problems

DISSON DOS PRAZERES AND EDUARDO V. TEIXEIRA

Abstract. In this work we establish local $C^{2,\alpha}$ regularity estimates for flat solutions to non-convex fully nonlinear elliptic equations provided the coefficients and the source function are of class $C^{0,\alpha}$. For problems with merely continuous data, we prove that flat solutions are locally $C^{1,\text{Log-Lip}}$.

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1. Introduction

The goal of this paper is to obtain optimal estimates for flat solutions to a class of non-convex fully nonlinear elliptic equations of the form

$$F(X, D^2u) = \mathscr{G}(X, u, \nabla u). \tag{1.1}$$

Under continuous differentiability with respect to the matrix variable and appropriate continuity assumptions on the coefficients and on the source function, we present a Schauder type regularity result for flat solutions, namely for solutions with small enough norm, $|u| \ll 1$.

The nonlinear operator $F\colon B_1 \times \text{Sym}(n) \to \mathbb{R}$ is assumed to be uniformly elliptic, namely, there exist constants $0 < \lambda \leq \Lambda$ such that for any $M, P \in \text{Sym}(n)$, with $P \geq 0$ and all $X \in B_1 \subset \mathbb{R}^n$ there holds

$$\lambda \|P\| \leq F(X, M + P) - F(X, M) \leq \Lambda \|P\|. \tag{1.2}$$

Under such condition it follows as a consequence of the Krylov-Safonov Harnack inequality that solutions to the homogeneous, constant coefficient equation

$$F(D^2h) = 0 \tag{1.3}$$

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are locally of class $C^{1,\alpha}$, for some $0 < \alpha < 1$. Under appropriate hypotheses on $\mathscr{G}$: $B_1 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, the same conclusion is obtained, *i.e.*, viscosity solutions are of class $C^{1,\alpha}$. Thus, insofar as the regularity theory for equation of the form (1.1) is concerned, one can regard the right hand side $\mathscr{G}(X, u, \nabla u)$ as an $\tilde{\alpha}-$Hölder continuous source, $f(X)$. Therefore, within this present work, we choose to look at the right-hand side $\mathscr{G}(X, u, \nabla u)$ simply as a source term $f(X)$, and equation (1.1) will be written as

$$F(X, D^2u) = f(X). \tag{1.4}$$

Regularity theory for heterogeneous equations (1.4) has been a central target of research for the past three decades. While a celebrated result due to Evans and Krylov assures that solutions to convex equations are classical, *i.e.*, $C^{2,\alpha}$ for some $\alpha > 0$, the problem of establishing continuity of the Hessian of solutions to general equations of the form (1.3) challenged the community for over twenty years. The problem has been settled in the negative by Nadirashvili and Vladut, [8, 9], who exhibit solutions to uniform elliptic equations whose Hessian blows-up.

In view of the impossibility of a general existence theory for classical solutions to all fully nonlinear equations (1.3), it becomes a central topic of research the study of reasonable conditions on F and on u as to assure that the Hessian of the solution is continuous. In such perspective the works [6] and [2] on interior $C^{2,\alpha}$ estimates for a particular class of non-convex equations are highlights. A decisive contribution towards Hessian estimates of solutions to fully nonlinear elliptic equations was obtained by Savin in [10]. By means of a robust approach, Savin shows in [10] that small solutions are classical, provided the operator is of class C^2 in all of its arguments.

Inspired by problems of the form (1.1), in the present work we obtain regularity estimates for flat solutions to heterogeneous equation (1.4), under continuity conditions on the media. We show that if $X \mapsto (F(X, \cdot), f(X))$ is α-Hölder continuous, then flat solutions are locally $C^{2,\alpha}$. In the case $\alpha = 0$, namely when the coefficients and the source are known to be just continuous, we show that flat solutions are locally $C^{1,\text{Log-Lip}}$.

The proofs of both results mentioned above, which are going to be stated properly in Theorem 2.2 and Theorem 2.3 respectively, are based on a combination of geometric tangential analysis and perturbation arguments inspired by compactness methods in the theory of elliptic PDEs.

We conclude this introduction explaining the heuristics of the geometric tangential analysis behind our proofs. Given a fully nonlinear elliptic operator F, we look at the family of elliptic scalings

$$F_\mu(M) := \frac{1}{\mu} F(\mu M), \text{ for } \mu > 0.$$

This is a continuous family of operators preserving the ellipticity constants of the original equation. If F is differentiable at the origin (recall that, by normalization $F(0) = 0$), then indeed

$$F_\mu(M) \to \partial_{M_{ij}} F(0) M_{ij}, \quad \text{as } \mu \to 0.$$

In other words, the linear operator $M \mapsto \partial_{M_{ij}} F(0) M_{ij}$ is the *tangential equation* of F_μ as $\mu \to 0$. Now, if u solves an equation involving the original operator F, then $u_\mu := \frac{1}{\mu} u$ is a solution to a related equation for F_μ. However, if in addition it is known that the norm of u is at most μ, then it amounts to saying that u_μ is a normalized solution to the μ-related equation, and hence we can access the universal regularity theory available for the (linear) tangential equation by compactness methods. In the sequel we transport such good *limiting* estimates towards u_μ, properly adjusted by the geometric tangential path used to access the *tangential* linear elliptic regularity theory. Similar reasoning has been recently employed in [11–13,15].

The paper is organized as follows. In Section 2 we state all the hypotheses, mathematical set-up and notions to be used throughout the whole paper. In that section we also state properly the two main Theorems proven in the work. In Section 3 we rigorously develop the heuristics of the geometric tangential analysis explained in the previous paragraph. The proof of $C^{2,\alpha}$ estimates, Theorem 2.2, will be delivered in Section 4. Two applications of such a result will be discussed in Section 5. Theorem 2.3 will be proven in Section 6.

2. Hypotheses and main results

Let us start off by discussing the hypotheses, set-up and notation used in this article. For B_1 we denote the open unit ball in the Euclidean space $\mathbb{R}^n$. The space of $n \times n$ symmetric matrices will be denoted by $\mathrm{Sym}(n)$. By modulus of continuity we mean an increasing function $\varpi : [0, +\infty) \to [0, +\infty)$, with $\varpi(0^+) = 0$.

Hereafter we shall assume the following conditions on the operator $F : B_1 \times \mathrm{Sym}(n) \to \mathbb{R}$ and $f : B_1 \to \mathbb{R}$:

(H1) There exist constants $0 < \lambda \le \Lambda$ such that for any $M, P \in \mathrm{Sym}(n)$, with $P \ge 0$ and all $X \in B_1$, there holds

$$\lambda \|P\| \le F(X, M + P) - F(X, M) \le \Lambda \|P\|. \tag{2.1}$$

(H2) $F(X, M)$ is differentiable with respect to M and for a modulus of continuity ω there holds

$$\|D_M F(X, M_1) - D_M F(X, M_2)\| \le \omega(\|M_1 - M_2\|), \tag{2.2}$$

for all $(X, M_i) \in B_1 \times \mathrm{Sym}(n)$.

(H3) For another modulus of continuity τ, there holds

$$|F(X, M) - F(Y, M)| \le \tau(|X - Y|) \cdot \|M\|, \tag{2.3}$$

$$|f(X) - f(Y)| \le \tau(|X - Y|), \tag{2.4}$$

for all $X, Y \in B_1$ and $M \in \mathrm{Sym}(n)$.

The following normalization condition will also be enforced hereafter in this paper:

$$F(0, 0_{n\times n}) = f(0) = 0 \tag{2.5}$$

though such hypothesis is not restrictive, as one can always reduce the problem as to verify that.

Condition (H1) concerns the notion of uniform ellipticity. Under such a structural condition, the theory of viscosity solutions provides an appropriate notion of weak solutions to such equations.

Definition 2.1. A continuous function $u \in C^0(B_1)$ is said to be a viscosity subsolution to (1.4) in B_1 if whenever one touches the graph of u from above with a smooth function φ at $X_0 \in B_1$ (*i.e.* $\varphi - u$ has a local minimum at X_0), there holds

$$F(X_0, D^2\varphi(X_0)) \geq f(X_0).$$

Similarly, u is a viscosity supersolution to (1.4) if whenever one touches the graph of u from below by a smooth function ϕ at $Y_0 \in B_1$, there holds

$$F(Y_0, D^2\phi(Y_0)) \leq f(Y_0).$$

We say u is a viscosity solution to (1.4) if it is a subsolution and a supersolution of (1.4).

Condition (H2) fixes a modulus of continuity ω to the derivative of F. The regularity estimates proven in this paper depend upon ω. Condition (H3) sets the continuity of the media. When $\tau(t) \approx t^\alpha$, $0 < \alpha < 1$, the coefficients and the source function are said to be α-Hölder continuous. In such a scenario we prove that flat solutions are locally of class $C^{2,\alpha}$ – a sharp Schauder type of estimate for non-convex fully nonlinear equations.

Theorem 2.2 ($C^{2,\alpha}$ regularity). *Let $u \in C^0(B_1)$ be a viscosity solution to*

$$F(X, D^2u) = f(X) \text{ in } B_1,$$

where F and f satisfy (H1)–(H3) *with $\tau(t) = Ct^\alpha$ for some $0 < \alpha < 1$. There exists a $\overline{\delta} > 0$, depending only upon $n, \lambda, \Lambda, \omega, \alpha$, and $\tau(1)$, such that if*

$$\sup_{B_1} |u| \leq \overline{\delta}$$

then $u \in C^{2,\alpha}(B_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq M \cdot \overline{\delta},$$

where M depends only upon $n, \lambda, \Lambda, \omega$, and $(1-\alpha)$.

We should emphasize that the Hölder exponent obtained in Theorem 2.2 is sharp, as it is the same one from the Hölder continuity of the medium and the source function f. If f is merely continuous, then even for the classical Poisson equation

$$\Delta u = f(X)$$

solutions may fail to be of class C^2. In connection to Theorem 5.1 in [12], in this paper we show that flat solutions in continuous media are locally of class $C^{1,\text{Log-Lip}}$, which corresponds to the optimal regularity estimate under such weaker conditions.

Theorem 2.3 ($C^{1,\textbf{Log-Lip}}$ estimates). *Let $u \in C^0(B_1)$ be a viscosity solution to*

$$F(X, D^2u) = f(X) \text{ in } B_1.$$

Assume (H1)–(H3). *Then there exist a $\overline{\delta} = \overline{\delta}(n, \lambda, \Lambda, \omega, \tau)$ such that if*

$$\sup_{B_1} \|u\| \leq \overline{\delta}$$

then $u \in C^{1,\text{Log}-\text{Lip}}(B_{\frac{1}{2}})$ and

$$|u(X) - [u(Y) + \nabla u(Y) \cdot (X - Y)]| \leq -M\overline{\delta} \cdot |X - Y|^2 \log(|X - Y|),$$

for a constant M that depends only upon $n, \lambda, \Lambda, \omega$, and $(1 - \alpha)$.

3. Geometric tangential analysis

In this section we provide a rigorous treatment of the heuristics involved in the geometric tangential analysis explained at the end of the Introduction. The next Lemmas are central for the proof of both Theorem 2.2 and Theorem 2.3.

Lemma 3.1. *Let $F\colon B_1 \times \text{Sym}(n) \to \mathbb{R}$ satisfy conditions* (H1) *and* (H2). *Given $0 \leq \gamma < 1$, there exists $\eta > 0$, depending only on $n, \lambda, \Lambda, \omega$, and γ, such that if u satisfies $|u| \leq 1$ in B_1 and solves $\mu^{-1}F(X, \mu D^2u) = f(X)$ in B_1, for*

$$0 < \mu \leq \eta, \qquad \sup_{M \in \text{Sym}(n)} \frac{|F(X, M) - F(0, M)|}{\|M\|} \leq \eta \quad \text{and} \quad \|f\|_{L^\infty(B_1)} \leq \eta,$$

then one can find a number $0 < \sigma < 1$, depending only on n, λ and Λ, and a quadratic polynomial P satisfying

$$\mu^{-1}F(0, \mu D^2P) = 0, \quad \text{with} \quad \|P\|_{L^\infty(B_1)} \leq C(n, \lambda, \Lambda),$$

for a universal constant $C(n, \lambda, \Lambda) > 0$, such that

$$\sup_{B_\sigma} |u - P| \leq \sigma^{2+\gamma}.$$

Proof. Let us suppose, by contradiction, that the lemma fails to hold. If so, there would exist a sequence of elliptic operators, $F_k(X, M)$, satisfying hypotheses (H1) and (H2), a sequence $0 < \mu_k = \mathrm{o}(1)$ and sequences of functions

$$u_k \in C(B_1) \text{ and } f_k \in L^\infty(B_1),$$

all linked through the equation

$$\frac{1}{\mu_k} F_k(X, \mu_k D^2 u_k) = f_k(X) \text{ in } B_1, \tag{3.1}$$

in the viscosity sense, such that

$$\|u_k\|_\infty \le 1, \;\; \mu_k \le \frac{1}{k}, \quad \sup_{M \in \mathrm{Sym}(n)} \frac{|F_k(X,M) - F_k(0,M)|}{\|M\|} \le \frac{1}{k} \;\text{ and }\; \|f_k\|_\infty \le \frac{1}{k}; \tag{3.2}$$

however for some $0 < \sigma_0 < 1$

$$\sup_{B_{\sigma_0}} |u_k - P| > \sigma_0^{2+\gamma}, \tag{3.3}$$

for all quadratic polynomials P that satisfy

$$\frac{1}{\mu_k} F_k(0, \mu_k D^2 P) = 0.$$

Passing to a subsequence if necessary, we can assume $F_k(X, M) \to F_\infty(X, M)$ locally uniformly in $\mathrm{Sym}(n)$. From uniform C^1 estimates on F_k and the coefficient oscillation hypothesis in (3.2), we deduce

$$\frac{1}{\mu_k} F_k(X, \mu_k M) \to D_M F_\infty(0, 0) \cdot M, \tag{3.4}$$

locally uniformly in $\mathrm{Sym}(n)$. Also, by Krylov-Safonov $C^{0,\gamma}$ bounds for equation (3.1), up to a subsequence, $u_k \to u_\infty$ locally uniformly in B_1. Thus, by stability of viscosity solutions, we conclude

$$D_M F_\infty(0, 0) \cdot D^2 u_\infty = 0, \qquad \text{in } B_1. \tag{3.5}$$

Since u_∞ solves a linear elliptic equation with constant coefficients , u_∞ is smooth. Define

$$P := u_\infty(0) + Du_\infty(0) \cdot X + \frac{1}{2} X \cdot D^2 u_\infty(0) X.$$

Since $\|u_\infty\| \le 1$, it follows from C^3 estimates on u_∞ that

$$\sup_{B_r} |u_\infty - P| \le C r^3,$$

for a constant C that depends only upon the dimension n and the ellipticity constants, λ and Λ. Thus, if we select

$$\sigma := \sqrt[1-\gamma]{\frac{1}{2C}},$$

a choice that depends only on n, λ, Λ and γ, we readily have

$$\sup_{B_\sigma} |u_\infty - P| \le \frac{1}{2}\sigma^{2+\gamma},$$

Also, from equation (3.5), we obtain

$$D_M F_\infty(0,0) \cdot D^2 P = 0$$

which implies that

$$|\mu_k^{-1} F_k(0, \mu_k D^2 P| = \mathrm{o}(1).$$

Now, since F_k is uniformly elliptic in $B_1 \times \mathrm{Sym}(n)$ and $F_k(0,0) = 0$, it is possible to find a sequence of real numbers $(a_k) \subset \mathbb{R}$ with $|a_k| = \mathrm{o}(1)$, for which the quadratic polynomials

$$P_k := P + a_k |X|^2$$

do satisfy

$$\mu_k^{-1} F_k(0, \mu_k D^2 P_k) = 0.$$

Finally we have, for any point in B_σ and k large enough,

$$\begin{aligned}\sup_{B_\sigma} |u_k - P_k| &\le |u_k - u_\infty| + |u_\infty - P| + |P - P_k| \\ &\le \frac{1}{5}\sigma^{2+\gamma} + \frac{1}{2}\sigma^{2+\gamma} + |a_k|\sigma^2 \\ &< \sigma^{2+\gamma},\end{aligned}$$

which contradicts (3.3). Lemma 3.1 is proven. □

In the sequel, we transfer the geometric tangential access towards a smallness condition of the L^∞ of the solution.

Lemma 3.2. *Let F satisfy* (H1) *and* (H2) *and* $0 \le \alpha < 1$ *be given. There exist a small positive constant* $\delta > 0$ *depending on* $n, \lambda, \Lambda, \alpha$, *and a constant* $0 < \sigma < 1$ *depending only on* $n, \lambda, \Lambda, (1-\alpha)$ *such that if u is a solution to* (1.4),

$$\|u\|_{L^\infty(B_1)} \le \delta, \quad \sup_{M \in \mathrm{Sym}(n)} \frac{|F(X,M) - F(0,M)|}{\|M\|} \le \delta^{3/2} \;\; \textit{and} \;\; \|f\|_{L^\infty(B_1)} \le \delta^{3/2},$$

then one can find a quadratic polynomial P satisfying

$$F(0, D^2 P) = 0, \quad \textit{with} \quad \|P\|_{L^\infty(B_1)} \le \delta C(n, \lambda, \Lambda) \tag{3.6}$$

for a universal constant $C(n, \lambda, \Lambda) > 0$*, and*

$$\sup_{B_\sigma} |u - P| \leq \delta \cdot \sigma^{2+\alpha}$$

Proof. Define the normalized function $v = \delta^{-1} u$. We immediately check that

$$\delta^{-1} F(X, \delta D^2 v) = \frac{f(X)}{\delta}.$$

If η is the number from Lemma 3.1, we choose $\delta = \eta^2$ and the Lemma follows. □

4. $C^{2,\alpha}$ estimates in $C^{0,\alpha}$ media

In this section we show that if the coefficients and the source are α-Hölder continuous, then flat solutions are locally of class $C^{2,\alpha}$, *i.e.*, we assume

$$\tau(t) \lesssim C t^\alpha, \tag{4.1}$$

for some $0 < \alpha < 1$ and $C > 0$, where τ is the modulus of continuity of the coefficients and the source function appearing in (2.3) and (2.4). Under such condition, we aim to show that flat solutions are locally of class $C^{2,\alpha}$.

The idea of the proof is to employ Lemma 3.2 in an inductive process as to establish the aimed $C^{2,\alpha}$ estimate for flat solutions under an appropriate smallness regime for the oscillation of the coefficients and the source function.

Lemma 4.1. *Let* F*,* f *and* u *be as in the hypotheses of Lemma* 3.2*. Then there exists a* $\delta = \delta(n, \lambda, \Lambda, \omega) > 0$ *such that, if*

$$\sup_{B_1} |u| \leq \delta \quad \text{and} \quad \tau(1) \leq \delta^{3/2},$$

then $u \in C^{2,\alpha}$ *at the origin and*

$$|u - (u(0) + \nabla u(0) \cdot X + \frac{1}{2} X^t D^2 u(0) X)| \leq C \cdot \delta |X|^{2+\alpha},$$

where $C > 0$ *depends only upon* $n, \lambda, \Lambda, \omega$ *and* $(1 - \alpha)$.

Proof. The proof consists in iterating Lemma 3.2 in order to produce a sequence of quadratic polynomials

$$P_k = \frac{1}{2} X^t A_k X + b_k \cdot X + c_k \quad \text{with} \quad F(0, D^2 P_k) = 0, \tag{4.2}$$

that approximates u in a $C^{2,\alpha}$ fashion, *i.e.*,

$$\sup_{B_{\sigma^k}} |u(X) - P_k(X)| \leq \delta \sigma^{(2+\alpha)k}. \tag{4.3}$$

Furthermore, we aim to control the oscillation of the coefficients of P_k as

$$\begin{cases} |A_k - A_{k-1}| \le C\delta\sigma^{\alpha(k-1)} \\ |b_k - b_{k-1}| \le C\delta\sigma^{(1+\alpha)(k-1)} \\ |c_k - c_{k-1}| \le C\delta\sigma^{(2+\alpha)(k-1)} \end{cases} \tag{4.4}$$

where $C > 0$ is universal and σ and δ are the parameters from Lemma 3.2. The proof of existence of polynomials P_k verifying (4.2), (4.3) and (4.4) will be delivered by induction. The case $k = 1$ is precisely the statement of Lemma 3.2. Suppose now we have verified the kth step of induction, *i.e.*, there exists a quadratic polynomial P_k satisfying (4.2), (4.3) and (4.4). We define

$$\tilde{u}(X) := \frac{1}{\sigma^{(2+\alpha)k}}(u(\sigma^k X) - P_k(\sigma^k X)); \tag{4.5}$$

$$\tilde{F}(X, M) := \frac{1}{\sigma^{k\alpha}} F(\sigma^k X, \sigma^{k\alpha} \cdot M + D^2 P_k). \tag{4.6}$$

Notice that

$$\left| D_M \tilde{F}(X, M) - D_M \tilde{F}(X, N) \right| \le \omega(\sigma^{k\alpha} \|M - N\|) \le \omega(\|M - N\|),$$

that is, $\tilde{F}$ fulfills (H2). It readily follows from (4.3) that $\tilde{u}$ satisfies

$$|\tilde{u}|_{L^\infty(B_1)} \le \delta.$$

Moreover, $\tilde{u}$ solves

$$\tilde{F}(X, D^2\tilde{u}) = \frac{1}{\sigma^{k\alpha}} f(\sigma^k X) =: \tilde{f}(X)$$

in the viscosity sense. From τ-continuity of f and the coefficients of F, together with the smallness condition $\tau(1) \le \delta^{3/2}$, we verify

$$\|\tilde{f}\|_\infty \le \delta^{3/2},$$

and likewise,

$$\sup_{M \in \mathrm{Sym}(n)} \frac{|\tilde{F}(X, M) - \tilde{F}(0, M)|}{\|M\|} \le \delta^{3/2}.$$

Applying Lemma 3.2 to $\tilde{u}$ gives a quadratic polynomial $\tilde{P}$ satisfying $\tilde{F}(0, D^2\tilde{P}) = 0$ for which

$$|\tilde{u}(X) - \tilde{P}(X)| \le \delta\sigma^{2+\alpha}, \qquad \text{for } |X| \le \sigma.$$

The $(k + 1)$th step of induction is verified if we define

$$P_{k+1}(X) := P_k(X) + \sigma^{(2+\alpha)k} \tilde{P}(\sigma^{-k} X).$$

To conclude the proof of the lemma, notice that (4.4) implies that

$$\{A_k\} \subset \mathrm{Sym}(n), \quad \{b_k\} \subset \mathbb{R}^n, \quad \text{and} \quad \{c_k\} \subset \mathbb{R}$$

are Cauchy sequences. Let us label the limiting quadratic polynomial by

$$P_\infty(X) := \frac{1}{2} X^t A_\infty X + b_\infty X + c_\infty,$$

where $A_k \to A_\infty$, $b_k \to b_\infty$ and $c_k \to c_\infty$. It further follows from (4.4) that

$$|P_k(X) - P_\infty(X)| \le C\delta(\sigma^{\alpha k}|X|^2 + \sigma^{(1+\alpha)k}|X| + \sigma^{(2+\alpha)k}) \tag{4.7}$$

whenever $|X| \le \sigma^k$. Finally, for fixed $X \in B_\sigma$, take $k \in \mathbb{N}$ such that $\sigma^{k+1} < |X| \le \sigma^k$ and conclude, by means of (4.3) and (4.7), that

$$|u(X) - P_\infty(X)| \le C_1 \delta \sigma^{(2+\alpha)k} \le \frac{C_1 \delta}{\sigma^{2+\alpha}} |X|^{2+\alpha},$$

as desired. □

We conclude the proof of Theorem 2.2 by verifying that if $\tau(t) = \tau(1)t^\alpha$, the smallness condition of Lemma 4.1, namely

$$\tau(1) \le \delta^{3/2},$$

is not restrictive. In fact, if $u \in C^0(B_1)$ is a viscosity solution to

$$F(X, D^2u) = f(X) \text{ in } B_1, \tag{4.8}$$

the auxiliary function

$$v(X) := \frac{u(\mu X)}{\mu^2}$$

solves

$$F_\mu(X, D^2 v) = f_\mu(X),$$

where

$$F_\mu(X, M) := F(\mu X, M) \quad \text{and} \quad f_\mu(X) := f(\mu X).$$

Clearly the new operator F_μ satisfies the same assumptions (H1)–(H3) as F, with the same universal parameters λ, Λ and ω. Note however that

$$\max\left\{ |f_\mu(X) - f_\mu(Y)|, \frac{|F_\mu(X, M) - F_\mu(Y, M)|}{\|M\|} \right\} \le \tau(1)\mu^\alpha |X - Y|^\alpha,$$

for $M \in \mathrm{Sym}(n)$. Thus if τ_μ is the modulus of continuity for f_μ and F_μ,

$$\tau_\mu(1) = \tau(1)\mu^\alpha.$$

Finally, we take

$$\mu := \min\left\{1, \frac{\sqrt[2\alpha]{\delta^3}}{\sqrt[\alpha]{\tau(1)}}\right\},$$

where δ is the universal number from Lemma 3.2. In conclusion, if u solves (4.8) and satisfies the flatness condition

$$\|u\|_{L^\infty(B_1)} \leq \overline{\delta} := \delta\mu^2,$$

then Lemma 4.1 applied to v gives $C^{2,\alpha}$ estimates for v, which is accordingly transported to u. □

5. Applications

Probably an erudite way of understanding Theorem 2.2 is saying that, if u solves a fully nonlinear elliptic equation with C^α coefficients and source, then, if it is close enough to a $C^{2,\alpha}$ function, then indeed u is $C^{2,\alpha}$. This is particularly meaningful in problems involving some *a priori* set data.

In this intermediate section, we comment on two applications of Theorem 2.2. The first one concerns an improvement of regularity for classical solutions in Hölder continuous media.

Corollary 5.1 (**C^2 implies $C^{2,\alpha}$**). *Let $u \in C^2(B_1)$ be a classical, pointwise solution to*

$$F(X, D^2u) = f(X)$$

where $F(X,\cdot) \in C^1(\mathrm{Sym}(n))$ satisfies (H1)–(H2). *Assume further that condition* (H3) *holds with $\tau(t) = Ct^\alpha$ for some $0 < \alpha < 1$. Then $u \in C^{2,\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\left(n, \lambda, \Lambda, \alpha, \omega, \tau(1), \|u\|_{C^2(B_1)}\right).$$

Proof. We shall prove that u is $C^{2,\alpha}$ at the origin. To this end define, for an $r > 0$ to be chosen soon, $v\colon B_1 \to \mathbb{R}$ by

$$v(X) := \frac{1}{r^2}u(rX) - \left[\frac{1}{r^2}u(0) + \frac{1}{r}\nabla u(0)\cdot X + \frac{1}{2}X^t D^2u(0)X\right].$$

We clearly have

$$v(0) = |\nabla v(0)| = 0 \quad \text{and} \quad |D^2v(0)| \leq \varsigma(r), \tag{5.1}$$

where ς is the modulus of continuity for D^2u. Now, we choose $0 < r \ll 1$ so small that

$$\varsigma(r) \leq c_n\overline{\delta},$$

where c_n is a dimensional constant and $\overline{\delta}$ is the number appearing in Theorem 2.2. With such a choice, v is as in the hypotheses of Theorem 2.2, for $\tilde{F}(X, M) := F(rX, M + D^2u(0))$ and $\tilde{f}(X) = f(rX)$. □

Remark 5.2. We remark that in the proof of Corollary 5.1, we can estimate the absolute value of v using integral remainders of the Taylor expansion. Thus, the very same conclusion of that Corollary holds true if we start up only with VMO condition on D^2u. It is also interesting to highlight that Corollary 5.1 implies that if u is a viscosity solution in B_1 of a non-convex, fully nonlinear equation under hypotheses (H1)–(H3) and if u is C^2 at a point $p \in B_1$, then indeed u is $C^{2,\alpha}$ in a neighborhood of p.

The second application we explore here regards a mild extension of a recent result due to Armstrong, Silvestre, and Smart [1] on partial regularity for solutions to uniform elliptic PDEs.

Corollary 5.3 (Partial regularity). *Let $u \in C^0(B_1)$ be a viscosity solution to $F(D^2u) = f(X)$ where $F \in C^1(\mathrm{Sym}(n))$ satisfies $c \leq D_{u_iu_j}F(M) \leq c^{-1}$ for some constant $c > 0$ and the source function f is Lipschitz continuous. Then, $u \in C^{2,1^-}(B_1 \setminus \Sigma)$ for a closed set $\Sigma \subset B_1$, with Hausdorff dimension at most $(n-\epsilon)$ for a universal $\epsilon > 0$.*

Proof. The proof is obtained by following the analysis employed in [1]. The same conclusion of Lemma 5.2 from [1] follows by noticing that if $f \in C^{0,1}$, then

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(u_e)) \leq C \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(D^2(u_e)) \geq -C$$

where

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \inf_{\lambda I_n \leq A \leq \Lambda I_n} \mathrm{tr}(AM), \quad \mathcal{M}^+_{\lambda,\Lambda}(M) := \sup_{\lambda I_n \leq A \leq \Lambda I_n} \mathrm{tr}(AM)$$

are the Pucci extremal operators. Lemma 7.8 of [4] can still be employed. The very same conclusion of Lemma 5.3 from [1] also holds true for equations with Lipschitz sources. Indeed, using the same notation from that Lemma, if $Y \in B_{\frac{1}{2}}$ is such that there exist $M \in \mathrm{Sym}(n)$, $p \in \mathbb{R}^n$ and $Z \in B(Y,r)$ such that

$$|u(X) - u(Z) + p.(Z-X) + (Z-X).M(Z-X)| \leq \frac{1}{6} r^{-1}\overline{\delta}|Z-X|^3, \text{ for } X \in B_1,$$

we define

$$v(X) = \frac{1}{16r^2}(u(Z+4rX) - u(Z) + 4rp \cdot X + 16r^2 X \cdot MX)$$

and

$$\widetilde{F}(N) = F(N-M) - F(-M).$$

Notice that

$$\widetilde{F}(X, D^2v) = f(Z+4rX) - F(-M) = \widetilde{f}(X) \in C^{0,1}.$$

Thus, applying Theorem 2.2 to v shows that u is $C^{2,1^-}$ in $B(Y,r)$. The proof of Corollary 5.3 follows now exactly as in [1]. □

6. Log-Lipschitz estimates in continuous media

In this section we prove Theorem 2.3. Initially we show that under continuity assumption on the coefficients of F and on the source f, after a proper scaling, solutions are under the smallness regime requested by Lemma 3.2, with $\alpha = 0$. For this purpose, define

$$v(X) = \frac{u(\mu X)}{\mu^2}, \quad F_\mu(X, M) := F(\mu X, M) \quad \text{and} \quad f_\mu(X) := f(\mu X),$$

for a parameter μ to be determined. The equation

$$F_\mu(X, D^2 v) = f_\mu(X)$$

is satisfied in the viscosity sense. Now we choose μ so small that

$$\tau(\mu) \le \delta^{3/2},$$

where τ is the modulus of continuity of the media and $\delta > 0$ is the number sponsored by Lemma 3.2 with $\alpha = 0$. Define

$$\tau_\mu(t) := \tau(\mu t)$$

and note that

$$\max\left\{ |f_\mu(X|, \frac{|F_\mu(X, M) - F_\mu(0, M)|}{\|M\|} \right\} \le \tau_\mu(|X - Y|).$$

Thus,

$$\sup_{M \in \mathrm{Sym}(n)} \frac{|F_\mu(X, M) - F_\mu(0, M)|}{\|M\|} \le \delta^{3/2} \quad \text{and} \quad \|f_\mu\|_{L^\infty(B_1)} \le \delta^{3/2}.$$

Now if we take

$$\|u\|_{L^\infty(B_1)} \le \overline{\delta} := \delta \mu^2$$

then

$$\|v\|_{L^\infty(B_1)} \le \delta.$$

Estimates proven for v give the desired ones for u.

The conclusion of the above reasoning is that we can start off the proof of Theorem 2.3 using Lemma 3.2. That is, the proof of the current Theorem begins with the existence of a quadratic polynomial P_1 satisfying $F(0, D^2 P_1) = 0$ and a number $\sigma > 0$ for which the following estimate

$$\sup_{B_\sigma} |u - P_1| \le \sigma^2 \delta, \tag{6.1}$$

holds, provided δ is small enough, depending only on universal parameters. As in Lemma 4.1, we shall prove by induction the existence of a sequence of polynomials

$$P_k(X) = \frac{1}{2} X^t A_k X + b_k X + c_k$$

satisfying $F(0, D^2 P_k) = 0$ such that

$$|u(X) - P_k(X)| \le \delta \sigma^{2k} \quad \text{for } |X| \le \sigma^k. \tag{6.2}$$

Moreover, we have the following estimates on the coefficients

$$\begin{cases} |A_k - A_{k-1}| \le C\delta \\ |b_k - b_{k-1}| \le C\delta\sigma^{(k-1)} \\ |c_k - c_{k-1}| \le C\delta\sigma^{2(k-1)}. \end{cases} \tag{6.3}$$

The case $k = 1$ is precisely the conclusion expressed by (6.1). Assume we have verified the kth step of induction. Define the scaled function and the scaled operator

$$\tilde{u}(X) := \frac{1}{\sigma^{2k}} (u(\sigma^k X) - P_k(\sigma^k X)) \quad \text{and} \quad \tilde{F}(X, M) := F(\sigma^k X, M + D^2 P_k).$$

Easily one verifies that $\tilde{u}$ is a viscosity solution to

$$\tilde{F}(X, D^2\tilde{u}) = f(\sigma^k X) := \tilde{f}(X).$$

From the induction hypothesis (6.2), $\tilde{u}$ is flat, *i.e.*, $\|\tilde{u}\|_{L^\infty(B_1)} \le \delta$. Also, clearly

$$\sup_{M \in \mathrm{Sym}(n)} \frac{|\tilde{F}(X, M) - \tilde{F}(0, M)|}{\|M\|} \le \delta^{3/2} \quad \text{and} \quad \|\tilde{f}\|_{L^\infty(B_1)} \le \delta^{3/2}.$$

That is, for $\tilde{u}$ it is legitimate to conclude (6.1), thus there exists a quadratic polynomial $\tilde{P}$ with $\tilde{F}(0, D^2\tilde{P}) = 0$ and

$$|\tilde{u}(X) - \tilde{P}(X)| \le \delta\sigma^{2k} \quad \text{for } |X| \le \sigma.$$

The $(k+1)$th step of induction follows by defining

$$P_{k+1}(X) := P_k(X) + \sigma^{2k} \tilde{P}(\sigma^{-k} X).$$

In view of the coefficient oscillation control (6.3), we conclude b_k converges in $\mathbb{R}^n$ to a vector b_∞ and c_k converges in $\mathbb{R}$ to a real number c_∞. Also

$$|c_k - c_\infty| \le C\delta\sigma^{2k}, \tag{6.4}$$

$$|b_k - b_\infty| \le C\delta\sigma^k. \tag{6.5}$$

The sequence of matrices A_k may diverge; however, we can at least estimate

$$\|A_k\|_{\mathrm{Sym(n)}} \leq kC\delta. \tag{6.6}$$

In what follows, we define the tangential affine function

$$\ell_\infty(X) := c_\infty + b_\infty \cdot X$$

and estimate, in view of (6.4), (6.5) and (6.6), for $|X| \leq \sigma^k$,

$$\begin{aligned} |u(X) - \ell_\infty(X)| &\leq |u(X) - P_k(X)| + |c_k - c_\infty| \\ &\quad + |(b_k - b_\infty)||X| + |A_k||X|^2 \\ &\leq \delta\sigma^{2k} + 2C\delta\sigma^{2k} + kC\delta\sigma^{2k} \\ &\leq C\delta(k\sigma^{2k}). \end{aligned} \tag{6.7}$$

Finally, for fixed $X \in B_\sigma$, take $k \in \mathbb{N}$ such that $\sigma^{k+1} < |X| \leq \sigma^k$. From (6.7), we find

$$|u(X) - \ell_\infty(X)| \leq -(C_1\delta) \cdot |X|^2 \log |X|,$$

as desired. The proof of Theorem 2.3 is concluded. □

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Universidade Federal do Ceará
Departamento de Matemática
Campus do Pici - Bloco 914
Fortaleza, CE - Brazil 60.455-760
teixeira@mat.ufc.br