

## Varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$

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**Abstract.** We classify varieties with  $P_3(X) = 4$  and  $q(X) = \dim(X)$ .

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### 1. – Introduction

Let  $X$  be a smooth complex projective variety. When  $\dim(X) \geq 3$  it is very hard to classify such varieties in terms of their birational invariants. Surprisingly, when  $X$  has many holomorphic 1-forms, it is sometimes possible to achieve classification results in any dimension. In [Ka], Kawamata showed that: *If  $X$  is a smooth complex projective variety with  $\kappa(X) = 0$  then the Albanese morphism  $a : X \rightarrow A(X)$  is surjective. If moreover,  $q(X) = \dim(X)$ , then  $X$  is birational to an abelian variety.* Subsequently, Kollár proved an effective version of this result (cf. [Ko2]): *If  $X$  is a smooth complex projective variety with  $P_m(X) = 1$  for some  $m \geq 4$ , then the Albanese morphism  $a : X \rightarrow A(X)$  is surjective. If moreover,  $q(X) = \dim(X)$ , then  $X$  is birational to an abelian variety.* These results were further refined and expanded as follows:

**THEOREM 1.1** (cf. [CH1], [CH3], [HP], [Hac2]). *If  $P_m(X) = 1$  for some  $m \geq 2$  or if  $P_3(X) \leq 3$ , then the Albanese morphism  $a : X \rightarrow A(X)$  is surjective. If moreover  $q(X) = \dim(X)$ , then:*

- (1) *If  $P_m(X) = 1$  for some  $m \geq 2$ , then  $X$  is birational to an abelian variety.*
- (2) *If  $P_3(X) = 2$ , then  $\kappa(X) = 1$  and  $X$  is birational to a double cover of its Albanese variety.*
- (3) *If  $P_3(X) = 3$ , then  $\kappa(X) = 1$  and  $X$  is birational to a bi-double cover of its Albanese variety.*

In this paper we will prove a similar result for varieties with  $P_3(X) = 4$  and  $q(X) = \dim(X)$ . We start by considering the following examples:

**EXAMPLE 1.** Let  $G$  be a group acting faithfully on a curve  $C$  and acting faithfully by translations on an abelian variety  $\tilde{K}$ , so that  $C/G = E$  is an

elliptic curve and  $\dim H^0(C, \omega_C^{\otimes 3})^G = 4$ . Let  $G$  act diagonally on  $\tilde{K} \times C$ , then  $X := \tilde{K} \times C/G$  is a smooth projective variety with  $\kappa(X) = 1$ ,  $P_3(X) = 4$  and  $q(X) = \dim(X)$ . We illustrate some examples below:

- (1)  $G = \mathbb{Z}_m$  with  $m \geq 3$ . Consider an elliptic curve  $E$  with a line bundle  $L$  of degree 1. Taking the normalization of the  $m$ -th root of a divisor  $B = (m - a)B_1 + aB_2 \in |mL|$  with  $1 \leq a \leq m - 1$  and  $m \geq 3$ , one obtains a smooth curve  $C$  and a morphism  $g : C \rightarrow E$  of degree  $m$ . One has that

$$g_*\omega_C = \sum_{i=0}^{m-1} L^{(i)}$$

where  $L^{(i)} = L^{\otimes i}(-\lfloor \frac{iB}{m} \rfloor)$  for  $i = 0, \dots, m - 1$ .

- (2)  $G = \mathbb{Z}_2$ . Let  $L$  be a line bundle of degree 2 over an elliptic curve  $E$ . Let  $C \rightarrow E$  be the degree 2 cover defined by a reduced divisor  $B \in |2L|$ .
- (3)  $G = (\mathbb{Z}_2)^2$ . Let  $L_i$  for  $i = 1, 2$  be line bundles of degree 1 on an elliptic curve  $E$  and  $C_i \rightarrow E$  be degree 2 covers defined by disjoint reduced divisors  $B_i \in |2L_i|$ . Then  $C := C_1 \times_E C_2 \rightarrow E$  is a  $G$  cover.
- (4)  $G = (\mathbb{Z}_2)^3$ . For  $i = 1, 2, 3, 4$ , let  $P_i$  be distinct points on an elliptic curve  $E$ . For  $j = 1, 2, 3$  let  $L_j$  be line bundles of degree 1 on  $E$  such that  $B_1 = P_1 + P_2 \in |2L_1|$ ,  $B_2 = P_1 + P_3 \in |2L_2|$  and  $B_3 = P_1 + P_4 \in |2L_3|$ . Let  $C_j \rightarrow E$  be degree 2 covers defined by reduced divisors  $B_j \in |2L_j|$ . Let  $C$  be the normalization of  $C_1 \times_E C_2 \times_E C_3 \rightarrow E$ , then  $C$  is a  $G$  cover.

Note that (1) is ramified at 2 points. Following [Be] Section VI.12, one has that  $P_2(X) = \dim H^0(C, \omega_C^{\otimes 2})^G = 2$  and  $P_3(X) = \dim H^0(C, \omega_C^{\otimes 3})^G = 4$ . Similarly (2), (3), (4) are ramified along 4 points and hence  $P_2(X) = P_3(X) = 4$ .

EXAMPLE 2. Let  $q : A \rightarrow S$  be a surjective morphism with connected fibers from an abelian variety of dimension  $n \geq 3$  to an abelian surface. Let  $L$  be an ample line bundle on  $S$  with  $h^0(S, L) = 1$ ,  $P \in \text{Pic}^0(A)$  with  $P \notin \text{Pic}^0(S)$  and  $P^{\otimes 2} \in \text{Pic}^0(S)$ . For  $D$  an appropriate reduced divisor in  $|L^{\otimes 2} \otimes P^{\otimes 2}|$ , there is a degree 2 cover  $a : X \rightarrow A$  such that  $a_*(\mathcal{O}_X) = \mathcal{O}_A \oplus (L \otimes P)^\vee$ . One sees that  $P_i(X) = 1, 4, 4$  for  $i = 1, 2, 3$ .

EXAMPLE 3. Let  $q : A \rightarrow E_1 \times E_2$  be a surjective morphism from an abelian variety to the product of two elliptic curves,  $p_i : A \rightarrow E_i$  the corresponding morphisms,  $L_i$  be line bundles of degree 1 on  $E_i$  and  $P, Q \in \text{Pic}^0(A)$  such that  $P, Q$  generate a subgroup of  $\text{Pic}^0(A)/\text{Pic}^0(E_1 \times E_2)$  which is isomorphic to  $(\mathbb{Z}_2)^2$ . Then one has double covers  $X_i \rightarrow A$  corresponding to divisors  $D_1 \in |2(q_1^*L_1 \otimes P)|$ ,  $D_2 \in |2(q_2^*L_2 \otimes Q)|$ . The corresponding bi-double cover satisfies

$$a_*(\omega_X) = \mathcal{O}_A \oplus p_1^*L_1 \otimes P \oplus p_2^*L_2 \otimes Q \oplus p_1^*L_1 \otimes P \otimes p_2^*L_2 \otimes Q$$

One sees that  $P_i(X) = 1, 4, 4$  for  $i = 1, 2, 3$ .

We will prove the following:

**THEOREM 1.2.** *Let  $X$  be a smooth complex projective variety with  $P_3(X) = 4$ , then the Albanese morphism  $a : X \rightarrow A$  is surjective (in particular  $q(X) \leq \dim(X)$ ). If moreover,  $q(X) = \dim(X)$ , then  $\kappa(X) \leq 2$  and we have the following cases:*

- (1) *If  $\kappa(X) = 2$ , then  $X$  is birational either to a double cover or to a bi-double cover of  $A$  as in Examples 2 and 3 and so  $P_2(X) = 4$ .*
- (2) *If  $\kappa(X) = 1$ , then  $X$  is birational to the quotient  $\tilde{K} \times C/G$  where  $C$  is a curve,  $\tilde{K}$  is an abelian variety,  $G$  acts faithfully on  $C$  and  $\tilde{K}$ . One has that either  $P_2(X) = 2$  and  $C \rightarrow C/G$  is branched along 2 points with inertia group  $H \cong \mathbb{Z}_m$  with  $m \geq 3$  or  $P_2(X) = 4$  and  $C \rightarrow C/G$  is branched along 4 points with inertia group  $H \cong (\mathbb{Z}_2)^s$  with  $s \in \{1, 2, 3\}$ . See Example 1.*

**NOTATION AND CONVENTIONS.** We work over the field of complex numbers. We identify Cartier divisors and line bundles on a smooth variety, and we use the additive and multiplicative notation interchangeably. If  $X$  is a smooth projective variety, we let  $K_X$  be a canonical divisor, so that  $\omega_X = \mathcal{O}_X(K_X)$ , and we denote by  $\kappa(X)$  the Kodaira dimension, by  $q(X) := h^1(\mathcal{O}_X)$  the irregularity and by  $P_m(X) := h^0(\omega_X^{\otimes m})$  the  $m$ -th plurigenus. We denote by  $a: X \rightarrow A(X)$  the Albanese map and by  $\text{Pic}^0(X)$  the dual abelian variety to  $A(X)$  which parameterizes all topologically trivial line bundles on  $X$ . For a  $\mathbb{Q}$ -divisor  $D$  we let  $\lfloor D \rfloor$  be the integral part and  $\{D\}$  the fractional part. Numerical equivalence is denoted by  $\equiv$  and we write  $D < E$  if  $E - D$  is an effective divisor. If  $f: X \rightarrow Y$  is a morphism, we write  $K_{X/Y} := K_X - f^*K_Y$  and we often denote by  $F_{X/Y}$  the general fiber of  $f$ . A  $\mathbb{Q}$ -Cartier divisor  $L$  on a projective variety  $X$  is nef if for all curves  $C \subset X$ , one has  $L.C \geq 0$ . For a surjective morphism of projective varieties  $f : X \rightarrow Y$ , we will say that a Cartier divisor  $L$  on  $X$  is  $Y$ -big if for an ample line bundle  $H$  on  $Y$ , there exists a positive integer  $m > 0$  such that  $h^0(L^{\otimes m} \otimes f^*H^\vee) > 0$ . The rest of the notation is standard in algebraic geometry.

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## 2. – Preliminaries

### 2.1. – The Albanese map and the Iitaka fibration

Let  $X$  be a smooth projective variety. If  $\kappa(X) > 0$ , then the Iitaka fibration of  $X$  is a morphism of projective varieties  $f: X' \rightarrow Y$ , with  $X'$  birational to  $X$  and  $Y$  of dimension  $\kappa(X)$ , such that the general fiber of  $f$  is smooth,

irreducible, of Kodaira dimension zero. The Iitaka fibration is determined only up to birational equivalence. Since we are interested in questions of a birational nature, we usually assume that  $X = X'$  and that  $Y$  is smooth.

$X$  has *maximal Albanese dimension* if  $\dim(a_X(X)) = \dim(X)$ . We will need the following facts (cf. [HP], Propositions 2.1, 2.3, 2.12 and Lemma 2.14 respectively).

**PROPOSITION 2.1.** *Let  $X$  be a smooth projective variety of maximal Albanese dimension, and let  $f: X \rightarrow Y$  be the Iitaka fibration (assume  $Y$  smooth). Denote by  $f_*: A(X) \rightarrow A(Y)$  the homomorphism induced by  $f$  and consider the commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A(X) \\ f \downarrow & & f_* \downarrow \\ Y & \xrightarrow{a_Y} & A(Y). \end{array}$$

Then:

- a)  $Y$  has maximal Albanese dimension;
- b)  $f_*$  is surjective and  $\ker f_*$  is connected of dimension  $\dim(X) - \kappa(X)$ ;
- c) There exists an abelian variety  $P$  isogenous to  $\ker f_*$  such that the general fiber of  $f$  is birational to  $P$ .

Let  $K := \ker f_*$  and  $F = F_{X/Y}$ . Define

$$G := \ker \left( \text{Pic}^0(X) \rightarrow \text{Pic}^0(F) \right).$$

Then

**LEMMA 2.2.**  $G$  is the union of finitely many translates of  $\text{Pic}^0(Y)$  corresponding to the finite group

$$\overline{G} := G/\text{Pic}^0(Y) \cong \ker \left( \text{Pic}^0(K) \rightarrow \text{Pic}^0(F) \right).$$

## 2.2. – Sheaves on abelian varieties

Recall the following easy corollary of the theory of Fourier-Mukai transforms cf. [M]:

**PROPOSITION 2.3.** *Let  $\psi: \mathcal{F} \hookrightarrow \mathcal{G}$  be an inclusion of coherent sheaves on an abelian variety  $A$  inducing isomorphisms  $H^i(A, \mathcal{F} \otimes P) \rightarrow H^i(A, \mathcal{G} \otimes P)$  for all  $i \geq 0$  and all  $P \in \text{Pic}^0(A)$ . Then  $\psi$  is an isomorphism of sheaves.*

Following [M], we will say that a coherent sheaf  $\mathcal{F}$  on an abelian variety  $A$  is I.T. 0 if  $h^i(A, \mathcal{F} \otimes P) = 0$  for all  $i > 0$  and for all  $P \in \text{Pic}^0(A)$ . We will say that an inclusion of coherent sheaves on  $A$ ,  $\psi: \mathcal{F} \hookrightarrow \mathcal{G}$  is an I.T. 0 isomorphism if  $\mathcal{F}, \mathcal{G}$  are I.T. 0 and  $h^0(\mathcal{G}) = h^0(\mathcal{F})$ . From the above proposition, it follows that every I.T. 0 isomorphism  $\mathcal{F} \hookrightarrow \mathcal{G}$  is an isomorphism. We will need the following result:

LEMMA 2.4. *Let  $f : X \rightarrow E$  be a morphism from a smooth projective variety to an elliptic curve, such that  $K_X$  is  $E$ -big. Then, for all  $P \in \text{Pic}^0(X)_{\text{tors}}$ ,  $\eta \in \text{Pic}^0(E)$  and all  $m \geq 2$ ,  $f_*(\omega_X^{\otimes m} \otimes P \otimes f^*\eta)$  is I.T. 0. In particular*

$$\text{deg}(f_*(\omega_X^{\otimes m} \otimes P \otimes f^*\eta)) = h^0(\omega_X^{\otimes m} \otimes P \otimes f^*\eta).$$

The proof of the above lemma is analogous to the proof of Lemma 2.6 of [Hac2]. We just remark that it suffices to show that  $f_*(\omega_X^{\otimes m} \otimes P)$  is I.T. 0. The sheaf  $f_*(\omega_X^{\otimes m} \otimes P)$  is torsion free and hence locally free on  $E$ . By Riemann-Roch

$$h^0(\omega_X^{\otimes m} \otimes P) = h^0(f_*(\omega_X^{\otimes m} \otimes P)) = \chi(f_*(\omega_X^{\otimes m} \otimes P)) = \text{deg}(f_*(\omega_X^{\otimes m} \otimes P)).$$

### 2.3. – Cohomological support loci

Let  $\pi : X \rightarrow A$  be a morphism from a smooth projective variety to an abelian variety,  $T \subset \text{Pic}^0(A)$  the translate of a subtorus and  $\mathcal{F}$  a coherent sheaf on  $X$ . One can define the cohomological support loci of  $\mathcal{F}$  as follows:

$$V^i(X, T, \mathcal{F}) := \{P \in T \mid h^i(X, \mathcal{F} \otimes \pi^*P) > 0\}.$$

If  $T = \text{Pic}^0(X)$  we write  $V^i(\mathcal{F})$  or  $V^i(X, \mathcal{F})$  instead of  $V^i(X, \text{Pic}^0(X), \mathcal{F})$ . When  $\mathcal{F} = \omega_X$ , the geometry of the loci  $V^i(\omega_X)$  is governed by the following result of Green and Lazarsfeld (cf. [GL], [EL]):

THEOREM 2.5 (Generic Vanishing Theorem). *Let  $X$  be a smooth projective variety. Then:*

- a)  $V^i(\omega_X)$  has codimension  $\geq i - (\dim(X) - \dim(a_X(X)))$ ;
- b) Every irreducible component of  $V^i(X, \omega_X)$  is a translate of a sub-torus of  $\text{Pic}^0(X)$  by a torsion point (the same also holds for the irreducible components of  $V_m^i(\omega_X) := \{P \in \text{Pic}^0(X) \mid h^i(X, \omega_X \otimes P) \geq m\}$ );
- c) Let  $T$  be an irreducible component of  $V^i(\omega_X)$ , let  $P \in T$  be a point such that  $V^i(\omega_X)$  is smooth at  $P$ , and let  $v \in H^1(X, \mathcal{O}_X) \cong T_P \text{Pic}^0(X)$ . If  $v$  is not tangent to  $T$ , then the sequence

$$H^{i-1}(X, \omega_X \otimes P) \xrightarrow{\cup v} H^i(X, \omega_X \otimes P) \xrightarrow{\cup v} H^{i+1}(X, \omega_X \otimes P)$$

is exact. Moreover, if  $P$  is a general point of  $T$  and  $v$  is tangent to  $T$  then both maps vanish;

- d) If  $X$  has maximal Albanese dimension, then there are inclusions:

$$V^0(\omega_X) \supseteq V^1(\omega_X) \supseteq \dots \supseteq V^n(\omega_X) = \{\mathcal{O}_X\};$$

- e) Let  $f : Y \rightarrow X$  be a surjective map of projective varieties,  $Y$  smooth, then statements analogous to a), b), c) for  $P \in \text{Pic}^0_{\text{tors}}(Y)$  and d) above also hold for the sheaves  $R^i f_*\omega_X$ . More precisely we refer to [CH3], [CIH] and [Hac5].

When  $X$  is of maximal Albanese dimension, its geometry is very closely connected to the properties of the loci  $V^i(\omega_X)$ . We recall the following two results from [CH2]:

**THEOREM 2.6.** *Let  $X$  be a variety of maximal Albanese dimension. The translates through the origin of the irreducible components of  $V^0(\omega_X)$  generate a subvariety of  $\text{Pic}^0(X)$  of dimension  $\kappa(X) - \dim(X) + q(X)$ . In particular, if  $X$  is of general type then  $V^0(X, \omega_X)$  generates  $\text{Pic}^0(X)$ .*

**PROPOSITION 2.7.** *Let  $X$  be a variety of maximal Albanese dimension and  $G, Y$  defined as in Proposition 2.1. Then*

- a)  $V^0(X, \text{Pic}^0(X), \omega_X) \subset G$ ;
- b) For every  $P \in G$ , the loci  $V^0(X, \text{Pic}^0(X), \omega_X) \cap (P + \text{Pic}^0(Y))$  are non-empty;
- c) If  $P$  is an isolated point of  $V^0(X, \text{Pic}^0(X), \omega_X)$ , then  $P = \mathcal{O}_X$ .

The following result governs the geometry of  $V^0(\omega_X^{\otimes m})$  for all  $m \geq 2$ :

**PROPOSITION 2.8.** *Let  $X$  be a smooth projective variety of maximal Albanese dimension,  $f: X \rightarrow Y$  the Iitaka fibration (assume  $Y$  smooth) and  $G$  defined as in Proposition 2.1. If  $m \geq 2$ , then  $V^0(\omega_X^{\otimes m}) = G$ . Moreover, for any fixed  $Q \in V^0(\omega_X^{\otimes m})$ , and all  $P \in \text{Pic}^0(Y)$  one has  $h^0(\omega_X^{\otimes m} \otimes Q \otimes P) = h^0(\omega_X^{\otimes m} \otimes Q)$ .*

We will also need the following lemma proved in [CH2] Section 3.

**LEMMA 2.9.** *Let  $X$  be a smooth projective variety and  $D$  an effective  $a_X$ -exceptional divisor on  $X$ . If  $\mathcal{O}_X(D) \otimes P$  is effective for some  $P \in \text{Pic}^0(X)$ , then  $P = \mathcal{O}_X$ .*

The following result is due to Ein and Lazarsfeld (see [HP], Lemma 2.13):

**LEMMA 2.10.** *Let  $X$  be a variety such that  $\chi(\omega_X) = 0$  and such that  $a_X : X \rightarrow A(X)$  is surjective and generically finite. Let  $T$  be an irreducible component of  $V^0(\omega_X)$ , and let  $\pi_B : X \rightarrow B := \text{Pic}^0(T)$  be the morphism induced by the map  $A(X) \rightarrow \text{Pic}^0(\text{Pic}^0(X)) \rightarrow B$  corresponding to the inclusion  $T \hookrightarrow \text{Pic}^0(X)$ .*

*Then there exists a divisor  $D_T \prec R := \text{Ram}(a_X) = K_X$ , vertical with respect to  $\pi_B$  (i.e.  $\pi_B(D_T) \neq B$ ), such that for general  $P \in T$ ,  $G_T := R - D_T$  is a fixed divisor of each of the linear series  $|K_X + P|$ .*

We have the following useful Corollary:

**COROLLARY 2.11.** *In the notation of Lemma 2.10, if  $\dim(T) = 1$ , then for any  $P \in T$ , there exists a line bundle of degree 1 on  $B$  such that  $\pi_B^* L_P \prec K_X + P$ .*

**PROOF.** By [HP] Step 8 of the proof of Theorem 6.1, for general  $Q \in T$ , there exists a line bundle of degree 1 on  $B$  such that  $\pi_B^* L_Q \prec K_X + Q$ . Write  $P = Q + \pi_B^* \eta$  where  $\eta \in \text{Pic}^0(B)$ . Then, since

$$h^0(\omega_X \otimes P \otimes \pi_B^*(L_Q \otimes \eta)^\vee) = h^0(\pi_{B,*}(\omega_X \otimes Q) \otimes L_Q^\vee) \neq 0,$$

one sees that there is an inclusion  $\pi_B^*(L_Q \otimes \eta) \rightarrow \omega_X \otimes P$ . □

Recall the following result (cf. [Hac2], Lemma 2.17):

LEMMA 2.12. *Let  $X$  be a smooth projective variety, let  $L$  and  $M$  be line bundles on  $X$ , and let  $T \subset \text{Pic}^0(X)$  be an irreducible subvariety of dimension  $t$ . If for all  $P \in T$ ,  $\dim |L + P| \geq a$  and  $\dim |M - P| \geq b$ , then  $\dim |L + M| \geq a + b + t$ .*

LEMMA 2.13. *Let  $T$  be a 1-dimensional component of  $V^0(\omega_X)$ ,  $E := T^\vee$  and  $\pi : X \rightarrow E$  the induced morphism. Then  $P|_F \cong \mathcal{O}_F$  for all  $P \in T$ .*

PROOF. Let  $G_T, D_T$  be as in Lemma 2.10, then for  $P \in T$  we have  $|K_X + P| = G_T + |D_T + P|$  and hence the divisor  $D_T + P$  is effective. It follows that  $(D_T + P)|_F$  is also effective. However  $D_T$  is vertical with respect to  $\pi$  and hence  $D_T|_F \cong \mathcal{O}_F$ . By Lemma 2.9, one sees that  $P|_F \cong \mathcal{O}_F$ .  $\square$

### 3. – Kodaira dimension of Varieties with $P_3(X) = 4, q(X) = \dim(X)$

The purpose of this section is to study the Albanese map and Iitaka fibration of varieties with  $P_3 = 4$  and  $q = \dim(X)$ . We will show that: 1) the Albanese map is surjective, 2) the image of the Iitaka fibration is an abelian variety (and hence the Iitaka fibration factors through the Albanese map), 3) we have that  $\kappa(X) \leq 2$ .

We begin by fixing some notation. We write

$$V_0(X, \omega_X) = \cup_{i \in I} S_i$$

where  $S_i$  are irreducible components. Let  $T_i$  denote the translate of  $S_i$  passing through the origin and  $\delta_i := \dim(S_i)$ . For any  $i, j \in I$ , let  $\delta_{i,j} := \dim(T_i \cap T_j)$ .

Recall that  $V_0(X, \omega_X) \subset G \rightarrow \bar{G} := G/\text{Pic}^0(Y)$ . For any  $\eta \in \bar{G}$ , we fix once and for all  $S_\eta$  a maximal dimensional component which maps to  $\eta$ . In particular,  $T_0$  denotes the translate through the origin of a maximal dimensional component  $S_0 \subset V^0(X, \omega_X) \cap \text{Pic}^0(Y)$ . If  $X$  is of maximal Albanese dimension with  $q(X) = \dim(X)$ , then its Iitaka fibration image  $Y$  is of maximal Albanese dimension with  $q(Y) = \dim(Y) = \kappa(X)$ . Moreover, by Proposition 2.7, one has  $\delta_i \geq 1, \forall i \neq 0$ .

We denote by  $P_{m,\alpha} := h^0(X, \omega_X^{\otimes m} \otimes \alpha)$  for  $\alpha \in \text{Pic}^0(X)$ . Now let  $Q_i$  ( $Q_\eta$  resp.) be a general element in  $S_i$  ( $S_\eta$  resp.), we denote by  $P_{m,i} := h^0(X, \omega_X^{\otimes m} \otimes Q_i)$  ( $P_{m,\eta}$  resp.). We remark that it is convenient to choose  $Q_i$  ( $Q_\eta$  resp.) to be torsion so that the results of Kollár on higher direct images of dualizing sheaves will also apply to the sheaf  $\omega_X \otimes Q_i$ . Proposition 2.8 can be rephrased as

$$(1) \quad P_{m,\alpha} = P_{m,\alpha+\beta} \quad \forall \alpha \in \text{Pic}^0(X), \beta \in \text{Pic}^0(Y), m \geq 2.$$

Notice that if  $\alpha \notin G$  then also  $\alpha + \beta \notin G$  and so both numbers are equal to 0.

By Lemma 2.12 one has, for any  $\eta, \zeta \in \bar{G}$ ,

$$(2) \quad \begin{cases} P_{2,\eta+\zeta} \geq P_{1,\eta} + P_{1,\zeta} + \delta_{\eta,\zeta} - 1, \\ P_{2,2\eta} \geq 2P_{1,\eta} + \delta_\eta - 1, \\ P_{3,\eta+\zeta} \geq P_{1,\eta} + P_{2,\zeta} + \delta_\eta - 1. \end{cases}$$

Here  $\delta_\eta = \delta_i$  and  $\delta_{\eta,\zeta} = \delta_{i,j}$  if  $T_\eta, T_\zeta$  are represented by  $T_i, T_j$  respectively.

The following lemma is very useful when  $\kappa \geq 2$ .

LEMMA 3.1. *Let  $X$  be a variety of maximal Albanese dimension with  $\kappa(X) \geq 2$ . Suppose that there is a surjective morphism  $\pi : X \rightarrow E$  to an elliptic curve  $E$ , and suppose that there is an inclusion  $\varphi : \pi^*L \rightarrow \omega_X^{\otimes m} \otimes P$  for some  $m \geq 2$ ,  $P|_F = \mathcal{O}_F$  where  $F$  is a general fiber of  $\pi$  and  $L$  is an ample line bundle on  $E$ . Then the induced map  $L \rightarrow \pi_*(\omega_X^{\otimes m} \otimes P)$  is not an isomorphism,  $\text{rank}(\pi_*(\omega_X^{\otimes m} \otimes P)) \geq 2$  and  $h^0(X, \omega_X^{\otimes m} \otimes P) > h^0(E, L)$ .*

PROOF. By the easy addition theorem,  $\kappa(F) \geq 1$ . Hence by Theorem 1.1,  $P_m(F) \geq 2$  for  $m \geq 2$ . The sheaf  $\pi_*(\omega_X^{\otimes m} \otimes P)$  has rank equal to  $h^0(F, \omega_X^{\otimes m} \otimes P|_F) = h^0(F, \omega_F^{\otimes m}) \geq 2$ . Therefore,  $L \rightarrow \pi_*(\omega_X^{\otimes m} \otimes P)$  is not an isomorphism. Since they are non-isomorphic I.T.0 sheaves, it follows that  $h^0(\pi_*(\omega_X^{\otimes m} \otimes P)) > h^0(L)$ .  $\square$

COROLLARY 3.2. *Keep the notation as in Lemma 3.1. If there is a morphism  $\pi' : X \rightarrow E'$  and an inclusion  $\pi'^*L' \hookrightarrow \omega_X \otimes P^\vee$  for some ample line bundle  $L'$  on  $E'$  and  $P \in \text{Pic}^0(X)$  with  $P|_{F'} = \mathcal{O}_{F'}$ , then for all  $m \geq 2$*

$$P_{m+1}(X) \geq 2 + h^0(X, \omega_X^{\otimes m} \otimes P) > 2 + h^0(E', L').$$

PROOF. The inclusion  $\pi'^*L' \hookrightarrow \omega_X \otimes P^\vee$  induces an inclusion

$$\pi'^*L' \otimes \omega_X^{\otimes m} \otimes P \hookrightarrow \omega_X^{\otimes m+1}.$$

By Riemann-Roch, one has

$$P_{m+1}(X) \geq h^0(E', L' \otimes \pi'_*(\omega_X^{\otimes m} \otimes P)) \geq h^0(E', \pi'_*(\omega_X^{\otimes m} \otimes P)) + \text{rank}(\pi'_*(\omega_X^{\otimes m} \otimes P)).$$

By Proposition 2.7, there exists  $\alpha \in \text{Pic}^0(Y)$  such that  $h^0(\omega_X^{\otimes m-1} \otimes P^{\otimes 2} \otimes \alpha) \neq 0$  and hence there is an inclusion

$$\pi'^*L' \hookrightarrow \omega_X^{\otimes m} \otimes P \otimes \alpha.$$

By Proposition 2.8 and Lemma 3.1,

$$h^0(X, \omega_X^{\otimes m} \otimes P) = h^0(X, \omega_X^{\otimes m} \otimes P \otimes \alpha) > h^0(E', L'). \quad \square$$

REMARK 3.3. Let  $X$  be a variety with  $\kappa(X) \geq 2$ . Suppose that there is a 1-dimensional component  $S_i \subset V^0(\omega_X)$ . We often consider the induced map  $\pi : X \rightarrow E := T_i^\vee$ . It is easy to see that  $\pi$  factors through the Iitaka fibration. By Corollary 2.11 and Lemma 2.13, there is an inclusion  $\varphi : \pi^*L \rightarrow \omega_X \otimes P$  for some  $P \in \text{Pic}^0(X)$  with  $P|_F = \mathcal{O}_F$  and some ample line bundle  $L$  on  $E$ . In what follows, we will often apply Lemma 3.1 and Corollary 3.2 to this situation.

LEMMA 3.4. *Let  $X$  be a variety of maximal Albanese dimension with  $\kappa(X) \geq 2$  and  $P_3(X) = 4$ . Then for any  $\zeta \neq 0 \in \bar{G}$ , one has  $P_{2,\zeta} \leq 2$ .*



PROOF. If  $P_{2,\zeta} \geq 3$ , then by (2) and Proposition 2.7, one sees that  $\delta_{-\zeta} = 1$ . Let  $\pi : X \rightarrow E := T_{-\zeta}^\vee$  be the induced morphism. Then there is an ample line bundle  $L$  on the elliptic curve  $E$  and an inclusion  $L \rightarrow \pi_*(\omega_X \otimes Q_{-\zeta})$ . By Corollary 3.2,  $P_3(X) \geq 2 + P_{2,\zeta} \geq 5$  which is impossible.  $\square$

THEOREM 3.5. *Let  $X$  be a smooth projective variety with  $P_3(X) = 4$ , then the Albanese morphism  $a : X \rightarrow A$  is surjective.*

PROOF. We follow the proof of Theorem 5.1 of [HP]. Assume that  $a : X \rightarrow A$  is not surjective, then we may assume that there is a morphism  $f : X \rightarrow Z$  where  $Z$  is a smooth variety of general type, of dimension at least 1, such that its Albanese map  $a_Z : Z \rightarrow S$  is birational onto its image. By the proof of Theorem 5.1 of [HP], it suffices to consider the cases in which  $P_1(Z) \leq 3$  and hence  $\dim(Z) \leq 2$ . If  $\dim(Z) = 2$ , then  $q(Z) = \dim(S) \geq 3$  and since  $\chi(\omega_Z) > 0$ , one sees that  $V^0(\omega_Z) = \text{Pic}^0(S)$ . By the proof of Theorem 5.1 of [HP], one has that for generic  $P \in \text{Pic}^0(S)$ ,

$$P_3(X) \geq h^0(\omega_Z \otimes P) + h^0(\omega_X^{\otimes 3} \otimes f^* \omega_Z^\vee \otimes P) + \dim(S) - 1 \geq 1 + 2 + 3 - 1 \geq 5.$$

This is a contradiction, so we may assume that  $\dim(Z) = 1$ . It follows that  $g(Z) = q(Z) = P_1(Z) \geq 2$  and one may write  $\omega_Z = L^{\otimes 2}$  for some ample line bundle  $L$  on  $Z$ . Therefore, for general  $P \in \text{Pic}^0(Z)$ , one has that  $h^0(\omega_Z \otimes L \otimes P) \geq 2$  and proceeding as in the proof of Theorem 5.1 of [HP], that  $h^0(\omega_X^{\otimes 3} \otimes f^*(\omega_Z \otimes L)^\vee \otimes P) \geq 2$ . It follows as above that

$$P_3(X) \geq h^0(\omega_Z \otimes L \otimes P) + h^0(\omega_X^{\otimes 3} \otimes f^*(\omega_Z \otimes L)^\vee \otimes P) + \dim(S) - 1 \geq 2 + 2 + 2 - 1 \geq 5.$$

This is a contradiction and so  $a : X \rightarrow A$  is surjective.  $\square$

PROPOSITION 3.6. *Let  $X$  be a smooth projective variety with  $P_3(X) = 4$ ,  $q(X) = \dim(X)$ , then*

- (1)  $X$  is not of general type and
- (2) if  $\kappa(X) \geq 2$ , then

$$V^0(\omega_X) \cap f^* \text{Pic}^0(Y) = \{\mathcal{O}_X\}.$$

PROOF. If  $\kappa(X) = 1$ , then clearly  $X$  is not of general type as otherwise  $X$  is a curve with  $P_3(X) = 5g - 5 > 4$ . We thus assume that  $\kappa(X) \geq 2$ . It suffices to prove (2) as then (1) will follow from Theorem 2.6.

If all points of  $V^0(\omega_X) \cap f^* \text{Pic}^0(Y)$  are isolated, then the above statement follows from Proposition 2.7. Therefore, it suffices to prove that  $\delta_0 = 0$ . (Recall that  $\delta_0$  is the maximal dimension of a component in  $\text{Pic}^0(Y)$ .)

Suppose that  $\delta_0 \geq 2$ . Then by (2) and Proposition 2.8, one has

$$P_2 \geq 1 + 1 + \delta_0 - 1 \geq 3, \quad P_3 \geq 3 + 1 + \delta_0 - 1 \geq 5$$

which is impossible.

Suppose now that  $\delta_0 = 1$ , i.e. there is a 1-dimensional component  $S_0 \subset V^0(\omega_X) \cap f^* \text{Pic}^0(Y)$ . Let  $\pi : X \rightarrow E := T_0^\vee$  be the induced morphism. By

Corollary 2.11, for some general  $P \in S_0$ , there exists a line bundle of degree 1 on  $E$  and an inclusion  $\pi^*L \rightarrow \omega_X \otimes P$ . By Lemma 2.13,  $P|_{F_X/E} \cong \mathcal{O}_{F_X/E}$ .

We consider the inclusion  $\varphi : L^{\otimes 2} \rightarrow \pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})$ . By Lemma 3.1, one sees that  $h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) \geq 3$ , and  $\text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})) \geq 2$ . So

$$\begin{aligned} P_3(X) &= h^0(\omega_X^{\otimes 3} \otimes P^{\otimes 3}) \geq h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2} \otimes \pi^*L) \\ &= h^0(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) \otimes L) \geq \text{deg}(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})) + \text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})) \\ &\geq 3 + 2 \end{aligned}$$

and this is the required contradiction. □

**PROPOSITION 3.7.** *Let  $X$  be a smooth projective variety with  $P_3(X) = 4$ ,  $q(X) = \dim(X)$ , and  $f : X \rightarrow Y$  be a birational model of its Iitaka fibration. Then  $Y$  is birational to an abelian variety.*

**PROOF.** Since  $X, Y$  are of maximal Albanese dimension,  $K_{X/Y}$  is effective. If  $h^0(\omega_Y \otimes P) > 0$ , it follows that  $h^0(\omega_X \otimes f^*P) > 0$  and so by Proposition 3.6,  $f^*P = \mathcal{O}_X$ . By Proposition 2.1, the map  $f^* : \text{Pic}^0(Y) \rightarrow \text{Pic}^0(X)$  is injective and hence  $P = \mathcal{O}_Y$ . Therefore  $V^0(\omega_Y) = \{\mathcal{O}_Y\}$  and by Theorem 2.6, one has  $\kappa(Y) = 0$  and hence  $Y$  is birational to an abelian variety. □

We are now ready to describe the cohomological support loci of varieties with  $\kappa(X) \geq 2$  explicitly. Recall that by Proposition 2.7, for all  $\eta \neq 0 \in \bar{G}$ ,  $\delta_\eta \geq 1$ .

**THEOREM 3.8.** *Let  $X$  be a smooth projective variety with  $P_3(X) = 4$ ,  $q(X) = \dim(X)$  and  $\kappa(X) \geq 2$ . Then  $\kappa(X) = 2$  and  $\bar{G} \cong (\mathbb{Z}_2)^s$  for some  $s \geq 1$ .*

**PROOF.** The proof consists of following claims:

**CLAIM 3.9.** If  $\kappa(X) \geq 2$  and  $T \subset V^0(\omega_X)$  is a positive dimensional component, then  $T + T \subset \text{Pic}^0(Y)$ , i.e.  $\bar{G} \cong (\mathbb{Z}_2)^s$ .

**PROOF OF CLAIM 3.9.** It suffices to prove that  $2\eta = 0$  for  $0 \neq \eta \in \bar{G}$ . Suppose that  $2\eta \neq 0$ , we will find a contradiction.

We first consider the case that  $\delta_\eta \geq 2$  and  $\delta_{-2\eta} \geq 2$ . Then by (2),  $P_{2,2\eta} \geq 1 + 1 + \delta_\eta - 1 \geq 3$ , and  $P_3 \geq 3 + 1 + \delta_{-2\eta} - 1 \geq 5$  which is impossible.

We then consider the case that  $\delta_\eta \geq 2$  and  $\delta_{-2\eta} = 1$ . Again we have  $P_{2,2\eta} \geq 3$ . We consider the induced map  $\pi : X \rightarrow E := T_{-2\eta}^\vee$  and the inclusion  $\varphi : \pi^*L \rightarrow \omega_X \otimes Q_{-2\eta}$  where  $E$  is an elliptic curve and  $L$  is an ample line bundle on  $E$ . It follows that there is an inclusion

$$\pi^*L \otimes (\omega_X \otimes Q_\eta)^{\otimes 2} \rightarrow \omega_X^{\otimes 3} \otimes Q_\eta^{\otimes 2} \otimes Q_{-2\eta}.$$

By Lemma 3.1, one has that  $\text{rank}(\pi_*(\omega_X \otimes Q_\eta)^{\otimes 2}) \geq 2$ . By Proposition 2.8, Riemann-Roch and Lemma 2.4

$$\begin{aligned} P_3(X) &= h^0(\omega_X^{\otimes 3} \otimes Q_\eta^{\otimes 2} \otimes Q_{-2\eta}) \geq h^0(\pi^*L \otimes (\omega_X \otimes Q_\eta)^{\otimes 2}) \\ &= h^0((\omega_X \otimes Q_\eta)^{\otimes 2}) + \text{rank}(\pi_*(\omega_X \otimes Q_\eta)^{\otimes 2}) \geq P_{2,2\eta} + 2 \geq 5, \end{aligned}$$

which is impossible.

Lastly, we consider the case that  $\delta_\eta = 1$ . There is an induced map  $\pi : X \rightarrow E := T_\eta^\vee$  and an inclusion  $\pi^*L \rightarrow \omega_X \otimes Q_\eta$ . Hence there is an inclusion  $\varphi : \pi^*L^{\otimes 2} \rightarrow (\omega_X \otimes Q_\eta)^{\otimes 2}$ . By Lemma 3.1, we have  $P_{2,2\eta} \geq 3$ . We now proceed as in the previous cases.

Therefore, any element  $\eta \in \bar{G}$  is of order 2 and hence  $\bar{G} \cong (\mathbb{Z}_2)^s$ . □

CLAIM 3.10. If there is a surjective map with connected fibers to an elliptic curve  $\pi : X \rightarrow E$  and an inclusion  $\pi^*L \rightarrow \omega_X \otimes P$  for an ample line bundle  $L$  on  $E$  and  $P \in \text{Pic}^0(X)$  (in particular if  $\delta_i = 1$  for some  $i \neq 0$  cf. Corollary 2.11). Then  $\kappa(X) = 2$ .

PROOF OF CLAIM 3.10. Since  $K_X$  is effective, there is also an inclusion  $L \rightarrow \pi_*(\omega_X^{\otimes 2} \otimes P)$ . By Lemma 3.1, one has  $\text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2$ ,  $h^0(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2$ . Consider the inclusion

$$\pi_*(\omega_X^{\otimes 2} \otimes P) \otimes L \rightarrow \pi_*(\omega_X^{\otimes 3} \otimes P^{\otimes 2}).$$

Since

$$\begin{aligned} P_3(X) &= h^0(\pi_*(\omega_X^{\otimes 3} \otimes P^{\otimes 2})) \geq h^0(\pi_*(\omega_X^{\otimes 2} \otimes P) \otimes L) \\ &\geq \text{deg}(\pi_*(\omega_X^{\otimes 2} \otimes P)) + \text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)), \end{aligned}$$

it follows that

$$\text{deg}(\pi_*(\omega_X^{\otimes 2} \otimes P)) = \text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)) = 2$$

and the above homomorphism of sheaves induces an isomorphism on global sections and hence is an isomorphism of sheaves (cf. Proposition 2.3). Therefore,

$$P_3(F) = h^0(\omega_F^{\otimes 3} \otimes P^{\otimes 2}) = 2.$$

By Theorem 1.1, it follows that  $\kappa(F) = 1$  and by easy addition, one has that

$$\kappa(X) \leq \kappa(F) + \dim(E) = 2. \quad \square$$

CLAIM 3.11. For all  $i \neq 0$ ,  $P_{1,i} = 1$ .

PROOF OF CLAIM 3.11. If  $P_{1,i} \geq 2$ , then by (2),

$$4 \geq P_2 \geq 2P_{1,i} + \delta_i - 1.$$

It follows that  $\delta_i = 1$ . Let  $E = T^\vee$  and  $\pi : X \rightarrow E$  be the induced morphism. We follow Lemma 2.10 and let  $L := \pi_*(\mathcal{O}_X(D_T) \otimes Q_i)$ . The sheaf  $L$  is torsion free and hence locally free. Since  $D_T$  is vertical,  $L$  is of rank 1, i.e. a line bundle. There is an inclusion  $\pi^*L \rightarrow \omega_X \otimes Q_i$  and one has  $h^0(E, L) = h^0(\omega_X \otimes Q_i) \geq 2$ . Consider the inclusion  $\pi^*L^{\otimes 2} \rightarrow \omega_X^{\otimes 2} \otimes Q_i^{\otimes 2}$ . By Lemma 3.1, one sees that

$$P_3 \geq P_{2,2i} = h^0(\omega_X^{\otimes 2} \otimes Q_i^{\otimes 2}) > h^0(E, L^{\otimes 2}) \geq 4,$$

which is impossible. □

CLAIM 3.12. If  $\kappa(X) = \dim(S)$  for some component  $S$  of  $V^0(\omega_X)$ , then  $\kappa(X) = 2$ .

PROOF OF CLAIM 3.12. Let  $Q$  be a general point in  $S$ , and  $T$  be the translate of  $S$  through the origin. By Proposition 3.7, one sees that the induced map  $X \rightarrow T^\vee$  is isomorphic to the Iitaka fibration. We therefore identify  $Y$  with  $T^\vee$ . We assume that  $\dim(S) \geq 3$  and derive a contradiction. First of all, by (2)

$$P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q^{\otimes 2}) \geq h^0(\omega_X^{\otimes 2} \otimes Q) + \dim(S)$$

and so  $h^0(\omega_X^{\otimes 2} \otimes Q) = 1$  and  $\dim(S) = 3$ .

Let  $H$  be an ample line bundle on  $Y$  and for  $m$  a sufficiently big and divisible integer, fix a divisor  $B \in |mK_X - f^*H|$ . After replacing  $X$  by an appropriate birational model, we may assume that  $B$  has simple normal crossings support. Let  $L = \omega_X \otimes \mathcal{O}_X(-\lfloor B/m \rfloor)$ , then  $L \equiv f^*(H/m) + \{B/m\}$  i.e.  $L$  is numerically equivalent to the sum of the pull back of an ample divisor and a k.l.t. divisor and so one has

$$h^i(Y, f_*(\omega_X \otimes L \otimes Q) \otimes \alpha) = 0 \quad \text{for all } i > 0 \quad \text{and } \alpha \in \text{Pic}^0(Y).$$

Comparing the base loci, one can see that  $h^0(\omega_X \otimes L \otimes Q) = h^0(\omega_X^{\otimes 2} \otimes Q) = 1$  (cf. [CH1], Lemma 2.1 and Proposition 2.8) and so

$$h^0(Y, f_*(\omega_X \otimes L \otimes Q) \otimes \alpha) = h^0(f_*(\omega_X \otimes L \otimes Q)) = 1 \quad \forall \alpha \in \text{Pic}^0(Y).$$

Since  $f_*(\omega_X \otimes L \otimes Q)$  is a torsion free sheaf of generic rank one, by [Hac] it is a principal polarization  $M$ .

Since one may arrange that  $\lfloor \frac{B}{m} \rfloor < K_X$ , there is an inclusion  $\omega_X \otimes Q \hookrightarrow \omega_X \otimes L \otimes Q$ . Pushing forward to  $Y$ , it induces an inclusion

$$\varphi : f_*(\omega_X \otimes Q) \hookrightarrow M.$$

Therefore,  $f_*(\omega_X \otimes Q)$  is of the form  $M \otimes \mathcal{I}_Z$  for some ideal sheaf  $\mathcal{I}_Z$ . However,  $h^0(Y, f_*(\omega_X \otimes Q) \otimes P) = h^0(M \otimes P \otimes \mathcal{I}_Z) > 0$  for all  $P \in \text{Pic}^0(Y)$  and  $M$  is a principal polarization. It follows that  $\mathcal{I}_Z = \mathcal{O}_Y$  and thus  $f_*(\omega_X \otimes Q) = M$ . Therefore, one has an inclusion

$$f^*M^{\otimes 2} \hookrightarrow (\omega_X \otimes Q) \otimes (\omega_X \otimes L \otimes Q) \hookrightarrow \omega_X^{\otimes 3} \otimes Q^{\otimes 2}.$$

It follows that

$$4 = P_3(X) = h^0(X, \omega_X^{\otimes 3} \otimes Q^{\otimes 2}) \geq h^0(Y, M^{\otimes 2}) \geq 2^{\dim(S)}.$$

This is the required contradiction.  $\square$

CLAIM 3.13. Any two components of  $V^0(\omega_X)$  of dimension at least 2 must be parallel.

PROOF OF CLAIM 3.13. For  $i = 1, 2$ , let  $p_i : X \rightarrow T_i^\vee$  be the induced morphism. Assume that  $\delta_1, \delta_2 \geq 2$  and  $T_1, T_2$  are not parallel. By Lemma 2.10, one may write  $K_X = G_i + D_i$  where  $D_i$  is vertical with respect to  $p_i : X \rightarrow T_i^\vee$  and for general  $P \in S_i$ , one has  $|K_X + P| = G_i + |D_i + P|$  is a 0-dimensional linear system (see Claim 3.11).

Recall that we may assume that the image of the Iitaka fibration  $f : X \rightarrow Y$  is an abelian variety. Pick  $H$  an ample divisor on  $Y$  and for  $m$  sufficiently big and divisible integer, let

$$B \in |mK_X - f^*H|.$$

After replacing  $X$  by an appropriate birational model, we may assume that  $B$  has normal crossings support. Let

$$L := \omega_X \left( - \left\lfloor \frac{B}{m} \right\rfloor \right) \equiv \left\{ \frac{B}{m} \right\} + f^* \left( \frac{H}{m} \right).$$

It follows that

$$h^i(f_*(\omega_X \otimes L \otimes P) \otimes \alpha) = 0 \quad \text{for all } i > 0, \alpha \in \text{Pic}^0(Y), P \in \text{Pic}^0(X).$$

The quantity  $h^0(\omega_X \otimes L \otimes P \otimes f^*\alpha)$  is independent of  $\alpha \in \text{Pic}^0(Y)$ . For some fixed  $P \in S_1$  as above, and  $\alpha \in \text{Pic}^0(T_1^\vee)$ , one has a morphism

$$|D_1 + P + \alpha| \times |D_1 + P - \alpha| \rightarrow |2D_1 + 2P|$$

and hence  $h^0(\mathcal{O}_X(2D_1) \otimes P^{\otimes 2}) \geq 3$ . Similarly for some fixed  $Q \in S_2$ , and  $\alpha' \in \text{Pic}^0(T_2^\vee)$ , one has a morphism

$$|D_2 + Q + \alpha'| \times |K_X + L - Q + 2P - \alpha'| \rightarrow |K_X + L + D_2 + 2P|$$

and hence  $h^0(\omega_X(D_2) \otimes L \otimes P^{\otimes 2}) \geq 3$ . It follows that since  $h^0(\omega_X^{\otimes 3} \otimes P^{\otimes 2}) = 4$ , there is a 1 dimensional intersection between the images of the 2 morphisms above which are contained in the loci

$$|2D_1 + 2P| + 2G_1 + K_X, \quad |K_X + L + D_2 + 2P| + \left\lfloor \frac{B}{m} \right\rfloor + G_2.$$

It is easy to see that for all but finitely many  $P \in \text{Pic}^0(X)$ , one has  $h^0(\omega_X \otimes P) \leq 1$ . So there is a 1 parameter family  $\tau_2 \subset \text{Pic}^0(T_2^\vee)$  such that for  $\alpha' \in \tau_2$ , one has that the divisor  $D_{Q+\alpha'} = |D_2 + Q + \alpha'|$  is contained in  $D_{P+\alpha} + D_{P-\alpha} + 2G_1 + K_X$  where  $\alpha \in \tau_1$  a 1 parameter family in  $\text{Pic}^0(T_1^\vee)$ . Let  $D_{Q+\alpha'}^*$  be the components of  $D_{Q+\alpha'}$  which are not fixed for general  $\alpha' \in \tau_2$ , then  $D_{Q+\alpha'}^*$  is not contained

in the fixed divisor  $2G_1 + K_X$  and hence is contained in some divisor of the form  $D_{P+\alpha}^* + D_{P-\alpha}^*$  and hence is  $T_1^\vee$  vertical.

If  $\text{Pic}^0(T_1^\vee) \cap \text{Pic}^0(T_2^\vee) = \{\mathcal{O}_X\}$ , then  $D_{Q+\alpha'}^*$  is  $\alpha$ -exceptional, and this is impossible by Lemma 2.9.

If there is a 1-dimensional component  $\Gamma \subset \text{Pic}^0(T_1^\vee) \cap \text{Pic}^0(T_2^\vee)$ . Let  $E = \Gamma^\vee$  and  $\pi : X \rightarrow E$  be the induced morphism. The divisors  $D_{Q+\alpha'}^*$  are  $E$ -vertical. We may assume that  $\pi$  has connected fibers. Since the  $D_{Q+\alpha'}^*$  vary with  $\alpha' \in \tau_2$ , for general  $\alpha' \in \tau_2$ , they contain a smooth fiber of  $\pi$ . So for general  $\alpha' \in \tau_2$  there is an inclusion  $\pi^*M \rightarrow \omega_X \otimes \mathcal{Q} \otimes \pi^*\alpha'$  where  $M$  is a line bundle of degree at least 1. By Claim 3.10, one has  $\kappa(X) = 2$  and hence  $T_1, T_2$  are parallel.

If there is a 2-dimensional component  $\Gamma \subset \text{Pic}^0(T_1^\vee) \cap \text{Pic}^0(T_2^\vee)$ , then  $\delta_1 = \delta_2 \geq 3$ . By (2), one sees that  $P_{2, Q_1+Q_2} \geq 3$ . By Lemma 3.4, this is impossible.  $\square$

By Claim 3.10, if there is a one dimensional component, then  $\kappa(X) = 2$ . Therefore, we may assume that  $\delta_i \geq 2$  for all  $i \neq 0$ . By Claim 3.13, since  $\delta_i \geq 2$  for all  $i \neq 0$ , then  $S_i, S_j$  are parallel for all  $i, j \neq 0$ . By Theorem 2.6, for an appropriate  $i \neq 0$ ,  $\kappa(X) = \dim(S_i)$  and so by Claim 3.12, one has  $\kappa(X) = 2$ .  $\square$

**4. – Varieties with  $P_3(X) = 4, q(X) = \dim(X)$  and  $\kappa(X) = 2$**

In this section, we classify varieties with  $P_3(X) = 4, q(X) = \dim(X)$  and  $\kappa(X) = 2$ . The first step is to describe the cohomological support loci of these varieties. We must show that the only possible cases are the following (which corresponds to Examples 2 and 3 respectively):

- (1)  $\tilde{G} \cong \mathbb{Z}_2, V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\eta, \delta_\eta = 2$ .
- (2)  $\tilde{G} \cong \mathbb{Z}_2^2, V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\eta \cup S_\zeta \cup S_{\eta+\zeta}, \delta_\eta = \delta_\zeta = 1, \delta_{\eta+\zeta} = 2$ .

Using this information, we will determine the sheaves  $a_*(\omega_X)$  and this will enable us to prove the following:

**THEOREM 4.1.** *Let  $X$  be a smooth projective variety with  $P_3(X) = 4, q(X) = \dim(X)$  and  $\kappa(X) = 2$ , then  $X$  is one of the varieties described in Examples 2 and 3.*

**PROOF.** Recall that  $f : X \rightarrow Y$  is a morphism birational to the Iitaka fibration,  $Y$  is an abelian surface and  $f = q \circ a$  where  $q : A \rightarrow Y$ .

**CLAIM 4.2.** One has that  $f_*\omega_X = \mathcal{O}_Y$ .

**PROOF OF CLAIM 4.2.** By Proposition 3.6, one has that  $V^0(\omega_X) \cap f^*\text{Pic}^0(Y) = \{\mathcal{O}_X\}$ . By the proof of [CH3] Theorem 4, one sees that  $f_*\omega_X \cong \mathcal{O}_Y \otimes H^0(\omega_X)$ . Since  $h^0(\omega_X|_{F_{X/Y}}) = 1$ , it follows that  $\text{rank}(f_*\omega_X) = 1$  and hence  $f_*\omega_X \cong \mathcal{O}_Y$ .  $\square$

CLAIM 4.3. Let  $S_1, S_2$  be distinct components of  $V^0(\omega_X)$  such that  $S_1 \cap S_2 \neq \emptyset$ , then  $S_1 \cap S_2 = P$  and

$$f_*(\omega_X \otimes P) = L_1 \boxtimes L_2 \otimes \mathcal{I}_p$$

where  $Y = E_1 \times E_2$  and  $L_i$  are line bundles of degree 1 on the elliptic curves  $E_i$  and  $p$  is a point of  $Y$ .

PROOF OF CLAIM 4.3. Assume that  $P \in S_1 \cap S_2$ . Since  $\kappa(X) = 2$ , by Proposition 2.7, the  $T_i$  are 1-dimensional. Let  $\pi_i : X \rightarrow E_i := T_i^\vee$  be the induced morphisms. There are line bundles of degree 1,  $L_i$  on  $E_i$  and inclusions  $\pi_i^* L_i \rightarrow \omega_X \otimes P$  (cf. Corollary 2.11).

We claim that  $\text{rank}(\pi_{1,*}(\omega_X \otimes P)) = 1$ . If this were not the case, then by Lemma 2.13

$$P_1(F_{X/E_1}) = \text{rank}(\pi_{1,*}(\omega_X \otimes P)) \geq 2, \quad P_2(F_{X/E_1}) = \text{rank}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) \geq 3$$

and so

$$\begin{aligned} P_3(X) &= h^0(\omega_X^{\otimes 3} \otimes P^{\otimes 2}) \geq h^0(\omega_X^{\otimes 2} \otimes P \otimes \pi_1^* L_1) \\ &= h^0(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P) \otimes L_1) \\ &\geq \text{rank}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) + \text{deg}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) \end{aligned}$$

and therefore

$$\text{rank}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) = 3, \quad \text{deg}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) = 1.$$

Since  $\text{rank}(\pi_{1,*}(\omega_X)) = \text{rank}(\pi_{1,*}(\omega_X \otimes P))$ , one has

$$\text{deg}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) \geq \text{deg}(\pi_{1,*}(\omega_X) \otimes L_1) \geq \text{rank}(\pi_{1,*}(\omega_X)) \geq 2,$$

which is impossible. Therefore, we may assume that

$$\text{rank}(\pi_{i,*}(\omega_X \otimes P)) = 1 \quad \text{for } i = 1, 2.$$

For any  $P_i \in S_i$ , one has that  $P_i \otimes P^\vee = \pi_i^* \alpha_i$  with  $\alpha_i \in \text{Pic}^0(E_i)$ . One sees that

$$h^0(\omega_X \otimes P_i) = h^0(\pi_{i,*}(\omega_X \otimes P) \otimes \alpha_i) = h^0(\pi_{i,*}(\omega_X \otimes P)) = h^0(\omega_X \otimes P).$$

If  $h^0(\omega_X \otimes P) \geq 2$ , then we may assume that  $L_1 := \pi_{1,*}(\omega_X \otimes P)$  is an ample line bundle of degree at least 2. From the inclusion  $\phi : L_1^{\otimes 2} \rightarrow \pi_{1,*}(\omega_X^{\otimes 2} \otimes P^{\otimes 2})$ , one sees that  $h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) = 4$  and  $\phi$  is an I.T. 0 isomorphism (cf. Lemma 2.4) and so

$$P_2(F_{X/E_1}) = h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}|_F) = 1.$$

By Theorem 1.1,  $\kappa(F_{X/E_1}) = 0$  and hence by easy addition,  $\kappa(X) \leq 1$  which is impossible. Therefore we may assume that  $h^0(\omega_X \otimes P) = 1$ .

The coherent sheaf  $f_*(\omega_X \otimes P)$  is torsion free of generic rank 1 on  $Y$  and hence is isomorphic to  $L \otimes \mathcal{I}$  where  $L$  is a line bundle and  $\mathcal{I}$  is an ideal sheaf cosupported at finitely many points. Let  $q_i : Y \rightarrow E_i$ , so that  $\pi_i = q_i \circ f$ . Since

$$1 = \text{rank}(\pi_{i,*}(\omega_X \otimes P)) = \text{rank}(q_{i,*}(L \otimes \mathcal{I})) = \text{rank}(q_{i,*}L),$$

one sees that  $L.F_{Y/E_i} = 1$  and it easily follows that  $L = L_1 \boxtimes L_2$  where  $L_i = q_{i,*}(L)$  is a line bundle of degree 1 on  $E_i$ . Clearly,  $\mathcal{I}$  is the ideal sheaf of a point. □

We will now consider the case in which  $\bar{G} = \mathbb{Z}_2$ . Let  $B$  be the branch locus of  $a : X \rightarrow A$ . The divisor  $B$  is vertical with respect to  $q : A \rightarrow Y$  and hence we may write  $B = q^* \bar{B}$ . Let  $g \circ h : X \rightarrow Z \rightarrow A$  be the Stein factorization of  $a$ . Then  $Z$  is a normal variety and  $g$  is finite of degree 2 and so  $g_* \mathcal{O}_Z = \mathcal{O}_A \oplus M^\vee$  where  $M$  is a line bundle and the branch locus  $B$  is a divisor in  $|2M|$ . The map  $F_{Z/Y} \rightarrow F_{A/Y}$  is étale of degree 2 and so  $M = q^* L \otimes P$  where  $P$  is a 2-torsion element of  $\text{Pic}^0(X)$ . Let  $v : A' \rightarrow A$  be a birational morphism so that  $v^* B$  is a divisor with simple normal crossings support. Let  $B' = v^* B - 2 \lfloor \frac{v^* B}{2} \rfloor$  and  $M' = v^*(M) (-\lfloor \frac{v^* B}{2} \rfloor)$ . Let  $Z'$  be the normalization of  $Z \times_A A'$ , and  $g' : Z' \rightarrow A'$  be the induced morphism. Then  $g'$  is finite of degree 2,  $Z'$  is normal with rational singularities and  $g'_*(\mathcal{O}_{Z'}) = \mathcal{O}_{A'} \oplus (M')^\vee$ . Let  $\tilde{X}$  be an appropriate birational model of  $X$  such that there are morphisms  $\alpha : \tilde{X} \rightarrow A'$ ,  $v : \tilde{X} \rightarrow X$ ,  $\tilde{a} : \tilde{X} \rightarrow A$  and  $\beta : \tilde{X} \rightarrow Z'$ . For all  $n \geq 0$ , one has that  $\beta_*(\omega_{\tilde{X}}^{\otimes n}) \cong \omega_{Z'}^{\otimes n}$ . It follows that

$$\alpha_*(\omega_{\tilde{X}}^{\otimes m}) = \omega_{A'}^{\otimes m} \otimes (M'^{\otimes m-1} \oplus M'^{\otimes m}).$$

Therefore

$$\begin{aligned} a_*(\omega_X) &= \tilde{a}_*(\omega_{\tilde{X}}) \\ &= v_*(\omega_{A'} \oplus \omega_{A'} \otimes M') \\ &= \mathcal{O}_A \oplus v_* \left( \omega_{A'} \otimes v^*(q^* L) \left( - \left\lfloor \frac{v^* B}{2} \right\rfloor \right) \right) \\ &= \mathcal{O}_A \oplus q^* L \otimes P \otimes \mathcal{I} \left( \frac{B}{2} \right). \end{aligned}$$

CLAIM 4.4. If  $\bar{G} = \mathbb{Z}_2$ , then for any  $P \in V^0(\omega_X)$ , one has

$$f_*(\omega_X \otimes P) \neq L_1 \boxtimes L_2 \otimes \mathcal{I}_p$$

where  $Y = E_1 \times E_2$  and  $L_i$  are ample line bundles of degree 1 on  $E_i$  and  $p$  is a point of  $Y$ .

PROOF OF CLAIM 4.4. If  $f_*(\omega_X \otimes P) = L_1 \boxtimes L_2 \otimes \mathcal{I}_p$ , then  $\frac{B}{2}$  is not log terminal. By [Hac3] Theorem 1, one sees that since  $\frac{B}{2}$  is not log terminal, one has that  $\lfloor \frac{B}{2} \rfloor \neq 0$  and this is impossible as then  $Z$  is not normal.  $\square$

Combining Claim 4.3 and Claim 4.4, one sees that if  $\bar{G} = \mathbb{Z}_2$ , then  $V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\eta$  with  $\delta_\eta = 2$ . We then have the following:

CLAIM 4.5. If  $\bar{G} = \mathbb{Z}_2$ , then  $h^0(X, \omega_X \otimes P) = 1$  for all  $P \in S_\eta$ .

PROOF OF CLAIM 4.5. It is clear that  $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = h^0(A', \omega_{A'} \otimes M' \otimes P)$  for all  $P \in S_\eta$ , and  $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = 1$  for general  $P \in S_\eta$ .

If  $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes Q_0) \geq 2$  for some  $Q_0 \in S_\eta$ , then  $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes Q_0) = 2$  as otherwise  $h^0(\omega_{\tilde{X}}^{\otimes 2} \otimes Q_0^{\otimes 2}) \geq 3 + 3 - 1$  which is impossible.



Consider the linear series  $|K_{A'} + M' + Q_0|$ . Let  $\mu : \tilde{A} \rightarrow A'$  be a log resolution of this linear series. We have

$$\mu^*|K_{A'} + M' + Q_0| = |D| + F,$$

where  $|D|$  is base point free and  $F$  has simple normal crossings support. There is an induced map  $\phi_{|D|} : \tilde{A} \rightarrow \mathbb{P}^1$  such that  $|D| = \phi_{|D|}^*|\mathcal{O}_{\mathbb{P}^1}(1)|$ . We have an inclusion

$$\varphi_1 : \phi_{|D|}^*|\mathcal{O}_{\mathbb{P}^1}(2)| + G \hookrightarrow \mu^*|2K_{A'} + 2M' + 2Q_0|.$$

For all  $\alpha \in \text{Pic}^0(Y)$ , there is a morphism

$$\varphi_2 : \mu^*|K_{A'} + M' + Q_0 + \alpha| + \mu^*|K_{A'} + M' + Q_0 - \alpha| \longrightarrow \mu^*|2K_{A'} + 2M' + 2Q_0|.$$

Notice that  $h^0(A', \omega_{A'}^{\otimes 2} \otimes M'^{\otimes 2} \otimes Q_0^{\otimes 2}) \leq h^0(X, \omega_X^{\otimes 2} \otimes Q_0^{\otimes 2}) \leq 4$ .

Since  $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = 3$ ,  $\varphi_1$  has a 2-dimensional image. Since  $\alpha$  varies in a 2-dimensional family,  $\varphi_2$  also has 2-dimensional image. In particular, there is a positive dimensional family  $\mathcal{N} \subset \text{Pic}^0(Y)$  such that for general  $\alpha \in \mathcal{N}$ , one has

$$D_{\pm\alpha} + F_{\pm\alpha} \in \mu^*|K_{A'} + M' + Q_0 \pm \alpha|$$

where  $G = F_\alpha + F_{-\alpha}$  and  $D_\alpha + D_{-\alpha} \in \phi_{|D|}^*|\mathcal{O}_{\mathbb{P}^1}(2)|$ . Since  $G$  is a fixed divisor, it decomposes in at most finitely many ways as the sum of two effective divisors and so we may assume that  $F_\alpha, F_{-\alpha}$  do not depend on  $\alpha \in \mathcal{N}$ .

Take any  $\alpha \neq \alpha' \in \mathcal{N}$  with  $F_\alpha = F_{\alpha'}$ . One has that  $D_\alpha = \phi_{|D|}^*H$  is numerically equivalent to  $D_{\alpha'} = \phi_{|D|}^*H'$ . It follows that  $H$  and  $H'$  are numerically equivalent on  $\mathbb{P}^1$  hence linearly equivalent. Thus  $D_\alpha$  and  $D_{\alpha'}$  are linearly equivalent which is a contradiction.  $\square$

CLAIM 4.6. If  $\bar{G} = \mathbb{Z}_2$ , then a  $\lambda : X \rightarrow A$  has generic degree 2 and is branched over a divisor  $B \in |2f^*\Theta|$  where  $\mathcal{O}_Y(\Theta)$  is an ample line bundle of degree 1. Furthermore,  $a_*(\mathcal{O}_X) \cong \mathcal{O}_A \oplus q^*\mathcal{O}_Y(\Theta) \otimes P$  where  $P \notin \text{Pic}^0(Y)$  and  $P^{\otimes 2} = \mathcal{O}_A$ . See Example 2.

PROOF OF CLAIM 4.6. For all  $\alpha \in \text{Pic}^0(Y)$  and  $P \in S_\eta$ , one has that

$$h^0(\omega_X \otimes P \otimes \alpha) = h^0(\omega_{A'} \otimes M' \otimes P \otimes \alpha) = 1.$$

The sheaf  $q_*v_*(\omega_{A'} \otimes M' \otimes P)$  is torsion free of generic rank 1 and

$$h^0(q_*v_*(\omega_{A'} \otimes M' \otimes P) \otimes \alpha) = 1 \quad \text{for all } \alpha \in \text{Pic}^0(Y).$$

Following the proof of Proposition 4.2 of [HP], one sees that higher cohomologies vanish. By [Hac],  $q_*v_*(\omega_{A'} \otimes M' \otimes P)$  is a principal polarization  $\mathcal{O}_Y(\Theta)$ . From the isomorphism  $v_*(\omega_{A'} \otimes M' \otimes P) \cong \bar{L} \otimes \mathcal{I}(\frac{\bar{B}}{2})$ , one sees that  $\bar{L} = \mathcal{O}_Y(\Theta)$  and  $\mathcal{I}(\frac{\bar{B}}{2}) = \mathcal{O}_Y$ . Therefore,  $v_*(\omega_{A'} \otimes M' \otimes P) \cong q^*\mathcal{O}_Y(\Theta)$ . It follows that

$$a_*(\omega_X) \cong \mathcal{O}_A \oplus q^*\mathcal{O}_Y(\Theta) \otimes P. \quad \square$$

From now on we therefore assume that  $\bar{G} \neq \mathbb{Z}_2$ .

CLAIM 4.7.  $V^0(\omega_X)$  has at most one 2-dimensional component.

PROOF OF CLAIM 4.7. Let  $S_\eta, S_\zeta$  be 2-dimensional components of  $V^0(\omega_X)$  with  $\eta \neq \zeta$ . Since  $\kappa(X) = 2$ , one has  $\delta_{\eta, \zeta} = 2$ . Thus by (2),  $P_{2, \eta + \zeta} \geq 3$ . By Lemma 3.4, this is impossible.  $\square$

CLAIM 4.8. Let  $S_1, S_2$  be two parallel 1-dimensional components of  $V^0(\omega_X)$ , then  $S_1 + \text{Pic}^0(Y) = S_2 + \text{Pic}^0(Y)$ .

PROOF OF CLAIM 4.8. Let  $P_i \in S_i$ ,  $\pi : X \rightarrow E := T_1^\vee = T_2^\vee$  the induced morphism and  $L_i$  ample line bundles on  $E_i$  with inclusions  $\phi_i : \pi^*L_i \rightarrow \omega_X \otimes P_i$ . By Lemma 2.12, one sees that  $h^0(\omega_X^{\otimes 2} \otimes P_1 \otimes P_2) \geq 2$ . If it were equal, then the inclusion

$$L_1 \otimes L_2 \rightarrow \pi_*(\omega_X^{\otimes 2} \otimes P_1 \otimes P_2)$$

would be an I.T. 0 isomorphism and this would imply that  $P_2(F_{X/E}) = 1$  and hence that  $\kappa(X) \leq 1$ . So  $h^0(\omega_X^{\otimes 2} \otimes P_1 \otimes P_2) \geq 3$ . By Lemma 3.4, this is impossible.  $\square$

CLAIM 4.9. If  $\bar{G} \neq \mathbb{Z}_2$ , let  $S_\eta$  be a 2-dimensional component of  $V^0(\omega_X)$ , then  $h^0(\omega_X \otimes P) = 1$  for all  $P \in S_\eta$ . In particular  $f_*(\omega_X \otimes P)$  is a principal polarization.

PROOF OF CLAIM 4.9 Let  $f : X \rightarrow (T_\eta)^\vee$  be the induced morphism. Then  $f$  is birational to the Iitaka fibration of  $X$  i.e.  $(T_\eta)^\vee = Y$ . By Claim 4.7,  $V^0(\omega_X)$  has at most one 2-dimensional component, and so there must exist a 1-dimensional component  $S_\zeta$  of  $V^0(\omega_X)$ . Let  $\pi : X \rightarrow E := T_\zeta^\vee$  be the induced morphism. There is an ample line bundle  $L$  on  $E$  and an inclusion  $\pi^*L \rightarrow \omega_X \otimes Q_\zeta$  for some general  $Q_\zeta \in S_\zeta$ .

Assume that  $P \in S_\eta$  and  $h^0(\omega_X \otimes P) \geq 2$ . If  $\text{rank}(\pi_*(\omega_X \otimes P)) = 1$ , then  $\pi_*(\omega_X \otimes P)$  is an ample line bundle of degree at least 2 and hence  $h^0(\pi_*(\omega_X \otimes P) \otimes \alpha) \geq 2$  for all  $\alpha \in \text{Pic}^0(E)$ . It follows that

$$h^0(\omega_X^{\otimes 2} \otimes P \otimes Q_\zeta) \geq h^0(\omega_X \otimes P \otimes \pi^*L) = h^0(\pi_*(\omega_X \otimes P) \otimes L) \geq 3.$$

By Lemma 3.4, this is impossible.

Therefore, we may assume that  $\text{rank}(\pi_*(\omega_X \otimes P)) \geq 2$ . Proceeding as above, since

$$h^0(\pi_*(\omega_X \otimes P) \otimes L) \geq \text{rank}(\pi_*(\omega_X \otimes P)) + \deg(\pi_*(\omega_X \otimes P)),$$

it follows that  $\pi_*(\omega_X \otimes P)$  is a sheaf of degree 0. Since  $h^0(\pi_*(\omega_X \otimes P) \otimes \alpha) > 0$  for all  $\alpha \in \text{Pic}^0(E)$ , By Riemann-Roch one sees that also  $h^1(\pi_*(\omega_X \otimes P) \otimes \alpha) > 0$  for all  $\alpha \in \text{Pic}^0(E)$ . By Theorem 2.5, this is impossible.

Finally, the sheaf  $f_*(\omega_X \otimes P)$  is torsion free of generic rank 1 on  $Y$  and hence, by [Hac], it is a principal polarization.  $\square$

CLAIM 4.10. Assume that  $\bar{G} \neq \mathbb{Z}_2$ . Then, for any  $P \in V^0(\omega_X) - \text{Pic}^0(Y)$  one has that  $f_*(\omega_X \otimes P)$  is either:

- i) a principal polarization on  $Y$ ,
- ii) the pull-back of a line bundle of degree 1 on an elliptic curve or
- iii) of the form  $L \boxtimes L' \otimes \mathcal{I}_p$  where  $L, L'$  are ample line bundles of degree 1 on  $E, E'$ ,  $Y = E \times E'$  and  $p$  is a point of  $Y$ .

In particular, there are no 2 distinct parallel components of  $V^0(\omega_X)$ .

PROOF OF CLAIM 4.10. By Claim 4.9, we only need to consider the case in which all the components of  $(P + \text{Pic}^0(Y)) \cap V^0(\omega_X)$  are 1-dimensional. By Claim 4.3, we may also assume that these components are parallel.

For any 1 dimensional component  $S_i$  of  $(P + \text{Pic}^0(Y)) \cap V^0(\omega_X)$ ,  $P_i \in S_i$  and corresponding projection  $\pi_i : X \rightarrow E_i := T_i^\vee$ , one has  $\text{rank}(\pi_{i,*}(\omega_X \otimes P_i)) = 1$  and hence  $\pi_{i,*}(\omega_X \otimes P_i) = L_i$  is an ample line bundle of degree at least 1 on  $E_i$ . If this were not the case, then By Lemma 2.13,

$$\text{rank}(\pi_{i,*}(\omega_X \otimes P_i)) = h^0(\omega_F) \geq 2$$

and so

$$\text{rank}(\pi_{i,*}(\omega_X^{\otimes 2} \otimes P_i)) = h^0(\omega_F^{\otimes 2}) \geq 3.$$

From the inclusion (cf. Corollary 2.11)

$$\pi_i^* L_i \rightarrow \omega_X \otimes P_i \rightarrow \omega_X^{\otimes 2} \otimes P_i,$$

one sees that  $h^0(\omega_X^{\otimes 2} \otimes P_i) \geq 2$  (cf. Lemma 3.1).

By Lemma 2.4,  $\text{deg}(\pi_{i,*}(\omega_X^{\otimes 2} \otimes P_i)) \geq 2$ . By Riemann-Roch, one has

$$h^0(L \otimes \pi_{i,*}(\omega_X^{\otimes 2} \otimes P_i)) \geq \text{deg}(\pi_{i,*}(\omega_X^{\otimes 2} \otimes P_i)) + \text{rank}(\pi_{i,*}(\omega_X^{\otimes 2} \otimes P_i)) \geq 5.$$

This is a contradiction and so  $\text{rank}(\pi_{i,*}(\omega_X \otimes P_i)) = 1$ .

Since we assumed that all components of  $V^0(\omega_X) \cap (P + \text{Pic}^0(Y))$  are parallel, then one has  $\pi_i = \pi$ ,  $E = E_i$  are independent of  $i$ . Let  $q : Y \rightarrow E$ . Since there are injections

$$\text{Pic}^0(E) + P_1 = S_1 \hookrightarrow P_1 + \text{Pic}^0(Y) \hookrightarrow \text{Pic}^0(X),$$

we may assume that  $q$  has connected fibers. The sheaf  $f_*(\omega_X \otimes P_1)$  is torsion free of rank 1, and hence we may write  $f_*(\omega_X \otimes P_1) \cong M \otimes \mathcal{I}$  where  $M$  is a line bundle and  $\mathcal{I}$  is supported in codimension at least 2 (i.e. on points). Since  $\text{rank}(\pi_*(\omega_X \otimes P_1)) = 1$ , one has that  $h^0(M|_{F_{Y/E}}) = 1$ .

For general  $\alpha \in \text{Pic}^0(Y)$ , one has that  $V^0(\omega_X) \cap P_1 + \alpha + \text{Pic}^0(E) = \emptyset$  and so the semi-positive torsion free sheaf  $\pi_*(\omega_X \otimes P_1 \otimes \alpha)$  must be the 0-sheaf. In particular  $h^0(M \otimes \alpha|_{F_{Y/E}}) = 0$ . It follows that  $\text{deg}(M|_{F_{Y/E}}) = 0$  and hence  $M|_{F_{Y/E}} = \mathcal{O}_{F_{Y/E}}$ . One easily sees that  $h^0(M \otimes \alpha) = 0$  for all  $\alpha \in \text{Pic}^0(Y) - \text{Pic}^0(E)$  and hence

$$V^0(\omega_X) \cap (P_1 + \text{Pic}^0(Y)) = P_1 + \text{Pic}^0(E) = T_1.$$

By Proposition 2.3, one has that  $q^* L_1$  and  $f_*(\omega_X \otimes P_1)$  are isomorphic if and only if the inclusion  $q^* L_1 \rightarrow f_*(\omega_X \otimes P_1)$  induces isomorphisms

$$H^i(Y, q^* L_1 \otimes \alpha) \rightarrow H^i(Y, f_*(\omega_X \otimes P_1) \otimes \alpha)$$

for  $i = 0, 1, 2$  and all  $\alpha \in \text{Pic}^0(Y)$ . If  $\alpha \in \text{Pic}^0(Y) - \text{Pic}^0(E)$ , then both groups vanish and so the isomorphism follows. If  $\alpha \in \text{Pic}^0(E)$ , we proceed as follows: Let  $p : A \rightarrow E$  and  $W \subset H^1(A, \mathcal{O}_A)$  a linear subspace complementary to the tangent space to  $T_1$ . By Proposition 2.12 of [Hac2], one has isomorphisms

$$\begin{aligned} H^i(a_*(\omega_X \otimes P_1) \otimes p^* \alpha) &\cong H^0(a_*(\omega_X \otimes P_1) \otimes p^* \alpha) \otimes \wedge^i W \\ &\cong H^0(q^*(L_1 \otimes \alpha)) \otimes \wedge^i W \\ &\cong H^i(q^*L_1 \otimes \alpha). \end{aligned}$$

Pushing forward to  $Y$ , one obtains the required isomorphisms. □

CLAIM 4.11. If  $\bar{G} \neq \mathbb{Z}_2$ , then  $\bar{G} = (\mathbb{Z}_2)^2$  and

$$V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\alpha \cup S_\zeta \cup S_\xi$$

with  $\delta_\alpha = 2, \delta_\zeta = \delta_\xi = 1$ .

PROOF OF CLAIM 4.11. We have seen that  $V^0(\omega_X)$  has at most one 2-dimensional component and there are no parallel 1-dimensional components. Since  $\bar{G} \neq \mathbb{Z}_2$ , then there are at least two 1-dimensional components of  $V^0(\omega_X)$ . We will show that given two one dimensional components contained in  $Q_1 + \text{Pic}^0(Y) \neq Q_2 + \text{Pic}^0(Y)$ , then

$$(Q_1 + Q_2 + \text{Pic}^0(Y)) \cap V^0(\omega_X)$$

does not contain a 1-dimensional component. Grant this for the time being. Then, by Proposition 2.7, it follows that  $Q_1 + Q_2 + \text{Pic}^0(Y)$  is a 2-dimensional component of  $V^0(\omega_X)$ . If  $|\bar{G}| > 4$ , this implies that there are at least two 2-dimensional components, which is impossible, and so  $|\bar{G}| = 4$  and the claim follows.

Suppose now that there are three 1-dimensional components of  $V^0(\omega_X)$ , say  $S_1, S_2, S_3$ , contained in  $Q_1 + \text{Pic}^0(Y), Q_2 + \text{Pic}^0(Y), Q_3 + \text{Pic}^0(Y)$  respectively with  $Q_1 + Q_2 + Q_3 \in \text{Pic}^0(Y)$ . By Claim 4.10, these components are not parallel to each other. We may assume that  $\pi_i : X \rightarrow E_i := S_i^\vee$  factors through  $f : X \rightarrow Y$  and that  $Y$  is an abelian surface. Let  $q_i : Y \rightarrow E_i$  be the induced morphisms.

Let  $Q_1, Q_2, Q_3$  be general torsion elements in  $S_1, S_2, S_3$  and

$$\mathcal{G} := f_*(\omega_X^{\otimes 2} \otimes Q_2 \otimes Q_3), \quad \mathcal{F} := f_*(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3).$$

From the inclusions  $\pi_i^*L_i \rightarrow \omega_X \otimes Q_i$ , one sees that we have inclusions

$$\varphi : q_2^*L_2 \otimes q_3^*L_3 \rightarrow \mathcal{G}, \quad \psi : q_1^*L_1 \otimes q_2^*L_2 \otimes q_3^*L_3 \rightarrow \mathcal{F}$$

where  $L_i$  are ample line bundles on  $E_i$  respectively. Since  $\mathcal{F}$  is torsion free of generic rank one, we may write

$$\mathcal{F} = q_1^*L_1 \otimes q_2^*L_2 \otimes q_3^*L_3 \otimes N \otimes \mathcal{I}$$

where  $N$  is a semi-positive line bundle on  $Y$  and  $\mathcal{I}$  is an ideal sheaf cosupported at points. If  $N$  is not numerically trivial (or if  $F_{Y/E_1} \cdot q_i^* L_i > 1$  for  $i = 2$  or  $i = 3$ ), then  $N$  is not vertical with respect to one of the projections  $q_i$ , say  $q_1$ . Then

$$\text{rank}(q_{1,*}(\mathcal{F})) = F_{Y/E_1} \cdot (q_1^* L_1 + q_2^* L_2 + q_3^* L_3 + N) \geq 3.$$

On the other hand, from the inclusion  $\varphi$ , one sees that  $\text{rank}(q_{1,*}(\mathcal{G})) \geq 2$ . Consider the inclusion of I.T. 0 sheaves  $L_1 \rightarrow q_{1,*}(\mathcal{G} \otimes \alpha)$  with  $\alpha = Q_1 \otimes Q_2^\vee \otimes Q_3^\vee \in \text{Pic}^0(Y)$ . Since it is not an isomorphism, one sees that

$$h^0(\mathcal{G}) = h^0(\mathcal{G} \otimes \alpha) > h^0(L_1) \geq 1.$$

From the inclusion

$$\rho : L_1 \otimes q_{1,*}(\mathcal{G}) \rightarrow q_{1,*}(\mathcal{F}) = \pi_{1,*}(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3)$$

one sees that by Riemann-Roch

$$h^0(\mathcal{G}) + \text{rank}(q_{1,*}(\mathcal{G})) \leq h^0(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3) = P_3(X)$$

and therefore

$$h^0(\mathcal{G}) = 2, \quad \text{rank}(q_{1,*}(\mathcal{G})) = 2.$$

In particular,  $\rho$  is an I.T. 0 isomorphism. So,  $\text{rank}(q_{1,*}(\mathcal{F})) = \text{rank}(q_{1,*}(\mathcal{G})) = 2$  which is a contradiction. Therefore, we have that

$$N \in \text{Pic}^0(Y) \quad \text{and} \quad q_2^* L_2 \cdot F_{Y/E_1} = q_3^* L_3 \cdot F_{Y/E_1} = 1.$$

Since  $\deg(L_i) = 1$ , one has  $q_i^* L_i \equiv F_{Y/E_i}$ . Since  $(q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3)^2 \geq 8$ , we have that  $q_2^* L_2 \cdot q_3^* L_3 \geq 2$ . Since

$$h^0(q_2^* L_2 \otimes q_3^* L_3) \leq h^0(\mathcal{G}) = 2,$$

one sees that  $q_2^* L_2 \cdot q_3^* L_3 = 2$  and hence  $\mathcal{I} = \mathcal{O}_Y$ .

Now let  $\mathcal{G}' := f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_3)$ . Proceeding as above, one sees that

$$\text{rank}(q_{2,*} \mathcal{G}') \geq F_{Y/E_2} \cdot (q_1^* L_1 + q_3^* L_3) = 3, \quad h^0(q_{2,*} \mathcal{G}') > h^0(L_2) = 1.$$

By Riemann Roch, one has that

$$P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3) \geq h^0(L_2 \otimes q_{2,*} \mathcal{G}') \geq 5$$

which is the required contradiction. □

CLAIM 4.12. If  $\bar{G} \cong (\mathbb{Z}_2)^2$ , then  $Y = E_1 \times E_2$  and there are line bundles  $L_i$  of degree 1 on  $E_i$ , projections  $p_i : A \rightarrow E_i$  and 2-torsion elements  $Q_1, Q_2 \in \text{Pic}^0(X)$  that generate  $\bar{G}$ , such that

$$a_*(\mathcal{O}_X) \cong \mathcal{O}_A \oplus M_1^\vee \oplus M_2^\vee \oplus M_1^\vee \otimes M_2^\vee$$

with

$$M_1 = p_1^*L_1 \otimes Q_1^\vee, \quad M_2 = p_2^*L_2 \otimes Q_2^\vee \quad \text{and} \quad M_3 = M_1 \otimes M_2.$$

In particular  $X$  is birational to the fiber product of two degree 2 coverings  $X_i \rightarrow A$  with  $P_3(X_i) = 2$ .

PROOF OF CLAIM 4.12. By Claim 4.11, the degree of  $a : X \rightarrow A$  is  $|\bar{G}| = 4$  and there are two non parallel 1-dimensional components of  $V^0(\omega_X)$  say  $S_1, S_2$  such that  $S_1 + \text{Pic}^0(Y) \neq S_2 + \text{Pic}^0(Y)$ . Let  $E_i := T_i^\vee$  and  $q_i : Y \rightarrow E_i$ ,  $\pi_i : X \rightarrow E_i$  be the induced morphisms. Then there are inclusions  $\pi_i^*L_i \rightarrow \omega_X \otimes Q_i$  where  $Q_i \in S_i$ . Moreover, by Claim 4.11,  $Q_1 + Q_2 + \text{Pic}^0(Y) \subset V^0(\omega_X)$ . By Claim 4.9, one has that

$$L := f_*(\omega_X \otimes Q_1 \otimes Q_2)$$

is an ample line bundle of degree 1. Moreover,

$$V^0(\omega_X) = \{\mathcal{O}_X\} \cup S_1 \cup S_2 \cup (Q_1 + Q_2 + \text{Pic}^0(Y)).$$

From the inclusion

$$q_1^*L_1 \otimes q_2^*L_2 \otimes L \rightarrow f_*(\omega_X^{\otimes 3} \otimes Q_1^{\otimes 2} \otimes Q_2^{\otimes 2})$$

and the equality  $4 = P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q_1^{\otimes 2} \otimes Q_2^{\otimes 2})$ , one sees that

$$L^2 = 2, \quad L \cdot q_i^*L_i = q_1^*L_1 \cdot q_2^*L_2 = 1.$$

By the Hodge Index Theorem, one sees that since

$$L^2(q_1^*L_1 + q_2^*L_2)^2 = (L \cdot (q_1^*L_1 + q_2^*L_2))^2$$

then the principal polarization  $L$  is numerically equivalent to  $q_1^*L_1 + q_2^*L_2$ . Therefore,

$$(Y, q_1^*L_1 \otimes q_2^*L_2) \cong (E_1, L_1) \times (E_2, L_2),$$

and one sees that

$$L = q_1^*(L_1 \otimes P_1) \otimes q_2^*(L_2 \otimes P_2), \quad P_i \in \text{Pic}^0(E_i).$$

We have inclusions

$$L \rightarrow f_*(\omega_X \otimes Q_1 \otimes Q_2) \rightarrow f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2),$$

$$q_1^*L_1 \otimes q_2^*L_2 \rightarrow f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2).$$

Let  $\mathcal{G} := \omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2$ . If  $h^0(\mathcal{G}) = 1$ , then  $L = q_1^* L_1 \otimes q_2^* L_2$  as required. If  $h^0(\mathcal{G}) \geq 2$ , then one sees that

$$h^0(\pi_{1,*}(\mathcal{G}) \otimes L_1 \otimes P_1) \geq \text{rank}(\mathcal{G}) + \text{deg}(\mathcal{G}) \geq 1 + 2.$$

Since

$$\text{rank}(\pi_{2,*}(\mathcal{G} \otimes \pi_1^*(L_1 \otimes P_1))) \geq \text{rank}(q_{2,*}(q_1^*(L_1^{\otimes 2} \otimes P_1) \otimes q_2^*(L_2))) = 2,$$

one sees that

$$P_3(X) \geq h^0(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2 \otimes L) = h^0(\pi_{2,*}(\mathcal{G} \otimes \pi_1^*(L_1 \otimes P_1)) \otimes L_2 \otimes P_2) \geq 2 + 3$$

and this is impossible. Let  $M_i := p_i^* L_i \otimes Q_i^\vee$ . By Claim 4.10, one has

$$a_*(\omega_X) \cong \mathcal{O}_A \oplus M_1 \oplus M_2 \oplus M_1 \otimes M_2$$

and hence by Groethendieck duality,

$$a_*(\mathcal{O}_X) \cong \mathcal{O}_A \oplus M_1^\vee \oplus M_2^\vee \oplus M_1^\vee \otimes M_2^\vee.$$

Let  $X \rightarrow Z \rightarrow A$  be the Stein factorization. Following [HM] Section 7, one sees that the only possible nonzero structure constants defining the  $4 - 1$  cover  $Z \rightarrow A$  are  $c_{1,4} \in H^0(M_1 \otimes M_2 \otimes M_3^\vee)$ ,  $c_{1,6} \in H^0(M_1 \otimes M_2^\vee \otimes M_3)$  and  $c_{4,6} \in H^0(M_1^\vee \otimes M_2 \otimes M_3)$ . So,  $Z \rightarrow A$  is a bi-double cover. It is determined by two degree 2 covers  $a_i : X_i \rightarrow A$  defined by  $a_{i,*}(\mathcal{O}_{X_i}) = \mathcal{O}_A \oplus p_i^* L_i \otimes Q_i^\vee$  and sections  $-c_{1,4} c_{1,6} \in H^0(M_1^{\otimes 2})$  and  $c_{1,4} c_{4,6} \in H^0(M_2^{\otimes 2})$ . It is easy to see that  $X_1, X_2, Z$  are smooth. □

This completes the proof. □

**5. – Varieties with  $P_3(X) = 4$ ,  $q(X) = \dim(X)$  and  $\kappa(X) = 1$**

**THEOREM 5.1.** *Let  $X$  be a smooth projective variety with  $P_3(X) = 4$ ,  $q(X) = \dim(X)$  and  $\kappa(X) = 1$  then  $X$  is birational to  $(C \times \tilde{K})/G$  where  $G$  is an abelian group acting faithfully by translations on an abelian variety  $\tilde{K}$  and faithfully on a curve  $C$ . The Iitaka fibration of  $X$  is birational to  $f : (C \times \tilde{K})/G \rightarrow C/G = E$  where  $E$  is an elliptic curve and  $\dim H^0(C, \omega_C^{\otimes 3})^G = 4$ .*

**PROOF.** Let  $f : X \rightarrow Y$  be the Iitaka fibration. Since  $\kappa(X) = 1$ , and  $a : X \rightarrow A$  is generically finite, one has that  $Y$  is a curve of genus  $g \geq 1$ . If  $g = 1$ , then  $Y$  is an elliptic curve and by Proposition 2.1,  $Y \rightarrow A(Y)$  is of degree 1 (i.e. an isomorphism). By Proposition 2.1 one sees that if  $g \geq 2$ , then  $q(X) \geq \dim(X) + 1$  which is impossible.

From now on we will denote the elliptic curve  $A(Y)$  simply by  $E$  and  $f : X \rightarrow E$  will be the corresponding algebraic fiber space. Let  $X \rightarrow \tilde{X} \rightarrow A$  be the Stein factorization of the Albanese map. Since  $\tilde{X} \rightarrow E$  is isotrivial, there is a generically finite cover  $C \rightarrow E$  such that  $\tilde{X} \times_E C$  is birational to  $C \times \tilde{K}$ . We may assume that  $C \rightarrow E$  is a Galois cover with group  $G$ .  $G$  acts by translations on  $\tilde{K}$  and we may assume that the action of  $G$  is faithful on  $C$  and  $\tilde{K}$ . Since  $G$  acts freely on  $C \times \tilde{K}$ , one has that

$$H^0(X, \omega_X^{\otimes 3}) = H^0(C \times \tilde{K}, \omega_{C \times \tilde{K}}^{\otimes 3})^G = [H^0(\tilde{K}, \omega_{\tilde{K}}^{\otimes 3}) \otimes H^0(C, \omega_C^{\otimes 3})]^G.$$

Since  $G$  acts on  $\tilde{K}$  by translations,  $G$  acts on  $H^0(\tilde{K}, \omega_{\tilde{K}}^{\otimes 3})$  trivially. It follows that

$$4 = P_3(X) = \dim H^0(C, \omega_C^{\otimes 3})^G.$$

Similarly, one sees that  $q(X) = q(C/G) + q(\tilde{K}/G)$  and so  $q(C/G) = 1$ .  $\square$

We now consider the induced morphism  $\pi : C \rightarrow C/G =: E$ . By the argument of [Be], Example VI.12, one has

$$4 = \dim H^0(C, \omega_C^{\otimes 3})^G = h^0\left(E, \mathcal{O}\left(\sum_{P \in E} \left[3\left(1 - \frac{1}{e_P}\right)\right]\right)\right).$$

Where  $P$  is a branch points of  $\pi$ , and  $e_P$  is the ramification index of a ramification point lying over  $P$ . Note that  $|G| = e_P s_P$ , where  $s_P$  is the number of ramification points lying over  $P$ .

It is easy to see that since

$$\left[3\left(1 - \frac{1}{e_P}\right)\right] = 1 \text{ (resp. } = 2) \text{ if } e_P = 2 \text{ (resp. } e_P \geq 3),$$

we have the following cases:

CASE 1. 4 branch points  $P_1, \dots, P_4$  with  $e_{P_i} = 2$ .

CASE 2. 3 branch points  $P_1, P_2, P_3$  with  $e_{P_1} \geq 3, e_{P_2} = e_{P_3} = 2$ .

CASE 3. 2 branch points  $P_1, P_2$  with  $e_{P_i} \geq 3$ .

We will follow the notation of [Pa]. Let  $\pi : C \rightarrow E$  be an abelian cover with abelian Galois group  $G$ . There is a splitting

$$\pi_* \mathcal{O}_C = \bigoplus_{\chi \in G^*} L_\chi^\vee.$$

In particular, if  $d_\chi := \deg(L_\chi)$ , then

$$g = 1 + \sum_{\chi \in G^*, \chi \neq 1} d_\chi.$$



For every branch point  $P_i$  with  $i = 1, \dots, s$ , the inertia group  $H_i$ , which is defined as the stabilizer subgroup at any point lying over  $P_i$ , is a cyclic subgroup of order  $e_i := e_{P_i}$ . We also associate a generator  $\psi_i$  of each  $H_i^*$  which corresponds to the character of  $P_i$ . For every  $\chi \in G^*$ ,  $\chi|_{H_i} = \psi_i^{n(\chi)}$  with  $0 \leq n(\chi) \leq |H_i| - 1$ . And define

$$\epsilon_{\chi, \chi'}^{H_i, \psi_i} := \left\lfloor \frac{n(\chi) + n(\chi')}{|H_i|} \right\rfloor.$$

Following [Pa], one sees that there is an abelian cover  $C \rightarrow E$  with group  $G$  with building data  $L_\chi$  if and only if the line bundles  $L_\chi$  satisfy the following set of linear equivalences:

$$(3) \quad L_\chi + L_{\chi'} = L_{\chi\chi'} + \sum_{i=1, \dots, s} \epsilon_{\chi, \chi'}^{H_i, \psi_i} P_i.$$

If  $\chi|_{H_i} = \psi_i^{n_i(\chi)}$ , then

$$(4) \quad d_\chi + d_{\chi'} = d_{\chi\chi'} + \sum_{i=1, \dots, s} \left\lfloor \frac{n_i(\chi) + n_i(\chi')}{e_i} \right\rfloor.$$

Let  $H$  be the subgroup of  $G$  generated by the inertia subgroups  $H_i$  and let  $Q = G/H$ . One sees that there is an exact sequence of groups

$$1 \longrightarrow Q^* \longrightarrow G^* \longrightarrow H^* \longrightarrow 1.$$

The generators  $\psi_i$  of  $H_i^*$  define isomorphisms  $H_i^* \cong \mathbb{Z}_{e_i}$  where  $e_i := |H_i|$ . Therefore, we have an induced injective homomorphism

$$\varphi : H^* \hookrightarrow \prod_{i=1, \dots, s} \mathbb{Z}_{e_i}$$

such that the induced maps  $\varphi_i : H^* \rightarrow \mathbb{Z}_{e_i}$  are surjective. By abuse of notation, we will also denote by  $\varphi$  the induced homomorphism  $\varphi : G^* \rightarrow \prod_{i=1, \dots, s} \mathbb{Z}_{e_i}$ . We will write

$$\varphi(\chi) = (n_1(\chi), \dots, n_s(\chi)) \quad \forall \chi \in G^*.$$

Let  $\mu(\chi)$  be the order of  $\chi$ . By [Pa] Proposition 2.1,

$$d_\chi = \sum_{i=1, \dots, s} \frac{n_i(\chi)}{e_i}.$$

We will now analyze all possible inertia groups  $H$ .

CASE 1:  $s = 4$ , and  $e := e_i = 2$ . Then  $H^* \subset \mathbb{Z}_2^4$ . Note that  $H^* \neq \mathbb{Z}_2^4$  since  $(1, 0, 0, 0) \notin H^*$ . Thus  $H^* \cong (\mathbb{Z}_2)^s$  with  $1 \leq s \leq 3$ .

By Example 1, all of these possibilities occur.

CASE 2:  $s = 3$  and  $e_1 \geq 3$ ,  $e_2 = e_3 = 2$ . There must be a character  $\chi$  with  $\varphi(\chi) = (1, n_2, n_3)$ , and so

$$d_\chi = \frac{1}{e_1} + \frac{n_2}{2} + \frac{n_3}{2}$$

which is not an integer. Therefore this case is impossible.

CASE 3:  $s = 2$  and  $e_1, e_2 \geq 3$ . Assume that  $e_1 > e_2$ . Since  $G^* \rightarrow \mathbb{Z}_{e_1}$  is surjective, there is  $\chi \in H^*$  with  $\varphi(\chi) = (1, n_2)$ . Then

$$d_\chi = \frac{1}{e_1} + \frac{n_2}{e_2} < 1$$

which is impossible. So we may assume that  $e = e_1 = e_2 \geq 3$  and  $H^* \subset \mathbb{Z}_e^2$ . Let  $\varphi(\chi) = (n_1, n_2)$ . One has  $d_\chi = \frac{n_1+n_2}{e}$ . Thus  $n_2 = e - n_1$  for any  $\chi \neq 1$ . Therefore,  $H^* = \{(i, e - i) | 0 \leq i \leq e - 1\} \cong \mathbb{Z}_e$ . By Example 1, all of these possibilities occur.

From the above discussion, it follows that:

**PROPOSITION 5.2.** *Let  $\phi : C \rightarrow E$  be a  $G$ -cover with  $E$  an elliptic curve and  $\dim H^0(\omega_C^{\otimes 3})^G = 4$ . Then either  $\phi$  is ramified over 4-points and the inertia group  $H$  is isomorphic to  $(\mathbb{Z}_2)^s$  with  $s \in \{1, 2, 3\}$  or  $\phi$  is ramified over 2-points and the inertia group  $H$  is isomorphic to  $\mathbb{Z}_m$  with  $m \geq 3$ .*

## REFERENCES

- [Be] A. BEAUVILLE, “Complex Algebraic Surfaces”, London Math. Soc. Student Texts 34, Cambridge University Press, 1996.
- [CH1] J. A. CHEN – C. D. HACON, *Characterization of Abelian Varieties*, Invent. Math. **143** (2001), 435-447.
- [CH2] J. A. CHEN – C. D. HACON, *Pluricanonical maps of varieties of maximal Albanese dimension*, Math. Ann. **320** (2001), 367-380.
- [CH3] J. A. CHEN – C. D. HACON, *On Algebraic fiber spaces over varieties of maximal Albanese dimension*, Duke Math. J. **111** (2002), 159-175.
- [CIH] H. CLEMENS – C. D. HACON, *Deformations of the trivial line bundle and vanishing theorems*, Amer. J. Math. **124** (2002), 769-815.
- [EL] L. EIN – R. LAZARSEFELD, *Singularities of theta divisors and birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), 243-258.

- [GL] M. GREEN – R. LAZARSFELD, *Deformation theory, generic vanishing theorems and some conjectures of Enriques*, Catanese and Beauville, Invent. Math. **90** (1987), 389-407.
- [Hac] C. D. HACON, *Fourier transforms, generic vanishing theorems and polarizations of abelian varieties*, Math. Z. **235** (2000), 717-726.
- [Hac2] C. D. HACON, *Varieties with  $P_3 = 3$  and  $q = \dim(X)$* , to appear in Math. Nachr.
- [Hac3] C. D. HACON, *Divisors on principally polarized varieties*, Compositio Math. **119** (1999), 321-329.
- [Hac4] C. D. HACON, *Effective criteria for birational morphisms*, J. London Math. Soc. **67** (2003), 337-348.
- [Hac5] C. D. HACON, *A derived category approach to generic vanishing theorems*, to appear in J. Reine Angew. Math.
- [HP] C. D. HACON – R. PARDINI, *On the birational geometry of varieties of maximal albanese dimension*, J. Reine Angew. Math. **546** (2002), 177-199.
- [Ka] Y. KAWAMATA, *Characterization of abelian varieties*, Compositio Math. **43** (1981), 253-276.
- [Ko1] J. KOLLÁR, *Higher direct images of dualizing sheaves I*, Ann. of Math. **123** (1986), 11-42.
- [Ko2] J. KOLLÁR, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math. **113** (1993), 177-215.
- [HM] D. W. HAHN – D. MIRANDA, *Quadruple covers of algebraic varieties*, J. Algebraic Geom. **8** (1999), 1-30.
- [M] S. MUKAI, *Duality between  $D(X)$  and  $D(\hat{X})$  with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153-175.
- [Pa] R. PARDINI, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191-213.

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